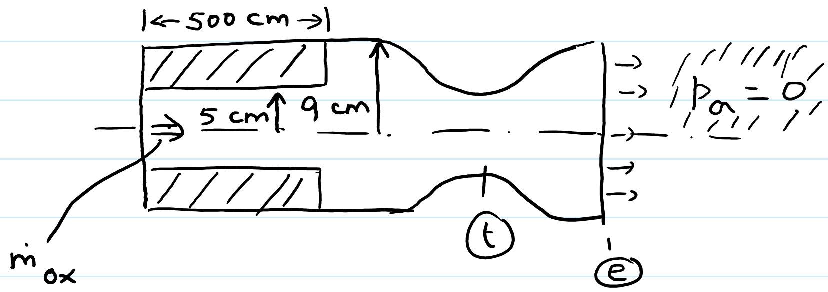
Example

A small-scale HRE with cylindrical fuel geometry is designed for operation in vacuum ($p_a = 0$).



The fuel is HTPB and oxidizer is LOX. The regression rate of HTPB in oxygen stream is known to be

$$r = 1.6086\, G_{ox}^{0.681},$$

where G_{ox} is in $\frac{kg}{cm^2 - s}$ and r is in cm/s. The mass flow rate of the oxidizer ($\dot{m}_{ox}$) is constant at 6.283 kg/s. The HRE generates an initial specific impulse of 250 s. The burn characteristics are given in the following table for HTPB – oxygen reaction*.

$(\dot{m}_{ox}/\dot{m}_f)$	c^* (ft/s)	γ	$\bar{M}_{prod}$ #
1.0	4825	1.308	18
1.2	5180	1.282	19
1.4	5543	1.239	20
1.6	5767	1.201	20
1.8	5882	1.171	21
2.0	5912	1.152	21
2.2	5885	1.143	22
2.4	5831	1.138	23
2.6	5768	1.135	25
2.8	5703	1.133	26
3.0	5639	1.132	27

* Table 16.2, "Rocket Propulsion Elements," Sutton, Biblarz

This data column is not from Table 16.2 of Sutton, et. al, text.

a) What is the initial oxidizer flux ($G_{ox,i}$)?

b) Find the initial fuel regression rate (r_i) and the initial fuel mass flow rate ($\dot{m}_{f,i}$) {$SG_f = 0.915$}

c) what is the initial ox-f ratio ($\dot{m}_{ox,i}/\dot{m}_{f,i}$)?

d) what is the initial thrust generated?

e) What are the initial values of c^* and C_T?

f) Calculate the initial value of the flame temperature.

g) What is the nozzle area ratio (A_e/A_t) and the exit Mach number (M_e)?

h) If the initial chamber pressure is 1.5 MPa, what is the throat area (A_t) of the nozzle?

i) What is the HRE's burn time?

a) $$G_{ox} = \frac{\dot{m}_{ox}}{A_p} = \frac{\dot{m}_{ox}}{\pi R^2} \quad (A_p \text{ - port area})$$

$$G_{ox,i} = \frac{\dot{m}_{ox,i}}{\pi R_i^2} = \frac{6.283}{\pi (5)^2} = 0.08 \ \frac{kg}{cm^2\text{-}s}$$

b) $$r_i = a G_{ox,i}^n = 1.6086 (0.08)^{0.681} = 0.288 \ cm/s$$

$$\dot{m}_{f,i} = \rho_f A_{b,i} r_i = \rho_f \, 2\pi R_i L r_i$$

$$\underset{g/cm^3}{} \quad \underset{cm}{} \quad \underset{cm}{} \quad \underset{cm/s}{}$$

$$= (0.915)(2)(\pi)(5)(500)(0.288)$$

$$= 4139 \ g/s = 4.139 \ kg/s$$

c) $$\frac{\dot{m}_{ox,i}}{\dot{m}_{f,i}} = \frac{6.283}{4.139} = 1.52$$

d) $$T = \dot{m} u_{eq} = \dot{m} I_{sp} g_e$$

$$T_i = \dot{m}_i I_{sp,i} g_e$$

$$\dot{m}_i = (\dot{m}_{ox,i} + \dot{m}_{f,i}) = (6.283 + 4.139) = 10.422 \ kg/s$$

