Problem 3 - Quadratic equation

(b) The discrepancy arises because in the quadratic equation $ax^2 + bx + c = 0$ both a and c are small (i.e., $b^2 \gg 4ac$), so that in the numerator of

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and the denominator of

$$x = \frac{2c}{-b \mp \sqrt{b^2 - 4ac}}$$

computing the (+) root involves subtracting b from a quantity nearly equal to that of b,

$$\sqrt{b^2 - 4ac} \simeq b, \qquad b \gg 4ac.$$

Thus, this computation (either formula) can become quite inaccurate for small enough a, c, depending on the sign of b.

(c) We want to avoid subtracting quantities which are almost equal; thus, we want to use the regular quadratic equation for the (+) root if $b < 0$, and the (−) root if $b \geq 0$. Similarly, we want to use

$$x = \frac{2c}{-b \mp \sqrt{b^2 - 4ac}}$$

for the (−) root if $b \geq 0$ and the (+) root if $b < 0$. This can be implemented concisely by defining

$$\mathcal{Q} \equiv -\frac{1}{2}\left(b + \mathrm{sign}(b)\sqrt{b^2 - 4ac}\right),$$

whence

$$x_1 = \frac{\mathcal{Q}}{a}, \qquad x_2 = \frac{c}{\mathcal{Q}} \qquad \text{are the two roots.}$$