Planar Vehicle Dynamics Notes

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1 Preface

This document serve as the derivation/documentation of the 15dof planar vehicle dynamics model as implemented in Simulink (+a derivation of a linear bicycle model, for reference).

Some notable difference between this and a typical EOM is that I use a kind of ”inverted SAE” convention. I.e. positive X is the forward longitudinal direction, positive Y is the port side lateral direction, and positive Z is pointing upwards. This was done because it felt more natural to me.

2 Linear Bicycle Model

2.1 Lateral Dynamics

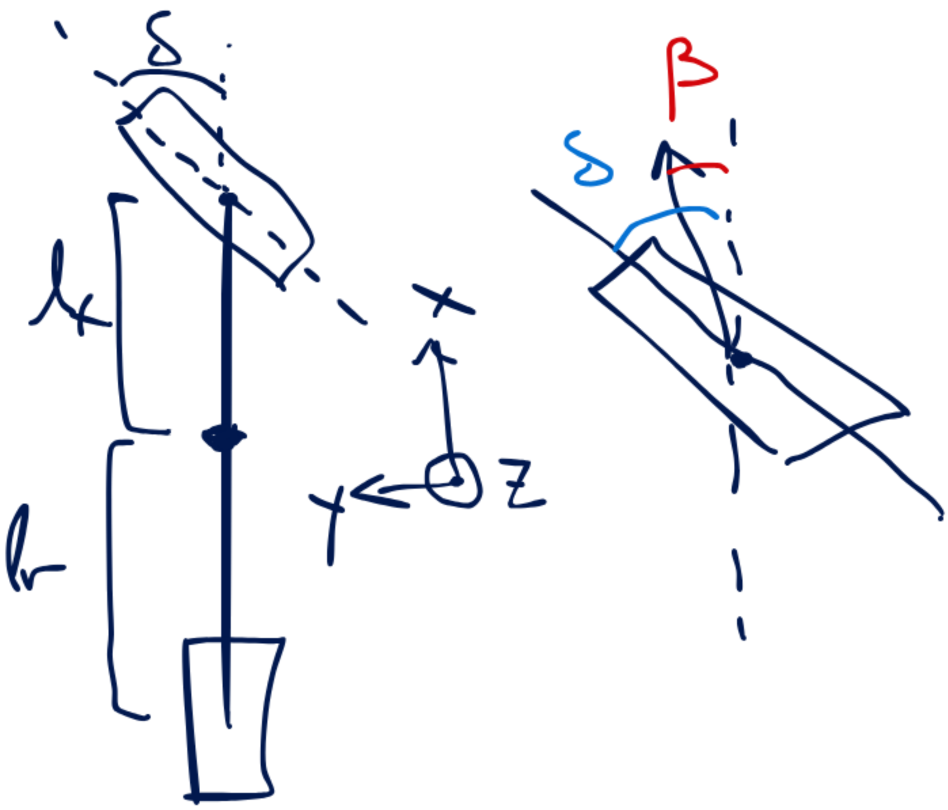

Figure 1. Linear Bicycle Model.

$$
\begin{aligned}
F_x &= m(\ddot{x} - \dot{\phi}\dot{y}) = F_{xf} + F_{xr} && \text{(included for completeness)} \\
F_y &= m(\ddot{y} + \dot{\phi}\dot{x}) = F_{yf} + F_{yr} \\
M_z &= I_z z\ddot{\phi} = F_{yf} l_f - F_{yr} l_r
\end{aligned}
$$

Note: If you are unsure about the $\dot{\phi}\dot{y}$ and the $\dot{\phi}\dot{x}$ terms in the above formula, refer to the appendix.

For the sake of linearity, the forces generated by the tire are represented using small angle approximation:

$$
\begin{aligned}
F_{yf} &= C_f(\delta - \frac{\dot{y} + \dot{\phi} l_f}{\dot{x}}) \\
F_{yr} &= C_r(0 - \frac{\dot{y} - \dot{\phi} l_r}{\dot{x}})
\end{aligned}
$$

where C is the cornering stiffness of the tire, assumed to be a constant and the same front and rear. Substituting back into the state equations:

$$
\begin{aligned}
F_y &= C_f(\delta - \frac{\dot{y} + \dot{\phi} l_f}{\dot{x}}) + C_r(-\frac{\dot{y} - \dot{\phi} l_r}{\dot{x}}) \\
F_y &= C_f\delta - (C_f + C_r)\frac{\dot{y}}{\dot{x}} - (C_f l_f - C_r l_r)\frac{\dot{\phi}}{\dot{x}} \\
M_z &= C_f(\delta - \frac{\dot{y} + \dot{\phi} l_f}{\dot{x}}) l_f - C_r(-\frac{\dot{y} - \dot{\phi} l_r}{\dot{x}}) l_r \\
M_z &= C_F \delta l_f - (C_f l_f + C_r l_r)\frac{\dot{y}}{\dot{x}} - (C_f l_f^2 + C_f l_r^2)\frac{\dot{\phi}}{\dot{x}}
\end{aligned}
$$

