Classical vs. Data Driven Regularization in Imaging Methods in Imaging

Part I: Classical Regularization Methods

First Tandem Tutorial of Thematic Einstein Semester
Mathematics of Imaging in Real-World Challenges

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29 October 2021

WIAS Research group 8:
Nonsmooth Variational Problems
and Operator Equations

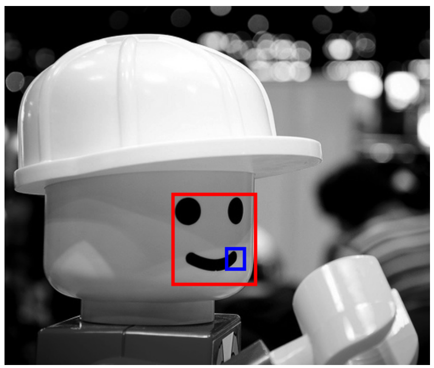
Leibniz
Association

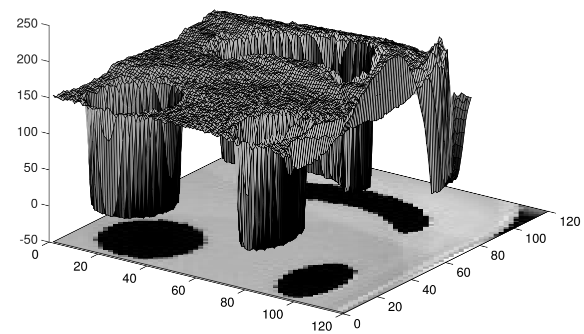
Mathematical Imaging

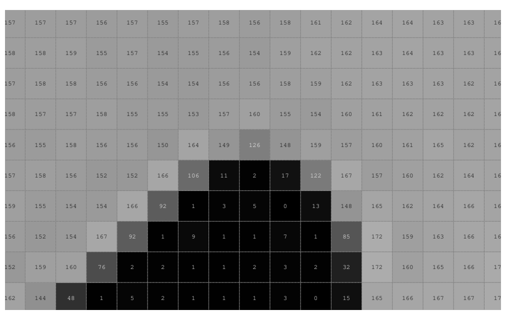
- A modern and rich field of mathematics whose target is to solve complex image processing tasks, using rigorous mathematical theories.
- A variety of mathematical tools are employed such as
  - *Calculus of Variations*
  - *Partial Differential equations*
  - *Optimisation*
  - *Convex Analysis*
  - *Numerical Analysis*
  - *Scientific Computing*
  - ...
- Great variety of real world applications, spanning from digital image enhancement to medical imaging, e.g. Magnetic Resonance Imaging (MRI) and Positron Emission Tomography (PET).

The mathematical set up

What is an digital image u?
It is function from a finite grid Ω to $\{0, \ldots, 255\}$







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3	5	1	2	0	4	1	2	2	0	3	2	35	183	171	174
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167	173	174	177	182	177	178	179	179	181	178	178	179	179	177	178

The mathematical set up

What is an "mathematical" image u?
It is an L^1 function $u : \Omega \to \mathbb{R}$, $\Omega \subseteq \mathbb{R}^2$

Why do we need the continuous setting?

- Powerful analytical tools.
- Better understanding of the image processing methods, independently of discretisation or image resolution.
- Image reconstruction tasks lead to the development of novel mathematical theories.

The mathematical set up

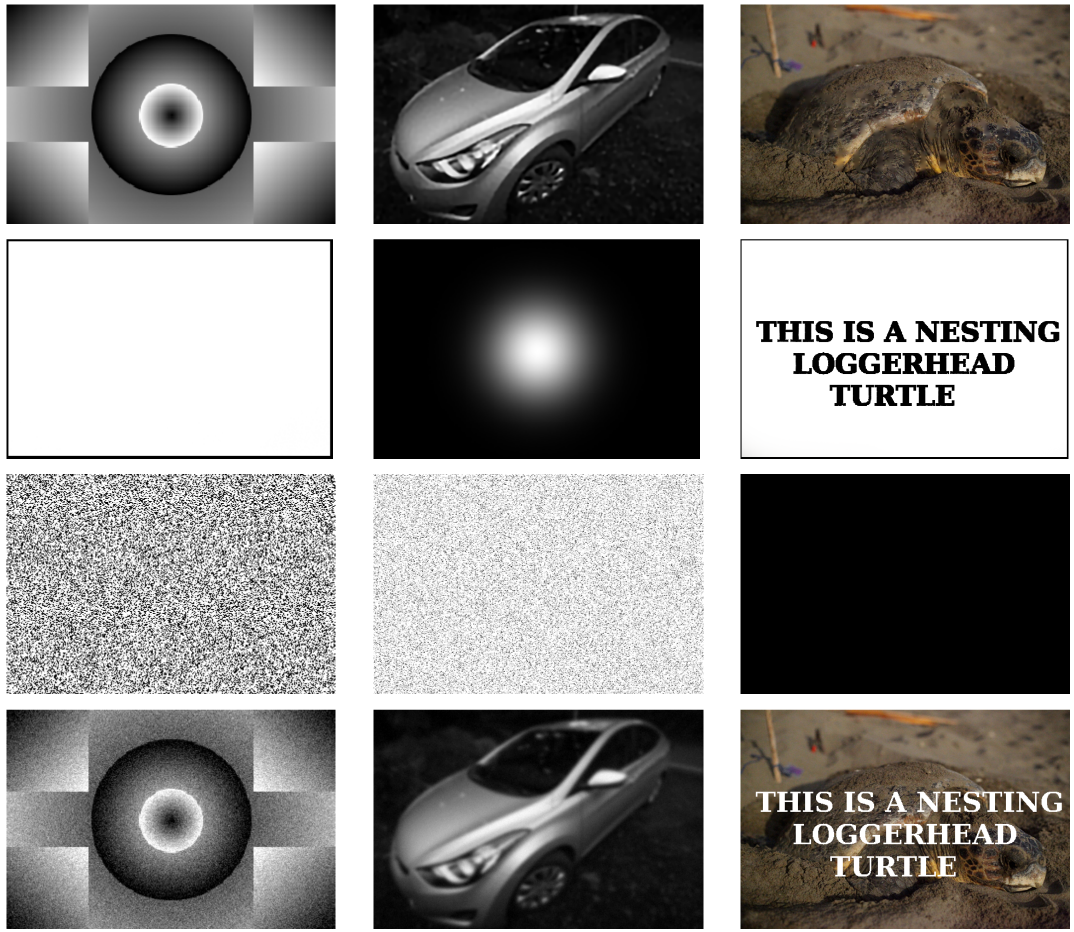
$$f = Tu_{true} + \eta$$

u_{true}: The original clean image

T: Degradation operator, (bounded, linear)

η: Random noise, e.g. Gaussian, Impulse, Poisson...

f: Corrupted data



Random noise

- Measurements are often **non-accurate/noisy** due to errors/limitations during the acquisition process.
- Such types of interferences cannot be predicted and eliminated through machine calibrations (**random noise**).

Random noise typically adversely effects any **reconstruction** process.

Image Deblurring: A case study.

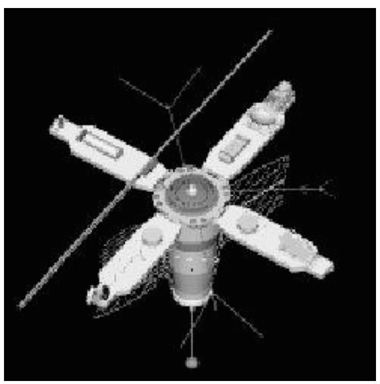
Random Noise

Blur occurs due to...

Camera / patient movement, light refraction, atmospheric turbulence,...

