Lüscher's formula in 3, 2, 1 (lift off!) dimensions

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Abstract

We derive Lüscher's formula for a torus in 3, 2, and 1 dimensions.

1 Setup

We consider two interacting particles of equal mass m. We assume a pure contact interaction (regulated if needed depending on dimension),

$$V(\vec{r}) = C_0(\Lambda)\delta(\vec{r}) \ , \tag{1}$$

where $\vec{r}$ represents the relative distance between the two particles and $C_0(\Lambda)$ is the strength of interaction that in principle depends on the regulator and whose units depend on the spatial dimension d. We also assume the system has zero CM motion (i.e. $\vec{P}_{cm} = 0$) to make the problem simpler. The relevant diagram to calculate, both for infinite and finite volume, is the bubble sum shown in Figure 1:

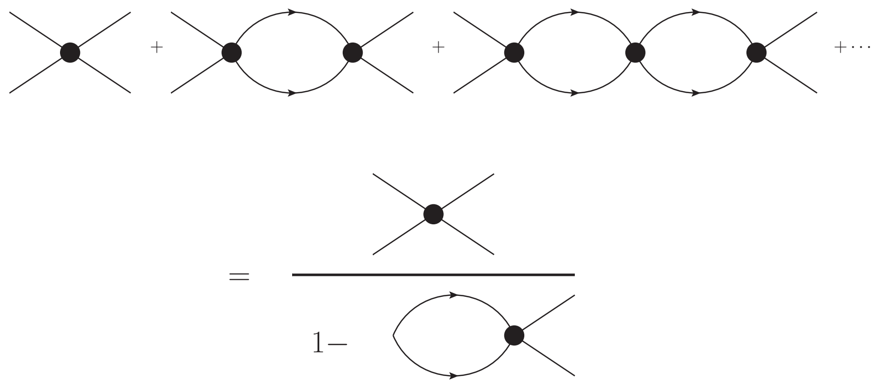

Figure 1: Bubble sum. Each vertex represents $-iC_0$ and the bubble represents I_0.

The sum is a geometric series and gives[1]

$$\frac{-iC_0(\Lambda)}{1 - I_0(p,\Lambda)C_0(\Lambda)} \equiv iT(p) \ , \tag{2}$$

where T is the standard T-matrix, p is the relative momentum, and $I_0(p,\Lambda)$ is a d-dependent function (more on this later). Since we assume a momentum-independent contact interaction, this problem only affects s-wave systems, and the T-matrix in d spatial dimensions can be related to the s-wave phase shift δ_0 by

$$T = \frac{4}{m}\mathcal{F}_d \frac{1}{\cot\delta_0(p) - i} \ , \tag{3}$$

where

$$\mathcal{F}_d = \begin{cases} \pi/p & (d=3) \\ 1 & (d=2) \\ p/2 & (d=1) \end{cases} . \tag{4}$$

Therefore we have

$$\boxed{\frac{4}{m}\mathcal{F}_d \frac{1}{\cot\delta_0(p) - i} = \frac{-C_0(\Lambda)}{1 - I_0(p,\Lambda)C_0(\Lambda)}} \ . \tag{5}$$

[1]Note that the red minus sign in eq. (2) is missing in the archive version of [1], but is present in [2].

