Mathematica code to accompany:

Ranjan R, Koffel T, Klausmeier CA. The three-species problem: incorporating competitive asymmetry, intransitivity and founder control in modern coexistence theory.

In[∘]:=
```
(* Mathematica version *)
$Version
```

Out[∘]=
```
13.3.1 for Mac OS X ARM (64-bit) (July 24, 2023)
```

In[∘]:=
```
(* requires EcoEvo package <https://
 github.com/cklausme/EcoEvo> -- run this once (ever) to install *)
PacletInstall[
 "https://github.com/cklausme/EcoEvo/raw/master/Paclets/EcoEvo-1.7.2.paclet"]
```

Out[∘]=
```
PacletObject[ Name: EcoEvo
              Version: 1.7.2 ]
```

Note: tested with EcoEvo v1.7.2 — compatibility with future versions not guaranteed.

Run this first!

Load EcoEvo package, define model

In[∘]:=
```
<< EcoEvo`
```

Out[∘]=
```
EcoEvo Package Version 1.7.2 (September 1, 2023)
Christopher A. Klausmeier <christopher.klausmeier@gmail.com>
```

In[∘]:=
```
SetModel[{
   Pop[n1] → {Equation :→ (r1 - α11 n1 - α12 n2 - α13 n3) n1, Color → Red},
   Pop[n2] →
    {Equation :→ (r2 - α21 n1 - α22 n2 - α23 n3) n2, Color → RGBColor[{255, 211, 0} / 255]},
   Pop[n3] → {Equation :→ (r3 - α31 n1 - α32 n2 - α33 n3) n3, Color → Blue},
   Parameters :→ {r1 ≥ 0, r2 ≥ 0, r3 ≥ 0, α11 ≥ 0,
     α12 ≥ 0, α13 ≥ 0, α21 ≥ 0, α22 ≥ 0, α23 ≥ 0, α31 ≥ 0, α32 ≥ 0, α33 ≥ 0}
  }]
```

In[∘]:=
```
Off[Power::infy]
```

Various functions (plotting, making *α*-matrix, etc.)

Analysis (run this too!)

Table A2 — Outcomes and their conditions

In[ ]:=
```
(* from Table A1 *)
n1stable = (λ21 ≤ 0) && (λ31 ≤ 0);
n2stable = (λ12 ≤ 0) && (λ32 ≤ 0);
n3stable = (λ13 ≤ 0) && (λ23 ≤ 0);
n12stable = (λ12 > 0) && (λ21 > 0) && (λ312 ≤ 0);
n23stable = (λ23 > 0) && (λ32 > 0) && (λ123 ≤ 0);
n31stable = (λ31 > 0) && (λ13 > 0) && (λ231 ≤ 0);
```

In[ ]:=
```
(* i wins *)
n1wins = FullSimplify[n1stable && ! n2stable && ! n3stable && ! n23stable]
n2wins = FullSimplify[n2stable && ! n1stable && ! n3stable && ! n31stable]
n3wins = FullSimplify[n3stable && ! n1stable && ! n2stable && ! n12stable]
```

Out[ ]=
```
(λ12 > 0 || λ32 > 0) && (λ123 > 0 || λ23 ≤ 0 || λ32 ≤ 0) &&
 (λ13 > 0 || λ23 > 0) && λ21 ≤ 0 && λ31 ≤ 0
```

Out[ ]=
```
λ12 ≤ 0 && (λ13 > 0 || λ23 > 0) &&
 (λ13 ≤ 0 || λ231 > 0 || λ31 ≤ 0) && (λ21 > 0 || λ31 > 0) && λ32 ≤ 0
```

Out[ ]=
```
(λ12 > 0 || λ32 > 0) && (λ12 ≤ 0 || λ21 ≤ 0 || λ312 > 0) &&
 λ13 ≤ 0 && (λ21 > 0 || λ31 > 0) && λ23 ≤ 0
```

In[ ]:=
```
(* i&j coexist *)
n12coex = FullSimplify[n12stable && ! n3stable && ! n31stable && ! n23stable]
n23coex = FullSimplify[n23stable && ! n1stable && ! n31stable && ! n12stable]
n31coex = FullSimplify[n31stable && ! n2stable && ! n23stable && ! n12stable]
```

Out[ ]=
```
λ12 > 0 && λ21 > 0 && λ312 ≤ 0 && (λ13 > 0 || λ23 > 0) &&
 (λ13 ≤ 0 || λ231 > 0 || λ31 ≤ 0) && (λ123 > 0 || λ23 ≤ 0 || λ32 ≤ 0)
```

Out[ ]=
```
λ23 > 0 && λ32 > 0 && λ123 ≤ 0 && (λ21 > 0 || λ31 > 0) &&
 (λ13 ≤ 0 || λ231 > 0 || λ31 ≤ 0) && (λ12 ≤ 0 || λ21 ≤ 0 || λ312 > 0)
```

Out[ ]=
```
λ31 > 0 && λ13 > 0 && λ231 ≤ 0 && (λ12 > 0 || λ32 > 0) &&
 (λ123 > 0 || λ23 ≤ 0 || λ32 ≤ 0) && (λ12 ≤ 0 || λ21 ≤ 0 || λ312 > 0)
```

In[ ]:=
```
(* i or j *)
fc1x2 = FullSimplify[n1stable && n2stable && ! n3stable]
fc2x3 = FullSimplify[n2stable && n3stable && ! n1stable]
fc3x1 = FullSimplify[n3stable && n1stable && ! n2stable]
```

Out[ ]=
```
λ12 ≤ 0 && (λ13 > 0 || λ23 > 0) && λ21 ≤ 0 && λ31 ≤ 0 && λ32 ≤ 0
```

Out[ ]=
```
λ12 ≤ 0 && λ13 ≤ 0 && (λ21 > 0 || λ31 > 0) && λ23 ≤ 0 && λ32 ≤ 0
```

Out[ ]=
```
(λ12 > 0 || λ32 > 0) && λ13 ≤ 0 && λ21 ≤ 0 && λ23 ≤ 0 && λ31 ≤ 0
```

In[ ]:=
```
(* i or j&k coexist *)
fc1x23 = FullSimplify[n1stable && n23stable]
fc2x31 = FullSimplify[n2stable && n31stable]
fc3x12 = FullSimplify[n3stable && n12stable]
```

Out[ ]=
```
λ21 ≤ 0 && λ31 ≤ 0 && λ23 > 0 && λ32 > 0 && λ123 ≤ 0
```

Out[ ]=
```
λ12 ≤ 0 && λ32 ≤ 0 && λ31 > 0 && λ13 > 0 && λ231 ≤ 0
```

Out[ ]=
```
λ13 ≤ 0 && λ23 ≤ 0 && λ12 > 0 && λ21 > 0 && λ312 ≤ 0
```

In[ ]:=
```
(* i&j coexist or i&k coexist *)
fc12x23 = FullSimplify[n12stable && n23stable]
fc12x31 = FullSimplify[n12stable && n31stable]
fc23x31 = FullSimplify[n23stable && n31stable]
```

Out[ ]=
```
λ12 > 0 && λ21 > 0 && λ312 ≤ 0 && λ23 > 0 && λ32 > 0 && λ123 ≤ 0
```

Out[ ]=
```
λ12 > 0 && λ21 > 0 && λ312 ≤ 0 && λ31 > 0 && λ13 > 0 && λ231 ≤ 0
```

Out[ ]=
```
λ23 > 0 && λ32 > 0 && λ123 ≤ 0 && λ31 > 0 && λ13 > 0 && λ231 ≤ 0
```

In[ ]:=
```
(* 1 or 2 or 3 *)
fc1x2x3 = FullSimplify[n1stable && n2stable && n3stable]
```

Out[ ]=
```
λ21 ≤ 0 && λ31 ≤ 0 && λ12 ≤ 0 && λ32 ≤ 0 && λ13 ≤ 0 && λ23 ≤ 0
```

In[ ]:=
```
(* stable heteroclinic cycle *)
stablehc = FullSimplify[
  λ12 > 0 && λ21 < 0 && λ23 > 0 && λ32 < 0 && λ31 > 0 && λ13 < 0 && λ12 λ23 λ31 < -λ21 λ32 λ13]
```

Out[ ]=
```
λ12 > 0 && λ21 < 0 && λ23 > 0 && λ32 < 0 && λ31 > 0 && λ13 < 0 && λ12 λ23 λ31 + λ13 λ21 λ32 < 0
```

Equilibria & invasion rates

In[ ]:=
```
(* solve for equilibria *)
eq = SolveEcoEq[];
eq // NumberedGridForm
```

Out[ ]=

1 $\{n1 \to 0, n2 \to 0, n3 \to 0\}$

2 $\left\{n1 \to \frac{r1}{\alpha 11}, n2 \to 0, n3 \to 0\right\}$

3 $\left\{n1 \to 0, n2 \to \frac{r2}{\alpha 22}, n3 \to 0\right\}$

4 $\left\{n1 \to 0, n2 \to 0, n3 \to \frac{r3}{\alpha 33}\right\}$

5 $\left\{n1 \to -\frac{-r2\,\alpha 12+r1\,\alpha 22}{\alpha 12\,\alpha 21-\alpha 11\,\alpha 22}, n2 \to -\frac{-r2\,\alpha 11+r1\,\alpha 21}{-\alpha 12\,\alpha 21+\alpha 11\,\alpha 22}, n3 \to 0\right\}$

6 $\left\{n1 \to -\frac{-r3\,\alpha 13+r1\,\alpha 33}{\alpha 13\,\alpha 31-\alpha 11\,\alpha 33}, n2 \to 0, n3 \to -\frac{-r3\,\alpha 11+r1\,\alpha 31}{-\alpha 13\,\alpha 31+\alpha 11\,\alpha 33}\right\}$

7 $\left\{n1 \to 0, n2 \to -\frac{-r3\,\alpha 23+r2\,\alpha 33}{\alpha 23\,\alpha 32-\alpha 22\,\alpha 33}, n3 \to -\frac{-r3\,\alpha 22+r2\,\alpha 32}{-\alpha 23\,\alpha 32+\alpha 22\,\alpha 33}\right\}$

8 $\left\{n1 \to -\frac{-r3\,\alpha 13\,\alpha 22+r3\,\alpha 12\,\alpha 23+r2\,\alpha 13\,\alpha 32-r1\,\alpha 23\,\alpha 32-r2\,\alpha 12\,\alpha 33+r1\,\alpha 22\,\alpha 33}{\alpha 13\,\alpha 22\,\alpha 31-\alpha 12\,\alpha 23\,\alpha 31-\alpha 13\,\alpha 21\,\alpha 32+\alpha 11\,\alpha 23\,\alpha 32+\alpha 12\,\alpha 21\,\alpha 33-\alpha 11\,\alpha 22\,\alpha 33},\right.$
$n2 \to -\frac{r3\,\alpha 13\,\alpha 21-r3\,\alpha 11\,\alpha 23-r2\,\alpha 13\,\alpha 31+r1\,\alpha 23\,\alpha 31+r2\,\alpha 11\,\alpha 33-r1\,\alpha 21\,\alpha 33}{\alpha 13\,\alpha 22\,\alpha 31-\alpha 12\,\alpha 23\,\alpha 31-\alpha 13\,\alpha 21\,\alpha 32+\alpha 11\,\alpha 23\,\alpha 32+\alpha 12\,\alpha 21\,\alpha 33-\alpha 11\,\alpha 22\,\alpha 33},$
$\left.n3 \to -\frac{r3\,\alpha 12\,\alpha 21-r3\,\alpha 11\,\alpha 22-r2\,\alpha 12\,\alpha 31+r1\,\alpha 22\,\alpha 31+r2\,\alpha 11\,\alpha 32-r1\,\alpha 21\,\alpha 32}{-\alpha 13\,\alpha 22\,\alpha 31+\alpha 12\,\alpha 23\,\alpha 31+\alpha 13\,\alpha 21\,\alpha 32-\alpha 11\,\alpha 23\,\alpha 32-\alpha 12\,\alpha 21\,\alpha 33+\alpha 11\,\alpha 22\,\alpha 33}\right\}$

In[ ]:=
```
(* 2-species invasion rates *)
λ21 = Inv[eq[[2]], n2]
λ12 = Inv[eq[[3]], n1]
λ32 = Inv[eq[[3]], n3]
λ23 = Inv[eq[[4]], n2]
λ13 = Inv[eq[[4]], n1]
λ31 = Inv[eq[[2]], n3]
```

Out[ ]=
$$r2 - \frac{r1\,\alpha 21}{\alpha 11}$$

Out[ ]=
$$r1 - \frac{r2\,\alpha 12}{\alpha 22}$$

Out[ ]=
$$r3 - \frac{r2\,\alpha 32}{\alpha 22}$$

Out[ ]=
$$r2 - \frac{r3\,\alpha 23}{\alpha 33}$$

Out[ ]=
$$r1 - \frac{r3\,\alpha 13}{\alpha 33}$$

Out[ ]=
$$r3 - \frac{r1\,\alpha 31}{\alpha 11}$$

In[∘]:=
```
(* 3-species invasion rates *)
λ312 = Inv[eq⟦5⟧, n3]
λ123 = Inv[eq⟦7⟧, n1]
λ231 = Inv[eq⟦6⟧, n2]
```

Out[∘]=

$$\frac{r3\,\alpha12\,\alpha21 - r3\,\alpha11\,\alpha22 - r2\,\alpha12\,\alpha31 + r1\,\alpha22\,\alpha31 + r2\,\alpha11\,\alpha32 - r1\,\alpha21\,\alpha32}{\alpha12\,\alpha21 - \alpha11\,\alpha22}$$

Out[∘]=

$$\frac{r3\,\alpha13\,\alpha22 - r3\,\alpha12\,\alpha23 - r2\,\alpha13\,\alpha32 + r1\,\alpha23\,\alpha32 + r2\,\alpha12\,\alpha33 - r1\,\alpha22\,\alpha33}{\alpha23\,\alpha32 - \alpha22\,\alpha33}$$

Out[∘]=

$$\frac{r3\,\alpha13\,\alpha21 - r3\,\alpha11\,\alpha23 - r2\,\alpha13\,\alpha31 + r1\,\alpha23\,\alpha31 + r2\,\alpha11\,\alpha33 - r1\,\alpha21\,\alpha33}{-\alpha13\,\alpha31 + \alpha11\,\alpha33}$$

In[∘]:=
```
(* condition for feasible n1-n2-n3 equilibrium *)
feasible123 = FullSimplify[n1 > 0 && n2 > 0 && n3 > 0 /. eq⟦8⟧]
```

Out[∘]=

$$(r3\,\alpha13\,\alpha22 - r3\,\alpha12\,\alpha23 - r2\,\alpha13\,\alpha32 + r1\,\alpha23\,\alpha32 + r2\,\alpha12\,\alpha33 - r1\,\alpha22\,\alpha33)\,(\alpha13\,\alpha22\,\alpha31 - \alpha12\,\alpha23\,\alpha31 - \alpha13\,\alpha21\,\alpha32 + \alpha11\,\alpha23\,\alpha32 + \alpha12\,\alpha21\,\alpha33 - \alpha11\,\alpha22\,\alpha33) > 0 \;\&\&$$
$$(r3\,\alpha13\,\alpha21 - r3\,\alpha11\,\alpha23 - r2\,\alpha13\,\alpha31 + r1\,\alpha23\,\alpha31 + r2\,\alpha11\,\alpha33 - r1\,\alpha21\,\alpha33)\,(-\alpha13\,\alpha22\,\alpha31 + \alpha12\,\alpha23\,\alpha31 + \alpha13\,\alpha21\,\alpha32 - \alpha11\,\alpha23\,\alpha32 - \alpha12\,\alpha21\,\alpha33 + \alpha11\,\alpha22\,\alpha33) > 0 \;\&\&$$
$$(r3\,\alpha12\,\alpha21 - r3\,\alpha11\,\alpha22 - r2\,\alpha12\,\alpha31 + r1\,\alpha22\,\alpha31 + r2\,\alpha11\,\alpha32 - r1\,\alpha21\,\alpha32)\,(\alpha13\,\alpha22\,\alpha31 - \alpha12\,\alpha23\,\alpha31 - \alpha13\,\alpha21\,\alpha32 + \alpha11\,\alpha23\,\alpha32 + \alpha12\,\alpha21\,\alpha33 - \alpha11\,\alpha22\,\alpha33) > 0$$

