A primer to numerical simulations: The perihelion motion of Mercury

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Abstract. Numerical simulations are playing an increasingly important role in modern science. In this work it is suggested to use a numerical study of the famous perihelion motion of the planet Mercury (one of the prime observables supporting Einsteins General Relativity) as a test case to teach numerical simulations to high school students. The paper includes details about the development of the code as well as a discussion of the visualization of the results. In addition a method is discussed how to estimate the size of the effect a priori.

1. Introduction

Numerical simulations play a key role in modern physics for they allow one to tackle theoretical problems not accessible otherwise, e.g., since there are too many particle participating in the system (as in simulations for weather predictions) or the interactions are too complicated to allow for a systematic, perturbative approach (as in theoretical descriptions of nuclear particles at the fundamental level). The goal of this paper is it to introduce a project that could be used to make numerical simulations themselves the topic of the class. On the example of the perihelion motion of the planet Mercury the students are supposed to get in touch with

- the importance of differential equations in theoretical physics;
- the numerical implementation of Newtonian dynamics;
- systematic ~~test~~ tests and optimization of computer codes;
- effective tools to estimate the result a priori as an important cross check;
- the visualization of numerical results using V-Phython.

The course as well as this paper is structured as follows: after an introduction to Newtonian dynamics and the concepts of differential equations their discretization is discussed based on Newtons law of gravitation and possible extensions thereof. Afterwards the visualization of the resulting trajectories using V-Phython is introduced and applied to the problem at hand. In particular tools are developed to extract the

relevant quantity from the result of the simulation. Finally the principle of dimensional analysis is presented as a tool to cross check, if the results of the simulation are sensible.

We are convinced that in order to excite the students for numerical simulations it is compulsory to demonstrate their power on an example that they catches their interest. This purpose is served perfectly by the case chosen here, since Einsteins equations of General Relativity are fascinating to a very broad public. While their detailed study needs deeper knowledge in Differential Geometry, for the study outlined here very little math is necessary, such that high school students from about 10$^{\text{th}}$ grade up should benefit from it. This was already demonstrated in the 'Schülderakademie Teilchenphysik', where an early version of this course was tested successfully on a goup of 10 German high school students from 10$^{\text{th}}$ to 13$^{\text{th}}$ grade.

2. Trajectories, velocities, accelerations and Newtons second law

We can say that we understand a physical system, if we can demonstrate that ~~an~~ the assumed force leads to the trajectories observed~~, where here trajectory means~~. In other words, we need to show that we can calculate the location in space of the object of interest at any point in time, once the initial conditions are fixed properly. If we can neglect the finite size of this object (and in particular its orientation in space) the location is parametrized by a single three vector $\vec{x}(t)$. As will become clear below in order to describe and control the dynamics of a physical object in addition we need to be able to calculate its velocity, $\vec{v}(t)$, related to the location via

$$\vec{x}(t + \Delta t) = \vec{x}(t) + \vec{v}(t)\Delta t + ... \, , \tag{1}$$

for some infinitesimally small Δt and the acceleration, $\vec{a}(t)$ that describes the change of the velocity

$$\vec{v}(t + \Delta t) = \vec{v}(t) + \vec{a}(t)\Delta t + ... \, . \tag{2}$$

The dots in the above expressions indicate that there are in general additional terms that may be expressed with higher powers in Δt, however, for sufficiently small Δt those can be safely neglected. Thus we may define the time derivative via

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{x}(t)}{\Delta t} =: \frac{d\vec{x}(t)}{dt} = \dot{\vec{x}}(t) \, , \tag{3}$$

where $\Delta \vec{x}(t) = \vec{x}(t + \Delta t) - \vec{x}(t)$ and we introduced with the last expression a common short hand notation for time derivatives. Analogously we get

$$\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}(t)}{\Delta t} =: \frac{d\vec{v}(t)}{dt} = \dot{\vec{v}}(t) \, , \tag{4}$$

$$= \frac{d^2\vec{x}(t)}{dt^2} = \ddot{\vec{x}}(t) \, , \tag{5}$$

where we introduced in the second line the second derivative.

