Motor Modeling and Control Notes

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Abstract

This document serves as personal notes on motor math as well as documentation for the implementation of a motor simulator written in Matlab.

1 References

These are the main references I used to learn how to model and control 3 phase PMSM/BLDC motor.

- Ben Katz's work:

  His blog. link

  His undergraduate thesis. link

  His Github. link

- James Mavey's thesis on BLDC motor modeling and control. link
- Kirtley's permanent magnet motor MIT class notes. link
- Kundur, P. Power System Stability and Control. New York, NY: McGraw Hill, 1993. link

2 Equation of (Electron) Motion

2.1 Single phase derivation

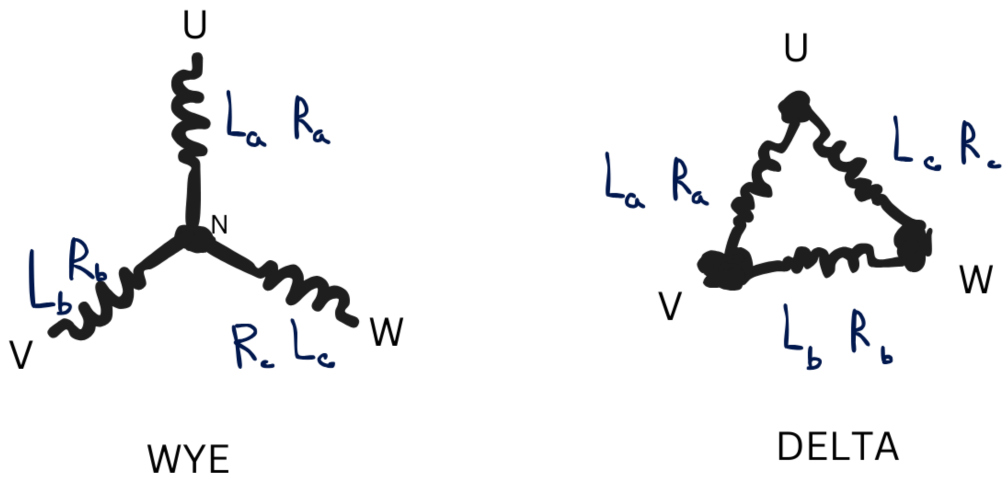

Figure 1. 3 phase model configurations

The motor electrical model can be modelled as a sort of RL circuit where R is the phase resistance and λ is the flux linkage:

$$V = Ri + \frac{\mathrm{d}\lambda}{\mathrm{d}t}$$

The flux linkage can be decomposed into 3 parts (using phase a as an example):

$$\lambda_a = \lambda_{ra} + L_a i_a + L_{ab} i_b + L_{ac} i_c$$

- λ_r is the flux linked by the rotor magnets.
- $L_a i_a$ is the flux linked by the self inductance of the phases.
- $L_{ab} i_b + L_{ac} i_c$ is the flux linked by the other phases where L_{ab} is the mutual inductance of phase a and b.

In reality, most of these values are not constant, they could be a function of many things such as rotor position and phase current. For generality, inductances are not assumed to be constant for the derivation of the following equations. Expanding the voltage equation with the components of the flux linkage:

$$\begin{aligned}
V_a &= R_a i_a + \frac{\mathrm{d}\lambda_a}{\mathrm{d}t} \\
V_a &= R_a i_a + \frac{d}{dt}(\lambda_{ra} + L_a i_a + L_{ab} i_b + L_{ac} i_c) \\
V_a &= R_a i_a + \frac{\mathrm{d}\lambda_{ra}}{\mathrm{d}t} + \frac{\mathrm{d}L_a}{\mathrm{d}t} i_a + L_a \frac{\mathrm{d}i_a}{\mathrm{d}t} + \frac{\mathrm{d}L_{ab}}{\mathrm{d}t} i_b + L_{ab} \frac{\mathrm{d}i_b}{\mathrm{d}t} + \frac{\mathrm{d}L_{ac}}{\mathrm{d}t} i_c + L_{ac} \frac{\mathrm{d}i_c}{\mathrm{d}t}
\end{aligned}$$