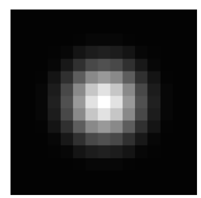
Blurring may be the result of **weighted averaging** of the data via a **point spread function (PSF)**:

Blurry Data = (PSF) $*$ (Ideal Data) + Random Noise

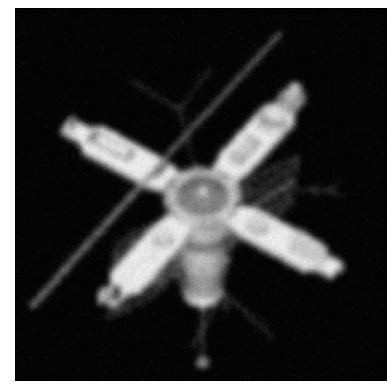


original

averaged by



Gaussian PSF



blurred

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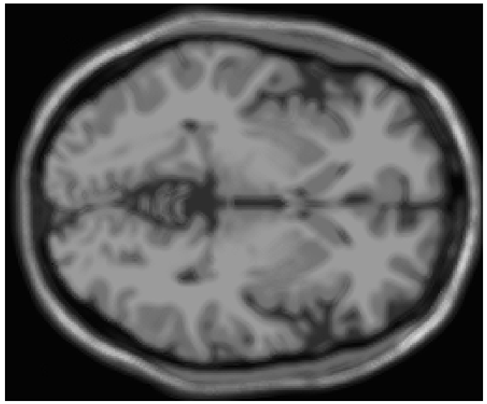
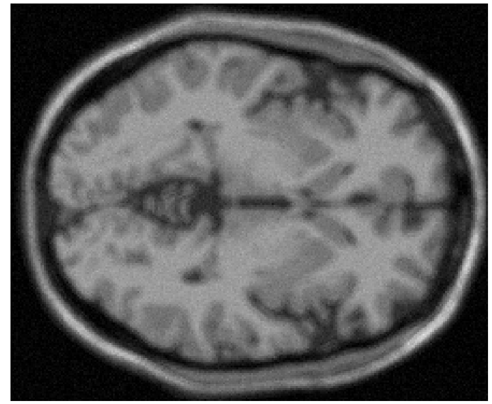
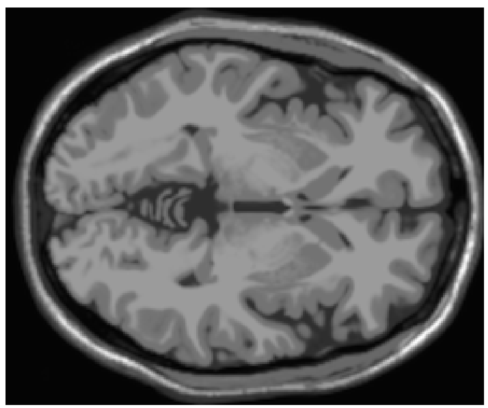


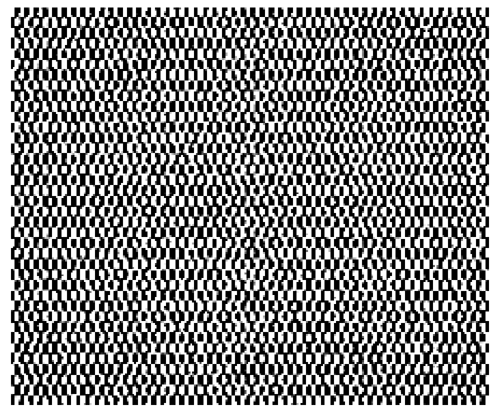
Blur without noise



Blur with noise



Reconstruction



Reconstruction (fails)

This is an ill-posed problem (as are many imaging problems)!

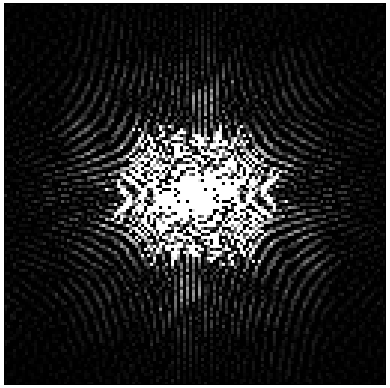
Incomplete Measurements in Medical Imaging

Magnetic Resonance Imaging (MRI)

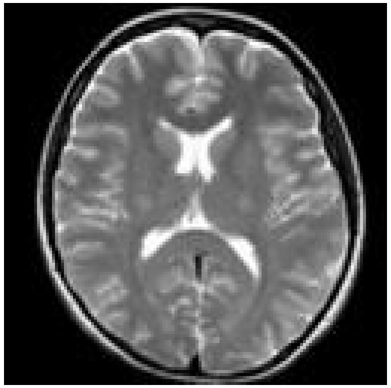
Measurement: $T = S \circ \mathcal{F}$

...an **incomplete** sample of Fourier (frequency) coefficients of an emitted signal which contains information of the measured object.

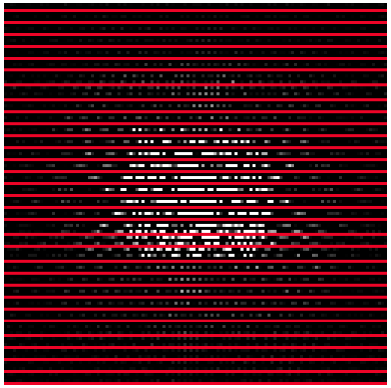
Wikipedia, CC BY-SA 3.0



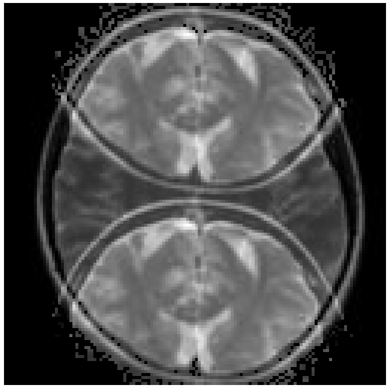
Highly sampled data



Reconstruction



Low sampled data



Reconstruction

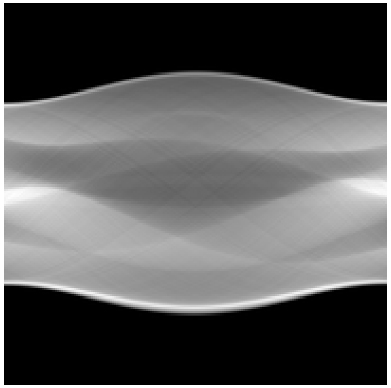
Incomplete Measurements in Medical Imaging

Tomography (CT scan, PET scan...)

Measurement: $T = \mathcal{R}$, Radon transform

The attenuation of wave energy gone through the tissue under a specific angle.
However the number of angles is limited resulting in **incomplete** data collection.

Wikipedia, CC BY 2.0



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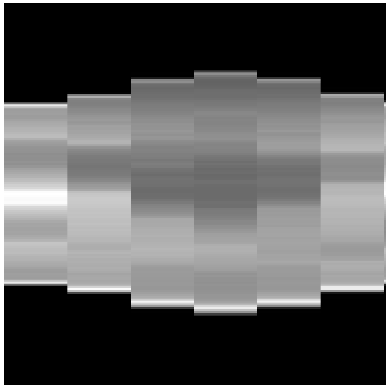

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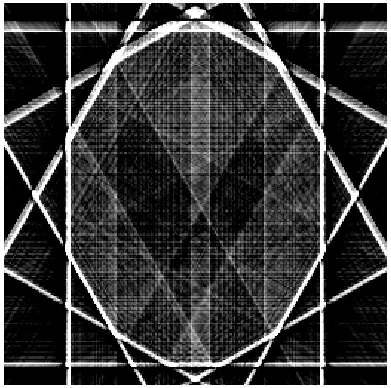

Reconstruction

Variational approach in image reconstruction

The **inversion process** of the forward operation T needs to be **regularized**.