It was Newton who observed that if a body is at rest it will remain at rest, and if it is in motion it will remain in motion in a constant velocity in a straight line, unless it is acted upon by some force — this is known as Newtons first law. Said differently: a

force $\vec{F}$ shows up by changing the motion of some object. This is quantified in Newton's second law

$$\vec{F}(\vec{x}, t) = \frac{d}{dt}(m\vec{v}) \ . \tag{6}$$

If the mass does not change ‡ with time this reduces to the well known

$$\vec{F}(\vec{x}) = m\vec{a}(t) = m\dot{\vec{v}}(t) = m\ddot{\vec{x}}(t) \ . \tag{7}$$

Note that in general the force could depend also on the time or the velocity~~, we~~. We here restrict ourselves to the case relevant for our example where the force depends on the location only. Therefore, as soon as the force $\vec{F}(\vec{x})$ is known for all $\vec{x}$ one can in principle ~~can~~solve, e.g., Eq. (**??**) for $\vec{x}(t)$. Sometimes this requires some advanced knowledge in math, sometimes no closed form solution exists. However, alternatively one can calculate the whole trajectory of some test body that experiences this force by a successive application of the rules given in Eqs. (**??**) and (**??**):

(i) For a given time t, where $\vec{x}(t)$ and $\vec{v}(t)$ are known, use Eq. (**??**) to calculate $\vec{a}(t)$.

(ii) Use Eq. (**??**) to calculate $\vec{x}(t + \Delta t)$ and

(iii) then use Eq. (**??**) to calculate $\vec{v}(t + \Delta t)$.

(iv) Go back to (i) with $t \to t + \Delta t$.

Clearly to initiate the procedure at some time t_0 both $\vec{x}(t_0)$ as well as $\vec{v}(t_0)$ must be known — the trajectories depend on these initial conditions. §

Clearly for this procedure to work Δt must be sufficiently small. What this means depends on the process studied. One way to ~~check~~estimate, if Δt is small enough, is to verify, if the ~~realation~~ relation

$$|\vec{v}(t)| \gg \frac{1}{2}|\vec{a}(t)|\Delta t = \frac{1}{2m}\vec{F}(x)\vec{\Delta t} \ . \tag{8}$$

holds, for the next term neglected in Eq. (**??**) reads $(1/2)a(t)(\Delta t)^2$. Eq. (**??**) also shows that small (large) time steps are necessary (sufficient), if the force is strong (weak), since the time steps need to be small enough that all changes induced by the force get resolved. Clearly, a relation as Eq. (**??**) can only provide guidance and can not replace a careful numerical check of the solutions: A valid result has to be insensitive the concrete value Δt chosen — in particular replacing Δt by $\Delta t/2$ should not change the result significantly.

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‡ A well known example where m does change with time is a rocket, whose mass decreases as the rocket rises.

§ In general a differential equation of nth degree (where the highest derivative is n) needs n initial conditions specified. For $n = 2$ those are often chosen as location and velocity at some defined time, but also to pick two location at different times is possible.

3. Example: ~~The~~ General Relativity and the perihelion motion of Mercury

For this concrete example the starting point for the force is Newtons law of gravitation

$$F_N(\vec{x}) = \frac{G_N m M_\odot}{r^2} \ , \tag{9}$$

where $G_N = 6.67 \times 10^{-11}$ m³kg⁻¹s-2 is the Newtonian constant of gravitation, m is the mass of Mercury and $M_\odot = 2 \times 10^{30}$ kg is the mass of the sun. In addition $r = |\vec{x}(t)|$ denotes the distance between sun and Mercury, when we assume that the sun is infinitely more heavy than the planet and located at the center of the coordinate system. Although this is not exact, since $m/M_\odot \sim 10^{-8}$ this is a pretty good approximation. For later convenience we introduce the Schwarzschildradius of the sun

$$r_S = \frac{2 G_N M_\odot}{c^2} = 3 \text{ km} \ , \tag{10}$$

where $c = 3 \times 10^8$ m/s denotes the speed of light. Note the r_S is the characteristic length scale of the gravitational field of the sun for — up to the prefactor — one can not form another quantity with dimensions of a length from G_N, $M_\odot$ and c. This is what we need r_S for later in the discussion. That the Schwarzschild radius is also an important quantity to characterize black holes is not relevant for the discussion at hand. With this Newtons second law reads

