

Li model - Chordnormal CoM displacement

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1. Introduction

Li *et al.* (2022) defines a two-dimensional quasi-steady aerodynamic system of equations for falling plates with displaced CoM, suitable for modelling gliding seeds such as *Alsomitra macrocarpa*. CoM displacement is only considered in the chordwise direction; this report aims to extend the model to include CoM displacement in the chordnormal direction. The changes to the equations assume that the effect of moving the CoM in this direction are similar (only perpendicular) to the existing chordwise direction, and for a chordnormal displacement of 0 the equations reduce to their original definitions.

2. Non-dimensional terms

pics/CoM_diagram.png

FIGURE 1. Model parameters of interest. (a) thin plate with chord ℓ , thickness $h \ll \ell$, and chordwise CoM displacement ℓ_{CM} (Li *et al.* (2022), fig 7a). (b) proposed definitions including chordnormal CoM displacement h_{CM}

Certini (2023) defines the normalised chordwise CoM displacement e_x as the relationship between the chordwise CoM displacement ℓ_{CM} , and chord length ℓ , as shown in fig. 1a.

$$e_x = \frac{\ell_{CM}}{\ell} \quad (2.1)$$

A similar definition can be made for the normalised chordnormal CoM displacement, with a chordnormal CoM displacement (below the plate) h_{CM} , and ℓ , shown in fig. 1b. I propose this definition since for dandelion-inspired flyers, $h_{CM} \gg h$:

$$e_y = \frac{h_{CM}}{\ell} \quad (2.2)$$

3. CoV angle-of-attack, α

Li *et al.* (2022) calculates aerodynamic forces about the center of volume (CoV), but rigid body dynamics about the center of mass (CoM). Since these centers are displaced, it necessitates converting velocities between these points. α about the CoV is calculated from the CoM velocities ($v_{x'}$ and $v_{y'}$), and another term with the angular velocity ω and ℓ_{CM} to account for the chordwise displacement (Li *et al.* (2022), p15); shown in fig 2a.

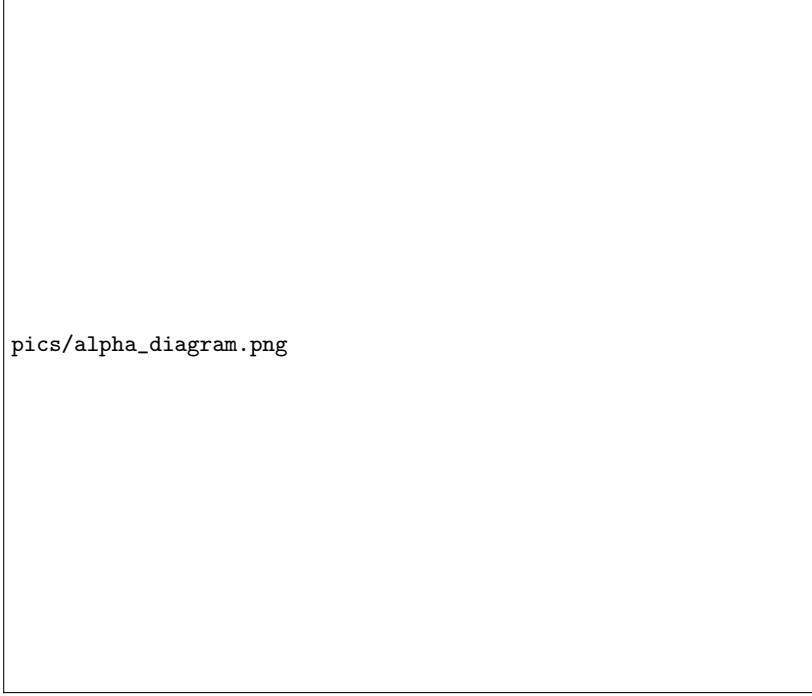


FIGURE 2. Parameters relevant to α (a); non-inertial reference frame (x,y) , inertial frame (x',y',θ,ω) , CoM velocity v^{CM} and CoV velocity v_{CV} (Li *et al.* (2022), fig 7d). From 2d we can see that the and v^{CM} only differ due to ω and ℓ_{CM} . (b) new inertial frame definition including h_{CM} . (c) Effect of ω and ℓ_{CM} on the CoV relative to the CoM - the tangential velocity at the CoV is in the negative y' direction. (d) Effect of ω and h_{CM} on the CoV relative to the CoM - the tangential velocity at the CoV is in the negative x' direction.