In[∘]:=
```
(* condition for stable n1-n2-n3 equilibrium — too hideous to look at *)
stable123 = RouthHurwitzCriteria[EcoJacobian[eq⟦8⟧]];
```

A calculations

In[∘]:=
```
(* pairwise asymmetries *)
A12 := Sqrt[α22 α21 / (α11 α12)];
A23 := Sqrt[α33 α32 / (α22 α23)];
A31 := Sqrt[α11 α13 / (α33 α31)];
```

In[∘]:=
```
(* average asymmetry *)
A̅ = Simplify[GeometricMean[{A12, A23, A31}]]
```

Out[∘]=

$$\left(\frac{\alpha13\,\alpha21\,\alpha32}{\alpha12\,\alpha23\,\alpha31}\right)^{1/6}$$

In[∘]:=
```
Simplify[A12 / A̅ == A12^(2/3) / (A23^(1/3) A31^(1/3))]
```

Out[∘]=

```
True
```

In[ ]:=
```
Simplify[Ā A12^(2/3) / (A23^(1/3) A31^(1/3))]
```

Out[ ]=

$$\sqrt{\frac{\alpha 21\,\alpha 22}{\alpha 11\,\alpha 12}}$$

In[ ]:=
```
(* 3-sp recentering factors — for Fig. S3-4 *)
A⁄ = Simplify[{(A12/A31)^(1/3), (A23/A12)^(1/3), (A31/A23)^(1/3)} /.
   {α11 → 1, α22 → 1, α33 → 1}]
```

Out[ ]=

$$\left\{\left(\frac{\alpha 21\,\alpha 31}{\alpha 12\,\alpha 13}\right)^{1/6},\ \frac{1}{\left(\frac{\alpha 21\,\alpha 23}{\alpha 12\,\alpha 32}\right)^{1/6}},\ \left(\frac{\alpha 13\,\alpha 23}{\alpha 31\,\alpha 32}\right)^{1/6}\right\}$$

Results

Figures

Fig. 1 — Outcome legend

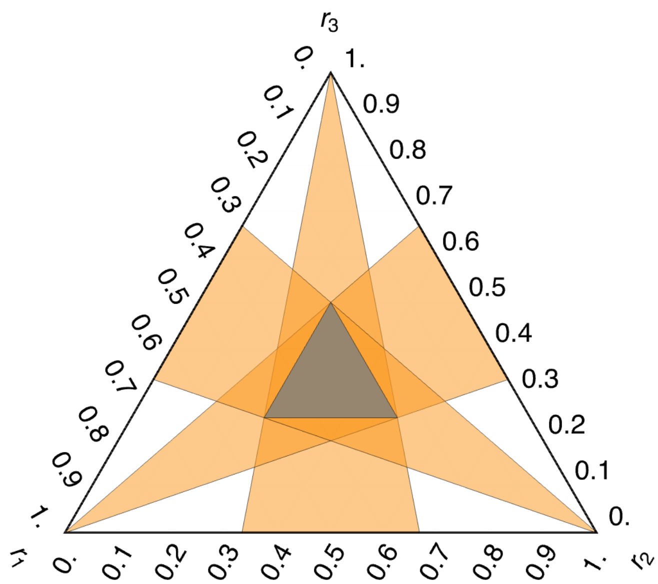
Fig. 2 — Effect of pairwise coexistence vs founder control

In[ ]:=
```
(* Fig. 2A-B - three pairs coexist *)
α = ρAtoα[{0.5, 0.5, 0.5}, 1];
α // MatrixForm
FeasibilityPlot[α]
OutcomePlot[α]
```

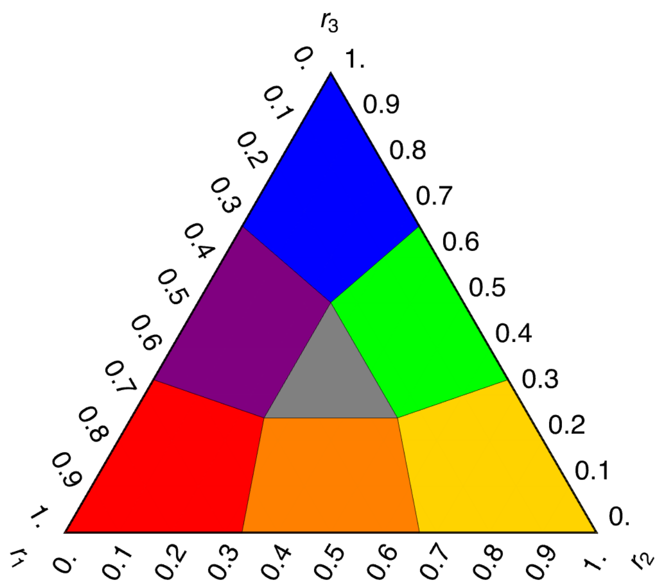
Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix}$$

Out[ ]=



Out[ ]=

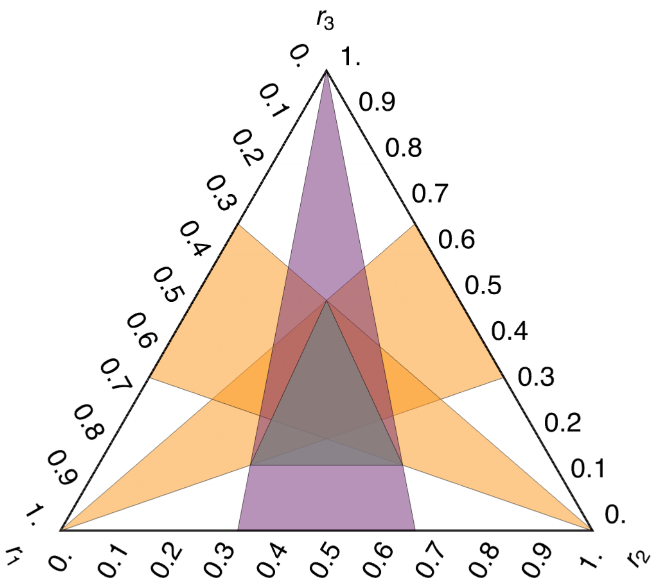


In[ ]:=
```
(* Fig. 2C-D - two pairs coexist, one pair founder control *)
α = ρAtoα[{2, 0.5, 0.5}, 1];
α // MatrixForm
FeasibilityPlot[α]
OutcomePlot[α]
```

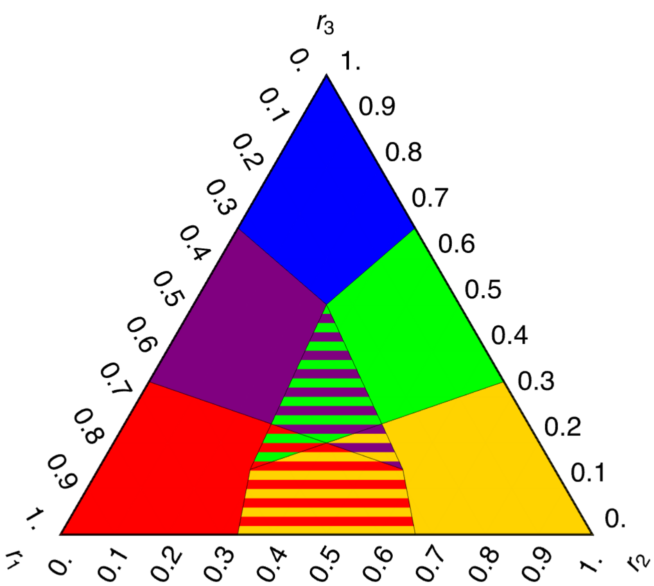
Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 0.5 \\ 2 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix}$$

Out[ ]=



Out[ ]=

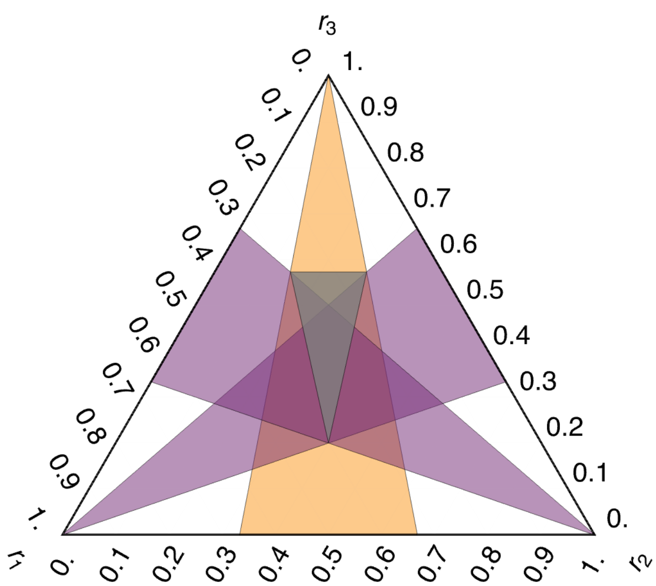


In[∘]:=
```
(* Fig. 2E-F - one pair coexists, two pairs founder control *)
α = ρAtoα[{0.5, 2, 2}, 1];
α // MatrixForm
FeasibilityPlot[α]
OutcomePlot[α]
```

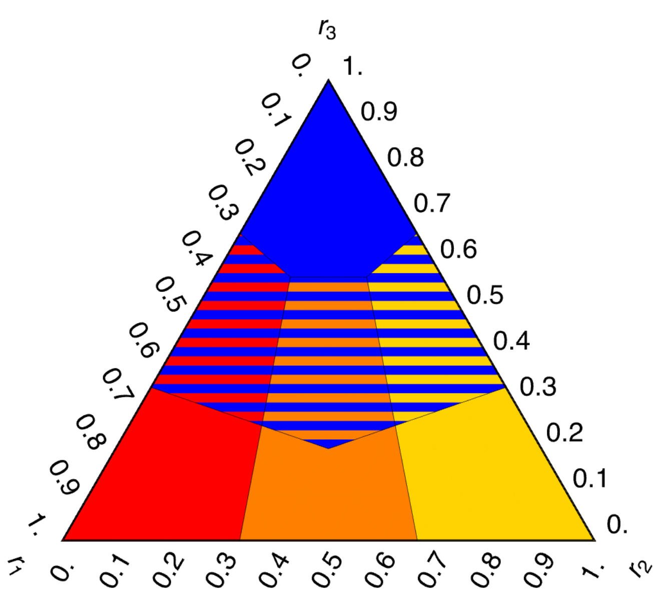
Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0.5 & 2 \\ 0.5 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

