Lüscher's formula in 3, 2, 1 (lift off!) dimensions

Evan Berkowitz[1] and Thomas Luu[1,2]

[1]Institute for Advanced Simulation 4, Forschungszentrum Jülich, Germany
[2]Rheinische Friedrich-Williams-Universität Bonn, Germany

email: t.luu@fz-juelich.de

Abstract

We derive Lüscher's formula for a torus in 3, 2, and 1 dimensions.

1 Setup

We consider two interacting particles of equal mass m. We assume a pure contact interaction (regulated if needed depending on dimension),

$$V(\vec{r}) = C_0(\Lambda)\delta(\vec{r}) \ , \tag{1}$$

where $\vec{r}$ represents the relative distance between the two particles and $C_0(\Lambda)$ is the strength of interaction that in principle depends on the regulator and whose units depend on the spatial dimension d. We also assume the system has zero CM motion (i.e. $\vec{P}_{cm} = 0$) to make the problem simpler. The relevant diagram to calculate, both for infinite and finite volume, is the bubble sum shown in Figure 1:

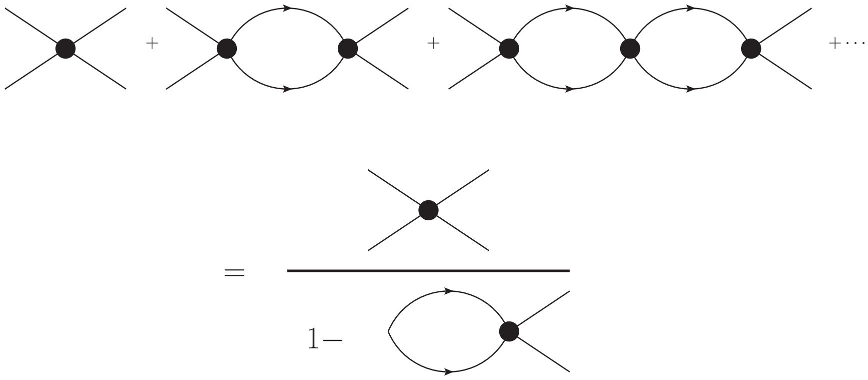

Figure 1: Bubble sum. Each vertex represents $-iC_0$ and the bubble represents I_0.

The sum is a geometric series and gives[1]

$$\frac{-iC_0(\Lambda)}{1 - I_0(p,\Lambda)C_0(\Lambda)} \equiv iT(p) \ , \tag{2}$$

where T is the standard T-matrix, p is the relative momentum, and $I_0(p,\Lambda)$ is a d-dependent function (more on this later). Since we assume a momentum-independent contact interaction, this problem only affects s-wave systems, and the T-matrix in d spatial dimensions can be related to the s-wave phase shift δ_0 by

$$T = \frac{4}{m}\mathcal{F}_d \frac{1}{\cot\delta_0(p) - i} \ , \tag{3}$$

where

$$\mathcal{F}_d = \begin{cases} \pi/p & (d=3) \\ 1 & (d=2) \\ p/2 & (d=1) \end{cases} \ . \tag{4}$$

Therefore we have

$$\boxed{\frac{4}{m}\mathcal{F}_d \frac{1}{\cot\delta_0(p) - i} = \frac{-C_0(\Lambda)}{1 - I_0(p,\Lambda)C_0(\Lambda)}} \ . \tag{5}$$

[1] Note that the red minus sign in Eq. (2) is missing in the archive version of [1], but is present in [2].

