In Depth Analysis of The Three Dimensional Lorenz System

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Why the Lorenz System?:

The Lorenz system developed by Edward Lorenz back in 1963 at MIT is a simplified model on weather convection derived from the the Boussinesq approximations. Lorenz attempted to model the system numerically but accidentally discovered chaos when his computer rounded the initial conditions, changing them by a slight amount and thus the outcome of the solution was dramatically different. Lorenz thought there was a hardware problem but instead he had discovered chaos. After further investigations, he wrote a paper describing what he observed but in the beginning, no scientists or mathematicians paid much attention to his paper, not believing such chaotic behavior can exist under these conditions or not interested in an abstract simplified model modeling real world behavior. It took many years for scientists and mathematicians to pay attention to what Lorenz has discovered and now, it is widely accepted by many that chaotic system can arise from seemingly simple nonlinear systems. Different chaotic systems have been discovered but Lorenz pioneered the theory of chaos and we can now appreciate the significance of his discovery.

The beauty of the Lorenz system involves the fact that the system exhibits all aspects of nonlinear dynamics as the parameters are varied and the system simulated. Lorenz discovered a thing of beauty in the world of nonlinear dynamics, as his famed system contains bifurcations, limit cycles, creation and destruction of stability, and the famous chaotic system responses. It is an area that must be studied in the field of nonlinear dynamics with all that it offers people to learn.

System Basics:

The Lorenz equations are a three dimensional differential equations given as follows

$$\frac{dx}{dt} = \sigma(y-x)$$
$$\frac{dy}{dt} = x(\rho-z)-y$$
$$\frac{dz}{dt} = xy-\beta z$$

It can be seen the system is dependent on three different external parameters. The parameters σ and ρ are the Prandtl number and ratio of Rayleigh Number to critical Rayleigh number respectively. In fluid mechanics, each of these numbers has an important physical meaning. The Prandtl number is the ratio between the kinematic viscosity and thermal diffusivity of the fluid. The Rayleigh number is a similar parameter, however it deals mainly with buoyancy driven flows caused by a density gradient in a fluid. A Rayleigh number less than the critical value indicates conduction dominates the heat transfer of the fluid, whereas a value greater than the critical value indicates convection dominates.

It can be seen that the system itself is nonlinear, as both dy/dt and dz/dt contain nonlinear functions of x, y and z. The Lorenz system also has symmetry about the z-axis, as a simple substitution shows that the system will be the same if (x, y) are replaced with (-x, -y).

Shrinking Volumetric System:

Analysis can be done on the Lorenz system to show that it is volumetrically bounded in its solutions. This can be shown by calculating the Lyapunov exponent of the system volume. For a three dimensional system $f(\overline{x})$, we select a closed surface of volume $V(t)$ in the system phase space. The derivative of this volume function can be equated to the the integral of the gradient of the system with respect to the volume.

$$V'(t) = \int_V \nabla f \, dV$$

Using the Lorenz Equations in this formula, the following result can be calculated for the volume growth of the system

$$\nabla f = [\sigma(y-x)]_x + (\rho x - y - xz)_y + (xy - \beta z)_z = -\sigma - 1 - b$$
$$V'(t) = -(\sigma + 1 + \beta)V$$
$$V(t) = V(0)e^{-(\sigma+1+\beta)t}$$

This result shows that the Lorenz system has a negative Lyapunov exponent. This means that the volume of a sphere of initial conditions will rapidly decrease to a limiting set of zero volume. All trajectories that start within this sphere will stay within the sphere as time goes to infinity.

Bounded Trajectories:

Similarly, it can be shown that trajectories are bounded within a spherical surface. A hypothetical sphere of radius *R* is considered.

$$x^2 + y^2 + (z - \rho - \sigma)^2 = R^2$$

The derivative rate of change of trajectories on this surface can then be expressed as

$$\frac{d}{dt}[x^2+y^2+(z-\rho-\sigma)^2] = 2xx' + 2yy' + 2(z-\rho-\sigma)z'$$
$$= -2[\sigma x^2 + y^2 + \beta(z - \frac{\rho+\sigma}{2})^2 - \frac{\beta(\rho+\sigma)^2}{4}]$$

Setting this equal to zero, a bounding region for an ellipsoid of conditions that always keep a negative rate of change of the spherical surface is found

$$\sigma x^2 + y^2 + \beta(z - \frac{\rho+\sigma}{2})^2 = \frac{\beta(\rho+\sigma)^2}{4}$$

So if if the radius R is selected large enough to enclose this ellipsoid, the rate of change of the spherical volume will always be less than zero. This indicates Lyapunov stability and that trajectories starting on the spherical surface will always enter it and not leave, indicating bounded initial condition trajectories.