2.2 3-phase model

Combining all 3 phases into matrix form:

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} R_a & 0 & 0 \\ 0 & R_b & 0 \\ 0 & 0 & R_c \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + \begin{bmatrix} L_a & L_{ab} & L_{ac} \\ L_{ba} & L_b & L_{bc} \\ L_{ca} & L_{cb} & L_c \end{bmatrix} \begin{bmatrix} \frac{\mathrm{d}i_a}{\mathrm{d}t} \\ \frac{\mathrm{d}i_b}{\mathrm{d}t} \\ \frac{\mathrm{d}i_c}{\mathrm{d}t} \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} L_a & L_{ab} & L_{ac} \\ L_{ba} & L_b & L_{bc} \\ L_{ca} & L_{cb} & L_c \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + \frac{d}{dt} \begin{bmatrix} \lambda_{ra} \\ \lambda_{rb} \\ \lambda_{rc} \end{bmatrix}$$

Simplifying the matrices:

$$\left[V\right] = \left[R\right]\left[i\right] + \left[\tfrac{\mathrm{d}\lambda_r}{\mathrm{d}t}\right] + \left[L\right]\left[\tfrac{\mathrm{d}i}{\mathrm{d}t}\right] + \left[\tfrac{\mathrm{d}L}{\mathrm{d}t}\right]\left[i\right]$$

In differential equation form:

$$\left[\tfrac{\mathrm{d}i}{\mathrm{d}t}\right] = \left[L\right]^{-1}\left(\left[V\right] - \left[R\right]\left[i\right] - \left[\tfrac{\mathrm{d}L}{\mathrm{d}t}\right]\left[i\right] - \left[\tfrac{\mathrm{d}\lambda_r}{\mathrm{d}t}\right]\right)$$

2.3 System identification

- Resistance can be measured between two terminals. Assuming all phases have identical construction, the resistance is half of the measured value.
- Rotor permanent magnet flux linkage can be found by the back-emf waveform.
- Phase inductance can be calculated from d-q inductance:

  $$L_{ph} = \frac{1}{2}(L_d + L_q + (L_d - L_q)cos(2\theta))$$

  Derivation is given in Kundar. I am not 100$\%$ about this derivation but for now it is used naively.

In the above, P_d and P_q are the permeance coefficients of the d- and q-axis, respectively. In addition to the actual permeance, they include factors required to relate flux per pole with peak value of the mmf wave.

The total air-gap flux linking phase a is

$$
\begin{aligned}
\Phi_{gaa} &= \Phi_{gad}\cos\theta - \Phi_{gaq}\sin\theta \\
&= N_a i_a (P_d \cos^2\theta + P_q \sin^2\theta) \\
&= N_a i_a \left(\frac{P_d + P_q}{2} + \frac{P_d - P_q}{2}\cos 2\theta \right)
\end{aligned}
\tag{3.34}
$$

The self-inductance l_{gaa} of phase a due to air-gap flux is

$$
\begin{aligned}
l_{gaa} &= \frac{N_a \Phi_{gaa}}{i_a} \\
&= N_a^2 \left(\frac{P_d + P_q}{2} + \frac{P_d - P_q}{2}\cos 2\theta \right) \\
&= L_{g0} + L_{aa2}\cos 2\theta
\end{aligned}
\tag{3.35}
$$

Figure 2. Phase inductance derivation from Kundar

2.4 Back emf

Assuming the inductance $[L]$ and rotor permanent magnet flux linkages $[\lambda_r]$ are both function of the rotor position θ, the back emf terms can be presented in a form where it is a function of the rotor velocity ω:

$$\frac{\mathrm{d}\lambda_r(\theta)}{\mathrm{d}t} = \frac{\mathrm{d}\lambda_r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega\frac{\mathrm{d}\lambda_r}{\mathrm{d}\theta}$$

$$\frac{\mathrm{d}L(\theta)}{\mathrm{d}t} = \frac{\mathrm{d}L}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = \omega\frac{\mathrm{d}L}{\mathrm{d}\theta}$$

The inductance term is the reluctance back emf term.

