The negative binomial Indian buffet process

Creighton Heaukulani
University of Cambridge

Joint work with **Daniel M. Roy** (Cambridge)

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Outline

1. The Indian buffet process (IBP; [GG06, GGS07]) induces a distribution on *allocations of features* to individuals. We are interested in **count extensions of the IBP**, i.e., each individual may have *multiple copies* of each feature.

2. ([TJ07]). The IBP is the combinatorial structure of an exchangeable sequence of Bernoulli processes directed by a beta process base measure.

3. We are interested in the combinatorial structure of an exchangeable sequence of negative binomial processes ([BMPJ11, ZHDC12]), directed by a beta process, which we describe as the negative binomial Indian buffet process, a *count* extension of the IBP.

Plan

1. Review the IBP and connect it to the theory of completely random measures.

2. Develop a different continuum of Pólya urn schemes perspective for sampling the IBP.

3. Develop a method to **sample** a beta negative binomial process using the continuum of Pólya urn schemes intuition.

4. Present the corresponding negative binomial IBP.

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- The $n + 1$-st customer:
  - takes $\text{Bernoulli}\left(\frac{m_k}{c+n}\right)$ servings of every previously sampled dish k, where m_k is the total number of servings taken of dish k by the previous n customers;
  - tastes $\text{Poisson}\left(\alpha \frac{c}{c+n}\right)$ new dishes.

IBP: matrix perspective

rows = individuals/customers
columns = features/dishes

$n = 1$	1	1				
$n = 2$		1	1	1		
$\vdots$	1		1		1	
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$$
\begin{array}{c|c|c|c|c|c|c|}
 & \omega_1 & \omega_2 & \omega_3 & \omega_4 & \cdots & \\
\hline
X_1 & 1 & 1 & & & & \\
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$$X_{n+1} \mid X_1, \ldots, X_n \;\sim\; \mathrm{BeP}\Big(\frac{c}{c+n}\widetilde{B}_0 + \frac{1}{c+n}\sum_{i=1}^{n} X_i\Big)$$

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Thm ([TJ07]): Then there exists a beta process

$$B \;\sim\; \mathrm{BP}(c, \widetilde{B}_0)$$

such that

$$(X_n)_{n\in\mathbb{N}} \mid B \;\overset{iid}{\sim}\; \mathrm{BeP}(B).$$

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Bernoulli(p) $\rightarrow$ geometric(p): # successes before the **first failure**

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NB(r, p): # successes before the r-th failure

Urn scheme perspective

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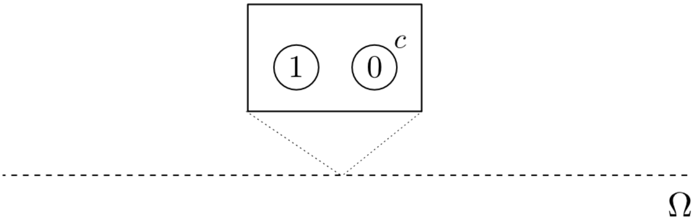
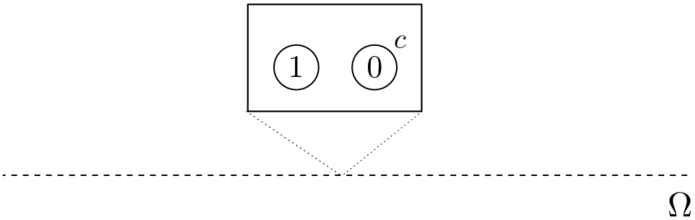
Imagine an infinite number of independent Pólya urn schemes, each with two tables labelled 1 and 0