Regularization ensures **stable** data recovery and also **noise elimination** (*filtering*). Typically it takes a form of an optimization problem:

$$\underset{\substack{\text{all possible} \\ \text{reconstructions } u}}{\text{minimize}} \quad \mathrm{distance}(T(u), data) + \mathrm{regularization}(u)$$

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or a bit more mathematically...

$$f = Tu_{true} + \eta$$

$\rightarrow$ Find a good approximation of u_{true} by solving a problem of the type:

$$\min_{u \in X} \quad \underbrace{D(Tu, f)}_{\text{data discrepancy}} \quad + \quad \underbrace{J_{\boldsymbol{\alpha}}(u)}_{\text{regularisation with weight } \boldsymbol{\alpha}}$$

Data discrepancy functionals

Their choice is typically dictated by the statistics of the noise:

- $\frac{1}{2}\int_\Omega |Tu - f|^2\,dx$
- $\int_\Omega |Tu - f|\,dx$
- $\int_\Sigma Tu - f\log(Tu)\,d\sigma$
- $\min_v \lambda_1 \|v\|_{L^1(\Omega)} + \frac{\lambda_2}{2}\|f - Tu - v\|^2_{L^2(\Omega)}$
- $\ldots$

Gaussian noise

[Rudin, Osher, Fatemi (1992)], [Chambolle, Lions (1997)]

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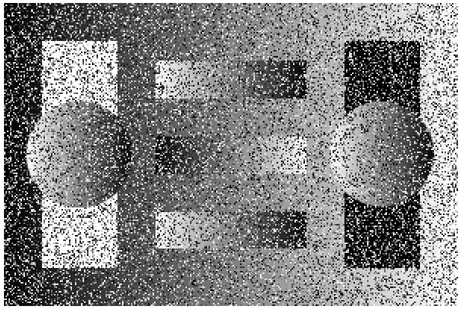
Impulse noise



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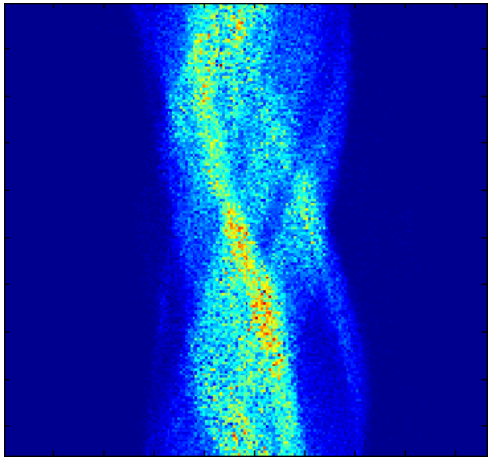
Poisson noise



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Mixed noise

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How to measure distance, matters!

Salt & Pepper noise

$D(u,f) = \frac{1}{2}\int_\Omega (u-f)^2 dx$

$D(u,f) = \int_\Omega |u-f|\, dx$

Total Variation Regularization

Functions of bounded variation $\mathrm{BV}(\Omega)$

Let $\Omega \subseteq \mathbb{R}^d$, bounded, open with Lipschitz boundary.

The space of functions of bounded variation $\mathrm{BV}(\Omega)$ consists of all the $L^1(\Omega)$ functions whose distributional derivative Du can be represented by a Radon measure.

$u \in \mathrm{BV}(\Omega)$ if and only if $\mathrm{TV}(u) < \infty$ where
$\mathrm{TV}(u) = \sup \left\{ \int_\Omega u \, \mathrm{div}\phi \, dx : \ \phi \in C_c^\infty(\Omega, \mathbb{R}^d), \ \|\phi\|_\infty \leq 1 \right\}$
In that case $\mathrm{TV}(u) = |Du|(\Omega)$.

$$Du = D^a u + D^s u = \nabla u \mathcal{L}^d + D^s u$$

If $u \in W^{1,1}(\Omega)$ then $|Du|(\Omega) = \int_\Omega |\nabla u| \, dx$.

Banach space under the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |Du|(\Omega)$.

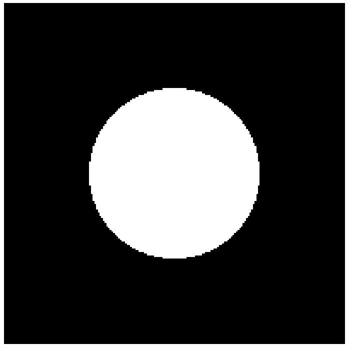
Functions of bounded variation $\mathrm{BV}(\Omega)$

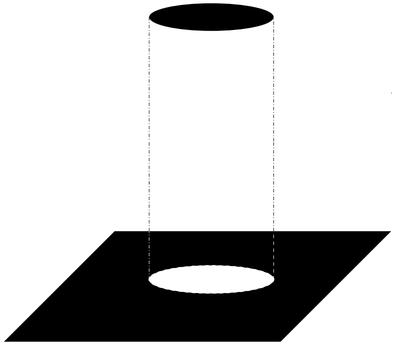
Why is BV useful from the analysis point of view?

Unlike $W^{1,1}(\Omega)$ is has useful compactness properties: Bounded BV-norm sequences have weakly* convergent subsequences.

Why is BV useful from the modelling point of view?

BV functions can have **jump discontinuities** along hypersurfaces, that models **sharp edges** in images.





Total variation in imaging

In 1992, Rudin, Osher and Fatemi proposed to use the (dicretised version of) total variation as a regulariser for denoising:

$$\min_{u\in \mathrm{BV}(\Omega)} \frac{1}{2}\int_\Omega (\quad u - f)^2 dx + \alpha \mathrm{TV}(u)$$

- The problem is well-posed, i.e., there exists unique solution u.
- Since the solution $u \in \mathrm{BV}(\Omega)$, we expect to obtain a reconstruction with sharp edges
- The positive parameter α balances the straight of the regulariser and the fidelity term.

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- The positive parameter α balances the straight of the regulariser and the fidelity term.
- TV can be used a regulariser for general reconstruction tasks.
  - Image denoising
  - Image deblurring
  - Image inpainting
  - Image zooming
  - Cartoon-texture decomposition
  - MRI and PET reconstruction
  - JPEG decompression
  - Video processing
  - Optical flow estimation
  - ...

The influence of the regularisation parameter α

$$\min_{u \in \mathrm{BV}(\Omega)} \frac{1}{2} \int_\Omega (u-f)^2 dx + \alpha \mathrm{TV}(u), \quad \alpha > 0$$

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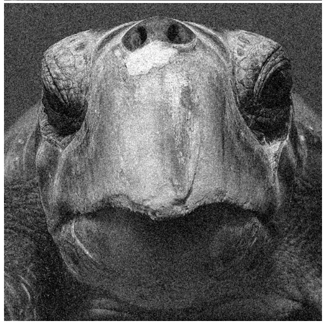

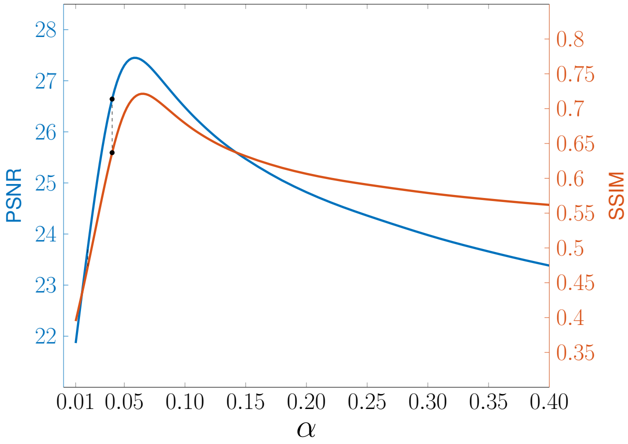

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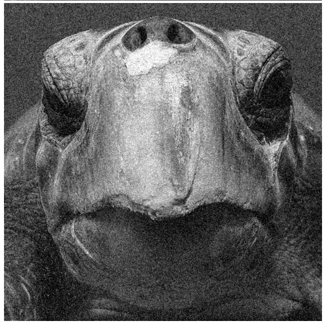

