# AE 434 – Spacecraft Control

## Chapter 1:
## Mathematical modeling

### 1.1 Linearization and Laplace transforms

(Dorf & Bishop: *Ch.2.3 – 2.4*)

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EMBRY-RIDDLE AERONAUTICAL UNIVERSITY

# Course overview

- **Modeling a physical system**
  - Linearization of non-linear models
  - ODEs and their solutions – with Laplace transforms
  - Transfer functions
  - Block diagrams
  - State-space models

- **Analysis**
  - Time response
  - Steady-state error and PID control
  - Stability: Routh-Hurwitz criterion

- **Design control laws**
  - Root Locus
  - Frequency response
  - Pole placement
  - Introduction to Optimal control

Lead/Lag compensators

# Linear Approximation

- Nature tends to behave **nonlinearly**
- A great majority of physical systems are linear within some range of the variables
  - Behavior can often be **modeled as linear** within certain domains

- Linear systems satisfy **principle of superposition**
  - For a linear system, it is necessary that the excitation $x1(t) + x2(t)$ result in a response $y1(t) + y2(t)$
- The magnitude scale factor must be preserved in a linear system
  - This is the property of **homogeneity**

# Linearization via Taylor Series

- We will use the **Taylor Series** expansion of functions
- Consider nonlinear function $f(x)$ and suppose that $f(\bar{x}) = 0$
- $\bar{x}$ is an **equilibrium point** of $\dot{x} = f(x)$
- The Taylor Series expansion around equilibrium point becomes:

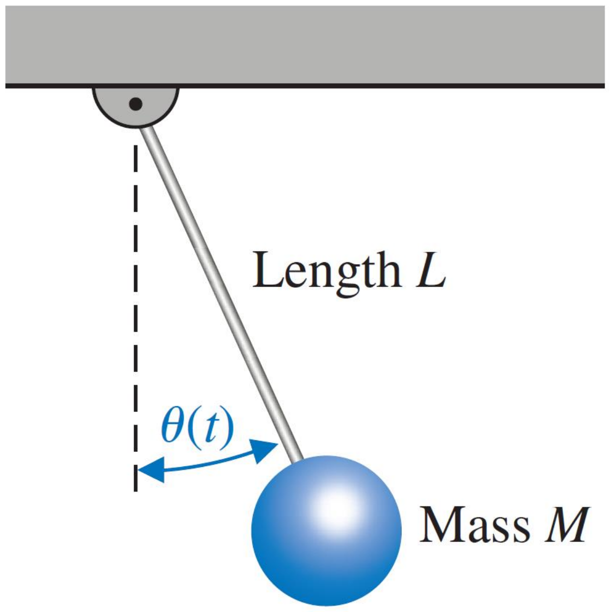
$$f(x) = f(\bar{x}) + \left.\frac{df}{dx}\right|_{x=\bar{x}} (x - \bar{x}) + \frac{1}{2} \left.\frac{d^2 f}{dx^2}\right|_{x=\bar{x}} (x - \bar{x})^2 + \frac{1}{6} \left.\frac{d^3 f}{dx^3}\right|_{x=\bar{x}} (x - \bar{x})^3 + \cdots$$

