PHILIP G. HILL
Department of Mechanical Engineering
Massachusetts Institute of Technology

CARL R. PETERSON
Research and Development, Ingersoll-Rand Company
Bedminster, New Jersey

MECHANICS AND THERMODYNAMICS OF PROPULSION

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Preface

This book is based on an introductory course in propulsion which has been given for several years to students in aeronautical and mechanical engineering at Massachusetts Institute of Technology. It deals mainly with the propulsion of aircraft and space vehicles.

The objective of the book is to show how basic principles may be used to predict the behavior of propulsion devices. There are, of course, many methods of propulsion, but the underlying physical principles are relatively few. Often there is a large gap between theoretical studies and their useful application to actual engineering problems. This book does not pretend to close such a gap, but it does stress the application of the student's theoretical knowledge to several real devices, pointing toward the engineering objectives of propulsion. Experience has shown that the application of fundamentals to these problems can reward the student not only with a valuable appreciation of their practical significance but also with a deeper understanding of the principles themselves. To this end, the book attempts to show how basic physical laws both describe and limit the performance of particular devices.

With the present trend toward increasingly basic engineering curricula there remains a definite need for at least one course in which a variety of fundamental ideas are brought together and made to interact so that their engineering implications can be clearly seen. Propulsion is an admirable subject for this purpose, since it significantly involves so many disciplines. This text may therefore serve a more general purpose than simply conveying knowledge of propulsion devices to the student.

In general, the most satisfactory performance of a given propulsion engine is determined not only by the laws of mechanics and thermodynamics, but also, to a large extent, by the behavior of materials. The main emphasis in this book is on the former, though material limitations are identified and discussed.

The mathematical portions of the text assume a knowledge of calculus. In addition, an introductory course in vector calculus is desirable, but not essential. An attempt has been made to minimize mathematical complication so that physical principles can be more readily discerned, and the problem sections at the ends of

the chapters are designed to give the reader an idea of the practical applications of these principles.

Throughout the text much attention has been given to the "working fluid" of the propulsion engine. Generally it is considered to be a continuum, since the principal interest lies in its macroscopic behavior. Two exceptions to this rule occur. First, in the study of plasma, it is necessary to examine microscopic processes in order to understand and predict the macroscopic behavior. Second, in the study of ion or electron engines, it is often necessary to discuss the behavior of discrete particles.

The control volume method of analysis, whose usefulness for fluid flow has been demonstrated so thoroughly by Shapiro, has been used extensively. One-dimensional approximations are frequently employed, but certain two-dimensional phenomena like heat transfer and other boundary layer behavior are also discussed.

Basically, two classes of propulsion devices are considered: (1) air-breathing engines, and (2) rockets.

Air-breathing engines have been under very intensive development over the past 25 years. They are particularly instructive examples of engineering achievement, because of both the magnitude and the success of the development effort. It is most illuminating to see the power and the limitations of theoretical methods and to observe the mixture of art and science in their evolution. Although it has now approached some degree of completion, development continues toward better performance and new applications, so that air-breathing engines will remain a source of challenging engineering problems for some time to come.

Rocket engines, by comparison, are currently under much more active development. Much is now known about chemical rockets, but nuclear and electrical rockets are only entering the experimental stage. The latter are nonetheless discussed in this text, since current developments indicate that nuclear and electrical phenomena will be important in future propulsion engines. There is need for an introductory discussion of the application of these phenomena to propulsion.

Part 1 is concerned mainly with a review of those topics in thermodynamics, combustion, and fluid mechanics which are relevant to propulsion engines. One of the purposes of this review is to introduce necessary concepts and laws in the notation used in subsequent chapters. In Chapter 2 the thermodynamics of equilibrium combustion reactions are discussed in some detail because of their relevance to chemical rockets especially. Since propulsion systems generally deal with accelerating fluids, the gas dynamics of isentropic expansion are reviewed in Chapter 3, along with the modifying effects of friction at solid boundaries, and shocks. In many engines, e.g., the turbojet, the viscous boundary layer exercises a controlling effect on the performance of the engine; hence the question of boundary layer behavior is discussed in some detail in Chapter 4. Convective heat transfer is also briefly reviewed, chiefly because of its relevance to the cooling of rocket motors. Electrostatic and electromagnetic acceleration are reviewed in Chapter 5 because they are centrally important in the rising generation of propulsion engines for space flight. We also give an elementary description of the behavior of charged particle beams and conducting fluids, in order to aid understanding of relevant phenomena, though it is not entirely clear at present which phenomena will be of greatest significance in working electrical-thrust engines.

Part 2 is confined to air-breathing engines. Chapter 6 demonstrates that the performance of any air-breathing engine can be predicted from the laws of thermodynamics, given the maximum permissible temperature and the appropriate efficiencies of engine components, e.g., the compressor. Chapter 7 deals with the components of the ramjet (diffuser, combustor, and nozzle) and shows how their efficiencies depend principally on aerodynamic phenomena. The essential difference between the ramjet and the turbojet is simply the turbomachinery, which is the subject of Chapters 8 and 9. Fortunately the turbojet is so highly developed that several interesting examples of actual engine configurations and performance can be cited by way of illustration.

Part 3 is confined to rockets. Chapter 10 presents an elementary treatment of rocket vehicle performance in order to show the significance of specific impulse and other variables, and also to indicate the relative advantages of low- and high-thrust engines for various missions. The trajectories of space vehicles are briefly discussed in order to show what they require of the propulsion system. Chemical rockets are the subject of Chapters 11, 12, 13 and 14, which take up a series of pertinent topics. The combustion process, the gas dynamics of the nozzle, and the cooling of the wall are all examined in some detail, as well as auxiliary equipment such as the turbopump. Chapter 15 is concerned with the nuclear rocket, in which a nuclear reactor is used to heat a gas such as hydrogen to high temperature before it expands in the nozzle. The limitations of this engine and the phenomena that govern its performance are briefly discussed. Chapter 16 focuses on electrostatic and electromagnetic rockets, although these are in early stages of development. Considerable experimental work has been done on the components of the electrostatic rocket, and it appears relatively promising for future application, provided suitable electrical energy conversion machinery can be developed. The electrothermal and three kinds of possible electromagnetic rockets are briefly described.

This text may be used for either a one- or two-term course in propulsion. For a two-term sequence, the first term could be used for either air-breathing or rocket engines. A one-term course could be based on Chapters 6, 10, and 11, with selected portions of Parts 2 and 3 and appropriate review material.

The authors would like to record their indebtedness to Professor Edward S. Taylor, who has been our teacher, critic, and friend during our years of association in the M.I.T. Gas Turbine Laboratory. The text has been written largely as a result of his encouragement, and it may be that his influence will be recognizable in some of the more worthwhile portions. We are certainly grateful to him for writing the introductory chapter.

Throughout the text, there are a number of references to the work of Professors J. H. Keenan and Ascher H. Shapiro. Their expositions of thermodynamics

and fluid mechanics were so enlightening to us as students that we could not help drawing heavily from them in presenting the subject of propulsion.

The preparation of earlier editions of classroom notes upon which the book is based was supported in part by a grant made to the Massachusetts Institute of Technology by the Ford Foundation for the purpose of aiding in the improvement of engineering education. This support is gratefully acknowledged.

The authors would also like to acknowledge the diligent and efficient work of Mrs. Madelyn Euvrard and Mrs. Rose Tedmon in piecing together many fragments and revisions of the text.

Cambridge, Massachusetts P.G.H.
October 1964 C.R.P.

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PART 1

Introduction and Review of Fundamental Sciences

Kitty Hawk, North Carolina, 1903: man's first plane flight. The Wright brothers' famous flying machine was the first plane to lift a man into the air. Orville Wright took the first airplane ride, lying face downward on the lower wing of the biplane. His ride, which was only a few feet above the ground, lasted 12 seconds and covered 120 feet before running into the ground. Photograph courtesy the Library of Congress and United Airlines.

1

Propulsion*

1–1 THEORY OF PROPULSION

The verb "propel" means to drive or push forward or onward. Clearly this definition implies the existence of a body, a force (drive or push), and a preferred direction (forward or onward). According to Newton's third law of motion, forces occur only in equal and opposite pairs, and the existence of a force on a body therefore entails a reaction. This reaction cannot be on the body itself, since a pair of equal and opposite forces on the same body will not "drive or push forward or onward." Therefore one or more additional bodies are essential to propulsion. For a self-propelled vehicle like an automobile, the second body is the earth itself, and the act of propulsion entails a transfer of momentum between the automobile and the earth. In cases of particular interest in this text, the second body is usually a fluid medium. In the instance of a propeller-driven airplane, this fluid is the surrounding air, while in the case of a rocket it is a fluid carried along with, and discharged from, the rocket itself. In a jet airplane, the major part of the fluid used in propulsion is the surrounding atmosphere, but the fuel, which may constitute 2% of the propulsive fluid, is carried along and discharged as in a rocket.

If we consider an airplane in straight level flight at uniform speed, it is clear that the horizontal component of the resultant force acting on the airplane must be zero, in order to satisfy Newton's second law of motion. We know that to propel the airplane it is necessary to overcome the drag (frictional and induced) of the fluid on the body. If this is done by means of a propeller, the force of the propeller on the airplane must be sufficient to cancel the drag forces. The propeller can produce this force only by transferring momentum to the stream of air which passes through it. The drag forces also transfer the momentum to the surrounding air; and indeed, the horizontal momentum change due to the propeller thrust and the horizontal momentum change due to drag forces must be equal and opposite,

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* An introduction by Edward S. Taylor, Professor of Flight Propulsion, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology.

whether we are speaking of the momentum of the airplane or of the fluid medium surrounding it. Thus, although the air is disturbed by the passage of such an airplane, there is no net change in its horizontal momentum before and after the passage.

If we consider an accelerating rocket (assumed, for the purpose of simplifying the argument, to be in gravity-free flight outside the atmosphere), the total momentum of the rocket and the material which it has discharged must remain constant. It is inconvenient to keep track of the momentum of the propellant after it leaves the rocket, and we shall see (Chapter 2) that in order to calculate acceleration of the rocket, it is necessary only to observe the momentum of the jet leaving the rocket at the nozzle discharge. Clearly the principles of propulsion are merely applications of Newton's laws of motion.

Long before Newton announced his laws of motion (1687) which now make the principles of propulsion comprehensible to us, devices for effecting propulsion were known and used. The Chinese had rockets as early as the twelfth century, and screw propellers are visible in some of Leonardo da Vinci's drawings. About 200 B.C., the Egyptian philosopher Hero invented and constructed a turbine which clearly shows the principle of jet propulsion. In propulsion, as in most other technological advances, the art preceded the science. However, it is highly improbable that man could ever have produced supersonic aircraft or rockets capable of putting vehicles in orbit around the earth without a very good theory of propulsion. By "theory" we mean a method of generalizing the results of experiments so that we can, with some confidence, predict the results of new experiments.

A good theory enables one to see the main points without getting confused by details, and makes it possible to perform limit analysis; e.g., to determine what could be done if the machinery could be made without losses. Furthermore, a theory facilitates communication between workers. Moreover, even poor theory may stimulate thought and experiment which will lead to new ways of accomplishing things. A good theory can prevent wasting effort on unworkable devices.

Many erroneous ideas of propulsion by jets were prevalent as late as the 1920's. It was well known at this time that a fireboat directing its streams of water at a fire would gradually move away from the fire. Eminent professors of engineering have been heard to scoff at this observation, saying that "you can't push on a rope." This statement, of course, reflects a profound ignorance of the problem, which has nothing to do with whether the streams from the fire nozzle ever reach any target.

Goddard, at about this time, found it necessary to demonstrate that his rockets had thrust even when fired in a vacuum. This points up the prevalence of the same fallacy as existed in the fireboat problem. It apparently was difficult for people to conceive of something propelling itself without some medium or body to push on. The idea of the device pushing on a medium which was being discarded was understood by only a few. Now it is a part of the knowledge of most if not all newly graduated mechanical and aeronautical engineers.

1–2 DEVELOPMENT OF PROPULSIVE DEVICES

The history of propulsion illustrates the interaction between economic or military need and technological advances. Although da Vinci's sketches of screw propellers date from the fifteenth (or possibly early sixteenth) century, the need for such a device did not arise until the development of the steam engine provided an appropriate driving mechanism. The first successful steamboats were driven by paddle wheels, but the paddle wheel must operate at the interface of two media (air and water) and is therefore clearly unsuitable for submarines, lighter-than-air ships, and airplanes. (It might also be said to be unsuitable for *any* ship when the sea is rough and the interface is shifting about.) The first successful submarine, the first airship, and the first airplane all had screw propellers. The airplane propeller still serves well when the speed of the aircraft is less than 350 miles per hour. When man began to want to fly at higher speeds (originally for military reasons), a lighter and smaller power plant was essential. The jet engine answered this need, while at the same time avoiding the difficulties encountered by propellers operating at supersonic tip velocities. It is interesting to note that there was no incentive for developing a jet engine until the art of airplane design had advanced enough to make high-speed flight practical. As a matter of fact, the jet is inappropriate to low-speed flight, and the propeller provided an essential first step without which we would almost certainly not have developed airplanes at all. An interesting history of the development of aircraft engines is to be found in Reference 4.

The jet engine is a good example of the solution of a technical problem—that of high-speed flight—which required that many conditions exist simultaneously. First the need had to exist; then there had to be sufficient mastery of the art of airplane design, including aerodynamics, structure, and control; and finally there had to be the knowledge of turbomachinery design and the availability of materials capable of withstanding high temperatures. These are only some of the major hurdles. Of course the solution of the problem also depended on a sophisticated manufacturing technique—forging, casting, cold-forming, welding, and machining—and on the availability of countless "minor" but essential items: bearings, seals, lubricants, pumps, control devices, etc. In any development as complex as the jet engine, special circumstances are bound to arise which require changes and advancement of the technology in these "minor" items. However, without a reasonable state of development in these as well as the major items, the development period would be likely to stretch out until it became clear that the project had better be postponed until the necessary supporting technology was able to catch up. John Barber patented a gas turbine in 1791. There is little doubt that Barber had insufficient knowledge of basic mechanics, and certainly insufficient knowledge of fluid mechanics to develop a successful gas turbine; but even if he had had all the fundamental knowledge we have today, his job would still have been impossible until the general technological level had risen enormously. Materials and manufacturing techniques were not adequate even to begin this task until well over a

hundred years later. Furthermore, had he been able to build a gas turbine, there would have been no economically feasible use for it.

History is full of such examples of inventions made too soon. They emphasize the importance of the engineering task of determining when a need exists and when supporting technology, both in theory and practice, are sufficiently advanced to begin a new development. To determine these facts involves a knowledge or at least an estimate of what extensions of theory and what refinements of practice are essential to its success.

The first successful aircraft propulsion device was Henry Gifford's steam-engine, propeller-driven dirigible balloon, which went from Paris to Trappes in 1851 at an average speed of about seven miles per hour. However, for an airplane power plant, the necessary water supply (or condenser) and boiler would be so unreasonably heavy that even if the steam engine weighed nothing, only very poor airplane performance would be possible. The successful airplane, in a practical sense, had to wait for the development of the internal combustion engine, which eliminated the necessity for either a water supply (or condenser) or a boiler. At the time of the Wright brothers' experiments, available internal combustion engines were still too heavy, but they did exist, and the time was ripe for their development to a reasonable engine weight per horsepower. The Wright brothers had to take on this problem [1], along with the design and construction of the first powered airplanes to demonstrate controlled flight. Langley, who was facing the same problem at the same time, found an extraordinary engineer in Manley [2], who handled this part of the problem with outstanding success.

The possibility of propulsion by means of a jet had been well known [3] for many years before it was usefully applied by von Ohain (first flight August 27, 1939) and Whittle (first flight May 15, 1941). Before jet propulsion could become feasible, it was essential that airplanes be designed and built for high speed, and also it was necessary that the art of design and construction of high-temperature turbines and efficient compressors be advanced considerably to meet the stringent requirements (primarily light weight combined with high efficiency) of a useful aircraft engine.

The development of the sophisticated rocket is an illustration of a similar series of events. The Chinese seem to have used the rocket mostly for fireworks, although there is some indication of military use. In the early days of artillery, accuracy was so poor that even a crude unguided rocket could compete, and about the beginning of the nineteenth century the British officer Congreve was instrumental in developing rockets as a substitute for artillery. There is no doubt that these rockets were useful in frightening cavalry horses, and perhaps were not much less accurate than the artillery of the time. The "rocket's red glare" of the national anthem refers to Congreve rockets in the War of 1812.

Subsequently artillery was greatly improved and military rockets became obsolete. Rockets, in order to be effective, would have to be guided. The necessary understanding of control problems came about in the 1930's, and the requisite development of this technology came soon after.

While the German V-2 rocket, with a 2000-pound warhead and a range of about a hundred miles, had considerable value as a morale-builder for the Germans, the damage done by these rockets was minor and the accuracy was poor compared with the effort involved. However, with the advent of nuclear devices, the potential explosive power of a warhead of given size was multiplied by an enormous factor, and the effectiveness of a long-range missile with even poor accuracy became enormous. This, then, was the motivating force behind the development of rocket propulsion; it was inevitable that billions of dollars would be poured into the development of such devices. Except for the guidance devices (where enormous advantages were yet to be obtained by subsequent improvements), the art of rocketry was reasonably ready and the science had been ready for some time, with the exception of one factor: the reentry of a rocket at high speed. This problem was not sufficiently understood until much later, but practically speaking it proved to be less difficult than was feared. Thus in the forties the need was evident and the technology and science existed. With reasonable luck and not-unreasonable improvement in the technology, the job could be done.

1–3 FUTURE POSSIBILITIES

At the present writing (1964) we see successful chemical rockets, and the need, if we are to travel far in the solar system, for vastly improved means of propulsion. The possibilities now appear to be:

(1) Improvement in chemical rockets, which undoubtedly will take place, but may be insufficient to answer the needs for exploring the solar system.

(2) Nuclear rockets. These hinge on our ability to make nuclear reactors which will operate at very much higher temperatures than they do at present.

(3) Some form of electrical propulsion. This appears to require better understanding of the properties and behavior of high-temperature ionized gases, and considerable improvement in the means for producing electric power in space.

What route we shall take from here is unclear. It may be that improvement in conventional chemical rockets will offer the least expensive alternative. If we can judge from history, it may even be that we shall wait for new scientific discoveries, new materials, and new improvements in the technology of manufacturing before the task can be accomplished. In any case we must remember that a continuing strong desire to do the job will be essential to its accomplishment.

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2

Mechanics and Thermodynamics of Fluid Flow

2–1 INTRODUCTION

An understanding of fluid mechanics and thermodynamics is perhaps the most important prerequisite for the study of propulsion. In nearly all the propulsion methods discussed in this volume, thrust is developed by imparting momentum to fluid streams. Most of the methods involve thermal effects in one way or another. The purpose of this chapter is to set forth a concise statement of the principles of mass, momentum, energy, and entropy in a form which is suitable to the treatment of fluid streams.

