

Random partition-based inference schemes for feature allocations

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Joint work with

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Overview

1. We connect **random partitions** to **random feature allocations**.
2. Construct the Indian buffet process (IBP) from Chinese restaurant processes (CRPs).
3. Posterior inference on CRPs \Rightarrow inference on IBP.
4. Generalize to any exchangeable partition, results in broad class of feature allocations.

Random partitions and feature allocations

A partition of $[n] := (1, \dots, n)$ is a set of subsets of $[n]$, called blocks, whose union is $[n]$.

$$\Pi_5 = \left\{ \{1, 5\}, \{2, 4\}, \{3\} \right\}$$

A feature allocation of $[n]$ is a (finite) set of subsets of $[n]$, called features (See [4, 3]).

$$F_5 = \left\{ \{1, 2, 5\}, \{2, 5\}, \{3\}, \{3\} \right\}$$

Feature allocations allows the subsets to overlap.

Chinese restaurant process

$\text{CRP}(\theta)$

Chinese restaurant process

$\text{CRP}(\theta)$

- ▶ The first customer sits at the first table.

Chinese restaurant process

CRP(θ)

- ▶ The first customer sits at the first table.
- ▶ The $n + 1$ -st customer sits at table k with probability

$$\begin{cases} \frac{m_k}{\theta+n}, & \text{if } k \text{ old,} \\ \frac{\theta}{\theta+n}, & \text{if } k \text{ new.} \end{cases}$$

Chinese restaurant process

$$\text{CRP}(\theta)$$

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m_j is the # in cluster j

Chinese restaurant process

$$\text{CRP}(\theta)$$

Indian buffet process

$$\text{IBP}(\gamma, \theta); [6, 4]$$

- ▶ The first customer sits at the first table.
- ▶ The $n + 1$ -st customer sits at table k with probability

$$\begin{cases} \frac{m_k}{\theta + n}, & \text{if } k \text{ old,} \\ \frac{\theta}{\theta + n}, & \text{if } k \text{ new.} \end{cases}$$

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Indian buffet process

IBP(γ, θ); [6, 4]

- ▶ The first customer takes Poisson(γ) dishes.

m_j is the # in cluster j

Chinese restaurant process

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- ▶ The first customer sits at the first table.
- ▶ The $n + 1$ -st customer sits at table k with probability

$$\begin{cases} \frac{m_k}{\theta+n}, & \text{if } k \text{ old,} \\ \frac{\theta}{\theta+n}, & \text{if } k \text{ new.} \end{cases}$$

Indian buffet process

IBP(γ, θ); [6, 4]

- ▶ The first customer takes $\text{Poisson}(\gamma)$ dishes.
- ▶ The $n + 1$ -st customer
 - ▶ Takes previous dish k with probability $\frac{m_k}{\theta+n}$;

m_j is the # in cluster j

Chinese restaurant process

CRP(θ)

- ▶ The first customer sits at the first table.
- ▶ The $n + 1$ -st customer sits at table k with probability

$$\begin{cases} \frac{m_k}{\theta + n}, & \text{if } k \text{ old,} \\ \frac{\theta}{\theta + n}, & \text{if } k \text{ new.} \end{cases}$$

Indian buffet process

IBP(γ, θ); [6, 4]

- ▶ The first customer takes Poisson(γ) dishes.
- ▶ The $n + 1$ -st customer
 - ▶ Takes previous dish k with probability $\frac{m_k}{\theta + n}$;
 - ▶ Takes Poisson($\gamma \frac{\theta}{\theta + n}$) new dishes.

m_j is the # in cluster j

Construct an IBP from i.i.d. CRPs

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- First customer enters buffet and takes $\text{Poisson}(\gamma)$ dishes

Construct an IBP from i.i.d. CRPs

- First customer enters buffet and takes $\text{Poisson}(\gamma)$ dishes

$$\overbrace{\Pi_1^{(1)} \quad \Pi_1^{(2)}}^{\text{Poisson}(\gamma)} \quad \overset{iid}{\sim} \quad \text{CRP}(\theta)$$

Construct an IBP from i.i.d. CRPs

- First customer enters buffet and takes $\text{Poisson}(\gamma)$ dishes

$$\begin{array}{ccc} & \text{Poisson}(\gamma) & \\ & \underbrace{\hspace{1.5cm}} & \\ \Pi_1^{(1)} & & \Pi_1^{(2)} \\ \parallel & & \parallel \\ \{1\} & & \{1\} \end{array} \quad \begin{array}{c} iid \\ \sim \end{array} \quad \text{CRP}(\theta)$$