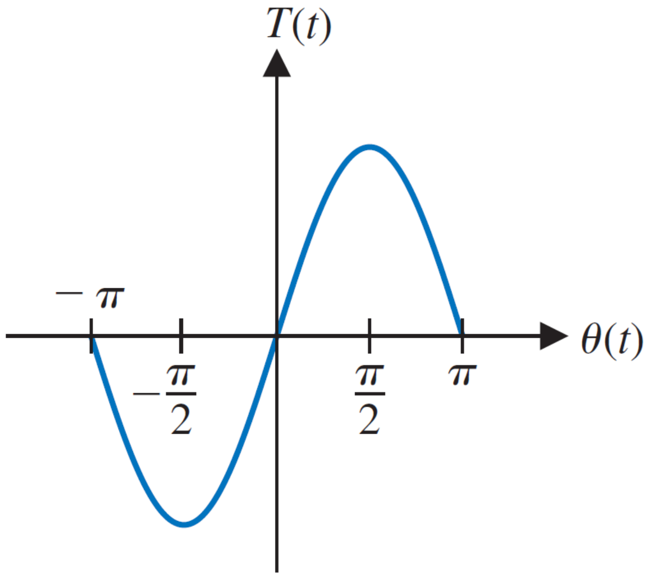
$$f(x) = f(\bar{x}) + \underbrace{\left.\frac{df}{dx}\right|_{x=\bar{x}}}_{a} (x - \bar{x}) + \text{ higher order terms.}$$

$$\boxed{\delta\dot{x} = a\delta x}$$

- Note that this linear model is valid **only** near the equilibrium point (how "near" depends on how nonlinear the function is).

# Linearization via Taylor Series




$$T(t) = MgL\theta(t)$$

- This approximation is reasonably accurate for $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$
- The response of the linear model for the swing through ±30° is within 5% of the actual nonlinear pendulum response.

# Laplace transform

- To understand and control complex systems, one must obtain quantitative **mathematical models**
- Systems under consideration are dynamic in nature, the descriptive equations are usually **differential equations.**
- DEs are inconvenient to work with from a controls perspective
- If these equations can be **linearized**, then the **Laplace transform** can be used to simplify the method of solution

# Laplace transform definition

- In order to solve an ordinary differential equation (ODE), transforming it into an algebraic equation (AE) might be required.

- This is done by using the Laplace transform $(\mathcal{L})$ :

$$F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\, dt = \mathscr{L}\{f(t)\} \qquad \text{with } s = \sigma + j\omega \ \text{(complex variable)}$$

**$F(s)$ is the Laplace transform of $f(t)$**

- Using the Laplace transform, we can find the solution to a complicated ODE easily.

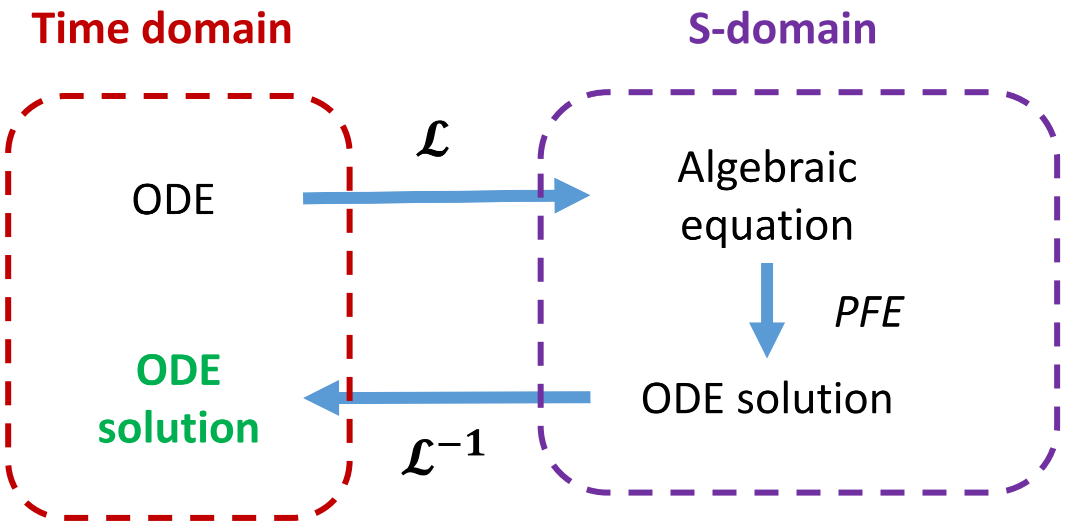
- The inverse Laplace $(\mathcal{L}^{-1})$ is then applied to convert the ODE solution from the s-domain to the time domain

