Final Project

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1 Introduction

Nearly all electronic devices have printed circuit boards (PCBs) as their foundation. A PCB consists of alternating layers of insulating material and etched metal. The etched metal consists of a multitude of "traces." These traces, which are analogous to metal wires, provide electric current paths that connect various components (resistors, inductors, capacitors, connectors, silicon chips, etc.) together. PCBs are commonly used because they are small, compact, and can easily be mass-manufactured.

Most circuits are digital circuits, which are synchronized to clock cycles. Clock cycles dictate how fast a digital circuit performs. Designers prefer higher circuit operating frequencies, because it means that more clock cycles can fit in a given time interval; in other words, the circuit can become 'faster' if the operating frequency is increased. Unfortunately, at higher frequencies, non-ideal circuit effects begin to appear.

Compared to most other physical systems, electronic circuits are incredibly ideal: they only contain two dimensions, amplitude and time. Components such as resistors, capacitors and inductors have well-defined properties and can be modeled easily. The circuit model of a given component can be derived by taking the first few terms of the Taylor Series approximation of the time-harmonic electromagnetic field in the component. As frequency increases, the circuit model begins to lose accuracy; additional terms in the Taylor Series become necessary.

The wave equation is a useful PDE that can be used to model all sorts of physical phenomena, ranging from water waves to violin strings to light and sound. In electromagnetics, the first, second, and third dimensional wave equations all have practical applications. This paper will focus on the two-dimensional wave equation, which can be used to model the non-ideal effects of a via in a PCB environment.

Nearly all modern electronic devices have printed circuit boards (PCBs) as their foundation. A PCB consists of alternating layers of insulating material and etched metal. The etched metal contains a multitude of "traces." These traces, which are analogous to metal wires, provide electric current paths that connect various components (resistors, inductors, capacitors, connectors, silicon chips, etc.) together. PCBs are commonly used because they are small, compact, and easily mass-manufactured.

A via is a vertical, hollow metal cylinder that connects two metal layers or traces together in a PCB. At high frequencies, electric currents traveling through a via induce electric and magnetic fields in the region between the metal layers. These induced fields introduce non-ideal noise into the PCB, which can sometimes cause the circuit to fail. PCB designers need ways to easily determine what kinds of impact the induced fields caused by vias will have on their circuits, so that they can determine effective countermeasures for them. One way to figure this out is to simulate the circuit using a full-wave electromagnetic computer simulator. However, this method is computationally expensive, especially if the PCB is more than a few layers thick. A preferred alternative would be some sort of analytical solution for the expected field distribution.

Incidentally, the induced fields caused by a via can be modeled by the two-dimensional wave equation. Fortunately, with some simplifications, this wave equation and its boundary conditions (imposed by the PCB environment) has an actual, nontrivial solution that can be found using the Green's Functions method.

1.1 Model

Figure 1 (below) shows a picture of the physical model used in [1]. Figure 2 is a picture of the physical model that I solved for in this paper. While the disk problem that I solved for is a little less physical in terms of the boundary conditions (nearly all PCBs are rectangles or squares, not circles), it provides

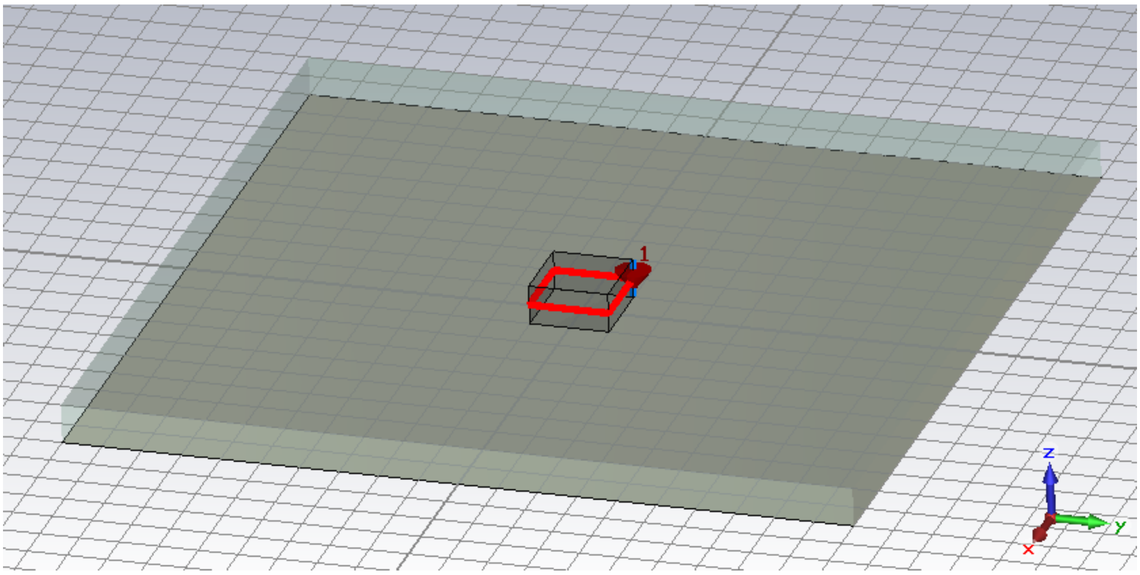

Figure 1: A dielectric square with side length 2a, with a source (representing a via) in the center. Metal planes sandwich the dielectric on the top and bottom.

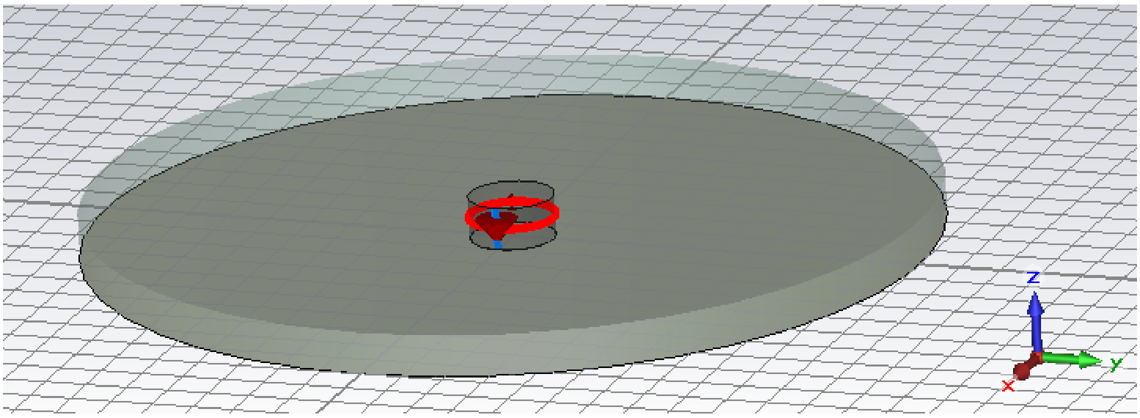

Figure 2: A dielectric disk with radius a, with a source (representing a via) in the center. Metal planes sandwich the dielectric on the top and bottom.

an accurate model for a round via. It also resulted in an interesting solution that was in terms of Bessel functions instead of the usual sines and cosines.