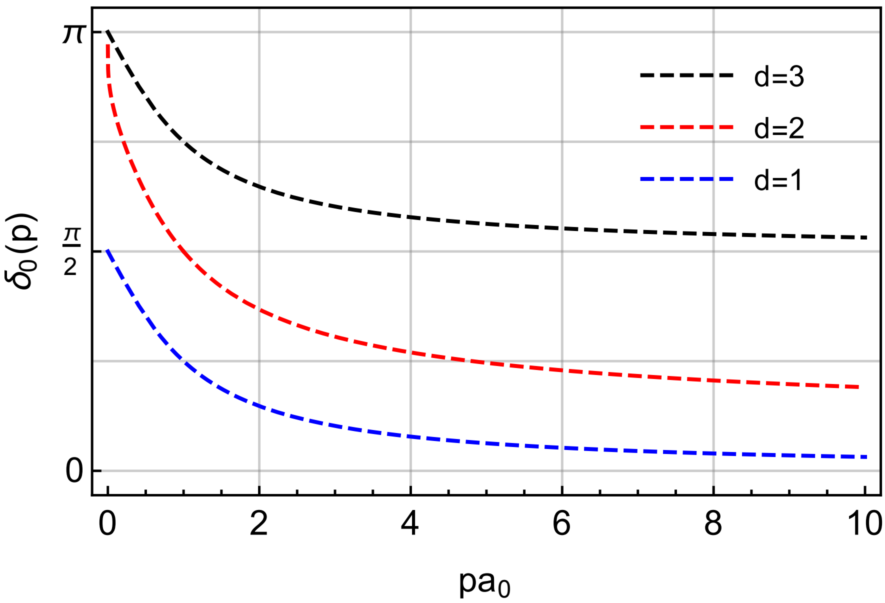

Figure 2: Phase shift $\delta(p)$, modulo π, as a function of pa_0 for the contact interaction in different dimensions d.

We can recast Eq. (5),

$$\cot\delta_0(p) - i = \frac{4}{m}\mathcal{F}_d\left(\frac{-1}{C_0(\Lambda)} + I_0(p,\Lambda)\right) \ . \tag{6}$$

For the special case of a delta function interaction, one has [3]

$$\cot\delta_0(p) = \begin{cases} -\frac{1}{a_0 p} & d=3 \\ \frac{2}{\pi}\log\left(pa_0\right) & d=2 \\ pa_0 & d=1 \end{cases} , \tag{7}$$

where a_0 is the scattering length. Figure 2 plots these phase shifts for the different dimensions. For $d = 1, 2$ the phase shift asymptotes to zero as $pa_0 \to \infty$, while for $d = 3$ the phase shift asymptotes to $\pi/2$, corresponding to the (unphysical) fact that the delta function interaction is 'felt' at all momentum scales.

But the delta function is felt at all momenta in $d = 1, 2$ cases too, right?

1.1 Infinite volume I_0

Consider the following diagram defining I_0:

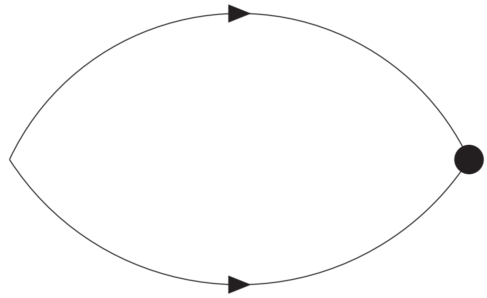

Figure 3: Loop diagram contributing to I_0.

The loop integral with $E = p^2/m$ (on-shell relation) gives

$$\begin{aligned}
I_0(p) &= -i\int^{\Lambda} \frac{\mathrm{d}q_0 \mathrm{d}^d\mathbf{q}}{(2\pi)^{d+1}} \left(\frac{i}{\frac{E}{2}+q_0-\frac{\vec{q}^2}{2m}+i\epsilon}\right)\left(\frac{i}{\frac{E}{2}-q_0-\frac{\vec{q}^2}{2m}+i\epsilon}\right) \\
&= \int^{\Lambda} \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\left(\frac{1}{E-\frac{\vec{q}^2}{m}+i\epsilon}\right) = \frac{\Omega_d}{(2\pi)^d}\int^{\Lambda} \mathrm{d}q\ q^{d-1}\left(\frac{1}{E-\frac{\vec{q}^2}{m}-i\epsilon}\right) \\
&= \frac{\Omega_d}{(2\pi)^d}\int^{\Lambda} \mathrm{d}q\ q^{d-1}\left[\mathcal{P}\left(\frac{1}{E-\frac{\vec{q}^2}{m}}\right) - i\pi\delta(E-\vec{q}^2/m)\right] \\
&= \frac{\Omega_d}{(2\pi)^d}\int^{\Lambda} \mathrm{d}q\ q^{d-1}\left[\mathcal{P}\left(\frac{1}{E-\frac{\vec{q}^2}{m}}\right) - i\frac{\pi m}{2q}\delta(q-\sqrt{mE})\right] \quad , \qquad (8)
\end{aligned}$$

where $\mathcal{P}$ refers to Principle (Cauchy) Value and

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} = \begin{cases} 4\pi & d=3 \\ 2\pi & d=2 \\ 2 & d=1 \end{cases} , \qquad (9)$$

and d is the dimension and Λ represents a regulator if needed (for 3-dimensions). We will now do the integral for each dimension and determine $C_0(\Lambda)$ for each case.