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{+st}\, ds$$

# Laplace transforms

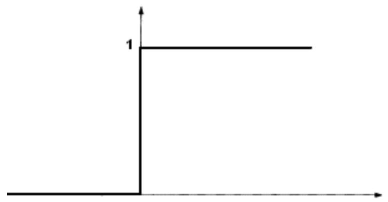
How to use them to solve ODEs:

1. Obtain the linearized differential equations
2. Obtain the Laplace transformation of the ODE
3. Solve the resulting algebraic equation for the transform of the variable of interest (Partial fraction expansion)
4. Apply to inverse Laplace to obtain ODE solution in time domain



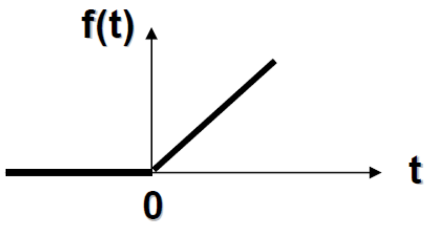
# Main Laplace transforms (1)

- Unit step function : $f(t) = u_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$




$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} 1 \cdot e^{-st}dt = -\frac{1}{s}[e^{-st}]_0^{\infty} = \boldsymbol{\frac{1}{s}}$$

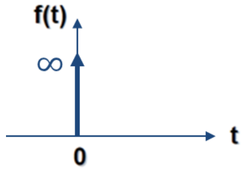
- Unit ramp function : $f(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases}$



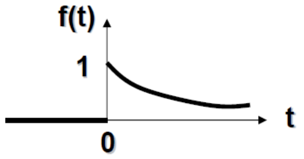

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}dt = \int_0^{\infty} t \cdot e^{-st}dt = -\frac{1}{s}[te^{-st}]_0^{\infty} + \frac{1}{s}\int_0^{\infty} e^{-st}dt = \boldsymbol{\frac{1}{s^2}}$$

# Main Laplace transforms (2)

- Unit impulse function : $\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases} \longrightarrow \boldsymbol{F(s) = 1}$




- Exponential function : $f(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases} \longrightarrow \boldsymbol{F(s) = \frac{1}{s + \alpha}}$




- Sine function : $f(t) = sin(\omega t) \longrightarrow \boldsymbol{F(s) = \frac{\omega}{s^2 + \omega^2}}$

- Cosine function : $f(t) = cos(\omega t) \longrightarrow \boldsymbol{F(s) = \frac{s}{s^2 + \omega^2}}$

# LT table

From Dorf & Bishop

| $f(t)$ | $F(s)$ |
|---|---|
| Step function, $u(t)$ | $\frac{1}{s}$ |
| $e^{-at}$ | $\frac{1}{s+a}$ |
| $\sin \omega t$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos \omega t$ | $\frac{s}{s^2+\omega^2}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ |
| $f^{(k)}(t) = \frac{d^k f(t)}{dt^k}$ | $s^k F(s) - s^{k-1} f(0^-) - s^{k-2} f'(0^-) - \ldots - f^{(k-1)}(0^-)$ |
| $\int_{-\infty}^{t} f(t)\, dt$ | $\frac{F(s)}{s} + \frac{1}{s} \int_{-\infty}^{0} f(t)\, dt$ |
| Impulse function $\delta(t)$ | 1 |
| $e^{-at} \sin \omega t$ | $\frac{\omega}{(s+a)^2+\omega^2}$ |
| $e^{-at} \cos \omega t$ | $\frac{s+a}{(s+a)^2+\omega^2}$ |
| $\frac{1}{\omega}\left[(\alpha-a)^2+\omega^2\right]^{1/2} e^{-at} \sin(\omega t+\phi)$, $\phi = \tan^{-1}\frac{\omega}{\alpha-a}$ | $\frac{s+\alpha}{(s+a)^2+\omega^2}$ |
| $\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n\sqrt{1-\zeta^2}\,t,\ \zeta<1$ | $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ |
| $\frac{1}{a^2+\omega^2} + \frac{1}{\omega\sqrt{a^2+\omega^2}} e^{-at} \sin(\omega t-\phi)$, $\phi = \tan^{-1}\frac{\omega}{-a}$ | $\frac{1}{s\left[(s+a)^2+\omega^2\right]}$ |
| $1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\phi\right)$, $\phi = \cos^{-1}\zeta,\ \zeta<1$ | $\frac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$ |
| $\frac{\alpha}{a^2+\omega^2} + \frac{1}{\omega}\left[\frac{(\alpha-a)^2+\omega^2}{a^2+\omega^2}\right]^{1/2} e^{-at} \sin(\omega t+\phi)$. $\phi = \tan^{-1}\frac{\omega}{\alpha-a} - \tan^{-1}\frac{\omega}{-a}$ | $\frac{s+\alpha}{s\left[(s+a)^2+\omega^2\right]}$ |