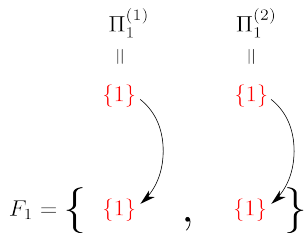
Construct an IBP from i.i.d. CRPs

- First customer enters buffet and takes $\text{Poisson}(\gamma)$ dishes
- Mark blocks as special

$$\begin{array}{ccc} & \text{Poisson}(\gamma) & \\ & \underbrace{\hspace{1.5cm}} & \\ \Pi_1^{(1)} & & \Pi_1^{(2)} \\ \parallel & & \parallel \\ \{1\} & & \{1\} \end{array} \quad \begin{array}{c} iid \\ \sim \end{array} \quad \text{CRP}(\theta)$$

Construct an IBP from i.i.d. CRPs

- First customer enters buffet and takes $\text{Poisson}(\gamma)$ dishes
- Mark blocks as special
- Create feature allocation from special blocks

$$F_1 = \left\{ \begin{array}{c} \Pi_1^{(1)} \\ \parallel \\ \{1\} \end{array} \right\}, \left\{ \begin{array}{c} \Pi_1^{(2)} \\ \parallel \\ \{1\} \end{array} \right\}$$


Construct an IBP from i.i.d. CRPs

- First customer enters buffet and takes $\text{Poisson}(\gamma)$ dishes
- Mark blocks as special
- Create feature allocation from special blocks

$$\begin{array}{cc} \Pi_1^{(1)} & \Pi_1^{(2)} \\ \parallel & \parallel \\ \{1\} & \{1\} \end{array}$$

$$F_1 = \left\{ \begin{array}{c} \{1\} \\ \{1\} \end{array} \right\} \sim \text{IBP}(\gamma, \theta)$$

Clearly.

Construct an IBP from i.i.d. CRPs

$$\begin{array}{cc} \Pi_1^{(1)} & \Pi_1^{(2)} \\ \parallel & \parallel \\ \{1\} & \{1\} \end{array}$$

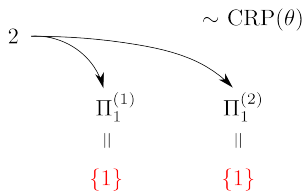
Construct an IBP from i.i.d. CRPs

- 2nd customer enters buffet and decides on previous dishes

$$\begin{array}{cc} \Pi_1^{(1)} & \Pi_1^{(2)} \\ \parallel & \parallel \\ \{1\} & \{1\} \end{array}$$

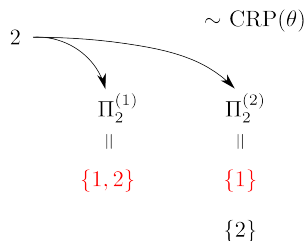
Construct an IBP from i.i.d. CRPs

- 2nd customer enters buffet and decides on previous dishes



Construct an IBP from i.i.d. CRPs

- 2nd customer enters buffet and decides on previous dishes



Construct an IBP from i.i.d. CRPs

- 2nd customer enters buffet and decides on previous dishes

$$\begin{array}{cc} \Pi_2^{(1)} & \Pi_2^{(2)} \\ \parallel & \parallel \\ \{1, 2\} & \{1\} \\ & \{2\} \end{array}$$

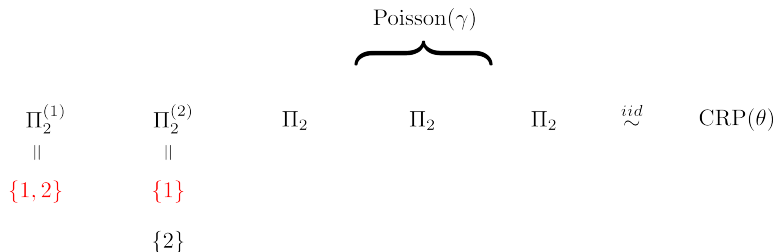
Construct an IBP from i.i.d. CRPs

- 2nd customer enters buffet and decides on previous dishes
- Samples new dishes

$$\begin{array}{cc} \Pi_2^{(1)} & \Pi_2^{(2)} \\ \parallel & \parallel \\ \{1, 2\} & \{1\} \\ & \{2\} \end{array}$$

