Chemical Rockets

Single stage rocket

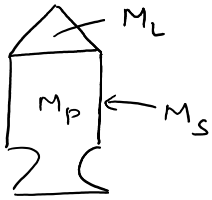
M_L - payload mass

M_S - structural mass

M_p - propellant mass

M_0 - initial mass

M_b - burnout mass



$$M_0 = M_L + M_S + M_p \quad (10.19)$$

$$M_b = M_0 - M_p = M_L + M_S \quad (10.20)$$

$$\lambda \overset{def}{=} \frac{M_L}{M_0 - M_L} \quad (10.22)$$

payload ratio

$$\epsilon \overset{def}{=} \frac{M_S}{M_S + M_p} = \frac{M_S}{M_0 - M_L} \quad (10.23)$$

structural coefficient

$$R = \frac{M_0}{M_b} = \frac{1+\lambda}{\epsilon + \lambda} \quad (10.24)$$

verify

mass ratio

$$\Delta u_{b,imp} = u_{eq} \ln\left(\frac{M_0}{M_b}\right) = u_{eq} \ln \frac{1+\lambda}{\epsilon+\lambda}$$

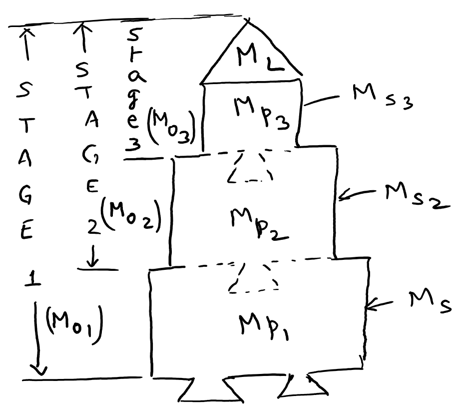
$$\left\{ \text{Recall } \Delta u_b = \Delta u_{b,imp} - \overline{g\cos\theta}\, t_b - \overline{\left(\frac{D}{M}\right)} t_b \right\}$$

Multistage rocket

stage 1:

$$M_{0_1} = M_{L_1} + M_{S_1} + M_{p_1}$$

The payload mass for stage 1 is the initial mass of stage 2.



$$M_{0_1} = M_{0_2} + M_{S_1} + M_{p_1}$$

$$M_{b1} = M_{o1} - M_{p1} = M_{o2} + M_{s1}$$

$$Q_1 = \frac{M_{o1}}{M_{b1}}$$

$$\lambda_1 = \frac{M_{L1}}{M_{o1} - M_{L1}} = \frac{M_{o2}}{M_{o1} - M_{o2}}, \quad \epsilon_1 = \frac{M_{s1}}{M_{s1} + M_{p1}} = \frac{M_{s1}}{M_{o1} - M_{L1}} = \frac{M_{s1}}{M_{o1} - M_{o2}}$$

$$\Delta u_{b,imp1} = u_{eq_1} \ln Q_1 = u_{eq_1} \ln \frac{1+\lambda_1}{\epsilon_1 + \lambda_1}$$

Similarly, for stage 2,

$$M_{o2} = M_{o3} + M_{s2} + M_{p2}, \quad M_{b2} = M_{o3} + M_{s2}$$

$$Q_2 = \frac{M_{o2}}{M_{b2}}, \quad \lambda_2 = \frac{M_{o3}}{M_{o2} - M_{o3}}, \quad \epsilon_2 = \frac{M_{s2}}{M_{o2} - M_{o3}},$$

$$\Delta u_{b,imp2} = u_{eq_2} \ln Q_2 = u_{eq_2} \ln \frac{1+\lambda_2}{\epsilon_2 + \lambda_2},$$

and, for stage 3,

$$M_{o3} = \cancel{M_{o4}}\, M_L + M_{s3} + M_{p3}, \quad M_{b3} = \cancel{M_{o4}}\, M_L + M_{s3}$$