2.5 Termination Type

2.5.1 Wye

Most typical mass produced BLDC motors are terminated in the wye (or star) pattern. In this configuration the phase voltages are the voltages between the neutral point and the terminal:

$$V_a = V_u - V_N$$
$$V_b = V_v - V_N$$
$$V_c = V_w - V_N$$

To solve for the extra parameter V_N. An additional equation is needed, and it can be obtained using Kirchhoff's current law at the neutral point, where the currents must satisfy (assuming balanced operation):

$$i_a + i_b + i_c = 0$$

The differential equation then becomes:

$$\begin{bmatrix} V_u \\ V_v \\ V_w \\ 0 \end{bmatrix} = \begin{bmatrix} & & & 1 \\ & [L] & & 1 \\ & & & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \left[\frac{\mathrm{d}i}{\mathrm{d}t}\right] \\ V_N \end{bmatrix} + \begin{bmatrix} & & & 0 \\ & [R] & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} [i] \\ 0 \end{bmatrix} + \begin{bmatrix} & & & 0 \\ & \left[\frac{\mathrm{d}L}{\mathrm{d}t}\right] & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} [i] \\ 0 \end{bmatrix} + \begin{bmatrix} \left[\frac{\mathrm{d}\lambda_r}{\mathrm{d}t}\right] \\ 0 \end{bmatrix}$$

2.5.2 Delta

Delta configuration does not have a neutral point, and the voltage relation is much simpler:

$$V_a = V_u - V_v$$
$$V_b = V_v - V_w$$
$$V_c = V_w - V_u$$

2.6 Power and Torque

2.6.1 Electrical Power

The electrical power supplied to the motor is the phase voltages times phase currents:

$$P_{elec} = V_a I_a + V_b I_b + V_c I_c$$

Electrical resistive power dissipated in the windings:

$$P_{res} = R_a I_a^2 + R_b I_b^2 + R_c I_c^2$$

Electrical power absorbed by the inductance in the windings:

$$P_{ind} = L \left[\frac{\mathrm{d}i}{\mathrm{d}t}\right] \cdot [i]$$

There are two electromechanical coupling terms. The reluctance term:

$$P_{reluc} = L\left[\frac{\mathrm{d}L}{\mathrm{d}t}\right][i] \cdot [i]$$

and the permanent magnet term:

$$P_{pm} = \left[\frac{\mathrm{d}\lambda_r}{\mathrm{d}t}\right] \cdot [i]$$

The total electrical power equals the summation of all the above terms:

$$P_{elec} = P_{res} + P_{ind} + P_{reluc} + P_{pm}$$

2.6.2 Mechanical Power

Mechanical power is the sum of the above two electromechanical coupling terms:

$$P_{mech} = P_{reluc} + P_{pm}$$

Mechanical torque is then the quotient of mechanical power and angular velocity:

$$T = P_{mech}/\dot{\theta}$$

3 Field Oriented Control

In essence, field oriented control (FOC) is a technique used to transform a motor's dynamics into the rotor's reference frame (the dq frame), such that the dynamics appear linear and the AC signal is transformed into a DC signal. This is particularly useful when using a controller such as a PI current controller, which has infinite DC gain at 0Hz frequency. FOC can transform the rotor dynamics to the d-q frame where current and inductances are constant. This means the controller can control phase currents with much higher frequency than the bandwidth of the controller.