Urn scheme perspective

Imagine an infinite number of independent Pólya urn schemes, each with two tables labelled 1 and 0

i.e., a **continuum of Pólya urn schemes**



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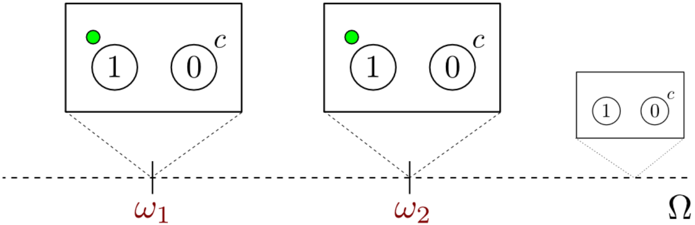

$\widetilde{B}_0(\Omega) = \alpha$

Urn scheme perspective

How to sample $X_1 \sim \mathrm{BeP}(\widetilde{B}_0)$

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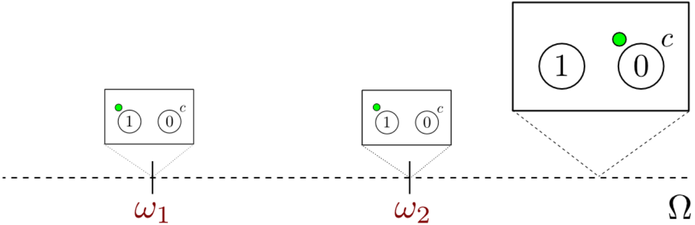
- The remaining urn schemes on the continuum fail.



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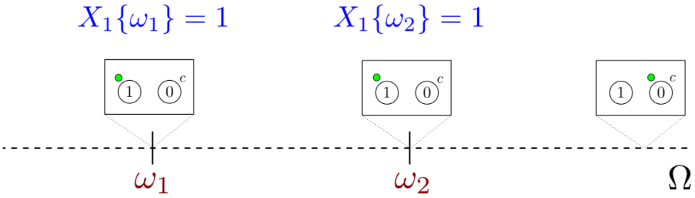
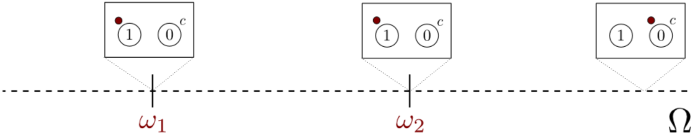


X_1 done.

How to sample $X_2 \mid X_1$



$$X_2 \mid X_1 \;\sim\; \mathrm{BeP}\Big(\tfrac{c}{c+1}\widetilde{B}_0 + \tfrac{1}{c+1}X_1\Big)$$

How to sample $X_2 \mid X_1$

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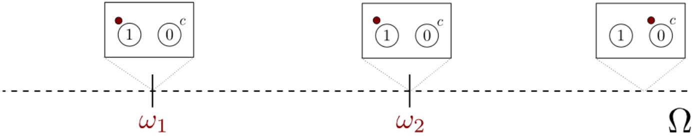

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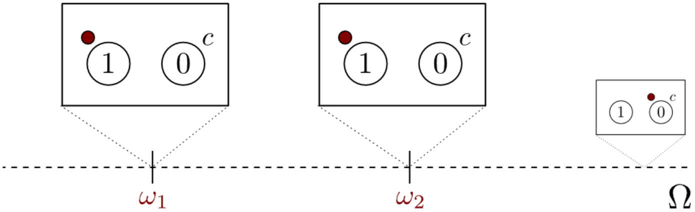

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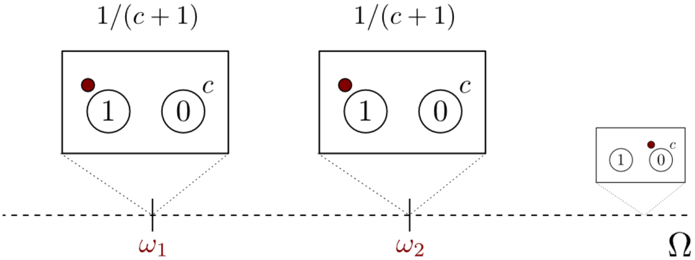