2 The PDE for the model in Fig. 2 and its boundary conditions

Electromagnetics are defined by Maxwell's Equations, three of which are the following:

$$\nabla \times \underline{E} = -\frac{\partial}{\partial t}(\mu \underline{H}) \tag{1}$$

$$\nabla \times \underline{H} = \frac{\partial}{\partial t}(\epsilon \underline{E}) + \underline{J} \tag{2}$$

$$\nabla \cdot \underline{E} = \frac{\rho_s}{\epsilon} = 0 \ \text{ (Sourceless Region)} \tag{3}$$

The underline notation describes vector fields. μ and ϵ are physical properties, and are usually considered to be constant/isotropic. $\underline{J}$ represents current density; it is nonzero in the center, at the location of the via. ρ_s is electric charge density (typically as a function of position), and is zero for this problem because the region of interest is a dielectric that contains no charge.

Assume $\underline{E}$, $\underline{H}$, and $\underline{J}$ are C^2 in time and space. It is possible to obtain the wave equation directly from Maxwell's Equations. Begin by taking the curl of (1):

$$\nabla \times \nabla \times \underline{E} = \nabla \times \left(-\frac{\partial}{\partial t}(\mu \underline{H})\right)$$

Due to continuity assumptions on the vector fields, the curl and the time derivative are interchangeable. Also, note that μ is assumed constant, so it can pass through any differential operator.

$$\begin{aligned}
\nabla \times \nabla \times \underline{E} &= -\frac{\partial}{\partial t}(\nabla \times \mu \underline{H}) = -\frac{\partial}{\partial t}(\mu \nabla \times \underline{H}) \\
&= -\frac{\partial}{\partial t}\left(\mu \frac{\partial}{\partial t}(\epsilon \underline{E}) + \underline{J}\right) \quad \text{(Substituted in (2))} \\
&= -\mu\epsilon \frac{\partial^2}{\partial t^2}(\underline{E}) - \mu \frac{\partial}{\partial t}(\underline{J}) \\
\nabla \times \nabla \times \underline{E} &= \nabla(\nabla \cdot \underline{E}) - \Delta \underline{E} \quad \text{(Vector Identity)} \\
\nabla(\nabla \cdot \underline{E}) - \Delta \underline{E} &= -\mu\epsilon \frac{\partial^2}{\partial t^2}(\underline{E}) - \mu \frac{\partial}{\partial t}(\underline{J}) \\
\nabla(0) - \Delta \underline{E} &= -\mu\epsilon \frac{\partial^2}{\partial t^2}(\underline{E}) - \mu \frac{\partial}{\partial t}(\underline{J}) \quad \text{(Substituted (3))}
\end{aligned}$$

Rearranging things and getting rid of the gradient of zero:

$$\Delta \underline{E} - \mu\epsilon \frac{\partial^2}{\partial t^2}(\underline{E}) = \mu \frac{\partial}{\partial t}(\underline{J}) \tag{4}$$

(4) is the non-homogeneous wave equation with forcing function $\mu \frac{\partial}{\partial t}(\underline{J})$. Something commonly done (at least in my Emag classes) is to move to the "frequency domain" by assuming that E and J are time-harmonic in the following sense:

$$\begin{aligned}
\underline{E} &= E(\mathbf{x}) \cos(\omega t + \theta_E) \\
\underline{J} &= J(\mathbf{x}) \cos(\omega t + \theta_J)
\end{aligned}$$

Additionally, the phase differences can be accounted for by switching to complex numbers, and absorbing the phase into the front term. For example, looking at E:

$$\begin{aligned}
\underline{E} &= E(\mathbf{x}) \cos(\omega t + \theta_E) \\
&= Re\{E(\mathbf{x}) e^{j(\omega t + \theta_E)}\} \\
&= Re\{E(\mathbf{x}) e^{j\theta_E} e^{j\omega t}\} \\
&= Re\{\underline{\widetilde{E}}(\mathbf{x}) e^{j\omega t}\}
\end{aligned}$$

j represents the unit imaginary number, $\sqrt{-1}$.
By working with $\underline{\widetilde{E}}(\mathbf{x})$ instead of $\underline{E}(\mathbf{x}, t)$, (4) can be simplified into a simpler, non-time-varying form:

$$\begin{aligned}
\Delta \underline{E} - \mu\epsilon \frac{\partial^2}{\partial t^2}(\underline{E}) &= \mu \frac{\partial}{\partial t}(\underline{J}) \\
\Delta \left(Re\{\underline{\widetilde{E}}(\mathbf{x}) e^{j\omega t}\}\right) - \mu\epsilon \frac{\partial^2}{\partial t^2}\left(Re\{\underline{\widetilde{E}}(\mathbf{x}) e^{j\omega t}\}\right) &= \mu \frac{\partial}{\partial t}\left(Re\{\underline{\widetilde{J}}(\mathbf{x}) e^{j\omega t}\}\right) \\
e^{j\omega t} \Delta \underline{\widetilde{E}} - \mu\epsilon \underline{\widetilde{E}} \frac{\partial^2}{\partial t^2}(e^{j\omega t}) &= \mu \underline{\widetilde{J}} \frac{\partial}{\partial t}(e^{j\omega t}) \\
e^{j\omega t} \Delta \underline{\widetilde{E}} - \mu\epsilon \underline{\widetilde{E}} (j\omega)^2 (e^{j\omega t}) &= \mu \underline{\widetilde{J}} (j\omega)(e^{j\omega t}) \\
\Delta \underline{\widetilde{E}} - \mu\epsilon \underline{\widetilde{E}} (j\omega)^2) &= \mu \underline{\widetilde{J}} (j\omega) \\
\Delta \underline{\widetilde{E}} + \omega^2 \mu\epsilon \underline{\widetilde{E}} &= j\omega\mu \underline{\widetilde{J}}
\end{aligned} \tag{5}$$

And for simplicity:

$$k^2 = \omega\sqrt{\mu\epsilon}$$

Which results in the nonhomogeneous Helmholtz equation:

$$\Delta \underline{\widetilde{E}} + k^2 \underline{\widetilde{E}} = j\omega\mu \underline{\widetilde{J}} \tag{6}$$

This is the equation that will be examined in the following sections.

