Isentropic Flow

$s_2 = s_1$, $s_{o_2} = s_{o_1}$, $q = 0$ (adiabatic, reversible)

With work ($w \neq 0$)

$$c_p\left(T_{o_2} - T_{o_1}\right) = -w$$

$$T_{o_2} \neq T_{o_1}$$

$$\frac{p_{o_2}}{p_{o_1}} = \left(\frac{T_{o_2}}{T_{o_1}}\right)^{\frac{\gamma}{\gamma-1}} \Rightarrow p_{o_2} \neq p_{o_1}$$

models ideal behavior of flows through rocket turbomachinery and gas turbine engine turbomachinery

Without work ($w = 0$)

$$c_p\left(T_{o_2} - T_{o_1}\right) = 0$$

$$T_{o_2} = T_{o_1}$$

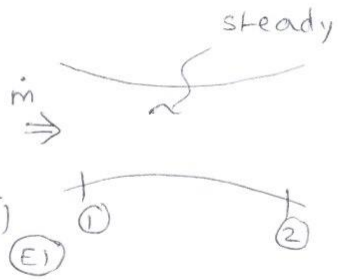
$$\frac{p_{o_2}}{p_{o_1}} = \left(\frac{T_{o_2}}{T_{o_1}}\right)^{\frac{\gamma}{\gamma-1}} = 1 \Rightarrow p_{o_2} = p_{o_1}$$

models ideal behavior of flows through rocket nozzles, gas turbine engine nozzles, and gas turbine engine inlets

Isentropic flow w/o work, $q = 0$, $w = 0$

$$T_{o_2} = T_{o_1} \overset{def}{=} T_o$$

$$p_{o_2} = p_{o_1} \overset{def}{=} p_o$$



$$\dot{m} = \sqrt{\frac{\gamma}{R}}\,\frac{p_o}{\sqrt{T_o}}\,M\left(1 + \frac{\gamma-1}{2}M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} A \qquad (E1)$$

Station where $M = 1$ is said to be critical station. Properties there are said to be critical properties and are designated with a superscript $*$: p^*, T^*, A^*, etc.

Evaluating $\dot{m}$ at the critical station ($M = 1$, $A = A^*$)

$$\dot{m} = \sqrt{\frac{\gamma}{R}}\ \frac{p_0}{\sqrt{T_0}}\ \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} A^* \qquad \text{(E2)}$$

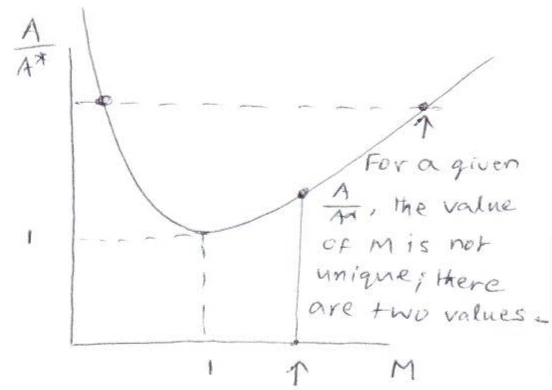
$$\Rightarrow\ A^* = \dot{m}\sqrt{\frac{R}{\gamma}}\ \frac{\sqrt{T_0}}{p_0}\ \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \qquad \text{(E3)} \quad [\text{defines } A^*]$$

For a given steady flow, $\dot{m}$ is constant

For a given isentropic flow w/o work, T_0 and p_0 are constant.

Thus, for a given steady, isentropic flow, A^* is a constant, and is given by (E3).

$$\frac{\text{(E1)}}{\text{(E2)}} \Rightarrow \frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \qquad (3.15)$$

