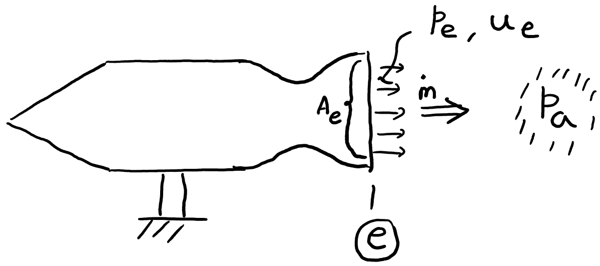
Rockets



$$\tau = \dot{m}u_e + (p_e - p_a)Ae \quad (10\text{-}4)$$

u_{eq} - equivalent exhaust velocity
(aka) effective exhaust velocity

$$ueq = \frac{\tau}{\dot{m}} = u_e + \frac{(p_e - p_a)Ae}{\dot{m}} \quad (10.5)\ \&\ (10.6)$$

since $\dot{m} = \rho_e u_e Ae$,

$$u_{eq} = u_e + \frac{(p_e - p_a)A_e}{\rho_e u_e A_e} = u_e + \frac{(p_e - p_a)}{\rho_e u_e}$$

Using the ideal gas equation $p = \rho RT$,

$$u_{eq} = u_e + \frac{RT\mathrm{e}}{u_e}\left(1 - \frac{p_e}{p_a}\right)$$ [verify]

The specific impulse (I_{sp}) of a rocket is

$$I_{sp} \stackrel{\text{def}}{=} \frac{\tau}{\dot{m}g_e}$$ [thrust per unit weight flow rate of propellant]

sea-level, earth ($9.81\,m/s^2$)

$$I_{sp} = \frac{u_{eq}}{g_e} \quad (10.7)$$ [units of seconds]

specific impulse of chemical rockets varies from 200 s to 450 s.
" " " nuclear " is around 900 s
" " " electrostatic thrusters is greater than 2000 s.

Example Isp values:

RD-170 (Atlas) : 309 s @ SL

RS-25 (SSME) : 453 s in vac.

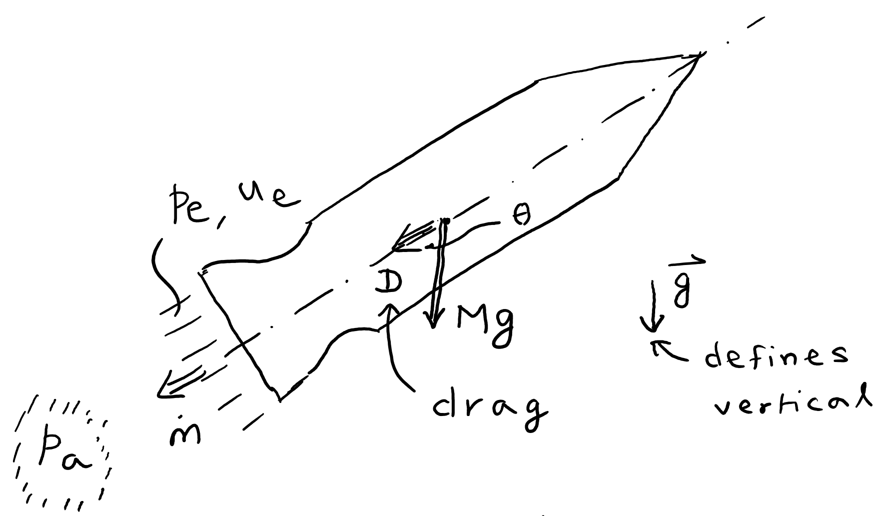
solid motor

R SRM : 267 s @ SL

reusable rocket

<u>Vehicle acceleration</u>, (sec 10.3)

Fig 10.3



initial ($t=0$) mass

Mass : $\dfrac{dM}{dt} = -\dot{m} \implies M = M_o - \dot{m}t$

mass of rocket at time t

Burnout mass is M_b

$M_b = M_0 - \dot{m}t_b$ burn time

Also, observe $M_p = \dot{m}t_b$

$M_b = M_0 - M_p$

Mass ratio $\mathcal{R} \stackrel{\text{def}}{=} \dfrac{M_0}{M_b}$

Momentum Equation :

accln. due to gravity

$$M \frac{du}{dt} = \dot{m} u_{eq} - Mg \cos\theta - D$$

inclination of flight direction to vertical

Acceleration (a) :

$$a = \frac{du}{dt} = \frac{\dot{m} u_{eq}}{M} - g \cos\theta - \frac{D}{M}$$

Launch accln (a_o) :

