Problem 2 - Monte Carlo Integration

Calculate a value for the integral

$$I = \int_0^1 \frac{x^{-1/2}\mathrm{d}x}{\mathrm{e}^x + 1}$$

using the weight $w(x) = x^{-1/2}$.

(a) *Show that the probability distribution $p(x)$ from which the sample points should be drawn is given by $p(x) = \frac{1}{2}x^{-1/2}$, and derive a transformation formula for generating random numbers between zero and one from this distribution.*

Solution: We have by definition

$$p(x) = \frac{w(x)}{\int_a^b w(x)\mathrm{d}x}.$$

Using the standard integration

$$\int_0^1 x^{-1/2}\mathrm{d}x = 2x^{1/2}\Big|_0^1 = 2,$$

we see that

$$p(x) = \frac{x^{-1/2}}{2} = \boxed{\frac{1}{2\sqrt{x}}}.$$

To find the transformation formula, we recall that in general the relation between the probability of finding a value of x within the interval $[x, x+\mathrm{d}x]$ and the probability of our source of random floating-point numbers with probability density $q(z)$ generating a value of z within the interval $[z, z+\mathrm{d}z]$ is equal. For the special case of **rand()**, $q(z) = 1$ in the interval $[0,1]$ and vanishes everywhere else. Integration then leads to

$$\int_{-\infty}^{x(z)} p(x')\mathrm{d}x' = \int_0^z \mathrm{d}z' = z.$$

For our case, we can perform the integration on the left hand side (the lower limit of integration must be zero in this case, since the probability becomes imaginary for negative x), resulting in

$$\int_0^{x(z)} \frac{\mathrm{d}x'}{2\sqrt{x'}} = \sqrt{x} = z,$$

or

$$\boxed{x = z^2},$$

which is the required transformation formula.