1.1.1 $d = 3$

Here we must choose a regulator to tame the integral in I_0. The coefficient $C_0(\Lambda)$ will depend on this regulator such that observables are *independent* of it. Here we choose a simple hard cutoff in maximum momentum Λ. There are other possibilities; we could have chosen a Pauli-Villars (Gaussian) regulator, or dim-reg (e.g. see [2]). As with all calculations, at the end we will formally take the $\Lambda \to \infty$ limit. Equation (8) becomes

$$\frac{1}{2\pi^2}\int_0^\Lambda \mathrm{d}q\, q^2 \left[\mathcal{P}\left(\frac{1}{E-\frac{\vec{q}^2}{m}}\right) i\frac{\pi m}{2q}\delta(q-\sqrt{mE})\right]$$
$$= \frac{1}{2\pi^2}\mathcal{P}\int_0^\Lambda \mathrm{d}q\, q^2\left(\frac{1}{E-\frac{\vec{q}^2}{m}}\right) - i\frac{m}{4\pi}p \ , \quad (10)$$

where we have used the on-shell condition $\sqrt{mE}=p$. If we plug this into Eq. (6) we get

$$p\cot\delta_0(p) - ip = \frac{4\pi}{m}\left(\frac{-1}{C_0(\Lambda)} + \frac{1}{2\pi^2}\mathcal{P}\int_0^\Lambda \mathrm{d}q\, q^2\left(\frac{1}{E-\frac{\vec{q}^2}{m}}\right)\right) - ip \ . \quad (11)$$

For general interactions (not just contact), we can determine $C_0(\Lambda)$ by looking at the $E=0$ threshold. In this limit one has

$$-\frac{1}{a_0} = \frac{4\pi}{m}\left(\frac{-1}{C_0(\Lambda)} + \frac{1}{2\pi^2}\int_0^\Lambda \mathrm{d}q\, q^2\left(\frac{1}{-\frac{\vec{q}^2}{m}}\right)\right) = \frac{4\pi}{m}\left(\frac{-1}{C_0(\Lambda)} - \frac{m\Lambda}{2\pi^2}\right) \quad (12)$$

Solving for $C_0(\Lambda)$ gives

$$C_0(\Lambda) = \frac{4\pi/m}{1/a_0 - 2\Lambda/\pi} \ . \quad (13)$$

1.1.2 $d=2$

We also have to regulate in this dimension, and use the same hard cutoff as in $d=3$. Equation (8) becomes

$$\frac{1}{2\pi}\int_0^\Lambda \mathrm{d}q\, q \left[\mathcal{P}\left(\frac{1}{E-\frac{\vec{q}^2}{m}}\right) - i\frac{\pi m}{2q}\delta(q-\sqrt{mE})\right]$$
$$= \frac{1}{2\pi}\mathcal{P}\int_0^\Lambda \mathrm{d}q\, q\left(\frac{1}{E-\frac{\vec{q}^2}{m}}\right) - i\frac{m}{4} \ . \quad (14)$$

Plugging this into Eq. (6) gives

$$\cot\delta_0(p) - i = \frac{4}{m}\frac{-1}{C_0(\Lambda)} + \frac{2}{m\pi}\mathcal{P}\int_0^\Lambda \mathrm{d}q\, q\left(\frac{1}{E-\frac{\vec{q}^2}{m}}\right) - i$$
$$\implies \frac{2}{\pi}\log(pa_0) = \frac{4}{m}\frac{-1}{C_0(\Lambda)} + \frac{2}{m\pi}\mathcal{P}\int_0^\Lambda \mathrm{d}q\, q\left(\frac{1}{\frac{p^2}{m}-\frac{\vec{q}^2}{m}}\right) \ , \quad (15)$$

where we plugged in the relevant expression from Eq. (7). We now perform the integral,

$$\frac{2}{\pi}\log(pa_0) = \frac{4}{m}\frac{-1}{C_0(\Lambda)} + \frac{2}{\pi}\log\left(\frac{p}{\sqrt{\Lambda^2-p^2}}\right)$$
$$= \frac{4}{m}\frac{-1}{C_0(\Lambda)} + \frac{2}{\pi}\log\left(\frac{p}{\Lambda}\right) + \mathcal{O}(p^2) \quad (16)$$

and solve for $C_0(\Lambda)$,

$$C_0(\Lambda) = -\frac{2\pi}{m\log(a_0\Lambda)} \ . \tag{17}$$

1.1.3 $d = 1$

No regulator is needed in this case. We have

$$\frac{1}{\pi}\int_0^\infty \mathrm{d}q \left[\mathcal{P}\left(\frac{1}{E-\frac{\vec{q}^2}{m}}\right) - i\frac{\pi m}{2q}\delta(q-\sqrt{mE})\right] = -i\frac{m}{2p} \ , \tag{18}$$