2.1 Boundary Conditions on Edges of Disk and Simplifications that Result

The region under consideration is

$$D = \{(x, y) : x^2 + y^2 < a\}, \quad a \in \Re$$

$$\partial D = \{(x, y) : x^2 + y^2 = a\}$$

Certain boundary conditions derived from Maxwell's Equations (which won't be derived here) dictate that there is no tangential (x-y) component of the electric field on the top and bottom surfaces of the dielectric, where it is touching metal.
If the thickness of the dielectric is thin enough, then the E field can be assumed to be z-invariant, and then because of the boundary conditions on the top and bottom of the dielectric disk, the assumption

$$\underline{\widetilde{E}} = \underline{\widetilde{E}}(x, y) = \widetilde{E}_z(x, y) \cdot u_z$$

can be made throughout the disk. (u_z is the unit vector in the z direction). Assume the same thing for $\underline{\widetilde{J}}$ as well.

With these simplifications, the vectors in (6) get cut down to scalars:

$$\Delta\widetilde{E}_z + k^2\widetilde{E}_z = j\omega\mu\widetilde{J}_z \tag{7}$$

And then, for simpler notation, let

$$\widetilde{F} = j\omega\mu\widetilde{J}_z$$

Then,

$$\Delta\widetilde{E}_z(x, y) + k^2\widetilde{E}_z(x, y) = \widetilde{F}(x, y) \tag{8}$$

Another "boundary" condition is that the E-field doesn't diverge at the center of the disk. This will be important to note when I switch to polar coordinates later.

$$\widetilde{E}_z(0, 0) < \infty \tag{9}$$

Finally, there is an 'open' boundary condition on the outer edge of the disk. This is kind of similar to having insulating boundary equations to the heat equation. They result in the Neumann boundary condition:

$$\frac{\partial}{\partial n}\widetilde{E}_z(x, y) = 0 \tag{10}$$

With (8), (9), and (10), we have enough information to sufficiently describe our problem.

3 Green's function of the NH Wave Equation

Given a function $G(\mathbf{x}, \mathbf{x}')$ with $\mathbf{x} = (x, y)$, $\mathbf{x}' = (x', y')$ such that

$$\Delta_x G(\mathbf{x}, \mathbf{x}') + k^2 G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}')$$

$\delta(\mathbf{x} - \mathbf{x}')$ represents the Dirac Delta (distribution) in two dimensions. and, on ∂D, $G(\mathbf{x}, \mathbf{x}')$ has the same boundary conditions as $\widetilde{E}_z$:

$$\frac{\partial}{\partial n} G(\mathbf{x}, \mathbf{x}') = 0$$

Where $\frac{\partial}{\partial n}$ represents the outward normal derivative (in $\mathbf{x}$ and $\mathbf{x}'$).
It can be shown that $\widetilde{E}_z$ can be recovered from just G and $\widetilde{F}$:

$$\widetilde{E}_z(\mathbf{x}') = \iint\limits_D G(\mathbf{x}, \mathbf{x}')\widetilde{F}(\mathbf{x})\, \mathrm{d}S \tag{11}$$

3.1 Proof of (11)

For simpler notation, in this section, let $E = \widetilde{E}_z$ and $F = \widetilde{F}$

$$\Delta_x E(\mathbf{x}) + k^2 E(\mathbf{x}) = F(\mathbf{x})$$

Multiplying by $G(\mathbf{x}, \mathbf{x}')$:

$$G(\mathbf{x}, \mathbf{x}')\Delta_x E(\mathbf{x}) + k^2 G(\mathbf{x}, \mathbf{x}')E(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}')F(\mathbf{x})$$

Integrating over area $\iint\limits_D \mathrm{d}S$

$$\iint\limits_D G(\mathbf{x}, \mathbf{x}')\Delta_x E(\mathbf{x})\,\mathrm{d}S + \iint\limits_D k^2 G(\mathbf{x}, \mathbf{x}')E(\mathbf{x})\,\mathrm{d}S = \iint\limits_D G(\mathbf{x}, \mathbf{x}')F(\mathbf{x})\,\mathrm{d}S$$

By Green's Second Identity in Two Dimensions:

$$\iint\limits_D u\Delta_x v\,\mathrm{d}S - \iint\limits_D v\Delta_x u\,\mathrm{d}S = \int\limits_{\partial D} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,\mathrm{d}\ell$$

Using functions of interest:

$$\iint\limits_D G(\mathbf{x}, \mathbf{x}')\Delta_x E(\mathbf{x})\,\mathrm{d}S - \iint\limits_D E(\mathbf{x})\Delta_x G(\mathbf{x}, \mathbf{x}')\,\mathrm{d}S = \int\limits_{\partial D} G(\mathbf{x}, \mathbf{x}')\frac{\partial E(\mathbf{x})}{\partial n} - E(\mathbf{x})\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n}\,\mathrm{d}\ell$$

On the boundary, the outward normal derivatives for both $G(\mathbf{x}, \mathbf{x}')$ and $E(\mathbf{x})$ are zero, so the entire right side is zero, leading to the equality:

$$\iint\limits_D G(\mathbf{x}, \mathbf{x}')\Delta_x E(\mathbf{x})\,\mathrm{d}S = \iint\limits_D E(\mathbf{x})\Delta_x G(\mathbf{x}, \mathbf{x}')\,\mathrm{d}S$$

The Laplacian now operates on G instead of E.
Plugging this identity back into the original triple integral:

$$\iint\limits_D E(\mathbf{x})\Delta_x G(\mathbf{x}, \mathbf{x}')\,\mathrm{d}S + \iint\limits_D k^2 G(\mathbf{x}, \mathbf{x}')E(\mathbf{x})\,\mathrm{d}S = \iint\limits_D G(\mathbf{x}, \mathbf{x}')F(\mathbf{x})\,\mathrm{d}S$$