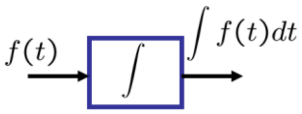
# Properties of the Laplace transform

- Linearity $\mathcal{L}\{\alpha_1 f_1(t) + \alpha_2 f_2(t)\} = \alpha_1 F_1(s) + \alpha_2 F_2(s)$

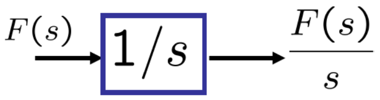
- Differentiation $\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)$

$$\mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

- Integration $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{F(s)}{s}$




*t-domain*




*s-domain*

# In-class example

Example: Solve the following differential eq.: $\ddot{x} + 2\dot{x} + 5x = 3$

for $x(0) = \dot{x}(0) = 0$.

applying LT: $s^2 X(s) + 2s X(s) + 5 X(s) = 3/s$

$$X(s) = \frac{3}{s} \cdot \frac{1}{s^2+2s+5} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$$

Solving: $(A+B)s^2 + (2A+C)s + 5A = 3$

$\rightarrow A = 3/5$

$2A + C = 0 \rightarrow C = -6/5$

$A + B = 0 \rightarrow B = -3/5$

$$X(s) = \frac{3}{5}\frac{1}{s} - \frac{3}{5}\left[\frac{s+2}{s^2+2s+5}\right] = \frac{3}{5}\cdot\frac{1}{s} - \frac{3}{5}\frac{(s+2)}{(s+1)^2+2^2}$$

$$X(s) = \frac{3}{5}\cdot\frac{1}{s} - \frac{3}{5}\frac{(s+1)+1}{(s+1)^2+2^2} = \frac{3}{5}\frac{1}{s} - \frac{3}{5}\frac{(s+1)}{(s+1)^2+2^2} + \frac{3}{5}\frac{1}{(s+1)^2+2^2}$$

$$X(s) = \frac{3}{5}\frac{1}{s} - \frac{3}{5}\frac{(s+1)}{(s+1)^2+2^2} - \frac{3}{5\times 2} \times \frac{2}{(s+1)^2+2^2}$$

$$x(t) = \mathcal{L}^{-1}[X(s)] = \frac{3}{5} - \frac{3}{5}e^{-t}\cos 2t - \frac{3}{10}e^{-t}\sin 2t \quad //$$

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# Final Value Theorem

- It is usually desired to determine the steady-state or final value of the response
- Final value theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ **iff** $\lim_{t\to\infty} f(t)$ exists

***Example :*** $F(s) = 1 * \dfrac{1}{s+1}$ *Impulse response ($*$ 1)*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s * \frac{1}{s+1} = 0$$

*The system returns to zero after being disturbed by a short impulse.*

- Initial value theorem: $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$