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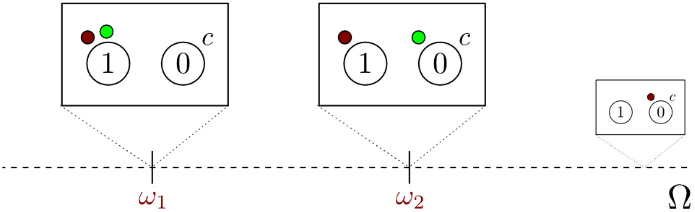

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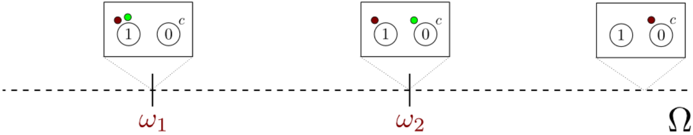

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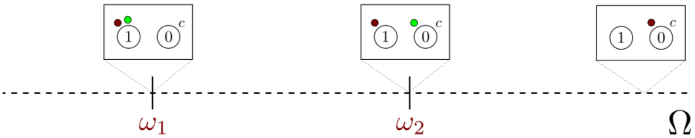

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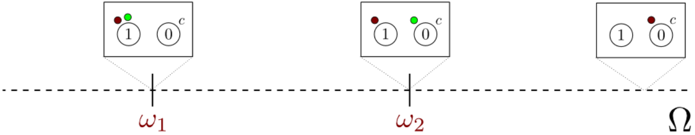

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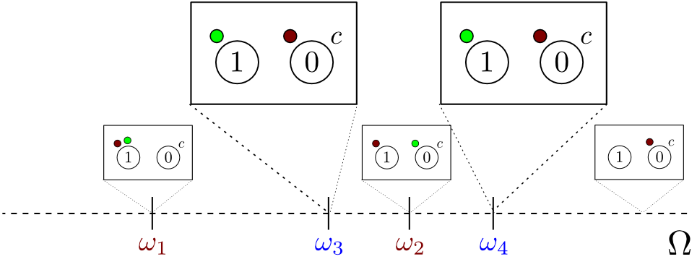

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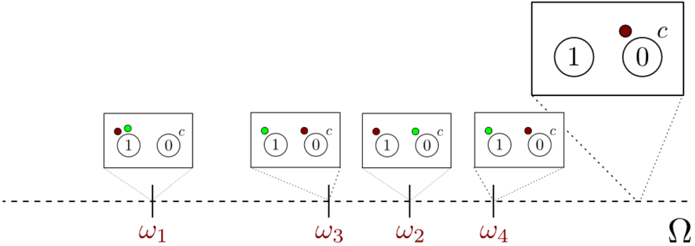

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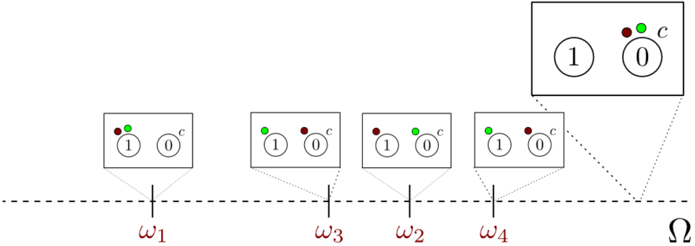

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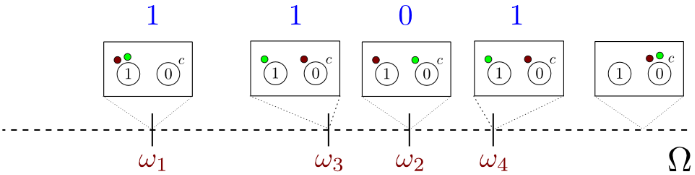
X_2 done.



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A count extension

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NB(r, p): distribution of # successes until r failures.

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NB(r, p): distribution of # successes until r failures.

Intuition: The Bernoulli process runs one trial at every urn scheme. So the *negative binomial process* continues running trials until r failures at each urn scheme.

Key point: Instead of binary indicators at each urn scheme, we get integer-valued counts.