Out[∘]=



Out[∘]=

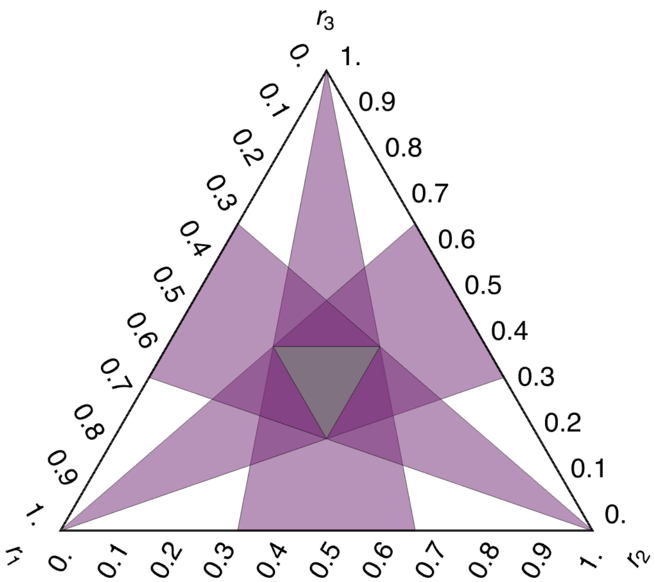


In[ ]:=
```
(* Fig. 2G-H - three pairs founder control *)
α = ρAtoα[{2, 2, 2}, 1];
α // MatrixForm
FeasibilityPlot[α]
OutcomePlot[α]
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

Out[ ]=



Out[ ]=

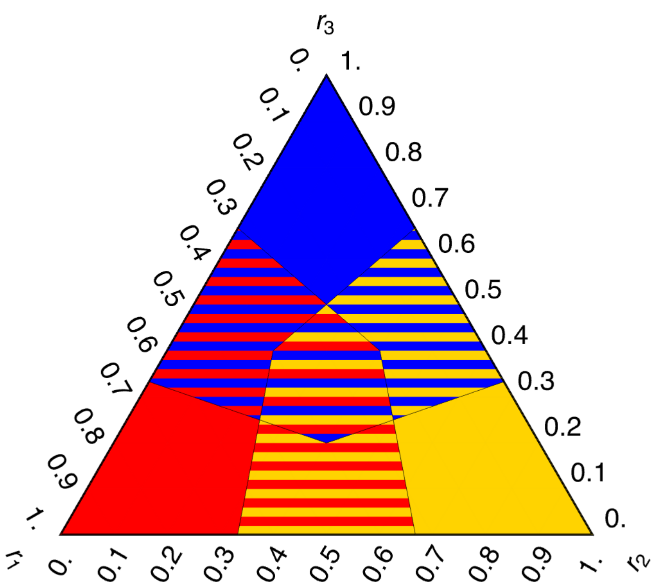


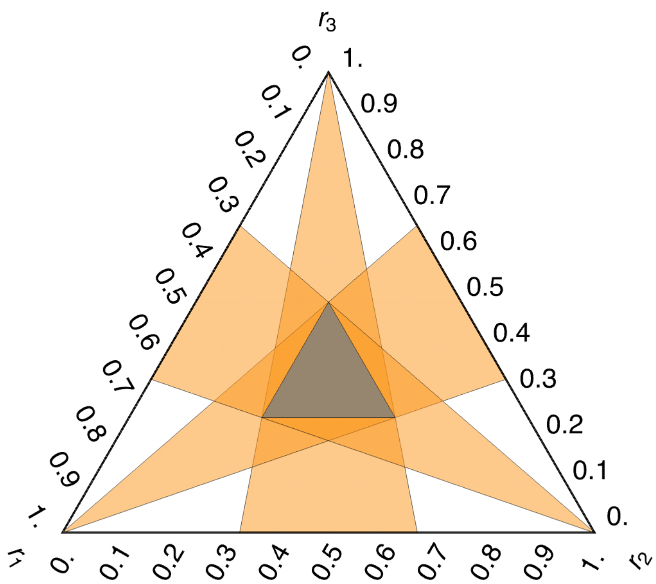
Fig. 3 — Effect of intransitivity

In[∘]:=
```
(* Fig. 3A-B - A=1 *)
α = ρAtoα[0.5, 1];
α // MatrixForm
FeasibilityPlot[α]
OutcomePlot[α]
```

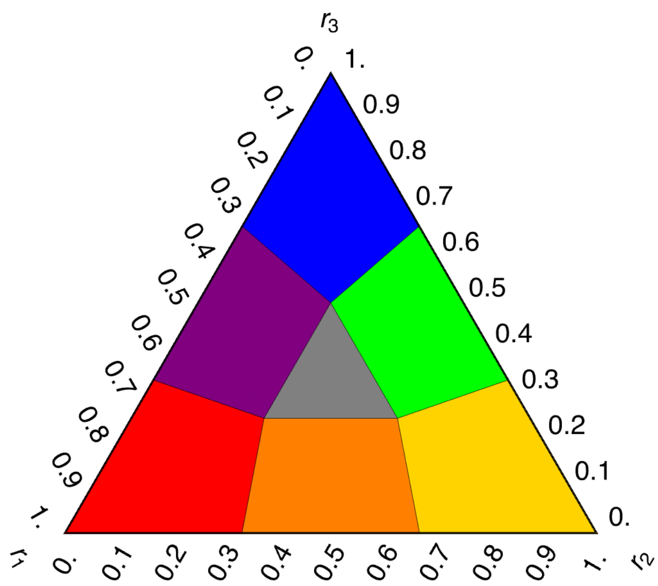
Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0.5 & 0.5 \\ 0.5 & 1 & 0.5 \\ 0.5 & 0.5 & 1 \end{pmatrix}$$

Out[∘]=



Out[ ]=

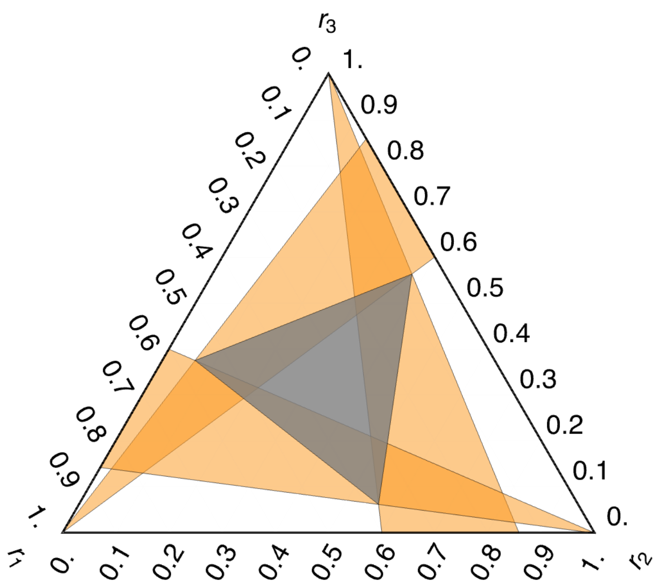


In[ ]:=
```
(* Fig. 3C-D - A=3 *)
α = ρAtoα[0.5, 3];
α // MatrixForm
FeasibilityPlot[α]
OutcomePlot[α]
```

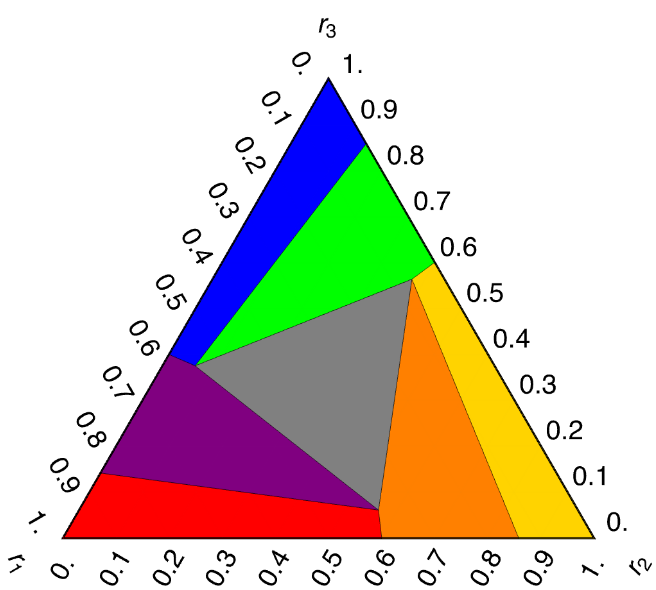
Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0.166667 & 1.5 \\ 1.5 & 1 & 0.166667 \\ 0.166667 & 1.5 & 1 \end{pmatrix}$$

Out[ ]=



Out[ ]=

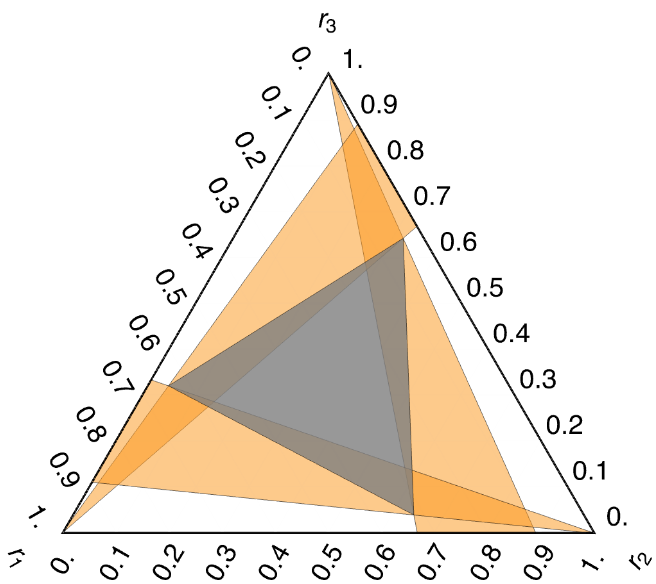


In[∘]:=
```
(* Fig. 3E-F - A=4 *)
α = ρAtoα[0.5, 4];
α // MatrixForm
FeasibilityPlot[α]
OutcomePlot[α]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0.125 & 2. \\ 2. & 1 & 0.125 \\ 0.125 & 2. & 1 \end{pmatrix}$$

Out[∘]=



Out[∘]=

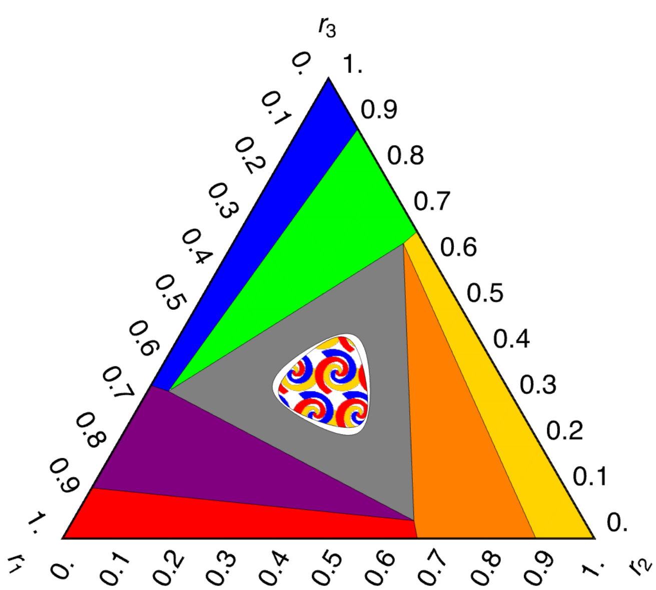


Fig. 4 — Example time series

In[∘]:=
```
Setαs[ρAtoα[0.5, 4]] // MatrixForm
```

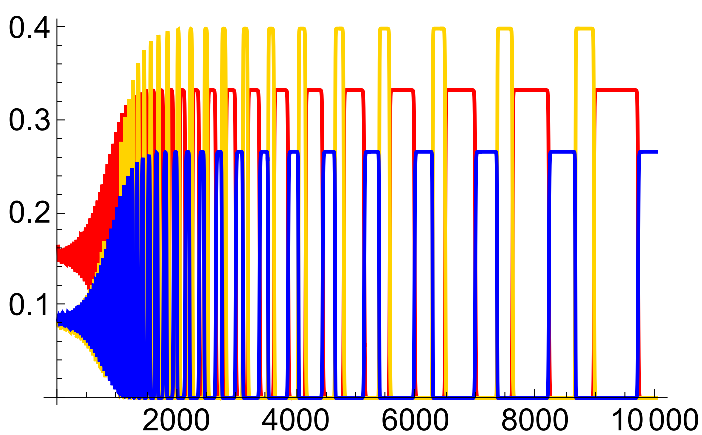
Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0.125 & 2. \\ 2. & 1 & 0.125 \\ 0.125 & 2. & 1 \end{pmatrix}$$

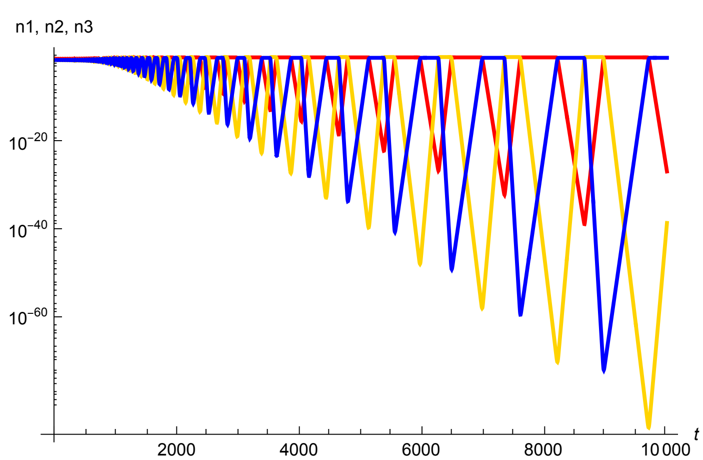
In[∘]:=
```
r1 = 1 / 3;
r3 := 1 - r1 - r2;
```

In[∘]:=
```
(* Fig. 4A – heteroclinic cycle *)
r2 = 0.4;
sol =
  EcoSim[RuleListAdd[eq⟦8⟧, {n1 → 0.01}], 10 000, NDSolveOpts → {AccuracyGoal → ∞}];
PlotDynamics[sol, AxesLabel → None, TicksStyle → 16]
PlotDynamics[sol, Logged → True]
PlotTrajectory[sol, PlotStyle → Gray, ViewPoint → {1, 1, 0.8}, PlotPoints → 200]
```

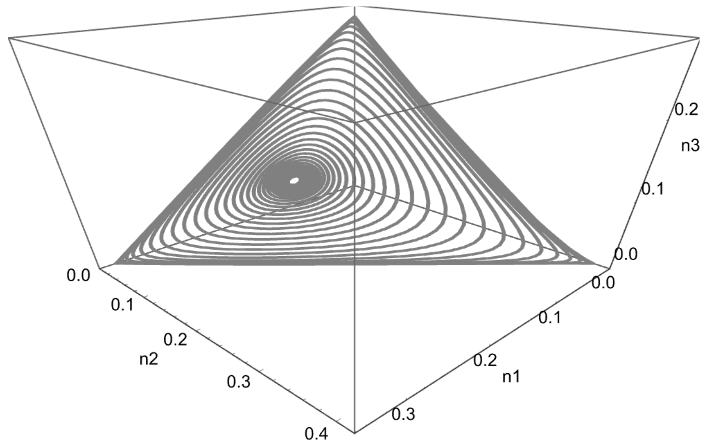
Out[∘]=



Out[∘]=

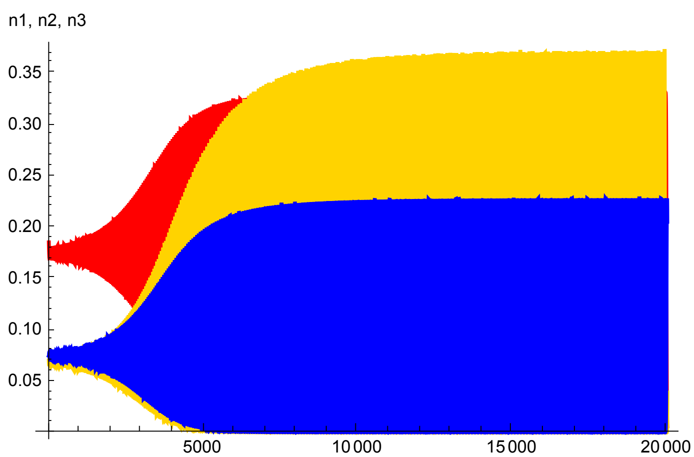


Out[ ]=

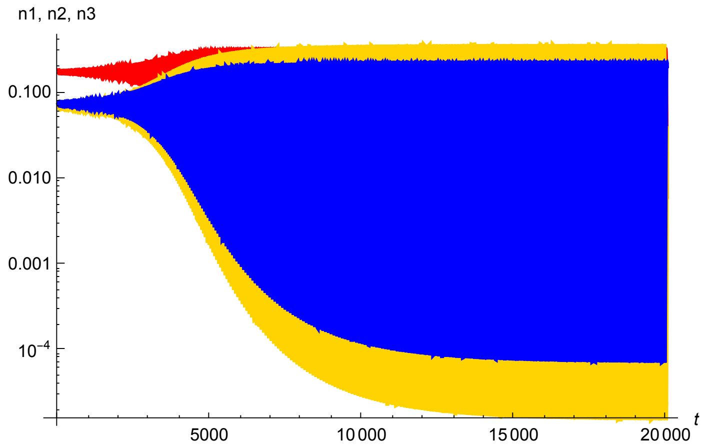


In[ ]:=
```
(* Fig. 4B – limit cycle *)
r2 = 0.43;
sol = EcoSim[RuleListAdd[eq⟦8⟧, {n1 → 0.01}], 20 000];
PlotDynamics[sol, PlotPoints → 200]
PlotDynamics[sol, Logged → True, PlotPoints → 200]
PlotDynamics[FinalSlice[sol, 1000], AxesLabel → None, TicksStyle → 16]
PlotTrajectory[FinalSlice[sol, 1000],
 PlotStyle → Gray, ViewPoint → {1, 1, 0.8}, PlotPoints → 200]
```