State space ($\dot{x}$ assumed constant):

$$
\begin{bmatrix} \ddot{y} \\ \ddot{\phi} \end{bmatrix} = \begin{bmatrix} \frac{-(C_f + C_r)}{m\dot{x}} & -\frac{(C_f l_f - C_r l_r)}{m\dot{x}} - \dot{x} \\ -\frac{(C_f l_f + C_r l_r)}{I_z \dot{x}} & -\frac{(C_f l_f^2 + C_f l_r^2)}{I_z \dot{x}} \end{bmatrix} \begin{bmatrix} \dot{y} \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} \frac{C_f}{m} \\ \frac{C_f l_f}{I_z} \end{bmatrix} [\delta]
$$

One of the more important information obtained from the linear bicycle model is the $-(C_f l_f^2 + C_f l_r^2)\frac{\dot{\phi}}{\dot{x}}$ term in the M_z equation. This term represents the "yaw damping" of the vehicle's cornering dynamics (since it is a negative moment proportional to the yaw velocity). This yaw damping dictates the over/under steer behavior of the vehicle. Notably this yaw damping is inversely proportional to the vehicle's longitudinal velocity $\dot{x}$, meaning the faster the car is going, the harder it is to induce turn the car.

Obviously this linear bicycle model is very much simplified and not at all representative of actual vehicle dynamics of a car, however lessons such as the yaw damping holds true even in a full vehicle model and this bicycle model can give a very simple yet insightful understanding of certain vehicle dynamic behaviours.

3 15dof Model

3.1 Degrees of Freedom

1. $\ddot{x}$ vehicle longitudinal acceleration
2. $\ddot{y}$ vehicle lateral acceleration
3. $\ddot{\phi}$ vehicle yaw acceleration
4. $Z\ddot{u}_{fl}$ unsprung mass front left acceleration
5. $Z\ddot{u}_{fr}$ unsprung mass front right acceleration
6. $Z\ddot{u}_{rl}$ unsprung mass rear left acceleration
7. $Z\ddot{u}_{rr}$ unsprung mass rear right acceleration
8. $Z\ddot{s}_{cg}$ sprung mass heave acceleration
9. $\ddot{\psi}_f$ front roll acceleration
10. $\ddot{\psi}_r$ rear roll acceleration
11. $\ddot{\rho}$ pitch acceleration
12. $\ddot{\theta}_{fl}$ front left wheel rotational acceleration
13. $\ddot{\theta}_{fr}$ front right wheel rotational acceleration
14. $\ddot{\theta}_{rl}$ rear left wheel rotational acceleration
15. $\ddot{\theta}_{rr}$ rear right wheel rotational acceleration

Vehicle equation of motion:

$$\ddot{x} = \frac{(F_{xb,fl} + F_{xb,fr} + F_{xb,rl} + F_{xb,rr})}{m} + \dot{\phi}\dot{y}$$

$$\ddot{y} = \frac{(F_{yb,fl} + F_{yb,fr} + F_{yb,rl} + F_{yb,rr})}{m} - \dot{\phi}\dot{x}$$

$$\ddot{\phi} = \frac{F_{xb,fl}(-r_{fl}) + F_{yb,fl}(l_f) + F_{xb,fr}(r_{fr}) + F_{yb,fr}(l_f) + F_{xb,rl}(-r_{rl}) + F_{yb,rl}(-l_r) + F_{xb,rr}(r_{rr}) + F_{yb,rr}(-l_r)}{I_{zz}}$$