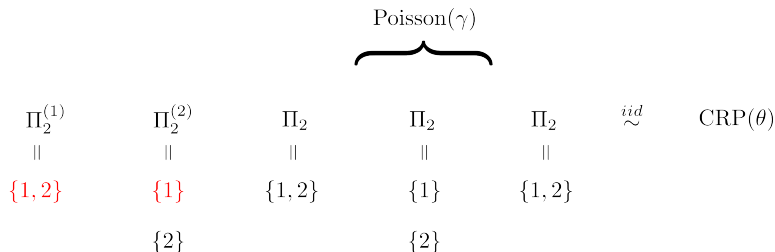
Construct an IBP from i.i.d. CRPs

- 2nd customer enters buffet and decides on previous dishes
- Samples new dishes



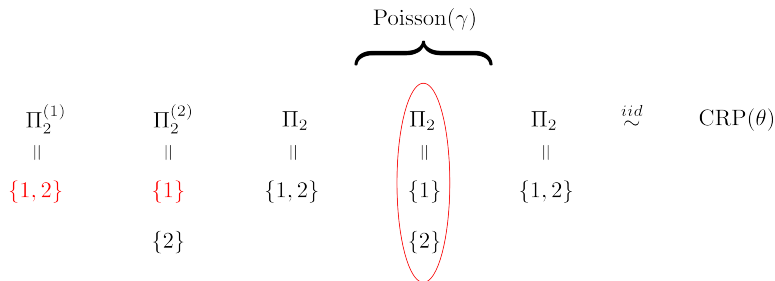
Construct an IBP from i.i.d. CRPs

- 2nd customer enters buffet and decides on previous dishes
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Construct an IBP from i.i.d. CRPs

- 2nd customer enters buffet and decides on previous dishes
- Samples new dishes
- Keep partition if 2 forms a singleton; marked as special



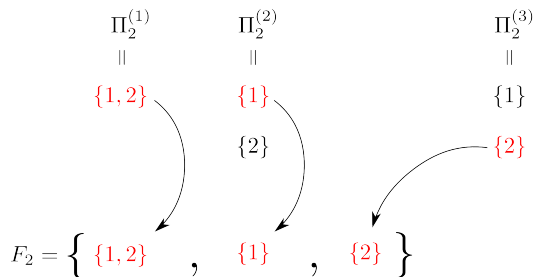
Construct an IBP from i.i.d. CRPs

- 2nd customer enters buffet and decides on previous dishes
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$\Pi_2^{(1)}$	$\Pi_2^{(2)}$	$\Pi_2^{(3)}$
\parallel	\parallel	\parallel
$\{1, 2\}$	$\{1\}$	$\{1\}$
	$\{2\}$	$\{2\}$

Construct an IBP from i.i.d. CRPs

- 2nd customer enters buffet and decides on previous dishes
- Samples new dishes
- Keep partition if 2 forms a singleton; marked as special
- Form a feature allocation from the special blocks



Construct an IBP from i.i.d. CRPs

- 2nd customer enters buffet and decides on previous dishes
- Samples new dishes
- Keep partition if 2 forms a singleton; marked as special
- Form a feature allocation from the special blocks

$\Pi_2^{(1)}$	$\Pi_2^{(2)}$	$\Pi_2^{(3)}$
\parallel	\parallel	\parallel
$\{1, 2\}$	$\{1\}$	$\{1\}$
	$\{2\}$	$\{2\}$

$$F_2 = \left\{ \{1, 2\}, \{1\}, \{2\} \right\} \sim \text{IBP}(\gamma, \theta) \quad \text{Easy to show}$$

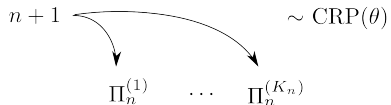
Construct an IBP from i.i.d. CRPs

- $n + 1$ -st customer enters buffet and decides on previous dishes

$$\Pi_n^{(1)} \quad \dots \quad \Pi_n^{(K_n)}$$

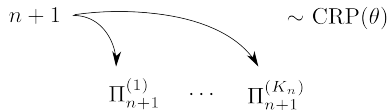
Construct an IBP from i.i.d. CRPs

- $n + 1$ -st customer enters buffet and decides on previous dishes



Construct an IBP from i.i.d. CRPs

- $n + 1$ -st customer enters buffet and decides on previous dishes



Construct an IBP from i.i.d. CRPs

- $n + 1$ -st customer enters buffet and decides on previous dishes
- Samples new dishes

$$\Pi_{n+1}^{(1)} \quad \cdots \quad \Pi_{n+1}^{(K_n)} \quad \underbrace{\Pi_{n+1} \quad \cdots \quad \Pi_{n+1}}^{\text{Poisson}(\gamma)} \quad \overset{iid}{\sim} \quad \text{CRP}(\theta)$$

Construct an IBP from i.i.d. CRPs

- $n + 1$ -st customer enters buffet and decides on previous dishes
- Samples new dishes

$$\begin{array}{ccccccc} & & & \text{Kept Singletons} & & & \\ & & & \underbrace{\hspace{10em}} & & & \\ \Pi_{n+1}^{(1)} & \cdots & \Pi_{n+1}^{(K_n)} & \{\Pi_n^{(K_n+1)}, \textcolor{red}{n+1}\} & \cdots & \{\Pi_n^{(K_{n+1})}, \textcolor{red}{n+1}\} \end{array}$$