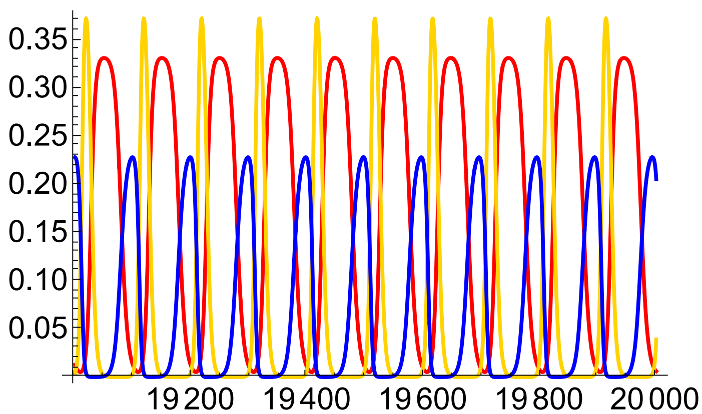
Out[ ]=



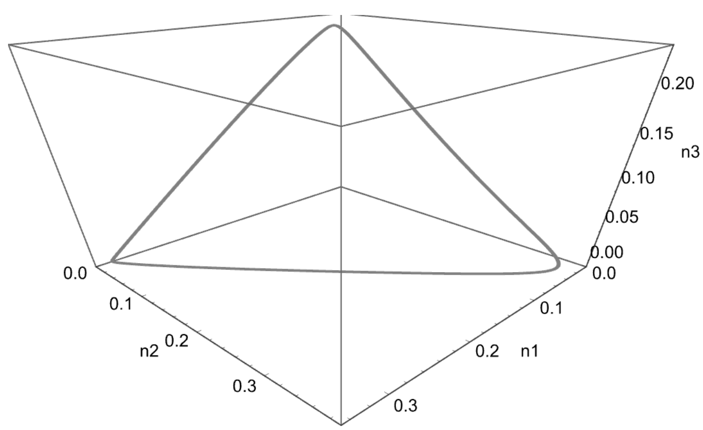
Out[ ]=


n1, n2, n3
0.100
0.010
0.001
10^-4
5000
10000
15000
20000
t

Out[ ]=

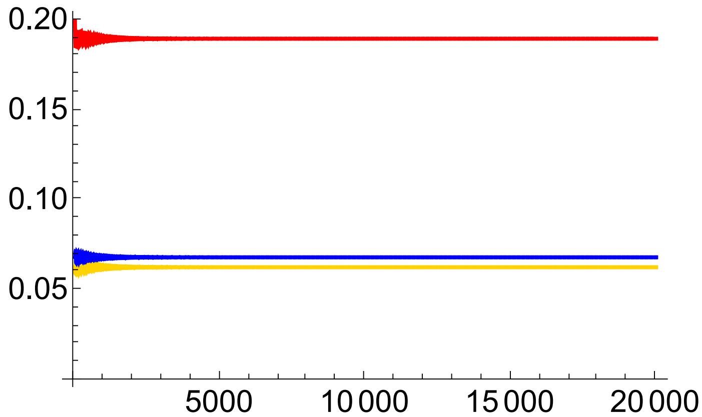

0.35
0.30
0.25
0.20
0.15
0.10
0.05
19200
19400
19600
19800
20000

Out[ ]=


0.20
0.15
n3
0.10
0.05
0.00
0.0
0.0
0.1
n2
0.2
0.3
0.1
0.2
n1
0.3

In[ ]:=
```
(* Fig. 4C – stable equilibrium *)
r2 = 0.45;
sol = EcoSim[RuleListAdd[eq⟦8⟧, {n1 → 0.01}], 20 000];
PlotDynamics[sol, AxesLabel → None, PlotPoints → 200, TicksStyle → 16]
```

Out[ ]=



Figs. 5, S2 — Outcome plot grids

In[ ]:=
```
Clear[α, α12, α23, α31, α21, α32, α13, α11, α22, α33, ρ, A];
```

In[ ]:=
```
ρ := 2 ^ ρp;
A := 2 ^ Ap;
```

In[ ]:=
```
Dynamic[{Ap, ρp}]
```

Out[ ]=

{Ap, ρp}

In[ ]:=
```
(* Fig. 5 *)
plots = Table[OutcomePlot[ρAtoα[ρ, A], Axes → False,
    GridSpacing → 1 / 30, BoundaryStyle → Black], {Ap, 3, 0, -1}, {ρp, -3, 3, 1}];
```

Greater: Invalid comparison with ComplexInfinity attempted.

Greater: Invalid comparison with ComplexInfinity attempted.

Greater: Invalid comparison with ComplexInfinity attempted.

General: Further output of Greater::nord will be suppressed during this calculation.

LessEqual: Invalid comparison with ComplexInfinity attempted.

LessEqual: Invalid comparison with ComplexInfinity attempted.

LessEqual: Invalid comparison with ComplexInfinity attempted.

General: Further output of LessEqual::nord will be suppressed during this calculation.

Infinity: Indeterminate expression 0 ComplexInfinity encountered.

Infinity: Indeterminate expression 0 ComplexInfinity encountered.

Infinity: Indeterminate expression 0. ComplexInfinity encountered.

General: Further output of Infinity::indet will be suppressed during this calculation.

```
(* Fig. 5 *)
(*plots=Table[Rasterize[OutcomePlot[ρAtoα[ρ,A],Axes→False,GridSpacing→1/30,
    BoundaryStyle→Black],ImageResolution→300],{Ap,3,0,-1},{ρp,-3,3,1}];*)
```

Greater: Invalid comparison with ComplexInfinity attempted.

Greater: Invalid comparison with ComplexInfinity attempted.

Greater: Invalid comparison with ComplexInfinity attempted.

General: Further output of Greater::nord will be suppressed during this calculation.

LessEqual: Invalid comparison with ComplexInfinity attempted.

LessEqual: Invalid comparison with ComplexInfinity attempted.

LessEqual: Invalid comparison with ComplexInfinity attempted.

General: Further output of LessEqual::nord will be suppressed during this calculation.

Infinity: Indeterminate expression 0 ComplexInfinity encountered.

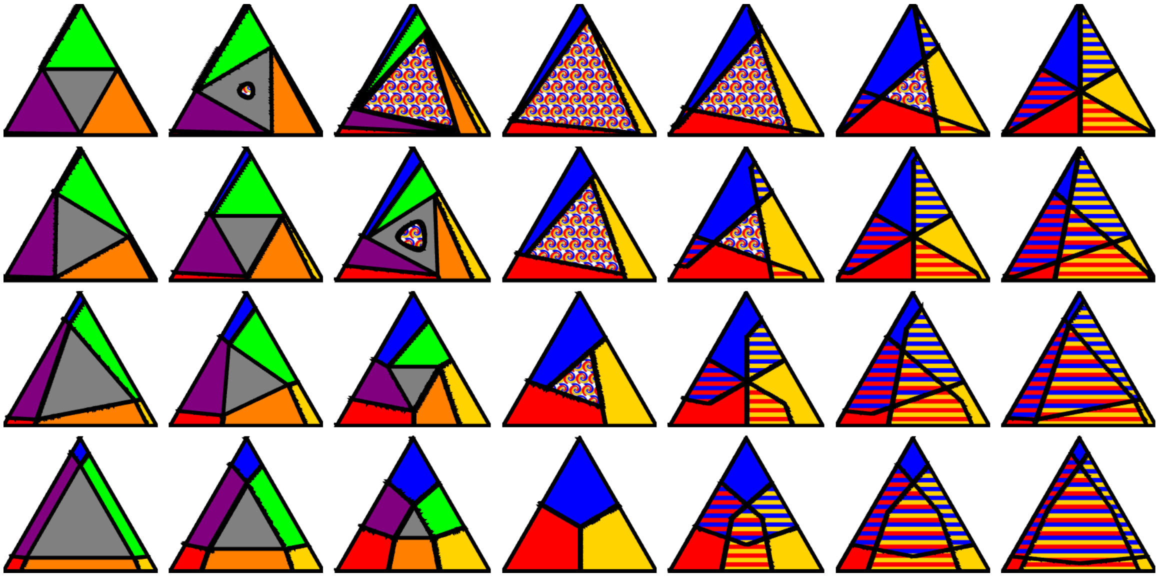
Infinity: Indeterminate expression 0 ComplexInfinity encountered.

Infinity: Indeterminate expression 0. ComplexInfinity encountered.

General: Further output of Infinity::indet will be suppressed during this calculation.

In[ ]:=
```
GraphicsGrid[plots]
```

Out[ ]=



In[ ]:=
```
GraphicsGrid[plots, ImageSize → 3000];
```

In[ ]:=
```
(* Fig. S2 *)
plots2 = Table[OutcomePlot[ρAtoα[{2^ρp, 2^ρp, 2^-ρp}, 2^Ap], Axes → False,
     GridSpacing → 1 / 30, BoundaryStyle → Black],
    {Ap, 1, 0, -0.25}, {ρp, -0.75, 0.75, 0.25}];
```

Greater: Invalid comparison with ComplexInfinity attempted.

Greater: Invalid comparison with ComplexInfinity attempted.

Greater: Invalid comparison with ComplexInfinity attempted.

General: Further output of Greater::nord will be suppressed during this calculation.

LessEqual: Invalid comparison with ComplexInfinity attempted.

LessEqual: Invalid comparison with ComplexInfinity attempted.

LessEqual: Invalid comparison with ComplexInfinity attempted.

General: Further output of LessEqual::nord will be suppressed during this calculation.

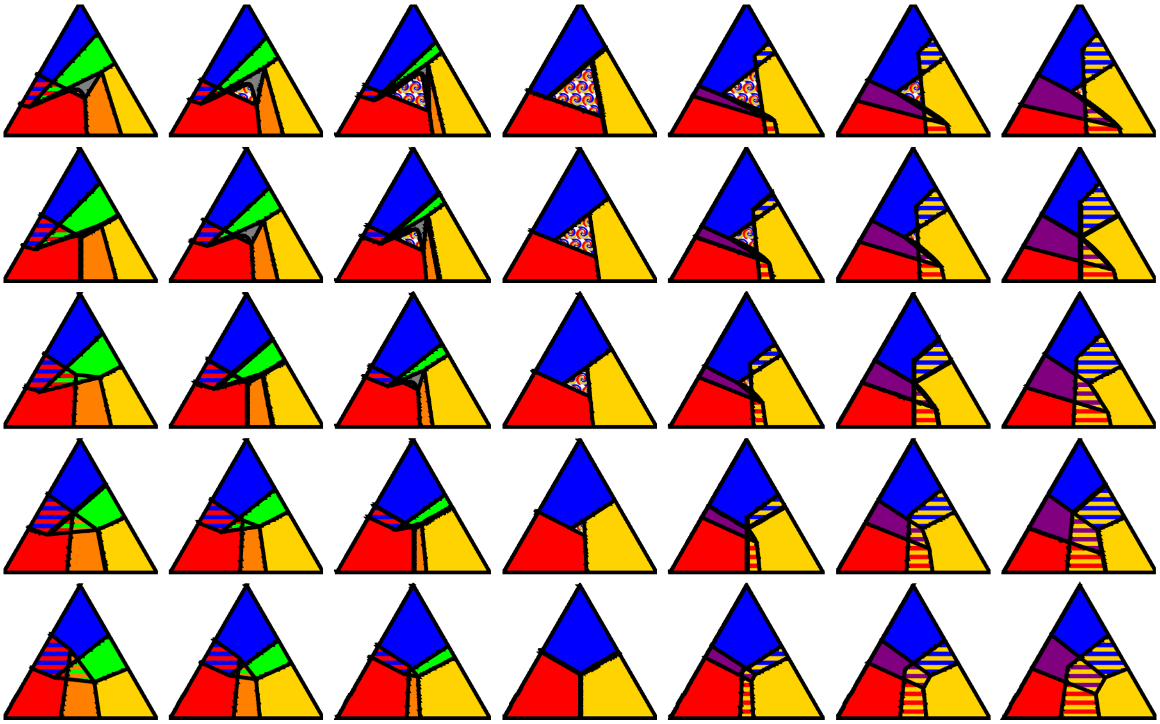
Infinity: Indeterminate expression 0. ComplexInfinity encountered.

Infinity: Indeterminate expression 0. ComplexInfinity encountered.

Infinity: Indeterminate expression 0. ComplexInfinity encountered.

General: Further output of Infinity::indet will be suppressed during this calculation.

In[ ]:=
```
GraphicsGrid[plots2]
```
Out[ ]=



```
GraphicsGrid[plots2, ImageSize → 3000];
```

Fig. 6 — Size of coexistence region

```
(* note: p works,
ρ doesn't <https://mathematica.stackexchange.com/questions/272646> *)
```

In[ ]:=
```
Clear[α12, α13, α21, α23, α31, α32, ρ, A, r1, r2, r3, p]
```

In[ ]:=
```
Setαs[ρAtoα[p, A]] // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{p}{A} & A\,p \\ A\,p & 1 & \frac{p}{A} \\ \frac{p}{A} & A\,p & 1 \end{pmatrix}$$

In[ ]:=
```
r3 := 1 - r1 - r2;
```

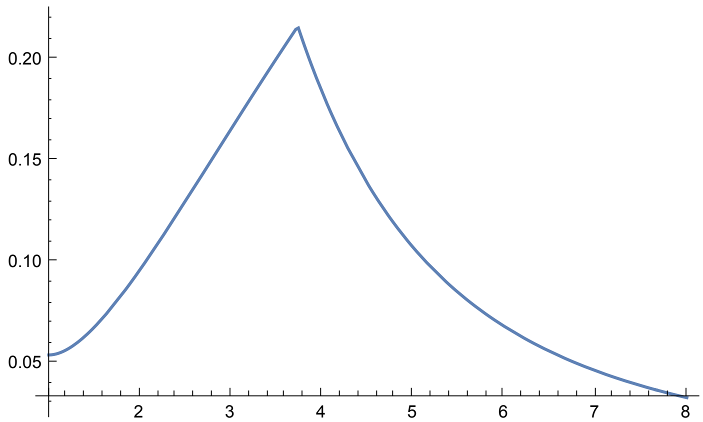
In[ ]:=
```
ineq = Simplify[feasible123 && stable123[[1, 1, 2]]]
```
Out[ ]=

$$A\left(p^3+A^6 p^3+A^3\left(1-3 p^2\right)\right)\left(A^2\left(r1-p^2\, r1\right)-A\, p\, r2+A^4 p^2\, r2+A^3 p\,(-1+r1+r2)-p^2\,(-1+r1+r2)\right)>0 \;\&\&\; A\left(p^3+A^6 p^3+A^3\left(1-3 p^2\right)\right)\left(A^3 p\, r1-p^2\, r1+A^2\left(-1+p^2\right) r2-A\, p\,(-1+r1+r2)+A^4 p^2\,(-1+r1+r2)\right)<0 \;\&\&\; A\left(p^3+A^6 p^3+A^3\left(1-3 p^2\right)\right)\left(-A\, p\, r1+A^4 p^2\, r1-A^3 p\, r2+p^2\, r2+A^2\left(-1+p^2\right)(-1+r1+r2)\right)>$$