Fixed Point Analysis:

While these facts are important, the true beauty of the Lorenz system is what happens within the volume of these initial conditions. This nonlinear system can encompass stable and unstable critical points, limit cycles, bifurcations and even chaos depending on the selection of system parameters. It truly covers all the possibilities of nonlinear dynamics in a single system.

To determine what occurs in any nonlinear dynamic system, it is important to find the fixed points of the system. Setting the derivative of each state of the system to zero, the system can be solved simultaneously and the fixed points of the Lorenz system can be found.

$$\begin{aligned} &y - x = 0 \\ &\rho x - zx - y = 0 \\ &xy - \beta z = 0 \end{aligned} \qquad\qquad \begin{aligned} &x = 0, \pm\sqrt{\beta(\rho-1)} \\ &y = 0, \pm\sqrt{\beta(\rho-1)} \\ &z = 0, \rho-1 \end{aligned}$$

It can be seen that the system contains only one fixed point if the ratio of the Rayleigh Number to the critical Rayleigh number is greater than one. Linearizing the system equations, the following result is obtained and can be used to analyze the stability of the fixed points.

$$\begin{aligned} \frac{dx}{dt} &= \sigma(y-x) \\ \frac{dy}{dt} &= \rho x - y \\ \frac{dz}{dt} &= -\beta z \end{aligned}$$

Visually it can be seen that that *z* will exponentially decay to zero as time goes to infinity. Two dimensional analysis can then be performed on the remaining two dimensions. The two dimensional jacobian of the system is

$$J = \begin{bmatrix} -\sigma & \sigma \\ \rho & -1 \end{bmatrix}$$

With eigenvalues given by

$$\lambda_{1.2} = \frac{-(1+\sigma) \pm \sqrt{(\sigma-1)^2 + 4\sigma\rho}}{2}$$

If rho is bounded between zero and one, the result is two negative eigenvalues and a stable node at the origin, where as a rho greater than one produces one positive and one negative eigenvalue, producing an unstable saddle node. Physically, this can be interpreted as the origin being a stable fixed point when the rayleigh number is less than the critical rayleigh number for the fluid. This implies that the origin is stable in conduction dominated heat flows and loses stability as convection begins to dominate.

When ρ is bounded in the region from zero to one, the origin is actually a globally stable fixed point that all trajectories are attracted to as time goes to infinity. Consider the following potential function

$$V = \frac{1}{\sigma}x^2 + y^2 + z^2$$

Taking the derivative and substituting the Lorenz equations yields the following result

$$V' = \frac{2}{\sigma}xx' + 2yy' + 2zz' = -2(x - \frac{\rho+1}{2}y)^2 - 2(1 - (\frac{\rho+1}{2})^2)y^2 - 2\beta z^2$$

V' is strictly negative when ρ is bounded between zero and one and $(x, y, z) \neq (0, 0, 0)$. Therefore, all initial trajectories will go to zero over time. The origin is a globally stable attractor.

A three dimensional linearized jacobian can be used to analyze the stability of the other two fixed points that could appear. It should be noted that these fixed points only occur if ρ is greater than one as these fixed points will be undefined for values below this. The three dimensional jacobian for this system is as shown and will be evaluated at the following fixed points.

$$J = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - z & -1 & -x \\ y & x & -\beta \end{bmatrix}$$

$$X_1^* = (\sqrt{\beta(\rho-1)}, \sqrt{\beta(\rho-1)}, \rho - 1)$$
$$X_2^* = (-\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}, \rho - 1)$$

Evaluation of the jacobian at these critical points results in the following matrices.

$$J_{x_1^*} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{\beta(\rho-1)} \\ \sqrt{\beta(\rho-1)} & \sqrt{\beta(\rho-1)} & -\beta \end{bmatrix} \quad J_{x_2^*} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \sqrt{\beta(\rho-1)} \\ -\sqrt{\beta(\rho-1)} & -\sqrt{\beta(\rho-1)} & -\beta \end{bmatrix}$$