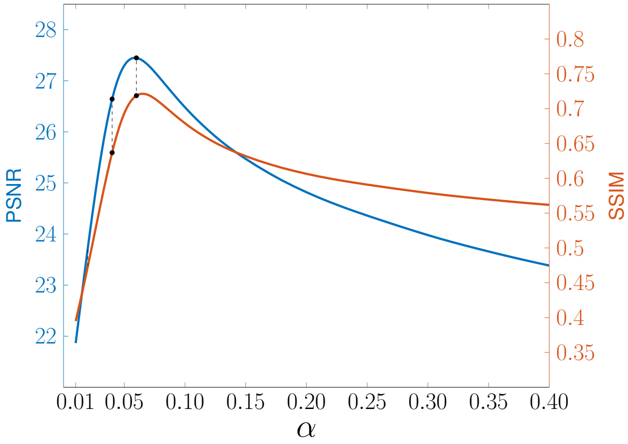

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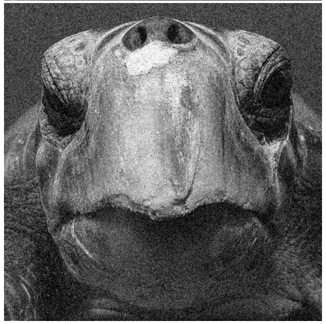

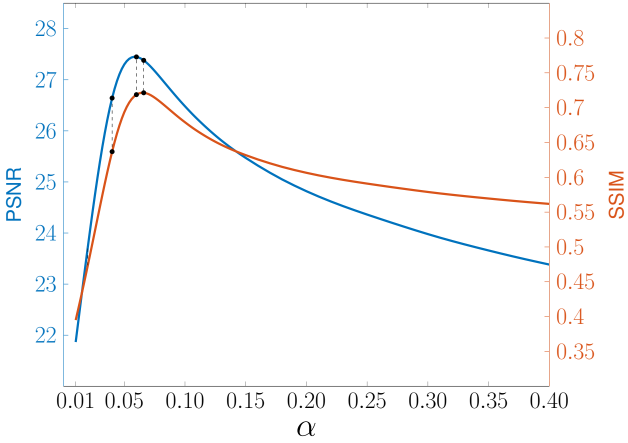

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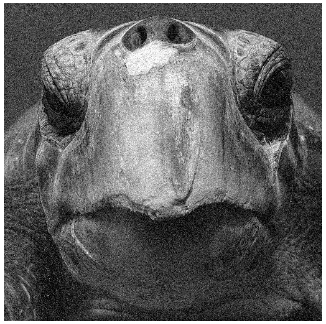

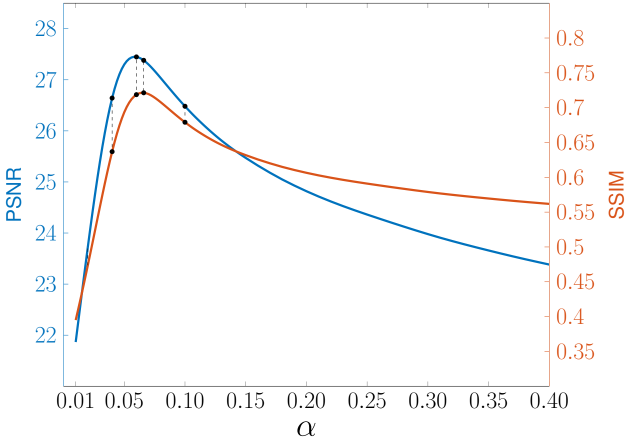

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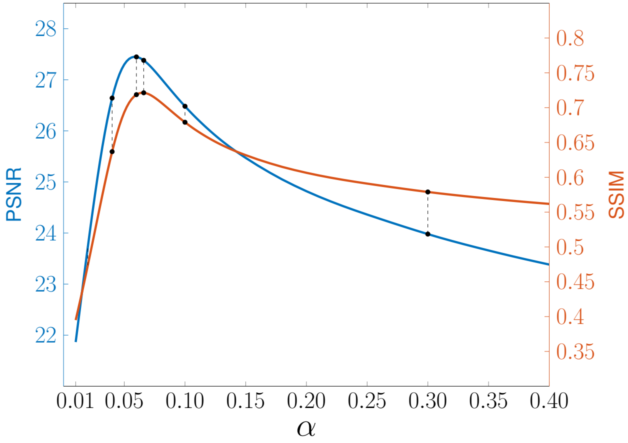

Discontinuities and staircasing: Theory

$$(P) \quad \min_{u \in \mathrm{BV}(\Omega)} \frac{1}{s} \int_\Omega |f - u|^s dx + \alpha \mathrm{TV}(u), \qquad s = 1 \text{ or } 2.$$

References: Caselles, Chambolle, Novaga (2007), Valkonen (2015), Ring (2000), Jalalzai (2014), Chambolle, Duval, Peyré, Poon (2016)

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Theorem (No new discontinuities)

If $\mathbf{s} = \mathbf{2}$, $f \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ *and* u *solves* (P) *then*
$\mathcal{H}^{d-1}(J_u \setminus J_f) = 0$.

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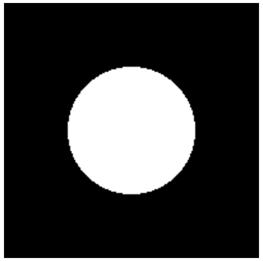

References: Caselles, Chambolle, Novaga (2007), Valkonen (2015), Ring (2000), Jalalzai (2014), Chambolle, Duval, Peyré, Poon (2016)

Discontinuities and staircasing: Theory

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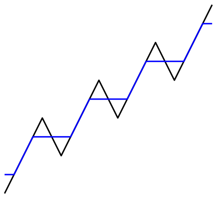
In dimension one u is constant in the areas where $u \neq f$. In higher dimensions, flat zones are greated at global extrema of f, all extrema of u.



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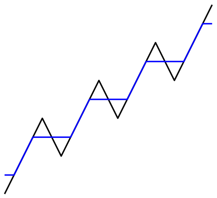

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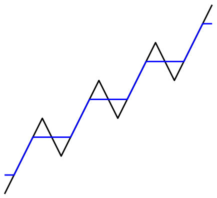
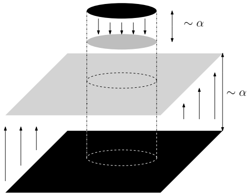




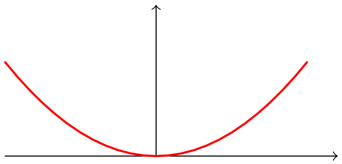
In general: *Preservation of discontinuities, loss of contrast, piecewise constant structures*