Negative binomial process

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Negative binomial processes are parameterized by a “base measure”

$$B_0 = \widetilde{B}_0 + \sum_{k=1}^{\kappa} b_k \delta_{\omega_k}. \tag{1}$$

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Defn. We call $X \sim \mathrm{NBP}(r, B_0)$ a <u>negative binomial process</u>, when it is a completely random measure with

- fixed component (as defined by [BMPJ11, ZHDC12]):

$$\sum_{k=1}^{\kappa} \zeta_k \delta_{\omega_k}, \qquad \zeta_k \overset{ind}{\sim} \mathrm{NB}(r, b_k), \quad \kappa \in \mathbb{N} \cup \{\infty\}, \tag{2}$$

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- ordinary component (our definition): a Poisson process with intensity $r\widetilde{B}_0$.

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Our goal: give a method to directly sample X without representing the underlying beta process.

Start simple

Consider a $\mathrm{NB}(r, p)$ distribution when $r = 1$, i.e., a geometric distribution

$$\mathrm{NB}(1, p) = \mathrm{geometric}(p), \tag{4}$$

which counts the number of successful trials (with success probability p) before the **first failure**.

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Intuition: Run the urn scheme until the first failure.

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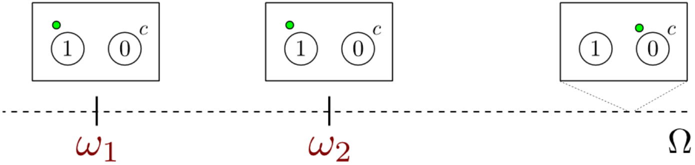

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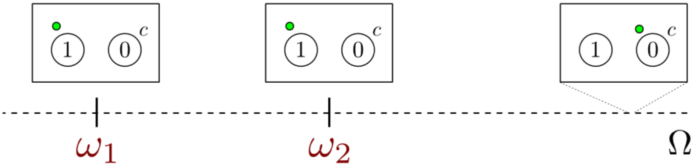
We continue until all urn schemes have failed once!



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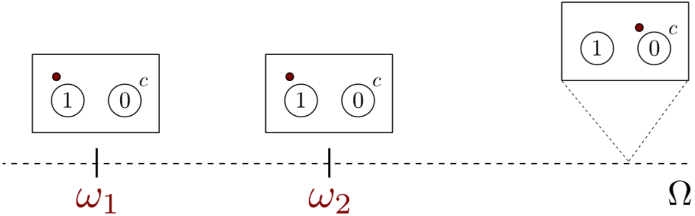
$\Rightarrow$ The remaining urns on the continuum have already failed



How to sample $X \sim \mathrm{BGP}(c, \widetilde{B}_0)$

We continue until all urn schemes have failed once!

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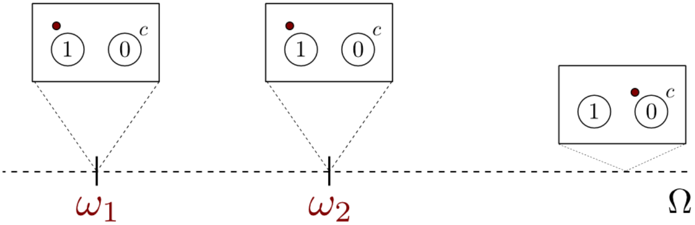
$\Rightarrow$ We **complete** the atoms



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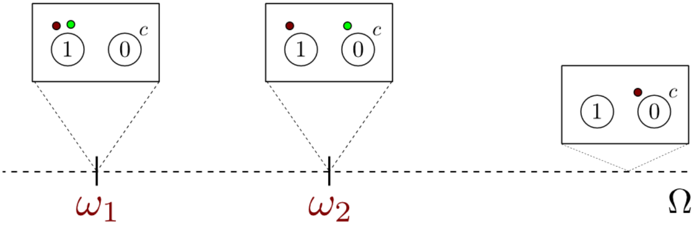

