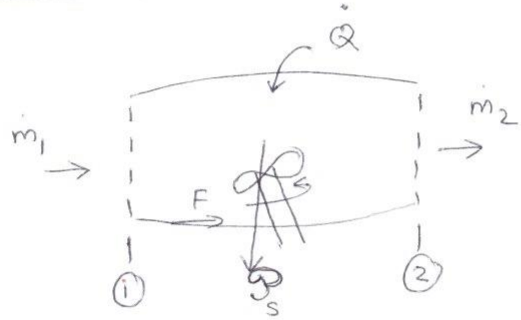
Sec 3.2, General 1-D steady flow of a perfect gas

Mass: $\dot{m}_2 = \dot{m}_1 \equiv \dot{m}$

[Integrated form of (3.1)]



Momentum: $\dot{m}_2 u_2 = \dot{m}_1 u_1 + p_1 A_1 - p_2 A_2 + F$

[Integrated form of (3.2), where F represents the sum of the forces exerted on the flow by the surfaces of the device it interacts with and any body forces]

$$F = (\dot{m}_2 u_2 + p_2 A_2) - (\dot{m}_1 u_1 + p_1 A_1)$$

The group $(\dot{m} u + pA)$ is referred to as impulse function (I): $I \overset{def}{=} (\dot{m}u + pA)$

$$\Rightarrow F = (I_2 - I_1)$$

The force exerted on the device by the flow is of the same magnitude, but in the opposite direction

$$F_{\text{on device}} = (I_1 - I_2)$$

I Law of thermodynamics (aka the energy equation):

$$\dot{m}\left(h_2 + \frac{u_2^2}{2}\right) = \dot{m}\left(h_1 + \frac{u_1^2}{2}\right) + \dot{Q} - \wp_s$$

[Integrated form of (3.3a)]

Dividing by $\dot{m}$,

$$\left(h_2 + \frac{u_2^2}{2}\right) = \left(h_1 + \frac{u_1^2}{2}\right) + \frac{\dot{Q}}{\dot{m}} - \frac{\mathcal{P}_s}{\dot{m}}$$

$$q \stackrel{def}{=} \frac{\dot{Q}}{\dot{m}}$$ (heat transfer per unit mass) $\left(\frac{J}{kg}\right)$

$$w = \frac{\mathcal{P}_s}{\dot{m}}$$ (work done per unit mass) $\left(\frac{J}{kg}\right)$

$$\left(h_2 + \frac{u_2^2}{2}\right) = \left(h_1 + \frac{u_1^2}{2}\right) + q - w$$

[Integrated form of (3-3b)]

II Law of thermodynamics:

$$\dot{m}\,(s_2 - s_1) \geq \iint_A \frac{\dot{q}\, dA}{T}$$ $\left(\dot{q} \stackrel{def}{=} \frac{d\dot{Q}}{dA}\right.$, heat transfer rate per unit area)

A ← area over which heat transfer occurs

[(2.11) for 1-D, steady flow]

Inequality applies for irreversible process, equality applies for reversible process

For an adiabatic process, $\dot{q} = \dot{Q} = 0$

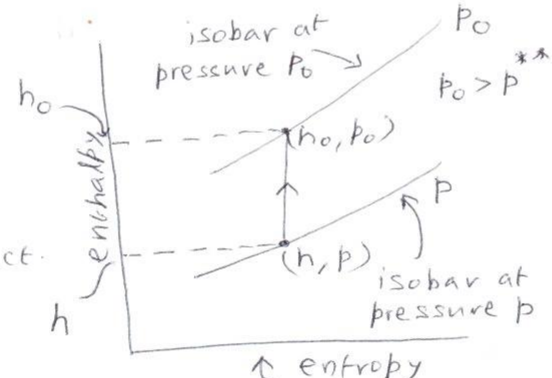
$$\Rightarrow s_2 \geq s_1$$

Again, for a reversible, adiabatic process, $s_2 = s_1$

(isentropic process)

<u>Stagnation state</u> is defined as the state attained by a fluid, brought to rest ($u = 0$) adiabatically, reversibly, and without work. Applying this definition,

$$\boxed{\begin{aligned} h_0 &= h + \frac{u^2}{2} \\ s_0 &= s \end{aligned}} \qquad [\text{see } (3\text{-}5)]$$



** (higher pressure isobars lie above lower pressure isobars in the enthalpy-entropy plane)

An ideal gas with constant c_p is said to be calorically perfect.

