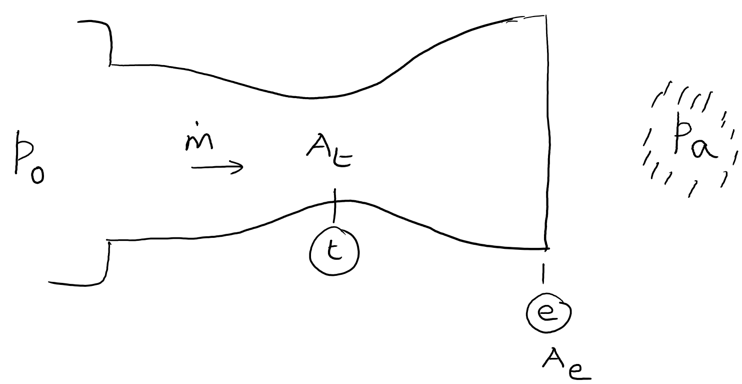
Flow behavior in c-d nozzles

The Mach number at the exit of a c-d nozzle is governed by two factors

1. the difference between the pressure in the regions just upstream and downstream of the nozzle; more precisely, the ratio of the pressures
2. the nozzle area ratio

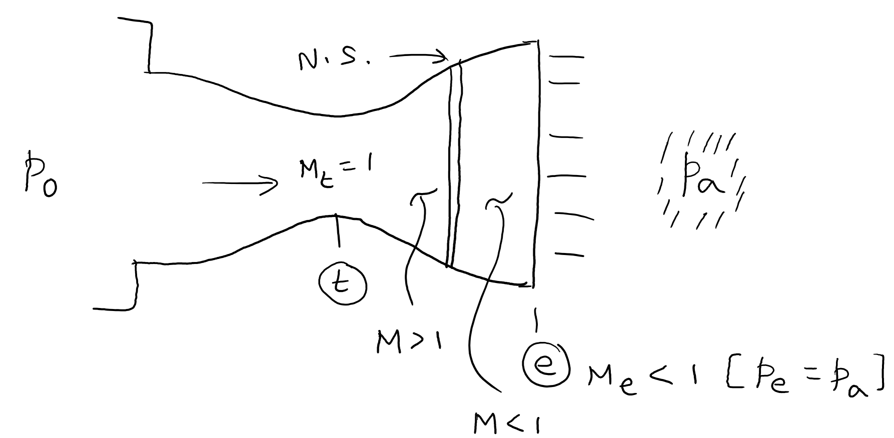


In the figure, the c-d nozzle is between regions with pressures p_0 and p_a, and has area ratio $\frac{A_e}{A_t}$.

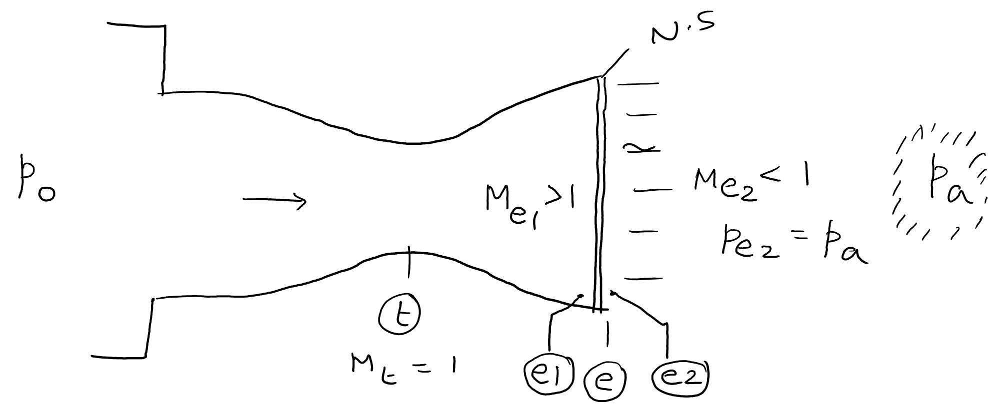
Thus, the values of (p_0/p_a) and (A_e/A_t) govern the Mach number at the nozzle exit (M_e).