From 2c, we can see that the tangential velocity of the CoV about the CoM as a result of ω and ℓ_{CM} is in the opposite direction of y' , so it is subtracted in the equation:

$$\alpha_{\text{CoV}} = \arctan((v_{y'} - \omega\ell_{\text{CM}})/v_{x'}) \quad (3.1)$$

This can be extended to include the chordnormal displacement, as shown in 2b. From 2d we can see that the tangential velocity of the CoV about the CoM as a result of ω and h_{CM} is in the opposite direction of x' , so it is also subtracted:

$$\alpha_{\text{CoV}} = \arctan((v_{y'} - \omega\ell_{\text{CM}})/(v_{x'} - \omega h_{\text{CM}})) \quad (3.2)$$

Assuming these equations are correct, the addition of h_{CM} to other equations in the model is straightforward: by replacing $v_{x'}$ with $v_{x'} - h_{\text{CM}}$.

4. Translational lift force, L_{T}

(Li *et al.* (2022), 4.10)

$$L_{\text{T}} = \frac{1}{2}\rho_f\ell_{\text{CL}}\sqrt{v_{x'}^2 + (v_{y'} - \omega\ell_{\text{CM}})^2}(v_{y'} - \omega\ell_{\text{CM}}, -v_{x'}) \quad (4.1)$$

Extended to include the chordnormal displacement:

$$L_T = \frac{1}{2} \rho_f \ell C_L \sqrt{(v_{x'} - \omega h_{CM})^2 + (v_{y'} - \omega \ell_{CM})^2} (v_{y'} - \omega \ell_{CM}, -(v_{x'} - \omega h_{CM})) \quad (4.2)$$

5. Rotational lift force, L_R

(Li *et al.* (2022), 4.11)

$$L_R = -\frac{1}{2} \rho_f \ell^2 C_R \omega (v_{y'} - \omega \ell_{CM}, -v_{x'}) \quad (5.1)$$

Extended to include the chordnormal displacement:

$$L_R = -\frac{1}{2} \rho_f \ell^2 C_R \omega (v_{y'} - \omega \ell_{CM}, -(v_{x'} - \omega h_{CM})) \quad (5.2)$$

6. Drag force, D

(Li *et al.* (2022), 4.13)

$$D = -\frac{1}{2} \rho_f \ell C_D \sqrt{v_{x'}^2 + (v_{y'} - \omega \ell_{CM})^2} (v_{x'}, v_{y'} - \omega \ell_{CM}) \quad (6.1)$$

Extended to include the chordnormal displacement:

$$D = -\frac{1}{2} \rho_f \ell C_D \sqrt{(v_{x'} - \omega h_{CM})^2 + (v_{y'} - \omega \ell_{CM})^2} (v_{x'} - \omega h_{CM}, v_{y'} - \omega \ell_{CM}) \quad (6.2)$$

7. Torque from translational forces, τ_T

(Li *et al.* (2022), 4.14)

$$\begin{aligned} \tau_T &= (L_{Ty'} + D_{y'}) (\ell_{CP} - \ell_{CM}) \\ &= -\frac{1}{2} \rho_f \ell \sqrt{v_{x'}^2 + (v_{y'} - \omega \ell_{CM})^2} [C_L v_{x'} + C_D (v_{y'} - \omega \ell_{CM})] (\ell_{CP} - \ell_{CM}) \end{aligned} \quad (7.1)$$

Extended to include the chordnormal displacement:

$$\begin{aligned} \tau_T &= -\frac{1}{2} \rho_f \ell \sqrt{(v_{x'} - \omega h_{CM})^2 + (v_{y'} - \omega \ell_{CM})^2} [C_L (v_{x'} - \omega h_{CM}) \\ &\quad + C_D (v_{y'} - \omega \ell_{CM})] \sqrt{(\ell_{CP} - \ell_{CM})^2 + h_{CM}^2} \end{aligned} \quad (7.2)$$

I assume that $(\ell_{CP} - \ell_{CM})$ is the distance over which the torque is applied, and expand it to be the hypotenuse of $\ell_{CP} - \ell_{CM}$ and $h_{CM} - 0$; both are assumed positive in eq. 7.2. However, if $\ell_{CP} > \ell_{CM}$ and $h_{CM} = 0$, the last term in eq. 7.1 will be negative, but positive in eq. 7.2. This can be fixed by multiplying with an absolute:

$$\begin{aligned} \tau_T &= -\frac{1}{2} \rho_f \ell \sqrt{(v_{x'} - \omega h_{CM})^2 + (v_{y'} - \omega \ell_{CM})^2} [C_L (v_{x'} - \omega h_{CM}) \\ &\quad + C_D (v_{y'} - \omega \ell_{CM})] \sqrt{|\ell_{CP} - \ell_{CM}| (\ell_{CP} - \ell_{CM}) + h_{CM}^2} \end{aligned} \quad (7.3)$$