Construct an IBP from i.i.d. CRPs

- $n + 1$ -st customer enters buffet and decides on previous dishes
- Samples new dishes
- Form a feature allocation from the special blocks

$$\begin{array}{ccccccc} \Pi_{n+1}^{(1)} & \cdots & \Pi_{n+1}^{(K_n)} & \{\Pi_n^{(K_n+1)}, \{n+1\}\} & \cdots & \{\Pi_n^{(K_{n+1})}, \{n+1\}\} \\ & \searrow & & \bullet \quad \bullet \quad \bullet & & \swarrow \\ F_{n+1} & = & \left\{ \Pi_{n+1,\star}^{(1)}, \cdots, \Pi_{n+1,\star}^{(K_{n+1})} \right\} \end{array}$$

Construct an IBP from i.i.d. CRPs

- $n + 1$ -st customer enters buffet and decides on previous dishes
- Samples new dishes
- Form a feature allocation from the special blocks

$$\begin{array}{ccccccc} \Pi_{n+1}^{(1)} & \cdots & \Pi_{n+1}^{(K_n)} & \{\Pi_n^{(K_n+1)}, \{n+1\}\} & \cdots & \{\Pi_n^{(K_{n+1})}, \{n+1\}\} \\ & \searrow & & \bullet \quad \bullet \quad \bullet & & \swarrow \\ F_{n+1} = \left\{ \Pi_{n+1,\star}^{(1)}, \cdots, \Pi_{n+1,\star}^{(K_{n+1})} \right\} & \sim & \text{IBP}(\gamma, \theta) \end{array}$$

Posterior Inference

- ▶ Maintain random partitions as latent variables
 - ▶ Gibbs sample assignments to $\Pi_{n+1}^{(k)}$ from posterior distribution
 - ▶ \Rightarrow resamples assignments to F_{n+1}

$$\begin{array}{ccccccc} \Pi_{n+1}^{(1)} & \cdots & \Pi_{n+1}^{(K_n)} & \{\Pi_n^{(K_n+1)}, \{n+1\}\} & \cdots & \{\Pi_n^{(K_{n+1})}, \{n+1\}\} \\ & & \searrow & \bullet \quad \bullet \quad \bullet & \nearrow & \\ F_{n+1} & = & \left\{ \Pi_{n+1,\star}^{(1)}, \cdots, \Pi_{n+1,\star}^{(K_{n+1})} \right\} \end{array}$$

Posterior Inference

$\Pi_5^{(1)}$	$\Pi_5^{(2)}$	$\Pi_5^{(3)}$
\parallel	\parallel	\parallel
$\{1, 3, 5\}$	$\{1, 2\}$	$\{1, 4\}$
$\{2, 4\}$	$\{3\}$	$\{2, 5\}$
	$\{4, 5\}$	$\{3\}$

$$F_5 = \left\{ \{1, 3, 5\} \quad , \quad \{1, 2\} \quad , \quad \{3\} \quad \right\}$$

Posterior Inference

- Resample assignment of 3 in $\Pi_5^{(k)} \Rightarrow$ Resample allocation in F_5

$\Pi_5^{(1)}$	$\Pi_5^{(2)}$	$\Pi_5^{(3)}$
$\{1, 3, 5\}$	$\{1, 2\}$	$\{1, 4\}$
$\{2, 4\}$	$\{3\}$	$\{2, 5\}$
	$\{4, 5\}$	$\{3\}$

$$F_5 = \left\{ \{1, 3, 5\} \quad , \quad \{1, 2\} \quad , \quad \{3\} \quad \right\}$$

Posterior Inference

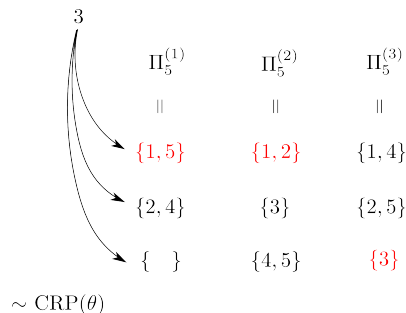
- Resample assignment of 3 in $\Pi_5^{(k)} \Rightarrow$ Resample allocation in F_5

$\Pi_5^{(1)}$	$\Pi_5^{(2)}$	$\Pi_5^{(3)}$	Remove 3 from $\Pi_5^{(1)}$
{1, 5}	{1, 2}	{1, 4}	
{2, 4}	{3}	{2, 5}	
	{4, 5}	{3}	

$$F_5 = \left\{ \{1, 5\} \quad , \quad \{1, 2\} \quad , \quad \{3\} \quad \right\}$$

Posterior Inference

- Resample assignment of 3 in $\Pi_5^{(k)} \Rightarrow$ Resample allocation in F_5