$$\ddot{\vec{x}} = \frac{c^2}{2} \left(\frac{r_S}{r^2} \right) \ . \tag{11}$$

~~Now the students should implement the program outlined in Sec.~~ **??** ~~and visualize the resulting trajectories of Mercury for different start parameters. Why is it sufficient to work in two dimensions using simply $\vec{x}(t) = (x(t), y(t))^T$? When the parameters are chosen appropriately close orbits should emerge that are fixed in time — in particular the perihelion (the~~ In general an attractive force that vanishes at large distances leads, depending on the initial conditions, either to closed orbits or open orbits — in the latter case the planet simply disappears from the sun. Depending on time available for the course it might be helpful to explain to the students the origin of this, however, since the most transparent reasoning needs an integration as well as the concept of angular momentum not necessarily familiar to the whole class (details are given in the appendix **to be done**), it appears in general more appropriate to work out together with the students that a moving body can not be kept by a force if its momentum is too large. Then they may try to find closed orbits in the simulation, e.g., by varying the start velocity while keeping the start location fixed.

The closed orbits that emerge from a force that scales as $1/r^2$ are elliptic and in particular fixed in space. In particular the point of closest approach of ~~planet and sun)~~ ~~stays fixed in space. It is a well known feature of a $1/r^2$ force that elliptic orbits with fixed perihelionemerge~~the planet to the sun, the perihelion, does not move. However, as soon as the potential is different, the perihelion moves. ~~This will be studied next: Now the students should add a new term to the right hand side of~~ Because of this the behavior of the perihelion is a very sensitive probe of the gravitational potential.

The observed perihelion motion of Mercury is nowadays determined as $(574.10 \pm 0.65)''$ per 100 earth years ‖, but it should be noted that the bulk of this number can be understood by the presence of other planets within the Newtonian theory, since their gravitational force also acts on Mercury. However, a residual motion of $(43)''$ in 100 earth years remained unexplained, until Einstein quantified the predictions of General Relativity to this particular observable.

Here should be a brief description of the relation between Newtons dynamics and General Relativity

To allow for a movement of the perihelion also in the study presented here the potential of Eq. (**??**). ~~In particular the equation should now read~~ We do this using the following ansatz

$$\ddot{\vec{x}} = \frac{c^2}{2} \left(\frac{r_S}{r^2} \right) \left(1 + \alpha \left(\frac{r_S}{r} \right) \right) , \tag{12}$$

where α is some parameter. ~~What happens to the trajectories as α is varied? Finally the start values need to be chosen appropriately such that the simulation indeed mimicks the trajectories of Mercury~~ The idea is that the correction must be dimension less and should scale with some inverse power of r, for we still need to demand that the potential vanishes at large distances. Since the only parameter of the system with dimension of a length is r_S, the correction should scale as (r_S/r). When replacing r by a typical distance of Mercury to the sun, $(r_S/r) \sim 10^{-7}$, it should be sufficient to include this kind of term to first order only. For comparison: the corresponding number for the venus, the neighbor of Mercury, reads ...., which naturally explains why the effect is observable for Mercury only (IS IT - CHECK!)

4. Numerical Implementation

In this section we present the numerical implementation ~~of~~ as well as the visualization of planetary trajectories and in particular the perihelion motion of Mercury. We select `Python` ~~as the programming languageof choice [] ~~ [] as programming language, because `Python` is easy to learn, intuitive to understand and an open source language¶. No prior knowledge of `Python` or any other programming language is required because we explain all the ~~essential steps in order~~ steps necessary to create the simulation. We also provide a working example [] ~~which~~ that can be used as template.

~~The first step of a numerical simulation is the extraction of the relevant physical parameters and quantities in order to translate the problem to mathematical equations. The relevant objects for this example are the sun and mercury. Thus, to describe their motion, one needs their~~ To start the simulation one needs the "initial" distances ~~, velocities and accelerations which are the~~ and velocities, which are needed as input to their equation of motions (Eq. (**??**) and Eq. (**??**)). ~~In the case of this problem,~~ As described above, here one can simplify the ~~implementation~~ problem by placing one

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‖ The symbol $''$ denotes 'arc seconds': $1'' = (1/3600)^o$.