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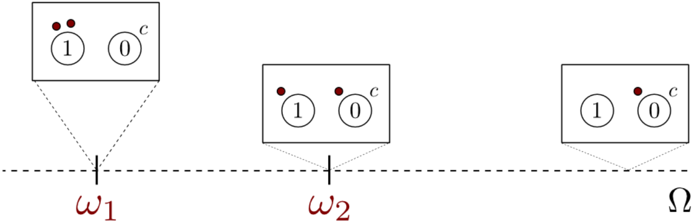

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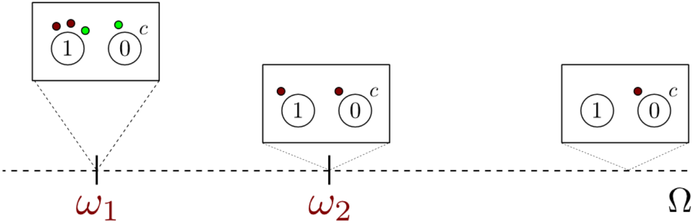

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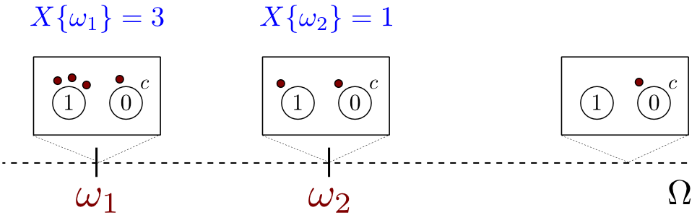

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Ordinary component: BGP

Thm ([HR13]). Let $Y \sim \mathrm{PP}(\widetilde{B}_0)$ and let

$$\zeta_s \overset{ind}{\sim} \text{beta-geometric}(1, c), \quad s \in \Omega,$$

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$$X = \sum_{s \in Y} (1 + \zeta_s)\delta_s \sim \mathrm{BGP}(c, \widetilde{B}_0).$$

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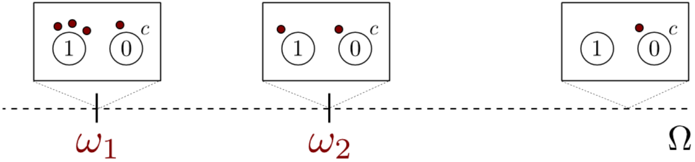

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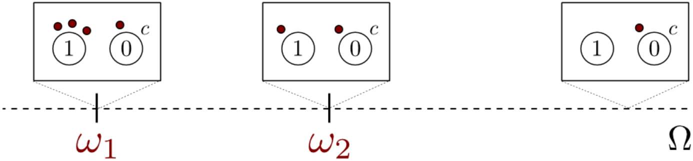

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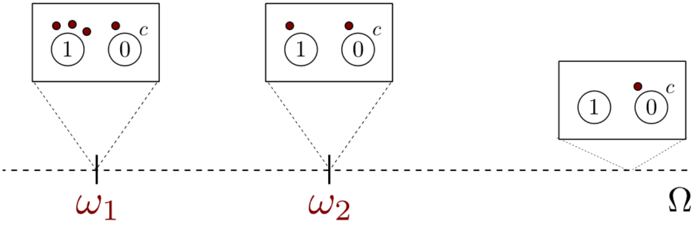

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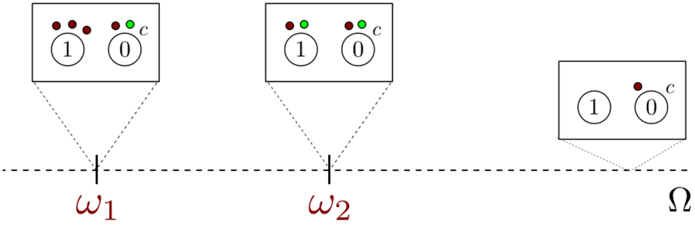
E.g., if $r = 2$, we stop.