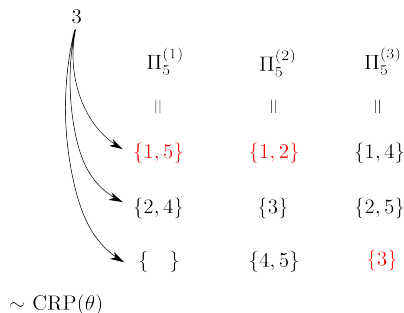


Resample 3 in $\Pi_5^{(1)}$
from posterior

$$F_5 = \left\{ \{1, 5\}, \{1, 2\}, \{3\} \right\}$$

Posterior Inference

- Resample assignment of 3 in $\Pi_5^{(k)} \Rightarrow$ Resample allocation in F_5



Resample 3 in $\Pi_5^{(1)}$
from posterior

Typically only 2
likelihood terms!

$$F_5 = \left\{ \{1, 5\}, \{1, 2\}, \{3\} \right\}$$

Posterior Inference

- Resample assignment of 3 in $\Pi_5^{(k)} \Rightarrow$ Resample allocation in F_5

$\Pi_5^{(1)}$	$\Pi_5^{(2)}$	$\Pi_5^{(3)}$
{1, 5}	{1, 2}	{1, 4}
{2, 4}	{3}	{2, 5}
{3}	{4, 5}	{3}

$$F_5 = \left\{ \{1, 5\} \quad , \quad \{1, 2\} \quad , \quad \{3\} \quad \right\}$$

Posterior Inference

- Resample assignment of 3 in $\Pi_5^{(k)} \Rightarrow$ Resample allocation in F_5
- Resample singletons

$\Pi_5^{(1)}$	$\Pi_5^{(2)}$	$\Pi_5^{(3)}$
{1, 5}	{1, 2}	{1, 4}
{2, 4}	{3}	{2, 5}
{3}	{4, 5}	{3}

$$F_5 = \left\{ \{1, 5\} \quad , \quad \{1, 2\} \quad , \quad \{3\} \quad \right\}$$

Posterior Inference

- Resample assignment of 3 in $\Pi_5^{(k)} \Rightarrow$ Resample allocation in F_5
- Resample singletons

$\Pi_5^{(1)}$	$\Pi_5^{(2)}$	$\Pi_5^{(3)}$
\parallel	\parallel	\parallel
$\{1, 5\}$	$\{1, 2\}$	$\{1, 4\}$
$\{2, 4\}$	$\{3\}$	$\{2, 5\}$
$\{3\}$	$\{4, 5\}$	$\{3\}$

Propose removing
singletons

$$F_5 = \left\{ \{1, 5\}, \{1, 2\}, \{3\} \right\}$$

Posterior Inference

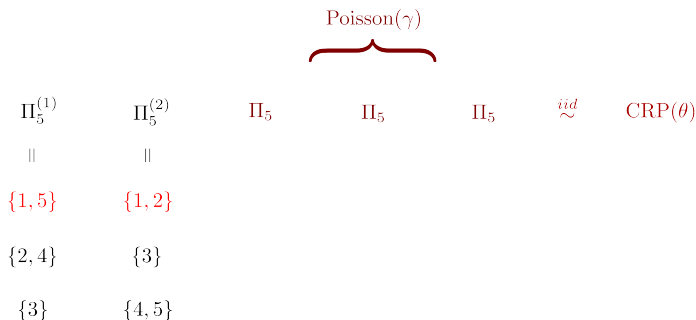
- Resample assignment of 3 in $\Pi_5^{(k)} \Rightarrow$ Resample allocation in F_5
- Resample singletons

$\Pi_5^{(1)}$	$\Pi_5^{(2)}$
\parallel	\parallel
$\{1, 5\}$	$\{1, 2\}$
$\{2, 4\}$	$\{3\}$
$\{3\}$	$\{4, 5\}$

$$F_5 = \left\{ \{1, 5\}, \{1, 2\} \right\}$$

Posterior Inference

- Resample assignment of 3 in $\Pi_5^{(k)} \Rightarrow$ Resample allocation in F_5
- Resample singletons



$$F_5 = \left\{ \{1, 5\}, \{1, 2\} \right\}$$

Posterior Inference

- Resample assignment of 3 in $\Pi_5^{(k)} \Rightarrow$ Resample allocation in F_5
- Resample singletons