The Laplacian of G is known from the initial assumption of G:

$$\begin{aligned}
\iint\limits_D E(\mathbf{x})(\delta(\mathbf{x} - \mathbf{x}') - k^2 G(\mathbf{x}, \mathbf{x}'))\,\mathrm{d}S + \iint\limits_D k^2 G(\mathbf{x}, \mathbf{x}')E(\mathbf{x})\,\mathrm{d}S &= \iint\limits_D G(\mathbf{x}, \mathbf{x}')F(\mathbf{x})\,\mathrm{d}S \\
\iint\limits_D E(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}')\,\mathrm{d}S - \iint\limits_D E(\mathbf{x})k^2 G(\mathbf{x}, \mathbf{x}')\,\mathrm{d}S + \iint\limits_D k^2 G(\mathbf{x}, \mathbf{x}')E(\mathbf{x})\,\mathrm{d}S &= \iint\limits_D G(\mathbf{x}, \mathbf{x}')F(\mathbf{x})\,\mathrm{d}S \\
\iint\limits_D E(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}')\,\mathrm{d}S &= \iint\limits_D G(\mathbf{x}, \mathbf{x}')F(\mathbf{x})\,\mathrm{d}S
\end{aligned}$$

The Dirac Delta allows one to jump from the double integral to just E in the final step:

$$E(\mathbf{x}') = \iint\limits_D G(\mathbf{x}, \mathbf{x}')F(\mathbf{x})\,\mathrm{d}S$$

Now we know that E can be found in terms of the function G and the forcing function F. All that is left to do is to find an expression for G.

4 Finding the Green's function

4.1 In terms of eigenfunctions

If you assume that G is an infinite sum of eigenfunctions:

$$G(\mathbf{x}, \mathbf{x}') = \sum_m \sum_n a_{nm}(\mathbf{x}') \cdot \psi_{nm}(\mathbf{x})$$

With the $\psi_{nm}(\mathbf{x})$ functions as solutions to

$$\Delta \psi_{nm}(\mathbf{x}) + \gamma^2 \psi_{nm}(\mathbf{x}) = 0$$

Where γ is a real constant, which is dependent on m and n.
Then you can show that

$$a_{nm}(\mathbf{x}) = \frac{\psi_{nm}(\mathbf{x}')}{(k^2 - \gamma^2)||\psi_{nm}||^2}$$

So in all,

$$G(\mathbf{x}, \mathbf{x}') = \sum_m \sum_n \frac{\psi_{nm}(\mathbf{x}')\psi_{nm}(\mathbf{x})}{(k^2 - \gamma^2)||\psi_{nm}||^2} \tag{12}$$

4.1.1 Setting up for proof of (12)

Beginning with

$$G(\mathbf{x}, \mathbf{x}') = \sum_m \sum_n a_{nm}(\mathbf{x}') \cdot \psi_{nm}(\mathbf{x})$$

Multiplying both sides by $\psi_{jk}(\mathbf{x})$ and integrating over S:

$$\begin{aligned}
\iint_D G(\mathbf{x}, \mathbf{x}')\psi_{jk}(\mathbf{x})\,\mathrm{d}S &= \iint_D \sum_m \sum_n a_{nm}(\mathbf{x}') \cdot \psi_{nm}(\mathbf{x})\psi_{jk}(\mathbf{x})\,\mathrm{d}S \\
&= \sum_m \sum_n a_{nm}(\mathbf{x}') \cdot \iint_D \psi_{nm}(\mathbf{x})\psi_{jk}(\mathbf{x})\,\mathrm{d}S \qquad (13)
\end{aligned}$$

Where, in order to pass the integral through the sum, it's assumed that the sum terms don't diverge:

$$\lim_{n,m\to\infty} a_{nm}(\mathbf{x}')\psi_{nm}(\mathbf{x})\psi_{jk}(\mathbf{x}) = 0 \quad \text{on D}$$

Since $\psi_{nm}(\mathbf{x})$ are eigenfunctions, they are orthogonal with relationship

$$\iint_D \psi_{nm}(\mathbf{x})\psi_{jk}(\mathbf{x})\,\mathrm{d}S = \delta_{nm,jk} \iint_D (\psi_{nm}(\mathbf{x}))^2\,\mathrm{d}S = \delta_{nm,jk}||\psi_{nm}||^2$$

The $\delta_{nm,jk}$ represents the Kronecker Delta, which is one when $n = j$ and $m = k$, but zero otherwise.
The $||\psi_{nm}||^2$ is the square of the l^2 norm of ψ_{nm} on D.
Plugging back to (13):

$$\begin{aligned}
\iint_D G(\mathbf{x}, \mathbf{x}')\psi_{jk}(\mathbf{x})\,\mathrm{d}S &= \sum_m \sum_n a_{nm}(\mathbf{x}') \cdot \iint_D \psi_{nm}(\mathbf{x})\psi_{jk}(\mathbf{x})\,\mathrm{d}S \\
&= \sum_m \sum_n a_{nm}(\mathbf{x}') \cdot \delta_{nm,jk}||\psi_{nm}||^2
\end{aligned}$$

The Kronecker Delta selects only the jth and kth terms from the series:

$$\iint_D G(\mathbf{x}, \mathbf{x}')\psi_{jk}(\mathbf{x})\,\mathrm{d}S = a_{jk}(\mathbf{x}')||\psi_{jk}||^2 \tag{14}$$

This relationship will be useful in the next section.

