The Lorenz System

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$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

Introduction

- Developed by Ed Lorenz to model weather convection
  - Wanted to test new computer, but discovered the system is chaotic by accident
- σ is the Prandtl number
  - Ratio between kinematic viscosity and thermal diffusion
- ρ is the ratio of the Rayleigh number to the critical Rayleigh number
  - Rayleigh < Rayleigh critical means conduction dominates
  - Rayleigh > Rayleigh critical means convection dominates
  - Analysis assumes all constants are greater than zero

Volumetrically Bounded

- Calculate Lyapunov exponent for system volume

1. Select a closed surface of volume V(t) in the system phase space
2. Derivative of this volume is equated to the integral of the gradient of the system with respect to the volume
3. Doing this for the Lorenz system:
4. Integrate results:
5. The volume of a sphere of initial conditions will rapidly decrease to a limiting set of zero volume

$$\nabla f = [\sigma(y-x)]_x + (\rho x - y - xz)_y + (xy - \beta z)_z = -\sigma - 1 - b$$

$$V'(t) = \int_V \nabla f \, dV$$

$$V'(t) = -(\sigma + 1 + \beta)V$$

$$V(t) = V(0)e^{-(\sigma+1+\beta)}$$

Fixed Points

- Set derivative of each individual equation to zero

$$\begin{aligned} y - x &= 0 \\ \rho x - zx - y &= 0 \\ xy - \beta z &= 0 \end{aligned} \qquad \begin{aligned} x &= 0, \pm\sqrt{\beta(\rho-1)} \\ y &= 0, \pm\sqrt{\beta(\rho-1)} \\ z &= 0, \rho-1 \end{aligned}$$

- Only one fixed point if the ratio of the Rayleigh Number to the critical Rayleigh number is greater than one
- Two fixed points are created at $\rho > 1$
  - Pitchfork Bifurcation

Stability of Fixed Points

- Linearizing the system of equations gives:
- We can see that Z will decay to 0 exponentially over time
- Now we can produce a 2D Jacobian:

$$J = \begin{bmatrix} -\sigma & \sigma \\ \rho & -1 \end{bmatrix}$$

- With eigenvalues:

$$\lambda_{1.2} = \frac{-(1+\sigma) \pm \sqrt{(\sigma-1)^2 + 4\sigma\rho}}{2}$$

- Stability
- ρ bounded between a 0 and 1: two negative eigenvalues and a stable node
- ρ greater than 1 produces negative and positive eigenvalues – saddle node
- Means origin is a stable point when Rayleigh number is less than Rayleigh critical

3D Jacobian Analysis

- Jacobian evaluated at the second and third fixed points
  - Only created if $\rho > 1$

$$J_{x_1^*} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{\beta(\rho-1)} \\ \sqrt{\beta(\rho-1)} & \sqrt{\beta(\rho-1)} & -\beta \end{bmatrix} \quad J_{x_2^*} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \sqrt{\beta(\rho-1)} \\ -\sqrt{\beta(\rho-1)} & -\sqrt{\beta(\rho-1)} & -\beta \end{bmatrix}$$

- Eigenanalysis shows:
  - Fixed points are stable if ρ is between one and a critical value ρ_{crit}
  - This is a supercritical pitchfork bifurcation since the two stable fixed points are generated when ρ equals one
  - Origin loses stability as the two other fixed points are created
  - ρ_{crit} can be expressed as

$$\rho_{crit} = \frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-1}$$

Eigenvalue Analysis Cont.