¶ See also appendix **??** for an instruction how to set up python on different operating systems.

planet (the "infinitely" heavy sun) in the center of the coordinate system and describe the motion of the other planet (mercury) in a two-dimensional plane. *Why is this simplified picture sufficient to describe the motion?*

Because the problem does not depend on the initial position of mercury on its orbit around the sun, we extract the parameters at the perihelion with $|\vec{r}_{MS}(0)| = 46 \cdot 10^6 km$ and $|\vec{v}_M(0)| = 39 km/s$ $|\vec{v}_M(0)| = 39$ km/s [+]. Since the computer does not understand physical units, one has to express each variable in ~~their respective~~ an appropriate unit. Both, for the numerical treatment and the intuitive understanding, it is useful to select parameters in a "natural range". A particular useful choice for this problem is given by expressing time intervals in days $T_0 = 1d$ **These are earth days, I assume, right?** and distances in $R_0 = 10^{10}m$ $R_0 = 10^{10}$ m. With this choice, the initial distance of mercury to the sun, the size of the initial velocity of mercury and the acceleration prefactor become

$$r_{MS}(0) = 4.6 R_0 \,, \quad v_M(0) = 0.34 \frac{R_0}{T_0} \,, \quad a_M = 0.99 \frac{R_0}{T_0^2} \frac{1}{(r_{MS}/R_0)^2} \,. \quad (13)$$
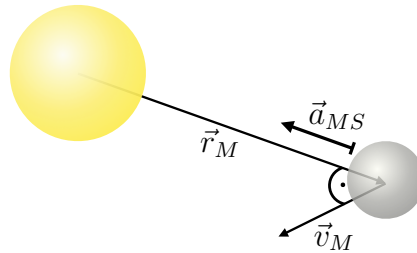

Figure 1. Sun mercury system with relevant vectors. Because the mercury is at its perihelion, its velocity is perpendicular to its direct connection vector with the sun.

In python, this reads

```
# Definition of parameters
rM0 = 4.6    # in units of R0
vM0 = 0.34   # in units of R0/T0
aM  = 0.99   # in units of R0/T0**2
```

So far we only fixed the length of the vectors. Next, we set up the initial ~~vectors~~ directions, which will describe the motion in the two-dimensional plane. We will build on the existing Python module ~~:~~ Vpython, which provides ~~implementation for the vectorial implementation and animation~~ an implementation for treating vectors as well as their visualization. The first object of interest is a vector, which takes three-dimensional coordinates as its input. Because of our choice of initial conditions (we picked the initial vectors in the perihelion), the velocity of mercury is perpendicular to the vector which connects mercury and the sun (see figure **??**):

[+] https://nssdc.gsfc.nasa.gov/planetary/factsheet/mercuryfact.html

```
# Import the class vector from python
from vpython import vector
# Initialize distance and velocity vectors
vec_rM0 = vector(0, rM0, 0)
vec_vM0 = vector(vm0, 0, 0)
```

According to Eq. (**??**), the force which acts on mercury changes the velocity of mercury which eventually changes the position. ~~In first approximation we can numerically express the updates of the position and the velocity by~~ However, we first need to fix the time step Δt and use the estimate of Eq. (**??**) as a guidance **Careful: Check units of dt**

```
# \DIFaddbegin \DIFadd{Definition of the time step
    dt = 2 * vm0 / aM / 100
}
```

Here the factor $1/100$ makes sure that Δt is indeed consistent with Eq. (**??**). **We need to make sure that 'dt' is known to the subroutines** Now we are in the position to calculate location and velocity of the planet at $t_0 + \Delta t$ using the following comands

```
\DIFaddend Compute the strength of the acceleration
aMS = aM * ( 1 + alpha * rS / vec_rM_old.mag  ) / vec_rM_old.mag**2
# Multiply by the direction
vec_aMS = - aMS * ( vec_rM_old / vec_rM_old.mag )
# Update velocity vector
vec_vM_new = vec_vM_old + vec_aMS * dt
# Update position vector
vec_rM_new = vec_rM_old + vec_vM_new * dt
```

Note the beauty of working with the predefined `vector` class: the basic vector operations are already implemented. The difference and sum of two vectors, or the scalar vector multiplication return vectors themselves. Also the magnitude of a vector – `vector.mag` – is a property of the vector and can be easily extracted.