Eigenanalysis of these fixed points shows that they are stable if ρ is between one and a critical value ρ_{crit}. It can also be noted that the system must have a supercritical pitchfork bifurcation as two stable fixed are points generated when rho is equal to one. The origin loses stability when these two fixed points are created, indicative of a supercritical pitchfork bifurcations. The critical rho value can be expressed in terms of sigma and beta and is as follows.

$$\rho_{crit} = \frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-1}$$

If ρ is equal to this critical value, the eigenvalues can more easily be analytically obtained and are as follows.

$$\lambda_{1,2} = \pm\sqrt{\beta(\sigma+\rho)}i \ , \ \lambda_3 = -(\sigma+\beta+1)$$

The eigenvalues of the three dimensional system at the critical value of rho consist of a complex pair of eigenvalues and a single eigenvalue with varying magnitude and a nonzero real part. This indicates that a hopf bifurcation occurs at this point. Mardsen and McCracken proved in their 1976 book that the limit cycle is in fact subcritical and that unstable limit cycles only exist for $\rho < \rho_{crit}$ and are destroyed as the system passes the critical value. A basic bifurcation diagram of the system is shown in Fig. 1

So what happens when the system passes the critical value of rho? All three of the fixed points of the system have become unstable, all limit cycles have been destroyed due to the subcritical hopf bifurcation, trajectories are bounded, and the volume of initial conditions contracts exponentially. This is where Lorenz analyzed the system and found the famed strange attractor.

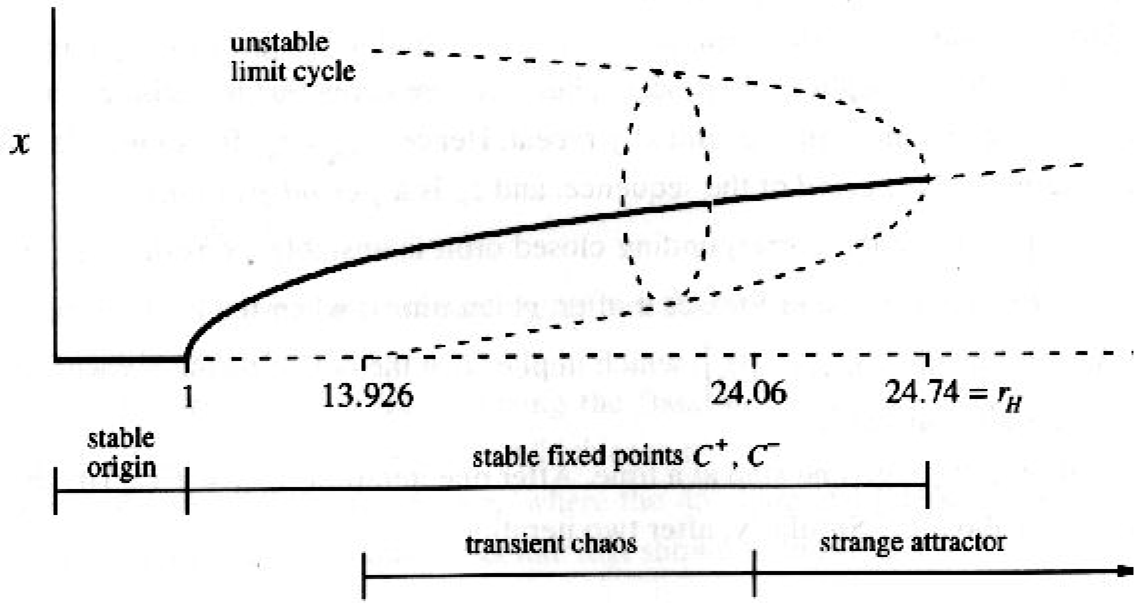

Fig. 1: Lorenz System Bifurcation Diagram

Analysis of Lorenz Simulations:

Lorenz defined chaos as “When the present determines the future, but the approximate present does not approximately determine the future.” Lorenz found that the system settles into an aperiodic motion that oscillates in an erratic fashion for infinite time but never exactly repeats. This result is shown later in Fig. 7.

