PHYS 50733 - COMPUTATIONAL PHYSICS

HOMEWORK 5

3.(a) Planck's theory of blackbody radiation gives the spectral radiance per unit area as

$$I(\omega) = \frac{\hbar}{4\pi^2 c^2} \frac{\omega^3}{\mathrm{e}^{\hbar\omega/kT} - 1}.$$

The total energy per unit area is then

$$W = \int_0^\infty I(\omega)\mathrm{d}\omega = \frac{\hbar}{4\pi^2 c^2} \int_0^\infty \frac{\omega^3 \mathrm{d}\omega}{\mathrm{e}^{\hbar\omega/kT} - 1}.$$

Upon making the substitution

$$x = \frac{\hbar\omega}{kT}, \qquad \mathrm{d}x = \frac{\hbar}{kT}\mathrm{d}\omega,$$

this becomes

$$\boxed{W = \frac{k^4 T^4}{4\pi^2 c^2 \hbar^3} \int_0^\infty \frac{x^3 \mathrm{d}x}{\mathrm{e}^x - 1}}.$$

3.(b) I happen to know that this integral may in fact be evaluated using the Riemann Zeta function $\zeta(s)$, which admits the following integral representation

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}\mathrm{d}x}{\mathrm{e}^x - 1},$$

where $\Gamma(s)$ is the usual Gamma function (which is, for integer s, simply $(s-1)!$). Thus,

$$\int_0^\infty \frac{x^3 \mathrm{d}x}{\mathrm{e}^x - 1} = \Gamma(4)\zeta(4) = \frac{3!\pi^4}{90} = \frac{\pi^4}{15} \simeq 6.49393940,$$

where I have used the useful fact $\zeta(4) = \sum_{n=0}^\infty n^{-4} = \frac{\pi^4}{90}$. To perform the numerical quadrature, we must first perform the substitution $z = x(1+x)^{-1}$, yielding

$$\int_0^\infty \frac{x^3 \mathrm{d}x}{\mathrm{e}^x - 1} = \int_0^1 \frac{z^3/(1-z)}{\mathrm{e}^{z/(1-z)} - 1}\mathrm{d}z.$$

We choose Gaussian quadrature to perform the integration. For $N = 30$ sample points, the result is $\boxed{I \simeq 6.493947}$. Thus the chosen method performs well.

3.(c) Using this value for our integral, we can approximate the Stefan-Boltzmann constant as

$$\boxed{\sigma = \frac{\pi^2 k_B^4}{60 c^2 \hbar^3} \simeq 5.6703 \times 10^{-8}\ \mathrm{Jm^{-2}s^{-1}K^{-4}}}.$$