```
0 && (p^3 + A^6 p^3 + A^3 (1 - 3 p^2))^2
  (A^2 (r1 - p^2 r1) - A p r2 + A^4 p^2 r2 + A^3 p (-1 + r1 + r2) - p^2 (-1 + r1 + r2))
  (A^3 p r1 - p^2 r1 + A^2 (-1 + p^2) r2 - A p (-1 + r1 + r2) + A^4 p^2 (-1 + r1 + r2))
  (-A p r1 + A^4 p^2 r1 - A^3 p r2 + p^2 r2 + A^2 (-1 + p^2) (-1 + r1 + r2)) < 0 &&
(A + p + A^2 p) (A p + A^3 p - p^2 - A^4 p^2 + A^2 (-1 + p^2))^2 (p^7 r1 r2 (-1 + r1 + r2) +
    A^14 p^7 r1 r2 (-1 + r1 + r2) + A p^6 (-r1^3 + (-1 + r2)^2 r2 + r1^2 (1 + r2) + r1 r2 (-3 + 4 r2)) +
    A^13 p^6 (r1^3 - (-1 + r2) r2^2 + r1^2 (-2 + 4 r2) + r1 (1 - 3 r2 + r2^2)) +
    A^12 p^5 (-((-3 + p^2) r1^3) + (2 + p^2 (-1 + r2) - 3 r2) (-1 + r2) r2 +
        r1 (1 + (2 - 3 p^2) r2 + (-6 + 4 p^2) r2^2) + r1^2 (-4 + 3 r2 + p^2 (1 + r2))) +
    A^7 ((-1 - 4 p^4 + 2 p^6) (-1 + r2) r2 + r1^2 (-1 + p^6 (2 - 4 r2) + r2 + 6 p^2 r2 + 4 p^4 (-1 + 3 r2)) -
        r1 (-1 + r2) (1 + p^4 (4 - 12 r2) - r2 - 6 p^2 r2 + p^6 (-2 + 4 r2))) +
    A^4 p^3 ((2 - 3 p^2 + 3 p^4) r1^3 + r1^2 (-3 p^2 (-1 + r2) - 7 r2 + p^4 (-4 + 3 r2)) -
        (-1 + r2) r2 (2 (-2 + r2) - 3 p^2 (-1 + r2) + p^4 (-2 + 3 r2)) +
        r1 (-2 + 13 r2 - 13 r2^2 + 3 p^2 r2 (-1 + 2 r2) + p^4 (1 + 2 r2 - 6 r2^2))) +
    A^2 p^5 ((-3 + p^2) r1^3 - (-1 + r2) r2 (1 + (-3 + p^2) r2) + r1^2 (5 - 6 r2 + p^2 (-2 + 4 r2)) +
        r1 (-2 + 2 r2 + 3 r2^2 + p^2 (1 - 3 r2 + r2^2))) + A^6 p (-1 + r1^3 + r2 - r2^3 + r1^2 (-3 + 4 r2) +
        r1 (2 - 4 r2 + r2^2) + p^2 (1 + 3 r1^3 - 3 r2 + 6 r2^2 - 3 r1 r2^2 - 3 r2^3 + r1^2 (-3 + 6 r2)) +
        p^6 (1 + 2 r1^3 + r1^2 (1 - 7 r2) - 5 r2 + 7 r2^2 - 2 r2^3 + r1 (-3 + 14 r2 - 13 r2^2)) +
        p^4 (-(1 - 2 r2)^2 + 4 r1^2 (-1 + 3 r2) + 4 r1 (1 - 4 r2 + 3 r2^2))) +
    A^10 p^3 ((-2 + 3 p^2 - 3 p^4) r1^3 + r1^2 (6 + p^4 (5 - 6 r2) + 6 p^2 (-1 + r2) - 13 r2) +
        (-1 + r2) r2 (-3 p^2 r2 + 2 (1 + r2) + p^4 (-1 + 3 r2)) +
        r1 (-4 + 13 r2 - 7 r2^2 - 3 p^2 (-1 + r2 + r2^2) + p^4 (-2 + 2 r2 + 3 r2^2))) +
    A^9 p^2 (1 - 3 r1^3 + r1^2 (5 - 6 r2) + r2 - 4 r2^2 + 3 r2^3 + r1 (-2 + 2 r2 + 3 r2^2) -
        3 p^4 (r1^3 + r1^2 (-1 + r2) - (-1 + r2)^2 r2 + r1 (r2 - 2 r2^2)) -
        p^2 (1 + r2 - r2^2 + r1^2 (-1 + 4 r2) + r1 (1 - 5 r2 + 4 r2^2))) +
    A^5 p^2 (1 + 3 r1^3 - 2 r2 + 5 r2^2 - 3 r2^3 + r1^2 (-4 + 3 r2) + r1 (1 + 2 r2 - 6 r2^2) +
        3 p^4 (r1^3 + 2 r1^2 (-1 + r2) - (-1 + r2) r2^2 - r1 (-1 + r2 + r2^2)) -
        p^2 (1 + r2 - r2^2 + r1^2 (-1 + 4 r2) + r1 (1 - 5 r2 + 4 r2^2))) +
    A^8 p (-1 - r1^3 + r1 (1 - 2 r2)^2 + 2 r2 + r1^2 r2 - 3 r2^2 + r2^3 +
        p^6 (1 - 2 r1^3 + r1^2 (7 - 13 r2) - 3 r2 + r2^2 + 2 r2^3 + r1 (-5 + 14 r2 - 7 r2^2)) +
        p^4 (-(1 - 2 r2)^2 + 4 r1^2 (-1 + 3 r2) + 4 r1 (1 - 4 r2 + 3 r2^2)) +
        p^2 (1 - 3 r1^3 - 3 r1^2 (-2 + r2) - 3 r2^2 + 3 r2^3 + r1 (-3 + 6 r2^2))) +
    A^11 p^4 (1 + 2 r1^3 - 4 r2 - 7 r1^2 r2 + 6 r2^2 - 2 r2^3 + r1 (-2 + 13 r2 - 13 r2^2) +
        p^2 (-1 + 3 r2 - 3 r2^2 + r1^2 (-3 + 6 r2) + r1 (3 - 9 r2 + 6 r2^2))) +
    A^3 p^4 (1 - 2 r1^3 + r1^2 (6 - 13 r2) - 2 r2 + 2 r2^3 + r1 (-4 + 13 r2 - 7 r2^2) +
        p^2 (-1 + 3 r2 - 3 r2^2 + r1^2 (-3 + 6 r2) + r1 (3 - 9 r2 + 6 r2^2)))) > 0
```

```
In[•]:= reg = ImplicitRegion[ineq, {{r1, 0, 1}, {r2, 0, 1}}];
```

In[∘]:=
```
(* 12 s *)
Clear[A]; p = 0.5;
Plot[Area[reg] Sqrt[3], {A, 1, 8}, PlotPoints → 5]
```
Out[∘]=



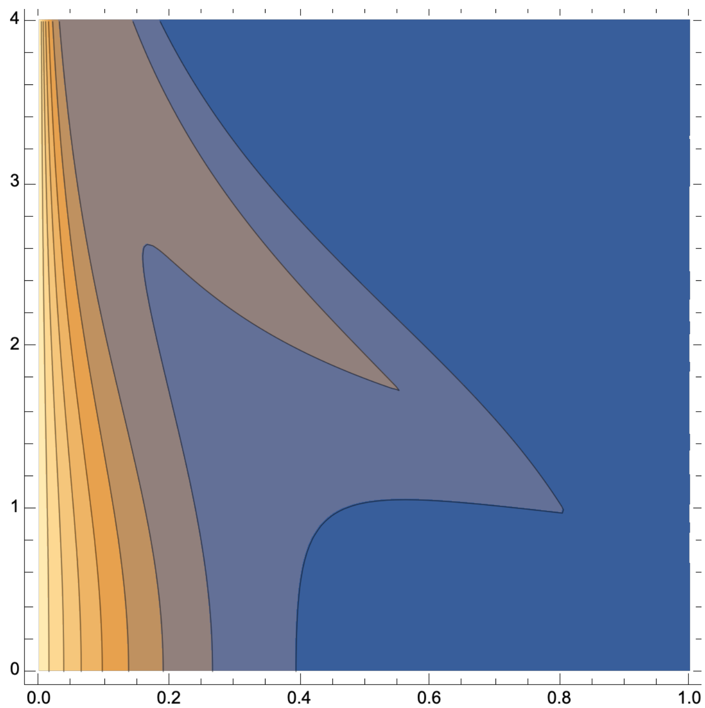
In[∘]:=
```
Clear[A, p]
```

In[∘]:=
```
Dynamic[{Ap, p}]
```
Out[∘]=
```
{Ap, p}
```

```
(* didn't finish overnight... *)
A := 2^Ap; p := 2^pp;
ContourPlot[Area[reg] Sqrt[3], {pp, 0, 1}, {Ap, 0, 4}, PlotPoints → 5,
 PlotRange → {0, 1}, MaxRecursion → 4, Contours → Table[x, {x, 0, 1, 0.1}],
 FrameTicks →
  {{Table[{Ap, 2^Ap}, {Ap, 0, 4}], None}, {Table[{pp, 2^pp}, {ρp, -4, 0}], None}}]
```
Out[∘]=
```
$Aborted
```

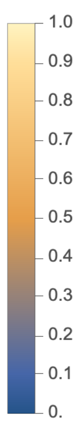
```
(* 7637 sec *)
A := 2^Ap;
ContourPlot[Area[reg] Sqrt[3], {p, 0, 1}, {Ap, 0, 4}, PlotPoints → 5,
 PlotRange → {0, 1}, MaxRecursion → 4, Contours → Table[x, {x, 0, 1, 0.1}]]
```

Out[ ]=



In[ ]:=
```
BarLegend[{"M10DefaultDensityGradient", {0, 1}}, Ticks → Table[x, {x, 0, 1, 0.1}]]
```

Out[ ]=



In[ ]:=
```
Clear[A, p]
```

In[ ]:=
```
stablehc
```

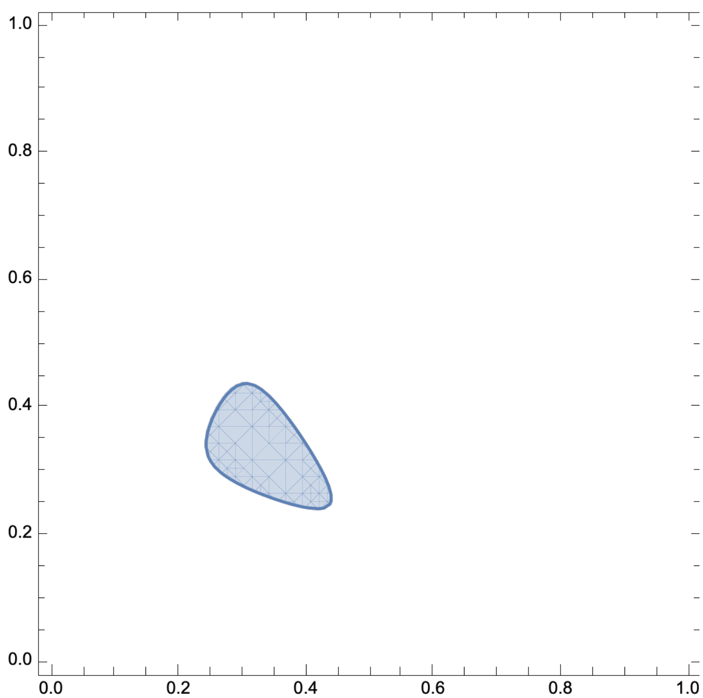
Out[ ]=

$$r1 - \frac{p\,r2}{A} > 0 \;\&\&\; -A\,p\,r1 + r2 < 0 \;\&\&\; -\frac{p\,(1 - r1 - r2)}{A} + r2 > 0 \;\&\&$$
$$1 - r1 - r2 - A\,p\,r2 < 0 \;\&\&\; 1 - r1 - \frac{p\,r1}{A} - r2 > 0 \;\&\&\; r1 - A\,p\,(1 - r1 - r2) < 0 \;\&\&$$
$$\left(1 - r1 - \frac{p\,r1}{A} - r2\right)\left(-\frac{p\,(1 - r1 - r2)}{A} + r2\right)\left(r1 - \frac{p\,r2}{A}\right) +$$
$$(r1 - A\,p\,(1 - r1 - r2))\,(-A\,p\,r1 + r2)\,(1 - r1 - r2 - A\,p\,r2) < 0$$

In[ ]:=
```
reg = ImplicitRegion[stablehc, {{r1, 0, 1}, {r2, 0, 1}}];
```

In[ ]:=
```
A = 4;
p = 0.5;
RegionPlot[{r1, r2} ∈ reg, {r1, 0, 1}, {r2, 0, 1}]
```

Out[ ]=



In[ ]:=
```
Clear[A, p]
```

In[ ]:=
```
Dynamic[{Ap, p}]
```

Out[ ]=

{Ap, p}

```
(* crashed kernel *)
A := 2^Ap;
ContourPlot[Area[reg] Sqrt[3], {p, 0, 1}, {Ap, 0, 4}, PlotPoints → 5,
 PlotRange → {0, 1}, MaxRecursion → 4, Contours → Table[x, {x, 0, 1, 0.1}]]
```

Fig. S1 — Stability of symmetric equilibrium a la May & Leonard

In[ ]:=
```
Clear[ρ, A]
```

In[ ]:=
```
(* assume no fitness differences, equal ρ and equal A *)
r1 = r2 = r3 = 1;
Setαs[ρAtoα[ρ, A]] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & \frac{\rho}{A} & A\,\rho \\ A\,\rho & 1 & \frac{\rho}{A} \\ \frac{\rho}{A} & A\,\rho & 1 \end{pmatrix}$$

Note: May & Leonard's α & β correspond to our ρ/A & Aρ.

In[ ]:=
```
(* constructed with our parameterization -- note: A ranges from 1/8 to 8 *)
Clear[ρ, A];
Show[
 RegionPlot[True, {ρ, 1 / 8, 8}, {A, 1 / 8, 8},
  MaxRecursion → 3, ScalingFunctions → {"Log2", "Log2"},
  PlotStyle → Texture[spirals2], TextureCoordinateScaling → False,
  TextureCoordinateFunction → ({#1 * 2, #2 * 2} &), BoundaryStyle → Black],
 RegionPlot[stable123, {ρ, 1 / 8, 8}, {A, 1 / 8, 8}, MaxRecursion → 3,
  ScalingFunctions → {"Log2", "Log2"}, PlotStyle → Gray, BoundaryStyle → Black],
 RegionPlot[! stable123 && 1 / ρ < A < ρ, {ρ, 1 / 8, 8},
  {A, 1 / 8, 8}, MaxRecursion → 3, ScalingFunctions → {"Log2", "Log2"},
  Mesh → {Table[x, {x, -3, 3, 3 / 60}]}, MeshStyle → None, MeshFunctions → {#2 &},
  MeshShading → {Color[n1], Color[n2], Color[n3]}, BoundaryStyle → Black],
 FrameTicks →
  {{Table[{Ap, 2 ^ Ap}, {Ap, -3, 3}], None}, {Table[{ρp, 2 ^ ρp}, {ρp, -3, 3}], None}},
 AspectRatio → 1, ImageSize → Large, FrameTicksStyle → 16]
```

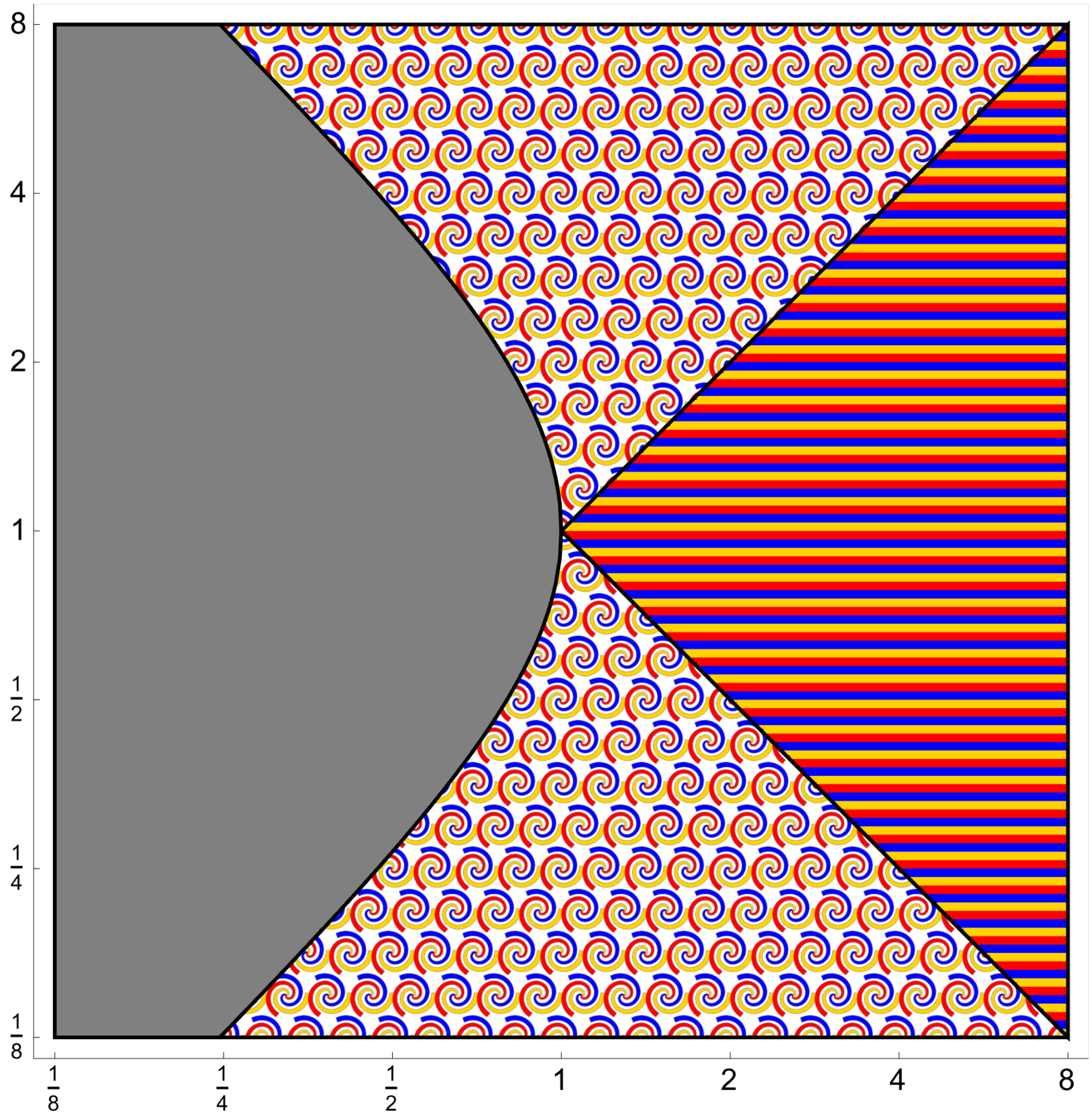
Infinity: Indeterminate expression 0. ComplexInfinity encountered.