Unsprung masses:

$$Z\ddot{u}_{fl} = \frac{-Zf_{fl} - K_{t,fl}(Zu_{fl} - Z0_{fl}) - C_{t,fl}(Z\dot{u}_{fl} - Z\dot{0}_{fl})}{m_{u,fl}}$$

$$Z\ddot{u}_{fr} = \frac{-Zf_{fr} - K_{t,fr}(Zu_{fr} - Z0_{fr}) - C_{t,fr}(Z\dot{u}_{fr} - Z\dot{0}_{fr})}{m_{u,fr}}$$

$$Z\ddot{u}_{rl} = \frac{-Zf_{rl} - K_{t,rl}(Zu_{rl} - Z0_{rl}) - C_{t,rl}(Z\dot{u}_{rl} - Z\dot{0}_{rl})}{m_{u,rl}}$$

$$\ddot{Zu}_{rr} = \frac{-Zf_{rr} - K_{t,rr}(Zu_{fl} - Z0_{rr}) - C_{t,rr}(\dot{Zu}_{rr} - \dot{Z0}_{rr})}{m_{u,rr}}$$

sprung mass roll, pitch, heave:

$$\ddot{Zs}_{cg} = \frac{Zf_{fl} + Zf_{fr} + Zf_{rl} + Zf_{rr} - m_s g}{m_s}$$

$$\ddot{\psi}_f = \frac{(Zf_{fl})(r_{fl}) + (Zf_{fr})(-r_{fr}) + M_{roll,f} - M_{cts}}{I_{xx,f}}$$

$$\ddot{\psi}_r = \frac{(Zf_{rl})(r_{rl}) + (Zf_{rr})(-r_{rr}) + M_{roll,r} + M_{cts}}{I_{xx,r}}$$

$$\ddot{\rho} = \frac{(Zf_{fl} + Zf_{fr})(-a) + (Zf_{rl} + Zfrr)(b) + M_{pitch}}{I_{yy}}$$

where the corner forces are:

$$Zf_{fl} = -K_{fl}(Zs_{fl} - Zu_{fl}) - C_{fl}(\dot{Zs}_{fl} - \dot{Zu}_{fl}) - F_{inelastic,fl} - F_{arb,fl}$$

$$Zf_{fr} = -K_{fr}(Zs_{fr} - Zu_{fr}) - C_{fr}(\dot{Zs}_{fr} - \dot{Zu}_{fr}) + F_{inelastic,fr} + F_{arb,fr}$$

$$Zf_{rl} = -K_{rl}(Zs_{rl} - Zu_{rl}) - C_{rl}(\dot{Zs}_{rl} - \dot{Zu}_{rl}) - F_{inelastic,rl} - F_{arb,rl}$$

$$Zf_{rr} = -K_{rr}(Zs_{rr} - Zu_{rr}) - C_{rr}(\dot{Zs}_{rr} - \dot{Zu}_{rr}) + F_{inelastic,rr} + F_{arb,rr}$$

wheel rotation (only open differential for now)

$$\ddot{\theta}_{fl} = \frac{-F_{xw,fl}(r_{w,fl}) - T_{brake,fl}}{I_{w,fl}}$$

$$\ddot{\theta}_{fr} = \frac{-F_{xw,fr}(r_{w,fr}) - T_{brake,fr}}{I_{w,fr}}$$

$$\ddot{\theta}_{rl} = \frac{\frac{1}{2}T_e G_r(r_{w,rl}) - T_{brake,rl} - F_{xw,rl}(r_{w,rl})}{I_{w,rl}}$$

$$\ddot{\theta}_{rr} = \frac{\frac{1}{2}T_e G_r(r_{w,rr}) - T_{brake,rr} - F_{xw,rr}(r_{w,rr})}{I_{w,rr}}$$

3.2 Tire Forces

Tire forces are generated in terms of longitudinal and lateral directions, in the wheel frame of reference.
Lateral force generated by the tire is a function of the slip angle and the normal force (IA ignored for now for simplicity).

$$F_{yw} = f(SA, Fz)$$

Longitudinal force generated by the tire is a function of slip ratio and normal force.

$$F_{xw} = f(SR, Fz)$$

Tire forces are calculated by using a lookup table generated from TTC data.
Slip angle is the angle between the wheel heading direction and the velocity vector of the wheel.