* If $\left(\frac{p_0}{p_a}\right)$ is "low," then the flow through the nozzle will be entirely subsonic, and both M_t and M_e will be less than 1. The highest Mach number will occur at the throat. Since $M_e < 1$, $p_e = p_a$.

* If $\left(\frac{p_o}{p_a}\right)$ is larger, $M_t = 1$, and the flow will accelerate to supersonic velocities in the diverging portion of the nozzle before encountering a normal shock. The flow then becomes subsonic and $M_e < 1$.



* If $\left(\frac{p_o}{p_a}\right)$ further increases, the normal shock will move downstream and will station itself at the exit plane at a specific $\left(\frac{p_o}{p_a}\right)$ value.



The value of $\left(\frac{p_o}{p_a}\right)$ at this condition is, say, $\left(\frac{p_o}{p_a}\right)_{crit}$,

and will be a function of nozzle area ratio $\left(\frac{A_e}{A_t}\right)$.

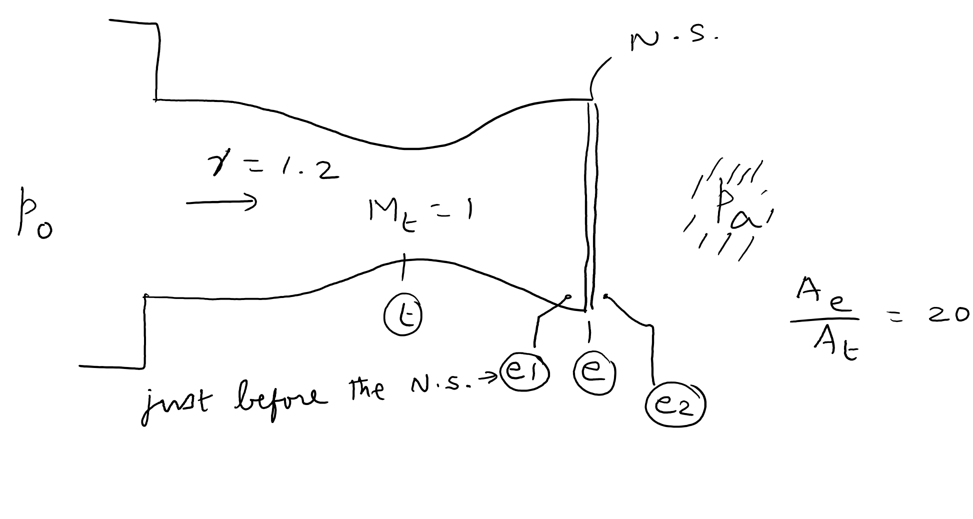
* If $\left(\frac{p_o}{p_a}\right)$ is increased further, the shock will move outside the nozzle, and $M_e > 1$. The flow will continuously accelerate from $M < 1$ at the upstream reservoir, reach $M_t = 1$ at the throat, and leave the nozzle at $M_e > 1$.

* Thus the minimum $\left(\frac{p_o}{p_a}\right)$ required to have a shock-free flow in the c-d nozzle is $\left(\frac{p_o}{p_a}\right)_{crit}$.

Example

A c-d nozzle has $\frac{A_e}{A_t} = 20$. Find $\left(\frac{p_o}{p_a}\right)_{crit}$.

Assume that the nozzle is operating with $\left(\frac{p_o}{p_a}\right) = \left(\frac{p_o}{p_a}\right)_{crit}$



$$A_{e_1} = A_e \Rightarrow \frac{A_{e_1}}{A_t} = 20$$

$$\frac{A_{e_1}}{A^*} = \frac{A_{e_1}}{A_t} = 20$$

$M_t = 1$

$$\frac{\cancel{A_{e_1}}^{\,20}}{A^*} = \frac{1}{M_{e_1}} \left[\frac{2}{\gamma+1} \left(1 + \frac{\gamma-1}{2} M_{e_1}^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Solve for M_e by trial-and-error, using MATLAB or online gas dynamic calculator. Observe $M_{e_1} > 1$.