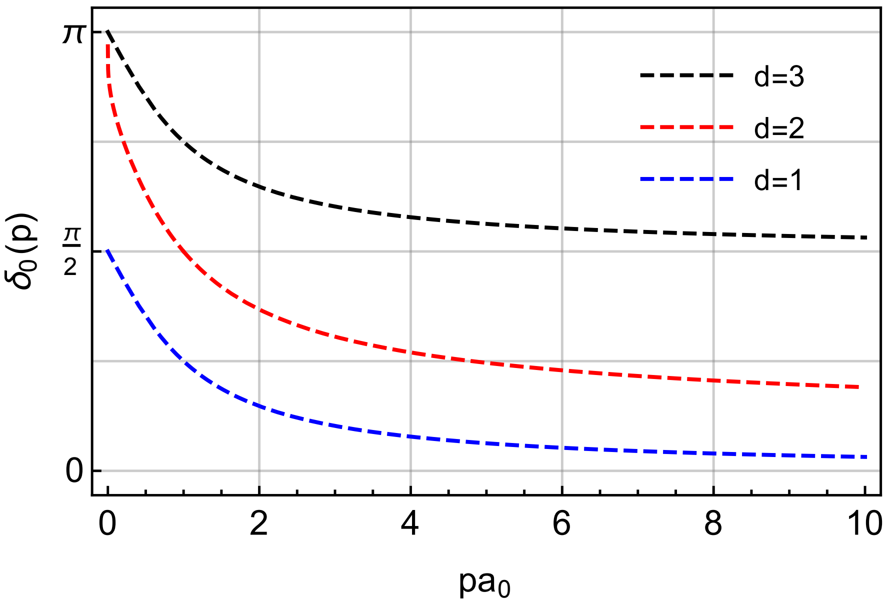

Figure 2: Phase shift $\delta(p)$, modulo π, as a function of pa_0 for the contact interaction in different dimensions d.

We can recast eq. (5),

$$\cot \delta_0(p) - i = \frac{4}{m}\mathcal{F}_d\left(\frac{-1}{C_0(\Lambda)} + I_0(p,\Lambda)\right) \ . \tag{6}$$

For the special case of a delta function interaction, one has [3]

$$\cot \delta_0(p) = \begin{cases} -\frac{1}{a_0 p} & d=3 \\ \frac{2}{\pi}\log(pa_0) & d=2 \\ pa_0 & d=1 \end{cases} , \tag{7}$$

where a_0 is the scattering length. Figure 2 plots these phase shifts for the different dimensions. For $d = 1, 2$ the phase shift asymptotes to zero as $pa_0 \to \infty$, while for $d = 3$ the phase shift asymptotes to $\pi/2$, corresponding to the (unphysical) fact that the delta function interaction is 'felt' at all momentum scales.

But the delta function is felt at all momenta in $d = 1, 2$ cases too, right?

1.1 Infinite volume I_0

Consider the following diagram defining I_0:

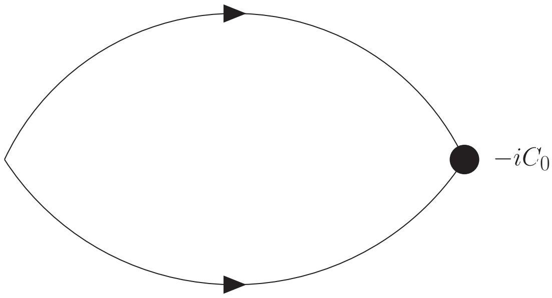

Figure 3: Loop diagram contributing to I_0.

The loop integral with $E = p^2/m$ (on-shell relation) gives

$$\begin{aligned} I_0(p) &= -i\int^{\Lambda} \frac{\mathrm{d}q_0 \mathrm{d}^d\mathbf{q}}{(2\pi)^{d+1}} \left(\frac{i}{\frac{E}{2}+q_0-\frac{\vec{q}^2}{2m}+i\epsilon}\right)\left(\frac{i}{\frac{E}{2}-q_0-\frac{\vec{q}^2}{2m}+i\epsilon}\right) \\ &= \int^{\Lambda} \frac{\mathrm{d}^d\mathbf{q}}{(2\pi)^d}\left(\frac{1}{E-\frac{\vec{q}^2}{m}+i\epsilon}\right) = \frac{\Omega_d}{(2\pi)^d}\int^{\Lambda}\mathrm{d}q\; q^{d-1}\left(\frac{1}{E-\frac{\vec{q}^2}{m}-i\epsilon}\right) \\ &= \frac{\Omega_d}{(2\pi)^d}\int^{\Lambda}\mathrm{d}q\; q^{d-1}\left[\mathcal{P}\left(\frac{1}{E-\frac{\vec{q}^2}{m}}\right) - i\pi\delta(E-\vec{q}^2/m)\right] \\ &= \frac{\Omega_d}{(2\pi)^d}\int^{\Lambda}\mathrm{d}q\; q^{d-1}\left[\mathcal{P}\left(\frac{1}{E-\frac{\vec{q}^2}{m}}\right) - i\frac{\pi m}{2q}\delta(q-\sqrt{mE})\right] , \end{aligned} \tag{8}$$