We shall assume that the student is thoroughly familiar with the first and second laws of thermodynamics and their corollaries as applied to ordinary nonreacting continuum substances in equilibrium. Hence, after careful definition of the concepts of system and control volume, these laws will merely be stated in forms which the reader will find useful in the remainder of the text. The thermodynamics of equilibrium chemical reactions will then be treated in some detail, because of its relevance to combustors in propulsion engines. Considerable literature is available as background material for the subject matter of both this chapter and Chapter 3. This summary largely follows the material of Keenan [1] and Shapiro [2].

The concepts of *system* and *control volume*, as applied in the following pages and throughout the text, have very specific meanings. The purpose of these concepts is the specification either of a definite collection of material or of a region in space which is to be analyzed.

System. A system is a collection of matter of fixed identity. It may be considered enclosed by an invisible, massless, flexible surface through which *no matter* can pass. The boundary of the system may change position, size, and shape.

Control volume. A control volume is a region of constant shape and size which is fixed in space relative to the observer. One can imagine an invisible, massless, rigid envelope (the control surface) enclosing the control volume. The control surface offers no resistance to the passage of mass and is arbitrarily located. Analyses to be developed later will show that the most suitable location of a control surface in any particular problem is a matter of convenience.

Most problems may be treated by either system or control volume analyses. However, the proper choice often leads to substantial simplification. For example, most fluid-flow problems, and especially steady flows, are much more conveniently handled in terms of an appropriately specified control volume.

Continuum. In the following discussions the fluids under consideration will be treated as continua. That is, their atomic structure will be ignored and they will be considered as capable of being subdivided into infinitesimal pieces of identical structure. In this way it is legitimate to speak of the properties of a continuum—e.g., density, pressure, velocity—as point properties.

2–2 THE LAWS OF MECHANICS AND THERMODYNAMICS FOR FLUID FLOWS

Mass conservation and the continuity equation

The law of mass conservation applied to a system simply requires that the mass of the system remain constant regardless of its size or shape, the number of pieces into which it is divided, or the length of the time interval during which it is observed (for "nonrelativistic" system velocities).

Here, as in most cases, the physical law is stated as it applies to a system. This is inconvenient for the stationary observer of fluid flow. Such an observer is unable to continuously measure the behavior of a moving fluid particle (a small system) and must, therefore, content himself with measurements taken at specific points in the space through which the fluid moves. The basic law must then be expressed in terms of the required relationships between fluid properties at various stationary points; i.e., at points within a control volume.

Consider a general control volume defined within a region of fluid flow as shown in Fig. 2–1. The conservation of mass simply requires that the rate of change of mass stored within the control volume be equal to the net inflow of mass:

$$\frac{d}{dt} m_{cv} = \dot{m}_{in} - \dot{m}_{out}, \tag{2–1}$$

where m_{cv} is the mass within the control volume and $\dot{m}$ is the indicated mass flow rate. These terms can be expressed in more convenient analytical forms. Consider the mass flow through a small surface element dA on the control surface. If the local velocity is $\mathbf{u}$ and a unit vector normal to dA is $\mathbf{n}$ (outward pointing) as in Fig. 2–1, the elemental mass flow rate is

$$d\dot{m} = \rho|\mathbf{u}| \cos \alpha \, dA,$$

FIG. 2–1. Fluid flow through a control volume.

where ρ is the local density and α is the angle between $\mathbf{u}$ and $\mathbf{n}$, the product $|\mathbf{u}| \cos \alpha$ being positive for outflow and negative for inflow. The mass within the control volume m_{cv} can be found by the integration

$$m_{cv} = \int_{cv} \rho \, d\mathcal{V},$$

where $d\mathcal{V}$ is a volume element and the integral extends throughout the control volume. Finally, writing the product $|\mathbf{u}| \cos \alpha$ as the usual *dot product* $\mathbf{u} \cdot \mathbf{n}$, the *continuity equation* may be written

$$\frac{d}{dt} \int_{cv} \rho \, d\mathcal{V} + \int_{cs} \rho \mathbf{u} \cdot \mathbf{n} \, dA = 0, \tag{2–2}$$

where the second integral extends over the entire control surface.

Newton's second law and the momentum equation

For a system of mass m acted on by a force $\mathbf{F}$, Newton's law requires that

$$\mathbf{F} = K \frac{d}{dt} (m\mathbf{u}), \tag{2–3}$$

where K is simply a proportionality constant whose numerical value depends entirely on the units selected for the three variables in the equation. It is possible to choose units of force, mass, length, and time such that K is unity, and this is done in all the following equations.

In a derivation very similar to the preceding (see, for example, Chapter 1 of Reference [2]), the consequences of this system equation can be written for a control volume in the following vector equation form, where the notation is that of Fig. 2–1:

$$\sum \mathbf{F} = \frac{d}{dt} \int_{cv} \rho \mathbf{u} \, d\mathcal{V} + \int_{cs} \rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n}) \, dA. \tag{2–4}$$

The term $\sum \mathbf{F}$ is the vector sum of *all* forces acting on the material within the control volume. Forces may be one of two kinds: surface forces (stresses) and

body forces. The two types of stresses are pressure, which is a stress normal to the surface, and shear, which is a stress tangential to the surface. Body forces are those which are proportional to the mass or volume of the fluid and include gravitational, electrostatic, and electromagnetic forces.

The first law of thermodynamics and the energy equation

The first law of thermodynamics relates *heat* and *work* interactions between a system and its environment to changes in the state of the system. To avoid difficulty in application of this law it is necessary to give each of these terms a rather careful definition. The following definitions are not the only ones which may be used, but they are the basis of the well-established method of expressing the principle of the first law used in this volume.

Heat. When two systems at different temperatures are brought together, an interaction occurs. This *interaction* is called *heat*. The mathematical sign of the heat is conveniently taken as positive for the lower-temperature (cold) system and negative for the higher-temperature (hot) system. It is common to speak of heat *transfer*, saying that heat transfers from the hotter system to the colder. Thus heat transfer *to* a system is positive and *from* a system is negative. If this terminology is used, we must carefully avoid the implication that heat is a storable quantity. Heat exists only as a transfer process or interaction and *cannot be stored.* A process which does not involve a heat transfer is said to be adiabatic.

Work. A system is said to do work on its surroundings when the sole effect external to the system could be the rise of a weight in a gravitational field. Work done by a system on its surroundings is, by convention, positive; work done on a system by its surroundings is negative. Work, like heat, is an interaction between systems, and is not a storable quantity. In general, the magnitude of the work depends on the motion of the observer.

The first law of thermodynamics may be stated for a system in the form

$$Q = \Delta E_0 + W, \tag{2–5}$$

in which ΔE_0 denotes the change in internal energy of a system subjected to heat and work interactions Q and W, respectively.*

The internal energy, as its name suggests, is a property of the system. The first law in a sense *defines* this property except for an arbitrary datum value. Internal energy *is* a storable quantity, in contrast to heat and work which may be considered energy transfers. The first law itself does not distinguish between heat and work;

* The notation throughout this text differs slightly from the more common fundamental texts in that it attempts to present a unified system for the several disciplines covered. E_0 (seldom used) refers to internal energy *including* kinetic and potential energies, while E refers to internal energy without these terms.

both contribute to changes in the internal energy. The second law, however, shows that they are fundamentally different.

It is customary to treat separately those portions of E_0 due to motion [kinetic energy $m(u^2/2)$], position in a gravitational field (potential energy mgz), charge in an electrical field (electrical potential energy qV), and other effects. For most ordinary fluid-mechanics problems it is sufficient to consider kinetic and potential energies only and write, for a system of mass m,

$$E_0 = E + m\frac{u^2}{2} + mgz, \tag{2–6}$$

where z is the height above some arbitrary datum and g is the (magnitude of the) local acceleration due to gravity. Further, for continua, it is convenient to denote the values of E_0 and E *per unit mass* as e_0 and e, respectively. Using this and Eq. (2–6), Eq. (2–3) can be written

$$Q = \Delta \int_s \left(e + \frac{u^2}{2} + gz\right) \rho \, d\mathcal{V} + W, \tag{2–7}$$

where $\int_s$ denotes integration throughout the entire system. To express the first law of thermodynamics for a control volume, it is often convenient to introduce heat and work transfer rates. Equation 2–7 may be transformed (see again Chapter 1 of Reference [2]) in the same way as Eqs. (2–2) and (2–4), with the result that

$$\int_{cs} \dot{\mathcal{Q}} \, dA = \frac{d}{dt}\int_{cv} (e_0\rho)\, d\mathcal{V} + \int_{cs} \left(h + \frac{u^2}{2} + gz\right) \rho \mathbf{u}\cdot\mathbf{n}\, dA + \mathcal{P}_s - \int_{cv} \mathbf{X}\cdot\mathbf{u}\, d\mathcal{V}, \tag{2–8}$$

where $\dot{\mathcal{Q}}$ = local heat transfer rate (per unit time per unit area), positive *to* control volume,

$h = e + p/\rho$ = enthalpy of fluid,

$\mathcal{P}_s$ = net "shear power" (work transfer rate), positive *from* control volume, and

$\mathbf{X}$ = body force per unit volume on fluid within control volume.

The shear power $\mathcal{P}_s$ is that work done by the material within the control volume on the surroundings at control surface points where both shear stress and material motion (parallel to the control surface) occur. Usually this term reduces to *shaft power* transmitted by shear stresses in rotating shafts. However, there are situations where the fluid may move along a control surface accompanied by a shear stress (at points within a boundary layer, for example). The last integral, which we shall not often have to consider, represents work transfer via body forces. The dot product $\mathbf{X}\cdot\mathbf{u}$ is the rate (per unit volume) at which work is done *on* the fluid *by* the environment through body force $\mathbf{X}$ (hence the negative sign preceding it).

It is instructive at this point to dwell on certain essential similarities of the continuity, momentum, and energy equations for a control volume. The common form of these equations may be seen by arranging them as follows:

Mass: $$\frac{d}{dt}\int_{cv} \rho \, d\mathcal{V} + \int_{cs} (\rho \mathbf{u} \cdot \mathbf{n}) \, dA = 0.$$

Momentum: $$\frac{d}{dt}\int_{cv} \rho \mathbf{u} \, d\mathcal{V} + \int_{cs} \mathbf{u}(\rho \mathbf{u} \cdot \mathbf{n}) \, dA = \sum \mathbf{F}.$$

Energy: $$\frac{d}{dt}\int_{cv} e_0 \rho \, d\mathcal{V} + \int_{cs} e_0(\rho \mathbf{u} \cdot \mathbf{n}) \, dA = \int_{cs} \dot{\mathcal{Q}} \, dA - \mathcal{P}_s + \int_{cv} \mathbf{X} \cdot \mathbf{u} \, d\mathcal{V} - \int_{cs} p\mathbf{u} \cdot \mathbf{n} \, dA.$$

In each equation the first term represents the rate of change of a certain quantity contained within the control volume. The second term is its rate of convection out of the control volume. The right-hand side of each equation may be called a production or source term. The zero on the right-hand side of the first equation means, of course, that mass can be neither produced nor destroyed. On the other hand, momentum can be brought into being by the application of force. Correspondingly, energy can be changed by heat transfer or work.

These equations will generally be greatly simplified for application in the following chapters, usually by assuming steady flow and single inlet and outlet states, in which case the integrations become trivially easy.

FIG. 2–2. Stationary jet engine.

To illustrate the use of these equations in a control-volume analysis, let us consider the generation of thrust by a stationary turbojet. Figure 2–2 shows the engine running steadily on a test stand. Air enters the inlet at a total mass flow rate $\dot{m}_a$ and fuel at a rate $\dot{m}_f$. Since the flow is steady, there cannot be a continuous accumulation of fluid within the engine, and the rate at which the fluid exhausts from the engine must be $(\dot{m}_a + \dot{m}_f)$. The exhaust velocity is u_e, and the force necessary to balance engine thrust is $\mathcal{T}$, as in Fig. 2–2.

In order to show how thrust depends on the properties of the jet, it is convenient to choose the control volume indicated in Fig. 2–2. Its surface intersects the exhaust jet in the plane of the nozzle, where the flow is nearly one-dimensional and the momentum flux can be measured more easily than at planes further downstream. The control surface extends far upstream of the engine and far from the floor on which it rests. Later we shall select another control volume, simply to illustrate the convenience of a suitable choice; but for the moment we proceed with the control volume of Fig. 2–2.

In applying Eq. (2–4) to this control volume, we confine our attention to those components of force and momentum parallel to the axis of the jet, so that the momentum equation may be written

$$\sum F_x = \int_{cs} \rho u_x(\mathbf{u} \cdot \mathbf{n}) \, dA,$$

in which the subscript x denotes the direction indicated in Fig. 2–2. The time-derivative term in Eq. (2–4) has been omitted because the flow is steady.

No body forces act in the x-direction, so that the total x-force on the control volume is due to pressure distributed over its surface and the reaction to the thrust $\mathcal{T}$. We assume the fuel hose to be flexible so that it does not transmit force to the control volume. The force $\mathcal{T}$ may be considered to be a normal force on the plane where the control surface intersects the structural member which holds the engine stationary. The pressure forces always act normal to the control surface, so that only the pressure forces on the two ends of the rectangular surface of Fig. 2–2 contribute to the x-summation of forces. Further, the pressure on these surfaces may be considered to have the constant value p_a, except possibly in the exhaust plane of the jet. There it may have a different value p_e, if the jet is sonic or supersonic. Thus the sum of the forces acting on the control volume in the x-direction can be written as

$$\sum F_x = \mathcal{T} + A_e p_a - A_e p_e,$$

in which A_e is the nozzle exhaust area. The momentum flux integral may be evaluated simply by recognizing that the momentum flux is infinitesimal except in the exhaust plane of the jet. There the velocity is normal to the surface, so that

$$|\mathbf{u} \cdot \mathbf{n}| = u_x = u_e,$$

and the momentum flux integral is therefore

$$\int_{cs} \rho u_x(\mathbf{u} \cdot \mathbf{n}) \, dA = \int_{A_e} \rho_e u_e^2 \, dA,$$

in which ρ_e is the fluid density at the exhaust plane. Considering the flow in this plane to be uniform (i.e., one-dimensional), the integral may be simplified to $\rho_e u_e^2 A_e$. The mass flow rate at this plane is $\rho_e u_e A_e$, and this has already been shown to equal $(\dot{m}_a + \dot{m}_f)$. Thus the momentum flux from the engine is

FIG. 2–3. Stationary jet engine.

$(\dot{m}_a + \dot{m}_f)u_e$, and the thrust may be obtained from Eq. (2–4) as

$$\mathcal{T} = (\dot{m}_a + \dot{m}_f)u_e + A_e(p_e - p_a).$$

Consider now the alternative control volume III indicated in Fig. 2–3. If we assume the flow in the engine inlet plane to be approximately one-dimensional, with pressure p_i and velocity u_i, then by reasoning very similar to the foregoing it may be established from control volume III that

$$\mathcal{T} = (\dot{m}_a + \dot{m}_f)u_e - \dot{m}_a u_i + A_e(p_e - p_a) - A_i(p_i - p_a).$$

This thrust equation is of course not so convenient as the previous one, because it contains three additional terms: A_i, u_i, and p_i. Other choices of control volume could lead to even more inconvenient results.

The two thrust equations can easily be shown to be equivalent, however. Using control volume II and Eq. (2–4) again, it may be shown in a similar way that

$$0 = \dot{m}_a u_i - A_i(p_a - p_i).$$

Using this equation, we can easily transform the second thrust equation to the first one.

The second law of thermodynamics and the entropy equation

The second law of thermodynamics states that it is impossible for a system to describe a cyclic process (i.e., to go through a process and return to its original state) which produces work and exchanges heat with a single reservoir of uniform temperature. Although we shall not use this statement directly, we shall use several of its important corollaries. The absolute temperature scale which we shall use is in fact based on this law [1]. Further, the second law permits the definition of the property entropy. For a system,

$$dS \equiv \left(\frac{dQ}{T}\right)_{\text{rev}}, \tag{2–9}$$

where dS is the change of entropy during a reversible heat exchange dQ at the temperature T. Keenan [1] and other thermodynamicists discuss the concept of thermodynamic reversibility extensively. For our purposes it is adequate to note that processes which involve any of the following are *not* reversible:

(1) Friction,
(2) Heat transfer with finite temperature gradient,
(3) Mass transfer with finite concentration gradient,
(4) Unrestrained expansion.

In general, irreversibility lessens the engineering usefulness of any given process. Though it may be impossible to absolutely eliminate irreversibility from any real process, its effects may often be substantially reduced in practical devices, so that a reversible process is the limit of possible performance. For any process it can be shown that

$$dS \geq \frac{dQ}{T}, \tag{2–10}$$

where the equality holds only for reversible processes. Thus if a process is reversible and adiabatic ($dQ = 0$), it must be *isentropic* ($dS = 0$), although the converse is not necessarily true. Finally, defining s as the entropy per unit mass of a continuum substance, the control-volume form of Eq. (2–10) can be written

$$\int_{\text{cs}} \frac{\dot{\mathcal{Q}}\,dA}{T} \leq \frac{d}{dt}\int_{\text{cv}} \rho s\, d\mathcal{V} + \int_{\text{cs}} s(\rho\mathbf{u}\cdot\mathbf{n})\, dA, \tag{2–11}$$

where again the equality holds only for reversible processes.

2–3 THERMODYNAMICS OF GASES

A pure substance may be defined, in the thermodynamic sense, as a substance which has only two independent static properties (in the absence of electricity, magnetism, and capillarity). Most engineering fluids, including gases like oxygen, nitrogen, and vapors in equilibrium with their liquid phase are pure substances.

Consider a small system composed of a pure substance in the absence of gravity and motion (of system relative to observer). If the properties are uniform throughout the system, the first law for an incremental change of state is

$$dq = de + dw,$$

where q and w are the heat and work per unit mass, respectively. Suppose the system experiences a reversible process for which $dw = p\,dv$, in which v is the specific volume. Then, using Eq. (2–9), we write

$$T\,ds = de + p\,dv. \tag{2–12}$$

Although this equation has been derived for a reversible process it must also be true for irreversible ones, since there are only two independent static properties for a pure substance, and all the terms in this equation are properties. The entropy change ds depends only on the change in state of the system, which is fixed by the values of de and dv. The increment ds is therefore independent of the process required to produce the increments de and dv. Thus, with the specified restrictions, Eq. (2–12) must hold true for any process whose end states could possibly be connected by a process for which the work is $\int p\,dv$.

The solution of a thermodynamic problem involving pure substances must include sufficient information on the relationships between the properties of the substance. The existence of only two independent static properties greatly simplifies the presentation of this information. If it is given algebraically, the equations are known as the *equations of state* of the substance. Equations of state are merely statements of experimentally observed relationships. Often the empirical relationships cannot be conveniently reduced to equation form and the equations of state must be represented by graphs or tables. Equations, graphs, or tables are available for the properties of fluids of common interest. One of the simplest of these is the perfect gas.