$$Q_3 = \frac{M_{o3}}{M_{b3}}, \quad \lambda_3 = \frac{\cancel{M_{o4}}\, M_L}{M_{o3} - \cancel{M_{o4}}\, M_L}, \quad \epsilon_3 = \frac{M_{s3}}{M_{o3} - \cancel{M_{o4}}\, M_L}$$

$$\Delta u_{b,imp3} = u_{eq_3} \ln Q_3 = u_{eq_3} \ln \frac{1+\lambda_3}{\epsilon_3 + \lambda_3}$$

$$\Delta u_{b,imp\,tot} = u_{eq_1} \ln \frac{1+\lambda_1}{\epsilon_1+\lambda_1} + u_{eq_2} \frac{1+\lambda_2}{\epsilon_2+\lambda_2} + u_{eq_3} \ln \frac{1+\lambda_3}{\epsilon_3+\lambda_3}$$

The text provides the same equations for stage i

$$M_{oi} = M_{o,i+1} + M_{si} + M_{pi}, \quad M_{bi} = M_{o,i+1} + M_{si}$$

$$\lambda_i = \frac{M_{o,i+1}}{M_{oi} - M_{o,i+1}} \qquad (10.27)$$

$$\epsilon_i = \frac{M_{si}}{M_{oi} - M_{o,i+1}} \qquad (10.28)$$

$$R_i = \frac{1+\lambda_i}{\epsilon_i+\lambda_i} \quad (10.31)$$

# of stages

$$\Delta u_{b,\,imp\,tot} = \sum_{i=1}^{n} u_{eq_i} \ln\left(\frac{1+\lambda_i}{\epsilon_i+\lambda_i}\right)$$

ASIDE (see page 484, text)

$$\lambda_1 = \frac{M_{02}}{M_{01}-M_{02}}$$

$$1+\lambda_1 = 1 + \frac{M_{02}}{M_{01}-M_{02}} = \frac{M_{01}}{M_{01}-M_{02}}$$

$$\Rightarrow \frac{1+\lambda_1}{\lambda_1} = \frac{M_{01}}{M_{02}}$$

Similarly,

$$\frac{1+\lambda_2}{\lambda_2} = \frac{M_{02}}{M_{03}}$$

$$\frac{1+\lambda_3}{\lambda_3} = \frac{M_{03}}{\cancel{M_{04}} \rightarrow M_L}$$

Combining,

$$\left(\frac{1+\lambda_1}{\lambda_1}\right)\left(\frac{1+\lambda_2}{\lambda_2}\right)\left(\frac{1+\lambda_3}{\lambda_3}\right) = \frac{M_{01}}{\cancel{M_{02}}} \cdot \frac{\cancel{M_{02}}}{\cancel{M_{03}}} \cdot \frac{\cancel{M_{03}}}{M_L} = \frac{M_{01}}{M_L}$$

The above equation is for a three-stage rocket. In general,

$$\frac{M_{01}}{M_L} = \prod_{i=1}^{n} \left(\frac{1+\lambda_i}{\lambda_i}\right) \quad (10.35)$$

Ex: Consider a three-stage rocket with the following characteristics:

	stage 1	stage 2	stage 3
M_0 (kg)	2,500,000	580,000	179,000
M_p (kg)	1,824,000	373,000	90,000
M_s (kg)	96,000	28,000	21,000
I_{sp} (s)	370	370	370
u_{eq} $\left(\frac{m}{s}\right)$	3629.7	3629.7	3629.7
M_L (kg)	580,000	179,000	68000
λ	0.3021	0.4464	0.6126
ϵ	0.05	0.0698	0.1892
$\Delta u_{b,imp}$ $\left(\frac{m}{s}\right)$	4747	3740	2536

$$\Delta u_{b,\,imp\,tot} = \left(\Delta u_{b,\,imp\,1} + \Delta u_{b,\,imp\,2} + \Delta u_{b,\,imp\,3}\right)$$

$$= 11023 \text{ m/s}$$

{ In the table, given values are in black, determined values are in red }

Now, consider a single stage rocket with the same payload, structural and propellant masses as the three-stage rocket, above.