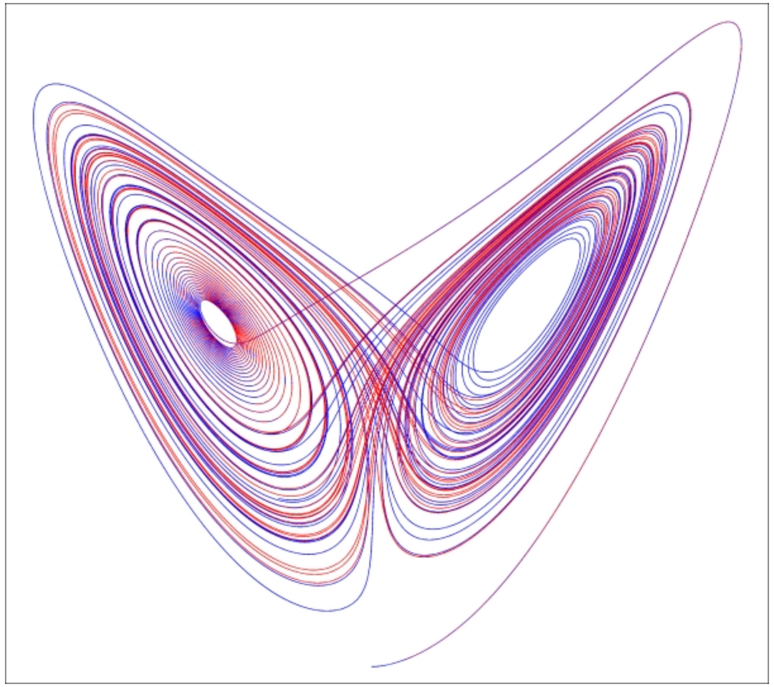
- If ρ equals critical value

$$\lambda_{1,2} = \pm\sqrt{\beta(\sigma+\rho)}i \;,\; \lambda_3 = -(\sigma+\beta+1)$$

- Consists of complex pair of eigenvalues and a single eigenvalue with varying magnitude and nonzero real part
  - Indicates Hopf bifurcation
  - Complicated analysis shows it is a Subcritical Hopf Birfurcation
    - Limit cycles exist below critical value of ρ
    - Are destroyed and fixed points become unstable as ρ increases past critical value
- So we know what happens as we vary ρ
  - What happens if we increase it past its critical value?
    - No stable fixed points and no limit cycles exist

Lorenz Attractor and Chaos

- When rho rises about the critical value of 24.74, chaos starts to occur
- Two trajectories with slight differences in initial values have dramatically different solutions



Lyapunov Exponent

- The Lyapunov exponent tells us the rate of separation of two trajectories: $$|\delta x(t)| = |x(t) - x'(t)| \approx \varepsilon e^{\lambda t}$$
- Positive exponent means the difference between two trajectories increases at an exponential rate
- Positive Lyapunov Exponent means chaos

How to Calculate

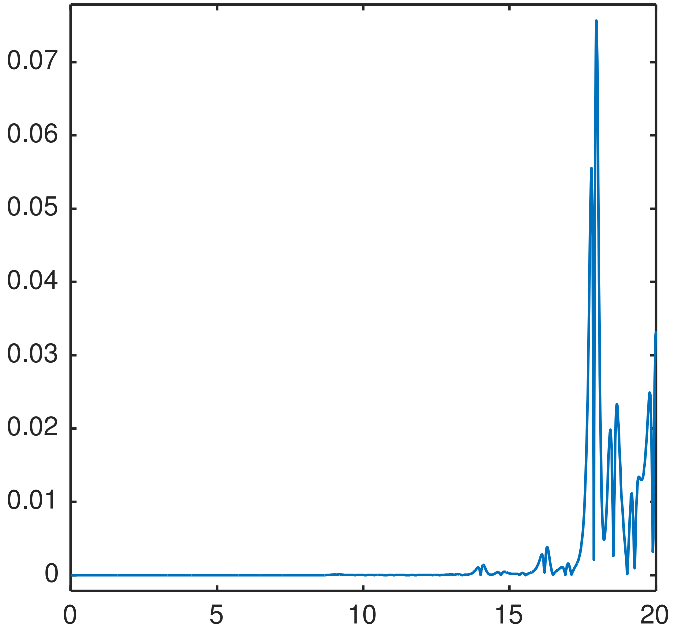
- Calculating Lyapunov Exponent for the Lorenz system analytically is rather difficult
  - Use numerical methods
- Ode45 from Matlab
  1. Use Ode45 to solve two Lorenz systems simultaneously with slightly different initial conditions (3, 3, 20) and (3+10^-19, 3, 20)
  2. Must use Ode45 to solve the two initial conditions simultaneously since Ode45 will use slightly different time steps when solving the equations separately

```
sigma=10;
beta=8/3;
rho=20;
f = @(t,a) [-sigma*L(1) + sigma*L(2); rho*L(1) - L(2) - L(1)*a(3); -beta*L(3) + L(1)*a(2); -sigma*L(4) + sigma*a(5);
rho*L(4) - a(5) - L(4)*L(6); -beta*L(6) + a(4)*L(5)];

% solution of orbit with initial condition (3, 3, 20) and (3+10^-9 3 20), solved simultaneously
[t,L] = ode45(f,[0 20],[3 3 20 3+10^-9 3 20]);
```