Perfect gas. A perfect gas is here defined by the following equation of state:

$$pv = RT, \tag{2–13}$$

in which p is the absolute pressure, v the volume per unit mass, T the absolute temperature and R a constant, called the gas constant. The gas constant depends only on the identity of the gas. More generally, it is observed that

$$pv = \frac{\overline{R}}{\overline{M}}T, \tag{2–14}$$

in which $\overline{M}$ is the molecular weight and $\overline{R}$ is the universal gas constant which has the value

$$\begin{aligned}\overline{R} &= 1545.43\ \text{ft}\cdot\text{lb}_f/^\circ\text{R}\cdot\text{lb}_m\text{-mole}\\ &= 1.9857\ \text{cal}/^\circ\text{K}\cdot\text{gm-mole}.\end{aligned}$$

Most real gases have a large range of temperatures and pressures over which they obey Eq. (2–14) very closely. By use of the Maxwell equations it may be shown [1] that the internal energy of any gas whose state can be so described depends only on temperature:

$$e = e(T).$$

Since the enthalpy h is defined by

$$h = e + pv,$$

Eq. (2–13) indicates that the enthalpy also depends only on temperature

$$h = h(T).$$

FIG. 2–4. Specific heats of gases [5].

In this case the relationship between internal energy, enthalpy, and temperature may be described by defining the "specific heats" at constant volume and temperature as follows (for a perfect gas):

$$c_v = \frac{de}{dT}, \qquad c_p = \frac{dh}{dT}. \tag{2–15}$$

These terms are actually misnomers, since heat is not a storable quantity. Also, the internal energy and the enthalpy can change in the complete absence of heat.

Using the definition of enthalpy and Eq. (2–13), we have

$$dh = de + R\,dT.$$

The relations (2–15) show that for a perfect gas the specific heats must be related by

$$c_p = c_v + R. \tag{2–16}$$

Typical variations with temperature of the specific heat of gases at constant pressure are shown in Fig. 2–4. Figure 2–5 illustrates the corresponding enthalpy-temperature dependence for a number of common gases where the zero has been arbitrarily set at 300°K.

Two types of perfect gas calculations may be demonstrated.

1. *Constant specific heats.* Figure 2–4 shows that for a sufficiently small range in temperature it may be acceptable to approximate the specific heats by constants. Using the perfect gas law, Eq. (2–12) may be written

$$ds = \frac{dh}{T} - \frac{dp}{\rho T}$$

or

$$ds = c_p\frac{dT}{T} - R\frac{dp}{p}. \tag{2–17}$$

FIG. 2–5. Enthalpy–temperature relationships for common gases [8].

For the particular case of constant specific heat and for an *isentropic* process, Eq. (2–17) may be integrated, with the results that

$$\frac{p}{p_1} = \left(\frac{T}{T_1}\right)^{c_p/R}. \tag{2–18}$$

Defining the specific heat ratio γ,

$$\gamma \equiv \frac{c_p}{c_v},$$

and using Eq. (2–16), Eq. (2–18) may be expressed in the form

$$\frac{p}{p_1} = \left(\frac{T}{T_1}\right)^{\gamma/(\gamma-1)}. \tag{2–19}$$

Using the perfect gas law, this can be transformed to

$$\frac{\rho}{\rho_1} = \left(\frac{T}{T_1}\right)^{1/(\gamma-1)} \tag{2–20}$$

or

$$\frac{p}{p_1} = \left(\frac{\rho}{\rho_1}\right)^{\gamma}, \tag{2–21}$$

in which $\rho = 1/v$ is the fluid density.

2. *Variable specific heats.* The functions $h(T)$ and $e(T)$ are tabulated for a number of gases (see, for example, References [4] and [8]). By defining a function ϕ according to

$$\phi = \int \frac{c_p\, dT}{T},$$

Eq. (2–17) may be integrated in the form

$$s - s_1 = \phi - \phi_1 - R \ln \frac{p}{p_1}, \tag{2–22}$$

in which the subscript 1 again denotes a reference value. The function ϕ depends only on temperature for a perfect gas, and hence may be tabulated with the enthalpy against the temperature argument for each gas. If the process is isentropic, Eq. (2–22) may be written

$$\frac{p}{p_1} = e^{(\phi-\phi_1)/R}.$$

For any given datum state 1 this *isentropic* pressure ratio p_r may be expressed by

$$p_r = \frac{p}{p_1} = e^{(\phi-\phi_1)/R},$$

which may also be tabulated as a function of temperature. Then if the temperature changes from any value T_A to another value T_B, the isentropic pressure change may be found from

$$\frac{p_B}{p_A} = \frac{p_r(T_B)}{p_r(T_A)},$$

in which $p_r(T_B)$ and $p_r(T_A)$ are obtained from tabulated functions.

The state of a gas having only two independent properties may be designated by a single point on a graph whose x- and y-axes correspond to any two properties. A process between states may be indicated by a line. The temperature-entropy diagram shown in Fig. 2–6 is an example of such a graph. The series of lines marked with various pressures correspond to possible constant-pressure processes. The shape of each line may be obtained from Eq. (2–17),

$$\left(\frac{\partial T}{\partial s}\right)_p = \frac{T}{c_p}.$$

FIG. 2–6. Constant-pressure lines on a T-s diagram.

From this it is clear that the lines have positive slopes, increasing with temperature. Further, if c_p is a function of T only, the slope of the curve is a function of T only. Hence the curves are of the same shape displaced horizontally.

From Eq. (2–17), it is also clear that

$$\left(\frac{\partial p}{\partial T}\right)_s = \frac{c_p}{R}\frac{p}{T}.$$

Since this derivative is always positive, the upper curves represent higher values of pressure.

2–4 EQUILIBRIUM COMBUSTION THERMODYNAMICS; CHEMICAL REACTIONS

Up to this point thermodynamic processes in which system components change their chemical identity have not been discussed. Since combustion is of vital importance to many propulsion methods, it is necessary to show how chemical reactions are governed by thermodynamic laws. Fortunately it is possible to simplify the discussion a great deal by adopting certain restrictions. First, chemical reactions of interest in propulsion systems nearly always occur between gases which can be closely approximated in behavior by a perfect gas. Second, with the exception of a few very rapid expansion processes, the important chemical processes usually take place between states which are approximately in equilibrium.

The basic problem may be stated as follows: Given a mixture of gases capable of a chemical reaction and a set of conditions under which the reaction will occur, what will be the result of that chemical reaction? For example, one might be asked to determine the composition and temperature of the exhaust products of a constant-pressure adiabatic combustion process. To formulate methods for answering this kind of question, it is convenient to consider three separate topics:

(1) Mixtures of gases,
(2) Chemical transformation,
(3) Composition of products.

Mixtures of gases

The reactants and products of a chemical process each generally exist as a gaseous mixture of chemical compounds. The properties of the mixture may be determined from the properties of the constituents by using the *Gibbs-Dalton Law*, which is expressed by Keenan [1] as follows:

1. The pressure of a mixture of gases is equal to the sum of the pressures of each constituent when each occupies alone the volume of the mixture at the temperature of the mixture.

2. The internal energy and the entropy of a mixture are equal, respectively, to the sums of the internal energies and the entropies of its constituents when each occupies alone the volume of the mixture at the temperature of the mixture.

Thus for a mixture of n constituents:

Temperature	$T_m = T_1 = T_2 = \cdots = T_n,$	(2–23a)
Pressure	$p_m = p_1 + p_2 + p_3 + \cdots + p_n,$	(2–23b)
Volume	$\mathcal{V}_m = \mathfrak{M}_m v_m = \mathfrak{M}_1 v_1 = \mathfrak{M}_2 v_2 = \cdots = \mathfrak{M}_n v_n,$	(2–23c)
Energy	$E_m = \mathfrak{M}_m e_m = \mathfrak{M}_1 e_1 + \mathfrak{M}_2 e_2 + \cdots + \mathfrak{M}_n e_n,$	(2–23d)
Entropy	$S_m = \mathfrak{M}_m s_m = \mathfrak{M}_1 s_1 + \mathfrak{M}_2 s_2 + \cdots + \mathfrak{M}_n s_n,$	(2–23e)
Enthalpy	$H_m = \mathfrak{M}_m h_m = \mathfrak{M}_1 h_1 + \mathfrak{M}_2 h_2 + \cdots + \mathfrak{M}_n h_n,$	(2–23f)

in which $\mathfrak{M}$ signifies mass, subscript m refers to the mixture and subscripts 1, 2, . . . , n refer to a series of constituents.

Since $de = c_v\, dT$, then

$$c_{vm} = \frac{\mathfrak{M}_1 c_{v1} + \mathfrak{M}_2 c_{v2} + \cdots + \mathfrak{M}_n c_{vn}}{\mathfrak{M}_m}. \tag{2–24}$$

Similarly,

$$c_{pm} = \frac{\mathfrak{M}_1 c_{p1} + \mathfrak{M}_2 c_{p2} + \cdots + \mathfrak{M}_n c_{pn}}{\mathfrak{M}_m}.$$

The pressures p_1, p_2, p_3, p_n that would be exerted by each constituent alone in the volume of the mixture at the temperature of the mixture are called the *partial pressures* of each constituent. The partial pressure is related to the number of moles of a constituent present. For any constituent i,

$$p_i v_i = R_i T_m$$

or, in terms of the universal gas constant $\overline{R}$,

$$p_i v_i = \frac{\overline{R}}{\overline{M}_i} T_m,$$

where $\overline{M}_i$ is the molecular weight of constituent i. Since

$$v_i = \frac{\mathcal{V}_m}{\mathfrak{M}_i},$$

$$p_i = \frac{\mathfrak{M}_i \overline{R} T_m}{\overline{M}_i \mathcal{V}_m} = n_i \frac{\overline{R} T_m}{\mathcal{V}_m},$$

in which n_i is the *number of moles* of constituent i present in the mixture. Since, according to the Gibbs-Dalton Law, the mixture pressure is the sum of the partial pressures,

$$\frac{p_i}{p_m} = \frac{n_i}{n_1 + n_2 + \cdots + n_n}. \tag{2–25}$$

Thus the ratio of the partial pressure to the mixture pressure is just the ratio of the number of moles of the constituent to the total number of moles. This ratio is called the *mole fraction*, χ:

$$\chi_i = \frac{n_i}{n_1 + n_2 + \cdots + n_n}. \tag{2–26}$$

The entropy change of the mixture is the sum of the entropy changes of all the constituents, each behaving as though the other constituents were not there. It is interesting to note that during a process which is isentropic for the mixture, the constituents do not necessarily undergo isentropic changes. There is generally a redistribution of entropy among the constituents.

In determining partial pressures, it is convenient to use moles rather than unit masses. For this purpose one can define molal specific heats by the equations

$$\bar{c}_p = \overline{M}c_p, \qquad \bar{c}_v = \overline{M}c_v,$$

To avoid confusion the values per mole of a property are usually denoted with a bar, as

$$d\bar{h} = \overline{M}dh = \bar{c}_p\, dT.$$

Chemical transformation

A general chemical reaction, in which α moles of reactant A and β moles of reactant B, etc., combine to form μ moles of product M plus ν moles of product N, etc., may be represented by

$$\alpha A + \beta B + \cdots \rightarrow \mu M + \nu N + \cdots. \tag{2–27}$$

The coefficients α, β, etc., are known from the mixture ratio of the reactants, and it may be assumed for the present that the product composition, that is, μ, ν, etc., is also known.

Accompanying a chemical reaction there is generally an exchange of energy with the surroundings. Those reactions which tend to give off heat are called *exothermic* and once started they will, under the proper conditions, proceed unaided. The other kind, called *endothermic*, require an energy input to proceed. Combustion processes, for instance, are exothermic.

The magnitude of these energy interactions can be determined from the composition and the heats of formation of the reactants and products. The constant-pressure heat of formation of a substance is the heat interaction which occurs when one mole of the substance is formed at constant pressure and temperature from its elements as they occur in nature. For example, the heat of formation of carbon dioxide (−94.052 kcal/gm-mole) is the heat interaction accompanying the following reaction from reactants to products at 300°K:

$$C_{\text{solid}} + O_{2\,\text{gas}} \rightarrow CO_{2\,\text{gas}}.$$

In this example, as in all exothermic reactions, the heat of formation is negative, since the system must give off heat to maintain (or return to) the reference temperature.

As another example, the formation of monatomic oxygen from naturally occurring oxygen is $\frac{1}{2}O_2 \rightarrow 0$. In this endothermic reaction the heat of formation (+59.162 kcal/gm-mole at 300°K) is positive, since the heat transfer to the system must be positive.

These heat quantities may be determined experimentally with a simple constant-pressure container. Practically, it may be difficult to produce a reaction at constant temperature and pressure. It is not necessary, in any event, since from the standpoint of thermodynamics only the end states of the reaction are of interest. The intermediate conditions do not matter (although they may be important to the preservation of laboratory equipment).

A *steady-flow process* is another experimental method for determining the heat of reaction Q_f. Equation (2–8) states that, for a steady-flow process with no shaft work, the heat transfer of the fluid flowing through a control volume is equal to the change in enthalpy (neglecting changes in velocity and elevation). Thus

$$Q = \Delta H$$

or, per *mole* of product formed,

$$Q = Q_f = H_{\text{product}} - H_{\text{reactants}}. \tag{2–28}$$

For example, the formation of carbon dioxide in steady flow is represented by Fig. 2–7.

FIG. 2–7. Schematic of steady-flow process for formation of CO_2, with T_f being the reference temperature.

Table 2–1 gives the heats of formation for some common products of combustion, T_f being the reference temperature at which Q_f is measured. The heats of formation of naturally occurring elements are, by definition, zero.

In contrast, most common reactions are complex in the sense that the reactants are not naturally occurring elements and the products include more than one substance. Also, they do not generally occur at constant temperature. Complex reactions can be treated as a combination of simpler reactions by the following method. Consider the steady-flow reaction indicated in Fig. 2–8, where Q is the heat interaction necessary to maintain constant temperature. It is convenient to consider this reaction to be the sum of distinct decomposition and formation reactions.

Table 2–1 Heats of Formation

Compound	Formula	Heat of formation, kcal/gm-mole	T_f, °K	Reference number
Aluminum oxide	AL_2O_3	−399.09 (g)	298	5
Ammonium nitrate	NH_4NO_3	− 87.27 (s)	298	5
Ammonium perchlorate	NH_4CLO_4	− 69.42 (s)	298	5
Aniline	$C_6H_5NH_2$	+ 6.11 (l)	300	9
Boron	B_2	+124.5 (g)	298	5
Boron atom	B	+ 97.2 (g)	298	5
Bromine atom	Br	+ 23.050 (g)	300	6
Carbon	C	+171.698 (g)	298	5
Carbon (graphite)	C	0 (s)	300	
Carbon dioxide	CO_2	− 94.052 (g)	300	7
Carbon monoxide	CO	− 26.413 (g)	300	7
Dimethylhydrazine (unsymmetrical) (UDMH)	$(CH_3)_2NNH_2$	+ 12.734 (l)	298	10
Fluorine	F_2	− 3.0 (l)		
		− 0 (g)	300	
Fluorine atom	F	+ 18.906 (g)	300	6
Hydrazine	N_2H_4	+ 12.000 (l)	298	10
Hydrogen	H_2	− 1.92 (l)	300	9
Hydrogen atom	H	+ 52.092 (g)	300	7
Hydrogen bromide	HBr	− 8.66 (g)	298	5
Hydrogen fluoride	HF	− 64.20 (g)	298	5
Hydrogen peroxide	H_2O_2	− 33.74 (g)	300	8
		− 44.84 (l)		
		− 47.36 (s)		
Hydrazine hydrate	$N_2H_4H_2O$	− 10.3 (l)	300	12
Hydroxyl	OH	+ 10.063 (g)	300	7
Isobutane (2-methylpropane)	C_4H_{10}	− 31.489 (g)	300	7
		− 36.169 (l)		
JP–3	$C_{7.366}H_{11.442}$	− 26.66 (g)	300	13
JP–4	H/C 1.93	− 0.423* (l)	300	8
Methyl alcohol	CH_3OH	− 48.08 (g)	298	5
		− 57.04 (l)		14
Monomethylhydrazine	CH_3NHNH_2	+ 13.109 (l)	298	10
Nitric acid	HNO_3	− 41.35 (l)	300	8
		− 31.92 (g)		

* kcal/gm in this case.

Table 2–1 Heats of Formation (*Continued*)

Compound	Formula	Heat of formation, kcal/gm-mole	T_f, °K	Reference number
Nitric acid, red fuming	14% NO_2 1% H_2O	− 45.3 (l)	300	8
Nitric oxide	NO	21.6 (g)	298	5
Nitrogen	N_2	0 (g)	300	
Nitrogen atom	N	+112.5 (g)	300	15
Nitrogen dioxide	NO_2	+ 8.091 (g)	298	5
Nitrogen pentoxide	N_2O_5	+ 3.6 (g)	298	5
Nitrogen tetroxide	N_2O_4	+ 2.30 (g)	300	8
Nitrogen trioxide	N_2O_3	+ 20.0 (g)	298	5
Nitrous oxide	N_2O	+ 19.55 (g)	298	5
N-octane	C_8H_{18}	− 49.88 (g)	300	7
		− 59.795 (l)		
Oxygen	O_2	0 (g)	300	
		+ 3.08 (l)		
Oxygen atom	O	+ 59.162 (g)	300	7
Ozone	O_3	+ 34.00 (g)	300	8
		+ 30.3 (l)		
Sulfur dioxide	SO_2	− 70.96 (g)	298	5
Water	H_2O	− 57.802 (g)	300	7

FIG. 2–8. Schematic of complex reaction at constant temperature, showing equivalent elementary reactions.

The decomposition reaction is just the reverse of the formation reaction; hence it is a simple one. However hypothetical the reverse process may be, the first law tells us that the decomposition heat interaction will be equal and opposite to the formation heat interaction. Thus, summing over all the decomposition reactions,

$$Q_1 = -\sum_i (n_i Q_{fi})_{\text{reactants}}, \tag{2–29}$$

where n_i is the number of moles of reactant i and Q_{fi} is its heat of formation per mole.

Once the reactants are in elemental form, the remainder of the reaction may be treated as the sum of separate formations, so that

$$Q_2 = \sum_j (n_j Q_{fj})_{\text{products}}, \tag{2–30}$$

where n_j is the number of moles of product j. Since the mixing process in the third chamber is adiabatic, the net heat interaction for the complex reaction at constant temperature T_f is

$$Q_R = Q_1 + Q_2 = \sum_j (n_j Q_{fj})_{\text{products}} - \sum_i (n_i Q_{fi})_{\text{reactants}}. \tag{2–31}$$

Also, using Eq. (2–28),

$$Q_R = H_{\text{products}} - H_{\text{reactants}} = \sum_j (n_j Q_{fj})_{\text{products}} - \sum_i (n_i Q_{fi})_{\text{reactants}}.$$

This difference in enthalpy is given the symbol $H_{\text{RP}f}$ and may be interpreted as the change in enthalpy from products to reactants at the constant reference temperature T_f. Hence

$$H_{\text{RP}f} = H_{\text{P}f} - H_{\text{R}f} = \sum_j (n_j Q_{fj})_{\text{products}} - \sum_i (n_i Q_{fi})_{\text{reactants}}. \tag{2–32}$$

The value of $H_{\text{RP}f}$ is tabulated for many common reactions, but if such a tabulation is not available for a particular reaction it can be calculated from Eq. (2–32) and a table such as Table 2–1. Note that this calculation requires knowledge of the composition of the product.