Plugging this into Eq. (6) gives

$$\cot\delta_0(p) - i = \frac{2p}{m}\frac{-1}{C_0(\Lambda)} - i \implies pa_0 = \frac{2p}{m}\frac{-1}{C_0(\Lambda)} \ , \tag{19}$$

which gives

$$C_0(\Lambda) = -\frac{2}{ma_0} \ . \tag{20}$$

Note that this coefficient is independent of Λ, which is a consequence of the fact that I_0 is well behaved.

1.2 Finite volume

The biggest change, when going to the finite volume is that the loop integral in I_0 becomes a sum over allowed momentum states. Each momentum component satisfies

$$p_n \equiv \frac{2\pi}{L}n \ , \tag{21}$$

where n is an integer and L is the length of one side of the volume. One has

$$\begin{aligned} I_0(p,L) &= -i\int\frac{\mathrm{d}q_0}{2\pi}\frac{1}{L^d}\sum_{\vec{q}}^{\Lambda}\left(\frac{i}{\frac{E}{2}+q_0-\frac{\vec{q}^2}{2m}+i\epsilon}\right)\left(\frac{i}{\frac{E}{2}-q_0-\frac{\vec{q}^2}{2m}+i\epsilon}\right) \\ &= \frac{1}{L^d}\sum_{\vec{q}}^{\Lambda}\frac{1}{E-\frac{\vec{q}^2}{m}} \ . \end{aligned} \tag{22}$$

Since we are interested in the eigenstates of the interacting system in the finite volume, we look for solutions that satisfy

$$\frac{1}{C_0(\Lambda)} - \mathrm{Re}\left(I_0(p,L)\right) = 0 \ . \tag{23}$$

This expression gives the poles of Eq. (2) but using a finite volume version of I_0. With the expressions for $C_0(\Lambda)$ derived in the previous section we can re-derive Lüscher's quantization conditions in the different dimensions.

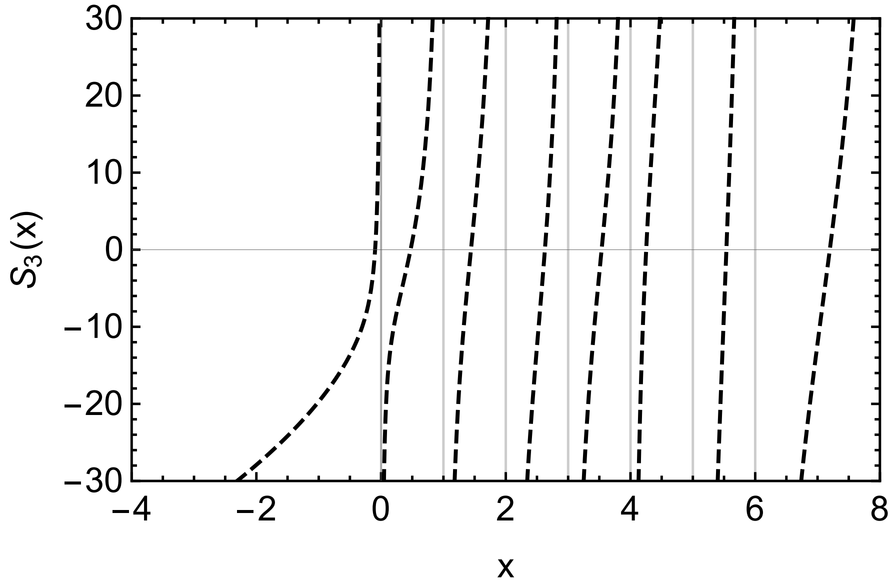

Figure 4: The function $S_3(x)$ defined in Eq. (27).