where $\mathcal{P}$ refers to Principle (Cauchy) Value and

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} = \begin{cases} 4\pi & d=3 \\ 2\pi & d=2 \\ 2 & d=1 \end{cases} , \tag{9}$$

and d is the dimension and Λ represents a regulator if needed (for 3-dimensions). We will now do the integral for each dimension and determine $C_0(\Lambda)$ for each case.

1.1.1 $d = 3$

Here we must choose a regulator to tame the integral in I_0. The coefficient $C_0(\Lambda)$ will depend on this regulator such that observables are *independent* of it. Here we choose a simple hard cutoff in maximum momentum Λ. There are other possibilities; we could have chosen a Pauli-Villars (Gaussian) regulator, or dim-reg (e.g. see [2]). As with all calculations, at the end we will formally take the $\Lambda \to \infty$ limit. Equation (8) becomes

$$\frac{1}{2\pi^2}\int_0^\Lambda \mathrm{d}q\; q^2 \left[\mathcal{P}\left(\frac{1}{E-\frac{q^2}{m}}\right) i\frac{\pi m}{2q}\delta(q-\sqrt{mE})\right]$$
$$= \frac{1}{2\pi^2}\mathcal{P}\int_0^\Lambda \mathrm{d}q\; q^2\left(\frac{1}{E-\frac{q^2}{m}}\right) - i\frac{m}{4\pi}p\ , \qquad (10)$$

where we have used the on-shell condition $\sqrt{mE} = p$. If we plug this into eq. (6) we get

$$p\cot\delta_0(p) - ip = \frac{4\pi}{m}\left(\frac{-1}{C_0(\Lambda)} + \frac{1}{2\pi^2}\mathcal{P}\int_0^\Lambda \mathrm{d}q\; q^2\left(\frac{1}{E-\frac{q^2}{m}}\right)\right) - ip\ . \qquad (11)$$

For general interactions (not just contact), we can determine $C_0(\Lambda)$ by looking at the $E = 0$ threshold. In this limit one has

$$-\frac{1}{a_0} = \frac{4\pi}{m}\left(\frac{-1}{C_0(\Lambda)} + \frac{1}{2\pi^2}\int_0^\Lambda \mathrm{d}q\; q^2\left(\frac{1}{-\frac{q^2}{m}}\right)\right) = \frac{4\pi}{m}\left(\frac{-1}{C_0(\Lambda)} - \frac{m\Lambda}{2\pi^2}\right) \qquad (12)$$

Solving for $C_0(\Lambda)$ gives

$$C_0(\Lambda) = \frac{4\pi/m}{1/a_0 - 2\Lambda/\pi}\ . \qquad (13)$$

1.1.1.1 The induced effective range and shape parameter in infinite volume

Instead of setting immediately $E = p^2/\mu = 0$ into eq. (11) to obtain eq. (12), one could start with eq. (11) and actually analytically perform the principal value integral,

$$p\cot\delta_0(p) - ip = \frac{4\pi}{m}\frac{-1}{C_0(\Lambda)} + \frac{2\Lambda}{\pi}\left(\frac{p}{\Lambda}\tanh^{-1}\left(\frac{\Lambda}{p}\right) - 1\right) - ip\ , \qquad (14)$$

where it is assumed that $p < \Lambda$. Now expand the result in powers of p/Λ,

$$p\cot\delta_0(p) - ip = \frac{4\pi}{m}\left(\frac{-1}{C_0(\Lambda)} - \frac{m\Lambda}{2\pi^2}\right) - ip + \frac{2}{\pi\Lambda}p^2 + \frac{2}{3\pi\Lambda^3}p^4 + \mathcal{O}(p^6)\ . \qquad (15)$$

