# Math 3302: Practice Exam 2

Oct, 2018

Name: ________________________

*On my honor, I have neither given nor received aid on this exam*

Signature: ________________________

**The following will be the instructions on your in-class exam (although the number of questions may vary):**

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- No calculators or other electronic devices are permitted without the explicit consent of the instructor. Please turn your cell phone off and keep it out of sight at all times
- This exam is open book and open notes.
- Circle or otherwise indicate your final answers.
- This test has x problems and is worth xxx points. It is your responsibility to make sure that you have all of the pages!

| # | Score |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 | |
| 5 | |
| 6 | |
| Total | |

**1.** Given $f(x,y) = x^2y^3$, find the equation of the tangent plane at $(1,1,1)$.

**2.** The following equation implicitely defines a surface $z = f(x,y)$. Find the equation of the tangent plane to this surface at the point $(1,-1,3)$

$$x^3 + y^3 + z^3 + 9xyz = \ln(x).$$

**3.** Consider the sphere $x^2 + y^2 + z^2 = 9$. Find the equation of the plane tangent to this sphere at

**(a)** $(-3,0,0)$

**(b)** $(2,2,1)$

**4.** Estimate $f(0.55,-0.01)$ if $f(x,y) = \sin\left(\pi\left(x^2 + xy\right)\right)$.

**5.** Suppose that $f(x,y) = e^{x-y}$ and $f(\ln 2, \ln 2) = 1$. Use the technique of linear approximation to estimate $f(\ln 2 + 0.1, \ln 2 + 0.04)$.

**6.** Let $w$ be a function of $x$, $y$, and $z$:

$$w = f(x,y,z).$$

Also assume that

$$\begin{aligned} x &= x(r,s) \\ y &= y(s,t) \\ z &= z(r,t) \end{aligned}$$

Write down expressions for $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial s}$, $\frac{\partial w}{\partial t}$.

**7.** Let $\phi$ be a function of the cartesian coordinates $x$ and $y$, and $z$:

$$\phi = x^2 + y^2.$$

**(a)** Compute the partial derivatives $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$.

**(b)** Now, suppose that $x$ and $y$ depend on underlying coordinates $(u,v)$ according to

$$x = 2u + v, \qquad y = u - 2v.$$

Compute the partial derivatives $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$, in terms of $u$ and $v$ only.

**(c)** Finally, suppose both $u$ and $v$ depend on $t$ by means of the relationships

$$u = 1 - t, \qquad v = t.$$

Compute the full derivative $\frac{d\phi}{dt}$, in terms of $t$ only.

**8.** If a sound with frequency $f_s$ is produced by a source traveling along a line with speed $v_s$ and an observer is traveling with speed $v_o$ along the same line from the opposite direction toward the source, then the frequency of the sound heard by the observer is

$$f_o = \left(\frac{c + v_o}{c - v_s}\right) f_s,$$

where $c$ is the speed of sound about 332m/s. (This is the Doppler effect.) Suppose that, at a particular moment, you are in a train traveling at 34 m/s and accelerating at 1.2 m/$s^2$. A train is approaching you from the opposite direction on the other track at 40 m/s, accelerating at 1.4 m/$s^2$, and sounds its whistle, which has frequency of 460 Hz. At that instant what is perceived frequency that you hear and how fast is changing? *Hint: speed is a function of time, and the acceleration is the rate of change of the speed w.r.t time.*

**9.** If $u = f(x, y)$, where $x = e^s \cos t$ and $y = e^s \sin t$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = e^{-2s}\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right].$$

**10.** Let

$$f(x, y) = x^2 + xy + y^2.$$

**(a)** Compute the gradient at the point $P = (2, 3)$.

**(b)** At the point $P$, what is the directional derivative in the direction $\mathbf{v} = \langle 3, 4\rangle$?

**(c)** At the point $P$, what is the equation of the linearization / tangent plane?

**(d)** Approximately, what is the value $f(1.9, 3.1)$?

**11.** Let

$$z = f(x, y) = xye^{x+y^2}$$

**(a)** Compute the gradient at the point $P(2, 1)$.

**(b)** At the point $P$, what is the directional derivative in the direction $\vec{\mathbf{v}} = \langle 1, 2\rangle$?

**(c)** At the point $P$, what is the equation of the tangent plane/linear approximation?

**12.** Find the directional derivative of the function $f(x, y, z) = \frac{x}{y+z}$ at the point $(4, 1, 1)$ in the direction of the vector $\langle 1, 2, 3\rangle$. In what direction is the derivative minimized?

**13.** Suppose $\vec{u} = \langle 1, 0\rangle$, $\vec{v} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle$, $D_{\vec{u}}(f(a, b)) = 3$ and $D_{\vec{v}}(f(a, b)) = \sqrt{2}$.