4.1.2 Proof of (12)

Starting with the definitions of G and ψ_{nm}:

$$\Delta_x G(\mathbf{x},\mathbf{x}') + k^2 G(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}') \tag{15}$$

$$\Delta\psi_{nm}(\mathbf{x}) + \gamma^2\psi_{nm}(\mathbf{x}) = 0 \tag{16}$$

Multiplying (15) by ψ_{nm} and (16) by $G(\mathbf{x},\mathbf{x}')$:

$$\psi_{nm}(\mathbf{x})\Delta_x G(\mathbf{x},\mathbf{x}') + \psi_{nm}(\mathbf{x})k^2 G(\mathbf{x},\mathbf{x}') = \psi_{nm}(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}') \tag{17}$$

$$G(\mathbf{x},\mathbf{x}')\Delta\psi_{nm}(\mathbf{x}) + G(\mathbf{x},\mathbf{x}')\gamma^2\psi_{nm}(\mathbf{x}) = 0 \tag{18}$$

Subtracting (17) and (18):

$$\psi_{nm}(\mathbf{x})\Delta_x G(\mathbf{x},\mathbf{x}') - G(\mathbf{x},\mathbf{x}')\Delta\psi_{nm}(\mathbf{x}) + \psi_{nm}(\mathbf{x})k^2 G(\mathbf{x},\mathbf{x}') - G(\mathbf{x},\mathbf{x}')\gamma^2\psi_{nm}(\mathbf{x}) = \psi_{nm}(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')$$

Integrating over area D:

$$\begin{aligned}\iint\limits_D \psi_{nm}(\mathbf{x})\Delta_x G(\mathbf{x},\mathbf{x}') - G(\mathbf{x},\mathbf{x}')\Delta\psi_{nm}(\mathbf{x})\,\mathrm{d}S + \iint\limits_D \psi_{nm}(\mathbf{x})k^2 G(\mathbf{x},\mathbf{x}') - G(\mathbf{x},\mathbf{x}')\gamma^2\psi_{nm}(\mathbf{x})\,\mathrm{d}S \\ = \iint\limits_D \psi_{nm}(\mathbf{x})\delta(\mathbf{x}-\mathbf{x}')\,\mathrm{d}S \\ = \psi_{nm}(\mathbf{x}')\end{aligned} \tag{19}$$

By Green's Second Identity:

$$\iiint\limits_D u\Delta_x v - v\Delta_x u\,\mathrm{d}\mathbf{x} = \iint\limits_{\partial D} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,\mathrm{d}S$$

Using functions of interest:

$$\iiint\limits_D \psi_{nm}(\mathbf{x})\Delta_x G(\mathbf{x},\mathbf{x}') - G(\mathbf{x},\mathbf{x}')\Delta\psi_{nm}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \iint\limits_{\partial D} \psi_{nm}(\mathbf{x})\frac{\partial G(\mathbf{x},\mathbf{x}')}{\partial n} - G(\mathbf{x},\mathbf{x}')\frac{\partial\psi_{nm}(\mathbf{x})}{\partial n}\,\mathrm{d}S$$

On the boundary, the outward normal derivatives for both $G(\mathbf{x},\mathbf{x}')$ and $\psi_{nm}(\mathbf{x})$ are zero, so the entire right side is zero, leading to the relationship:

$$\iiint\limits_D \psi_{nm}(\mathbf{x})\Delta_x G(\mathbf{x},\mathbf{x}') - G(\mathbf{x},\mathbf{x}')\Delta\psi_{nm}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$$

This simplifies the left hand side of (19) into:

$$\iint\limits_D \psi_{nm}(\mathbf{x})k^2 G(\mathbf{x},\mathbf{x}') - G(\mathbf{x},\mathbf{x}')\gamma^2\psi_{nm}(\mathbf{x})\,\mathrm{d}S = \psi_{nm}(\mathbf{x}')$$

$$\iint\limits_D \psi_{nm}(\mathbf{x})(k^2-\gamma^2)G(\mathbf{x},\mathbf{x}')\,\mathrm{d}S = \psi_{nm}(\mathbf{x}')$$

$$(k^2-\gamma^2)\iint\limits_D \psi_{nm}(\mathbf{x})G(\mathbf{x},\mathbf{x}')\,\mathrm{d}S = \psi_{nm}(\mathbf{x}')$$

Substituting in (14) for the integral, except with n and m instead of j and k:

$$(k^2-\gamma^2)a_{nm}(\mathbf{x}')||\psi_{nm}||^2 = \psi_{nm}(\mathbf{x}')$$

Solving for $a_{nm}(\mathbf{x}')$:

$$a_{nm}(\mathbf{x}') = \frac{\psi_{nm}(\mathbf{x}')}{(k^2-\gamma^2)||\psi_{nm}||^2}$$

And this directly leads to (12):

$$G(\mathbf{x},\mathbf{x}') = \sum_m\sum_n \frac{\psi_{nm}(\mathbf{x}')\psi_{nm}(\mathbf{x})}{(k^2-\gamma^2)||\psi_{nm}||^2}$$

4.2 Finding the eigenfunctions

Now all that needs to be done is to find the eigenfunctions, the solutions to the equation

$$\Delta\psi_{nm}(\mathbf{x}) + \gamma^2\psi_{nm}(\mathbf{x}) = 0$$

In polar coordinates (equations for the polar laplacian found in the Strauss text):

$$\psi_{nm}(\mathbf{x}) = \psi_{nm}(r,\phi)$$

$$\Delta\psi_{nm}(r,\phi) + \gamma^2\psi_{nm}(r,\phi) = \frac{1}{r}\frac{\partial}{\partial r}(\psi_{nm}) + \frac{\partial^2}{\partial r^2}(\psi_{nm}) + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}(\psi_{nm}) + \gamma^2\psi_{nm} = 0$$

This can be solved by separation of variables:

$$\psi_{nm} = R(r)\Phi(\phi)$$

$$\frac{1}{r}R'\Phi + R''\Phi + \frac{1}{r^2}R\Phi'' + \gamma^2 R\Phi = 0$$

$$\frac{R'}{rR} + \frac{R''}{R} + \frac{\Phi''}{r^2\Phi} + \gamma^2 = 0$$

$$\frac{rR'}{R} + \frac{r^2R''}{R} + \frac{\Phi''}{\Phi} + r^2\gamma^2 = 0$$

$$\frac{rR'}{R} + \frac{r^2R''}{R} + r^2\gamma^2 = -\frac{\Phi''}{\Phi}$$

The only way a function of r can equal a function of ϕ is if both functions equal a constant:

$$\frac{rR'}{R} + \frac{r^2R''}{R} + r^2\gamma^2 = -\frac{\Phi''}{\Phi} = \tilde{\lambda}$$

Note that all the eigenfunctions need to meet the Neumann boundary condition:

$$\frac{\partial}{\partial n}\psi_{nm}(a) = 0$$

$\frac{\partial}{\partial n}$ represents the outward normal, and we are using separation of variables, so

$$\frac{\partial}{\partial n}\psi_{nm}(a) = \frac{\partial}{\partial n}(R(a)\Phi(\phi)) = \frac{\partial}{\partial r}(R(a)\Phi(\phi)) = 0$$

Unless $\Phi(\phi) = 0$, there is a boundary condition for $R(r)$:

$$R'(a) = 0 \tag{20}$$

Also,

In addition, a "circular" boundary condition needs to be met for $\Phi(\phi)$, since ϕ is a measure of angle, not position:

$$\Phi(\phi) = \Phi(\phi + 2\pi)$$

4.2.1 Finding $\Phi(\phi)$

$\Phi(\phi)$ is a solvable ODE.

$$-\frac{\Phi''}{\Phi} = \tilde{\lambda}$$

$$\Phi'' = -\tilde{\lambda}\Phi$$

$$\Phi'' + \tilde{\lambda}\Phi = 0$$

The characteristic equation for this differential equation is:

$$\mu^2 + \tilde{\lambda} = 0$$

$$\mu = \pm\sqrt{\tilde{\lambda}}$$

Now the different possible values of λ need to be accounted for...