Infinity: Indeterminate expression 0. ComplexInfinity encountered.

Infinity: Indeterminate expression 0. ComplexInfinity encountered.

General: Further output of Infinity::indet will be suppressed during this calculation.

Out[ ]=



In[ ]:=
```
(* same, but created with May & Leonard'
 s stability conditions (see their Fig. 1) *)
{α, β} := {ρ / A, A ρ};

Show[
 RegionPlot[True, {ρ, 1 / 8, 8}, {A, 1 / 8, 8},
  MaxRecursion → 3, ScalingFunctions → {"Log2", "Log2"},
  PlotStyle → Texture[spirals2], TextureCoordinateScaling → False,
  TextureCoordinateFunction → ({#1 * 2, #2 * 2} &), BoundaryStyle → Black],
 RegionPlot[α + β < 2, {ρ, 1 / 8, 8}, {A, 1 / 8, 8}, MaxRecursion → 3,
  ScalingFunctions → {"Log2", "Log2"}, PlotStyle → Gray, BoundaryStyle → Black],
 RegionPlot[α > 1 && β > 1, {ρ, 1 / 8, 8}, {A, 1 / 8, 8},
  MaxRecursion → 3, ScalingFunctions → {"Log2", "Log2"},
  Mesh → {Table[x, {x, -3, 3, 3 / 60}]}, MeshStyle → None, MeshFunctions → {#2 &},
  MeshShading → {Color[n1], Color[n2], Color[n3]}, BoundaryStyle → Black
 ],
 FrameTicks →
  {{Table[{Ap, 2 ^ Ap}, {Ap, -3, 3}], None}, {Table[{ρp, 2 ^ ρp}, {ρp, -3, 3}], None}},
 AspectRatio → 1, ImageSize → Large, FrameTicksStyle → 16
]
```

Out[▪]=

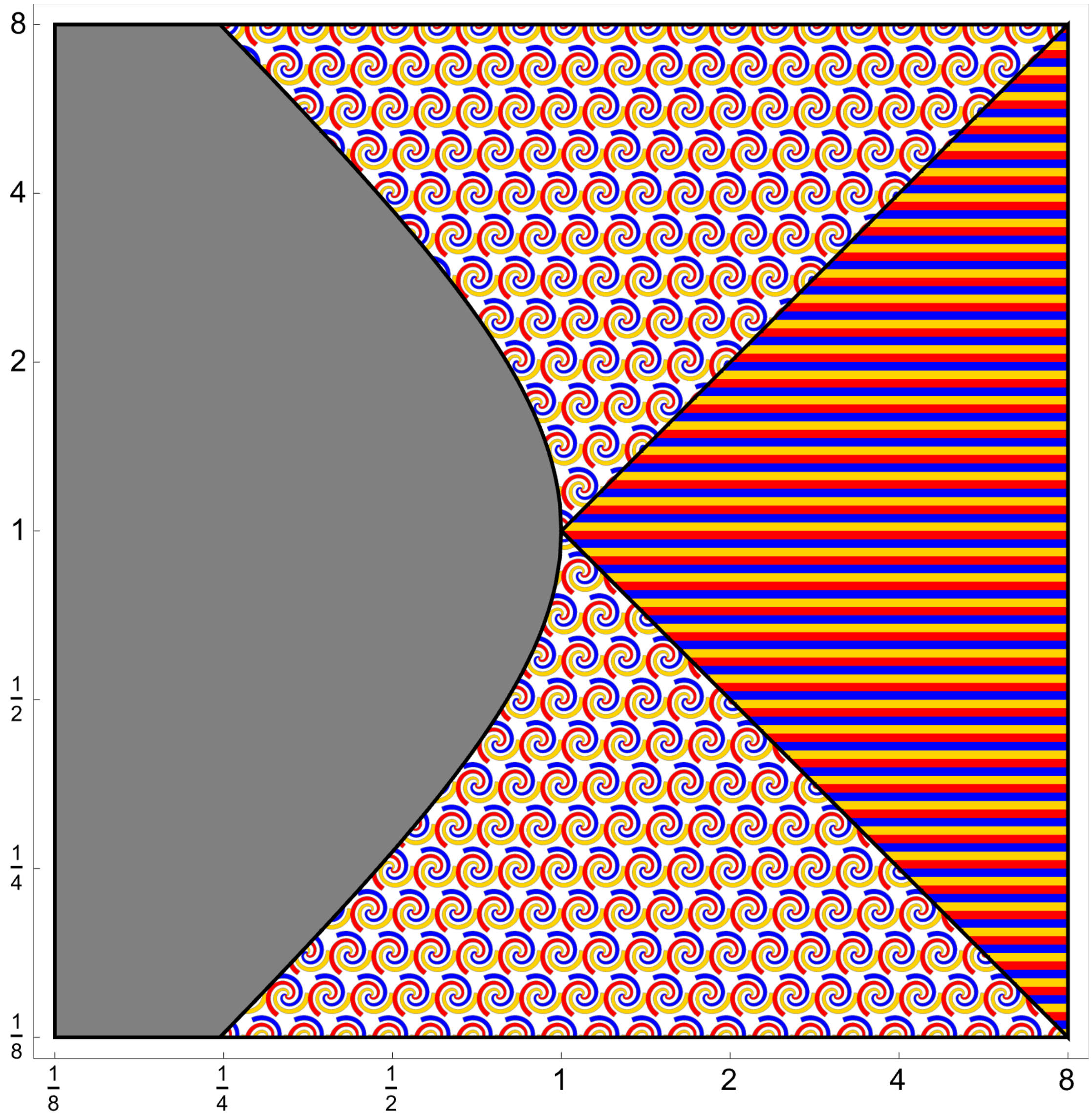


Fig. S3-5 — Recentering the r's

Fig. S3

In[ ]:=
```
ρAtoα[{0.5, 0.5, 0.5}, {2, 2, 2}] // MatrixForm
ρAtoα[{0.5, 0.5, 0.5}, {16, 2, 0.25}] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0.25 & 1. \\ 1. & 1 & 0.25 \\ 0.25 & 1. & 1 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0.03125 & 0.125 \\ 8. & 1 & 0.25 \\ 2. & 1. & 1 \end{pmatrix}$$

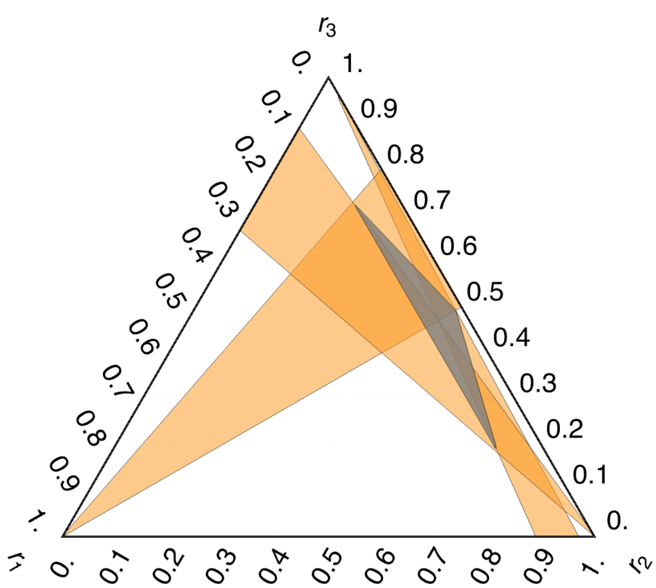
In[ ]:=
```
(* recentering factors *)
Setαs[ρAtoα[{0.5, 0.5, 0.5}, {16, 2, 0.25}]];
A⸝
```

Out[ ]=

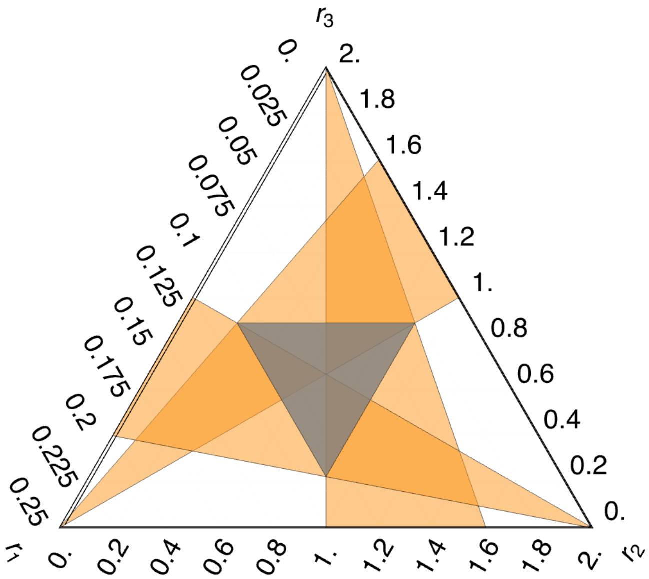
```
{4., 0.5, 0.5}
```

In[ ]:=
```
(* Fig. S3ACE *)
FeasibilityPlot[ρAtoα[{0.5, 0.5, 0.5}, {16, 2, 0.25}]]
FeasibilityPlot[ρAtoα[{0.5, 0.5, 0.5}, {16, 2, 0.25}], Recentered → True]
FeasibilityPlot[ρAtoα[{0.5, 0.5, 0.5}, {2, 2, 2}]]
```

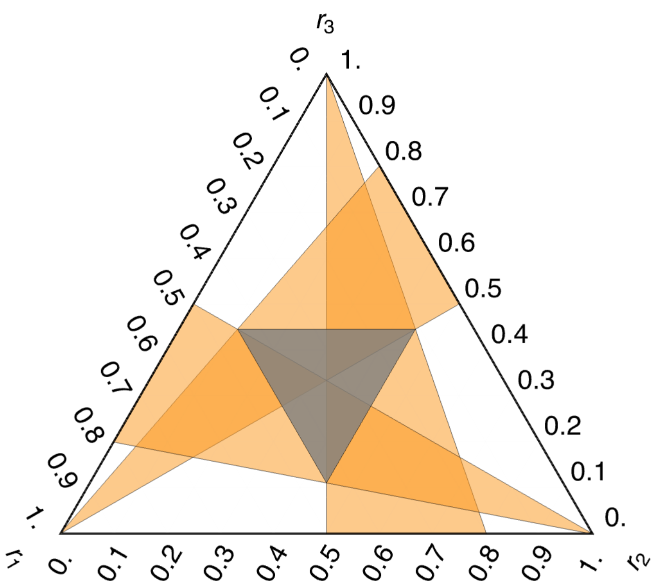
Out[ ]=



Out[ ]=

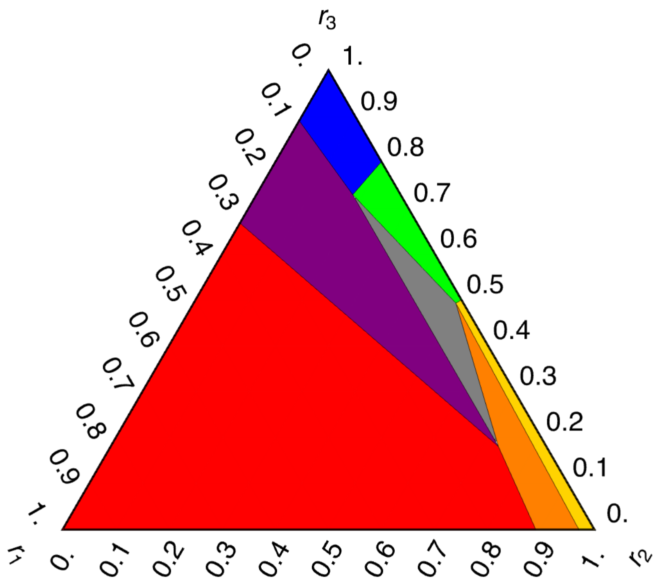


Out[ ]=

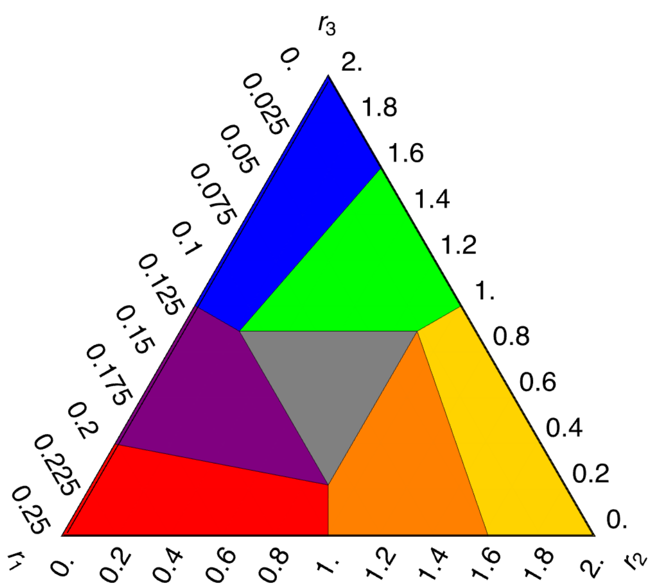


In[ ]:=
```
(* Fig S3BDF *)
OutcomePlot[ρAtoα[{0.5, 0.5, 0.5}, {16, 2, 0.25}]]
OutcomePlot[ρAtoα[{0.5, 0.5, 0.5}, {16, 2, 0.25}], Recentered → True]
OutcomePlot[ρAtoα[{0.5, 0.5, 0.5}, {2, 2, 2}]]
```

Out[ ]=


r_3
r_1
r_2
0.
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.

Out[ ]=


r_3
r_1
r_2
0.
0.025
0.05
0.075
0.1
0.125
0.15
0.175
0.2
0.225
0.25
0.2
0.4
0.6
0.8
1.
1.2
1.4
1.6
1.8
2.

Out[ ]=

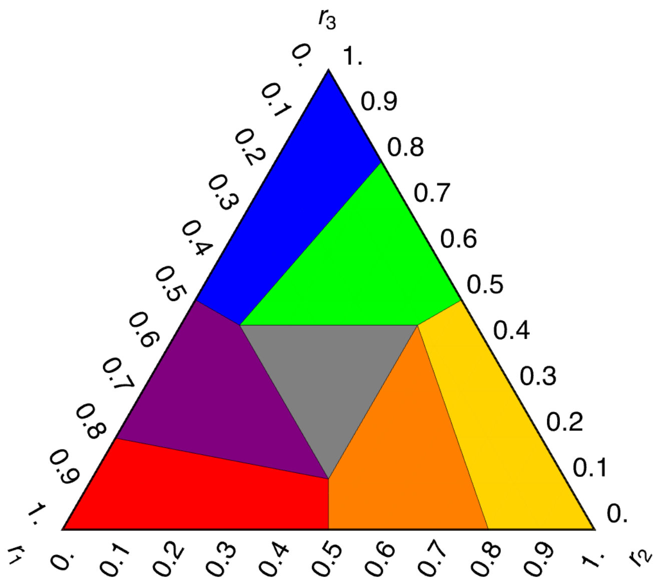


Fig. S4