Take the Difference

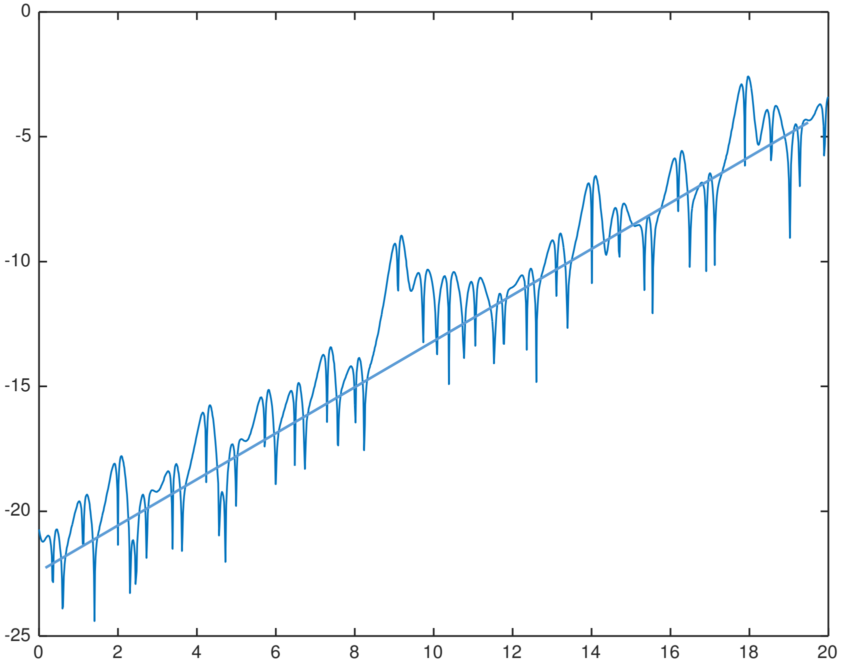
- Once we have obtained two solutions curves, plot the difference between the two solutions in the x direction vs time
- The absolute value of the difference of the two trajectories in the x direction vs time :



plot(t, log(abs(L(:, 4) - L(:, 1))))

Take the Natural Log

- Take the natural log of the difference
- This gives us a somewhat linear solution
    - The line of best fit should be the max lyapunov exponent
    - Use polyfit from matlab



polyfit(t, log(abs(L(:, 4) - L(:, 1))), 1)