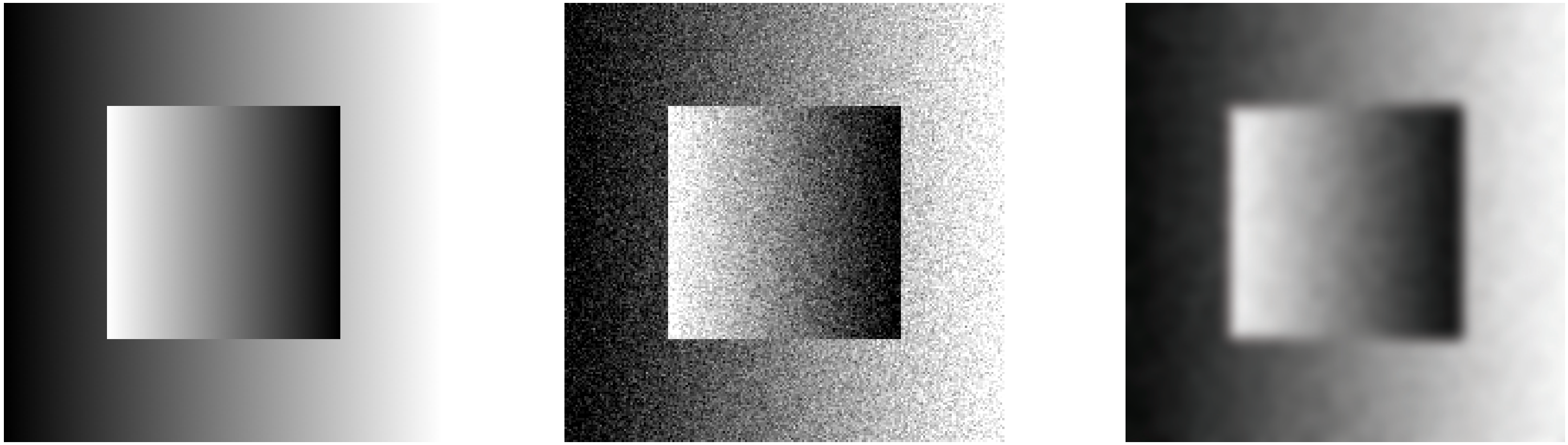
References: Caselles, Chambolle, Novaga (2007), Valkonen (2015), Ring (2000), Jalalzai (2014), Chambolle, Duval, Peyré, Poon (2016)

Discontinuities and staircasing

Tikhonov:

$$J(u) = \alpha \int_\Omega |\nabla u|^2 dx$$

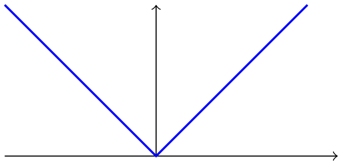


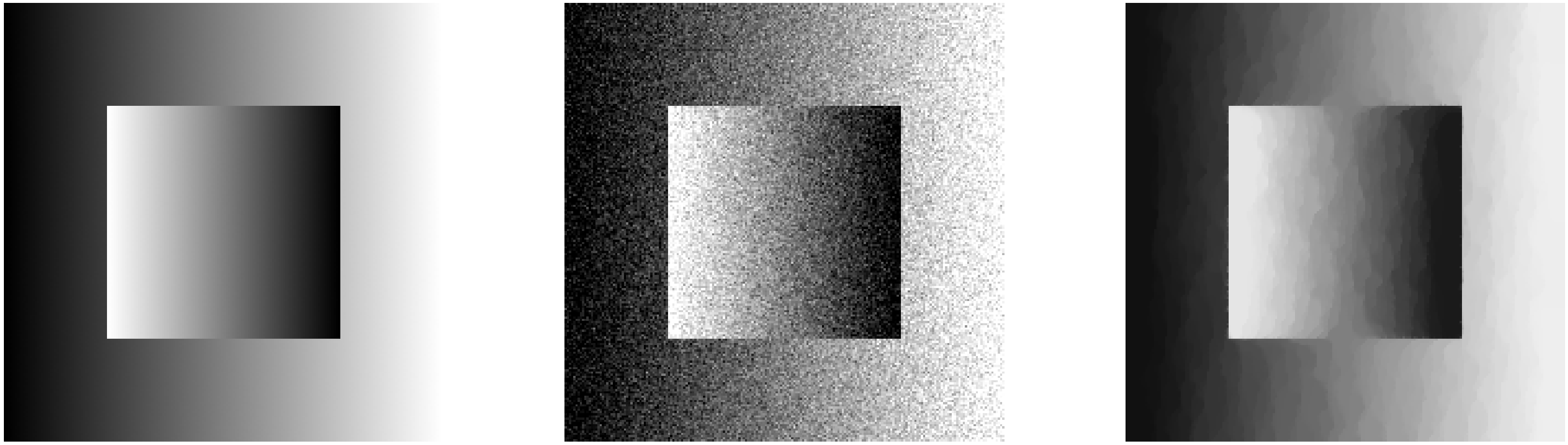


Discontinuities and staircasing

Total Variation:

$$\mathrm{TV}(u) = \alpha \int_\Omega d|Du|, \qquad u \in \mathrm{BV}(\Omega)$$
$$\Big(= \alpha \int_\Omega |\nabla u| \, dx, \quad u \in W^{1,1}(\Omega) \Big)$$



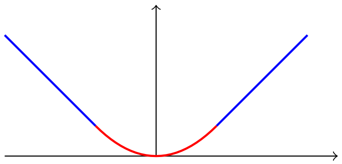


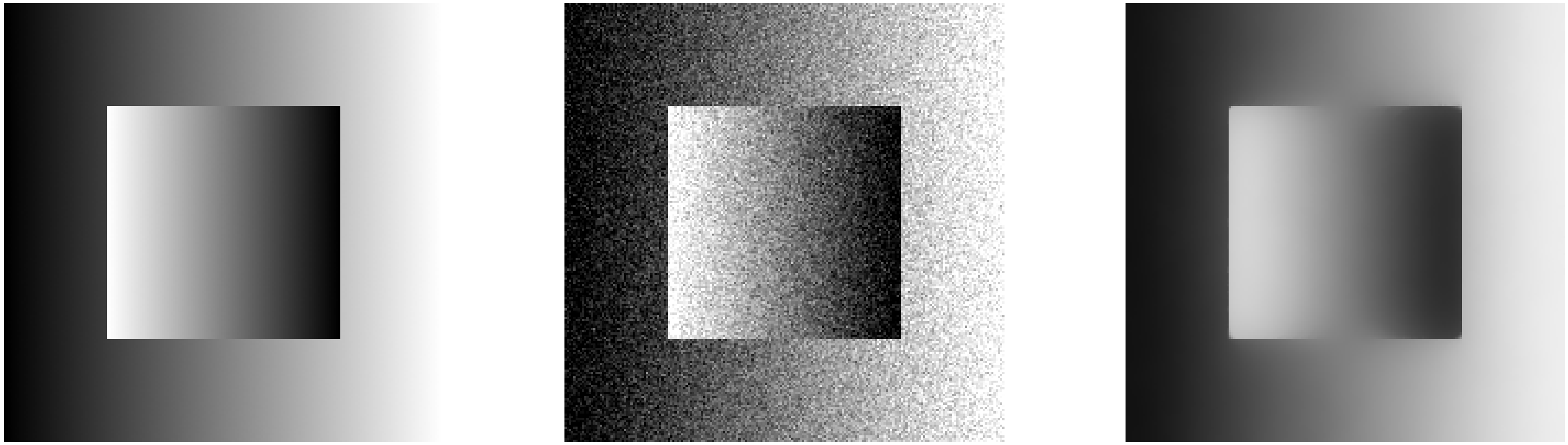
Discontinuities and staircasing

Huber Total Variation:

$$\mathrm{TV}_\varphi(u) = \alpha \int_\Omega \varphi(\nabla u)\, dx + \alpha |D^s u|(\Omega)$$

$$\varphi(v) = \begin{cases} |v| - \frac{1}{2}\gamma & \text{if } |v| \geq \gamma, \\ \frac{1}{2\gamma}|v(x)|^2 & \text{if } |v| < \gamma. \end{cases}$$