This does not fix the case where $h_{\text{CM}} < 0$, but I will ignore it since moving the CoM over the plate would cause it to flip and be below the plate

8. Added inertia about CoM, I_A

(Certini (2023))

$$I_A = e_x^2 + 1/32 \quad (8.1)$$

Extended to include the chordnormal displacement:

$$I_A = e_x^2 + e_y^2 + 1/32 \quad (8.2)$$

9. Aerodynamic resistance to rotation, τ_R

(Li *et al.* (2022), 4.15)

$$\tau_R = -\frac{1}{128}\rho_f\ell^4 C_D^{\pi/2}\omega|\omega|\left[\left(\frac{2\ell_{\text{CM}}}{\ell} + 1\right)^4 \pm \left(\frac{2\ell_{\text{CM}}}{\ell} - 1\right)^4\right] \quad (9.1)$$

Simplified to only consider $2e_x \leq 1$ (CoM lies within the plate), and substituting according to 2.1:

$$\tau_R = -\frac{1}{128}\rho_f\ell^4 C_D^{\pi/2}\omega|\omega|\left[(2e_x + 1)^4 + (2e_x - 1)^4\right] \quad (9.2)$$

To better understand this term, I consider the case where $e_x = 0$ and the equation reduces to that defined by Andersen *et al.* (2005):

$$\tau_R = -\frac{1}{64}\rho_f\ell^4 C_D^{\pi/2}\omega|\omega| \quad (9.3)$$

This term defines the resistance to rotation about the CoM, and since the CoM is always inside the plate this only involves broadside-on motion of each segment along the chord ($C_D^{\pi/2}$):



FIGURE 3. Calculation of τ_R with none or chordwise CoM displacement, according to 9.3 (a) and 9.2 (b). As the plate experiences pure rotation velocity about the CoM (ω), the drag force is integrated chordwise (shown in blue) for $-\frac{\ell}{2} \leq r \leq \frac{\ell}{2}$. The force at each point dr along the length of r depends on the point velocity v_n ; for (a) $v_n = \omega r$ and for (b) $v_n = \omega(r + \ell_{\text{CM}})$.

Introducing chordnormal CoM displacement makes this integration more complex, since the angle of motion of each segment relative to the fluid is no longer constant across the plate:

To derive the new integral, I will start from the definition of the derivative (local torque-induced drag) of 9.3 from (Andersen *et al.* (2005), p19):

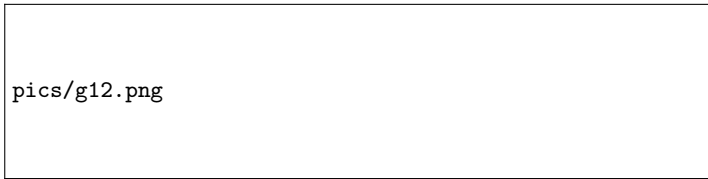


FIGURE 4. Calculation of τ_R with chordnormal CoM displacement.

$$d\tau_R = -\frac{1}{2}\rho_f C_D^{\pi/2} v_n |v_n| r dr \quad (9.4)$$

Where v_n is the velocity of each segment, and r is the position along the plate relative to the center. By integrating over ℓ we obtain the expression for τ_R :

$$\tau_R = -\frac{1}{2}\rho_f C_D^{\pi/2} \int_{\ell/2}^{-\ell/2} v_n |v_n| r dr \quad (9.5)$$

Substituting $v_n = \omega r$ simplifies down to 9.3 as required:

$$\tau_R = -\frac{1}{2}\rho_f C_D^{\pi/2} \int_{\ell/2}^{-\ell/2} \omega |\omega| r^2 |r| dr = -\frac{1}{64}\rho_f C_D^{\pi/2} \omega |\omega| \ell^4 \quad (9.6)$$

For chordnormal CoM displacement, $v_n = \omega \sqrt{r^2 + h_{CM}^2} \cos(\tan^{-1}(\frac{h_{CM}}{r}))$ which was found to be equivalent to $v_n = \omega r$. In other words, the chordnormal displacement of the CoM (by only considering the broadside-on velocity components) has no effect on the aerodynamic resistance to rotation. We can therefore use the chordwise definition 9.2 without incorporating the chordnormal displacement.

10. Results

The model equations have been implemented in a matlab script which can be found here. The aerodynamic coefficients have been optimised for *Alsomitra* diaspores, and the initial conditions are $[0; 0; 0; \pi/3; 0; 0]$ ($x, y, \theta, v_{x'}, v_{y'}, \omega$). The results comparing the Li model ($e_x = 0$) to the chordnormal model ($e_x = 0, e_y = 0.2$) are shown on the next page.

REFERENCES

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