**(a)** Find $\nabla f(a, b)$.

**(b)** What is the maximum possible value of $D_{\vec{w}}(f(a, b))$ for any $\vec{w}$?

**(c)** Find a unit vector $\vec{w} = \langle w_1, w_2\rangle$ such that $D_{\vec{w}}(f(a, b)) = 0$.

**14.** The temperature at a point $(x, y, z)$ is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2},$$

where $T$ is measured in $^\circ C$ and $x, y, z$ in meters.

**(a)** Find the rate of change of temperature at the point $P(2, -1, 2)$ in the direction toward the point $(3, -3, 3)$.

**(b)** In which direction does the temperature increase fastest at $P$?

**(c)** Find the maximum rate of increase at $P$.

**15.** The temperature in a region of the ocean is given by $T(x, y, z) = 20 + \sin(x) - \cos(y) - \ln(z + 1)$, where $T$ is measured in degrees Celsius, and $x$, $y$ and $z$ are measured in meters. If a fish is currently at the point $(0, \pi, 10)$, in which direction must the fish swim to decrease its temperature the fastest?

**16.** Find the absolute maximum and minimum values of $f(x, y) = x^2y$ on the set $D = \left\{(x, y) \,|x \geq 0, y \geq 0, x^2 + y^2 \leq 3\right\}$.

**17.** Find the maximum and minimum values of $f(x, y, z) = xyz$ over the region $x^2 + y^2 + z^2 \leq 3$.

**18.** Find the dimensions of the rectangular box with largest volume if the total surface are is given as $64\,cm^2$.

**19.** Write the system of equations you would solve to answer the following problem -- *do not solve these equations*. Find the points on the surface $z^2 - 3xy + 4x^2z = 6$ that are closest to the point $(2, 1, -3)$.

**20.** It is often necessary to decide among a variety of products with various costs $C$ and qualities $Q$. Typically, the function $Q = h(C)$ is concave down, meaning that increasingly expensive products bring increasingly small gains in quality (this is an application of the principle of diminishing returns). In such situations, a common purchasing strategy is to select not the product with the absolute highest quality, but rather the product with the greatest measure of quality per dollar spent. This problem can be posed as a constrained maximization problem: we seek to maximize the function

$$f = Q/C$$

subject to the constraint

$$g = Q - h(C) = 0.$$

What is the optimum amount of money to spend on a smartphone if

$$h(C) = \ln\left(\frac{C}{100}\right)?$$

**21.** Which point on the surface $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$, $x, y, z > 0$ is closest to the origin?

**22.** Suppose you are studying for two final exams. You learn two different subjects at different rates, and your total grade points for the two classes can be described by the function

$$G(x, y) = \ln(1 + 2x) + \ln(1 + 4y),$$

where $\{x, y\}$ are the lengths of time spent on each subject. In addition, the total number of days you have available is capped at 2:

$$T(x, y) = x + y = 2.$$

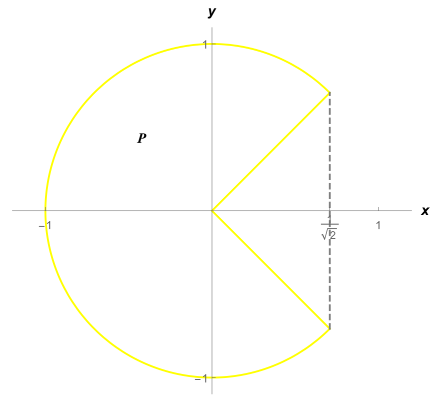
How many hours should you spend studying for each class?

**23.** Evaluate $\iint_R y\sin(xy)\,\mathrm{d}A$, where $R = [0, 1] \times [0, \pi]$.

**24.** Set up the iterated integral to compute the volume of the solid in the first octant under $z = 10+2e^x\cos(y)$ and above the region in the $xy$-plane bounded by the ellipse $4x^2 + y^2 = 4$. *Do not evaluate the integral.*

**25.** Consider the region enclosed by $y = x+1$, $y = -x+1$, and the $x-$axis. Compute the integral $\iint_R xydxdy$ using any method you know.

**26.** Observe the Pac-Man:



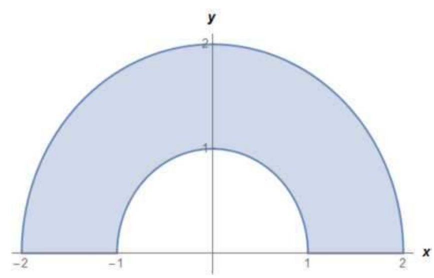

**(a)** Describe him in polar coordinates.

**(b)** Evaluate $\iint_{Pac-Man} x dA$ and $\iint_{Pac-Man} y dA$.

**27.** Consider the integral

$$\iint_R \frac{1}{9-(x^2+y^2)^{3/2}} dA$$

where $R$ is given by the region between the two semicircles pictured below:




**(a)** Compute the shaded area.

**(b)** Show that the function $\frac{1}{9-(x^2+y^2)^{3/2}}$ is constant on each of the two bounding semicircles.

**(c)** Give a lower bound and an upper bound for the double integral using the above information.