For $\tilde{\lambda} = \lambda > 0$:

$$\mu = \pm\sqrt{\lambda}$$

This results in exponentials,

$$\Phi(\phi) = Ae^{\lambda\phi} + Be^{-\lambda\phi}$$

Circular boundary condition on $\Phi : \Phi(\phi) = \Phi(\phi + 2\pi)$

$$Ae^{\lambda\phi} + Be^{-\lambda\phi} = Ae^{\lambda\phi+\lambda 2\pi} + Be^{-\lambda\phi-\lambda 2\pi}$$

$$Ae^{\lambda\phi} + Be^{-\lambda\phi} = (Ae^{\lambda 2\pi})e^{\lambda\phi} + (Be^{-\lambda 2\pi})e^{-\lambda\phi}$$

This equality will hold if

$$A = Ae^{\lambda 2\pi}; \quad B = Be^{-\lambda 2\pi}$$

Since strictly $\lambda > 0$ is assumed for this case, the above equalities can only hold if $A = B = 0$. Therefore the $\lambda > 0$ case leads to a trivial solution.

For $\tilde{\lambda} = 0$:

$$\mu = 0$$

This results in a polynomial,

$$\Phi(\phi) = A\phi + B$$

Again, circular boundary condition on $\Phi : \Phi(\phi) = \Phi(\phi + 2\pi)$ needs to be met.

$$\begin{aligned} A\phi + B &= A\phi + A \cdot 2\pi + B \\ 0 &= A \cdot 2\pi \\ A &= 0 \end{aligned}$$

This means that any constant function can meet the boundary condition on Φ.

$$\Phi(\phi) = B$$

For $\tilde{\lambda} = \lambda < 0$:

$$\mu = \pm i\sqrt{\lambda}$$

This results in sines and cosines.

$$\Phi(\phi) = A\cos(\sqrt{\lambda}\phi) + B\sin(\sqrt{\lambda}\phi)$$

Circular boundary condition on $\Phi : \Phi(\phi) = \Phi(\phi + 2\pi)$

$$A\cos(\sqrt{\lambda}(\phi)) + B\sin(\sqrt{\lambda}(\phi)) = A\cos(\sqrt{\lambda}(\phi + 2\pi)) + B\sin(\sqrt{\lambda}(\phi + 2\pi))$$

Using Trigonometric identities:

$$\begin{aligned} A\cos(\sqrt{\lambda}(\phi)) + B\sin(\sqrt{\lambda}(\phi)) &= A\cos(\sqrt{\lambda}(\phi + 2\pi)) + B\sin(\sqrt{\lambda}(\phi + 2\pi)) \\ A\cos(\sqrt{\lambda}(\phi)) + B\sin(\sqrt{\lambda}(\phi)) &= A\cos(\sqrt{\lambda}\phi)\cos(\sqrt{\lambda}2\pi) - A\sin(\sqrt{\lambda}\phi)\sin(\sqrt{\lambda}2\pi) \\ &\quad + B\cos(\sqrt{\lambda}\phi)\sin(\sqrt{\lambda}2\pi) + B\sin(\sqrt{\lambda}\phi)\cos(\sqrt{\lambda}2\pi) \end{aligned}$$

This can meet the boundary condition if $\sqrt{\lambda}2\pi = 2n\pi$, so $\sqrt{\lambda} = n$, where n is an integer zero or greater.

$$\begin{aligned} A\cos(n\phi) + B\sin(n\phi) &= A\cos(n\phi)\cos(2n\pi) - A\sin(n\phi)\sin(2n\pi) \\ &\quad + B\cos(n\phi)\sin(2n\pi) + B\sin(n\phi)\cos(2n\pi) \\ &= A\cos(n\phi) + B\sin(n\phi) \end{aligned}$$

So the solutions for Φ are

$$\Phi(\phi) = A\cos(n\phi) + B\sin(n\phi), \quad n = 1, 2, 3, \ldots$$

Note that the $n = 0$ case can also be encapsulated by this case.

Final solution for $\Phi(\phi)$

$$\Phi_n(\phi) = A_n \cos(n\phi) + B_n \sin(n\phi), \quad n = 0, 1, 2, 3, \ldots$$

4.2.2 Finding R(r)

$$\frac{rR'}{R} + \frac{r^2R''}{R} + r^2\gamma^2 = \lambda = n^2$$
$$r^2R'' + rR' + (r^2\gamma^2 - n^2)R = 0$$
$$r^2R_{rr} + rR_r + (r^2\gamma^2 - n^2)R = 0$$

Changing variables to $\rho = \gamma r$ so that $R_\rho = \gamma R_r$ and $R_{\rho\rho} = \gamma^2 R_{rr}$

$$r^2\gamma^2 R_{\rho\rho} + r\gamma R_\rho + (r^2\gamma^2 - n^2)R = 0$$

$$\rho^2 R_{\rho\rho} + \rho R_\rho + (\rho^2 - n^2)R = 0$$

$$R_{\rho\rho} + \frac{R_\rho}{\rho} + \left(1 - \frac{n^2}{\rho^2}\right) R = 0$$

The solutions to this equation are Bessel functions of the first order:

$$J_n(\rho) = \sum_{j=0}^{\infty} (-1)^j \frac{(\frac{1}{2}\rho)^{n+2j}}{j!(n+j)!}$$