Similarly, the Lorenz system exhibits a large dependency on initial conditions, as two initial conditions will rapidly diverge to different results no matter how close they are to each other to start. In order for the system to be accurately predicted over a long period of time the conditions must be known to an infinite level of accuracy for the entire time. Over time as error accumulates, the system becomes impossible to predict. This is what Lorenz discovered upon numerical solving of his system and helped define one of the main definitions of a chaotic system. Lorenz solved his system of equations with the parameters set as follows

$$\sigma = 10$$

$$\beta = \frac{8}{3}$$

$$\rho = 28$$

These values produced his famed butterfly attractor that can be seen in Fig. 2. Simulations were performed in this report to show how software can be used to create an accurate visual model of the Lorenz attractor. Calculations were also done to find the Lyapunov exponent of the Lorenz system to prove its dependency on initial conditions.

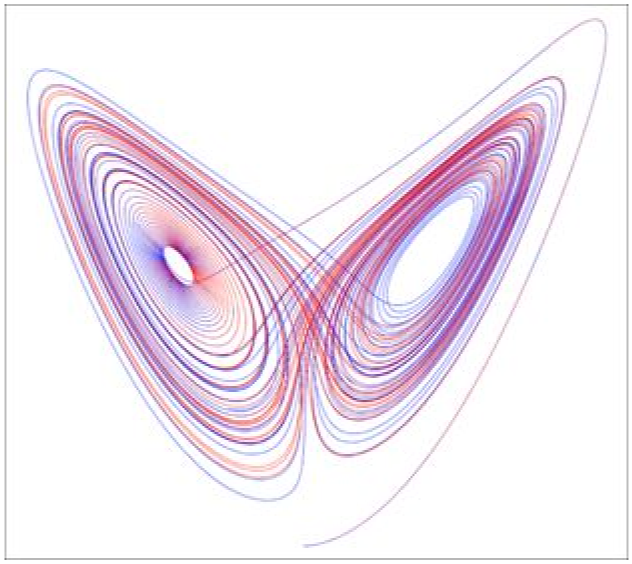

Fig. 2: Lorenz Butterfly Attractor

It should be noted that a positive lyapunov exponent of two trajectories with similar initial conditions does not always mean chaos. There are two other conditions. 1) Aperiodic long term behavior: there are solution trajectories that does not go towards any fixed points, periodic or quasi-periodic orbits as t goes to infinity. 2) Deterministic: no random or noisy input/parameters.

Here, two different graphs of the solutions are plotted with the same equations but slightly different initial conditions. Since the color is distributed in an uneven way throughout, that means the two solution curves diverges from each other.

Use of MATLAB to Find Lyapunov Exponent:

Two different solutions curves were generated with slightly different initial points (3, 3, 20) and (3+d0, 3+d0, 20+d0), d0 = 10^-5 using ode45 with time values [0 15]. We would then create a list of data points by taking the natural log of the magnitude of the difference between the two trajectories. It is important to note that for a 3D space, there are 3 Lyapunov exponents each corresponding to a particular direction and there is a largest value that dominate the overall behavior. We are interested in that largest value so would simply take the euclidean distance between the trajectories. Next, plot these data points vs the time steps used to solve the equation and take a line of best fit. The slope for that line should be an approximation for the maximum lyapunov exponent (MLE). It is important to note the ode45 uses different time steps for different solutions, thus the two solutions needs to be solved simultaneously to avoid errors.

When plotting the graphs over time to find the MLE, it is also ideal to choose an appropriate time value that correspond to the moment when the trajectories start separating rapidly. As shown in Figure 3, we are only interested in the slope of the solution from time values 0 to 15.

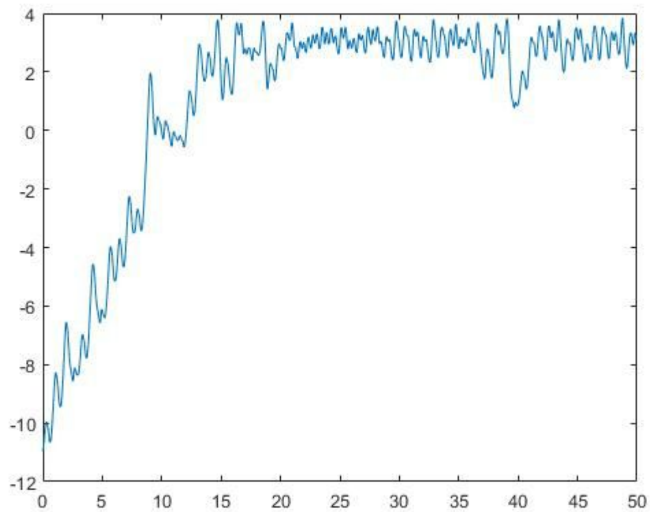

Fig. 3

(Corresponding MatLab code available)

Figure 3 is plotted on a logarithmic scale to enable calculation of the Lyapunov Exponent, with the y-axis being the natural log of initial condition separation distance, and the x-axis being the time.

