Math 3302: Exam 3 Practice problems

Nov 24, 2018

Name: ________________________

On my honor, I have neither given nor received aid on this exam

Signature: ________________________

The following will be the instructions on your in-class exam (although the number of questions may vary):

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- No calculators or other electronic devices are permitted without the explicit consent of the instructor. Please turn your cell phone off and keep it out of sight at all times.
- This exam is open book and open notes.
- Circle or otherwise indicate your final answers.
- This test has xx problems and is worth xxx points. It is your responsibility to make sure that you have all of the pages.
- Good luck!

#	Score
1	
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1. The boundary of a lamina consists of the semicircles $y = \sqrt{1-x^2}$ and $y = \sqrt{4-x^2}$, together with the portions of the x-axis that join them. If the density of the lamina is inversely proportional to its distance from the origin, find the location for its center of mass.

2. Bacteria is growing on a slide which corresponds to $[0, 20] \times [0, 5]$, where distances are in centimeters. If the density of the bacteria is determined to be

$$\rho(x, y) = 1000xy\,(20 - x)\,(5 - y)\ \frac{bacteria}{cm^2}$$

about how many bacteria are on the slide?

3. Rewrite the integral

$$\int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{z} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}y$$

in the form

$$\int_{?}^{?}\int_{?}^{?}\int_{?}^{?} f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

i.e. fill in the correct integration limits.

4. Set up the integral

$$\iiint_V x\,\mathrm{d}V,$$

where V is the tetrahedron bounded by the coordinate planes and the plane $6x + 3y + 2z = 6$. *Do not evaluate the integral, just set it up.*

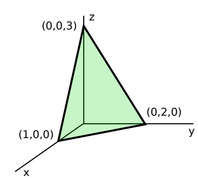

Figure 1: Problem 4

5. Set up the integral

$$\iiint_R (x^3z + xy^2z)\,\mathrm{d}V$$

in cylindrical coordinates, where R is the solid enclosed by the surface $z + x^2 + y^2 = 0$ and the plane $z = -9$. *Do not evaluate the integral, just set it up.*

6. What is the volume of the following region described in spherical coordinates: $1 \le \rho \le 3$, $0 \le \theta \le \frac{\pi}{2}$, $\frac{\pi}{6} \le \phi \le \frac{\pi}{4}$?

7. Find the gradient vector field of $f(x, y) = x\ln(y - 2z)$.

8. Which of the vector fields $\langle x, x-y\rangle$, $\langle y, x-y\rangle$, $\langle x, x+y\rangle$, and $\langle y, x+y\rangle$ describes the plot below?

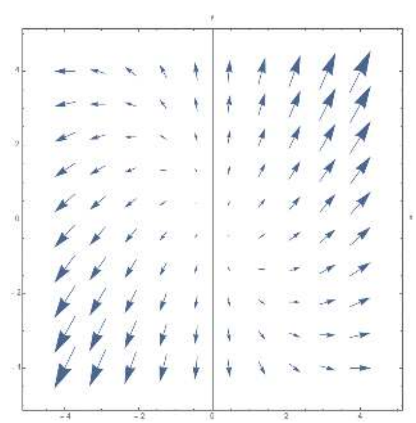

Figure 2: Problem 8

9. Find the work done by the force field $\vec{F}(x,y,z) = \langle z, x, y\rangle$ in moving a particle from the point $(3,0,0)$ to the point $(0,\pi/2,3)$ along

(a) a straight line

(b) the helix $x = 3\cos t$, $y = t$, $z = \sin t$

10. Show that $\vec{F}(x,y,z) = \langle e^y, xe^y + e^z, ye^z\rangle$ is conservative and use this fact to evaluate $\int_C \vec{F}\cdot d\vec{r}$ along the line segment from $(0,2,0)$ to $(4,0,3)$.

11. Consider the conservative vector field $\vec{F} = \left\langle 3y^2 - 4 + 4xy,\ 2 + 2x^2 + 6xy\right\rangle$.

(a) Find a scalar field f such that $\vec{F} = \nabla f$.

(b) Use your result from part (a) to compute

$$\int_C \vec{F}\cdot \mathrm{d}\vec{r},$$

where C is given by the space curve $\vec{r}(t) = \left\langle 2t^2,\ t-2\right\rangle$, with $0 \le t \le 1$.

(c) *Without using the fundamental theorem of line integrals*, compute

$$\int_C \vec{F}\cdot \mathrm{d}\vec{r}$$

as in part (b), i.e. you should use the standard approach for computing line integrals from section 13.2.

12. Compute

$$\int_C \left\langle y,\ -x,\ xy^2\right\rangle \cdot \mathrm{d}\vec{r},$$

where C is given by $\vec{r}(t) = \langle \sin(t),\ \cos(t),\ 3t\rangle$, $0 \le t \le 2$.

13. Use Green's theorem to evaluate

$$\int_C y^4\,\mathrm{d}x + 2xy^3\,\mathrm{d}y$$

where C is the triangle with line segments from $(0,0)$ to $(2,2)$, from $(2,2)$ to $(5,0)$, and from $(5,0)$ to $(0,0)$.

14. Let f be a scalar field and $\vec{F}$ be a vector field. State whether each expression is meaningful. If it is not meaningful explain why. If it is meaningful state whether the result is a scalar field or vector field.