($M = M_o$)

$\theta = 0$ [(typical) vertical launch]

$$a_o = \frac{\dot{m} u_{eq}}{M_o} - g_e \cos 0 - 0$$

@ SL, earth

$$\Rightarrow a_o = \frac{\dot{m} u_{eq}}{M_o} - g_e$$

Burnout accln (a_b) :

($M = M_b$)

$$a_{b^-} = \frac{\dot{m} u_{eq}}{M_b} - g \cos\theta_b - \frac{D_b}{M_b}$$

just before burnout

$$a_{b^+} = - g \cos\theta_b - \frac{D_b}{M_b}$$

just after burnout

a_b cannot be found without knowledge of burnout altitude (g depends on altitude), inclination angle at burnout (θ_b) and the drag (D_b) then. The magnitude of a_{b-}, however, will be significantly larger than that of a_0 since the propellant mass is a large fraction of the rocket initial mass (91 % for the lunar rocket, 86 % for the space shuttle, for example).

* short burn time results in high $\dot{m}$ $\Rightarrow$ large a_{b-}. Thus t_b should not be too small.

* Long burn time may result in insufficient a_0, and high gravity and drag losses (disussed later).

* Typical burn times for launch vehicles are on the order of several minutes {RSRM 2 min, SSME 8 min, lunar rocket 1.5 min (stage 1), 6 min (stage 2), 8 min (stage 3), Falcon 9 2.6 min (stage 1), 7 min (stage 2)}.

Velocity Rise

The function of a rocket is to provide velocity rise to the payload. To find it, integrate the acceleration equation.

$$\frac{du}{dt} = \frac{\dot{m}}{M} u_{eq} - g\cos\theta - \frac{D}{M}$$

since $\dot{m} = -\frac{dM}{dt}$,

$$\frac{du}{dt} = -\frac{u_{eq}}{M}\frac{dM}{dt} - g\cos\theta - \frac{D}{M}$$

$$\Rightarrow du = -u_{eq}\frac{dM}{M} - g\cos\theta\, dt - \frac{D}{M}dt \quad (10.9)$$

$$\int_{u=u_0}^{u=u_b} du = -\int_{M_0}^{M_b} u_{eq}\frac{dM}{M} - \int_{t=0}^{t=t_b} g\cos\theta\, dt - \int_{t=0}^{t=t_b}\frac{D}{M}dt$$

$$\Rightarrow (u_b - u_0) \equiv \Delta u_b = u_{eq}\ln\frac{M_0}{M_b} - \overline{g\cos\theta}\, t_b - \overline{\left(\frac{D}{M}\right)} t_b,$$

where $\overline{g\cos\theta} = \frac{\int_0^{t_b} g\cos\theta\, dt}{t_b}$, and $\overline{\left(\frac{D}{M}\right)} = \frac{\int_0^{t_b}\left(\frac{D}{M}\right)dt}{t_b}$

are just concise representations of the gravity and drag integrals, respectively.

Impulsive velocity rise ($\Delta u_{b,imp}$)

Impulsive velocity rise is defined to be the velocity rise attained with zero burn time ($t_b = 0$).

$$\Delta u_{b,imp} = u_{eq} \ln \frac{M_o}{M_b}$$

$$\Delta u_b = \Delta u_{b,imp} - \overline{g \cos\theta}\, t_b - \overline{\left(\frac{D}{M}\right)}\, t_b$$

It is clear that finite burn times decrease the attainable velocity rise. Longer the burn, greater are the gravity and drag losses (mentioned earlier).