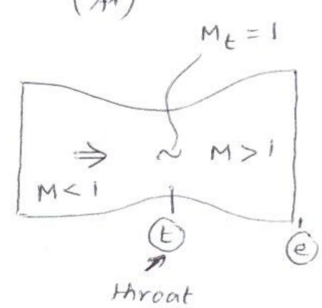


For a given M, there is a unique value for $\left(\frac{A}{A^*}\right)$

Comments

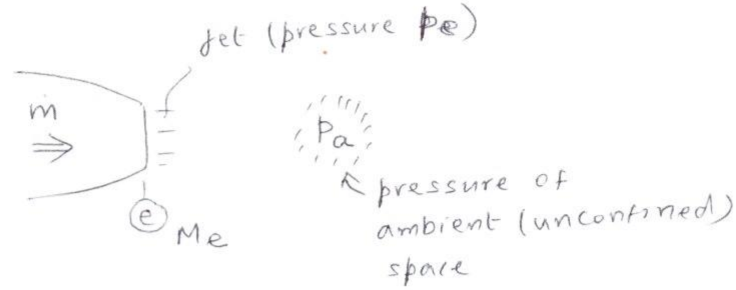
* To accelerate flow from $M<1$ to $M>1$, a c-d passage is required.
* A converging passage can accelerate flow from $M<1$ to a maximum $M=1$.
* The Mach number at the throat,** of a c-d nozzle in which the flow accelerates from $M<1$ to $M>1$, will be 1.

[** minimum area location]



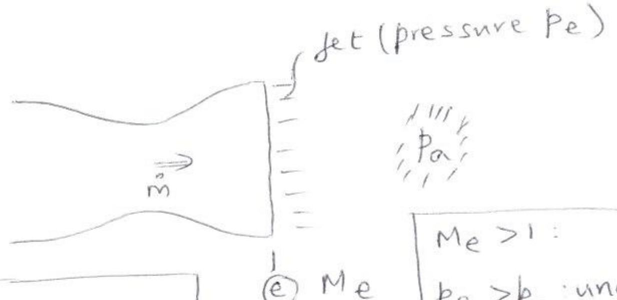
Important observation about subsonic jets

The pressure of a subsonic jet, issuing out of a confinement into unconfined space, is equal to the pressure in the unconfined space



If $M_e < 1$,

$$p_e = p_a$$



If $M_e < 1$,

$$p_e = p_a$$

Related results

If $M_e > 1$,

p_e may not be equal to p_a

$M_e > 1$:

$p_e > p_a$: underexpanded jet

$p_e = p_a$: fully expanded jet

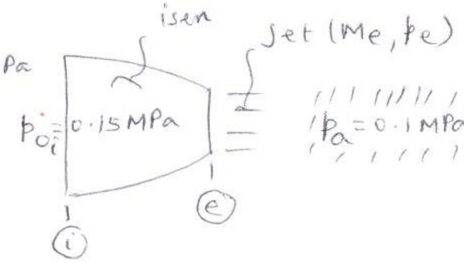
$p_e < p_a$: overexpanded jet

Ex: The pressure of a water jet issuing out of a garden hose is equal to the ambient pressure. [$M_{\text{water jet}} < 1$]

Ex: The pressure of the exhaust jet from a rocket nozzle is typically less than the ambient pressure at launch condition. ($p_e < p_a$, overexpanded jet) ($M_e > 1$)

[The rocket nozzle can be designed to form an underexpanded, fully expanded or overexpanded jet.]

Ex An aircraft engine has an isentropic, convergent nozzle. The stagnation pressure at its entrance is 0.15 MPa. Find the Mach number and the pressure of the jet in sea level operation ($p_a = 0.1$ MPa). $\gamma = 1.4$.



Isen flow w/o work: $p_{oe} = p_{o_i} = 0.15$ MPa

Let $p_e = p_a = 0.1$ MPa

$$\frac{p_{oe}}{p_e} = \frac{0.15}{0.1} = 1.5$$

$$\left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\frac{\gamma}{\gamma-1}} = 1.5 \Rightarrow M_e = 0.78$$

Since $M_e < 1$, $p_e = p_a$ is correct

$$\boxed{M_e = 0.78, \; p_e = 0.1 \text{ MPa}}$$

Ex Repeat for operation at 3-km altitude ($p_a = 70$ kPa)

$p_{oe} = p_{o_i} = 0.15$ MPa

Let $p_e = p_a = 70$ kPa $= 0.07$ MPa

$$\frac{p_{oe}}{p_e} = \frac{0.15}{0.07} = 2.143$$

$$\left(1 + \frac{\gamma - 1}{2} M_e^2\right)^{\frac{\gamma}{\gamma-1}} = 2.143 \Rightarrow M_e = 1.1$$