Now when we compare with the effective range expansion,

$$p\cot\delta_0(p) - ip = -\frac{1}{a_0} - ip + \frac{r_0}{2}p^2 - Pr_0^3p^4 + \mathcal{O}(p^6)\ ,$$

we can immediately identify the shape and range parameters,

$$-\frac{1}{a_0} = \frac{4\pi}{m}\left(\frac{-1}{C_0(\Lambda)} - \frac{m\Lambda}{2\pi^2}\right) \qquad (16)$$
$$r_0 = \frac{4}{\pi\Lambda} \qquad (17)$$
$$P = -\frac{\pi^2}{96}\ . \qquad (18)$$

Equation (16) agrees with eq. (13), as it should[2]. Obviously one can continue the expansion to higher orders in p/Λ to obtain higher order shape parameters. If one intepets $\Lambda = \pi/\epsilon$, where ϵ is the lattice spacing, then these results correspond to an *infinite volume* ($L = \infty$) discretized lattice. Only in the continuum limit does one truly have a zero range interaction.

1.1.1.2 Comment on spherical vs cartesian integration

The counter term in eq. (12),

$$\frac{2}{\pi}\Lambda \equiv \mathcal{L}_\circ \Lambda \ ,$$

has coefficient $\mathcal{L}_\circ$ originating from an integration in spherical coordinates. If one instead had done the integral in cartesian coordinates,

$$\frac{1}{2\pi^2}\int_0^\infty q^2 \left(\frac{1}{-q^2/m}\right) \Longrightarrow \frac{1}{(2\pi)^3}\int_{-\Lambda}^{\Lambda} dq_x dq_y dq_z \left(\frac{1}{-q^2/m}\right) \ ,$$

one would also have a linear-in-Λ counterterm but with a different coefficient[3],

$$\mathcal{L}_\Box \Lambda = 0.777551\ \Lambda \ .$$

This means that the matching should give us[4]

$$-\frac{1}{a_0} = \frac{4\pi}{m}\frac{-1}{C_0(\Lambda)} - \mathcal{L}_\Box \Lambda \tag{19}$$

$$r_0 = \frac{4}{\pi\Lambda} \tag{20}$$

$$P = -\frac{\pi^2}{96} \ . \tag{21}$$

1.1.2 $d = 2$

We also have to regulate in this dimension, and use the same hard cutoff as in $d = 3$. Equation (8) becomes

$$\begin{aligned}\frac{1}{2\pi}\int_0^\Lambda \mathrm{d}q\ q \left[\mathcal{P}\left(\frac{1}{E - \frac{\vec{q}^2}{m}}\right) - i\frac{\pi m}{2q}\delta(q - \sqrt{mE})\right] \\ = \frac{1}{2\pi}\mathcal{P}\int_0^\Lambda \mathrm{d}q\ q \left(\frac{1}{E - \frac{\vec{q}^2}{m}}\right) - i\frac{m}{4} \ .\end{aligned} \tag{22}$$

[2]For comparison, the effective range parameters for hard-sphere scattering of radius R are $a_0 = R$, $r_0 = 2R/3$, and $P = -3/40$.

[3]I was not smart enough to determine an analytic result for the coefficient.

[4]Though I do not derive it, I believe the effective range and shape parameters should be independent of how one does the integration, and thus should be the same as in the spherical integration case.

Plugging this into eq. (6) gives

$$\cot\delta_0(p) - i = \frac{4}{m}\frac{-1}{C_0(\Lambda)} + \frac{2}{m\pi}\mathcal{P}\int_0^\Lambda \mathrm{d}q\; q\left(\frac{1}{E-\frac{\vec{q}^2}{m}}\right) - i$$

$$\Longrightarrow \frac{2}{\pi}\log(pa_0) = \frac{4}{m}\frac{-1}{C_0(\Lambda)} + \frac{2}{m\pi}\mathcal{P}\int_0^\Lambda \mathrm{d}q\; q\left(\frac{1}{\frac{p^2}{m}-\frac{\vec{q}^2}{m}}\right)\;, \tag{23}$$