$$T_i = (10.422)(250)(9.81) = 25560 \text{ N}$$

e) From the data table, at $\left(\frac{\dot{m}_{ox}}{\dot{m}_f}\right) = 1.52$,

$$c_i^* = 5677 \text{ ft/s} = 1731 \text{ m/s}, \quad \gamma_i = 1.22, \quad \bar{M}_i = 20$$

$$u_{eq} = C_T c^* \quad \left\{ u_{eq} = \frac{T}{\dot{m}} = \frac{T}{p_0 A_t} \cdot \frac{p_0 A_t}{\dot{m}} = C_T c^* \right\}$$

$$\Rightarrow C_{T,i} = \frac{u_{eq,i}}{c_i^*} = \frac{I_{sp,i}\, g_e}{c_i^*} = \frac{(250)(9.81)}{1731} = 1.42$$

f) $$c^* = \sqrt{\frac{1}{\gamma}\left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{\gamma-1}} \frac{\bar{R} T_0}{\bar{M}}}$$

$$\underbrace{c_i^*}_{1731 \frac{m}{s}} = \sqrt{\frac{1}{\gamma_i}\left(\frac{\gamma_i+1}{2}\right)^{\frac{\gamma_i+1}{\gamma_i-1}} \frac{\overbrace{\bar{R}}^{8314.3 \frac{J}{kmol\text{-}K}} T_{0,i}}{\bar{M}_i}} \Rightarrow T_{0,i} = 3068 \text{ K}$$

g) $$C_T = \left\{ \frac{2\gamma^2}{\gamma-1}\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}} \left[1 - \left(\frac{p_e}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\right] \right\}^{\frac{1}{2}} + \frac{p_e - \cancelto{0}{p_a}}{p_0} \frac{A_e}{A_t}$$

Initially, $C_T = C_{T,i} = 1.42$, $\gamma = \gamma_i = 1.22$

Assume a value for $\frac{A_e}{A_t}$. Solve for $M_e\,(>1)$. Find corresponding $\left(\frac{p_e}{p_0}\right)$. Calculate C_T. Repeat until $C_T = 1.42$.

$$\Rightarrow \frac{A_e}{A_t} = 1.67, \quad M_e = 1.894$$

h) $$C_{T,i} = \frac{T_i}{p_{0,i} A_t} \Rightarrow p_{0,i} A_t = \frac{T_i}{C_{T,i}} = \frac{25560}{1.42} = 18000 \text{ N}$$

$$p_{0,i} = 1.5 \text{ MPa} \Rightarrow A_t = \frac{18000}{1.5(10^6)} = 0.012 \text{ m}^2 = 120 \text{ cm}^2$$

$\{ R_t = 6.18 \text{ cm} \}$

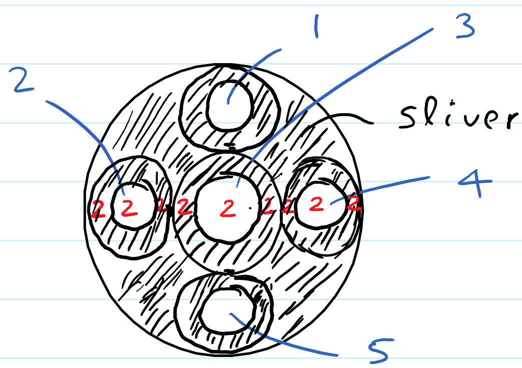
$$\text{(i)} \quad R = \left\{ a(2n+1)\left(\frac{\dot{m}_{ox}}{\pi}\right)^n t + R_i^{2n+1} \right\}^{\frac{1}{2n+1}}$$

$$R_f = \left\{ 1.6086\,(2.362)\left(\frac{6.283}{\pi}\right)^{0.681} t_b + 5^{2.362} \right\}^{\frac{1}{2.362}}$$