Being a convergent nozzle, M_e cannot exceed 1. $\Rightarrow M_e = 1$

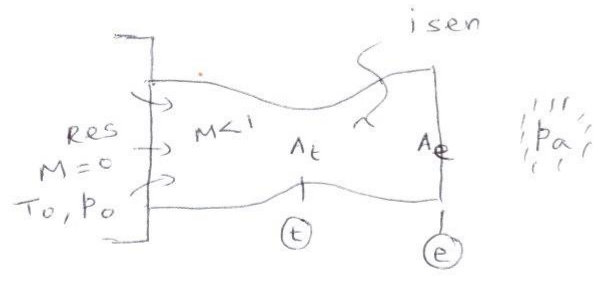
$$p_e = \frac{p_{oe}}{\left(1 + \frac{\gamma-1}{2} \underset{1}{M_e^2}\right)^{\frac{\gamma}{\gamma-1}}} = 0.079 \text{ MPa} = 79 \text{ kPa}$$

$$\underbrace{\overset{79\text{ kPa}}{p_e} > \overset{70\text{ kPa}}{p_a}}_{\text{underexpanded jet}}$$

$$\boxed{M_e = 1, \; p_e = 79 \text{ kPa}}$$

Ex An isentropic, c-d nozzle expands air in a reservoir at 3 MPa and 1000 K to ambient pressure p_a of 0.1 MPa. Find M_e, T_e, u_e, and the nozzle area ratio (A_e/A_t).

$\gamma = 1.34$, $R = 287 \frac{J}{kg-K}$.



isen flow w/o work.

$T_{oe} = T_o = 1000 K$

$p_{oe} = p_0 = 3 MPa$

$\underbrace{p_e}_{\text{given}} = p_a = 0.1 MPa$ [The nozzle is designed for full expansion]

$$\frac{p_{oe}}{p_e} = \frac{3}{0.1} = 30$$

$$\left(1 + \frac{\gamma-1}{2} M_e^2\right)^{\frac{\gamma}{\gamma-1}} = 30 \Rightarrow M_e = 2.84 \;\; [\gamma = 1.34]$$

$M_e > 1$

c-d nozzle accelerates flow from M < 1 to M > 1. $\Rightarrow M_t = 1$

since $M_t = 1$, $A_t = A^*$

$$\frac{A_e}{A_t} = \frac{A_e}{A^*} = \frac{1}{M_e}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2} M_e^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} = 4.00 \begin{bmatrix} \gamma = 1.34 \\ M_e = 2.84 \end{bmatrix}$$

The nozzle must be constructed with this area ratio to get the desired fully expanded flow.

$$T_e = \frac{T_{oe}}{1 + \frac{\gamma-1}{2} M_e^2} = 421.7 K$$

$$a_e = \sqrt{\gamma R T_e} = \sqrt{(1.34)(287)(421.7)} = 402.7 \text{ m/s}$$

$$u_e = M_e a_e = (2.84)(402.7) = 1143.7 \text{ m/s}$$

Isentropic flow w/work, $q = 0$, $w \neq 0$ (can be positive or negative. Work done by the turbine flow is positive, and work done by the compressor or pump* flow is negative.)

[* The ideal gas equation cannot be used for pump analysis, since the pump working fluid is a liquid]

Ideal pump (isentropic)

> A pump is a device that generates a pressure rise in liquids flowing through it

incompressible fluid (ρ constant)

Consider $T\,ds = dh - \dfrac{dp}{\rho}$

Apply this equation to stagnation states

$$T_0 \cancelto{0 \text{ (isentropic)}}{ds_0} = dh_0 - \frac{dp_0}{\rho} \Rightarrow dh_0 = \frac{dp_0}{\rho}$$