where we plugged in the relevant expression from eq. (7). We now perform the integral,

$$\frac{2}{\pi}\log(pa_0) = \frac{4}{m}\frac{-1}{C_0(\Lambda)} + \frac{2}{\pi}\log\left(\frac{p}{\sqrt{\Lambda^2-p^2}}\right)$$

$$= \frac{4}{m}\frac{-1}{C_0(\Lambda)} + \frac{2}{\pi}\log\left(\frac{p}{\Lambda}\right) + \mathcal{O}(p^2) \tag{24}$$

and solve for $C_0(\Lambda)$,

$$C_0(\Lambda) = -\frac{2\pi}{m\log(a_0\Lambda)}\;. \tag{25}$$

1.1.3 $d = 1$

No regulator is needed in this case. We have

$$\frac{1}{\pi}\int_0^\infty \mathrm{d}q\left[\mathcal{P}\left(\frac{1}{E-\frac{\vec{q}^2}{m}}\right) - i\frac{\pi m}{2q}\delta(q-\sqrt{mE})\right] = -i\frac{m}{2p}\;, \tag{26}$$

Plugging this into eq. (6) gives

$$\cot\delta_0(p) - i = \frac{2p}{m}\frac{-1}{C_0(\Lambda)} - i \;\Longrightarrow\; pa_0 = \frac{2p}{m}\frac{-1}{C_0(\Lambda)}\;, \tag{27}$$

which gives

$$C_0(\Lambda) = -\frac{2}{ma_0}\;. \tag{28}$$

Note that this coefficient is independent of Λ, which is a consequence of the fact that I_0 is well behaved.

1.2 Finite volume

The biggest change, when going to the finite volume is that the loop integral in I_0 becomes a sum over allowed momentum states. Each momentum component satisfies

$$p_n \equiv \frac{2\pi}{L}n\;, \tag{29}$$

where n is an integer and L is the length of one side of the volume. One has

$$I_0(p,L) = -i\int\frac{\mathrm{d}q_0}{2\pi}\frac{1}{L^d}\sum_{\vec{q}}^{\Lambda}\left(\frac{i}{\frac{E}{2}+q_0-\frac{\vec{q}^2}{2m}+i\epsilon}\right)\left(\frac{i}{\frac{E}{2}-q_0-\frac{\vec{q}^2}{2m}+i\epsilon}\right)$$

$$= \frac{1}{L^d}\sum_{\vec{q}}^{\Lambda}\frac{1}{E-\frac{\vec{q}^2}{m}}\;. \tag{30}$$

Since we are interested in the eigenstates of the interacting system in the finite volume, we look for solutions that satisfy

$$\frac{1}{C_0(\Lambda)} - \mathrm{Re}\left(I_0(p,L)\right) = 0 \ . \tag{31}$$

This expression gives the poles of eq. (2) but using a finite volume version of I_0. With the expressions for $C_0(\Lambda)$ derived in the previous section we can re-derive Lüscher's quantization conditions in the different dimensions.

1.2.1 $d = 3$

Using eq. (13) in eq. (31) we find

$$\frac{1/a_0 - 2\Lambda/\pi}{4\pi/m} - \frac{1}{L^3}\sum_{\vec{q}}^{\Lambda}\frac{1}{E-\frac{\vec{q}^2}{m}} = 0 \implies -\frac{1}{a_0} = -\frac{4\pi}{mL^3}\sum_{\vec{q}}^{\Lambda}\frac{1}{E-\frac{\vec{q}^2}{m}} - \frac{2\Lambda}{\pi} \ . \tag{32}$$

