Example

Consider a 3-stage rocket.

$\epsilon_1 = 0.0765$, $\epsilon_2 = 0.114$, $\epsilon_3 = 0.111$

$I_{sp_1} = 304\,s$, $I_{sp_2} = 421\,s$, $I_{sp_3} = 421\,s$.

$\Delta u_{b,imp} = 12.4$ km/s

optimize the vehicle for maximum $\left(\frac{M_L}{M_{01}}\right)$.

$$\left\{\frac{1}{\epsilon_1}\left[1 - \frac{1}{\alpha g_e I_{sp_1}}\right]\right\}^{\frac{g_e I_{sp_1}}{\Delta u_{b,imp}}} \otimes \left\{\frac{1}{\epsilon_2}\left[1 - \frac{1}{\alpha g_e I_{sp_2}}\right]\right\}^{\frac{g_e I_{sp_2}}{\Delta u_{b,imp}}} \otimes$$

$$\left\{\frac{1}{\epsilon_3}\left[1 - \frac{1}{\alpha g_e I_{sp_3}}\right]\right\}^{\frac{g_e I_{sp_3}}{\Delta u_{b,imp}}} = e \quad (E)$$

Use MATLAB to solve Equation (E) for αg_e

Then, get λ_i from Eqn. (D):

$$\lambda_i = \frac{\epsilon_i}{\alpha g_e I_{sp_i}(1-\epsilon_i) - 1}$$

$$\frac{M_L}{M_{01}} = \frac{\lambda_1}{1+\lambda_1} \cdot \frac{\lambda_2}{1+\lambda_2} \cdot \frac{\lambda_3}{1+\lambda_3} \quad \text{[from (10.35)]}$$

MATLAB solution:

$\alpha g_e = 0.0038966$

$\lambda_1 = 0.8142$, $\lambda_2 = 0.2514$, $\lambda_3 = 0.2421$

$$\frac{M_L}{M_{01}} = 0.01758$$

$$\left(\Delta u_{b,\,imp1} = 2127.5 \text{ m/s}, \; \Delta u_{b,\,imp2} = 5084.2 \text{ m/s}, \right.$$

$$\Delta u_{b,\,imp3} = 5194.3 \text{ m/s}$$

$$\left. \Delta u_{b,\,imp} = \sum_{i=1}^{3} \Delta u_{b,\,imp\,i} = 12400 \text{ m/s} \right)$$

The data of this problem correspond to the Saturn V lunar launch vehicle. Its initial mass (M_{01}) was 2,902,000 kg and the mass of the Apollo 11 payload (M_L) was 47000 kg, which gives $\frac{M_L}{M_{01}} = 0.0162$.

A vehicle with stages having equal ϵ_i and $I_{sp\,i}$ values is said to be similarly staged. It is clear from Eqn (D), that the stages will have equal values of λ_i as well. For such a vehicle, Eqn. (E) becomes

$$n\, g_e\, I_{sp} \ln\left(\frac{1+\lambda}{\epsilon+\lambda}\right) = \Delta u_{b,imp}, \quad \text{(F)}$$

where n is the number of stages, and

$$I_{sp} \overset{\text{def}}{=} I_{sp_1} = I_{sp_2} = I_{sp_3},$$

$$\lambda \overset{\text{def}}{=} \lambda_1 = \lambda_2 = \lambda_3, \text{ and}$$

$$\epsilon \overset{\text{def}}{=} \epsilon_1 = \epsilon_2 = \epsilon_3 .$$

In this case, λ can be found from Eqn. (F), without having to find α. If needed, α can be found from Eqn. (D) $\left\{ \lambda_i = \dfrac{\epsilon_i}{\alpha g_e I_{sp_i}(1-\epsilon_i) - 1} \right\}$ once λ has been determined.

Example

Reconsider the first three-stage example $\{M_{01} = 2{,}500{,}000$ kg, $\Delta u_b = 11023$ m/s$\}$. The stages all have the same I_{sp}. $M_s = 145{,}000$ kg, $M_p = 2{,}287{,}000$ kg. If the structural and propellant masses are redistributed such that $\epsilon_1 = \epsilon_2 = \epsilon_3 \overset{def}{=} \epsilon$,

$$\epsilon = \frac{M_{s1}}{M_{s_1}+M_{p1}} = \frac{M_{s2}}{M_{s_2}+M_{p2}} = \frac{M_{s3}}{M_{s_3}+M_{p3}} =$$

$$\frac{M_{s_1}+M_{s_2}+M_{s_3}}{(M_{s_1}+M_{p_1})+(M_{s_2}+M_{p_2})+(M_{s_3}+M_{p_3})} = \frac{M_{s_{tot}}}{M_{s_{tot}}+M_{p_{tot}}} =$$

$$\frac{145000}{145000+2287000} = 0.05962$$

With this value of ϵ, (F) $\Rightarrow$ $\lambda = 0.4772$ $[n = 3]$

$$\frac{M_L}{M_{01}} = \left(\frac{\lambda}{1+\lambda}\right)^3 = 0.0337$$ (larger than the 0.0294 value obtained with $\epsilon_1 = 0.05$, $\epsilon_2 = 0.0698$ and $\epsilon_3 = 0.1892$).

The value of αg_e (not really needed) can be found from (D):

$$\alpha g_e = \frac{1}{I_{sp}(1-\epsilon)}\left(1+\frac{\epsilon}{\lambda}\right) = 0.03233$$

The similarly-staged vehicle has the highest $\left(\frac{M_2}{M_{01}}\right)$.