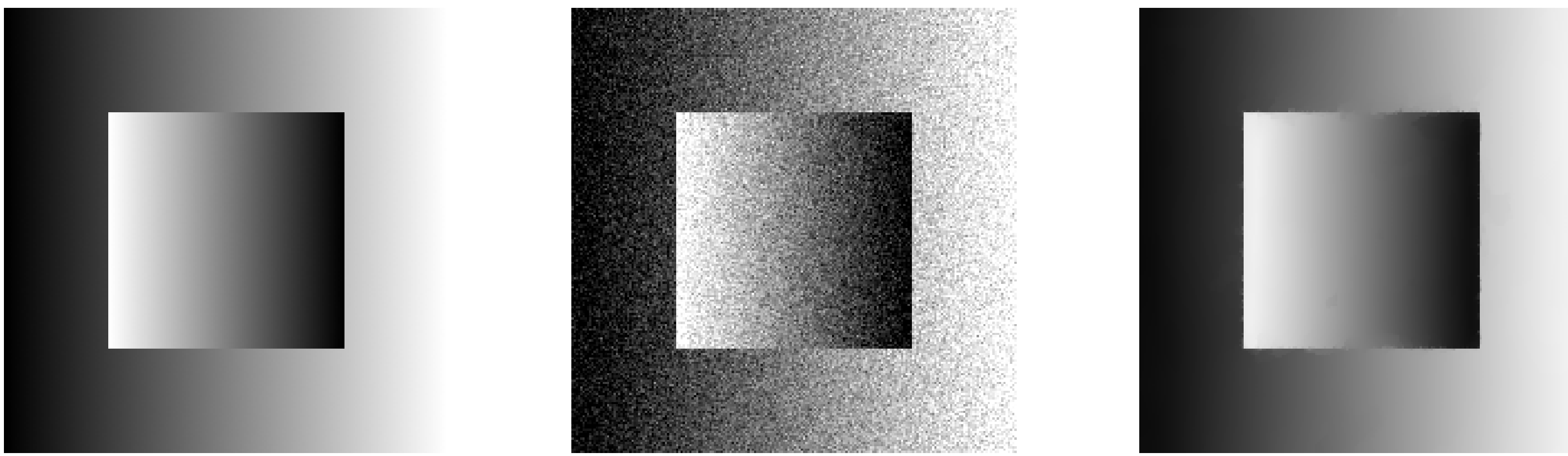


Discontinuities and staircasing

Total Generalised Variation:

[Bredies, Kunisch, Pock (2010)]

$$\mathrm{TGV}(u) = \min_{w \in \mathrm{BD}(\Omega)} \alpha \int_\Omega d|Du - w| + \beta \int_\Omega d|\mathcal{E}w|$$



Total Generalized Variation Regularization

Total generalised variation

$$\alpha \mathrm{TV}(u) = \sup \left\{ \int_\Omega u \,\mathrm{div}\phi \, dx : \ \phi \in C_c^1(\Omega, \mathbb{R}^d), \ \|\phi\|_\infty \le \alpha \right\}$$
$$\mathrm{TGV}_{\beta,\alpha}(u) = \sup \left\{ \int_\Omega u \,\mathrm{div}^2\phi \, dx : \ \phi \in C_c^2(\Omega, \mathcal{S}^{d\times d}), \ \|\phi\|_\infty \le \beta, \ \|\mathrm{div}\phi\|_\infty \le \alpha \right\}$$

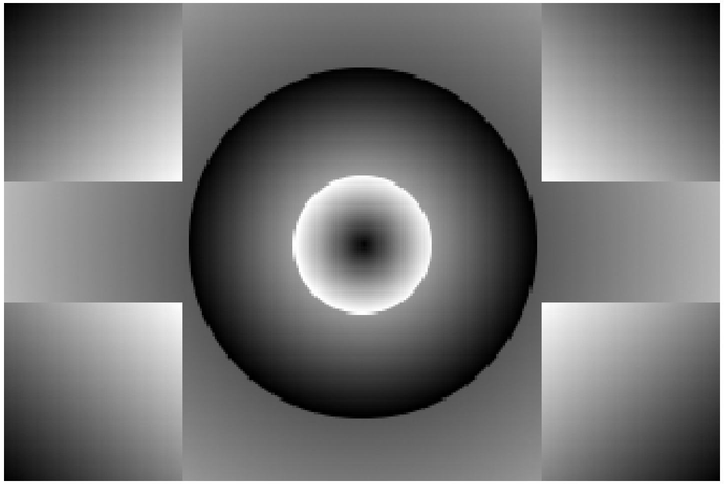
$$c\|u\|_{\mathrm{BV}(\Omega)} \le \|u\|_{L^1(\Omega)} + \mathrm{TGV}^2_{\beta,\alpha}(u) \le C\|u\|_{\mathrm{BV}(\Omega)} \quad \forall u \in L^1(\Omega)$$

$$\begin{aligned} \mathrm{TGV}^2_{\beta,\alpha}(u) &= \min_{w \in \mathrm{BD}(\Omega)} \alpha|Du - w|(\Omega) + \beta|\mathcal{E}w|(\Omega) \\ &\simeq \min_w \ \alpha \int_\Omega |\nabla u - w| \, dx + \beta \int_\Omega |Ew| \, dx \quad \left(Ew = (\nabla w + (\nabla w)^T)/2\right) \end{aligned}$$

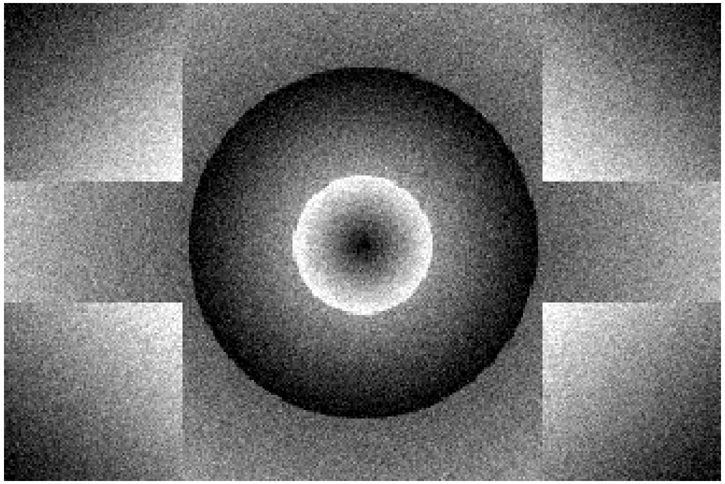
$$\min_{u \in \mathrm{BV}(\Omega)} \frac{1}{2} \int_\Omega (Tu - f)^2 dx + \mathrm{TGV}^2_{\beta,\alpha}(u)$$

References: Bredies, Kunisch, Pock (2010), Bredies, Valkonen (2011)