If the products or reactants are not at the reference temperature T_f, the foregoing procedure must be modified somewhat. Hypothetically, it is possible to bring the reactants to the reference temperature by heat transfer, allow the reaction to occur at constant temperature, and then bring the products to the final temperature by another heat transfer. The heat interactions required are indicated in Fig. 2–9.

The net heat interaction from reactants at T_1 to products at T_2 may be written as the sum of the three heat-transfer rates indicated in Fig. 2–9. Thus

$$Q = (H_{\text{P}2} - H_{\text{P}f}) - (H_{\text{R}1} - H_{\text{R}f}) + H_{\text{RP}f}. \tag{2–33}$$

Each of the parenthesized terms is a change in enthalpy from the actual state to

FIG. 2–9. Reaction at variable temperatures.

the reference state (T_f), and may be calculated from tables of the mixture properties or by the methods of Section 2–3. The reference temperature T_f is simply that temperature for which the heat of reaction, $H_{\text{RP}f}$, is available.

Let us take a typical problem: we wish to determine the adiabatic flame temperature for a given reactant mixture. Once we know the product composition and determine the reference temperature, we can calculate $H_{\text{R}1}$, $H_{\text{R}f}$, and $H_{\text{RP}f}$. Then, since $Q = 0$, we can determine the product enthalpy $H_{\text{P}2}$, and from this the product temperature.

FIG. 2–10. Enthalpy-temperature diagram for adiabatic combustion.

This can all be presented graphically on an enthalpy-temperature diagram which shows the relationship between enthalpy and temperature for both product and reactant mixtures. Figure 2–10 describes a hypothetical path of the process. The reactants at state 1 are brought to T_f at state A, after which conversion to products occurs at constant temperature along line AB. Then the products are heated to the outlet state at 2, and T_2 is the outlet temperature. Since there is no net heat transfer in this particular example, $H_{\text{R}1}$ must therefore equal $H_{\text{P}2}$ [as may be seen by inserting the definition of $H_{\text{RP}f}$ in Eq. (2–33) and setting Q equal to zero]. From this diagram it can be seen that if the specific heats* of reactant and product mixtures are unequal, then the heat of reaction $H_{\text{RP}f}$ must be a function of temperature.

* The specific heat of the mixture is given by the slope of its enthalpy-temperature curve in Fig. 2–9.

Composition of products

In all the aforementioned relationships, our calculations have depended on knowledge of the composition of the products of reaction. Once this composition is known, calculating the state of the product is a rather simple matter. However, determining the product composition is not always easy. The product composition depends on the temperature, which in turn has just been shown to depend on the composition. This suggests that a trial-and-error procedure may be required for the solution.

For certain reactions, it is relatively simple to determine product composition. For example, at sufficiently low temperature, the combustion of a particular nonstoichiometric mixture of octane and oxygen yields the following products:

$$C_8H_{18} + 15O_2 \rightarrow 8CO_2 + 9H_2O + 2.5O_2.$$

In this case, the composition is easily determined by requiring that the number of atoms of each element taking part in the reaction be constant.

Actually, at high temperatures this expression may be a considerable oversimplification, because the compounds CO_2, H_2O, and O_2 dissociate to some extent into free atoms and radicals. A more general statement is

$$C_8H_{18} + 15O_2 \rightarrow aCO_2 + bH_2O + cO_2 + dH_2 + eC + fCO + gH + iO + jOH + kC_8H_{18} + \text{other hydrocarbons}, \qquad (2\text{–}34)$$

where a, b, c, d, etc., are the unknown numbers of moles of each substance. A reasonable first approximation may be made by recognizing that for all practical purposes C_8H_{18} and the "other hydrocarbon" molecules are present in negligible quantities. However, there are still unknowns—a, b, c, d, e, f, g, i, j—for which we can obtain only three equations in terms of numbers of atoms. Atom balances yield:

For carbon, $a + e + f = 8$;

For hydrogen, $2b + 2d + g + j = 18$;

For oxygen, $2a + b + 2c + f + j + i = 30$.

Six more independent equations are needed in order to determine the product composition. They may be obtained from the requirements for chemical equilibrium.

Chemical equilibrium. Consider a reaction in which a stoichiometric combination of A and B reacts to form products M and N with no A and B left:

$$\alpha A + \beta B \rightarrow \mu M + \nu N. \qquad (2\text{–}35)$$

At the same time this reaction is taking place, there is generally some tendency for the reverse reaction,

$$\mu M + \nu N \rightarrow \alpha A + \beta B$$

FIG. 2–11. Reaction chamber for derivation of equilibrium constants.

to occur, especially at high temperatures. Equilibrium is that state at which both forward and reverse reactions are occurring at equal rates. When equilibrium obtains, the concentrations of A, B, M, and N are constant. Using the second law of thermodynamics, we can express the conditions for equilibrium in algebraic form.

Suppose that an equilibrium mixture resulting from the combination of A and B is contained in a chamber which communicates through semipermeable membranes with each individual constituent substance in a pure form, as in Fig. 2–11. The entire apparatus is surrounded by a constant-temperature bath.

A semipermeable membrane is a hypothetical surface through which one substance may pass freely, though all others are prevented from passing. Such membranes are only approximated by certain real substances, but they are nevertheless useful analytical concepts. For example, in Fig. 2–11, substance A may pass freely into or out of the mixture at cylinder A, and the pressure of the pure A will be equal to the partial pressure of A in the mixture, whereas B, M, and N cannot pass through this membrane. Similarly, the membranes of cylinders B, M, and N admit only substances B, M, and N respectively.

The amount of any particular substance within the mixture chamber can be controlled by simply pushing or pulling on the piston in the cylinder containing the pure substance. This control process is reversible if done slowly.

Starting with an equilibrium mixture resulting from the combination of A and B in any proportion, consider the introduction of additional small amounts of A and B through the semipermeable membranes into the mixture chamber, in the stoichiometric ratio $k\alpha A + k\beta B$, where k is some small number. Once in contact within the main chamber, the A and B tend to react, forming M and N. There are two ways to restore the contents of the main chamber to their original condition. One is to withdraw from the chamber $k\alpha A + k\beta B$. The other is to with-

draw $k\mu M + k\nu N$. In both cases the total number of atoms of each element in the chamber will have returned to its original value. After the mixture has again reached equilibrium, its composition will be restored to its former value. As a result of the reaction

$$k\alpha A + k\beta B \rightarrow k\mu M + k\nu N,$$

there will be a heat interaction of magnitude kH_{RP} with the constant-temperature bath surrounding the apparatus, since the reaction occurs at the constant temperature T of the bath. In the limit this heat interaction will be reversible. Then, for a system composed of all the gases in the mixture and in the cylinders,

$$dS = \left(\frac{dQ}{T}\right)_{\text{rev}} = \frac{kH_{RP}}{T}.$$

For perfect gases it may be shown that H_{RP} is a function of temperature only. Thus for a given value of k the change in entropy of the system is a function of temperature only: $dS = kf(T)$.

Since the temperature, volume, mass, and composition of the mixture do not change, there will be no change of state of the system in the mixture chamber. Thus

$$dS_{\substack{\text{mixture}\\ \text{chamber}}} = 0.$$

In addition, the partial pressure of each constituent does not change and hence the pressure in each cylinder does not change. Since the temperature is also constant in each cylinder, the entropy *per mole* of each pure substance must be constant and the change in entropy within the cylinders may be written

$$dS_A = -k\alpha\bar{S}_A, \qquad dS_M = +k\mu\bar{S}_M,$$
$$dS_B = -k\beta\bar{S}_B, \qquad dS_N = +k\nu\bar{S}_N,$$

where $\bar{S}$ is the entropy per mole of pure substance.

The total entropy change is equal to the sum of the changes in the mixture chamber and in the cylinders:

$$dS = -k\alpha\bar{S}_A - k\beta\bar{S}_B + k\mu\bar{S}_M + k\nu\bar{S}_N = k\frac{H_{RP}}{T}$$

or

$$\alpha\bar{S}_A + \beta\bar{S}_B - \mu\bar{S}_M - \nu\bar{S}_N = -\frac{H_{RP}}{T} = f(T). \tag{2–36}$$

The entropy of a perfect gas *per mole* can be obtained by integrating Eq. (2–17) with the result:

$$\bar{S} = -\bar{R}\ln p + \int_{T1}^{T} \bar{c}_p \frac{dT}{T} + \bar{R}\ln p_1,$$

where the subscript 1 refers to an arbitrary datum state. Thus, for a given datum

state, Eq. (2–36) can be written

$$-\alpha\bar{R}\ln p_A - \beta\bar{R}\ln p_B + \mu\bar{R}\ln p_M + \nu\bar{R}\ln p_N = g(T),$$

where $g(T)$ is another function of temperature only. Adding the logarithmic terms and writing $g(T) = \bar{R}\ln K_p(T)$ leads to the following relationship between the partial pressures:

$$\frac{p_M^{\mu}p_N^{\nu}}{p_A^{\alpha}p_B^{\beta}} = K_p(T). \tag{2–37}$$

The term K_p is called the *equilibrium constant* for this mixture in terms of partial pressures.

Figure 2–12 gives typical values of the equilibrium constant K_p. Several general statements should be made regarding equilibrium constants before their use is demonstrated in a detailed example.

$$K_4 = \frac{p(H_2)p(O)}{p(H_2O)}$$

$$K_1 = \frac{p(CO)p(H_2O)}{p(CO_2)p(H_2)} \qquad K_5 = \frac{p(H)}{[p(H_2)]^{1/2}} \qquad K_8 = \frac{p(CO)}{p(C)[p(O_2)]^{1/2}}$$

$$K_2 = \frac{p(NO)p(H_2)}{[p(N)]^{1/2}p(H_2O)} \qquad K_6 = \frac{[p(H_2)]^{1/2}p(OH)}{p(H_2O)} \qquad K_9 = \frac{p(CO_2)}{p(CO)[p(O_2)]^{1/2}}$$

$$K_3 = \frac{[p(H_2)]^2p(O_2)}{[p(H_2O)]^2} \qquad K_7 = \frac{p(N)^2}{p(N_2)} \qquad K_{10} = \frac{p(NO)}{[p(N_2)]^{1/2}[p(O_2)]^{1/2}}$$

FIG. 2–12. Equilibrium constants [8] in terms of partial pressures. Pressure in atmospheres.

First, it may be noted that the partial pressures of the product appear in the numerator and those of the reactants in the denominator, each with an exponent taken from Eq. (2–35). Qualitatively, then, one can expect a large K_p for sub-reactions which approach completion and small K_p for those that do not. The number of constituents considered in this derivation is four, but the result can easily be extended to include any number of reactants and products.

Second, the exponents are the *stoichiometric* coefficients. For example, consider the relationship between carbon dioxide, carbon monoxide, and oxygen which would be of interest in our octane-combustion example:

$$CO_2 \rightarrow CO + \tfrac{1}{2}O_2.$$

In this case the equilibrium equation is

$$\frac{p(CO)[p(O_2)]^{1/2}}{p(CO_2)} = K_p.$$

It may be noted that these exponents are not directly related to a, c, or f of Eq. (2–34) for the overall reaction. This equilibrium equation provides one of the six relationships necessary to the determination of product composition after the reaction (2–34). In a similar way we can state five other equilibrium requirements to govern the relative partial pressures of the products.

Third, since the object of this calculation is to determine composition, it is convenient to define an equilibrium constant K_n, based on *mole fractions* rather than partial pressures:

$$K_n = \frac{\chi_M^{\mu}\chi_N^{\nu}}{\chi_A^{\alpha}\chi_B^{\beta}}. \tag{2–38}$$

Since, according to the Gibbs-Dalton Law, $\chi_i = p_i/p_m$, it may be shown that

$$K_n = p_m^{\alpha+\beta-\mu-\nu}K_p, \tag{2–39}$$

where K_p is the equilibrium constant based on partial pressures. Note that K_n, while more convenient in calculating mole fractions, is a function of two variables, pressure and temperature, and is thus inconvenient to tabulate. In Eq. (2–39), p_m must be in the units for which K_p is tabulated.

The actual use of these concepts can best be demonstrated by an example which contains a considerably simplified description of an actual chemical reaction.

Example 1

3315.5°C

One mole of H_2O is raised in temperature to 6000°F, and it may be assumed that it dissociates to an equilibrium mixture of H_2O, H_2, and O_2 at a total pressure of 1 atm according to $H_2O \rightarrow H_2 + \frac{1}{2}O_2$. The equilibrium constant for this reaction at 6000°F is $K_p = 0.23\ \text{atm}^{1/2}$. The concentration of the mixture may be determined as follows. During the dissociation, x moles of H_2O break down

to form x moles of H and $x/2$ moles of O_2, according to the reaction

$$xH_2O \rightarrow xH_2 + \frac{x}{2}O_2.$$

The relative concentrations of the constituents of the mixture may be tabulated as shown below.

Constituents	Moles present at equilibrium	Mole fraction χ_i
H_2O	$1 - x$	$\dfrac{1-x}{1+x/2}$
H_2	x	$\dfrac{x}{1+x/2}$
O_2	$\dfrac{x}{2}$	$\dfrac{x}{2(1+x/2)}$
Total	$1 + x/2$	

The relative concentrations may be determined as follows:

$$K_n = \frac{\chi_{H_2}(\chi_{O_2})^{1/2}}{\chi_{H_2O}} = K_p p_m^{-1/2}.$$

Substituting the equilibrium concentrations, we obtain

$$\frac{\left(\dfrac{x}{1+x/2}\right)\left(\dfrac{x/2}{1+x/2}\right)^{1/2}}{\dfrac{1-x}{1+x/2}} = 0.23.$$

The quantity x may then be determined by trial and error and the mole fractions of H_2O, O_2, and H_2 at equilibrium obtained from the above tabulation.

The foregoing calculation oversimplifies the dissociation. Strictly speaking, the following three dissociation reactions are also of importance at 6000°F:

$$H_2O \rightarrow HO + \tfrac{1}{2}H_2, \qquad O_2 \rightarrow 2O, \qquad H_2 \rightarrow 2H.$$

Determination of the equilibrium concentrations of H_2O, H_2, O_2, O, and H then requires the simultaneous solution of three equilibrium conditions with known values of the three equilibrium constants for the above reactions.

Example 2 (from Reference [3])

Consider a complete set of six products for a hydrogen-oxygen reaction:

$$AH_2 + BO_2 \rightarrow n_{H_2O}H_2O + n_{O_2}O_2 + n_{H_2}H_2 + n_{O}O + n_{H}H + n_{OH}OH. \tag{2–40}$$

Since two equations can be obtained from the conservation of hydrogen and oxygen atoms, four equilibrium relationships are required.

The following method is due to Penner [3], who has shown that it is convenient to define yet another equilibrium constant $\mathfrak{K}_n$, based on *number of moles* rather than mole fractions. The use of this equilibrium constant results in simplified calculations, and it is introduced here because the technique has value beyond this example.

Using again the typical reaction (2–35),

$$\mathfrak{K}_n \equiv \frac{n_M^{\mu} n_N^{\nu}}{n_A^{\alpha} n_B^{\beta}}, \tag{2–41}$$

and using Eq. (2–25),

$$\mathfrak{K}_n = \left(\frac{n}{p_m}\right)^{\mu+\nu-\alpha-\beta} K_p. \tag{2–42}$$

Strictly speaking, the use of this equilibrium constant introduces another unknown, n, along with an additional equation which simply states that n is the total number of moles in the actual product mixture (not the subreaction). However, the simplification attained by this approach lies in the fact that n can be rather easily approximated, and in the fact that the results of the calculation are not strongly dependent on n. Thus, as illustrated in this example, n is at first assumed known and later checked. An iterative procedure can be used if greater accuracy is desired.

Returning to the example in Eq. (2–40), the two equations resulting from conservation of atoms are:

$$\text{For hydrogen,} \quad N_{\mathrm{H}} = 2n_{\mathrm{H_2O}} + 2n_{\mathrm{H_2}} + n_{\mathrm{H}} + n_{\mathrm{OH}} = 2A, \tag{2–43}$$

$$\text{For oxygen,} \quad N_{\mathrm{O}} = n_{\mathrm{H_2O}} + 2n_{\mathrm{O_2}} + n_{\mathrm{O}} + n_{\mathrm{OH}} = 2B, \tag{2–44}$$

where N_{H} and N_{O} are the *known* numbers of hydrogen and oxygen atoms, respectively.

The equilibrium information available is of the following form:

$$\tfrac{1}{2}\mathrm{O_2} \rightarrow \mathrm{O}, \qquad K_{p1} = \frac{p_{\mathrm{O}}}{p_{\mathrm{O_2}}^{1/2}},$$

$$\tfrac{1}{2}\mathrm{H_2} \rightarrow \mathrm{H}, \qquad K_{p2} = \frac{p_{\mathrm{H}}}{p_{\mathrm{H_2}}^{1/2}},$$

$$\tfrac{1}{2}\mathrm{H_2} + \tfrac{1}{2}\mathrm{O_2} \rightarrow \mathrm{OH}, \qquad K_{p3} = \frac{p_{\mathrm{OH}}}{p_{\mathrm{O_2}}^{1/2} p_{\mathrm{H_2}}^{1/2}},$$

$$\mathrm{H_2} + \tfrac{1}{2}\mathrm{O_2} \rightarrow \mathrm{H_2O}, \qquad K_{p4} = \frac{p_{\mathrm{H_2O}}}{p_{\mathrm{H_2}} p_{\mathrm{O_2}}^{1/2}},$$

and the corresponding relationships in terms of moles may be written from Eq. (2–41):

$$\mathfrak{K}_{n1} = \frac{n_{\mathrm{O}}}{n_{\mathrm{O_2}}^{1/2}}, \qquad \mathfrak{K}_{n2} = \frac{n_{\mathrm{H}}}{n_{\mathrm{H_2}}^{1/2}}, \qquad \mathfrak{K}_{n3} = \frac{n_{\mathrm{OH}}}{n_{\mathrm{O_2}}^{1/2} n_{\mathrm{H_2}}^{1/2}}, \qquad \mathfrak{K}_{n4} = \frac{n_{\mathrm{H_2O}}}{n_{\mathrm{H_2}} n_{\mathrm{O_2}}^{1/2}}.$$

A procedure for obtaining a solution for the six unknowns is now outlined.

