CSC411H1 , HW1

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①

a. Let $X, Y \sim \text{Uniform}(0,1)$.

Then, $E(X) = E(Y) = 1/2$.
$\text{Var}(X) = \text{Var}(Y) = 1/12$.

Let $Z = (X-Y)^2$.

Then, $E(Z) = E([X-Y]^2) = E(X^2+Y^2-2XY) =$

$= E(X^2) + E(Y^2) - 2E(XY) =$

$= \underbrace{(E(X))^2 + \text{Var}(X)}_{\text{From formula for variance}} + \underbrace{(E(Y))^2 + \text{Var}(Y)}_{\text{Same as } \leftarrow} - \underbrace{2E(X)E(Y)}_{X, Y \text{ are ind.}} =$

$= \frac{1}{4} + \frac{1}{12} + \frac{1}{4} + \frac{1}{12} - 2 \cdot \frac{1}{2} \cdot \frac{1}{2} = 1/6.$

$\text{Var}(Z) = E([Z - E(Z)]^2) = E([Z - \frac{1}{6}]^2) =$

$= E(Z^2 + \frac{1}{36} - \frac{2}{6}Z) = \frac{1}{36} - \frac{2}{6}E(Z) + E(Z^2) =$

$= -\frac{1}{36} + E([X-Y]^2)^2) =$

$= -\frac{1}{36} + E(X^4 - 4X^3Y + 6X^2Y^2 - 4XY^3 + Y^4) =$

$= -\frac{1}{36} + E(X^4) - 4E(X^3Y) + 6E(X^2Y^2) - 4E(XY^3) + E(Y^4)$

By properties of expectations,

$$E(X^4) = E(Y^4) = \int_{-\infty}^{\infty} x^4 f(x)\,dx, \quad \text{where } f(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$$

therefore, $E(X^4) = \int_0^1 x^4\,dx = \frac{x^5}{5}\Big|_0^1 = \frac{1}{5}$.

Similarly, $E(X^3) = E(Y^3) = \int_0^1 x^3\,dx = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$.

The finaly expression can be reduced to

$$Var(Z) = -\frac{1}{36} + \frac{1}{5} - 4\frac{1}{4}\cdot\frac{1}{2} + 6\cdot\frac{1}{3}\cdot\frac{1}{3} - 4\cdot\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{5} =$$

$$= \frac{7}{180}$$

Then $E(Z) = 1/6$ and $Var(Z) = 7/180$

b.

Let X_i, Y_i be independent uniform random variable sampled from $[0,1)$, for $i = 1, 2, \dots, d$.

Let $R = Z_1 + Z_2 + \cdots + Z_d$, where $Z_i = (X_i - Y_i)^2$.

Then, $E(R) = E\left(\sum_{i=1}^{d} Z_i\right) = \sum_{i=1}^{d} E(Z_i) = \underbrace{\frac{1}{6} + \frac{1}{6} + \cdots + \frac{1}{6}}_{d \text{ times}} =$

$= d/6$.

$$\text{Var}(R) = E\left(\left[R - E(R)\right]^2\right) = E\left(R^2 + \frac{d^2}{36} - \frac{2d}{6}R\right) =$$

$$= \frac{d^2}{36} - \frac{d^2}{18} + E(R^2) = -\frac{d^2}{36} + E(R^2) =$$

$$= -\frac{d^2}{36} + E\left(\left[Z_1 + \cdots + Z_d\right]^2\right) =$$

$$= -\frac{d^2}{36} + \underbrace{\sum_{i=1}^{d} E(Z_i^2) + 2\sum_{i \neq j}^{d} E(Z_i Z_j)}_{\text{Expanding the square of a poly.}}$$

$$= -\frac{d^2}{36} + \underbrace{\sum_{i=1}^{d}\left(\left[E(Z_i)\right]^2 + \text{Var}(Z_i)\right)}_{\text{Formula for Variance}} + 2\underbrace{\sum_{i \neq j}^{d} E(Z_i)E(Z_j)}_{\text{Independence.}}$$

$$= -\frac{d^2}{36} + \sum_{i=1}^{d}\left(\frac{1}{36} + \frac{7}{180}\right) + 2\sum_{i \neq j}^{d} \frac{1}{36}$$

$$= -\frac{d^2}{36} + \frac{d}{15} + 2\sum_{i \neq j}^{d} \frac{1}{36}$$

Now notice that $\sum_{i \neq j}^{d}$ just means to select two different indices from 1 to d. We can do this in $\binom{d}{2}$ ways.

$$\text{Var}(R) = \frac{d}{15} - \frac{d^2}{36} + 2\binom{d}{2}\cdot\frac{1}{36} = \frac{d}{15} - \frac{d^2}{36} + \frac{d!}{(d-2)!\,2!}\cdot\frac{2}{36} =$$

$$= \frac{d}{15} - \frac{d^2}{36} + \frac{d(d-1)}{36} = \frac{12d - 5d^2 + 5d^2 - 5d}{180} = \frac{7d}{180}$$

Then, $E(R) = d/6$ and $\text{Var}(R) = 7d/180$