Now if we re-express the LHS in terms of the scattering phase relation given by eq. (7) (which is valid for delta function interactions), and after a little manipulation, we get

$$p\cot\delta(p) = \frac{1}{\pi L}\sum_{\vec{n}}^{\Lambda L/2\pi}\frac{1}{\vec{n}^2 - \left(\frac{pL}{2\pi}\right)^2} - \frac{2\Lambda}{\pi} \tag{33}$$

$$\equiv \frac{1}{\pi L} S_3\left(\left(\frac{pL}{2\pi}\right)^2\right) \ , \tag{34}$$

where $\vec{n} = (n_x, n_y, n_z)$ is a triplet of integers,

$$S_3(x) \equiv \sum_{\vec{n}}^{\Lambda'}\frac{1}{\vec{n}^2 - x} - 4\pi\Lambda' \ , \tag{35}$$

and $\Lambda' = \frac{\Lambda L}{2\pi}$. The limit $\Lambda' \to \infty$ is implicit. Figure 4 shows this function. This reproduces Lüscher's formula.

1.2.1.1 3-d result in a Cartesian basis

Equation (35) is the result derived using a sum based on spherical coordinates. If one instead used a sum based in cartesian coordinates, which I think is what should be done when comparing with direct numerical calculations in a box, one would start with

$$\frac{1/a_0 - \mathcal{L}_\square\Lambda}{4\pi/m} - \frac{1}{L^3}\sum_{q_i\in\ (-\Lambda,\Lambda]}\frac{1}{E-\frac{\vec{q}^2}{m}} = 0$$

$$\implies -\frac{1}{a_0} = -\frac{4\pi}{mL^3}\sum_{q_i\in\ (-\Lambda,\Lambda]}\frac{1}{E-\frac{\vec{q}^2}{m}} - \mathcal{L}_\square\Lambda \ . \tag{36}$$

Now replace $E = p^2/m$ and $\vec{q} = 2\pi\vec{n}/L$ with $\vec{n} = (n_x, n_y, n_z)$. Then we have

$$-\frac{1}{a_0} = \frac{1}{\pi L}\sum_{-\frac{\Lambda L}{2\pi}<n_i\leq\frac{\Lambda L}{2\pi}}\frac{1}{n^2 - \left(\frac{pL}{2\pi}\right)^2} - \mathcal{L}_\square\Lambda \ . \tag{37}$$

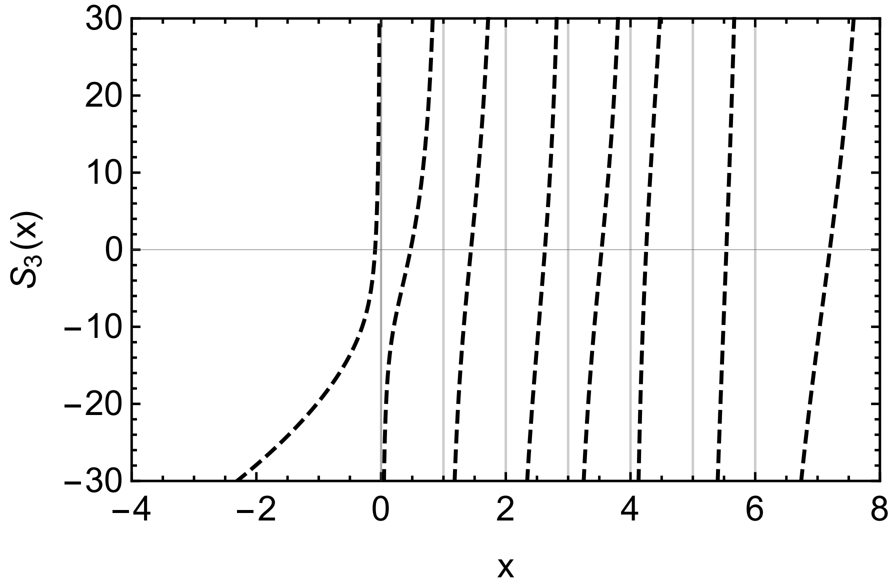

Figure 4: The function $S_3(x)$ defined in eq. (35).