For such a gas,

$$\frac{dh}{dT} = c_p \Rightarrow (h_0 - h) = c_p (T_0 - T)$$

Thus $\quad h_0 = h + \dfrac{u^2}{2} \Rightarrow \boxed{T_0 = T + \dfrac{u^2}{2 c_p}}$

From $(s_2 - s_1) = c_p \ln \dfrac{T_2}{T_1} - R \ln \dfrac{p_2}{p_1}$,

$$\cancelto{0}{(s_0 - s)} = c_p \ln \frac{T_0}{T} - R \ln \frac{p_0}{p} \Rightarrow \boxed{\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma - 1}}}$$

see (3-6)

h_0 – stagnation enthalpy	h – static enthalpy (or) enthalpy
s_0 – stagnation entropy	s – static entropy (or) entropy
T_0 – stagnation temperature	T – static temp (or) temp
p_0 – stagnation pressure	p – static pressure (or) pressure

Summary

I Law: $\underbrace{\left(h_2 + \frac{u_2^2}{2}\right)}_{h_{02}} = \underbrace{\left(h_1 + \frac{u_1^2}{2}\right)}_{h_{01}} + q - w$

$$\Rightarrow (h_{02} - h_{01}) = q - w$$

$$c_p (T_{02} - T_{01}) = q - w \quad [CPG]$$

II Law: $(s_2 - s_1) = c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$

$s_{02} = s_2$, $s_{01} = s_1$ (definition of stagnation state)

$$(s_{02} - s_{01}) = (s_2 - s_1)$$

$$(s_{02} - s_{01}) = c_p \ln \frac{T_{02}}{T_{01}} - R \ln \frac{p_{02}}{p_{01}}$$

For isentropic flow, $(s_2 - s_1) = 0$, $(s_{02} - s_{01}) = 0$

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} \leftarrow$$ relation between pressure ratio and temperature ratio in isentropic flow.

$$\frac{p_{02}}{p_{01}} = \left(\frac{T_{02}}{T_{01}}\right)^{\frac{\gamma}{\gamma-1}} \leftarrow$$ relation between stagnation pressure ratio and stagnation temperature ratio in isentropic flow.

Mach Number (M)

$$M \overset{\text{def}}{=} \frac{u}{a}, \quad a - \text{local speed of sound [see (3.7)]}$$

For an ideal gas $a = \sqrt{\gamma R T}$ [see (3.8)]

$$\boxed{\begin{array}{l} M<1 : \text{subsonic} \\ M>1 : \text{supersonic} \\ M=1 : \text{sonic} \end{array}}$$

Using the Mach number,

$T_0 = T + \frac{u^2}{2c_p}$ can be written as

$$T_0 = T + \frac{M^2 \gamma R T}{2 c_p} = T + \frac{M^2 \cancel{\gamma R} T}{2 \frac{\cancel{\gamma R}}{\gamma - 1}} = T\left(1 + \frac{\gamma-1}{2} M^2\right)$$

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2} M^2 \qquad \text{[see (3.10)]}$$

and $$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}} \qquad \text{[see (3.11)]}$$

Mass flow rate ($\dot{m}$) in terms of stagnation values

$$\dot{m} = \rho u A = \frac{p}{RT} M \sqrt{\gamma R T}\, A = \sqrt{\frac{\gamma}{R}} \frac{p}{\sqrt{T}} M A$$

substituting $T = \dfrac{T_0}{1 + \frac{\gamma-1}{2} M^2}$ and $p = \dfrac{p_0}{\left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}}$,

$$\dot{m} = \sqrt{\frac{\gamma}{R}} \frac{p_0}{\sqrt{T_0}} M \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} A \qquad \text{[see (3.13)]}$$

Ex

The heat transfer into a flow is 200 kJ/kg per unit mass. If the mass flow rate is 50 kg/s, what is the heat transfer rate?

$$q = 200 \frac{kJ}{kg}, \quad \dot{m} = 50 \frac{kg}{s}$$

$$\dot{Q} = \dot{m} q = (50)(200) = 10000 \frac{kJ}{s} = 10000 \, kW$$

Ex

The work done per unit mass on a flow is $150 \frac{kJ}{kg}$. If the mass flow rate is $50 \frac{kg}{s}$, what is the power?

$$w = -150 \frac{kJ}{kg}, \quad \dot{m} = 50 \frac{kg}{s}$$

$$\dot{W} = \dot{m} w = (50)(-150) = -7500 \frac{kJ}{s} = -7500 \, kW$$

Ex

Heat and work interactions occur as a flow moves from ① to ②. $q = 200$ kJ/kg, $w = 200 \frac{kJ}{kg}$. What is the change in the stagnation enthalpy of the fluid?

I Law:

$$\underbrace{\left(h_2 + \frac{u_2^2}{2}\right)}_{h_{02}} = \underbrace{\left(h_1 + \frac{u_1^2}{2}\right)}_{h_{01}} + q - w$$

$$\Rightarrow h_{02} - h_{01} = q - w, \text{(or)} \, (h_{02} - h_{01}) = 200 - 200 = 0$$

$$\Delta h_0 = 0$$

Ex

At a certain station in a flow, $h = 0.5 \frac{MJ}{kg}$ and $u = 400\ m/s$.