Poisson(γ)						
$\Pi_5^{(1)}$	$\Pi_5^{(2)}$	Π_5	Π_5	Π_5	$\stackrel{iid}{\sim}$	CRP(θ)
\parallel	\parallel	\parallel	\parallel	\parallel		
$\{1, 5\}$	$\{1, 2\}$	$\{1, 2, 3, 5\}$	$\{1\}$	$\{1, 2\}$		
$\{2, 4\}$	$\{3\}$	$\{4\}$	$\{2\}$	$\{3\}$		
$\{3\}$	$\{4, 5\}$		$\{3\}$	$\{4, 5\}$		
			$\{4, 5\}$			

$$F_5 = \left\{ \{1, 5\}, \{1, 2\} \right\}$$

Posterior Inference

- Resample assignment of 3 in $\Pi_5^{(k)} \Rightarrow$ Resample allocation in F_5
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Poisson(γ)						
$\Pi_5^{(1)}$	$\Pi_5^{(2)}$	Π_5	Π_5	Π_5	$\overset{iid}{\sim}$	CRP(θ)
{1, 5}	{1, 2}	{1, 2, 3, 5}	{1}	{1, 2}		
{2, 4}	{3}	{4}	{2}	{3}		
{3}	{4, 5}		{3}	{4, 5}		
			{4, 5}			

$$F_5 = \left\{ \{1, 5\}, \{1, 2\} \right\}$$

Posterior Inference

- Resample assignment of 3 in $\Pi_5^{(k)} \Rightarrow$ Resample allocation in F_5
- Resample singletons

$\Pi_5^{(1)}$	$\Pi_5^{(2)}$	$\Pi_5^{(3)}$	$\Pi_5^{(4)}$
$\{1, 5\}$	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$
$\{2, 4\}$	$\{3\}$	$\{2\}$	$\{3\}$
$\{3\}$	$\{4, 5\}$	$\{3\}$	$\{4, 5\}$
		$\{4, 5\}$	

$$F_5 = \left\{ \{1, 5\}, \{1, 2\} \right\}$$

Posterior Inference

- Resample assignment of 3 in $\Pi_5^{(k)} \Rightarrow$ Resample allocation in F_5
- Resample singletons

$\Pi_5^{(1)}$	$\Pi_5^{(2)}$	$\Pi_5^{(3)}$	$\Pi_5^{(4)}$
{1, 5}	{1, 2}	{1}	{1, 2}
{2, 4}	{3}	{2}	{3}
{3}	{4, 5}	{3}	{4, 5}
		{4, 5}	

$$F_5 = \left\{ \{1, 5\}, \{1, 2\}, \textcolor{red}{\{3\}}, \textcolor{red}{\{3\}} \right\}$$

Posterior Inference

- Resample assignment of 3 in $\Pi_5^{(k)} \Rightarrow$ Resample allocation in F_5
- Resample singletons

$\Pi_5^{(1)}$	$\Pi_5^{(2)}$	$\Pi_5^{(3)}$	$\Pi_5^{(4)}$
$\{1, 5\}$	$\{1, 2\}$	$\{1\}$	$\{1, 2\}$
$\{2, 4\}$	$\{3\}$	$\{2\}$	$\{3\}$
$\{3\}$	$\{4, 5\}$	$\{3\}$	$\{4, 5\}$
		$\{4, 5\}$	

$$F_5 = \left\{ \{1, 5\}, \{1, 2\}, \{3\}, \{3\} \right\} \quad \begin{array}{c} \text{Accept/Reject with} \\ \text{MH proposal} \end{array}$$

Posterior Inference

Key Points:

- ▶ Same complexity as typical Gibbs sampling for the IBP
- ▶ Only requires a posterior inference procedure for the CRP

Exchangeable partitions (more generally)

A random partition $\Pi_n = \{A_1, \dots, A_{B_n}\}$ of $[n]$ is exchangeable iff

$$\mathbb{P}\{B_n = k, |A_1| = n_1, \dots, |A_k| = n_k\} = f_\Pi(n_1, \dots, n_k)$$

for some symmetric function f_Π , called the exchangeable partition probability function, or **EPPF**.

The **EPPF** f_Π characterizes the distribution of Π_n .

Exchangeable partitions (more generally)

To sample f_{Π} -CRP:

Customer sits at table j with probability

$$\begin{cases} \frac{f_{\Pi}(N_1, \dots, N_j+1, \dots, N_k)}{f_{\Pi}(N_1, \dots, N_k)}, & \text{if } j \text{ old,} \\ \frac{f_{\Pi}(N_1, \dots, N_k, 1)}{f_{\Pi}(N_1, \dots, N_k)}, & \text{if } j \text{ new.} \end{cases}$$

N_j is number assigned to j -th block

k is number of blocks

Generalized IBPs

- First customer samples $\text{Poisson}(\gamma)$ i.i.d. CRP partitions

Generalized IBPs

- First customer samples $\text{Poisson}(\gamma)$ i.i.d. CRP f_{Π} partitions

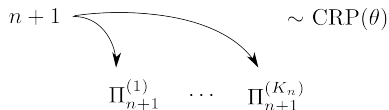
Generalized IBPs

- First customer samples $\text{Poisson}(\gamma)$ i.i.d. CRP f_{Π} partitions
- $n + 1$ -st customer samples previous dishes

$$\Pi_n^{(1)} \quad \dots \quad \Pi_n^{(K_n)}$$

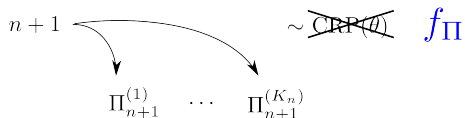
Generalized IBPs

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Generalized IBPs

- First customer samples $\text{Poisson}(\gamma)$ i.i.d. ~~CRP~~ f_{Π} partitions
- $n + 1$ -st customer samples previous dishes