Now define $\Lambda' = \frac{\Lambda L}{2\pi}$,

$$-\frac{1}{a_0} = \frac{1}{\pi L} \sum_{-\Lambda' < n_i \leq \Lambda'} \frac{1}{n^2 - \left(\frac{pL}{2\pi}\right)^2} - \mathcal{L}_\Box \frac{2\pi}{L}\Lambda' \tag{38}$$

$$\equiv \frac{1}{\pi L} S_\Box\left(\left(\frac{pL}{2\pi}\right)^2\right) , \tag{39}$$

where

$$S_\Box(x) \equiv \sum_{-\Lambda' < n_i \leq \Lambda'} \frac{1}{\vec{n}^2 - x} - \mathcal{L}_\Box 2\pi^2 \Lambda' . \tag{40}$$

This is what I call the "cartesian S-function". Now for a discretized lattice with lattice spacing ϵ, the allowed momenta in any of the cartesian directions (assuming symmetric volume) fall within $q_i \in (-\frac{\pi}{\epsilon}, \frac{\pi}{\epsilon}]$. So I identify $\Lambda = \pi/\epsilon$, which means that $\Lambda' = \frac{L}{2\epsilon}$. Obviously L/ϵ is simply the number of lattice points on a side.

1.2.2 $d = 2$

Using eq. (25) in eq. (31) we find

$$-\frac{m\log(a_0\Lambda)}{2\pi} - \frac{1}{L^2}\sum_{\vec{q}}^{\Lambda} \frac{1}{E - \frac{\vec{q}^2}{m}} = 0$$

$$\implies \frac{2}{\pi}\log(pa_0) + \frac{2}{\pi}\log(\Lambda/p) = -\frac{4}{mL^2}\sum_{\vec{q}}^{\Lambda}\frac{1}{E - \frac{\vec{q}^2}{m}} . \tag{41}$$

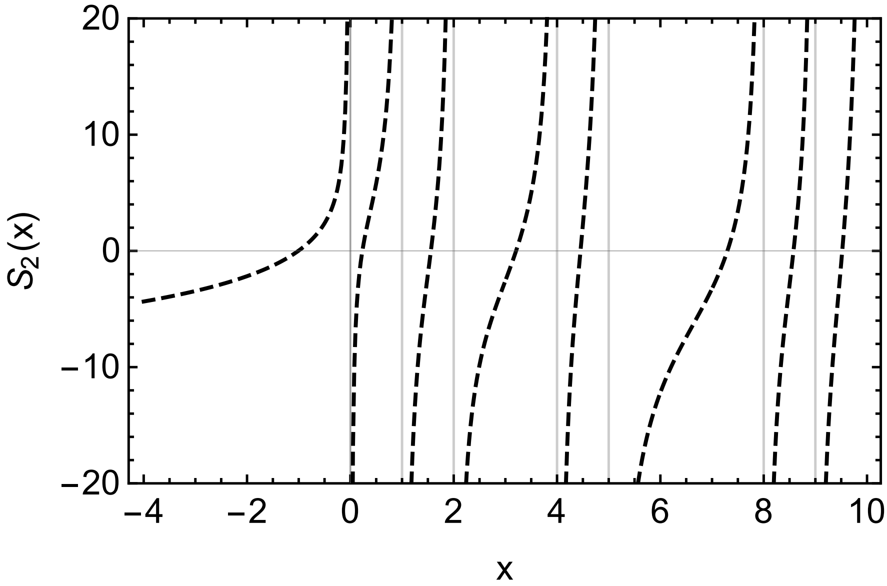

Figure 5: The function $S_2(x)$ defined in eq. (43).

Now as in the previous section, we use the phase shift relation of eq. (7), and after a little manipulation, We find

$$\cot\delta(p) = \frac{1}{\pi^2} S_2\left(\left(\frac{pL}{2\pi}\right)^2\right) + \frac{2}{\pi}\log\left(\frac{pL}{2\pi}\right) \ , \tag{42}$$

where

$$S_2(x) \equiv \sum_{\vec{n}}^{\Lambda'} \frac{1}{\vec{n}^2 - x} - 2\pi\log\left(\Lambda'\right) \ , \tag{43}$$

where $\Lambda' = \frac{\Lambda L}{2\pi}$. Here $\vec{n} = (n_x, n_y)$ is a doublet of integers and the limit $\Lambda' \to \infty$ is implicit. Figure 5 shows the functional form of eq. (43).