Integrating $\Delta h_0 = \dfrac{\Delta p_0}{\rho}$

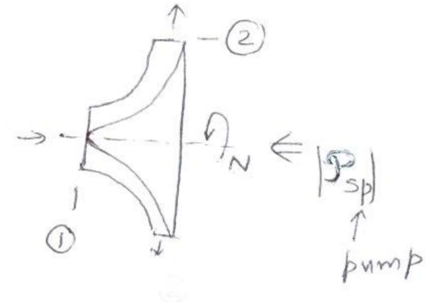
I Law: $\underbrace{(h_{02} - h_{01})}_{\Delta h_0} = \cancelto{0}{q} - w$

$$\Rightarrow -w = \Delta h_0 = \frac{\Delta p_0}{\rho} \Rightarrow w = \left(-\frac{\Delta p_0}{\rho}\right)$$

$$\mathcal{P}_{sp} = \dot{m} w = \dot{m}\left(-\frac{\Delta p_0}{\rho}\right)$$

$sp \leftarrow$ pump

$\Delta p_0 = (p_{02} - p_{01})$, pressure rise



Ex: A liquid oxygen (LOX) pump generates a pressure rise of 5 MPa. If the volume flow rate is 0.08 m^3/s, what is the ideal power required to drive the pump [$\rho_{LOX} = 1140\ kg/m^3$]

$$\mathcal{P}_{sp} = \dot{m}\left(-\frac{\Delta p_0}{\rho_{LOX}}\right) = \cancel{\rho_{LOX}}\dot{\mathcal{V}}\left(-\frac{\Delta p_0}{\cancel{\rho_{LOX}}}\right) = -\dot{\mathcal{V}}(\Delta p_0)$$

$$\dot{\mathcal{V}} = 0.08\ m^3/s,\quad \Delta p_0 = 5\ MPa = 5(10^6)\ Pa$$

$$\mathcal{P}_{sp} = -0.08(5)(10^6) = -4(10^5)\ J/s$$

indicates work done on fluid

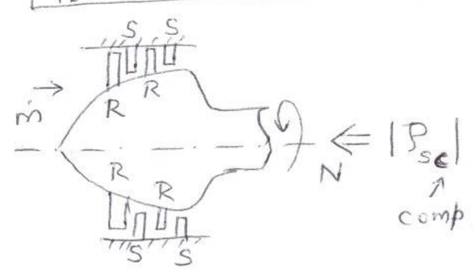
$$|\mathcal{P}_{sp}| = 400\ kW$$

Ideal compressor (isentropic)

A compressor is a device that generates a pressure increase in gases flowing through it.

Pressure rise is stated as a ratio

$$\pi_c \stackrel{def}{=} \frac{p_{02}}{p_{01}}$$



$$\frac{T_{02}}{T_{01}} = \left(\frac{p_{02}}{p_{01}}\right)^{\frac{\gamma-1}{\gamma}}$$

$$\text{I Law: } h_{02} - h_{01} = \cancel{q}^{\,0} - w_c$$

$$w_c = (h_{01} - h_{02}) = c_p(T_{01} - T_{02})$$

$$\mathcal{P}_{sc} = \dot{m}w_c = \dot{m}c_p(T_{01} - T_{02})$$

Ex: An isentropic compressor generates pressure ratio (π_c) of 20.

The entry stagnation conditions are 288K and 1 atm.

Find the exit stagnation conditions and the compressor specific work.

$\gamma = 1.35$, $R = 287 \frac{J}{kg\text{-}K}$

$$\pi_c = 20 \Rightarrow \frac{p_{02}}{p_{01}} = 20$$

$$p_{01} = 1\text{ atm} \Rightarrow \underline{p_{02} = 20\text{ atm}}$$

$$\frac{T_{02}}{T_{01}} = \left(\frac{p_{02}}{p_{01}}\right)^{\frac{\gamma-1}{\gamma}} = \pi_c^{\frac{\gamma-1}{\gamma}} = 20^{\frac{1.35-1}{1.35}} = 2.1742$$

$$T_{01} = 288\text{K} \Rightarrow \underline{T_{02} = 626.2\text{ K}}$$

$$\text{I Law: } w_c = c_p(T_{01} - T_{02}) \qquad c_p = \frac{\gamma R}{\gamma - 1} = 1107 \frac{J}{kg\text{-}K}$$

$$= 1107(288 - 626.2) = -3.744(10^5)\frac{J}{kg}$$

$$\underline{w_c = -374.4 \frac{kJ}{kg}}$$

Ex: If the mass flow rate through the compressor of the previous example is 20 kg/s, what is the power required to drive the compressor.

$$\text{kg/s} \quad \frac{kJ}{kg} \rightarrow kJ/s$$

$$P_{sc} = \dot{m}\, w_c = 20(-374.4) = -7488\text{ kW}$$

$$|P_{sc}| = 7488\text{ kW}$$ [see compressor schematic on previous page with the shaft power input.]