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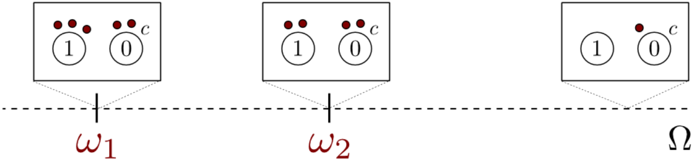

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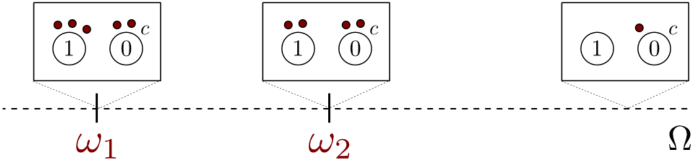

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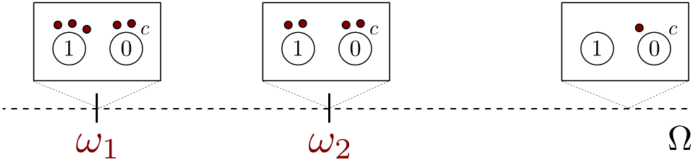

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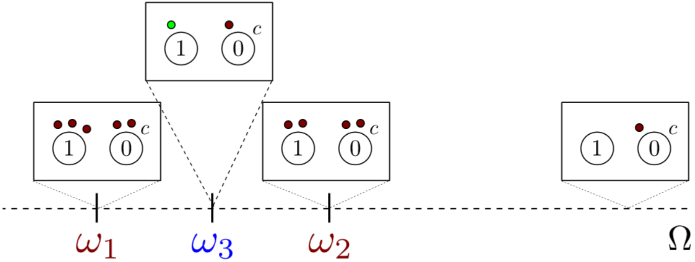

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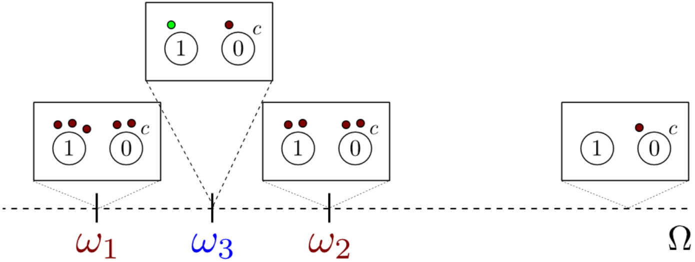

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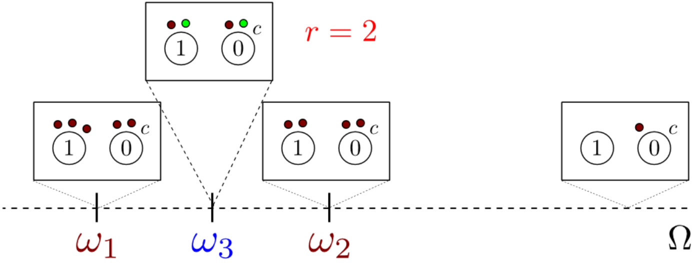

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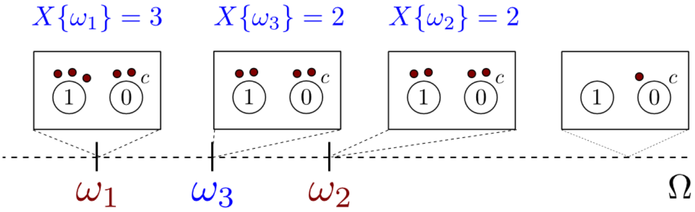

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Ordinary component: BNBP

Thm ([HR13]). Let

$$Y_\ell \overset{ind}{\sim} \mathrm{PP}\Big(\frac{c}{c+\ell-1}\widetilde{B}_0\Big), \qquad \ell \in [r] = (1,\dots,r), \tag{5}$$

and let

$$\zeta_{\ell,s} \overset{ind}{\sim} \text{beta-NB}(r-\ell+1, 1, c+\ell-1), \qquad \ell \in [r],\ s \in \Omega, \tag{6}$$

independent also from (Y_r). Then

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Key point: atoms arise in a sequence of r rounds.