$$M_0 = 2,500,000 \text{ kg}$$

$$M_s = (96,000 + 28,000 + 21,000) = 145,000 \text{ kg}$$

$$M_p = (1,824000 + 373,000 + 90,000) = 2,287,000 \text{ kg}$$

$M_L = 68{,}000$ kg

$I_{sp} = 370$ s, $u_{eq} = 3629.7$ m/s

$$\lambda = \frac{M_L}{M_0 - M_L} = 0.02796, \quad \epsilon = \frac{M_s}{M_0 - M_L} = 0.05962$$

$$\Delta u_{b,imp} = u_{eq} \ln \frac{1+\lambda}{\epsilon + \lambda} = 8939 \text{ m/s}$$

Staging has increased the velocity rise by 23.3 %.

$$\left\{ \frac{11023 - 8939}{8939} \otimes 100 \right\}$$

Optimization of multistage rocket

Known : $\Delta u_{b,imp}$ (from mission calculation)

Assumption : The number of stages, and the I_{sp} and ϵ values of each are known.

Problem : To find the values of λ_i such that the $\left(\frac{M_L}{M_{01}}\right)$ ratio is maximized.

$$\Delta u_{b,imp} = \sum_{i=1}^{n} g_e I_{spi} \ln\left(\frac{1+\lambda_i}{\epsilon_i + \lambda_i}\right) \quad \left\{ \begin{array}{l} I_{spi} \text{ and } \epsilon_i \\ \text{are known, see} \\ \text{assumption} \end{array} \right.$$

The value of $\Delta u_{b,imp}$ is known. Thus, the various λ_i should be changed in such a way that $\Delta u_{b,imp}$ stays fixed in the maximization of $\left(\frac{M_L}{M_{01}}\right)$. This is known as a constrained optimization problem. The constraint is mathematically written as

$$\delta\left[\Delta u_{b,imp}\right] = 0$$

$$\Rightarrow \delta\left[\sum^{n} g_e I_{spi} \ln\left(\frac{1+\lambda_i}{\epsilon_i + \lambda_i}\right)\right] = 0$$

$$\Rightarrow \delta\left[\sum_{i=1}^{n} g_e I_{sp_i} \ln\left(\frac{1+\lambda_i}{\epsilon_i+\lambda_i}\right)\right] = 0$$

$$(\text{or}) \quad \sum_{i=1}^{n} \delta\left[g_e I_{sp_i} \ln\left(\frac{1+\lambda_i}{\epsilon_i+\lambda_i}\right)\right] = 0$$

$$(\text{or}) \quad \sum_{i=1}^{n} \frac{\partial}{\partial\lambda_i}\left[g_e I_{sp_i} \ln\left(\frac{1+\lambda_i}{\epsilon_i+\lambda_i}\right)\right] \delta\lambda_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} g_e I_{sp_i} \frac{(\epsilon_i - 1)}{(1+\lambda_i)(\epsilon_i+\lambda_i)} \delta\lambda_i = 0 \quad \text{(A)}$$

Consider, now, the expression for $\left(\frac{M_{01}}{M_L}\right)$ [Eqn 10.35], which was obtained earlier. This can be written as

$$\Rightarrow \frac{M_L}{M_{01}} = \prod_{i=1}^{n}\left(\frac{\lambda_i}{1+\lambda_i}\right)$$