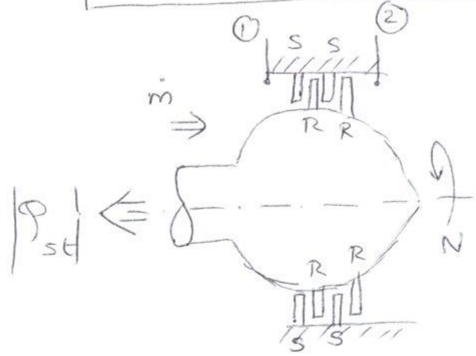
Ideal turbine (isentropic)

The pressure drop is stated as a ratio

A turbine is a device that utilizes the pressure drop available across it to generate shaft power

$$\pi_t \overset{def}{=} \frac{p_{o2}}{p_{o1}}$$

$$\frac{T_{o2}}{T_{o1}} = \left(\frac{p_{o2}}{p_{o1}}\right)^{\frac{\gamma-1}{\gamma}}$$



I Law: $(h_{o_2} - h_{o1}) = \cancel{q}^{\,0} - w_t$

$$w_t = (h_{o1} - h_{o2})$$

$$= c_p(T_{o1} - T_{o2})$$

$$\wp_{st} = \dot{m} w_t = \dot{m} c_p (T_{o1} - T_{o2})$$

Ex: An isentropic turbine has expansion ratio $\left(\frac{1}{\pi_t}\right)$ of 2. Hot gas at high pressure enters it at stagnation temperature of 960 K. Find the specific work of the turbine

$(\gamma = 1.3, \bar{M} = 12)$

$$\pi_t = \frac{1}{2} = 0.5 \Rightarrow \frac{p_{o2}}{p_{o1}} = 0.5$$

$$\frac{T_{o2}}{T_{o1}} = \left(\frac{p_{o2}}{p_{o1}}\right)^{\frac{\gamma-1}{\gamma}} = 0.5^{\frac{1.3-1}{1.3}} = 0.8522$$

$$R = \frac{\bar{R}}{\bar{M}} = \frac{8314.3}{12} = 692.9 \frac{J}{kg\text{-}K}$$

$$T_{o1} = 960\,K \Rightarrow T_{o2} = 818.1\,K$$

$$c_p = \frac{\gamma R}{\gamma - 1} = 3002.4 \frac{J}{kg\text{-}K}$$

$$w_t = c_p(T_{o1} - T_{o2}) = 3002.4(960 - 818.1)$$

$$= 4.26\,(10^5)\,\frac{J}{kg}$$

Ex: If the turbine of the previous example is used to drive the Lox pump of a few examples before, what is the required turbine mass flow rate?

$$\left|\mathcal{P}_{st}\right| = \left|\mathcal{P}_{sp}\right|$$

turbine

{power compatibility condition. The power required to drive the pump $[|\mathcal{P}_{sp}|]$ must be supplied by the turbine $[|\mathcal{P}_{st}|]$}

$$\left|\mathcal{P}_{sp}\right| = 400\,kW$$

$$= 4(10^5)\ \frac{J}{s}$$

$$\left|\mathcal{P}_{st}\right| = \left|\mathcal{P}_{sp}\right| = 4(10^5)\ \frac{J}{s}$$

Since work is done by the flow in the turbine, it has a positive value.

$$\mathcal{P}_{st} = 4(10^5)\ \frac{J}{s}$$

$$\dot{m}\, w_t = \mathcal{P}_{st} \Rightarrow \dot{m} = \frac{\mathcal{P}_{st}}{w_t} = \frac{4(10^5)}{4.26(10^5)} = 0.94\ \frac{kg}{s}$$