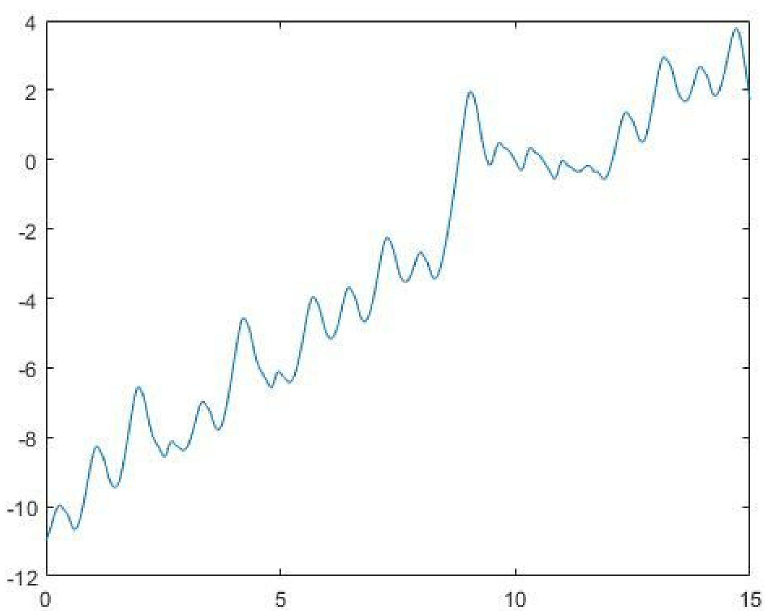

Fig. 4: Logarithmic Difference of Initial Conditions

Figure 4 has a Line of best fit slope equal to 0.9138, indicating a positive Lyapunov Exponent. This indicates a chaotic system response, as initial conditions will diverge away from each other at an exponential rate.

Figures 5 and 6 prove to show visually that the system is in fact not chaotic below the critical value of ρ. Figure 4 shows a three dimensional solution curve that does not form the traditional Lorenz Attractor. Figure 5 is another logarithmic difference plot, however this time it can be seen a negative Lyapunov Exponent is obtained, indicating solutions will not diverge from each other over time.