Because this is a second order ODE, there are two linearly independent solutions. For the Bessel functions of the first order, the two solutions are:

$$R_n(\rho) = J_n(\rho) + J_{-n}(\rho)$$

However, the $J_{-n}(\rho)$ term has a singularity at $\rho = \gamma r = 0$, so it will be omitted from the solution because this goes against the boundary conditions and is un-physical; this was brought up earlier via Equation (9).

$$R_n(\rho) = J_n(\rho)$$

$$R_n(r) = J_n(\gamma r)$$

Recall the other boundary condition on R, from (20):

$$R'(a) = 0$$

This has to hold for all R_n.

$$R_n'(r) = J_n'(\gamma r)\gamma$$

$$R_n'(a) = J_n'(\gamma a)\gamma = 0$$

So either $J_n'(\gamma a) = 0$ or $\gamma = 0$. For $J_n'(\gamma a)$:

Hopefully Figure 3 is convincing enough to say that for any n, $J_n'(x) = 0$ (slope of $J_n(x)$ is zero) at an infinite amount of places, because it has this sort of damped oscillating behavior. Let τ_{nm} represent the mth place where the nth Bessel function has slope zero:

$$J_n'(\tau_{nm}) = 0$$

Going back to the boundary condition

$$R_n'(a) = J_n'(\gamma a)\gamma = J_n'(\tau_{nm}) = 0$$

For $\gamma \neq 0$

$$J_n'(\gamma a) = J_n'(\tau_{nm})$$

$$\gamma a = \tau_{nm}$$

$$\gamma = \gamma_{nm} = \frac{\tau_{nm}}{a}$$

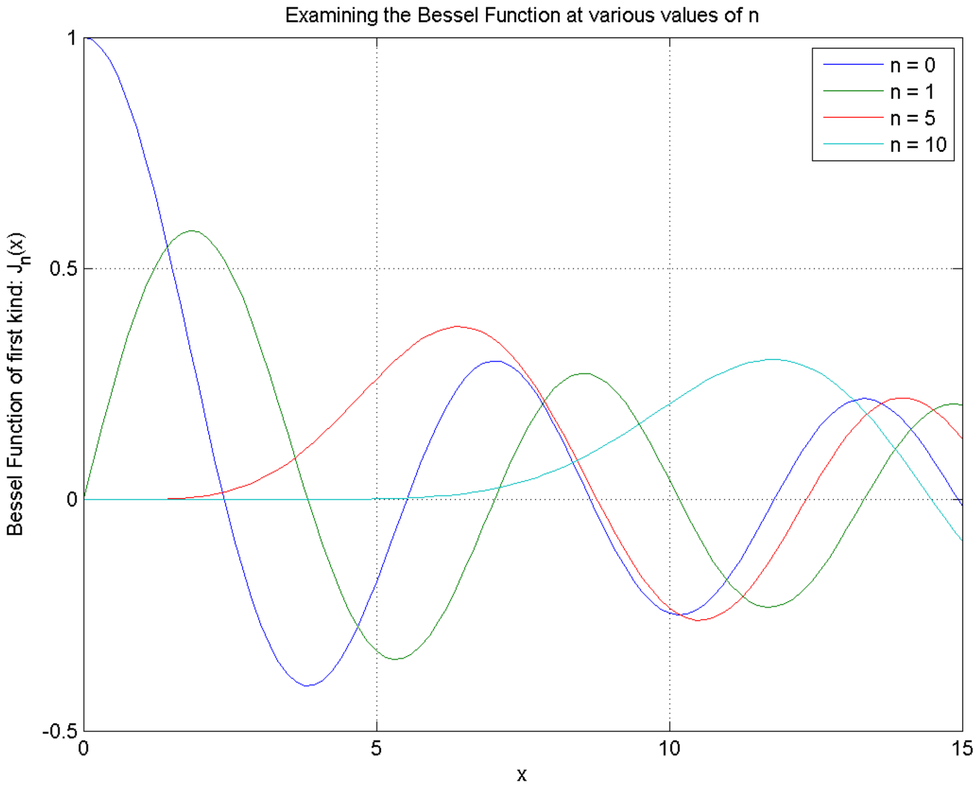

Figure 3: All Bessel functions of the first kind for various n.

4.2.3 Final solution for ψ_{nm}

The final solution for the eigenfunctions are:

$$\begin{aligned}\psi &= R(r)\Phi(\phi)\\ &= J_n(\gamma r)(A\cos(\sqrt{\lambda}\phi) + B\sin(\sqrt{\lambda}\phi))\\ \psi_{nm} &= J_n\left(\frac{\tau_{nm}}{a}r\right)(A_{nm}\cos(n\phi) + B_{nm}\sin(n\phi))\end{aligned}$$

In order to plug in this result into (12), $||\psi_{nm}||^2$ needs to also be calculated.

$$\begin{aligned}
\iint\limits_D (\psi_{nm})^2 \, \mathrm{d}S &= \int_0^{2\pi} \int_0^a (\psi_{nm})^2 r \, \mathrm{d}r \, \mathrm{d}\phi \\
&= \int_0^{2\pi} \int_0^a \left(J_n\left(\frac{\tau_{nm}}{a} r\right) (A_{nm}\cos(n\phi) + B_{nm}\sin(n\phi)) \right)^2 r \, \mathrm{d}r \, \mathrm{d}\phi \\
&= \int_0^{2\pi} [(A_{nm})^2 \cos^2(n\phi) + 2(B_{nm}A_{nm})\sin(n\phi)\cos(n\phi) + (B_{nm})^2 \sin^2(n\phi)] \\
&\quad \cdot \int_0^a J_n\left(\frac{\tau_{nm}}{a} r\right)^2 r \, \mathrm{d}r \, \mathrm{d}\phi \\
&= \int_0^{2\pi} [(A_{nm})^2 \cos^2(n\phi) + 2(B_{nm}A_{nm})\sin(n\phi)\cos(n\phi) + (B_{nm})^2 \sin^2(n\phi)] \, \mathrm{d}\phi \\
&\quad \cdot \int_0^a J_n\left(\frac{\tau_{nm}}{a} r\right)^2 r \, \mathrm{d}r \\
&= \int_0^{2\pi} \left[\frac{(A_{nm})^2}{2}(1+\cos(2n\phi)) + (B_{nm}A_{nm})\sin(2n\phi) + \frac{(B_{nm})^2}{2}(1-\cos(2n\phi)) \right] \mathrm{d}\phi \\
&\quad \cdot \int_0^a J_n\left(\frac{\tau_{nm}}{a} r\right)^2 r \, \mathrm{d}r \\
&= \left[\frac{(A_{nm})^2}{2}\left(2\pi + \frac{\sin(4n\pi)}{2n} - 0 - \frac{\sin(0)}{2n}\right) + (B_{nm}A_{nm})\left(-\frac{\cos(4n\pi)}{2n} + \frac{\cos(0)}{2n}\right) \right. \\
&\quad \left. + \frac{(B_{nm})^2}{2}\left(2\pi - \frac{\sin(4n\pi)}{2n} - 0 + \frac{\sin(0)}{2n}\right) \right] \cdot \int_0^a J_n\left(\frac{\tau_{nm}}{a} r\right)^2 r \, \mathrm{d}r \\
&= \pi[(A_{nm})^2 + (B_{nm})^2] \cdot \int_0^a J_n\left(\frac{\tau_{nm}}{a} r\right)^2 r \, \mathrm{d}r
\end{aligned}$$