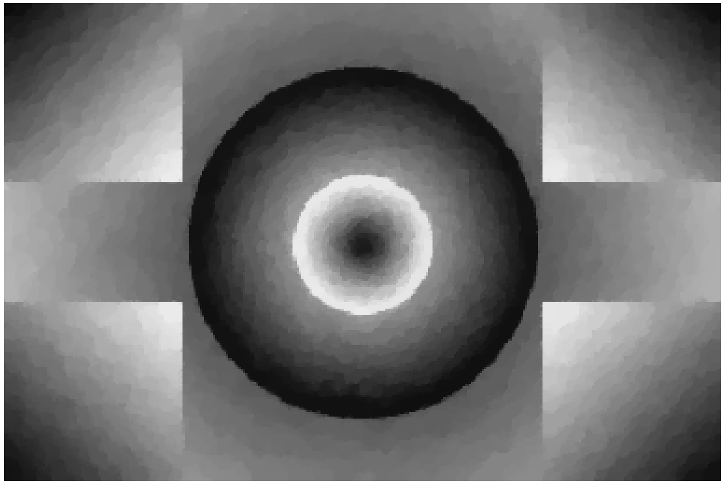
Total Generalised Variation



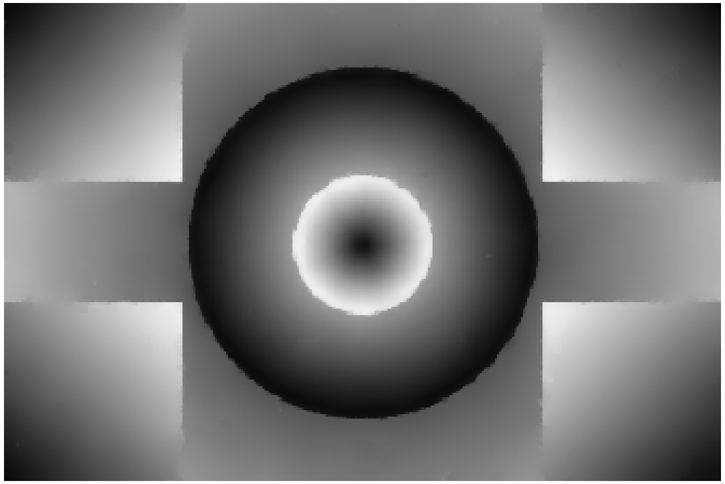
Original



Noisy

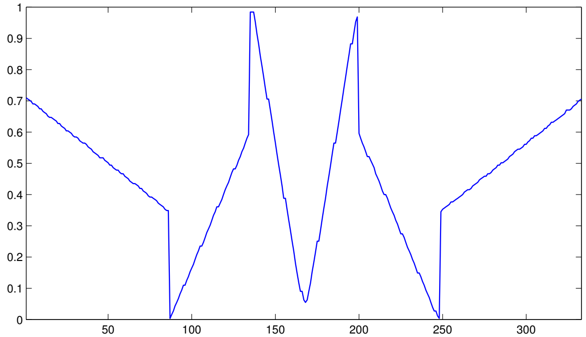


TV

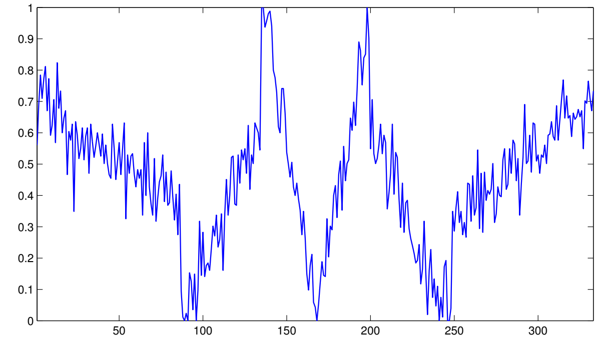


TGV

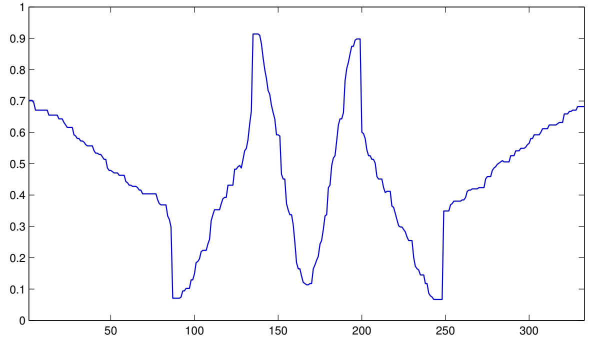
Total Generalised Variation



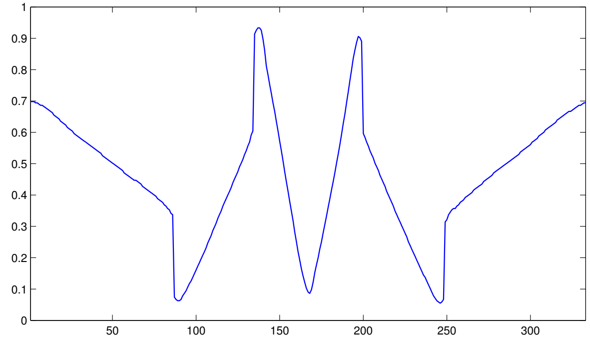
Original



Noisy

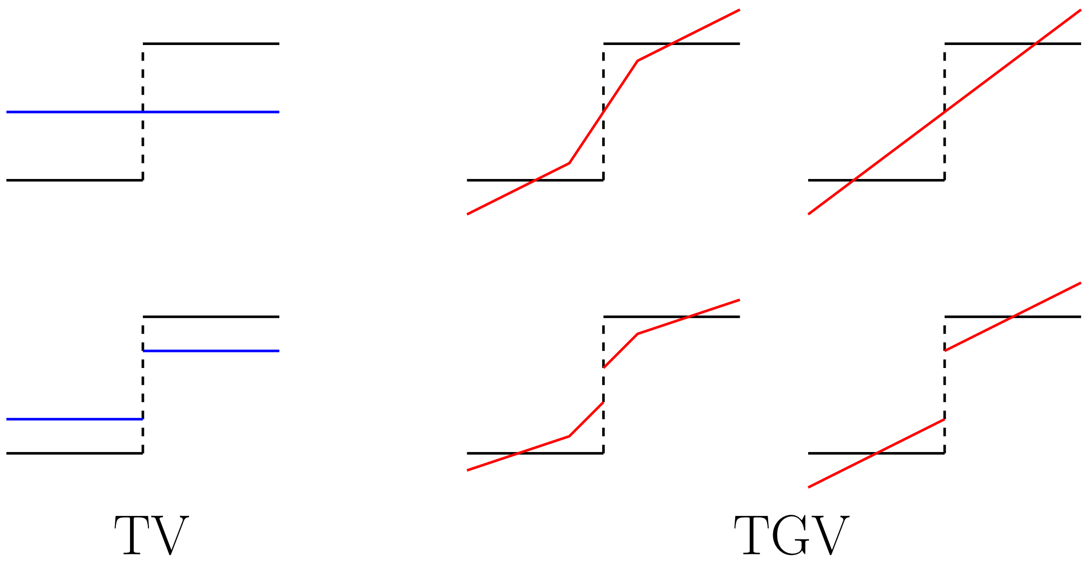


TV



TGV

Structure of TGV solutions



- For $d \geq 2$, the inclusion $\mathcal{H}^{d-1}(J_u \setminus J_f) = 0$ holds under additional technical assumptions.

References: Papafitsoros, Bredies (2015), Valkonen (2014)

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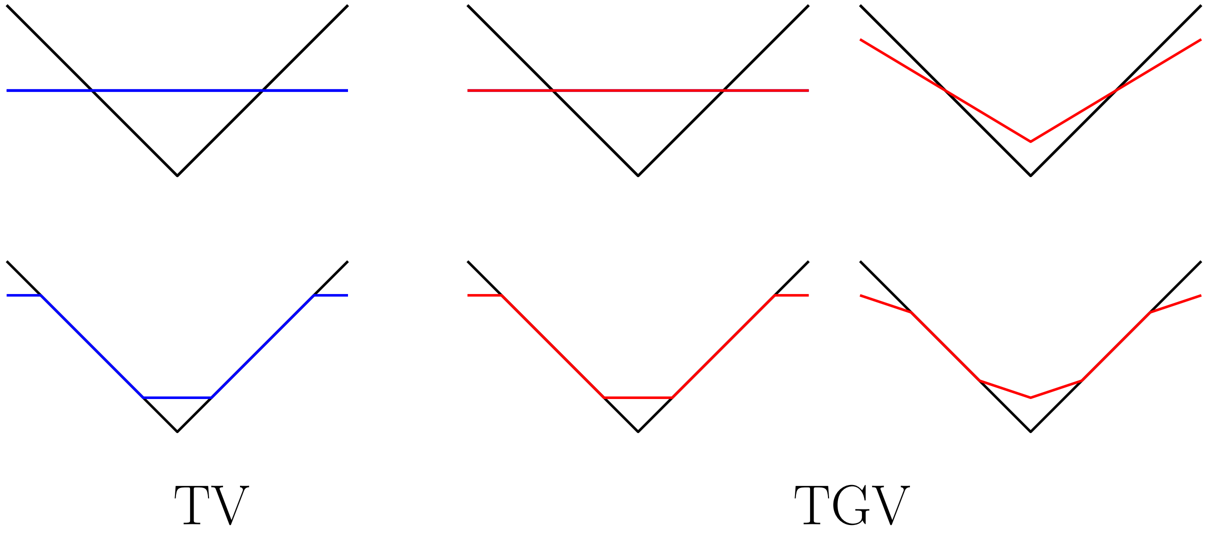