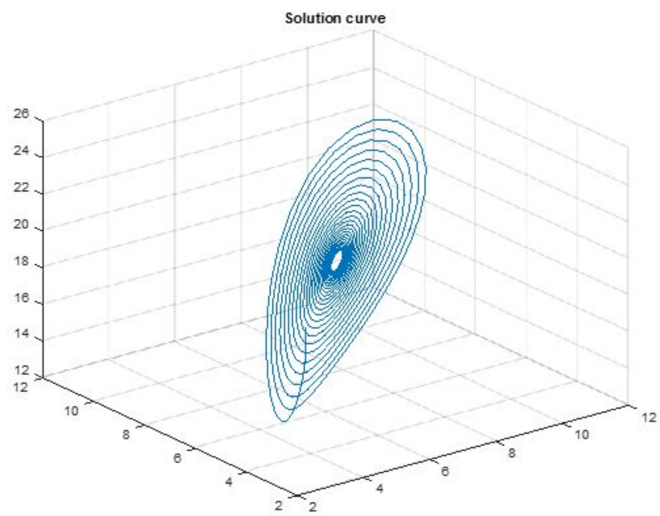

Fig. 5: Solution curve for ρ = 20

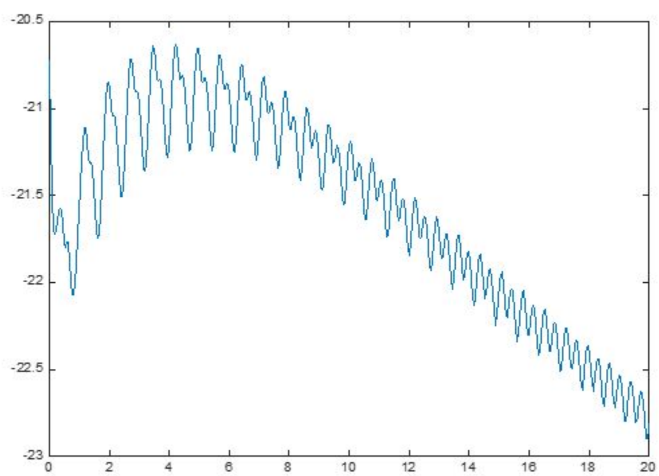

Fig. 6: MLE = -.0768

Changing ρ = 20, the calculated MLE was -0.0768 (Fig. 6), corresponding to a non chaotic system which is correct as shown in the solution curve (Fig. 5) below with. ρ = 20 is below the critical value of 24.74. Figure 7 shows the long term separation of the system of the system which can be seen to approach zero over time.

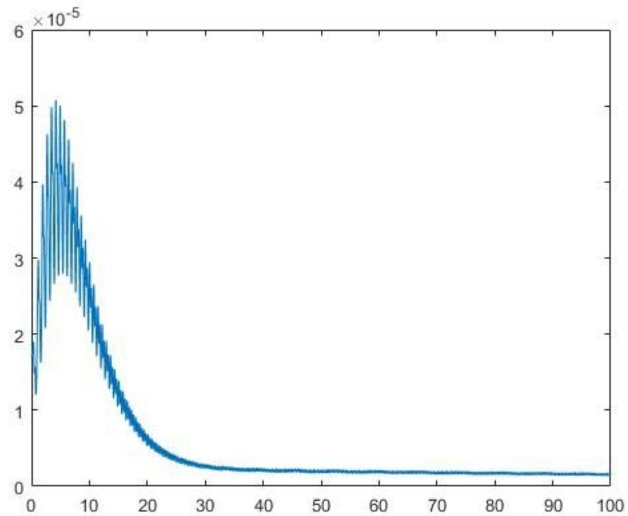

Fig. 7: Separation of Trajectories (non-chaotic)

The long term separation of the system approaching zero indicates that the system response is not chaotic when $\rho < \rho_{crit}$.

Application and Conclusion:

These equations were originally a simplified model of convection used for explaining weather patterns. The calculation of the lyapunov exponent shows that the system is chaotic and thus explains why weather is difficult to predict. Prediction becomes much more difficult in the future. For this reason, weather prediction only works up to a max of ten days in advance. In Fig. 8, the initial flat line corresponds to the time interval that weather prediction is actually feasible.

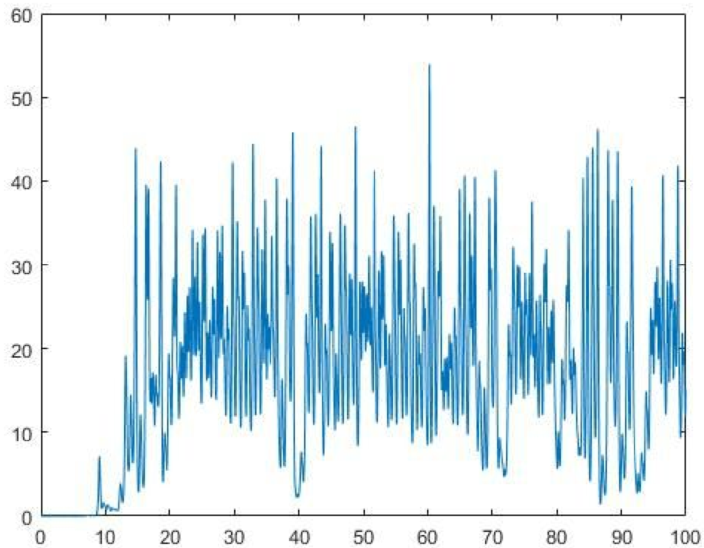

Fig. 8: Separation of trajectories over time

Ideally, a better way to calculate the Lyapunov exponent is desired but is rather difficult due to the complex nature of the Lorenz strange attractor. The results we found in the graphs are consistent with our background knowledge of the system. Changing ρ does give us results consistent with whether the solution is chaotic or nonchaotic.

References:

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