Gravity

$$\frac{g}{g_e} = \left(\frac{R_e}{R_e + h}\right)^2 \qquad (10.12)$$

g - accln. due to gravity at altitude h

g_e - " " " " at earth's surface

R_e - radius of the earth ($R_e = 6378$ km)

Drag

$$D = C_D \frac{1}{2} \rho u^2 A_f \qquad (10.14)$$

D – drag

ρ – density $\left\{\begin{array}{l}\rho = a\exp(-bh^{1.15}),\ a = 1.2, \\ b = 2.9(10^{-5}),\ h \text{ in m},\ \rho \text{ in } \frac{kg}{m^3}\end{array}\right\}$

u – velocity

A_f – frontal area

C_D – drag coefficient {see Fig 10.4}

Example

Consider a rocket

$\dot{m} = 8740$ kg/s, $u_e = 2600$ m/s, $p_e = 71$ kPa,

$A_e = 11.35\ m^2$, $M_0 = 1{,}720{,}000$ kg, $M_b = 680{,}000$ kg

vertical ascent ($\theta = 0$), $g = g_e$. Neglect drag.

Find T_{SL}, $u_{eq,SL}$, $I_{sp,SL}$, $u_{eq,vac}$, $I_{sp,vac}$,

a_0, a_{b-}, a_{b+}, Δu

$$T = \dot{m}u_e + (p_e - p_a)A_e$$

$p_a = 101$ kPa at SL.

$$T_{SL} = \dot{m}u_e + (p_e - p_{a,SL})A_e$$

$$= (8740)(2600) + (71-101)(1000)(11.35)$$

$$= 2.24(10^7)\ N$$

$$u_{eq,SL} = \frac{T_{SL}}{\dot{m}} = \frac{2.24(10^7)}{8740} = 2563\ m/s$$

$$I_{sp,SL} = \frac{u_{eq,SL}}{g_e} = \frac{2563}{9.81} = 261\ s$$

$p_a = 0$ at vacuum

$$T_{vac} = \dot{m} u_e + (p_e - p_{a,vac}) A_e$$

$$= (8740)(2600) + (71 - 0)(1000)(11.35)$$

$$= 2.35 (10^7) \text{ N}$$

{ p_e, u_e, $\dot{m}$ fixed for a rocket - important observation }

$$u_{eq,vac} = \frac{T_{vac}}{\dot{m}} = \frac{2.35(10^7)}{8740} = 2689 \text{ m/s}$$

$$I_{sp,vac} = \frac{u_{eq,vac}}{g_e} = \frac{2689}{9.81} = 274 \text{ s}$$

$$a_0 = \frac{\dot{m} u_{eq,SL}}{M_0} - g_e = \frac{(8740)(2563)}{1,720,000} - 9.81$$

$$= 3.21 \text{ m/s}^2 = 0.33 g_e$$

$$a_{b^-} = \frac{\dot{m} u_{eq,SL}}{M_b} - \underbrace{g_b}_{g_e \text{ (constant gravity)}} \overbrace{\cos \cancel{\theta_b}^{0}}^{1} - \cancel{\frac{D_b}{M_b}}^{0 \text{ (neglect drag)}}$$

(vertical ascent)

$$= \frac{(8740)(2563)}{680,000} - 9.81 = 23.1 \frac{\text{m}}{\text{s}^2} = 2.36 g_e$$

Burn time $t_b = \frac{M_p}{\dot{m}} = \frac{(M_0 - M_p)}{\dot{m}}$

$$= \frac{(1,720,000 - 680,000)}{8740} = 119 \text{ s}$$

$$\Delta u_b = u_{eq} \ln \frac{M_0}{M_b} - \int_0^{t_b} g \cos\theta \, dt - \cancel{\int_0^{t_b} \left(\frac{D}{M}\right) dt} \to 0$$

$$\left\{ \int_0^{t_b} g \cos\theta \, dt = \int_0^{t_b} g_e \cos(0) \, dt = g_e \int_0^{t_b} dt = g_e t_b \right.$$

$$\left. \overline{g \cos\theta} = \frac{\int_0^{t_b} g \cos\theta \, dt}{t_b} = \frac{g_e t_b}{t_b} = g_e \right\}$$

$$\Delta u_b = u_{eq} \ln \frac{M_0}{M_b} - \overline{g \cos\theta} \, t_b$$

$$= 2563 \ln \frac{1{,}720{,}000}{680{,}000} - (9.81)(119)$$

$$= 2378 - 1167 = 1211 \text{ m/s} = 1.21 \text{ km/s}$$

Observe the significant magnitude of the gravity loss. In a LEO launch, the gravity and drag losses can amount to 20 % of the Δu_b the rocket is designed to provide.