$$M_{e_1} = 3.76 \quad (\gamma = 1.2)$$

isen, no work

$$p_{e_1} = \frac{p_{oe_1}}{\left(1 + \frac{\gamma-1}{2} M_{e_1}^2\right)^{\frac{\gamma}{\gamma-1}}} = \frac{p_{oe_1}}{197.78} = \frac{p_o}{197.78}$$

Use the normal shock relation,

$$\frac{p_{e_2}}{p_{e_1}} = \frac{2\gamma}{\gamma+1} M_{e_1}^2 - \frac{\gamma-1}{\gamma+1},$$

online normal shock calculator to get

$$\frac{p_{e_2}}{p_{e_1}} = 15.33 \quad (M_{e_1} = 3.76, \gamma = 1.2)$$

$$p_{e_2} = 15.33\, p_{e_1} = 15.33 \cdot \frac{p_o}{197.78} = 0.0775\, p_o$$

$M_{e2} < 1 \Rightarrow p_{e2} = p_a$

$p_a = 0.0775\, p_o \Rightarrow \dfrac{p_o}{p_a} = \dfrac{1}{0.0775} = 12.90$

This is the $\left(\dfrac{p_o}{p_a}\right)_{crit}$ value : $\left(\dfrac{p_o}{p_a}\right)_{crit} = 12.90$

Observe that this result depends on the nozzle area ratio ; 20 in this case. If $\left(\dfrac{A_e}{A_t}\right)$ had been different, $\left(\dfrac{p_o}{p_a}\right)_{crit}$ would have been different as well.

Example: An ideal rocket thrust chamber has $p_o = 1$ MPa and nozzle area ratio of 1.2. Will the nozzle be shock-free during operation at 10-km altitude? What are the exit Mach number and the exit pressure? $\gamma = 1.2$.

t 10-km altitude, $p_a = 26.5$ kPa

$p_o = 1 \text{ MPa} = 1000 \text{ kPa}$

$$\left(\frac{p_o}{p_a}\right)_{actual} = \frac{1000}{26.5} = 37.74$$

The nozzle area ratio = 20, for which $\left(\dfrac{p_o}{p_a}\right)_{crit} = 12.90$

$\left(\dfrac{p_o}{p_a}\right)_{actual} > \left(\dfrac{p_o}{p_a}\right)_{crit} \Rightarrow$ shock free nozzle flow.

The exit Mach number is fixed by area ratio (20).

This was already found. $M_e = 3.76$

$$p_e = \frac{\overset{\text{isen}}{p_{oe}}}{\left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\frac{\gamma}{\gamma-1}}} = 5056 \text{ Pa} = 5.056 \text{ kPa}$$

$p_o = 1 \text{ MPa}$

$p_a = 26.5 \text{ kPa}$

$p_e < p_a$ (overexpanded jet)

Example: what is the lowest altitude where the thrust chamber of the previous example operate shock free?

$$\left(\frac{p_o}{p_a}\right)_{crit} = 12.90$$

$$p_o = 1 \text{ MPa} \Rightarrow p_a)_{crit} = \frac{10^6}{12.90} = 77519 \text{ Pa}$$

From atmospheric table (Table 1, p 700) and interpolating,

$$h_{crit} = 2211 \text{ m} \simeq 2.2 \text{ km}$$

The lowest altitude for shock-free operation is 2.2 km.

At $h < h_{crit}$, there will be a shock in the diverging portion of the c-d nozzle.

Comment

Observe that, for nozzle flow with $\left(\frac{p_o}{p_a}\right)_{actual}$ just greater than $\left(\frac{p_o}{p_a}\right)_{crit}$, $p_e < p_a$. To attain the fully expanded condition ($p_e = p_a$), $\left(\frac{p_o}{p_a}\right)$ has to be further increased.

Thus, if the flow is stated to be fully expanded or underexpanded ($p_e > p_a$), it follows that $\left(\frac{p_o}{p_a}\right)_{actual} > \left(\frac{p_o}{p_a}\right)_{crit}$, and there can be no shocks in the nozzle.