(a)	$(\nabla f)\times(\nabla\cdot\vec{F})$	**(e)**	$\nabla\vec{F}$	**(i)**	$\nabla\cdot(\nabla\cdot\vec{F})$
(b)	$\nabla(\nabla\cdot\vec{F})$	**(f)**	$\nabla(\nabla\cdot f)$	**(j)**	$\nabla\cdot\vec{F}$
(c)	$\nabla\cdot(\nabla f)$	**(g)**	∇f	**(k)**	$\nabla\times(\nabla f)$
(d)	$\nabla\cdot(\nabla\times(\nabla f))$	**(h)**	$\nabla\times f$	**(j)**	$\nabla\times(\nabla\times\vec{F})$

15. Consider the surface defined by the portion of the cone

$$z = \sqrt{x^2+y^2}$$

that lies over the rectangle $(x,y)\in[0,2]\times[1,4]$.

(a) Find parametric equations for this surface.

(b) Compute the area of this surface.

16.(a) Show that the parametric equations $x = a\sin u\cos v, y = b\sin u\sin v,\ z = c\cos u,\ 0 \le u \le \pi,\ 0 \le v \le 2\pi$, represent an ellipsoid.

16.(b) Set up, *but do not evaluate*, a double integral for the surface area of the ellipsoid with $a = 1$, $b = 2$, and $c = 3$.

17. Let the surface $\mathcal{S}$ be the portion of the cylinder $x^2 + y^2 = 1$, which lies between the planes $z = 0$ and $z = 1 + y$, with normal pointing *away* from the z-axis. Let the curve $\mathcal{C}$ be the intersection of the cylinder with the second plane (as pictured). Finally, let the vector field

$$\vec{\mathbf{v}} = 2\,(x-y)\,\vec{i} + 2\,(x+y)\,\vec{j} + z\vec{k}.$$

Now, set up integrals that give

(a) the length of the curve $\mathcal{C}$.

(b) the work done by $\vec{\mathbf{v}}$ around $\mathcal{C}$.

(c) the area of the surface $\mathcal{S}$.

(d) the flux of $\vec{\mathbf{v}}$ through $\mathcal{S}$.

Simplify the integrands, but *do not evaluate* the integrals.

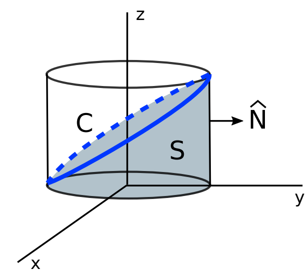

Figure 3: Problem 17

18. Evaluate the surface integral

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S},$$

where $\vec{F}(x, y, z) = \langle 0, z, z\rangle$ and S is the hemisphere $x = \sqrt{16 - y^2 - z^2}$, oriented in the direction of the positive x axis.

19. Use Stokes' Theorem to compute the line integral

$$\int_C \vec{F} \cdot \mathrm{d}\vec{r},$$

where $\vec{F}(x, y, z) = \langle xy,\ 2z,\ 3y\rangle$, and C is the curve of intersection of the plane $x + z = 5$ and the cylinder $x^2 + y^2 = 9$.

20. Use the Divergence Theorem to compute the surface integral

$$\iint_S \vec{F} \cdot \mathrm{d}\vec{S},$$

where $\vec{F}(x, y, z) = \left\langle x^2yz,\ xy^2z,\ xyz^2\right\rangle$ and S is the surface of the box enclosed by the planes $x = 0$, $x = 2$, $y = 0$, $y = 3$, $z = 0$ and $z = 5$.

21. Here you will show some interesting facts using integral theorems. Each of these demonstrations should take **only 2-3 lines**.

(a) A vector field $\vec{\mathbf{v}}$ is called *conservative* if it can be represented as the gradient of a scalar function ϕ:

$$\vec{\mathbf{v}} = \nabla\phi.$$

Here ϕ is called the *scalar potential.* Use a vector identify to show that conservative vector fields have no curl. Then, use an integral theorem to show that line integrals of conservative vector fields around any closed curve $\mathcal{C}$ are zero:

$$\oint_C \vec{\mathbf{v}} \cdot \vec{\mathbf{T}}\,\mathrm{d}s = 0$$

(b) A vector field $\vec{\mathbf{v}}$ is called *solenoidal* if it can be written as the curl of another vector field $\vec{\psi}$:

$$\vec{\mathbf{v}} = \nabla \times \vec{\psi}.$$

Here $\vec{\psi}$ is called the *vector potential*. Use a vector identity to show that solenoidal vector fields have no divergence. Then, use an integral theorem to show that surface integrals of solenoidal vector fields over any closed surface $\mathcal{S}$ are zero:

$$\oiint_{\mathcal{S}} \mathbf{v} \cdot \vec{\mathbf{N}} \, \mathrm{d}A = 0$$

22. Each of the following integrals is much simpler if you first apply an integral theorem. Compute all three values.

(a) $\oint_{\mathcal{C}} \vec{\mathbf{v}} \cdot \vec{\mathbf{T}} \, ds$, where $\mathcal{C}$ is the unit circle, and $\vec{\mathbf{v}} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

(b) $\oiint_{\mathcal{S}} \vec{\mathbf{n}} \cdot \vec{\mathbf{v}} \, dA$, where $\mathcal{S}$ is the surface of the unit sphere, and $\vec{\mathbf{v}} = 3z\vec{i} + 2y\vec{j} + x\vec{k}$.

(c) $\int_{\mathcal{S}} \vec{\mathbf{n}} \cdot (\nabla \times \vec{\mathbf{v}}) \, dA$, where $\mathcal{S}$ is the top half of the unit sphere, and $\vec{\mathbf{v}} = -y\vec{i} + x\vec{j}$.