3.1 Park's transform

Park's transform is used to transform the rotor dynamics from the stator (abc) to the rotor (dq) frame of reference. Park's transform in magnitude invariant form.

$$T = \frac{2}{3}\begin{bmatrix} \cos(\theta(t)) & \cos(\theta(t) - \frac{2\pi}{3}) & \cos(\theta(t) + \frac{2\pi}{3}) \\ -\sin(\theta(t)) & -\sin(\theta(t) - \frac{2\pi}{3}) & -\sin(\theta(t) + \frac{2\pi}{3}) \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$
$$T^{-1} = \begin{bmatrix} \cos(\theta(t)) & -\sin(\theta(t)) & 1 \\ \cos(\theta(t) - \frac{2\pi}{3}) & -\sin(\theta(t) - \frac{2\pi}{3}) & 1 \\ \cos(\theta(t) + \frac{2\pi}{3}) & -\sin(\theta(t) + \frac{2\pi}{3}) & 1 \end{bmatrix}$$

Park's transform in power invariant form.

$$T = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos(\theta(t)) & \cos(\theta(t) - \frac{2\pi}{3}) & \cos(\theta(t) + \frac{2\pi}{3}) \\ -\sin(\theta(t)) & -\sin(\theta(t) - \frac{2\pi}{3}) & -\sin(\theta(t) + \frac{2\pi}{3}) \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$
$$T^{-1} = \sqrt{\frac{2}{3}}\begin{bmatrix} \cos(\theta(t)) & -\sin(\theta(t)) & \frac{1}{\sqrt{2}} \\ \cos(\theta(t) - \frac{2\pi}{3}) & -\sin(\theta(t) - \frac{2\pi}{3}) & \frac{1}{\sqrt{2}} \\ \cos(\theta(t) + \frac{2\pi}{3}) & -\sin(\theta(t) + \frac{2\pi}{3}) & \frac{1}{\sqrt{2}} \end{bmatrix}$$

There are two main assumptions made when using the d-q frame. The first one is that magnetic flux from the permanent magnets are only produced in the d axis, that is:

$$\lambda_{dq} = \begin{bmatrix} \lambda_f \\ 0 \\ 0 \end{bmatrix}$$

where λ_f is the magnitude of the flux produced. The q axis is the axis orthogonal to the d axis, and no flux is produced there.
The second assumption is that the inductance in the d-q frame is independent of rotor position and have no mutual inductance:

$$L_{dq} = TL_{ph}T^{-1} = \begin{bmatrix} L_d & 0 & 0 \\ 0 & L_q & 0 \\ 0 & 0 & L_0 \end{bmatrix}$$

This assumption is most likely wrong to some extend but generally speaking it is close enough to being correct and allows for much simpler equation formulations. The third axis, the "zero" axis is not used and becomes zero in a balanced system.

3.2 D-Q frame EOM derivation

As shown before, the fluxes can be represented by:

$$\lambda_{ph} = \lambda_r + L_{ph} i_{ph}$$

where ph represents the phase elements and dq represents the dq frame elements. Using magnitude invariant form of the Park's transform, the armature voltage is (ignoring the resistive term for now):

$$\begin{aligned}
V_{ph} &= \frac{\mathrm{d}\lambda_{ph}}{\mathrm{d}t}, \\
V_{dq} &= T V_{ph} \\
V_{dq} &= T \frac{\mathrm{d}}{\mathrm{d}t} \lambda_{dq} \\
V_{dq} &= \frac{\mathrm{d}}{\mathrm{d}t} \lambda_{dq} + (T \frac{\mathrm{d}}{\mathrm{d}t} T^{-1}) \lambda_{dq}
\end{aligned}$$

It can be shown through manipulation that:

$$T \frac{\mathrm{d}}{\mathrm{d}t} T^{-1} = \begin{bmatrix} 0 & -\frac{\mathrm{d}\theta}{\mathrm{d}t} & 0 \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which shows a speed dependent magnitude in the back-emf, similar to the one shown in the normal 3 phase component previously. The d and q axis voltages are then:

$$\begin{aligned}
V_d &= R i_d + \frac{\mathrm{d}\lambda_d}{\mathrm{d}t} - \omega \lambda_q \\
V_q &= R i_q + \frac{\mathrm{d}\lambda_q}{\mathrm{d}t} + \omega \lambda_d
\end{aligned}$$

This voltage includes the resistive term. Generally the resistance used in d-q frame is assumed to be the same as the phase resistance. Expanding the flux linkage term in the equation:

$$\begin{aligned}
V_d &= R i_d + L_d \frac{\mathrm{d}i_d}{\mathrm{d}t} - \omega L_q i_q \\
V_q &= R i_q + L_q \frac{\mathrm{d}i_q}{\mathrm{d}t} + \omega(\lambda_r + L_d i_d)
\end{aligned}$$

These two equations are the essence of FOC, where a nonlinear 1st order dynamic system of 3 equations are turned into a pair of coupled linear 1st order equations.