```
In[ ]:= ρAtoα[{0.5, 0.5, 0.5}, {4, 4, 4}] // MatrixForm
ρAtoα[{0.5, 0.5, 0.5}, {32, 4, 0.5}] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0.125 & 2. \\ 2. & 1 & 0.125 \\ 0.125 & 2. & 1 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0.015625 & 0.25 \\ 16. & 1 & 0.125 \\ 1. & 2. & 1 \end{pmatrix}$$

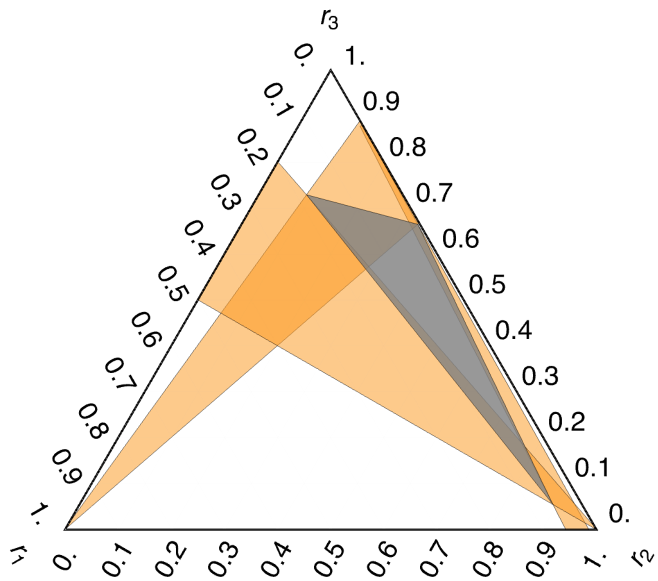
```
In[ ]:= (* recentering factors *)
Setαs[ρAtoα[{0.5, 0.5, 0.5}, {32, 4, 0.5}]];
A,
```

Out[ ]=

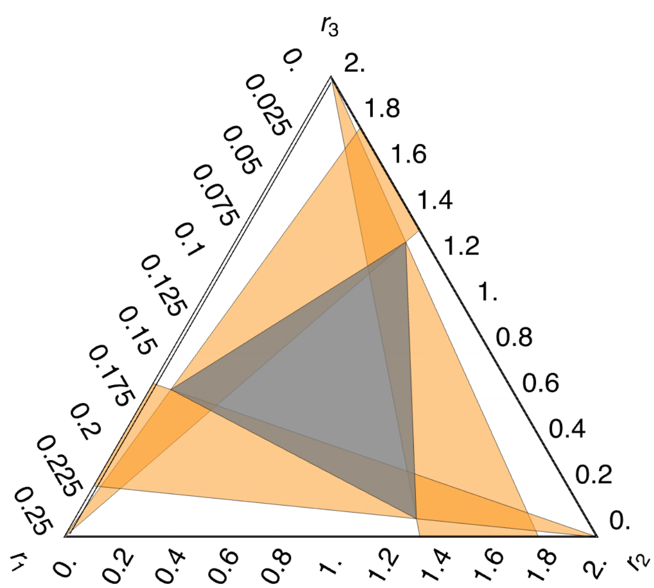
{4., 0.5, 0.5}

```
In[ ]:= (* Fig S4BDF *)
FeasibilityPlot[ρAtoα[{0.5, 0.5, 0.5}, {32, 4, 0.5}]]
FeasibilityPlot[ρAtoα[{0.5, 0.5, 0.5}, {32, 4, 0.5}], Recentered → True]
FeasibilityPlot[ρAtoα[{0.5, 0.5, 0.5}, {4, 4, 4}]]
```

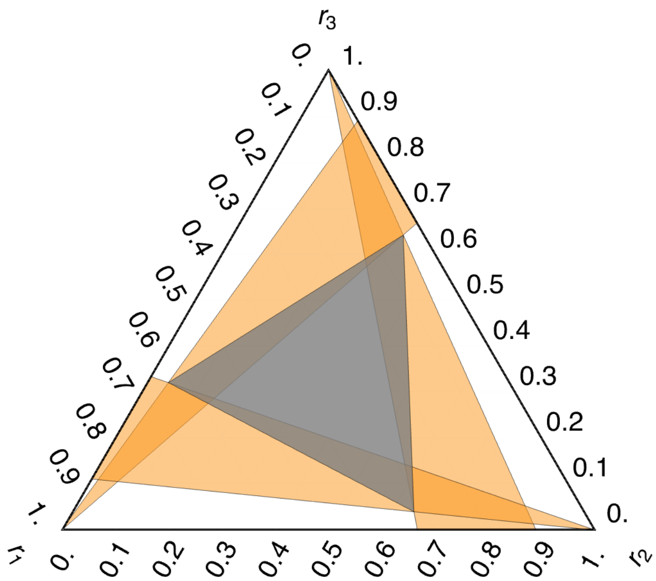
Out[ ]=


r3
r1
r2
0.
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.

Out[ ]=

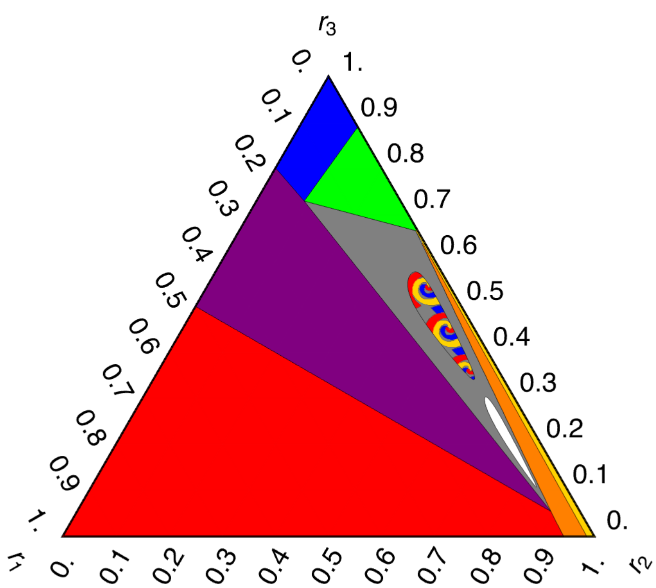

r3
r1
r2
0.
0.025
0.05
0.075
0.1
0.125
0.15
0.175
0.2
0.225
0.25
0.2
0.4
0.6
0.8
1.
1.2
1.4
1.6
1.8
2.

Out[ ]=

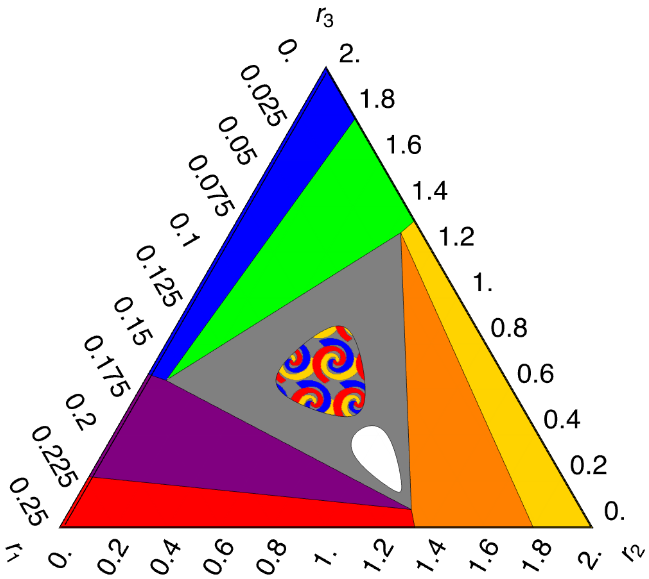


In[ ]:=
```
(* Fig. S4BDF *)
OutcomePlot[ρAtoα[{0.5, 0.5, 0.5}, {32, 4, 0.5}]]
OutcomePlot[ρAtoα[{0.5, 0.5, 0.5}, {32, 4, 0.5}], Recentered → True]
OutcomePlot[ρAtoα[{0.5, 0.5, 0.5}, {4, 4, 4}]]
```

Out[ ]=



Out[ ]=



Out[ ]=

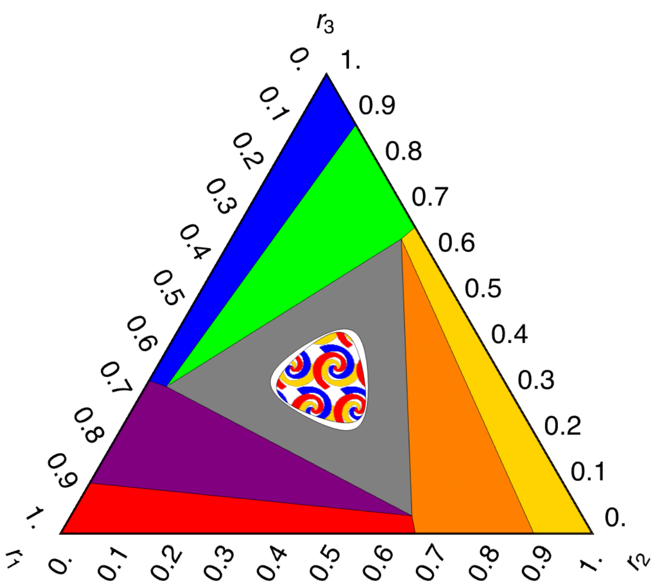


Fig. S5 — Some dynamics seen in Fig. S4 – three-species coexistence OR heteroclinic cycle (also a limit cycle)

In[ ]:=

```
Setαs[ρAtoα[{0.5, 0.5, 0.5}, {32, 4, 0.5}]] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0.015625 & 0.25 \\ 16. & 1 & 0.125 \\ 1. & 2. & 1 \end{pmatrix}$$

In[ ]:=
```
(* Fig. S5AB – three-species coexistence OR heteroclinic cycle *)
{r1, r2, r3} = {0.25 / 3, 2. / 3, 2. / 3};

eq⟦8⟧
Times[λ12, λ23, λ31] + Times[λ21, λ32, λ13]
EcoEigenvalues[eq⟦8⟧]

RuleListAdd[eq⟦8⟧, {n1 → 0.001}]
sol =
  EcoSim[RuleListAdd[eq⟦8⟧, {n1 → 0.001}], 500, NDSolveOpts → {AccuracyGoal → ∞}];
PlotDynamics[sol, AxesLabel → None, TicksStyle → 16]
FinalSlice[sol]