It is handy to use `Python`s `function`s to embed repeating structures ("DRY" – Don't Repeat Yourself)

```
# Define the function
def evolve_mercury(vec_rM_old, vec_vM_old\DIFaddbegin \DIFadd{,alpha}\DIFaddend ):
    <...Code...>
    return vec_rM_new, vec_vM_new

# Call the function
vec_rM_new, vec_vM_new = do_time_step(vec_rM_old, vec_vM_old)
```

Because of ~~Python~~ Python's syntax, it is necessary that the body of the function is indented relative to the definition statement **I do not understand this sentence**.

Finally, we can express the evolution by a `while`-loop

```
t = 0
\DIFaddbegin \DIFadd{alpha=0
}\DIFaddend # Execute the loop as long as t < T
```

```
    while  t  <  T:
        vec_rM,  vec_vM  =  evolve_mercury(vec_rM,  vec_vM\DIFaddbegin  \DIFadd{,alpha}\DIFad
        t  =  t  +  dt
```

where for the start we set the parameter α to zero in order first study the properties of the pure $1/r^2$ force. Note the required indent of the loop structure similar to the indent of a function. In each iteration of the `while`-loop, the previous distance and velocity are used to compute the new values – which directly overwrite the previous values and thus enter the next iteration. The ~~time step `dt` needs to be sufficiently small in order to avoid numerical errors. We recommend $dt = ...$ The~~ total runtime `T` is the amount of "virtual" days the simulations should run. To describe at least one full orbit, T needs to be larger than $T > ...$

Here we need to write something about visualizing the trajectories and how to run the code

5. Tests of stability

6. Tasks

Ideas for questions:

- Mass and motion of sun
- Estimate `dt` and also `T` in order to cover roughly two "mercury years".
- What does happen when you change those units?
- Compare values to the values in []. Where does the difference come from?
- Start with $\alpha = 0$. What do you observe?

7. ~~Visualization using V-Phython~~

7. Dimensional analysis

Dimensional analysis is not only a tool that allows one to cross check if the results of some simulation is of the right order of magnitude, it is also very helpful to identify unusual dynamics in some system. Especially the latter aspect should become clear from the discussion in this section.

The idea of dimensional analysis is that in a system that can be controlled by expanding the relevant quantities (like the force) in some small parameter(s), the coefficients in the expansion should turn out to be of order unity (that means anything between about 0.1 and 10 is fine - but 0.01 or 100 is irritating) — parameters in line with this are called 'natural'. Applied to the problem at hand given by Eq. (**??**) this statement implies that from naturalness one would expect that the parameter α is of

order 1 when one uses for r the average distance Mercury-sun, namely $\bar{r} = 6 \times 10^7$ km. Form this one estimates for the expected angular shift per orbit

$$\delta\phi \simeq 2\pi \left(\frac{r_S}{\bar{r}}\right) = (\pi \times 10^{-7}) \text{ rad} = (2 \times 10^{-5})^o = (7 \times 10^{-2})'' \,, \tag{14}$$

which leads to a shift of about $30''$ in 100 earth years to be compared to the empirical value of $43''$. Thus indeed the amount of perihelion motion of Mercury is in line with expectations, *if* — and this is an important 'if' — the Newtonian dynamics is simply the leading term of some more general underlying theory. In particular, no new scales enter in the correction terms.

Thanks to Einstein we indeed know that the conclusion formulated above is correct —- the more general underlying theory is General Relativity and indeed Einstein was able to quantitatively explain the perihelion motion of Mercury from his equations. On the other hand had we found a dramatic deviation of α from unity one would have concluded that there is probably some other dynamics going on that drives this difference.

Indeed, we now several of those hierarchy problems in modern physics: e.g. the so called QCD θ term, expected to be of natural size, is at present known to be at most 10^{-10}. This smallness, called the strong CP problem, is so irritating that physicists like S. Weinberg even proposed that there must exist an additional particle, the axion, whose interactions are in charge of pushing θ even to zero and there are now intense searches for this axion going on various labs.

8. Summary and possible extensions

8.1. Acknowledgments

Here they come

Appendix A. Instructions to install V-phython

Appendix B. The code

Here we should put the code