From Strauss Chapter 10.5, Eq. 8:

$$\int_0^a J_n(z)^2 z \, \mathrm{d}z = \frac{1}{2}a^2[J_n'(a)]^2 + \frac{1}{2}(a^2 - n^2)[J_n(a)]^2$$

Using u substitution:

$$z = \frac{\tau_{nm}}{a} r, \quad \mathrm{d}z = \frac{\tau_{nm}}{a} \, \mathrm{d}r, \quad r = \frac{a}{\tau_{nm}} z, \quad \mathrm{d}r = \frac{a}{\tau_{nm}} \, \mathrm{d}z$$

$$\begin{aligned}
\int_0^a J_n\left(\frac{\tau_{nm}}{a} r\right)^2 r \, \mathrm{d}r &= \int_0^{\tau_{nm}} J_n(z)^2 \left(\frac{a}{\tau_{nm}}\right)^2 z \, \mathrm{d}z \\
&= \left(\frac{a}{\tau_{nm}}\right)^2 \left[\frac{1}{2}(\tau_{nm})^2[J_n'(\tau_{nm})]^2 + \frac{1}{2}((\tau_{nm})^2 - n^2)[J_n(\tau_{nm})]^2\right] \\
&= a^2 \left[\frac{1}{2}[J_n'(\tau_{nm})]^2 + \frac{1}{2}\left(1 - \frac{n^2}{\tau_{nm}}\right)[J_n(\tau_{nm})]^2\right]
\end{aligned}$$

By the definition of τ_{nm}, the $J_n'(\tau_{nm})$ term is zero.

$$\int_0^a J_n\left(\frac{\tau_{nm}}{a} r\right)^2 r \, \mathrm{d}r = \frac{a^2}{2}\left(1 - \frac{n^2}{\tau_{nm}}\right)[J_n(\tau_{nm})]^2$$

So all in all:

$$\iint\limits_D (\psi_{nm})^2 \, \mathrm{d}S = \pi\left[(A_{nm})^2 + (B_{nm})^2\right] \frac{a^2}{2}\left(1 - \frac{n^2}{\tau_{nm}}\right)[J_n(\tau_{nm})]^2$$

And for the special case $n = 0$:

$$
\begin{aligned}
\iint\limits_D (\psi_{0m})^2 \,\mathrm{d}S &= \int_0^{2\pi} \int_0^a \left(J_0\left(\frac{\tau_{0m}}{a} r\right) (A_{0m}\cos(0\phi) + B_{0m}\sin(0\phi)) \right)^2 r \,\mathrm{d}r \,\mathrm{d}\phi \\
&= \int_0^{2\pi} \int_0^a (A_{0m})^2 J_0\left(\frac{\tau_{0m}}{a} r\right)^2 r \,\mathrm{d}r \,\mathrm{d}\phi \\
&= 2\pi (A_{0m})^2 \frac{a^2}{2} (1 - 0^2)[J_0(\tau_{0m})]^2 \\
&= \pi [a A_{0m} J_0(\tau_{0m})]^2
\end{aligned}
$$

Before the ψ_{nm} terms are summed up it should be noted that the n index starts at 0 (due to Φ), but the m index starts at 1.

5 Conclusion

Now, nearly everything has been found to obtain the solution for this problem:

$$
\begin{aligned}
E(r', \phi') &= \int_0^{2\pi} \int_0^a G(r, \phi, r', \phi') F(r, \phi) r \,\mathrm{d}r \,\mathrm{d}\phi \\
G(r, \phi, r', \phi') &= \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{\psi_{nm}(r', \phi') \psi_{nm}(r, \phi)}{(k^2 - (\frac{\tau_{nm}}{a})^2) ||\psi_{nm}||^2} \\
\psi_{nm}(r, \phi) &= J_n\left(\frac{\tau_{nm}}{a} r\right) (A_{nm}\cos(n\phi) + B_{nm}\sin(n\phi)) \\
||\psi_{nm}||^2 &= \pi \left[(A_{nm})^2 + (B_{nm})^2\right] \frac{a^2}{2} \left(1 - \frac{n^2}{\tau_{nm}}\right) [J_n(\tau_{nm})]^2 \\
\tau_{nm} &\text{ are the zeros to } J_n'(\tau_{nm})
\end{aligned}
$$

The only things that have not been solved for are A_{nm} and B_{nm}. In the original problem of [1], the A and B terms canceled each other out in 12, because the eigenfunctions were a product of sinusoids (in x and y). What I got for the disk was a sum of sinusoids (in ϕ) times the Bessel functions (in r), which didn't result in the A's and B's canceling out so nicely. They still do, though, provide a useful solution if the initial conditions are known. Using the concept of the generalized Fourier Series, one can calculate what the A's and B's should be for some given initial conditions, because the Bessel Functions are a complete, orthogonal set of functions. I believe that even further simplifications can be made if you assume F does not depend on ϕ. However, this paper is plenty long enough already, so I won't go further into that.

References

[1] Shringarpure, Ketan. *"The Study of a Model for Via Transition and the Multi-Layer Via Transition Tool GUI Design."* Master's Thesis, Electrical Engineering, Missouri University of Science and Technology, 2010.

[2] Franklin, J. *"Green's functions for Neumann boundary conditions"* ArXiv e-prints, Physics - Classical Physics, 2012.

[3] Strauss, Walter A.