$$SA_{fl}(\delta_{fl}, \dot{y}, \dot{\phi}, \dot{x}) = \delta_{fl} - tan^{-1}(\frac{\dot{y} + \dot{\phi}(a)}{\dot{x} - \dot{\phi}(r_{fl})}) + TC(F_{yb,fl})$$

$$SA_{fr}(\delta_{fr}, \dot{y}, \dot{\phi}, \dot{x}) = \delta_{fr} - tan^{-1}(\frac{\dot{y} + \dot{\phi}(a)}{\dot{x} + \dot{\phi}(r_{fl})}) + TC(F_{yb,fr})$$

$$SA_{rl}(\delta_{rl}, \dot{y}, \dot{\phi}, \dot{x}) = \delta_{rl} - tan^{-1}(\frac{\dot{y} - \dot{\phi}(b)}{\dot{x} - \dot{\phi}(r_{rl})}) + TC(F_{yb,rl})$$

$$SA_{rr}(\delta_{rr}, \dot{y}, \dot{\phi}, \dot{x}) = \delta_{rr} - tan^{-1}(\frac{\dot{y} - \dot{\phi}(b)}{\dot{x} + \dot{\phi}(r_{rr})}) + TC(F_{yb,rr})$$

where $TC(F)$ is the compliance of the corner, currently linearly proportional of the lateral force (body frame) of the corner only.
TODO section about compliance

δ is the steering angle of the wheel, including toe angle:

$$\delta_{fl} = \delta_{str,fl}(t) - toe_{fl}$$

$$\delta_{fr} = \delta_{str,fr}(t) + toe_{fr}$$

$$\delta_{rl} = \delta_{str,rl}(t) - toe_{rl}$$

$$\delta_{rr} = \delta_{str,rr}(t) + toe_{rr}$$

Toe angle is assumed to be symmetrical for left and right wheel, with positive toe = toe in.
Currently parallel steering is used.

Slip ratio is the ratio of the tire contact patch velocity to the vehicle velocity at the corner.

$$SR_{fl}(\dot{\theta_{fl}}, \dot{x_{w,fl}}) = \frac{\dot{\theta_{fl}}(tr_{fl})}{\dot{x_{w,fl}}} - 1$$

$$SR_{fr}(\dot{\theta_{fr}}, \dot{x_{w,fr}}) = \frac{\dot{\theta_{fr}}(tr_{fr})}{\dot{x_{w,fr}}} - 1$$

$$SR_{rl}(\dot{\theta_{rl}}, \dot{x_{w,rl}}) = \frac{\dot{\theta_{rl}}(tr_{rl})}{\dot{x_{w,rl}}} - 1$$

$$SR_{rr}(\dot{\theta_{rr}}, \dot{x_{w,rr}}) = \frac{\dot{\theta_{rr}}(tr_{rr})}{\dot{x_{w,rr}}} - 1$$

where $\dot{x_w}$ is the longitudinal velocity of the wheel in the wheel frame of reference. The rotation matrix to convert between body frame and wheel frame is:

$$\begin{bmatrix} \dot{x_{w,fl}} \\ \dot{y_{w,fl}} \end{bmatrix} = \begin{bmatrix} cos(\delta_{fl}) & sin(\delta_{fl}) \\ -sin(\delta_{fl}) & cos(\delta_{fl}) \end{bmatrix} \begin{bmatrix} \dot{x} - \dot{\phi}(r_{fl}) \\ \dot{y} + \dot{\phi}(a) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x_{w,fr}} \\ \dot{y_{w,fr}} \end{bmatrix} = \begin{bmatrix} cos(\delta_{fr}) & sin(\delta_{fr}) \\ -sin(\delta_{fr}) & cos(\delta_{fr}) \end{bmatrix} \begin{bmatrix} \dot{x} + \dot{\phi}(r_{fr}) \\ \dot{y} + \dot{\phi}(a) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x_{w,rl}} \\ \dot{y_{w,rl}} \end{bmatrix} = \begin{bmatrix} cos(\delta_{rl}) & sin(\delta_{rl}) \\ -sin(\delta_{rl}) & cos(\delta_{rl}) \end{bmatrix} \begin{bmatrix} \dot{x} - \dot{\phi}(r_{rl}) \\ \dot{y} - \dot{\phi}(b) \end{bmatrix}$$

$$\begin{bmatrix} \dot{x_{w,rr}} \\ \dot{y_{w,rr}} \end{bmatrix} = \begin{bmatrix} cos(\delta_{rr}) & sin(\delta_{rr}) \\ -sin(\delta_{rr}) & cos(\delta_{rr}) \end{bmatrix} \begin{bmatrix} \dot{x} + \dot{\phi}(r_{rr}) \\ \dot{y} - \dot{\phi}(b) \end{bmatrix}$$

Since forces are generated in the wheel frame, to convert back to body frame, simply use the tranpose of the above matrix:

$$\begin{bmatrix} cos(\delta) & -sin(\delta) \\ sin(\delta) & cos(\delta) \end{bmatrix}$$

3.2.1 Roll, Pitch, Heave Dynamics

The spring stiffness K used here is the wheel rate instead of the spring rate.

$$K = \frac{K_{spring}}{mr^2}$$

Similarly the damping coefficient c is also the damping ratio as experiencd at the wheel.

$$c = \frac{c_{spring}}{mr^2}$$

The motion ratio (mr) here is assumed to be a constant for simplicity.