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Probability function

Let $\mathcal{H}_n \equiv \mathbb{Z}_+^n \backslash \{0^n\}$, and for $h \in \mathcal{H}_n$, let M_h count the number of features k where every customer n has $h(n)$ copies of feature k.

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Claim. The probability function for the NB-IBP is given by

$$\frac{\alpha^{K_N}}{\prod_{h\in\mathcal{H}_N} M_h!} \exp\left(-\alpha \sum_{n=1}^{N}\sum_{\ell=1}^{r} \frac{c}{c+nr+\ell-1}\right) \prod_{h\in\mathcal{H}_N}\left[\frac{c}{c+Nr}\frac{\Gamma(s(h))\Gamma(c+Nr+1)}{\Gamma(c+Nr+s(h))} \prod_{n\in\mathbb{N}}\binom{r+h(n)-1}{r-1}\right]^{M_h}$$

where $s(h) \equiv \sum_{n=1}^{N} h(n)$.

General $r > 0$

- The previous urn scheme was only valid for $r \in \mathbb{N}$.
- Urn schemes for general $r > 0$ reduces to the case of **fractional** $r \in (0, 1)$, which is desirable for some applications.
  - We use Poisson process calculus to reduce the r rounds to one slightly more clever round.
  - Completions are no longer beta-NB variables, but what we call harmonic mixtures.

General $r > 0$

There is an analytical extension of the NB-IBP to general values of $r > 0$. The probability function looks similar:

Claim ([HR13]). Let $r > 0$. The probability function for the NB-IBP is given by

$$\begin{aligned} &\frac{(c\,\alpha)^{K_N}}{\prod_{h \in \mathcal{H}_N} M_h!} \exp\Big(-c\,\alpha \sum_{n=1}^{N} [\psi(c+(n+1)r) - \psi(c+nr)]\Big) \\ &\qquad \times \prod_{h \in \mathcal{H}_N} \left[\frac{\Gamma(s(h))\Gamma(c+Nr)}{\Gamma(c+Nr+s(h))} \prod_{n \in \mathbb{N}} \frac{\Gamma(r+h(n))}{h(n)!\,\Gamma(r)}\right]^{M_h}. \end{aligned} \tag{8}$$

Key difference: No longer a concept of *rounds*.

Extensions

We can also characterize the combinatorial structure of an exchangeable sequence of negative binomial processes, directed by a generalized beta process [Roy13], parametrized by a measurable family of EPPFs.

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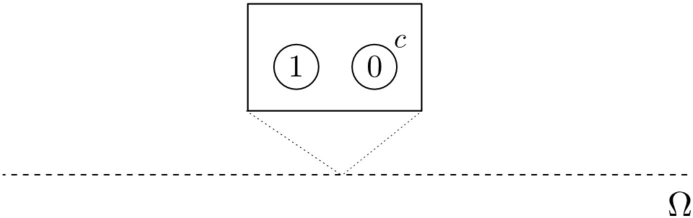

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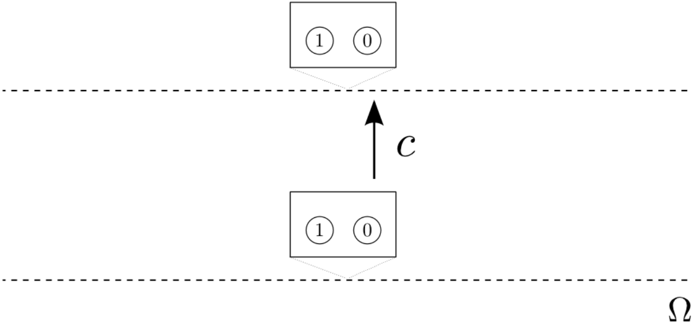