Generalized IBPs

- First customer samples $\text{Poisson}(\gamma)$ i.i.d. CRP f_{Π} partitions
- $n + 1$ -st customer samples previous dishes
- Samples new dishes

$$\Pi_{n+1}^{(1)} \quad \cdots \quad \Pi_{n+1}^{(K_n)} \quad \underbrace{\Pi_{n+1} \quad \cdots \quad \Pi_{n+1}}_{\text{Poisson}(\gamma)} \quad \overset{iid}{\sim} \quad \text{CRP}(\theta)$$

Generalized IBPs

- First customer samples $\text{Poisson}(\gamma)$ i.i.d. CRP f_Π partitions
- $n + 1$ -st customer samples previous dishes
- Samples new dishes

$$\begin{array}{ccccccc}
 & & & \text{Poisson}(\gamma) & & & \\
 & & & \underbrace{\hspace{2cm}} & & & \\
 \Pi_{n+1}^{(1)} & \cdots & \Pi_{n+1}^{(K_n)} & \Pi_{n+1} & \cdots & \Pi_{n+1} & \overset{iid}{\sim} \cancel{\text{CRP}(\theta)} \\
 & & & & & & f_\Pi
 \end{array}$$

Generalized IBPs

- First customer samples $\text{Poisson}(\gamma)$ i.i.d. CRP f_Π partitions
- $n + 1$ -st customer samples previous dishes
- Samples new dishes

$$\begin{array}{ccccccc}
 & & & \text{Kept Singletons} & & & \\
 & & & \underbrace{\hspace{10em}} & & & \\
 \Pi_{n+1}^{(1)} & \cdots & \Pi_{n+1}^{(K_n)} & \{\Pi_n^{(K_n+1)}, \{n+1\}\} & \cdots & \{\Pi_n^{(K_n+1)}, \{n+1\}\}
 \end{array}$$

Generalized IBPs

- First customer samples $\text{Poisson}(\gamma)$ i.i.d. CRP f_Π partitions
- $n + 1$ -st customer samples previous dishes
- Samples new dishes

$$\begin{array}{ccccccc}
 \Pi_{n+1}^{(1)} & \cdots & \Pi_{n+1}^{(K_n)} & \{\Pi_n^{(K_{n+1})}, \{n+1\}\} & \cdots & \{\Pi_n^{(K_{n+1})}, \{n+1\}\} \\
 & \searrow & & \bullet & \bullet & \bullet & \nearrow \\
 F_{n+1} & = & \left\{ \Pi_{n+1,\star}^{(1)}, \cdots, \Pi_{n+1,\star}^{(K_{n+1})} \right\}
 \end{array}$$

Generalized IBPs

- First customer samples $\text{Poisson}(\gamma)$ i.i.d. CRP f_{Π} partitions
- $n + 1$ -st customer samples previous dishes
- Samples new dishes

$$\begin{array}{c}
 \Pi_{n+1}^{(1)} \quad \cdots \quad \Pi_{n+1}^{(K_n)} \quad \{\Pi_n^{(K_{n+1})}, \{n+1\}\} \quad \cdots \quad \{\Pi_n^{(K_{n+1})}, \{n+1\}\} \\
 \searrow \qquad \qquad \qquad \bullet \quad \bullet \quad \bullet \qquad \qquad \qquad \swarrow \\
 F_{n+1} = \left\{ \Pi_{n+1,\star}^{(1)}, \quad \cdots, \quad \Pi_{n+1,\star}^{(K_{n+1})} \right\} \sim \text{IBP}(\gamma, \theta)
 \end{array}$$

Generalized IBPs

- First customer samples $\text{Poisson}(\gamma)$ i.i.d. CRP f_Π partitions
- $n + 1$ -st customer samples previous dishes
- Samples new dishes

$$\begin{array}{ccccccc}
\Pi_{n+1}^{(1)} & \cdots & \Pi_{n+1}^{(K_n)} & \{\Pi_n^{(K_{n+1})}, \{n+1\}\} & \cdots & \{\Pi_n^{(K_{n+1})}, \{n+1\}\} \\
\searrow & & & \bullet & \bullet & \bullet & \nearrow \\
F_{n+1} = \left\{ \Pi_{n+1,\star}^{(1)}, \dots, \Pi_{n+1,\star}^{(K_{n+1})} \right\} & \sim & \cancel{\text{IBP}(\gamma, \theta)}
\end{array}$$