STEP 1: From the equilibrium equations write the number of moles of each *new* species appearing in the product in terms of the reactants and the proper equilibrium constant. In terms of $\mathfrak{K}_n$:

$$\begin{aligned} n_{\mathrm{O}} &= \mathfrak{K}_{n1} n_{\mathrm{O_2}}^{1/2}, & n_{\mathrm{H}} &= \mathfrak{K}_{n2} n_{\mathrm{H_2}}^{1/2}, \\ n_{\mathrm{OH}} &= \mathfrak{K}_{n3} n_{\mathrm{O_2}}^{1/2} n_{\mathrm{H_2}}^{1/2}, & n_{\mathrm{H_2O}} &= \mathfrak{K}_{n4} n_{\mathrm{H_2}} n_{\mathrm{O_2}}^{1/2}. \end{aligned} \tag{2–45}$$

STEP 2: Use the equilibrium relationships (2–45) to eliminate the number of moles of new species in one of the atom-conservation equations. Thus, substituting Eq. (2–45) in (2–43), we have

$$2\mathfrak{K}_{n4} n_{\mathrm{H_2}} n_{\mathrm{O_2}}^{1/2} + 2n_{\mathrm{H_2}} + \mathfrak{K}_{n2} n_{\mathrm{H_2}}^{1/2} + \mathfrak{K}_{n3} n_{\mathrm{O_2}}^{1/2} n_{\mathrm{H_2}}^{1/2} = N_{\mathrm{H}}.$$

Then, solving for $n_{\mathrm{O_2}}$ in terms of $n_{\mathrm{H_2}}$:

$$n_{\mathrm{O_2}} = \left(\frac{N_{\mathrm{H}} - 2n_{\mathrm{H_2}} - \mathfrak{K}_{n2} n_{\mathrm{H_2}}^{1/2}}{2\mathfrak{K}_{n4} n_{\mathrm{H_2}} + \mathfrak{K}_{n3} n_{\mathrm{H_2}}^{1/2}}\right)^2. \tag{2–46}$$

STEP 3. Use this expression to eliminate $n_{\mathrm{O_2}}$ from Eqs. (2–45).

$$\begin{aligned} n_{\mathrm{O}} &= \mathfrak{K}_{n1}\left(\frac{N_{\mathrm{H}} - 2n_{\mathrm{H_2}} - \mathfrak{K}_{n2} n_{\mathrm{H_2}}^{1/2}}{2\mathfrak{K}_{n4} n_{\mathrm{H_2}} + \mathfrak{K}_{n3} n_{\mathrm{H_2}}^{1/2}}\right), \\ n_{\mathrm{H}} &= \mathfrak{K}_{n2} n_{\mathrm{H_2}}^{1/2}, \\ n_{\mathrm{OH}} &= \mathfrak{K}_{n3} n_{\mathrm{H_2}}^{1/2}\left(\frac{N_{\mathrm{H}} - 2n_{\mathrm{H_2}} - \mathfrak{K}_{n2} n_{\mathrm{H_2}}^{1/2}}{2\mathfrak{K}_{n4} n_{\mathrm{H_2}} + \mathfrak{K}_{n3} n_{\mathrm{H_2}}^{1/2}}\right), \\ n_{\mathrm{H_2O}} &= \mathfrak{K}_{n4} n_{\mathrm{H_2}}\left(\frac{N_{\mathrm{H}} - 2n_{\mathrm{H_2}} - \mathfrak{K}_{n2} n_{\mathrm{H_2}}^{1/2}}{2\mathfrak{K}_{n4} n_{\mathrm{H_2}} + \mathfrak{K}_{n3} n_{\mathrm{H_2}}^{1/2}}\right). \end{aligned} \tag{2–47}$$

STEP 4: Use these expressions along with Eq. (2–46) to eliminate all but $n_{\mathrm{H_2}}$ from the remaining atom conservation equation (2–44).

$$(\mathfrak{K}_{n4} n_{\mathrm{H_2}} + \mathfrak{K}_{n1} + \mathfrak{K}_{n3} n_{\mathrm{H_2}}^{1/2})\left(\frac{N_{\mathrm{H}} - 2n_{\mathrm{H_2}} - \mathfrak{K}_{n2} n_{\mathrm{H_2}}^{1/2}}{2\mathfrak{K}_{n4} n_{\mathrm{H_2}} + \mathfrak{K}_{n3} n_{\mathrm{H_2}}^{1/2}}\right) + 2\left(\frac{N_{\mathrm{H}} - 2n_{\mathrm{H_2}} - \mathfrak{K}_{n2} n_{\mathrm{H_2}}^{1/2}}{2\mathfrak{K}_{n4} n_{\mathrm{H_2}} + \mathfrak{K}_{n3} n_{\mathrm{H_2}}^{1/2}}\right)^2 = N_{\mathrm{O}}. \tag{2–48}$$

Equation (2–48) can now be solved for n_{H_2} if the various $\mathfrak{K}_n$ are known. The $\mathfrak{K}_n$'s may be evaluated from Eq. (2–42) for known values of K_p, p_m, and n. The total number of moles in the mixture is actually unknown, but it can be rather easily approximated, and the results are not very sensitive to n anyway. The actual product composition is not far from that resulting from "complete" reaction (unless the temperature is very high) so that n is approximately equal to the number of moles resulting from "complete" reaction. For this example, it is

$$AH_2 + BO_2 \rightarrow AH_2O + (2B - A)O_2$$

if there is excess O_2, or

$$2BH_2O + (A - 2B)H_2$$

if there is excess H_2. Hence n can be readily approximated in terms of the known reactant quantities.

For a numerical example, consider the equilibrium composition of $2H_2 + O_2$ at 3500°K (centigrade absolute) and 300 psia. Then $N_H = 4$ and $N_O = 2$. For complete reaction, the number of moles of products would be 2, and we write

$$2H_2 + O_2 \rightarrow 2H_2O.$$

However, since the temperature is rather high, n is assumed to be 2.2. Assumptions of this nature come from experience in this type of calculation; but as we shall see, iterations are easily made and the assumption is not critical. From the known K_p (pressure in atmospheres in this case) and pressure (20.42 atm), and the assumed n, the new equilibrium terms can be calculated. Using Eq. (2–42),

$$\mathfrak{K}_{n1} = K_{p1}\left(\frac{p_m}{n}\right)^{-1/2} = 0.17003, \qquad \mathfrak{K}_{n2} = K_{p2}\left(\frac{p_m}{n}\right)^{-1/2} = 0.19362,$$

$$\mathfrak{K}_{n3} = K_{p3} = 1.3046, \qquad \mathfrak{K}_{n4} = K_{p4}\left(\frac{p_m}{n}\right)^{1/2} = 15.020.$$

These values are then used with various assumed values of n_{H_2} to calculate the left-hand side of Eq. (2–48). The correct value is of course $N_O = 2$. Trial results are tabulated below.

Assumed n_{H_2}	Calculated N_O from Eq. (2–48)
0.2500	2.2561
0.3400	2.9549
0.3000	2.0224
0.3200	1.9917

By linear interpolation between the last two figures, n_{H_2} was found to be $n_{H_2} = 0.3170$. Then Eqs. (2–46) and (2–47) give

$$n_{H_2} = 0.3170, \qquad n_{O_2} = 0.1008, \qquad n_O = 0.05399,$$
$$n_H = 0.1090, \qquad n_{OH} = 0.2333, \qquad n_{H_2O} = 1.5119.$$

Table 2–2 Enthalpies and Heats of Formation for a Given Product Composition

Reactants	n	$\Delta\bar{h}$, kcal/gm-mole	$n\,\Delta\bar{h}$	Q_f, kcal/gm-mole	nQ_f
H_2	2	0	0	0	0
O_2	1	0	0	0	0
			0		0
Products					
H_2	0.317	25.693	8.145	0	0
O_2	0.1008	28.263	2.849	0	0
O	0.05399	16.028	0.865	59.162	3.194
H	0.1090	15.897	1.733	52.092	5.678
OH	0.2333	25.907	6.044	10.063	2.348
H_2O	1.5119	36.920	55.819	−57.802	−75.118
			75.455		−63.898

Adding these gives us the total number of moles (which was assumed to be 2.2):

$$n = 2.326.$$

The next step is to recalculate n_{H_2}, etc., assuming $n = 2.326$. When we do this, we find no appreciable change in composition. Hence even an inexperienced person could probably complete the calculation in two, or at most three, iterations.

This solution method is not limited to reactions of only two reactants. The generalization of the method for additional reactants is discussed in Reference [3].

In order to take 2 moles of H_2 and 1 mole of O_2 from 300°K and transform them at constant pressure to the above composition at 3500°K, it would be necessary to accompany the reaction by a heat transfer Q which can be determined by Eq. (2–33). If we use Table 2–1 to compute the heat of reaction, we must let $T_f = 300$°K. Since the inlet temperature is also 300°K in this example, $H_{R1} - H_{Rf} = 0$. Table 2–2 shows how the terms $(H_{P2} - H_{Pf})$ and H_{RPf} are computed from enthalpy-temperature information (e.g. Fig. 2–5), heats of formation (Table 2–2), and the given compositions. In the table, n is the number of moles and $\Delta\bar{h}$ the enthalpy per mole of each constituent above the datum temperature T_f. As before, Q_f is the heat of formation of each constituent, per mole. From Table 2–2, the heat of reaction H_{RPf} is −63.898 kcal/gm-mole of oxygen reactant. Also the enthalpy difference $(H_{P2} - H_{Pf})$ is 75.455 kcal/gm-mole of oxygen reactant. Therefore, from Eq. (2–33), the heat transfer necessary to take two moles of H_2 and one mole of O_2 at 300°K and transform them to the product composition of Table 2–2 at 3500°K is:

$$Q = (H_{P2} - H_{f2}) - (H_{R1} - H_{Rf}) + H_{RPf}$$
$$= 75.455 - 0 - 63.898 = 11.6 \text{ kcal/gm-mole}\cdot O_2.$$

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Problems

1. A perfect gas of molecular weight $\overline{M} = 20$ and specific heat ratio $\gamma = 1.2$ expands adiabatically from a pressure of 1000 psia, temperature 5000°R, and velocity 1000 fps to a pressure of 14.7 psia.

   (a) If the final temperature is 2600°R and the final velocity negligible, how much work has been done by the gas during expansion?

   (b) If no work is done by or on this gas during expansion to 14.7 psia, what is the maximum possible velocity at the end of the process?

   (c) If any adiabatic expansion process to the same final pressure is permissible, is it possible for the exit temperature to be 300°R? State your reasoning.

2. An inventor suggests a propulsion device shown schematically in the figure. Would such a scheme be feasible for steady operation over long periods of time? Discuss rigorously and briefly.

PROBLEM 2

PROBLEM 3

3. A strut of chord c is placed in an incompressible flow which is everywhere uniform except for the illustrated effect of the strut. Velocity measurements at a certain plane far downstream of the strut indicate the presence of a wake, as shown. The velocity distribution across the wake is approximately a cosine function; that is,

$$\frac{u}{U} = 1 - a\left(1 + \cos\frac{2\pi y}{b}\right) \quad \text{for} \quad -\frac{b}{2} < y < \frac{b}{2},$$

where U is the velocity of the uniform flow field far from the strut and its wake. If the measured values of a and b/c are 0.10 and 2.0, respectively, determine the drag coefficient $C_{\mathfrak{D}}$ of the strut. The drag coefficient is defined by $C_{\mathfrak{D}} = \mathfrak{D}/\frac{1}{2}\rho U^2 c$, where $\mathfrak{D}$ is the total drag per unit length on the strut due to pressure and viscous forces, and ρ is the fluid density.

PROBLEM 4

4. Calculate the static thrusts of the rocket and turbojet engines described in the figure. The thrust $\mathcal{T}$ is the force necessary to prevent horizontal movement of the engine. Both engines exhaust a mass flow of 120 lb/sec. The ratio of air and fuel mass flowing into the turbojet is 50:1 and in its exhaust plane the velocity is 1500 ft/sec and the pressure is the same as the ambient pressure. The rocket contains all its own propellant and exhausts it at a velocity of 9000 ft/sec through an area of 2 ft^2. The pressure in the exhaust plane of the rocket is 22.5 psia and ambient pressure is 14.7 psia.

5. An idealized supersonic ramjet diffuser consists of an axisymmetric center body located in a cylindrical duct as shown in the figure. The flow at stations (1) and (2) may be assumed uniform at the values indicated and the stream tube which enters the inlet has a diameter D far upstream.

   Show that the aerodynamic drag on the center body and struts is

$$\mathfrak{D} = -\frac{\pi}{4} D^2[\rho_1 u_1(u_2 - u_1) + p_2 - p_1].$$

PROBLEM 5

6. A mixture of gases containing 10 lb of nitrogen, 10 lb of hydrogen, and 15 lb of helium is contained at a pressure of 100 psia and temperature 500°R. If the constituents are taken to be perfect gases and the Gibbs-Dalton law holds, what is the molecular weight and specific heat ratio of the mixture?

Gas	Mol wt, $\overline{M}$	Specific heat ratio, γ
Nitrogen	28	1.4
Hydrogen	2	1.4
Helium	4	1.67

7. Calculate the combustion temperature of an oxygen-hydrogen mixture whose mass ratio is 3:1 (O_2:H_2). Since this means a considerable excess of hydrogen over the stoichiometric proportion, assume that the product temperature will be low enough to prevent dissociation. The temperature of the reactants entering the chamber is 20°C, and the heat transfer to the coolant from the chamber walls is 2 kcal per mole of product. As a first approximation, the mean specific heats may be taken as in the table below, and the heat of reaction as −57.802 kcal/gm-mole at 300°K (see Table 2–1).

	Average specific heat, cal/gm·mole·°C		
	H_2	O_2	H_2O
Reactants	6.5	7.5	
Products	7.4	8.3	11.0

8. A mixture of air and a hydrocarbon fuel whose average composition is indicated by $CH_{5.4}$ undergoes complete combustion. Twice as much air is present as needed to burn all the fuel. How high is the final temperature? The heats of formation are given in the table below. It may be assumed that air is 79% nitrogen and 21% oxygen by volume. The reactants have a temperature of 77°F before combustion. The adiabatic flame temperature may be found either by using average values of the specific heats, or enthalpy tables for each constituent. A very approximate estimate may be made by simply assuming that the product mixture has the same specific heat as air at the same temperature.

Constituent	Q_f, Btu/lb-mole at 14.7 psia and 77°F
$CH_{5.4}$	−130,000
CO_2 (g)	−169,300
O_2 (g)	0
N_2 (g)	0
H_2O	−104,000

9. Find the flame temperature of the products of combustion of hydrogen and oxygen whose composition and average specific heats are given in the table below. The reactants enter the adiabatic combustion chamber at 0°F. In this table, $\overline{C}_{pp}$ is the average molar specific heat of the component between 77°F and the flame temperature, $\overline{C}_{pr}$ is the average molar specific heat of the component between 0°F and 77°F, and Q_f is the heat of formation at 77°F.

Component	Moles	$\overline{C}_{pp}$, Btu/lb-mole·°R	$\overline{C}_{pr}$	Q_f, Btu/lb-mole
O_2	0.1008	8.8	7.0	0
H_2	0.3170	8.0	7.0	0
O	0.054	5.0		+105,400
H	0.109	5.0		+ 92,910
OH	0.233	8.0		+ 18,000
H_2O	1.512	11.2		−104,000

10. One mole of CO_2 is heated at atmospheric pressure to 5000°F. What is the equilibrium mixture composition if only CO, O_2, and CO_2 are present?

11. An inventor claims that a secret device shown figuratively below can take airstreams (1) and (2) and convert them into output streams (3) and (4) without energy exchange with its environment. Inside the device are mechanisms for mass, momentum, and energy exchange between the streams. Is it possible that such a device can be made to operate in the steady state? It may be assumed that velocities at all stations are negligible.

PROBLEM 11

12. Show that if any constant pressure line p_1 is given on a temperature-entropy plane, a line corresponding to any other constant pressure p can be obtained by horizontal shift of magnitude (for a perfect gas):

$$\Delta S = -R \ln \frac{p}{p_1}.$$

13. Two streams of air mix in a constant-area mixing tube. The primary stream enters the mixing tube at station (1) with a velocity of 1000 ft/sec and a temperature of 1500°R. The secondary stream enters with velocity 100 ft/sec and temperature 500°R. The flow at stations (1) and (2) may be assumed one-dimensional. The pressure at station (1) is 15 psia and the ratio of primary to secondary flow areas at station (1) is 1:3.

PROBLEM 13

(a) Using continuity, momentum, and energy equations, along with the perfect-gas law, show how the flow at (2) may be determined from conditions at (1).
(b) Determine the velocity, temperature, and pressure at station (2).

3

Steady One-Dimensional Flow of a Perfect Gas

3–1 INTRODUCTION

In most of the propulsion methods discussed in this volume, the working fluid can be quite satisfactorily approximated by a perfect gas. In this chapter the basic laws stated in the first chapter are applied to perfect gases flowing in channels. For algebraic convenience the specific heat of the gas is usually assumed constant herein.

Useful analyses of actual channel flows are relatively easy if it is assumed that the fluid conditions vary in the streamline direction only. In this case, the flow is said to be one-dimensional. The flow in the immediate vicinity of a wall can never be one-dimensional, since the velocity at the wall surface is zero and there are significant property variations across the streamlines. However, with the exception of a thin layer of fluid next to the wall, fluid conditions are often fairly uniform over a large part of a flow. Even if the bulk of the flow is not uniform on any surface normal to the streamlines, the one-dimensional approximation may still lead to useful solutions for the streamwise variation of average fluid properties. One-dimensional solutions exhibit the major characteristics of many important flows, including those in nozzles, diffusers, and combustors.

3–2 GENERAL ONE-DIMENSIONAL FLOW OF A PERFECT GAS

Fluid flows are governed by the equations of continuity, momentum, and energy, along with an appropriate equation of state. In this section these equations are applied to a one-dimensional differential control volume within a duct, as indi-

cated in Fig. 3–1. The rates of heat and work transfer are $d\dot{Q}$ and $d\mathcal{P}_s$, respectively. A body force of magnitude X per unit volume of fluid may act in the stream direction.

For steady flow the continuity requirement [from Eq. (2–2)] is

$$\int_{cs} \rho \mathbf{u} \cdot \mathbf{n}\, dA = 0.$$

Since the flow of Fig. 3–1 is assumed one-dimensional, this expression may be reduced to $(d/dx)(\rho u A) = 0$, or

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{du}{u} = 0 \tag{3–1}$$

for the control volume of infinitesimal length indicated in Fig. 3–1.

FIG. 3–1. One-dimensional flow through a control volume.