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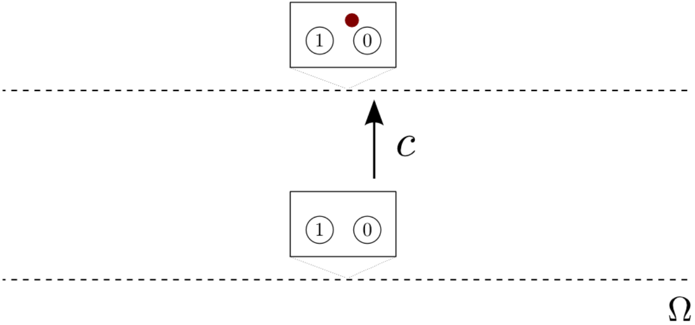

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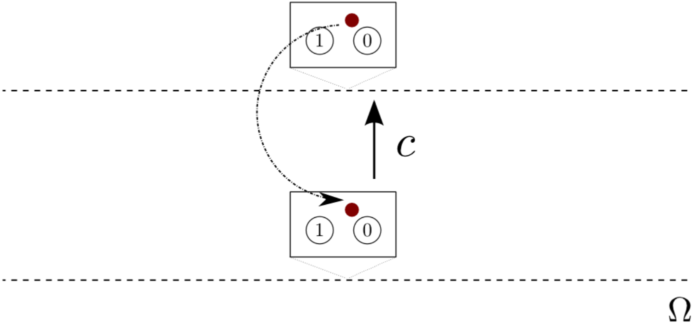

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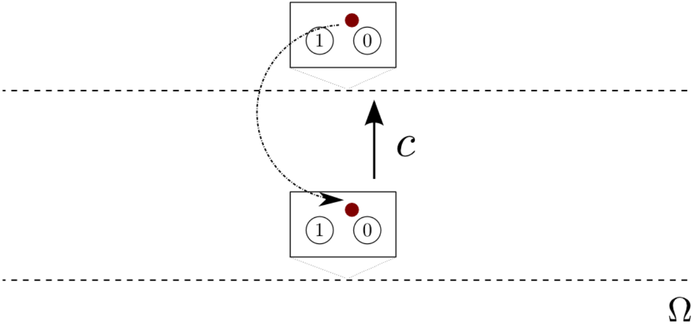
A continuum of generalized Blackwell-MacQueen urn schemes



Extensions

Special cases include:

- hierarchies of beta processes [TJ07]
- stable beta processes [TG09, BJP12]
  - corresponding power-law NB-IBP
- hierarchies of stable beta processes

Probability function

Claim. The probability function for the NB-IBP is given by

$$\frac{\alpha^{K_N}}{\prod_{h\in\mathcal{H}_N} M_h!}\exp\left(-\alpha\sum_{n=1}^{N}\sum_{\ell=1}^{r}\frac{c}{c+nr+\ell-1}\right)$$
$$\prod_{h\in\mathcal{H}_N}\left[\frac{c}{c+Nr}\frac{\Gamma(s(h))\Gamma(c+Nr+1)}{\Gamma(c+Nr+s(h))}\right.$$
$$\left.\prod_{n\in\mathbb{N}}\binom{r+h(n)-1}{r-1}\right]^{M_h}$$

Probability function from a generalized beta

Claim. The probability function for the generalized NB-IBP is given by

$$\frac{\alpha^{K_N}}{\prod_{h\in\mathcal{H}_N} M_h!}\exp\left(-\alpha\sum_{n=1}^{N}\sum_{\ell=1}^{r}\mathbb{P}(K_{nr+\ell} > K_{nr+\ell-1})\right)$$
$$\prod_{h\in\mathcal{H}_N}\Bigg[\mathbb{P}(K_{Nr+1} > K_{Nr}, Z_{Nr+s(h)} = \cdots = Z_{Nr+1})$$
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$\mathbb{P}(K_{n+1}>K_n)$: probability of a new table at step $n+1$ ...
$\mathbb{P}(K_{n+1}>K_n, Z_{n+m}=\cdots=Z_{n+1})$: probability of also staying there for next m steps

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