Find the stagnation enthalpy there. Also, what is $(s_o - s)$ there?

$$h_o = h + \frac{u^2}{2} = (0.5)(10^6) + \frac{400^2}{2} = 5.8(10^5)\frac{J}{kg}$$

$$= 0.58 \frac{MJ}{kg}$$

$(s_o - s) = 0$, by definition

Ex

At a certain station in the flow of a calorically perfect gas (CPG), $T = 300\,K$, $p = 120\ kPa$, and $u = 300\ m/s$. Find the stagnation temperature and stagnation pressure at this station [$c_p = 1107 \frac{J}{kg\text{-}K}$, $R = 287 \frac{J}{kg\text{-}K}$].

$$T_o = T + \frac{u^2}{2c_p} = 300 + \frac{300^2}{2(1107)} = 340.7\,K$$

$$\gamma = \frac{c_p}{c_v} = \frac{c_p}{c_p - R} = \frac{1107}{1107 - 287} = 1.35$$

$$\frac{p_o}{\underbrace{p}_{120\,kPa}} = \left(\frac{T_o}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{340.7}{300}\right)^{\frac{1.35}{1.35-1}} = 1.6335$$

$$p_o = (1.6335)(120) = 196\ kPa$$

Ex

H_2 ($\bar{M} = 2$) is heated in a nuclear rocket heat exchanger from stagnation temperature 300 K to stagnation temperature 2300K. Find q. ($\gamma_{H_2} = 1.29$)

$$R_{H_2} = \frac{\bar{R}}{\bar{M}_{H_2}} = \frac{8314.3}{2} = 4157.2 \frac{J}{kg\text{-}K}$$

$$c_{p_{H_2}} = \frac{\gamma_{H_2} R_{H_2}}{\gamma_{H_2} - 1} = 18492 \frac{J}{kg\text{-}K}$$

I Law:

$$\underbrace{\left(h_2 + \frac{u_2^2}{2}\right)}_{h_{02}} = \underbrace{\left(h_1 + \frac{u_1^2}{2}\right)}_{h_{01}} + q - \cancelto{0}{w}$$ (heat exchanger, no work done)

$$\Rightarrow q = (h_{o_2} - h_{o_1}) = c_p (T_{02} - T_{01})$$

[CPG since γ is given to be constant]

$$q = 18492(2300 - 300) = 3.6984 (10^7) \frac{J}{kg} = 36984 \frac{kJ}{kg}$$

Ex

In a flow across a device, the stagnation temperature remains constant and no work is done. The stagnation pressure drops from 7.8 atm to 5.6 atm across the device. Find (a) q and (b) the entropy change.
[$\gamma = 1.4$, $R = 287 \frac{J}{kg\text{-}K}$]

a)

I Law: $h_{o2} = h_{o1} + q - w \Rightarrow (h_{o2} - h_{o1}) = q - w$

CPG: $c_p\cancel{(T_{o2} - T_{o1})}^{0} = q - \cancel{w}^{0} \Rightarrow q = 0$

b) $(s_2 - s_1) = c_p \ln \dfrac{T_2}{T_1} - R \ln \dfrac{p_2}{p_1}$

$s_{o2} = s_2$ (by definition), $s_{o1} = s_1$ (by definition)

$(s_{o2} - s_{o1}) = (s_2 - s_1)$

$(s_{o2} - s_{o1}) = c_p \ln \dfrac{T_{o2}}{T_{o1}} - R \ln \dfrac{p_{o2}}{p_{o1}}$

$= c_p \ln 1 - 287 \ln \dfrac{5.6}{7.8} = 0 - (-95.1)$

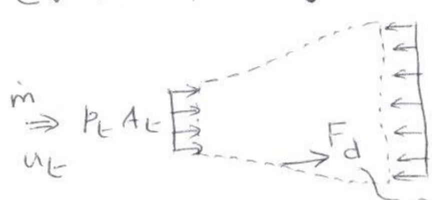
$= 95.1 \dfrac{J}{kg\text{-}K}$

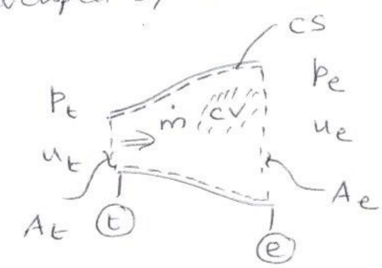
$\Delta s = (s_2 - s_1) = (s_{o2} - s_{o1}) = 95.1 \dfrac{J}{kg\text{-}K}$

Ex

Consider the diverging portion of a rocket nozzle, see Figure. Derive the expression for the thrust developed by the nozzle.

CV with forces ↓





force on fluid due to its interaction with the wall of the nozzle.

Momentum equation: $\dot m u_t + p_t A_t + F_d - p_e A_e = \dot m u_e$

$\Rightarrow F_d = \dot m(u_e - u_t) + p_e A_e - p_t A_t = (I_e - I_t)$

Nozzle thrust is F_d (to the left $\Leftarrow$).