1.2.1 $d = 3$

Using Eq. (13) in Eq. (23) we find

$$\frac{1/a_0 - 2\Lambda/\pi}{4\pi/m} - \frac{1}{L^3}\sum_{\vec{q}}^{\Lambda} \frac{1}{E - \frac{\vec{q}^2}{m}} = 0 \implies -\frac{1}{a_0} = -\frac{4\pi}{mL^3}\sum_{\vec{q}}^{\Lambda} \frac{1}{E - \frac{\vec{q}^2}{m}} - \frac{2\Lambda}{\pi} \ . \tag{24}$$

Now if we re-express the LHS in terms of the scattering phase relation given by Eq. (7) (which is valid for delta function interactions), and after a little manipulation, we get

$$p\cot\delta(p) = \frac{1}{\pi L}\sum_{\vec{n}}^{\Lambda L/2\pi} \frac{1}{\vec{n}^2 - \left(\frac{pL}{2\pi}\right)^2} - \frac{2\Lambda}{\pi} \tag{25}$$

$$\equiv \frac{1}{\pi L} S_3\left(\left(\frac{pL}{2\pi}\right)^2\right) , \tag{26}$$

where $\vec{n} = (n_x, n_y, n_z)$ is a triplet of integers,

$$S_3(x) \equiv \sum_{\vec{n}}^{\Lambda'} \frac{1}{\vec{n}^2 - x} - 4\pi\Lambda' \ , \tag{27}$$

and $\Lambda' = \frac{\Lambda L}{2\pi}$. The limit $\Lambda' \to \infty$ is implicit. Figure 4 shows this function. This reproduces Lüscher's formula.

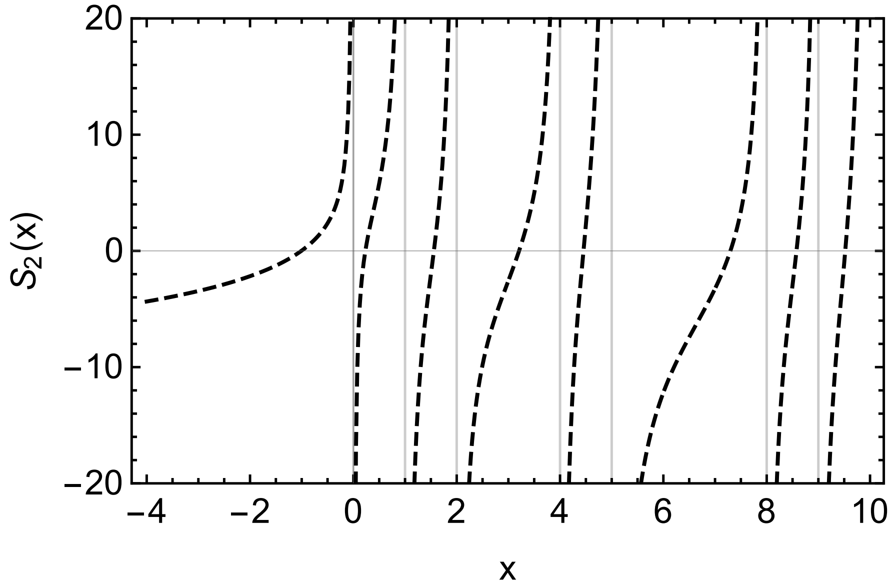

Figure 5: The function $S_2(x)$ defined in Eq. (30).

1.2.2 $d = 2$

Using Eq. (17) in Eq. (23) we find

$$
-\frac{m\log(a_0\Lambda)}{2\pi} - \frac{1}{L^2}\sum_{\vec{q}}^{\Lambda}\frac{1}{E-\frac{\vec{q}^2}{m}} = 0
$$

$$
\Longrightarrow \frac{2}{\pi}\log(pa_0) + \frac{2}{\pi}\log(\Lambda/p) = -\frac{4}{mL^2}\sum_{\vec{q}}^{\Lambda}\frac{1}{E-\frac{\vec{q}^2}{m}} \ . \tag{28}
$$

Now as in the previous section, we use the phase shift relation of Eq. (7), and after a little manipulation, We find

$$
\cot\delta(p) = \frac{1}{\pi^2}S_2\left(\left(\frac{pL}{2\pi}\right)^2\right) + \frac{2}{\pi}\log\left(\frac{pL}{2\pi}\right) \ , \tag{29}
$$

where

$$
S_2(x) \equiv \sum_{\vec{n}}^{\Lambda'}\frac{1}{\vec{n}^2 - x} - 2\pi\log\left(\Lambda'\right) \ , \tag{30}
$$

where $\Lambda' = \frac{\Lambda L}{2\pi}$. Here $\vec{n} = (n_x, n_y)$ is a doublet of integers and the limit $\Lambda' \to \infty$ is implicit. Figure 5 shows the functional form of Eq. (30).