The momentum equation for steady flow [from Eq. (2–4)] is

$$\sum \mathbf{F} = \int_{cs} \rho \mathbf{u}(\mathbf{u} \cdot \mathbf{n})\, dA.$$

Writing $\sum \mathbf{F}$ as the sum of pressure, wall shear, and body forces, and again considering the flow of Fig. 3–1, this equation can be written

$$-\left(A\, dp + \tau_0 c\, dx\right) + XA\, dx = \rho u A\, du$$

or

$$-\left(\frac{dp}{dx} + \frac{\tau_0 c}{A}\right) + X = \rho u \frac{du}{dx}, \tag{3–2}$$

where τ_0 is the wall shear stress and c is the duct circumference.

For steady flow the energy requirement [from Eq. (2–8)] is

$$d\dot{Q} = \int_{cs} \dot{\mathcal{Q}}\, dA_{cs} = \int_{cs} \left(h + \frac{u^2}{2}\right) \rho \mathbf{u} \cdot \mathbf{n}\, dA - \int_{cv} \mathbf{X} \cdot \mathbf{u}\, d\mathcal{V} + d\mathcal{P}_s,$$

where A_{cs} indicates the entire control surface area (as opposed to the area A of the duct cross section). The gravitational potential energy term has been neglected, as is usually permissible. Further, except for special cases to be considered in Chapters 5 and 16, the body force $\mathbf{X}$ will be zero. In this case the energy equation reduces to

$$\dot{m}(dh + u\, du) = d\dot{Q} - d\mathcal{P}_s \tag{3–3a}$$

or

$$dh + u\, du = dq - dw, \tag{3–3b}$$

where $\dot{m}$ is the mass flow rate $\rho u A$ (a constant) and q and w are the heat and work transfer per unit mass.

If the fluid can be assumed a perfect gas (as it will be in this chapter), the equation of state is

$$p = \rho R T. \tag{3–4}$$

If the flow area, heat transfer, work, shear stress, and body force are known, changes in the state of the fluid stream may be obtained from these four equations. In general, solutions cannot be expressed in simple algebraic form. There are a number of important special cases, however, which can be integrated in closed form. Before we proceed to these special cases, let us introduce two concepts useful for all cases: the stagnation state and the Mach number.

Stagnation state

The *stagnation state* is defined as that state which would be reached by a fluid if it were brought to rest reversibly, adiabatically, and without work. The energy equation for such a deceleration is obtained from Eq. (3–3b),

$$dh + u\, du = 0,$$

which may be integrated to

$$h_0 \equiv h + \frac{u^2}{2}. \tag{3–5}$$

The constant of integration, h_0, is called the stagnation enthalpy.

For any adiabatic flow viewed from a reference frame in which no work occurs, the stagnation enthalpy is constant. Note particularly that this conclusion holds for both reversible and irreversible flows. Unlike enthalpy, the pressure reached when there is adiabatic zero-work deceleration of a fluid is decreased by irreversible processes. This is illustrated in Fig. 3–2, an enthalpy-entropy diagram for a typical gas. The stagnation state corresponding to state 1 is 01 and the stagnation pressure is p_{01}, which is the pressure attained by the fluid after it has

FIG. 3–2. Definition of stagnation state.

been stopped isentropically. If the flow is brought to rest irreversibly, as at 03 (the increased entropy is a measure of the irreversibility), the stagnation enthalpy is the same as before but the pressure attained is less than p_{01}. For any process, such as 1 to 2, the stagnation state generally undergoes a change like the one from 01 to 02. The decrease in pressure from p_{01} to p_{02} is, in the absence of work, an indication of the irreversibility of the process. If the fluid is a perfect gas, the stagnation temperature is related to the stagnation enthalpy through the relation

$$h_0 - h = \int_T^{T_0} c_p \, dT.$$

The stagnation temperature T_0 of the fluid is that temperature which would be reached upon adiabatic, zero-work deceleration. For a perfect gas with constant specific heats the stagnation temperature and pressure are related [using Eq. (2–19)] by

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\gamma/(\gamma-1)}, \tag{3–6}$$

since the deceleration process is, by definition, isentropic. Other properties of the fluid at the stagnation state may be determined from p_0, T_0, and the equation of state. The stagnation conditions may be regarded as local fluid properties. Aside from analytical convenience, the definition of the stagnation state is useful experimentally since T_0 and p_0 are relatively easily measured. It is usually much more convenient to measure the stagnation temperature T_0 than the temperature T.

Mach number

A very convenient variable in compressible-flow problems is the Mach number M, defined as

$$\mathrm{M} = \frac{u}{a}, \tag{3–7}$$

where a is the local speed of sound in the fluid. The speed of sound is the speed of propagation of very small pressure disturbances. For a perfect gas it is given by [1]

$$a = \sqrt{\gamma R T}. \tag{3–8}$$

3–3 ISENTROPIC FLOW

Many actual processes, such as flows in nozzles and diffusers, are ideally isentropic. It is, therefore, worthwhile to study the isentropic flow of a perfect gas in the absence of work and body forces. The simple results obtained with the constant specific heat approximation are useful even for large temperature changes, so long as appropriate average values of c_p and γ are used.

For constant specific heat, Eq. (3–5) may be written

$$\frac{T_0}{T} = 1 + \frac{u^2}{2c_pT} \tag{3–9}$$

which, with the definition of Mach number, becomes

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2}\mathrm{M}^2. \tag{3–10}$$

Equation (3–6) may then be written

$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2}\mathrm{M}^2\right)^{\gamma/(\gamma-1)}. \tag{3–11}$$

From the perfect gas law and Eqs. (3–10) and (3–11),

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2}\mathrm{M}^2\right)^{1/(\gamma-1)}. \tag{3–12}$$

For *any* flow of a perfect gas these can be used to relate the local conditions (T, p, ρ) to the *local* stagnation conditions T_0, p_0, ρ_0, since this relationship is by *definition* isentropic whatever the actual flow considered may be.

The mass flow per unit area is

$$\frac{\dot{m}}{A} = \rho u.$$

Using Eqs. (3–7), (3–8), and (3–10), we may express the velocity as

$$u = \mathrm{M}\sqrt{\frac{\gamma R T_0}{1 + \frac{\gamma - 1}{2}\mathrm{M}^2}}.$$

FIG. 3–3. One-dimensional isentropic flow of a perfect gas. (After Shapiro [1].)

Then, using this expression with Eq. (3–12), we may write the mass flow rate as

$$\frac{\dot{m}}{A} = \frac{p_0\sqrt{\gamma}}{\sqrt{RT_0}}\,\mathrm{M}\left(\frac{1}{1+\dfrac{\gamma-1}{2}\mathrm{M}^2}\right)^{(\gamma+1)/2(\gamma-1)}. \tag{3–13}$$

For a given fluid (γ, R) and inlet state (p_0, T_0), it may readily be shown that the mass flow per unit area is maximum at M = 1. If we indicate those properties

of the flow at M = 1 with an asterisk, the maximum flow per unit area is, from Eq. (3–13),

$$\frac{\dot{m}}{A^*} = \frac{p_0}{\sqrt{RT_0}}\sqrt{\gamma}\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/2(\gamma-1)}. \tag{3–14}$$

Combining Eqs. (3–13) and (3–14), we have

$$\frac{A}{A^*} = \frac{1}{\mathrm{M}}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}\mathrm{M}^2\right)\right]^{(\gamma+1)/2(\gamma-1)}. \tag{3–15}$$

For a given isentropic flow (given γ, R, p_0, T_0, $\dot{m}$), it is clear that A^* is a constant so that it may be used as in Eq. (3–15) to normalize the actual flow area A. Equations (3–10), (3–11), and (3–12) relate the fluid properties to the Mach number M and Eq. (3–15) shows how the Mach number depends on the flow area. Expressing flow variables in terms of Mach number is a matter of great convenience, since the various flow functions can be plotted or tabulated as a function of M for the appropriate γ. Figure 3–3 gives the one-dimensional isentropic functions for a perfect gas of $\gamma = 1.4$. Tabulations for other γ are given by Shapiro [1], Keenan and Kaye [2], and others.

3–4 FRICTIONLESS CONSTANT-AREA FLOW WITH STAGNATION TEMPERATURE CHANGE

Another special case for which the basic equations may be solved in closed form is frictionless flow in a constant-area duct in which a stagnation enthalpy change occurs. Four processes of interest in which the stagnation temperature of a moving stream changes are:

1. Combustion,
2. Evaporation or condensation of liquid drops traveling with stream,
3. Flux of electric current through a fluid of finite conductivity (Joule heating), and
4. Wall heat transfer.

In any real flow, frictional effects are always present, especially near solid boundaries. As we shall discuss in Chapter 4, wall heat transfer and wall friction are in fact so closely related that it is not realistic to discuss the former without at the same time talking about the latter. Nevertheless, the study of frictionless flow in a constant-area duct in which a T_0 change occurs illustrates some important features of the real flows in which the first three effects above are present. Combustion and evaporation or condensation processes may result in variations of molecular weight and gas constants, but for the sake of simplicity these may be considered negligible. The stagnation-enthalpy change in the processes can be determined from

$$\Delta h_0 = q - w,$$

where q and w are net heat and work transfers per unit mass of fluid. For this case the continuity, momentum, and energy equations are from Eqs. (3–1), (3–2), and (3–3), respectively:

$$\text{Continuity:} \quad \frac{d\rho}{\rho} + \frac{du}{u} = 0,$$

$$\text{Momentum:} \quad dp = -\rho u\, du,$$

$$\text{Energy:} \quad dh_0 = dh + u\, du.$$

The first two of these can be readily integrated to

$$\rho u = \rho_1 u_1,$$

$$p - p_1 = -(\rho u)(u - u_1) = -(\rho_1 u_1)(u - u_1),$$

in which the subscript 1 signifies an initial condition. The second of these may also be expressed as

$$p - p_1 = \rho_1 u_1^2 - \rho u^2.$$

By introducing the Mach number, we can transform this to

$$\frac{p}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M^2}.$$

Further, by use of Eq. (3–11), we can derive the stagnation pressure ratio from this equation, with the result that

$$\frac{p_0}{p_{01}} = \left(\frac{1 + \gamma M_1^2}{1 + \gamma M^2}\right)\left(\frac{1 + \frac{\gamma - 1}{2} M^2}{1 + \frac{\gamma - 1}{2} M_1^2}\right)^{\gamma/(\gamma-1)}. \tag{3–16}$$

Thus the change in stagnation pressure is directly related to the Mach number.

The dependence of Mach number on stagnation enthalpy (or stagnation temperature) can in turn be obtained as follows: Using the perfect-gas law,

$$\frac{T}{T_1} = \frac{p}{p_1}\frac{\rho_1}{\rho},$$

or invoking the continuity condition, $\rho u = \rho_1 u_1$, it may be seen that

$$\frac{T}{T_1} = \frac{p}{p_1}\frac{u}{u_1}.$$

Now if the Mach number relation $M = u/\sqrt{\gamma RT}$ is introduced, it can easily be shown that

$$\frac{T}{T_1} = \left(\frac{p}{p_1}\frac{M}{M_1}\right)^2$$

and, using the above relation between static pressure ratio and Mach number,

$$\frac{T}{T_1} = \left[\frac{1 + \gamma M_1^2}{1 + \gamma M^2}\left(\frac{M}{M_1}\right)\right]^2.$$

From Eq. (3–10) and this expression, we can show that the stagnation temperature ratio is governed by

$$\frac{T_0}{T_{01}} = \left[\frac{1 + \gamma M_1^2}{1 + \gamma M^2}\left(\frac{M}{M_1}\right)\right]^2\left(\frac{1 + \frac{\gamma - 1}{2} M^2}{1 + \frac{\gamma - 1}{2} M_1^2}\right). \tag{3–17}$$

Thus the Mach number depends uniquely on the ratio of stagnation temperatures and the initial Mach number.

It is desirable in this case, as in the isentropic case, to simplify these relationships by the choice of a convenient reference state. Since stagnation conditions are *not* constant, the stagnation state is not suitable for this purpose. However, the state corresponding to unity Mach number is suitable since, as Eqs. (3–16) and (3–17) show, conditions there *are* constant for a given flow (i.e., for a given p_{01}, T_{01}, M_1). Again using an asterisk to signify properties at $M = 1$, it can be shown from the equations above that

$$\frac{T}{T^*} = \left(\frac{1 + \gamma}{1 + \gamma M^2}\right)^2 M^2, \tag{3–20}$$

$$\frac{T_0}{T_0^*} = \frac{2(\gamma + 1)M^2\left(1 + \frac{\gamma - 1}{2} M^2\right)}{(1 + \gamma M^2)^2}, \tag{3–21}$$

$$\frac{p}{p^*} = \frac{1 + \gamma}{1 + \gamma M^2}, \tag{3–22}$$

$$\frac{p_0}{p_0^*} = \left(\frac{2}{\gamma + 1}\right)^{\gamma/(\gamma-1)}\left(\frac{1 + \gamma}{1 + \gamma M^2}\right)\left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}. \tag{3–23}$$

In this way the fluid properties may be presented as a function of a single argument, the local Mach number. Equations (3–20) through (3–23) are shown graphically in Fig. 3–4 for the case $\gamma = 1.4$. Given particular entrance conditions, T_{01}, p_{01}, M_1, the exit conditions after a given change in stagnation temperature may be obtained as follows: The value M_1 fixes the value of T_{01}/T_0^* and thus the value T_0^*, since T_{01} is known. The exit state is then fixed by T_{02}/T_0^*, determined by

$$\frac{T_{02}}{T_0^*} = \frac{T_{01}}{T_0^*} + \frac{\Delta T_0}{T_0^*} \quad \text{or} \quad \frac{T_{02}}{T_0^*} = \frac{T_{01}}{T_0^*} + \frac{q - w}{c_p T_0^*}.$$

Then M_2, p_2/p^*, p_{02}/p_0^* are all fixed by the value of T_{02}/T_0^* (for a given value of γ).

FIG. 3–4. Frictionless flow of a perfect gas in a constant-area duct with stagnation temperature change. (After Shapiro [1].)

It may be seen from Fig. 3–4 that increasing the stagnation temperature drives the Mach number toward unity whether the flow is supersonic or subsonic. After the Mach number has approached unity in a given duct, further increase in stagnation enthalpy is possible only if the initial conditions change. The flow may be said to be thermally choked.

It is interesting to note the variations in stagnation pressure as the stream is subjected to energy transfer. The stagnation pressure always drops when energy is added to the stream and rises when energy is transferred from the stream. Thus, whether the flow is subsonic or supersonic, there may be significant loss of stagnation pressure due to combustion in a moving stream. Conversely, cooling tends to increase the stagnation pressure.

3–5 CONSTANT-AREA FLOW WITH FRICTION

Another solution of Eqs. (3–1) through (3–4) can be obtained for constant-area adiabatic flow with friction but no body force or work. In this case, from Eqs. (3–1), (3–2), and (3–3) the continuity, momentum, and energy relations are, respectively,

$$\text{Continuity:} \qquad \frac{d\rho}{\rho} + \frac{du}{u} = 0,$$

$$\text{Momentum:} \qquad \frac{dp}{p} = -\frac{\tau_0 c\, dx}{pA} - \frac{\rho u\, du}{p},$$

$$\text{Energy:} \qquad 0 = dh + u\, du.$$

For flow in long pipes it will be shown (Fig. 4–14) that the wall shear stresses τ_0 can be correlated as follows:

$$\frac{\tau_0}{\rho u^2/2} = f\left(\frac{\rho u D}{\mu}, \frac{\mathfrak{K}}{D}\right)$$

or

$$C_f = f\left(\mathrm{Re}, \frac{\mathfrak{K}}{D}\right),$$

in which C_f is the skin friction coefficient $\tau_0/(\rho u^2/2)$, Re is the Reynolds number $\rho u D/\mu$, $\mathfrak{K}$ is the average height of surface roughness elements, and D is the duct diameter. For Mach numbers greater than unity, the Mach number is also a significant variable. The effects of roughness (or at least of variations in roughness) are usually fairly unimportant for surfaces which have a reasonably smooth finish. This point is discussed in more detail in Chapter 4.

The way in which friction affects the Mach number may be shown as follows. First the conservation equations are transformed (by introducing the perfect gas law and the definition of Mach number) to read:

$$\text{Continuity:} \qquad \frac{dp}{p} - \frac{dT}{T} + \frac{1}{2}\frac{du^2}{u^2} = 0,$$

$$\text{Momentum:} \qquad \frac{dp}{p} = \frac{-\gamma \mathrm{M}^2}{2}\left(4\frac{C_f\, dx}{D}\right) - \frac{u\, du}{RT},$$

$$\text{Energy:} \qquad 0 = \frac{dT}{T} + \frac{\gamma - 1}{\gamma}\frac{u\, du}{RT}.$$

Since

$$\mathrm{M} = \frac{u}{\sqrt{\gamma R T}},$$

then

$$\frac{d\mathrm{M}^2}{\mathrm{M}^2} = \frac{du^2}{u^2} - \frac{dT}{T}.$$

And now, if we use the continuity equation to eliminate du^2/u^2, it follows that

$$\frac{d\mathrm{M}^2}{\mathrm{M}^2} = -2\frac{dp}{p} + \frac{dT}{T}.$$

Also, combining the momentum and energy equations, we can easily show that

$$\frac{dp}{p} = \frac{-\gamma\mathrm{M}^2}{2}\left(4\frac{C_f\,dx}{D}\right) + \frac{\gamma}{\gamma-1}\frac{dT}{T}.$$

Combining these last two expressions yields

$$\frac{d\mathrm{M}^2}{\mathrm{M}^2} = \gamma\mathrm{M}^2\left(4C_f\frac{dx}{D}\right) - \frac{\gamma+1}{\gamma-1}\frac{dT}{T}$$

which, for constant stagnation temperature [using Eq. (3–10)], becomes†

$$\frac{d\mathrm{M}^2}{\mathrm{M}^2} = \frac{\gamma\mathrm{M}^2\left(1+\frac{\gamma-1}{2}\mathrm{M}^2\right)}{1-\mathrm{M}^2}\left(\frac{4C_f\,dx}{D}\right). \tag{3–24}$$

Equation (3–24) shows that $d\mathrm{M}^2 > 0$ for $\mathrm{M} < 1$ and $d\mathrm{M}^2 < 0$ for $\mathrm{M} > 1$. That is, friction *always* changes the Mach number toward unity. For constant C_f, Eq. (3–24) can be integrated between the limits $\mathrm{M} = \mathrm{M}$ and $\mathrm{M} = 1$ to yield

$$\frac{4C_fL^*}{D} = \frac{1-\mathrm{M}^2}{\gamma\mathrm{M}^2} + \frac{\gamma+1}{2\gamma}\ln\frac{(\gamma+1)\mathrm{M}^2}{2\left(1+\frac{\gamma-1}{2}\mathrm{M}^2\right)}, \tag{3–25}$$

in which L^* is the length of duct necessary to change the Mach number of the flow from M to unity.