sol = EcoSim[{n1 → 0.0001, n2 → 0.0001, n3 → 0.0001},
   4000, NDSolveOpts → {AccuracyGoal → ∞}];
PlotDynamics[sol, AxesLabel → None, TicksStyle → 16]
PlotDynamics[sol, Logged → True]
FinalSlice[sol]
```

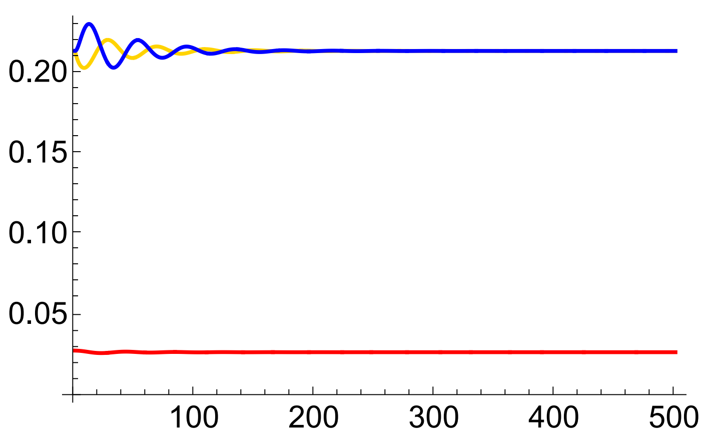
Out[ ]=
```
{n1 → 0.0266667, n2 → 0.213333, n3 → 0.213333}
```

Out[ ]=
```
-0.0122251
```

Out[ ]=
```
{-0.408886, -0.0222235 + 0.154916 i, -0.0222235 - 0.154916 i}
```

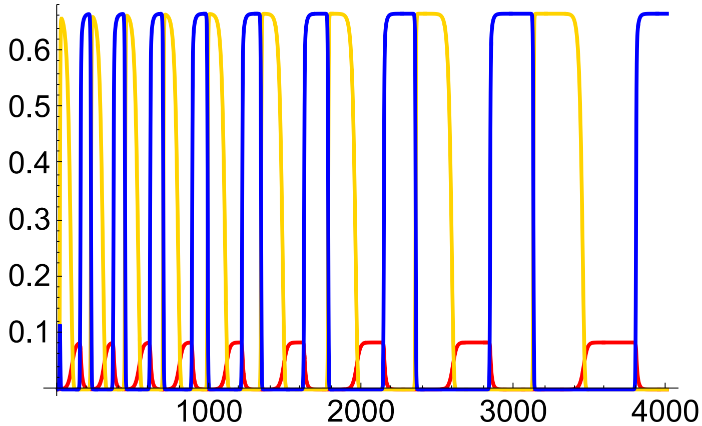
Out[ ]=
```
{n1 → 0.0276667, n2 → 0.213333, n3 → 0.213333}
```

Out[ ]=

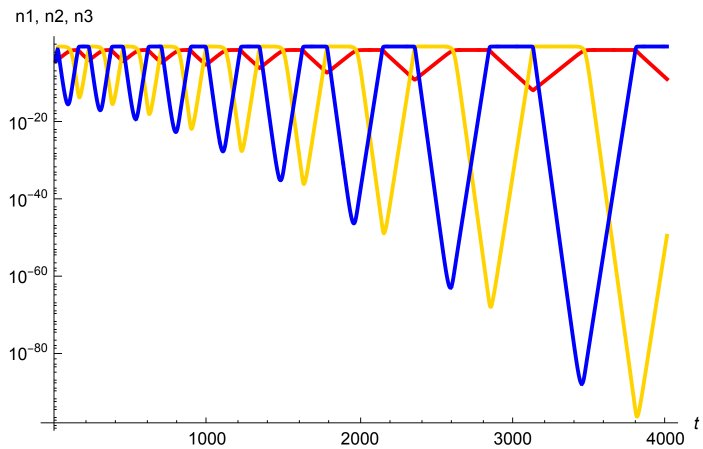


Out[ ]=
```
{n1 → 0.0266667, n2 → 0.213333, n3 → 0.213334}
```

Out[ ]=



Out[ ]=

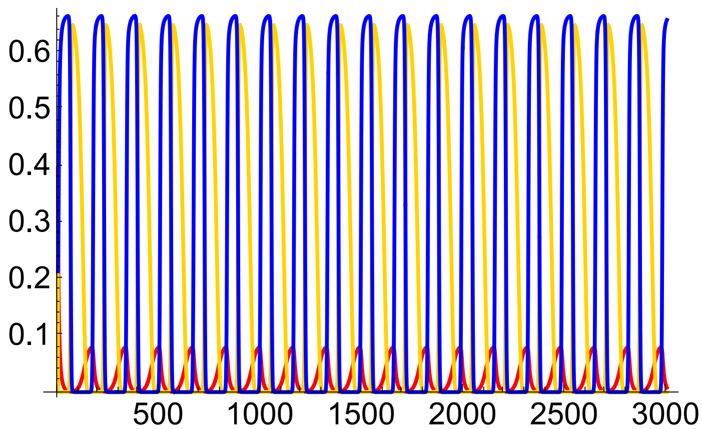


Out[ ]=

$\{n1 \to 2.25799 \times 10^{-9},\ n2 \to 4.1851 \times 10^{-50},\ n3 \to 0.666667\}$

In[∘]:=
```
sol = EcoSim[RuleListAdd[eq⟦8⟧, {n1 → 0.36841}],
   3000, NDSolveOpts → {AccuracyGoal → ∞}];
PlotDynamics[sol, AxesLabel → None, TicksStyle → 16]
FinalSlice[sol]
```

Out[∘]=

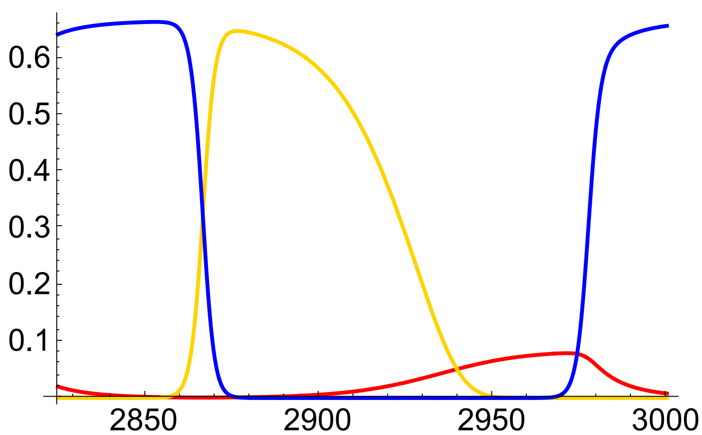


Out[∘]=

{n1 → 0.00796185, n2 → 2.48847×10^{-8}, n3 → 0.657437}

In[∘]:=
```
PlotDynamics[FinalSlice[sol, 175], AxesLabel → None, TicksStyle → 16]
```

Out[∘]=



In[∘]:=
```
ec = FindEcoCycle[FinalSlice[sol], Period → 175, Method → "FindRoot", Logged → True]
```

Out[∘]=

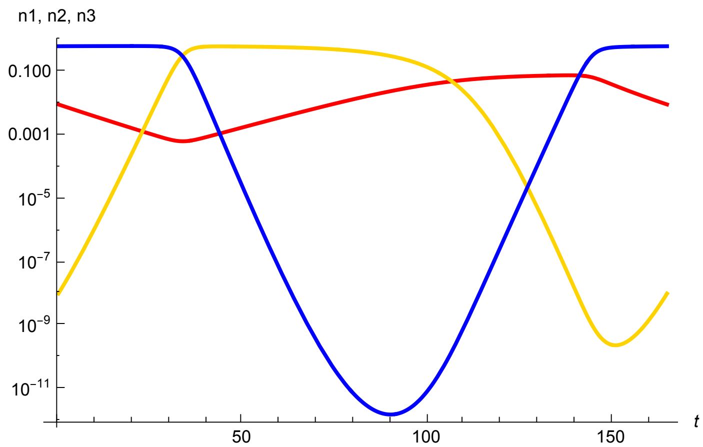
```
{n1 → InterpolatingFunction[ Domain: {{0., 165.}} Output: scalar ],
 n2 → InterpolatingFunction[ Domain: {{0., 165.}} Output: scalar ],
 n3 → InterpolatingFunction[ Domain: {{0., 165.}} Output: scalar ]}
```

In[ ]:= `FinalTime[ec]`

Out[ ]= 164.941

In[ ]:= `PlotDynamics[ec, Logged → True]`

Out[ ]=

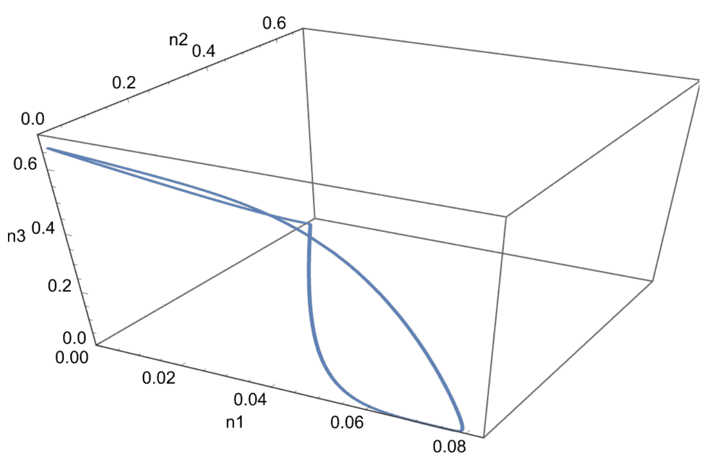


In[ ]:= `EcoEigenvalues[ec]`

Out[ ]= $\{0.00216717, 8.09715\times10^{-7}, -\infty\}$

In[ ]:= `PlotTrajectory[ec]`

Out[ ]=

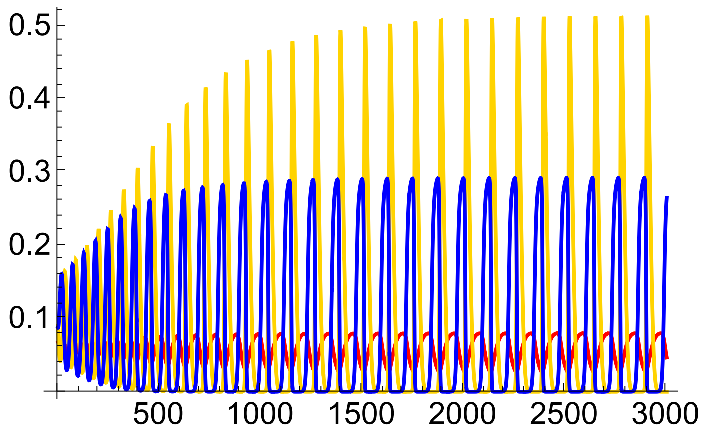


In[ ]:=
```
(* Fig. S5C – a limit cycle *)
{r1, r2, r3} = {0.081, 1.024, 0.326};

Times[λ12, λ23, λ31] + Times[λ21, λ32, λ13]
EcoEigenvalues[eq[[8]]]

sol =
  EcoSim[RuleListAdd[eq[[8]], {n1 → 0.01}], 3000, NDSolveOpts → {AccuracyGoal → ∞}];
PlotDynamics[sol, AxesLabel → None, TicksStyle → 16]
```

Out[ ]=
```
0.0154241
```

Out[ ]=
```
{-0.243287, 0.00380718 + 0.124405 i, 0.00380718 - 0.124405 i}
```

Out[ ]=



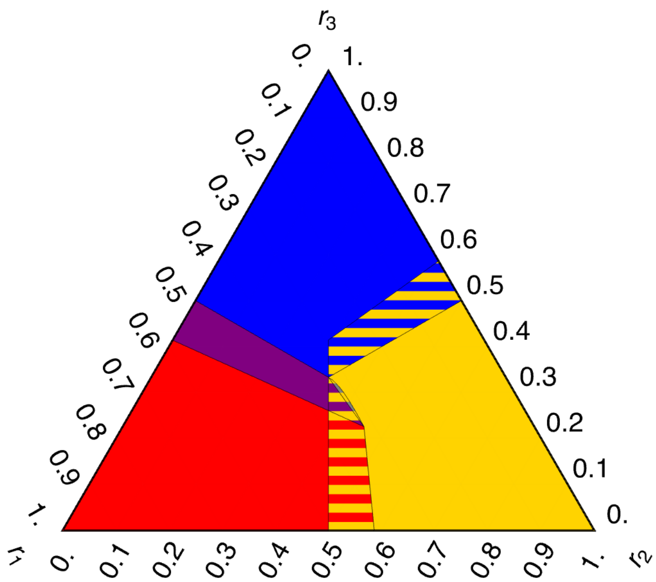
Misc stuff

1 or 1,2,3

In[ ]:=
```
α = ρAtoα[{2^0.25, 2^0.25, 2^-0.25}, 2^0.25];
```

In[∘]:= `OutcomePlot[α]`

Out[∘]=



In[∘]:= `{r1, r2, r3} = {0.32250303131912444`, 0.381025600763568, 0.29647136791730755`};`

In[∘]:= `Outcome[α, {r1, r2, r3}]`

Out[∘]=

{{n2}, {n1, n2, n3}}

In[∘]:= `NumberedGridForm[eq, EcoEigenvalues[eq], Header → True]`

Out[∘]=

#	eq	EcoEigenvalues[eq]
1	{n1 → 0, n2 → 0, n3 → 0}	{0.381026, 0.322503, 0.296471}
2	{n1 → 0.322503, n2 → 0, n3 → 0}	{-0.322503, -0.0750626, 0.0684273}
3	{n1 → 0, n2 → 0.381026, n3 → 0}	{-0.381026, -0.24238, -0.0585226}
4	{n1 → 0, n2 → 0, n3 → 0.296471}	{-0.296471, 0.0845542, 0.0260317}
5	{n1 → 0.141286, n2 → 0.181217, n3 → 0}	{-0.352582, -0.0597125, 0.0300789}
6	{n1 → 0.0888777, n2 → 0, n3 → 0.233625}	{-0.302391, 0.0217082, -0.0201119}
7	{n1 → 0, n2 → -0.204132, n3 → 0.585158}	{-0.190513 + 0.114815 i, -0.190513 - 0.114815 i, -0.0585226}
8	{n1 → 0.141286, n2 → 0.0370583, n3 → 0.144159}	{-0.318809, -0.0018471 + 0.0200697 i, -0.0018471 - 0.0200697 i}

In[∘]:= `{λ12, λ21, λ23, λ32, λ31, λ31}`

Out[∘]=

{-0.0585226, -0.0750626, 0.0845542, -0.24238, 0.0684273, 0.0684273}

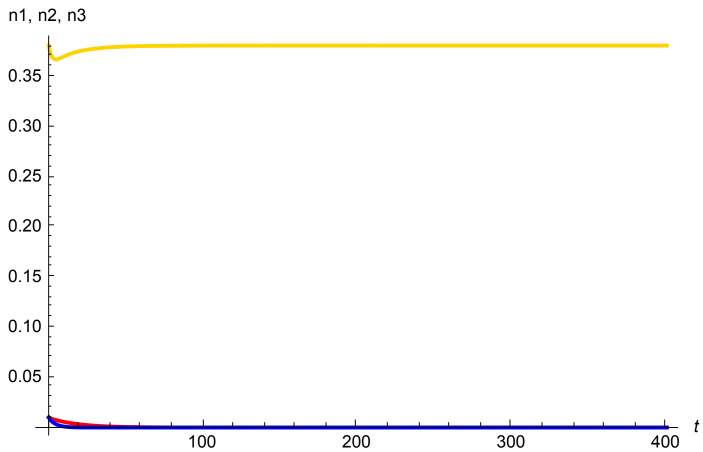
In[∘]:= `λ231`

Out[∘]=

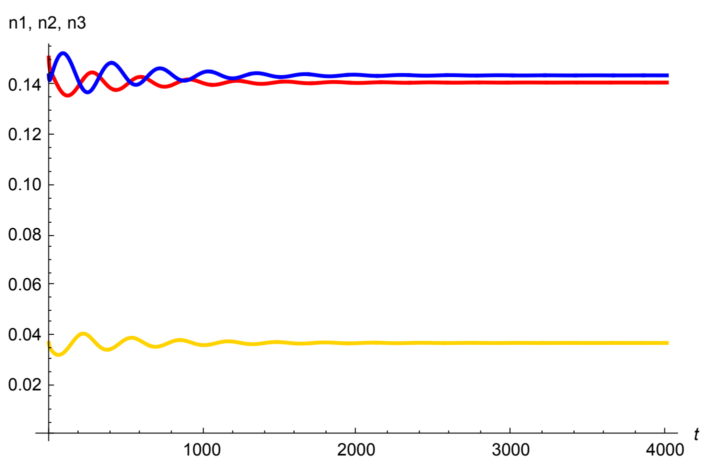
0.0217082

In[∘]:= `Setαs[α];`

In[∘]:=
```
eq⟦3⟧
EcoEigenvalues[eq⟦3⟧]
```
Out[∘]=
```
{n1 → 0, n2 → 0.381026, n3 → 0}
```
Out[∘]=
```
{-0.381026, -0.24238, -0.0585226}
```
In[∘]:=
```
sol = EcoSim[RuleListAdd[eq⟦3⟧, {n1 → 0.01, n3 → 0.01}], 400];
PlotDynamics[sol]
```
Out[∘]=



In[∘]:=
```
eq⟦8⟧
EcoEigenvalues[eq⟦8⟧]
```
Out[∘]=
```
{n1 → 0.141286, n2 → 0.0370583, n3 → 0.144159}
```
Out[∘]=
```
{-0.318809, -0.0018471 + 0.0200697 i, -0.0018471 - 0.0200697 i}
```
In[∘]:=
```
sol = EcoSim[RuleListAdd[eq⟦8⟧, {n1 → 0.01}], 4000];
PlotDynamics[sol]
```
Out[∘]=



```
In[•]:= {r1, r2, r3} = {0.342636782097654, 0.3763329626532089, 0.28103025524913705`};

In[•]:= Outcome[α, {r1, r2, r3}]
Out[•]=
{{n2}, {n1, n3}}

In[•]:= NumberedGridForm[eq, EcoEigenvalues[eq], Header → True]
Out[•]=
```

#	eq	EcoEigenvalues[eq]
1	{n1 → 0, n2 → 0, n3 → 0}	{0.376333, 0.342637, 0.28103}
2	{n1 → 0.342637, n2 → 0, n3 → 0}	{−0.342637, −0.108229, 0.0387495}
3	{n1 → 0, n2 → 0.376333, n3 → 0}	{−0.376333, −0.251185, −0.0336962}
4	{n1 → 0, n2 → 0, n3 → 0.28103}	{−0.28103, 0.0953027, 0.0616065}
5	{n1 → 0.0813498, n2 → 0.261287, n3 → 0}	{−0.36665, −0.146008, 0.024013}
6	{n1 → 0.210338, n2 → 0, n3 → 0.132299}	{−0.316919, −0.0534286, −0.0257179}
7	{n1 → 0, n2 → −0.230081, n3 → 0.606414}	{−0.188166 + 0.14962 i, −0.188166 − 0.14962 i, −0.0336962}
8	{n1 → 0.0813498, n2 → −0.0912083, n3 → 0.352495}	{−0.221681, −0.135856, 0.0149}

inverse jacobian exploration