This model splits the roll, pitch and heave dynamics into three different systems, each calculated seperately. The overall change in shock compression is the superposition of the shock compressions of all three systems. This assumes that the pitch and roll angles are small and therefore the translation of the sprung mass in the corner can be approximated as the arc length. This also assumes that the CG movement in roll and pitch is negligible and does not affect the load on the tire.

$$Zs_{fl} = \psi_f(r_{fl}) + Zs_{cg} - \rho(a)$$
$$Zs_{fr} = -\psi_f(r_{fr}) + Zs_{cg} - \rho(a)$$
$$Zs_{rl} = \psi_r(r_{rl}) + Zs_{cg} + \rho(b)$$
$$Zs_{rr} = -\psi_r(r_{rr}) + Zs_{cg} + \rho(b)$$

The anti-roll moment M_{arb} generated by the anti-roll bar is (NOTE this only applies to a typical U shaped ARB):

$$M_{arb,f} = \frac{TS_{arb,f}(r_{fr} + r_{fl})^2}{(mr_{arb,f})^2(lever_{arb,f})^2}[\frac{Nm}{rad}]$$

where:

- TS_{arb} is the torsional stiffness of the ARB.
- $(r_f r + r_f l)$ is the front track. This formula assumes the load distribution of the axle to be even or very near even.
- $mr_{arb,f}$ is the motion ratio of the ARB, which is the displacement of the wheel over the displacement of the arb lever arm attachement point.
- $lever_{arb,f}$ is the ARB lever arm length.

Therefore, the force acting on the unsprung and sprung mass due to the ARB can be calculated using:

$$F_{arb,fl} = \frac{TS_{arb}(r_{fr} + r_{fl})^2}{(mr_{arb,f})^2(lever_{arb,f})^2} \times \psi_f \times \frac{r_{fl}}{r_{fr}(r_{fl} + r_{fr})}$$

$$F_{arb,fr} = \frac{TS_{arb}(r_{fr} + r_{fl})^2}{(mr_{arb,f})^2(lever_{arb,f})^2} \times \psi_f \times \frac{r_{fr}}{r_{fl}(r_{fl} + r_{fr})}$$

Similar calculation for the rear.

Moment due to chassis torsional stiffness:

$$M_{cts} = CTS \times (\psi_f - \psi_r)$$

where CTS is the chassis torsional stiffness $[\frac{Nm}{rad}]$.

Load transfer moments and forces:

$$M_{roll,f} = (F_{yb,fl} + F_{yb,fr}) \times (cgh_f - rch_f)$$

$$M_{roll,r} = (F_{yb,rl} + F_{yb,rr}) \times (cgh_r - rch_r)$$

$$F_{inelastic,fl} = (F_{yb,fl} + F_{yb,fr}) \times \frac{(rch_f)r_{fl}}{r_{fr}(r_{fr} + r_{fl})}$$

$$F_{inelastic,fr} = (F_{yb,fl} + F_{yb,fr}) \times \frac{(rch_f)r_{fr}}{r_{fl}(r_{fr} + r_{fl})}$$

$$F_{inelastic,rl} = (F_{yb,fl} + F_{yb,fr}) \times \frac{(rch_r)r_{rl}}{r_{rr}(r_{rr} + r_{rl})}$$

$$F_{inelastic,rr} = (F_{yb,fl} + F_{yb,fr}) \times \frac{(rch_r)r_{rr}}{r_{rl}(r_{rr} + r_{rl})}$$

$$M_{pitch} = (F_{xb,fl} + F_{xb,fr} + F_{xb,rl} + F_{xb,rr}) \times cgh$$

3.3 Inputs

There are three driver input in this model,

1. T_e driving torque (motor).
2. T_{brake} braking torque.
3. δ steering angle.

There is the additional input in $Z0$, the ground profile.