Consider a duct of given cross-sectional area and variable length. If the inlet state, mass flow rate, and average skin friction coefficient are fixed, there is a maximum length of duct which can transmit the flow. Since the Mach number is unity at the exhaust plane in that case, the length is designated L^* and the flow may be said to be friction-choked. From Eq. (3–25), we can see that at any point 1 in the duct, the variable C_fL^*/D depends only on M_1 and γ. Since D is constant and C_f is assumed constant, then at some other point a distance x ($x < L^*$) downstream from point 1,

$$\left(\frac{4C_fL^*}{D}\right)_x = \left(\frac{4C_fL^*}{D}\right)_1 - \frac{4C_fx}{D}.$$

From this we can determine M_x. Other relationships which may be derived [1]

† For noncircular ducts, the hydraulic diameter $D_H = 4A/c$ can be used as an approximation in place of the diameter D.

FIG. 3–5. Adiabatic flow of a perfect gas in a constant-area duct with friction. (After Shapiro [1].)

by using the definition of Mach number and the continuity and energy relations are:

$$\frac{T}{T^*} = \frac{\gamma+1}{2\left(1+\frac{\gamma-1}{2}\mathrm{M}^2\right)}, \tag{3–26}$$

$$\frac{p}{p^*} = \frac{1}{\mathrm{M}}\left[\frac{\gamma+1}{2\left(1+\frac{\gamma-1}{2}\mathrm{M}^2\right)}\right]^{1/2}, \tag{3–27}$$

$$\frac{p_0}{p_0^*} = \frac{1}{\mathrm{M}}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}\mathrm{M}^2\right)\right]^{(\gamma+1)/2(\gamma-1)}. \tag{3–28}$$

FIG. 3–6. Typical temperature-entropy diagram for choking process (subsonic).

FIG. 3–7. Typical temperature-entropy diagram for choking process (supersonic).

Equations (3–26), (3–27), and (3–28) are plotted in Fig. 3–5 for $\gamma = 1.4$. Again the asterisk denotes properties at that point where $M = 1$.

In Sections 3–3, 3–4, and 3–5, three special problems have been treated to show the separate effects of area change, energy transfer, and friction. In each case, reference was made to a state where $M = 1$ and properties at that point were distinguished by an asterisk. Note that properties at that point are used as convenient normalizing quantities only, and the state need not actually exist in the real flow studied.

For a given inlet condition, the $M = 1$ conditions represent *three distinctly different states*. This is clearly indicated in Figs. 3–6 and 3–7. For two initial states, one subsonic and the other supersonic, three processes to reach a Mach number of one are shown on a typical *T-s* diagram. The energy-transfer path is called a Rayleigh line and the friction path is called a Fanno line. On both these lines the * condition occurs on the nose of the curve. The notation $*s$, $*f$, $*h$ indicates the $M = 1$ states reached by isentropic, friction, and heating processes, respectively. Subsonic states are above the nose of the appropriate curve, while supersonic states are below it. Note also that travel in either direction is possible along the isentropic and energy-transfer curves, while only one direction is possible on the friction curve.

If it is important to study combined effects of these processes, then simple solutions are not available. However, the complete equations may be integrated numerically according to procedures described by Shapiro [1].

3–6 SHOCKS

A shock is a discontinuity in a (partly) supersonic flow fluid. Fluid crossing a stationary shock front rises suddenly and irreversibly in pressure and decreases in velocity. It also changes its direction except when passing through a shock which is perpendicular (normal) to the approaching flow direction. Such plane normal shocks are the easiest to analyze.

Normal shocks

For flow through a normal shock, with no direction change, area change, or work done, the continuity, momentum, and energy equations are

$$\text{Continuity:} \quad \rho_1 u_1 = \rho_2 u_2,$$

$$\text{Momentum:} \quad p_1 - p_2 = \rho_1 u_1 (u_2 - u_1),$$

$$\text{Energy:} \quad T_{01} = T_{02},$$

where the subscripts 1 and 2 indicate initial and final states, respectively. Two solutions to these equations are possible, one of which states simply that no change

occurs. In terms of Mach number, the interesting solution may be stated [1] that

$$M_2^2 = \frac{M_1^2 + \dfrac{2}{\gamma - 1}}{\dfrac{2\gamma}{\gamma - 1} M_1^2 - 1},$$

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma + 1} M_1^2 - \frac{\gamma - 1}{\gamma + 1},$$

$$\frac{p_{02}}{p_{01}} = \frac{\left[\left(\dfrac{\gamma + 1}{2} M_1^2\right) \Big/ \left(1 + \dfrac{\gamma - 1}{2} M_1^2\right)\right]^{\gamma/(\gamma-1)}}{\left(\dfrac{2\gamma}{\gamma + 1} M_1^2 - \dfrac{\gamma - 1}{\gamma + 1}\right)^{1/(\gamma-1)}},$$

$$\frac{T_2}{T_1} = \frac{\left(1 + \dfrac{\gamma - 1}{2} M_1^2\right)\left(\dfrac{2\gamma}{\gamma - 1} M_1^2 - 1\right)}{\dfrac{(\gamma + 1)^2}{2(\gamma - 1)} M_1^2}.$$

In Fig. 3–8, M_2 and p_{02}/p_{01} are plotted for the case $\gamma = 1.4$. Note especially that rather large losses of stagnation pressure occur across Mach numbers greater than about 1.5. Tables of all the functions for various γ are provided by Keenan and Kaye [2].

FIG. 3–8. Normal shock functions. (After Shapiro [1].)

FIG. 3–9. Oblique shock geometry.

Oblique shocks

If a plane shock is inclined at an angle to the flow, the fluid passing through suffers not only a sudden rise in pressure and decrease in speed but also a sudden change of direction. The situation is illustrated in Fig. 3–9 for an oblique shock s-s' in one-dimensional flow. In passing through the shock the fluid is deflected through an angle δ. The basic equations applied to the indicated control volume are:

$$\text{Continuity:} \quad \rho_1 u_{1n} = \rho_2 u_{2n},$$
$$\text{Energy:} \quad T_{01} = T_{02}.$$

Two momentum equations may be written for this flow, one for changes in momentum perpendicular to the shock, the other for changes parallel to it:

(a) Momentum normal to the shock, $\quad p_1 - p_2 = \rho_2 u_{2n}^2 - \rho_1 u_{1n}^2,$

(b) Momentum parallel to the shock, $\quad 0 = \rho_1 u_{1n}(u_{2t} - u_{1t}),$

in which the subscripts n and t indicate directions normal to and parallel to the shock, respectively. From (b),

$$u_{2t} = u_{1t}.$$

Since the velocity component parallel to the shock is the same on both sides of it, it may be seen that an oblique shock becomes a normal shock relative to a coordinate system moving with velocity $u_{1t} = u_{2t}$. This fact may permit us to use the normal shock equations to calculate oblique shocks. For example, given the upstream (or initial) Mach number M_1 and the overall stagnation pressure ratio, the components M_{1n} and M_{2n} could be determined from Fig. 3–8. The pressure and temperature ratios p_2/p_1 and T_2/T_1 could then be obtained by replacing M_1 with M_{1n} in the appropriate normal shock equations. The shock angle σ could

FIG. 3–10. Shock angle versus inlet Mach number and turning angle [1]. Curves above dashed line hold for $M_2 < 1$ and curves below hold for $M_2 > 1$. (After Shapiro[1].)

FIG. 3–11. Stagnation pressure ratio versus inlet Mach number, with turning angle as parameter [1]. Curves above dashed line hold for $M_2 > 1$ and curves below hold for $M_2 < 1$. (After Shapiro [1].)

FIG. 3–12. Exit Mach number versus inlet Mach number, with turning angle as parameter [1]. Curves above dashed line correspond to small σ and curves below correspond to large σ. (After Shapiro [1].)

be determined from M_{1n} and M_1 and the downstream Mach number from M_{2n}, T_2/T_1 and the condition $u_{1t} = u_{2t}$.

Alternatively, the above four equations, along with the equation of state, can be reduced to a set of four independent equations relating the variables M_1, M_2, p_2/p_1, ρ_2/ρ_1, σ, and δ [1]. If any two variables are given, for example M_1 and the turning angle δ, then all others are determined and the downstream (or final) conditions can be found in terms of the upstream conditions.

The solution of these equations from Shapiro [1] is given in Figs. 3–10, 3–11 and 3–12 for the case $\gamma = 1.4$ and for various values of δ.

Axisymmetric oblique shocks are discussed by Shapiro [3], who presents a complete solution for the case of the conical shock as well as methods of treatment of the general axisymmetric shock problem.

References

1. SHAPIRO, ASCHER H., *The Dynamics and Thermodynamics of Compressible Fluid Flow*, Volume I. New York: The Ronald Press Company, 1953; Chapters 1–8, 16
2. KEENAN, J. H., and J. KAYE, *Gas Tables*. New York: John Wiley & Sons, 1957
3. SHAPIRO, ASCHER H., *op. cit.*, Volume II, Chapter 17

Problems

1. Show that for a pure substance the stagnation pressure cannot increase for an adiabatic zero-work process. During what physical processes might it increase for a zero-work process?

2. Show that, for a perfect gas, the Mach number at the nose of the Fanno and Rayleigh lines is indeed unity. The nose is defined by $ds/dT = 0$.

3. A perfect fluid expands in a frictionless nozzle from stagnation conditions $p_0 = 600$ psia, $T_0 = 5000°R$, to ambient pressure of 15 psia. If the expansion is isentropic, determine the following conditions at the final pressure: (1) velocity, (2) Mach number, (3) temperature, and (4) area per unit mass flow. How does the final flow area compare with the throat area for a given mass flow? The specific heat ratio γ is 1.4 and the molecular weight $\overline{M}$ is 30.

PROBLEM 4

4. Air flows through a cylindrical combustion chamber of diameter 1 ft and length 10 ft. Stations (1) and (2) are the inlet and outlet, respectively. The inlet stagnation temperature is 500°R and the inlet stagnation pressure is 200 psia.

(a) If the skin friction coefficient is $C_f = 0.0040$ (approximately uniform) and no combustion takes place, what will the inlet Mach number be if it is subsonic and the exit Mach number is unity? What are the values of static and stagnation pressures at the exit?

(b) If combustion takes place and the heat of reaction is 300 Btu per pound of mixture, neglecting frictional effects, and if the inlet Mach number is 0.25, what is the exit Mach number?

(c) If, in a similar problem, the effects of combustion and skin friction on the flow are of the same order of magnitude, how might their simultaneous effects be estimated?

The fluid may be considered a perfect gas with $\gamma = 1.4$ and $\overline{M} = 29$.

5. A Mach 2 flow passes through an oblique shock as shown, and deflects 10°. A second oblique shock reflects from the solid wall. What is the pressure ratio across the two-shock system? It may be assumed that there is no boundary layer near the wall; i.e., the flow is uniform in each of the regions bounded by the shocks.

PROBLEM 5

6. Air enters a constant-area duct at Mach 3 and stagnation conditions 1300°R and 185 psia. In the duct it undergoes a frictionless energy-transfer process such that the exit Mach number is unity. Consider two cases: (a) normal shock at inlet to the duct, and (b) shockfree supersonic heating. Determine the stagnation temperature and pressure at the exit in each case. Is there any reason why the total energy transfer should differ (or be equal) in the two cases?

7. A uniform mixture of very small solid particles and a perfect gas expands adiabatically from a given chamber temperature through a given pressure ratio in a nozzle. The particles are so small that they may be considered to travel with the local gas speed. Since the density of the solid material is much higher than the gas phase, the total volume of the particles is negligible. The ratio of solid to gas flow rates is μ, where $0 < \mu < 1$. Consider two cases:

(a) no heat transfer between solid and gaseous phases, and (b) the solid particles have the local gas temperature at all points in the flow. Qualitatively, how does the presence of the solid particles affect the velocity at the end of the expansion? Show that for both cases the mixture is equivalent to the adiabatic expansion of a homogeneous gas of different molecular weight $\overline{M}$ and specific heat ratio γ.

8. A 30-ft^3 rocket-propellant chamber is filled with combustion gases at 6000°R and 1000 psia at the instant combustion ceases. If the throat area of the nozzle is 1 ft^2, estimate the time it takes for the chamber pressure to drop to 100 psia. Assume that at the instant combustion ceases the chamber propellant supply stops completely, and also that $\gamma = 1.2$ and $\overline{M} = 20$.

9. (a) Compare the work of compression per unit mass of air for both reversible adiabatic and reversible isothermal compression through a pressure ratio of 10 with initial conditions 500°R and 14.7 psia. Is the importance of specific heat variation large? Why should cooling during compression reduce the work?

(b) In an effort to provide a continuous cooling process during compression, water is sprayed into an airstream entering the axial compressor of a gas turbine engine. The compression may be taken to be reversible and the evaporation rate is approximately $dw/dT = k$, where w is the water-air mass ratio, T is the mixture temperature, and k is a constant. If $w \ll 1$, the process may be simplified by considering the sole effect of the evaporation to be energy transferred from the gas phase. Show how the final temperature and the work of compression depend on k.

4

Boundary Layer Mechanics and Heat Transfer

4–1 THE VISCOUS BOUNDARY LAYER

Historically the development of boundary layer theory followed the development of mathematical solutions for flows assumed to be nonviscous. In the latter half of the nineteenth century—after Euler had formulated the basic equations of motion—Helmholtz, Kelvin, Lamb, Rayleigh, and others applied them to a number of physical situations. By neglecting the effects of viscosity they obtained elegant mathematical descriptions of various flow fields. It was argued that a fluid like air, for example, which is so rarefied as to be invisible, must have negligible viscosity. However, with some exceptions, the so-called perfect-fluid theory disagreed strongly with the results of experiment.

FIG. 4–1. Flow patterns about a cylinder.

One particular example, which came to be known as *d'Alembert's paradox*, attracted a great deal of attention. As an illustration, the flow of a fluid over a cylinder as predicted by nonviscous theory is shown in Fig. 4–1(a). The theory predicts that the velocity and pressure fields are both symmetrical about a plane normal to the upstream velocity vectors. Thus the fluid can exert no net forces (lift or drag) on the cylinder. Moreover, this conclusion is independent of the shape of the body; if the fluid is truly inviscid, it can be shown mathematically

FIG. 4–2. Pressure distribution on a cylinder. (After Flachsbart [11].)

that for any body shape the net integrated pressure force upon the body will be zero. Of course, this conclusion directly contradicts experience. Everyone knows that it takes a finite force to hold a body immersed in a moving stream or to propel it through a stationary mass of fluid; hence the paradox.

Observations of the actual streamline pattern around the cylinder reveal something like Fig. 4–1(b), which suggests that the streamline symmetry is grossly disturbed. On the rear side of the cylinder the fluid seems almost stagnant when the stream velocity is quite low. For higher speeds, fairly violent oscillations take place, which affect the entire streamline pattern near the cylinder. Under certain conditions it is possible to observe a periodic shedding of vortices alternately from the top and bottom sides, as suggested by Fig. 4–1(c). This downstream flow pattern has been called a *vortex street*.

The drag of the actual fluid on the cylinder is largely due to the asymmetry of the pressure distribution on its surface. Figure 4–2 shows measured pressure distributions on a cylinder and, for comparison, the prediction of perfect-fluid theory. The symbol ρ stands for fluid density and the pressure and velocity far upstream are denoted by p_∞ and u_∞. It can be seen that two experimental pressure distributions may be measured (the difference between the two will be discussed later) and that both differ greatly from the theoretical result. The fact that the average pressure on the cylinder is much higher on the upstream hemi-surface explains most of the drag. In addition to this, the viscous force of the fluid on the cylinder imposes a drag force which, in this particular case, is considerably smaller than the pressure drag. For "streamlined" bodies, which have much less pressure drag, the viscous force is relatively more important.

For quite a period the discrepancy between existing theory and physical results was unexplained, and a marked rift developed between the theoreticians, who continued to expound the perfect-fluid theory, and the engineering-oriented workers like Bernoulli and Hagen, who actively developed the empirical body of knowledge generally called hydraulics. The equations of motion which do take viscosity into account—the Navier-Stokes equations—were formulated but were

so very complicated that they could not, in general, be solved. It appeared exceedingly difficult to improve mathematically on the perfect-fluid theory except in a few very special cases.

Then Prandtl (in 1904) made a major contribution to the solution of the problem by introducing the concept of the viscous boundary layer. He showed that even for fluids of vanishingly small viscosity there is a thin region near the wall in which viscous effects cannot be neglected. Since the velocity of the fluid particles on the wall is zero, there may be, under certain conditions, a region of large velocity gradient next to it. Then, even though the viscosity μ may be small, the shear stress on a fluid layer, $\tau = \mu(\partial u/\partial y)$, may be large in that region since $\partial u/\partial y$ is large. This region is called the viscous boundary layer. A boundary layer of thickness δ is indicated in Fig. 4–3. Outside the layer, the velocity gradient $\partial u/\partial y$ will be so small that viscous shear may be quite unimportant, so that the fluid behaves as though it actually had zero viscosity.

FIG. 4–3. Boundary layer velocity distribution.

Thus Prandtl showed that the flow field may be broken into two parts; a thin viscous zone near the wall, and an outer zone where the nonviscous theory is adequate. This suggested the major simplification of the Navier-Stokes equations which is the basis of boundary layer theory. For fairly high-speed flow, in which the boundary layer is so thin that the pressure gradient normal to the wall is negligible, Prandtl and others showed how to treat the boundary layer mathematically for special cases, many of which have been solved in the past sixty years.

Given that a boundary layer exists on a body, it is easy to see qualitatively why the case of vanishing viscosity is fundamentally different from the case of zero viscosity. Figure 4–2 shows that for zero viscosity a particle on the surface streamline of the cylinder rises in pressure to the stagnation value when it impinges upon the cylinder ($\theta = 0$). Then the pressure falls to a minimum (and the velocity rises to a maximum) at the position $\theta = \pi/2$. From that point it travels to the rearward stagnation point ($\theta = \pi$), having just enough momentum to climb the "pressure hill" and arrive at the stagnation point.

However small the viscosity, the viscous case is fundamentally different. The fluid near the wall has been slowed by viscous action. When a particle in the boundary layer reaches $\theta = \pi/2$ it will have reduced momentum so that only a small pressure rise will stop its forward motion, or actually send it moving backward. For this reason the particle can no longer follow the contour of the wall. The accumulation of stagnant fluid on the back part of the cylinder deflects the outer streamlines, as suggested in Fig. 4–1(b). This in turn greatly modifies the wall pressure distribution, as shown in Fig. 4–2. When the streamlines in the vicinity of the wall cease to follow it, they are said to *separate*.

FIG. 4–4. Some practical devices in which separation may occur.

This accumulation of stagnant or near-stagnant fluid and the resultant gross distortion of the streamlines means that the boundary layer behavior can have an important influence on overall flow behavior even though the quantity of fluid directly affected by viscosity is a small fraction of the total flow. Three separated flows of practical significance are shown in Fig. 4–4. Both the compressor cascade and the subsonic diffuser are designed so that the mainstream velocity decreases in the flow direction; hence the flow is against an adverse pressure gradient. If the pressure rise is too great the boundary layer will separate, creating regions of near-stagnant fluid as shown. Within rocket nozzles the pressure gradient is usually favorable, so that separation need not be a problem. However, under certain off-design conditions shocks can occur within rocket nozzles and the sudden pressure rise across the shock can cause separation.