The λ_i are chosen to maximize $\left(\frac{M_L}{M_{01}}\right)$:

$$\delta\left(\frac{M_L}{M_{01}}\right) = 0 \quad \Rightarrow \quad \delta\left[\prod_{i=1}^{n}\left(\frac{\lambda_i}{1+\lambda_i}\right)\right] = 0$$

or, equivalently, $\delta\left[\sum_{i=1}^{n} \ln\left(\frac{\lambda_i}{1+\lambda_i}\right)\right] = 0$

$$\text{or,} \quad \sum_{i=1}^{n} \delta\left[\ln\left(\frac{\lambda_i}{1+\lambda_i}\right)\right] = 0$$

$$\text{or,} \quad \sum_{i=1}^{n} \frac{\partial}{\partial\lambda_i}\left[\ln\left(\frac{\lambda_i}{1+\lambda_i}\right)\right] = 0$$

$$\text{or,} \quad \sum_{i=1}^{n} \frac{1}{\lambda_i(1+\lambda_i)} \delta\lambda_i = 0 \quad \text{(B)}$$

Introduce Lagrange multiplier α: undetermined

B $+ \ \alpha \otimes$ A $\Rightarrow$

$$\sum_{i=1}^{n} \left[\frac{1}{\lambda_i(1+\lambda_i)} + \alpha g_e I_{sp_i} \frac{\epsilon_i - 1}{(1+\lambda_i)(\epsilon_i+\lambda_i)} \right] \delta\lambda_i = 0$$

Since λ_i are independent,

$$\frac{1}{\lambda_i(1+\lambda_i)} + \alpha g_e I_{sp_i} \frac{\epsilon_i - 1}{(1+\lambda_i)(\epsilon_i+\lambda_i)} = 0 \qquad \text{C}$$

C can be solved for λ_i :

$$\boxed{\lambda_i = \frac{\epsilon_i}{\alpha g_e I_{sp_i}(1-\epsilon_i) - 1}} \qquad \text{D}$$

The undetermined multiplier α is obtained from the known $\Delta u_{b,imp}$:

$$\sum_{i=1}^{n} g_e I_{sp_i} \ln\left(\frac{1+\lambda_i}{\epsilon_i+\lambda_i}\right) = \Delta u_{b,imp}$$

$$\Rightarrow \left\{\frac{1}{\epsilon_1}\left[1 - \frac{1}{\alpha g_e I_{sp_1}}\right]\right\}^{\frac{g_e I_{sp_1}}{\Delta u_{b,imp}}} \otimes \left\{\frac{1}{\epsilon_2}\left[1 - \frac{1}{\alpha g_e I_{sp_2}}\right]\right\}^{\frac{g_e I_{sp_2}}{\Delta u_{b,imp}}} \cdots \otimes$$

$$\left\{\frac{1}{\epsilon_n}\left[1 - \frac{1}{\alpha g_e I_{sp_n}}\right]\right\}^{\frac{g_e I_{sp_n}}{\Delta u_b}} = e \qquad \text{E}$$

2.71828

First obtain α from E, and, then, get λ_i from D.

$$\frac{\cancel{(1+\lambda)}}{\lambda\cancel{(1+\lambda)}} = -\alpha g_e I_{sp} \frac{(\epsilon-1)(1+\lambda)}{\cancel{(1+\lambda)}(\epsilon+\lambda)}$$

$$\frac{1+\lambda}{\epsilon+\lambda} = \left(1+\frac{1}{\lambda}\right) \frac{1}{(1-\epsilon)} \frac{1}{\alpha g_e I_{sp}}$$

$$1+\frac{1}{\lambda} = 1+\frac{\alpha g_e I_{sp}(1-\epsilon)-1}{\epsilon}$$

$$= \frac{\epsilon+\alpha g_e I_{sp}(1-\epsilon)-1}{\epsilon}$$

$$= \frac{(1-\epsilon)(\alpha g_e I_{sp}-1)}{\epsilon}$$

$$\frac{1+\lambda}{\epsilon+\lambda} = \frac{(1-\cancel{\epsilon)}(\alpha g_e I_{sp}-1)}{\epsilon} \cdot \frac{1}{(1-\cancel{\epsilon)}} \frac{1}{\alpha g_e I_{sp}}$$

$$= \frac{1}{\epsilon}\left[1-\frac{1}{\alpha g_e I_{sp}}\right]$$