1.2.3 $d = 1$

Using eq. (28) in eq. (31) We find

$$-\frac{ma_0}{2} - \frac{1}{L}\sum_{\vec{q}} \frac{1}{E - \frac{\vec{q}^2}{m}} = 0 \implies pa_0 = -\frac{2p}{mL}\sum_{\vec{q}} \frac{1}{E - \frac{\vec{q}^2}{m}} \ . \tag{44}$$

Again, we use eq. (7), and after a little manipulation, we get

$$\frac{1}{p}\cot\delta_0(p) = a_0 = \frac{L}{2\pi^2}\sum_{n=-\infty}^{\infty} \frac{1}{n^2 - \left(\frac{pL}{2\pi}\right)^2} \tag{45}$$

$$\equiv \frac{L}{2\pi^2} S_1\left(\left(\frac{pL}{2\pi^2}\right)^2\right) \ , \tag{46}$$

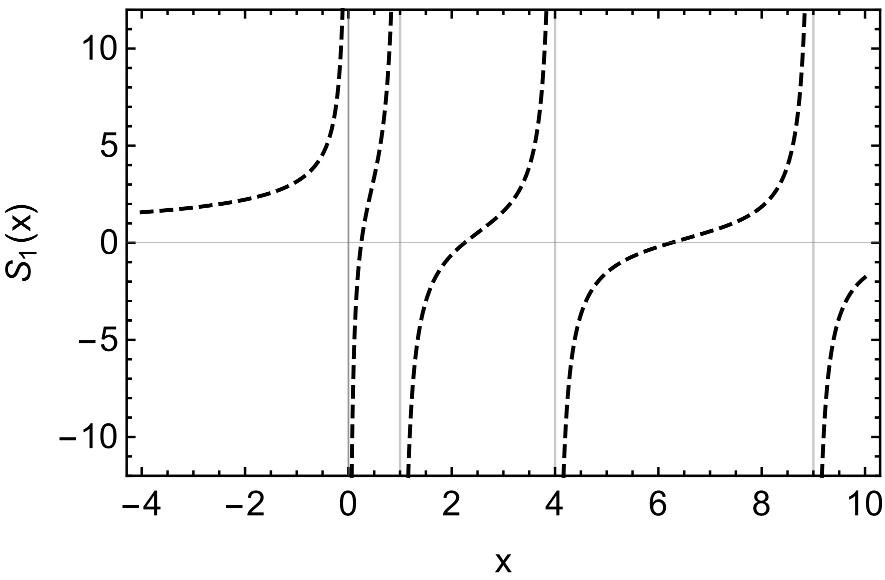

Figure 6: The function $S_1(x)$ defined in Eq.(47).

where

$$S_1(x) \equiv \sum_{n=-\infty}^{\infty} \frac{1}{n^2 - x} = -\pi \frac{\cot(\pi\sqrt{x})}{\sqrt{x}} \ . \tag{47}$$

This function is valid for x both positive and negative, and is shown in Figure 6.

- Bound state in $L \to \infty$ limit needs work
- Scattering states in $L \to \infty$ needs work

2 Recap

My findings for Lüscher's formula in the different dimensions are

$$p \cot\delta(p) = \frac{1}{\pi L} S_3\left(\left(\frac{pL}{2\pi}\right)^2\right) \qquad (d=3) \tag{48}$$

$$\cot\delta(p) = \frac{1}{\pi^2} S_2\left(\left(\frac{pL}{2\pi}\right)^2\right) + \frac{2}{\pi}\log\left(\frac{pL}{2\pi}\right) \qquad (d=2) \tag{49}$$

$$\frac{1}{p}\cot\delta_0(p) = \frac{L}{2\pi^2} S_1\left(\left(\frac{pL}{2\pi}\right)^2\right) \qquad (d=1) \tag{50}$$

where $S_3(x)$, $S_2(x)$, and $S_1(x)$ are given by Eqs. (35), (43), and (47), respectively.

References

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