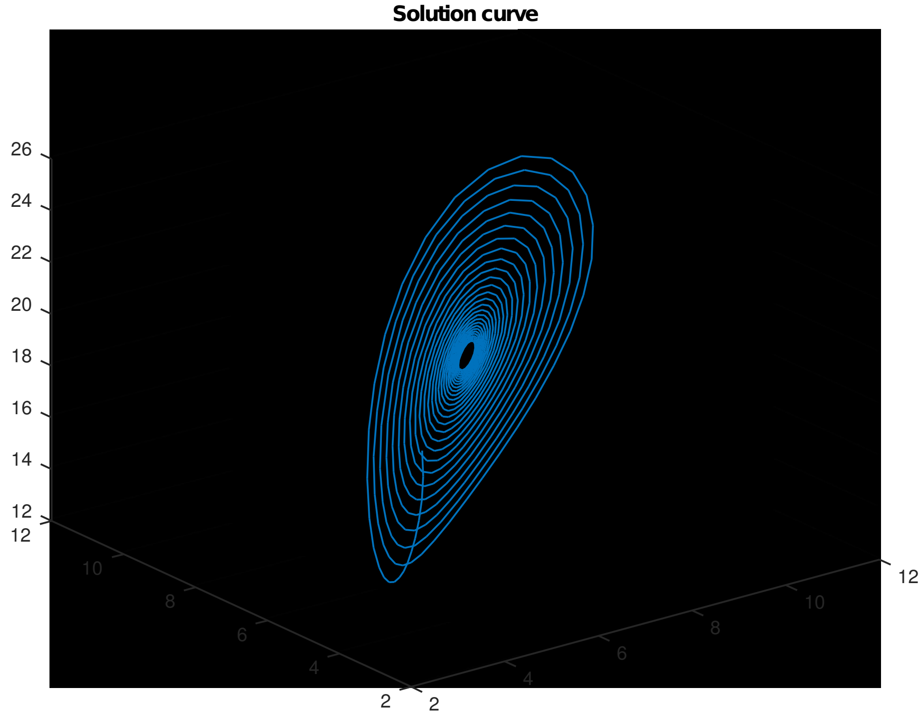
Slope: .9036

Online research says that the actual value is

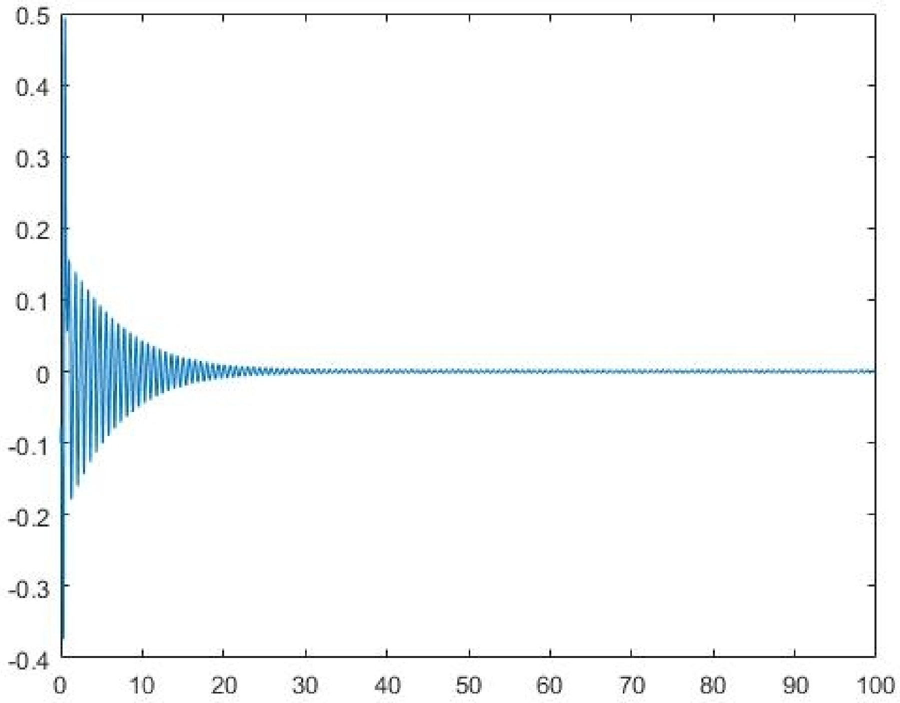
.906

When rho < 24.74

- When the system is not chaotic the solutions may resemble:

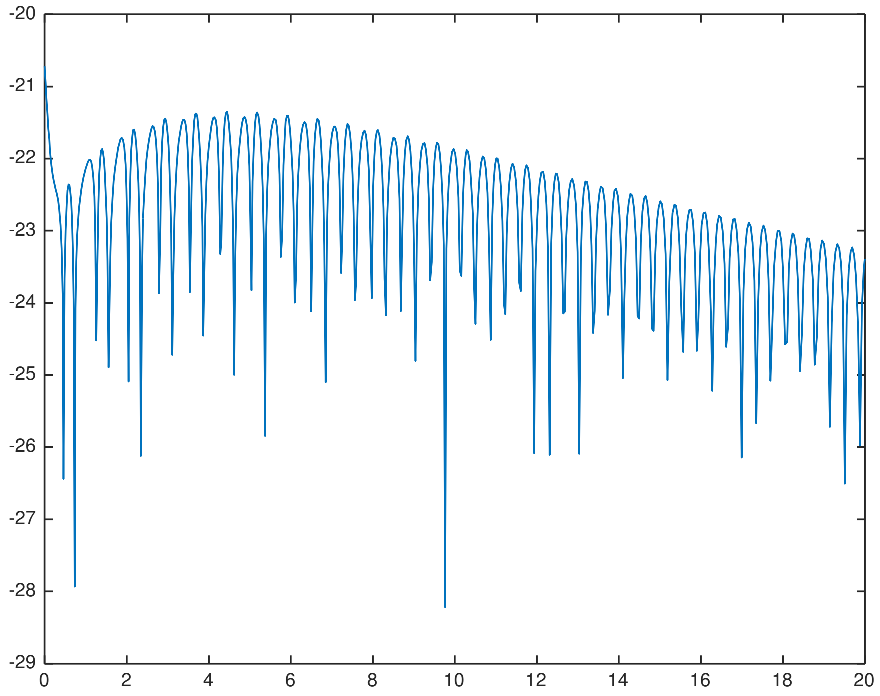


Difference Between Two Non-Chaotic Trajectories Over Time



Lyapunov Exp Again

- Using the same principles to calculate the Lyapunov exponent for a non chaotic version
- Here the line of best of fit has a negative slope, -0.0768
- Thus not chaotic



Applications/Questions

- Limit of Weather prediction
- The model was originated from simplified atmospheric convection models and thus give us some insight on weather prediction
- It is because of this chaotic nature, weather is extremely hard to predict farther than ten days in advance
- Method of calculation accurate? The results were consistent, but there might be a better way
- What about the other Lyapunov exp values?