$\nearrow$ 9

$$\Rightarrow t_b = 22.1 \text{ s}$$

Comment: Observe the length of the HRE. It is 500 cm (5 m = 16.4 ft). The thrust is not large 25560 N, which is about 5746 lb_f. To reduce the length (or generate larger thrust for a given length), multiple ports are utilized. Reconsider the previous example of the HRE, with the same outside radius of 9 cm, but with five ports of equal cross-sections, see Figure, below.



(port # in blue)

Since all the ports have the same cross-section, each port has an outside radius of 3 cm. If the initial radius of each port is 1 cm, then the fuel ring thickness in each port is 2 cm.

If the oxidizer flow is equally split, the following values represent quantities per port:

values represent quantities per port:

$$\dot{m}_{ox} = \frac{6.283}{5} = 1.257 \ \frac{kg}{s}$$

$$G_{ox,i} = \frac{\dot{m}_{ox}}{\pi R_i^2} = \frac{1.257}{\pi(1^2)} = 0.4 \ \frac{kg}{cm^2\text{-}s}$$

$$r_i = 1.6086\,(0.4)^{0.681} = 0.862 \text{ cm/s}$$

$$m_{f,i} = \rho_f \, 2\pi R_i L r_i = \overset{g/cm^3}{(0.915)}\,(2)\,(\pi)\,\overset{cm}{(1)}\,\overset{cm/s}{(0.862)}\,\overset{cm}{L}$$

$$= 4.956\,L \ \ (g/s)$$

Since the total initial fuel flow rate is 4139 g/s [part (b), previous example],

$$4.956\,L = \frac{4139}{5} \Rightarrow L = 167 \text{ cm}$$

Observe the two-thirds reduction in length.

$$G_{ox,f} = \frac{\dot{m}_{ox}}{\pi R_f^2} = \frac{1.257}{\pi(3^2)} = 0.0445 \ \frac{kg}{cm^2\text{-}s}$$

$$r_f = 1.6086\,(0.0445)^{0.681} = 0.193 \ \frac{cm}{s}$$

$$\dot{m}_{f,f} = \rho_f \, 2\pi R_f L r_f = (0.915)(2)(\pi)(3)(167)(0.193)$$

$$= 555.9 \ \frac{g}{s} = 0.556 \ \frac{kg}{s}$$

$$\frac{\dot{m}_{ox,f}}{\dot{m}_{f,f}} = \frac{1.257}{0.556} = 2.26$$

From the data table, at $\frac{\dot{m}_{ox,f}}{\dot{m}_{f,f}} = 2.26$,

$$c^*_f = 5869 \text{ ft/s} = 1789 \text{ m/s}, \quad \gamma = 1.14, \quad \bar{M} = 22.3$$

$$T_{o,f} = 3479 \text{ K}$$

$$\frac{A_e}{A_t} = 1.67, \quad \gamma = 1.14 \Rightarrow M_e = 1.85, \quad C_{T,f} = 1.42$$

$$u_{eq_f} = c^*_f \, C_{T,f} = (1789)(1.42) = 2540 \text{ m/s}$$

$$I_{sp,f} = \frac{u_{eq,f}}{g_e} = \frac{2540}{9.81} = 259 \text{ s}$$

$$R_f = \left\{ a(2n+1)\left(\frac{\dot{m}_{ox}}{\pi}\right)^n t_b + R_i^{2n+1} \right\}^{\frac{1}{2n+1}}$$

$$3 = \left\{ 1.6086\,(2.362)\left(\frac{1.257}{\pi}\right)^{0.681} t_b + 1^{2.362} \right\}^{\frac{1}{2.362}}$$

$\Rightarrow t_b = 6.1$ s (much less than the 22.1 s with single cylindrical fuel geometry)