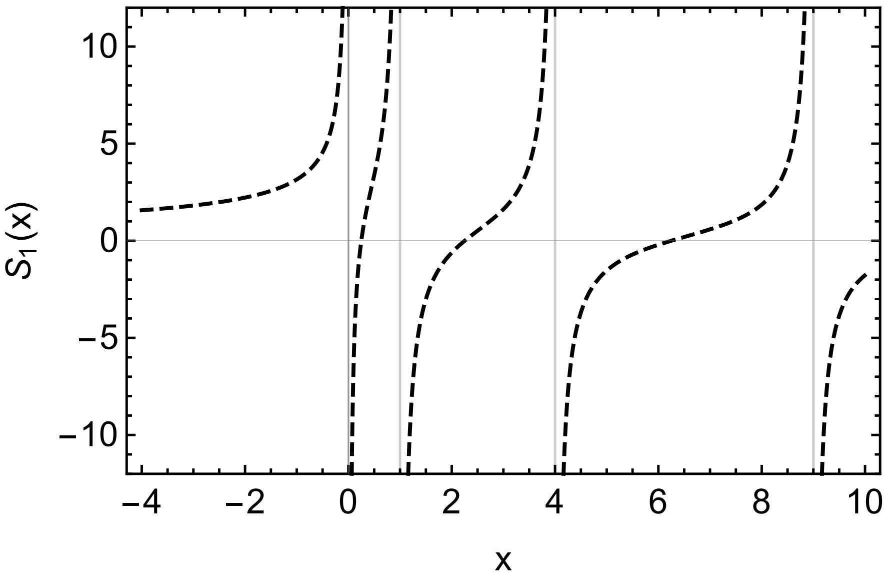

Figure 6: The function $S_1(x)$ defined in Eq.(34).

1.2.3 $d = 1$

Using Eq. (20) in Eq. (23) We find

$$-\frac{ma_0}{2} - \frac{1}{L}\sum_{\vec{q}} \frac{1}{E - \frac{\vec{q}^2}{m}} = 0 \implies pa_0 = -\frac{2p}{mL}\sum_{\vec{q}} \frac{1}{E - \frac{\vec{q}^2}{m}} \ . \tag{31}$$

Again, we use Eq. (7), and after a little manipulation, we get

$$\frac{1}{p}\cot\delta_0(p) = a_0 = \frac{L}{2\pi^2}\sum_{n=-\infty}^{\infty} \frac{1}{n^2 - \left(\frac{pL}{2\pi}\right)^2} \tag{32}$$

$$\equiv \frac{L}{2\pi^2} S_1\left(\left(\frac{pL}{2\pi^2}\right)^2\right) \ , \tag{33}$$

where

$$S_1(x) \equiv \sum_{n=-\infty}^{\infty} \frac{1}{n^2 - x} = -\pi\frac{\cot(\pi\sqrt{x})}{\sqrt{x}} \ . \tag{34}$$

This function is valid for x both positive and negative, and is shown in Figure 6.

- Bound state in $L \to \infty$ limit needs work
- Scattering states in $L \to \infty$ needs work

2 Recap

My findings for Lüscher's formula in the different dimensions are

$$p \cot \delta(p) = \frac{1}{\pi L} S_3 \left(\left(\frac{pL}{2\pi} \right)^2 \right) \qquad (d = 3) \tag{35}$$

$$\cot \delta(p) = \frac{1}{\pi^2} S_2 \left(\left(\frac{pL}{2\pi} \right)^2 \right) + \frac{2}{\pi} \log \left(\frac{pL}{2\pi} \right) \qquad (d = 2) \tag{36}$$

$$\frac{1}{p} \cot \delta_0(p) = \frac{L}{2\pi^2} S_1 \left(\left(\frac{pL}{2\pi} \right)^2 \right) \qquad (d = 1) \tag{37}$$

where $S_3(x)$, $S_2(x)$, and $S_1(x)$ are given by Eqs. (27), (30), and (34), respectively.

References

[1] S. R. Beane, P. F. Bedaque, A. Parreno, and M. J. Savage, Phys. Lett. **B585**, 106 (2004), hep-lat/0312004.

[2] D. B. Kaplan, M. J. Savage, and M. B. Wise, Nucl. Phys. **B534**, 329 (1998), nucl-th/9802075.

[3] T. Busch, B.-G. Englert, K. Rzazewski, and M. Wilkens, **28** (1998).