Generalized IBP; Roy (2014) [9]

Generalized IBPs

Generalized IBPs are parameterized by an EPPF

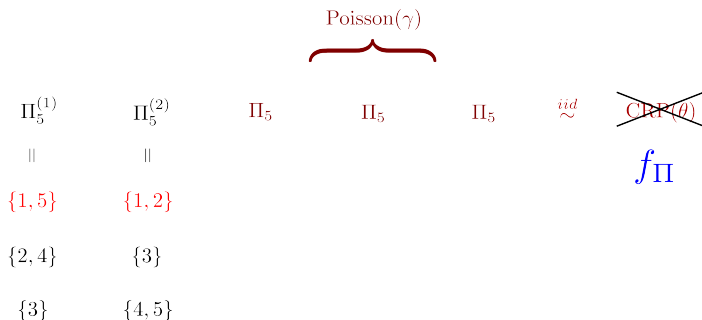
<u>EPPF</u>		<u>Feature allocation model</u>
Dirichlet process	→	IBP (G&G, 2006; GG&S, 2007) [6, 4]
Pitman–Yor process	→	stable IBP (T&G, 2009; BJ&P, 2012) [11, 1]
Gibbs-type prior (G&P, 2006) [5]	→	Gibbs-type IBP (H&R, 2015) [7]

Inference for an f_{Π} -IBP

Inference algorithm looks the same.

Inference for an f_{Π} -IBP

Inference algorithm looks the same.



$$F_5 = \left\{ \{1, 5\}, \{1, 2\} \right\}$$

Probability of a new species

EPPF	# singletons/new features
CRP(θ) (Dirichlet process)	$\text{Poisson}\left(\gamma \times \frac{\theta}{\theta+n}\right)$
CRP(θ, α) (Pitman–Yor process)	$\text{Poisson}\left(\gamma \times \frac{\Gamma(\theta+1)\Gamma(\theta+\alpha+n)}{\Gamma(\theta+n+1)\Gamma(\theta+\alpha)}\right)$
f_{Π}	$\text{Poisson}\left(\gamma \times \mathbb{E}\left[\frac{f_{\Pi}(N_1, \dots, N_k, 1)}{f_{\Pi}(N_1, \dots, N_k)}\right]\right)$

singletons/features is determined by probability of a new species

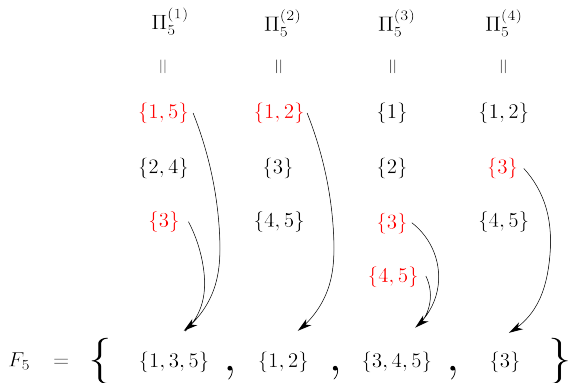
$$\frac{f_{\Pi}(N_1, \dots, N_k, 1)}{f_{\Pi}(N_1, \dots, N_k)}$$

Generalizations: Hierarchical beta processes

- Feature allocations induced by hierarchical beta processes
(T&J, 2009) [12]

Generalizations: Hierarchical beta processes

- Feature allocations induced by hierarchical beta processes (T&J, 2009) [12]
- Multiple blocks may be special (Roy, 2015) [9]



Generalizations: Dynamic IBPs

Replace with a Markov process of partitions (Fragmentation and Coagulation processes (Teh et al., 2010) [10]).

Generalizations: Dynamic IBPs

Replace with a Markov process of partitions (Fragmentation and Coagulation processes (Teh et al., 2010) [10]).

Produces a Markov process of feature allocations (current work with D. Roy and Z. Ghahramani).

$$\begin{array}{ccc} \Pi_n^{(1)}(t) & \dots & \Pi_n^{(K_n)}(t) \\ \downarrow & & \downarrow \\ F_n(t) = \left\{ \Pi_{n,\star}^{(1)}(t), \dots, \Pi_{n,\star}^{(K_n)}(t) \right\} & \sim & \text{IBP}(\gamma, f_\Pi) \end{array}$$

Generalizations: Multisets

- The urn scheme intuition extends to multisets (see my talk from BNP9).
- Negative binomial processes (Broderick et al., 2014) [2]; (Zhou et al., 2014) [13]; (H& R, 2015) [8]
- Partition perspective results in generalizations and inference procedures.

Summary

1. Construct feature allocations (FAs) from random partitions
2. Different random partitions result in different classes of FAs
3. Posterior inference on the partitions results in posterior inference in the FAs

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