To calculate the actual trajectory of the vehicle, transform the vehicle frame accelerations back into inertial reference frame:

$$\ddot{X_{inertial}} = (\ddot{x_{car}} - \dot{\phi}\dot{y_{car}})\cos(\phi) - (\ddot{y_{car}} + \dot{\phi}\dot{x_{car}})\sin(\phi)$$

$$\ddot{Y_{inertial}} = (\ddot{x_{car}} - \dot{\phi}\dot{y_{car}})\sin(\phi) + (\ddot{y_{car}} + \dot{\phi}\dot{x_{car}})\cos(\phi)$$

4 Appendix

4.1 Derivation of inertial reference frame equation of motion

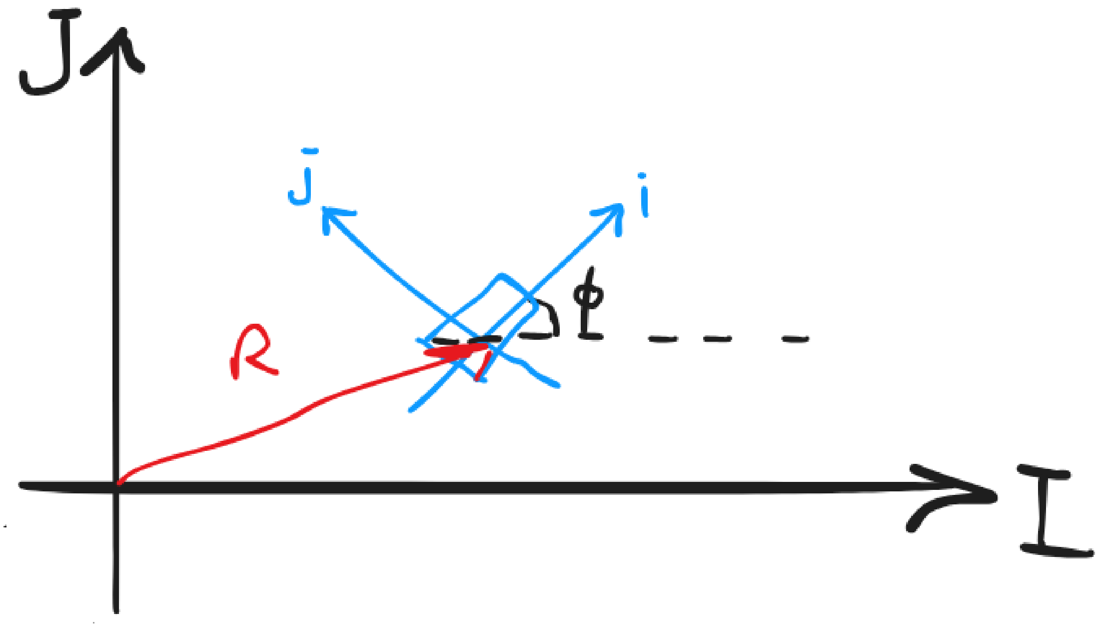

Figure 2. Graphical representation of the two reference frames.

Let I and J be the basis vectors of the inertial reference frame (ground frame) and i and j be the basis vectors of the vehicle body frame:

$$
\begin{aligned}
i &= \cos(\phi)I + \sin(\phi)J \\
j &= -\sin(\phi)I + \cos(\phi)J \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= -\sin(\phi)\frac{\mathrm{d}\phi}{\mathrm{d}t}I + \cos(\phi)\frac{\mathrm{d}\phi}{\mathrm{d}t}J = \frac{\mathrm{d}\phi}{\mathrm{d}t}j \\
\frac{\mathrm{d}j}{\mathrm{d}t} &= -\cos(\phi)\frac{\mathrm{d}\phi}{\mathrm{d}t}I - \sin(\phi)\frac{\mathrm{d}\phi}{\mathrm{d}t}J = -\frac{\mathrm{d}\phi}{\mathrm{d}t}i
\end{aligned}
$$

Let R be a vector in the inertial reference frame (ground frame), but representing its velocity in the vehicle body coordinate system:

$$
\begin{aligned}
\dot{R} &= \dot{x}i + \dot{y}j \\
\ddot{R} &= \ddot{x}i + \dot{x}\dot{i} + \ddot{y}j + \dot{y}\dot{j} \\
&= (\ddot{x} - \dot{y}\dot{\phi})i + (\ddot{y} + \dot{x}\dot{\phi})j
\end{aligned}
$$

Therefore $\ddot{R}$ is the representation of R's (the vehicle) acceleration as seen from the inertial reference frame, but using the vehicle body coordinate system, and $\ddot{x}$ and $\ddot{y}$ are the accelerations of the vehicle as seen from from the vehicle body reference frame.