Separation and its consequences in these devices will be discussed in the following chapters where appropriate. Our purpose in this chapter is to establish the nature and cause of separation and to indicate how one might predict its occurrence (although this is seldom possible with purely theoretical methods). Separation is, of course, only one phase of boundary layer behavior. We shall be interested also in the behavior of non-separated boundary layers, since they are of importance to heat transfer and other phenomena.

The classic picture of separation is illustrated in Fig. 4–5, which shows a series of velocity profiles in a boundary layer as it approaches and passes the separation point. As the pressure rises the free-stream velocity U falls, as the Bernoulli equation would predict. However, the effect of a given change in pressure is greater on the slow-moving fluid near the wall than on the free-stream fluid.

FIG. 4–5. A simplified picture of the development of separation.

The fluid near the wall can be slowed to a stagnant state and then be made to flow backward if the adverse pressure gradient continues.

The separation point is generally defined as that point where a streamline very near the wall leaves the wall. In two-dimensional flow this corresponds to the point at which the velocity profile has zero slope at the wall. That is,

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0 \quad \text{(two-dimensional separation).}$$

Figure 4–5 suggests that, past the separation point, there is a region of recirculating flow. In practice this region is highly unstable. It may be quite difficult to determine analytically, since once separation occurs there is a strong interaction between the body of stagnant fluid and the free stream. Often there results an adjustment of the free-stream geometry such that the adverse pressure gradient is reduced to practically zero, in which case there is no force to drive the recirculating fluid.

The effects of *pressure gradient* on the shape of the boundary layer, which are shown in Fig. 4–5, may be explained in a simplified manner as follows. Let the boundary layer be replaced by a hypothetical discontinuous flow shown in Fig. 4–6, in which the continuous variation in velocity normal to the wall is replaced by two step changes. Further, in order to focus on the effect of pressure gradient, let us for the moment neglect shear stresses. (The effects of shear stresses will be discussed subsequently.) With these assumptions, the momentum equation requires that

FIG. 4–6. Discontinuous boundary layer.

$$\frac{dp}{\rho} = -U\,dU \text{ (free stream)}, \quad \frac{dp}{\rho} = -u\,du \text{ (boundary layer)},$$

where, in keeping with boundary layer assumptions, the free stream and the boundary layer experience the *same* pressure change. Equating the two expressions for dp and defining the ratio $\alpha = u/U$, there results

$$d\alpha = -\frac{dp}{\rho U^2}\left(\frac{1-\alpha^2}{\alpha}\right).$$

Hence if the pressure rises ($dp > 0$), α decreases and the boundary layer fluid is decelerated more than the free stream. Noting that $-dp/\rho U^2$ is simply dU/U, this expression can be integrated (for constant density), yielding

$$\frac{1-\alpha^2}{1-\alpha_1^2} = \frac{U_1^2}{U^2},$$

where the subscript 1 refers to an initial condition. The thickness δ of the boundary layer can be found from the equation for continuity which requires that

$$\rho u \delta = \rho u_1 \delta_1$$

or, again for constant density,

$$\frac{\delta}{\delta_1} = \frac{u_1}{u} = \frac{\alpha_1 U_1}{\alpha U}.$$

The two quantities α and δ are plotted against U/U_1 in Fig. 4–7 for the particular case $\alpha_1 = \frac{1}{2}$. It may be seen that this boundary layer separates ($\alpha \to 0$, $\delta \to \infty$) when the free-stream velocity has decreased by less than 14%.

With this picture of the relatively large effect of a given pressure rise on the more slowly moving fluid, it would appear that *any* pressure rise applied to a real boundary layer would cause separation since, if the velocity goes continuously to zero at the wall, there will always be *some* fluid with insufficient momentum to negotiate the pressure rise. However, this tendency toward separation is resisted, up to a point, by viscous stress. That is, the slowly moving fluid can be "dragged" through a pressure rise by the faster fluid farther from the wall. An important method of reducing the separation tendency is to enhance this "drag" force by increasing the effective shear stress through turbulent motion.

The mention of turbulent motion introduces another important aspect of boundary layer behavior. In 1883 Reynolds showed that under certain conditions dye particles injected into a pipe flow moved smoothly along streamlines without lateral fluctuations. In the same pipe at higher velocities, this smooth laminar motion broke down into turbulent motion consisting of rapid random fluctua-

FIG. 4–7. "Growth" of step boundary layer near separation for $\alpha_1 = \frac{1}{2}$.

FIG. 4–8. Comparison of laminar and turbulent velocity profiles.

tions superposed on the mean motion. After the flow had become turbulent the dye lines rapidly mixed with the surrounding fluid, the macroscopic turbulent mixing process being at least an order of magnitude faster than molecular diffusion (the only agency for mixing in laminar flow).

The same transition behavior takes place in the boundary layer, resulting in a significant reduction in its tendency to separate. The reason for this is that the turbulent fluctuations constitute a mechanism for transporting high momentum from the outer part of the layer to the region near the wall. The effect is as if there were an increased shear stress or, as will be shown, an increased coefficient of viscosity. This tends to raise the wall shear stress, but it also means that the turbulent boundary fluid can climb higher up the "pressure hill" than the fluid in the laminar layer. A comparison of typical turbulent and laminar boundary layer velocity profiles is indicated in Fig. 4–8.

The fact that the turbulent layer tends to have much higher momentum (and shear stress) near the wall gives it an increased resistance to separation. As a method of specifying the resistance of a boundary layer to separation, it is useful to define a pressure coefficient C_p as

$$C_p = \frac{\Delta p}{\frac{1}{2}\rho U^2},$$

where U is the free-stream velocity at the point where the pressure begins to rise and Δp is the total increase in pressure up to the point in question.

For an approximate comparison of cases in which the adverse pressure gradient is not abrupt (i.e., where the shear stress has time to act) it may be said that the laminar boundary layer will support a pressure rise given by $0.15 < C_p < 0.2$, whereas the turbulent boundary layer can tolerate without separation a pressure rise corresponding to $0.4 < C_p < 0.8$, depending on the Reynolds number and the upstream pressure distribution of the flow.

The onset of turbulence, or transition as it is called, is dependent in a given device on fluid properties and velocity, the general or upstream turbulence level, and the geometry of the particular device. In the case of pipe flow, for example,

geometry is completely specified (far from the entrance) by pipe diameter D and wall roughness. The pertinent fluid properties are density ρ, viscosity μ, and the average velocity $\bar{U}$. These variables can be arranged in a dimensionless form known as the Reynolds number, Re, so that

$$\mathrm{Re} \equiv \frac{\rho \bar{U} D}{\mu}.$$

Reynolds found that for "smooth" pipes (commercial pipes and tubing are usually "smooth") and very quiet inlet conditions, the transition process could in fact be correlated with a critical Reynolds number below which the flow was laminar and above which it was turbulent.

More generally it is found that transition in any series of *geometrically similar* devices can be correlated with the Reynolds number, where D is replaced by some characteristic length L and $\bar{U}$ by some characteristic velocity U. Thus

$$\mathrm{Re} = \frac{\rho U L}{\mu}.$$

Obviously, a critical value for Re can have meaning only within the geometry for which it was defined. With these facts in mind, the explanation of the two pressure distributions observed on the cylinder of Fig. 4–2 becomes straightforward. For reasonably low velocities (below a "critical" Reynolds number) the boundary layer remains laminar and separates readily on the back of the cylinder before much pressure rise has taken place. At the critical Reynolds number, the boundary layer becomes turbulent and is able to flow considerably further around the back of the cylinder before separation. As it does so it rises in pressure, thus reducing the discrepancy between inviscid and actual pressure distributions. Since the drag on a blunt body such as a cylinder is primarily a pressure force (as opposed to skin friction), the onset of turbulent flow in this case is actually accompanied by decreased drag.

In most practical fluid machines to be discussed in this book the boundary layers are turbulent, and so long as this is true the influence of Reynolds-number variations on overall performance is usually minor. However, under extreme conditions, as for example at very high altitudes, the Reynolds number can become low enough that portions of the flow revert to laminar flow. There usually follows a rather substantial reduction in efficiency of performance, due to the increased tendency toward separation.

4–2 THE BOUNDARY LAYER EQUATIONS

The general equations for the flow of Newtonian viscous fluids are called the Navier-Stokes equations [1, 2]. Because of their mathematical complexity, it is possible to obtain exact solutions of them for only a few physical situations. In many cases, however, it is legitimate to make simplifying approximations and thereby obtain solutions which have a useful range of validity. The boundary

FIG. 4–9. Small control volume for development of boundary layer equations.

layer equations are much more tractable than the Navier-Stokes equations. They may be derived by neglecting certain terms in the Navier-Stokes equations which can be shown to be quite unimportant in boundary layer flows. The method is developed by Schlichting [1] in careful detail.

In the following discussion, equations will be developed less rigorously, but more directly, for the special case of the steady, two-dimensional, incompressible boundary layer. Shapiro [3] uses this approach to develop equations for compressible flow, and includes a good discussion of the range of validity of the simplifications. The object in view is to lay a foundation for a quantitative description of both laminar and turbulent boundary layers, with particular emphasis on skin friction, separation, and heat transfer. Although the solutions of the boundary layer equations which are available are seldom directly applicable to the complex flows in real machines, they at least illustrate the forces at work and the behavior typical of the boundary layer.

Consider a very small control volume within a boundary layer, as shown in Fig. 4–9. Velocities in the x- and y-directions are u and v, respectively, while the shear stresses in these directions are τ_x and τ_y. To apply the continuity and momentum equations to this control volume, we shall need to know the mass and momentum flux across each surface and the force (shear and pressure) acting on each surface.

The outward fluxes of mass and x-momentum are shown in Table 4–1. Since the flow is taken to be two-dimensional, no fluid crosses the surfaces parallel to the xy-plane.

The continuity equation for steady, two-dimensional, incompressible flow requires that

$$\rho\left(u + \frac{\partial u}{\partial x} dx\right) dy - \rho u\, dy + \rho\left(v + \frac{\partial v}{\partial y} dy\right) dx - \rho v\, dx = 0$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{4–1}$$

Table 4–1

Surface	Mass flux	Flux of x-momentum
①②	$\rho\left(u + \frac{\partial u}{\partial x} dx\right) dy$	$\rho\left(u + \frac{\partial u}{\partial x} dx\right)^2 dy$
③④	$-\rho u\, dy$	$-\rho u^2\, dy$
②③	$\rho\left(v + \frac{\partial v}{\partial y} dy\right) dx$	$\rho\left(v + \frac{\partial v}{\partial y} dy\right)\left(u + \frac{\partial u}{\partial y} dy\right) dx$
④①	$-\rho v\, dx$	$-\rho v u\, dx$

The momentum equation in the x-direction, using Table 4–1 to obtain the net outward momentum flux, is (applying Eq. (2–4))

$$\sum F_x = \rho\left(u + \frac{\partial u}{\partial x} dx\right)^2 dy - \rho u^2\, dy + \rho\left(v + \frac{\partial v}{\partial y} dy\right)\left(u + \frac{\partial u}{\partial y} dy\right) dx - \rho u v\, dx.$$

Expanding and disregarding second-order terms, we have

$$\sum F_x = \rho\left(2u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}\right) dx\, dy.$$

Using the continuity equation, we may simplify this to

$$\sum F_x = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) dx\, dy.$$

Since only pressure and shear forces act on the control surface, the force summation may be written, in accordance with Fig. 4–9, as

$$\sum F_x = p\, dy - \left(p + \frac{\partial p}{\partial x} dx\right) dy + \left(\tau_x + \frac{\partial \tau_x}{\partial y} dy\right) dx - \tau_x\, dx$$
$$= \left(-\frac{\partial p}{\partial x} + \frac{\partial \tau_x}{\partial y}\right) dx\, dy.$$

Thus the momentum equation may be written

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial \tau_x}{\partial y}.$$

The shear stress τ developed in a Newtonian fluid is proportional to the rate of shearing strain. For the particular case of the boundary layer, in which there is a

large velocity gradient close to the wall, the shear stress is very nearly equal to

$$\tau_x = \mu \frac{\partial u}{\partial y},$$

where μ is the coefficient of viscosity.

Using this definition, the momentum equation for the x-direction can be written

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}.$$

As a consequence of the thinness of the boundary layer, pressure changes normal to the wall may be neglected inside the layer, so that it may be assumed that p is a function of x only. Then the momentum equation may be written

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{4–2}$$

where $\nu \equiv \mu/\rho$ is the kinematic viscosity.

The momentum equation as derived here assumes a fluid of constant viscosity and constant density. Although these conditions may be satisfied within many low-speed laminar boundary layers, we shall see that the "effective" viscosity in turbulent flow may vary strongly through the boundary layer. Further, ρ and μ could vary as the result of temperature variation within the boundary layer, due either to high heat-transfer rates or, in compressible flow, to the variation of static temperature in response to velocity variation. In such cases a complete set of equations must include at least an energy equation and an equation of state. However, even in cases not requiring the simultaneous solution of an energy equation, Eqs. (4–1) and (4–2) can present considerable difficulties in the solution for u and v (where p is assumed a known function of x). It is desirable, indeed necessary in most cases, to utilize approximate methods for solving the boundary layer equations.

An extremely useful approximation, the *momentum integral method*, has been developed and widely applied. The object of the method is to find a solution which will satisfy the integral of the momentum equation across the boundary layer even though it may fail to satisfy the equation at particular points. The integration of the momentum equation (4–2) may proceed as follows:

$$\int_0^h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) dy = \int_0^h -\frac{1}{\rho}\frac{dp}{dx}\,dy + \int_0^h \nu\frac{\partial^2 u}{\partial y^2}\,dy,$$

where h is an undefined distance from the wall outside the boundary layer. In the free stream,

$$-\frac{1}{\rho}\frac{dp}{dx} = U\frac{dU}{dx}.$$

Thus this equation may be written

$$\int_0^h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{dU}{dx}\right) dy = \int_0^h \nu\frac{\partial^2 u}{\partial y^2}\,dy.$$

Integrating the right-hand term and noting that the wall shear stress is given by

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0},$$

while $\partial u/\partial y$ is zero outside the boundary layer, we can write (for constant ν)

$$\int_0^h \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - U\frac{dU}{dx}\right) dy = -\frac{\tau_0}{\rho}.$$

Integrating the second term in the integral by parts, we have

$$\int_0^h v\frac{\partial u}{\partial y}\,dy = vu\Big|_0^h - \int_0^h u\frac{\partial v}{\partial y}\,dy.$$

From the continuity equation, v at $y = h$ is given by

$$v = \int_0^h \frac{\partial v}{\partial y}\,dy = -\int_0^h \frac{\partial u}{\partial x}\,dy.$$

Thus

$$\int_0^h v\frac{\partial u}{\partial y}\,dy = -U\int_0^h \frac{\partial u}{\partial x}\,dy + \int_0^h u\frac{\partial u}{\partial x}\,dy.$$

The momentum equation may then be written

$$\int_0^h \left(2u\frac{\partial u}{\partial x} - U\frac{\partial u}{\partial x} - U\frac{dU}{dx}\right) dy = -\frac{\tau_0}{\rho}$$

or, rearranging,

$$\int_0^h \frac{\partial}{\partial x}[u(U-u)]\,dy + \frac{dU}{dx}\int_0^h (U-u)\,dy = +\frac{\tau_0}{\rho}. \tag{4–3}$$

Both the integrands are zero outside the boundary layer, so h can be indefinitely large.

Equation (4–3) is arranged in this particular form because of the physical significance (and the common usage) of the individual integrals. In the second integral, $(U - u)$ is the mass flux *defect* (actually $1/\rho$ times the defect) that occurs as the result of the deceleration of the boundary layer fluid. In the case of flow within a duct, for example, if there is a mass flux defect near the walls there must be an *increased* mass flux within the free stream. This effect on the free stream would be the same if there were no boundary layer but the duct walls were dis-

placed inward an amount δ^* so that

$$U\delta^* = \int_0^\infty (U - u)\, dy.$$

With respect to mass flux then, the boundary layer has the effect of making the thickness δ^* unavailable for free-stream flow; δ^* is called the *displacement thickness* of the boundary layer.

The first integral can be treated in a similar manner. Note first that since the limits of integration are not functions of x, the partial differentiation $\partial/\partial x$ can be carried outside the integral where it becomes d/dx. In this case we interpret u as $1/\rho$ times the actual mass flux, while $(U - u)$ is the *momentum defect* per unit mass of the boundary layer fluid. Hence the integral is the momentum defect in the boundary layer which, as above, could be accounted for by the loss of free-stream thickness θ such that

$$U^2\theta = \int_0^\infty u(U - u)\, dy.$$

Here θ is called the *momentum thickness* of the boundary layer. The two terms are then defined as follows (for incompressible flow):

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy, \tag{4–4a}$$

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) dy. \tag{4–4b}$$

With these definitions, the momentum integral equation, Eq. (4–3), may be written

$$\frac{d}{dx}(U^2\theta) + \frac{dU}{dx}(U\delta^*) = \frac{\tau_0}{\rho}$$

or

$$\frac{d\theta}{dx} + \left(2 + \frac{\delta^*}{\theta}\right)\frac{\theta}{U}\frac{dU}{dx} = \frac{\tau_0}{\rho U^2}. \tag{4–5}$$

As it stands this equation applies to both laminar and turbulent boundary layers. The solution, say for a given $U(x)$, requires knowledge of the relationship between δ^* and θ, which may be found if an approximate velocity profile shape is assumed (δ^*/θ is often called the *shape factor* H), and the skin friction τ_0 is known. Note that shear stress is evaluated only at the wall where, even in turbulent flow, it is given by $\mu(\partial u/\partial y)$, since turbulent fluctuations must go to zero at the wall.

4–3 LAMINAR BOUNDARY LAYER SOLUTIONS

A variety of exact† solutions of Eqs. (4–1) and (4–2) are available, two of which will be mentioned here.

† Exact in the sense that they do not neglect any of the terms in the boundary layer equations. Most solutions are, however, expressed in terms of series expansions.

FIG. 4–10. Velocity distribution in the laminar boundary layer on a flat plate at zero incidence, as measured by Nikuradse [1].

The simplest flow is that over a flat plate where dp/dx is zero. In this case, Eqs. (4–1) and (4–2) reduce to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

with the boundary conditions

$$y = 0: \qquad u = v = 0,$$

$$y = \infty: \qquad u = U = \text{constant}.$$

A stream function ψ may be defined such that

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}.$$

The convenience of this form is that it automatically satisfies the continuity equation so that the above two equations reduce to

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x\, \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu\frac{\partial^3 \psi}{\partial y^3}. \tag{4–6}$$

This equation may be considerably simplified by substituting

$$\eta = y\sqrt{U/\nu x} \qquad \text{and} \qquad \psi = \sqrt{\nu x U}\, f(\eta),$$