3.3 Power and Torque

Remember all the derivations here uses the magnitude invariant park's transform. The total power is:

$$\begin{aligned}
P &= V_a I_a + V_b I_b + V_c I_c \\
P &= \frac{3}{2} V_d I_d + \frac{3}{2} V_q I_q + 3 V_0 I_0 \\
P &= \omega \frac{3}{2} (\lambda_d I_q - \lambda_q I_d) + \frac{3}{2} (\frac{\mathrm{d}\lambda_d}{\mathrm{d}t} I_d + \frac{\mathrm{d}\lambda_q}{\mathrm{d}t} I_q) + 3 \frac{\mathrm{d}\lambda_0}{\mathrm{d}t} I_0
\end{aligned}$$

Similar to the previous analysis in the phases, this total power is comprised of the mechanical power (first term) and the rate of stored electrical energy (second term). Here resistance is ignored again and the I_0 term is 0 in a balanced system.

$$T = p\frac{3}{2}(\lambda_d I_q - \lambda_q I_d)$$
$$T = p\frac{3}{2}(\lambda_f + (L_d - L_q)I_d)I_q$$

Note that this torque calculation is identical to the ones calculated in the 3 phase frame of reference. The 3/2 term is dismissed if power invariant Park's transform is used instead.

4 Numerical Simulation

The simulation method used to simulate the motor is euler's method. Typically euler's method is not used since it introduces a large global and local error in the numerical simulation. Here the euler step size is the controller sampling period, typically in the range of 10 to 100 microseconds (sampling frequency of 10000 to 100000 Hz) for a hobbyist microcontroller-based motor controller. To investigate this issue, take a linear RL circuit with sinusoidal voltage input:

$$L\frac{\mathrm{d}i}{\mathrm{d}t} + Ri = V(t), i(0) = 0$$
$$V(t) = sin(t)$$

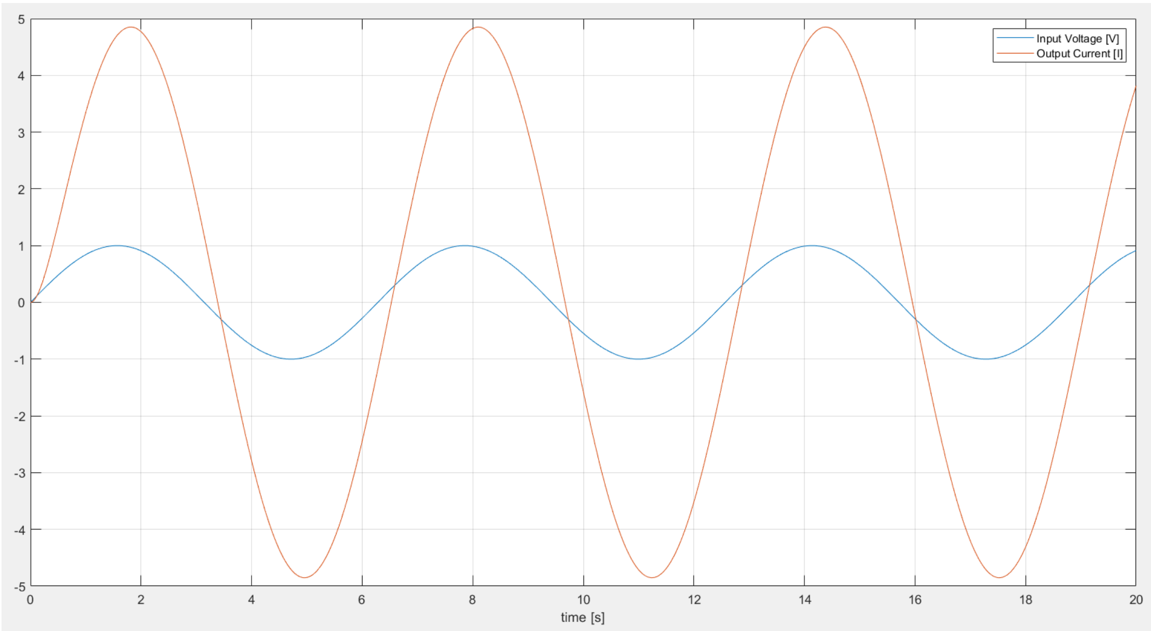

Figure 3. RL circuit analytical results

6 different sampling period/time step is tested on this model to figure out the limit of the error (100,500,1000,5000,10000,50000 Hz).

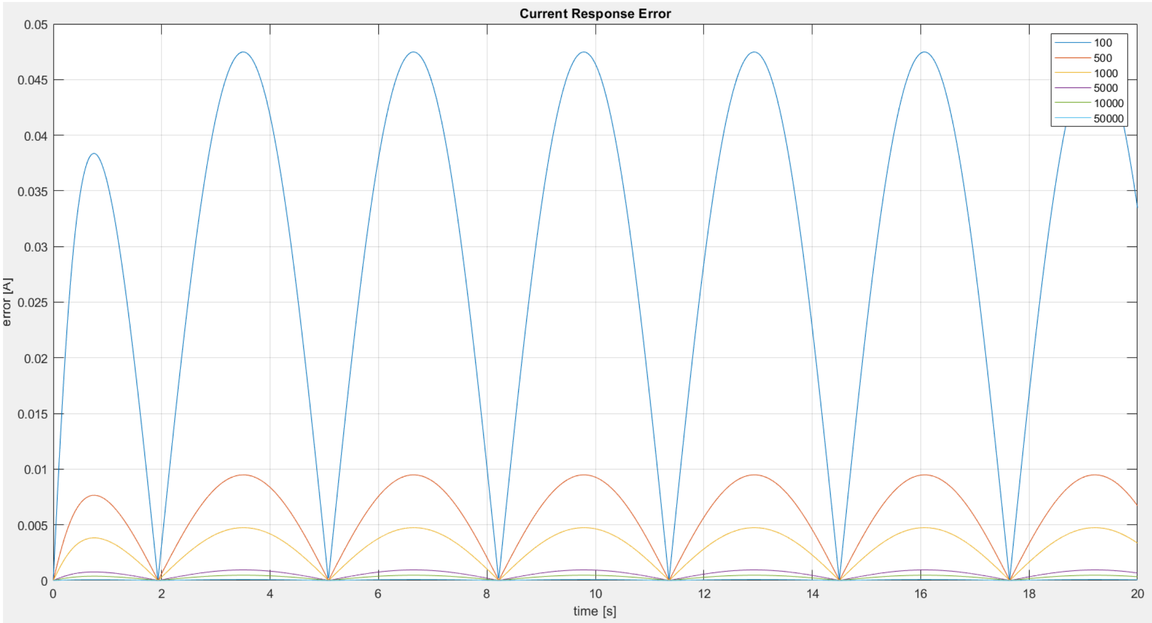

Figure 4. numerical integration error with different sampling frequencies.

	100 Hz	500 Hz	1000 Hz	5000 Hz	10000 Hz	50000 Hz
Euler Error (A)	3.017E-02	6.028E-03	3.014E-03	6.027E-04	3.013E-04	6.027E-05
Huen Error (A)	3.103E-02	6.203E-03	3.101E-03	6.203E-04	3.101E-04	6.203E-05

Figure 5. Average error using euler and huen methods.

Although the error for a sinusoidal output is ultimately bounded, the error varies significantly depending on the time step. It is therefore recommended to keep the euler time steps to at most 0.0002s (at least 5000Hz). A quick test with huen's method shows that it does not significantly improve the results. If a lower frequency controller is to be simulated, it is recommended to either decrease the time steps by simulating multiple time steps in 1 sampling period, or use a higher order method such as RK4 and ode45.