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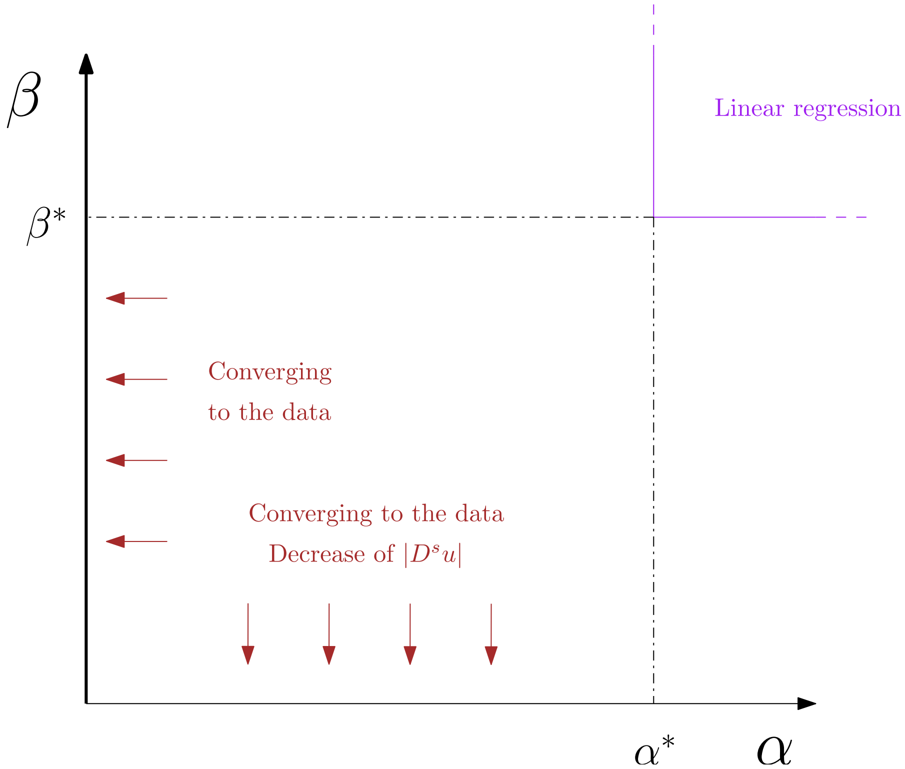
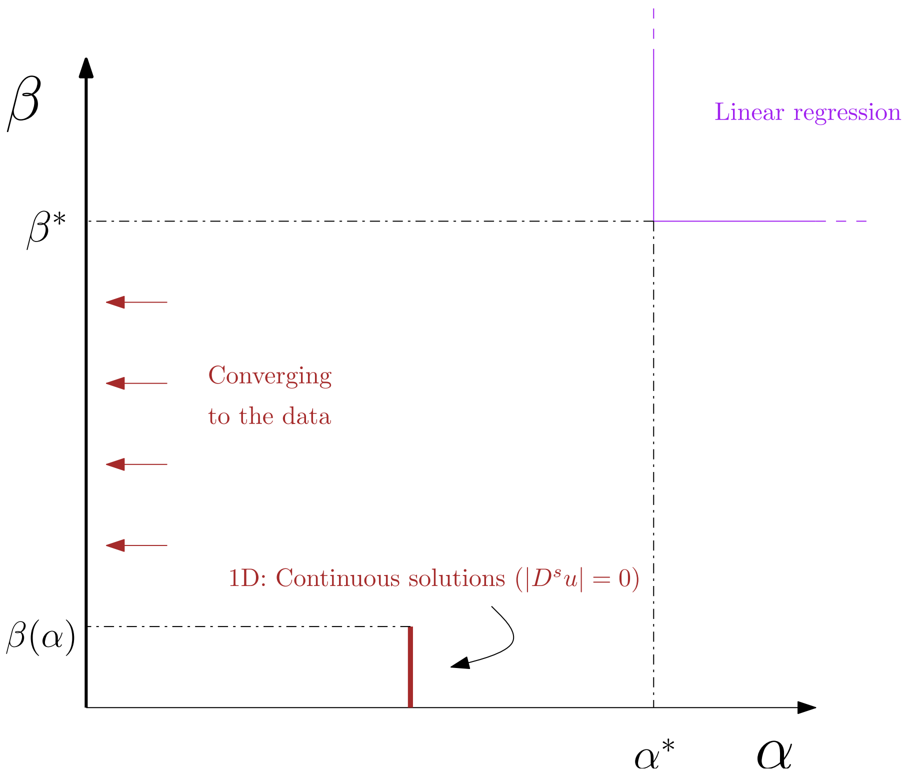
Asymptotic behaviour of TGV



Asymptotic behaviour



The influence of regularisation parameters

$$\min_{\substack{u\in \mathrm{BV}(\Omega)\\ w\in \mathrm{BD}(\Omega)}} \frac{1}{2}\|u-f\|^2_{L^2(\Omega)} + \alpha\int_\Omega d|Du-w| + \beta\int_\Omega d|\mathcal{E}w|, \quad \alpha>0, \beta>0$$

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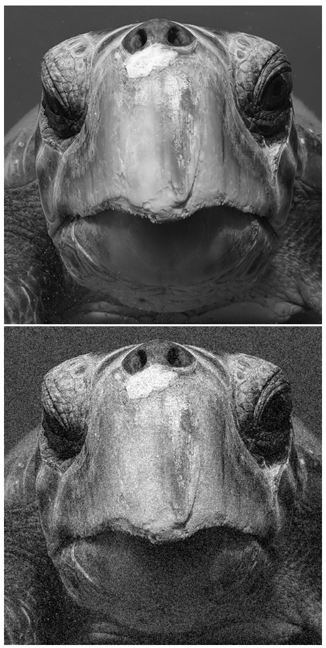

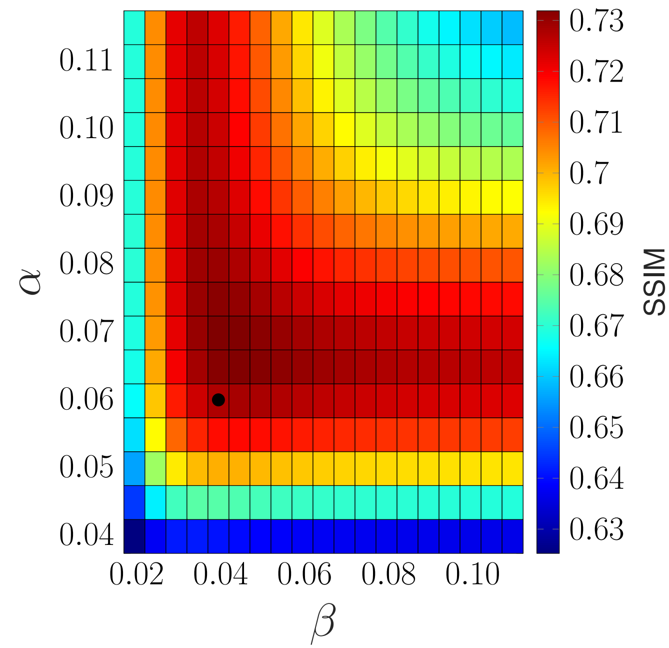

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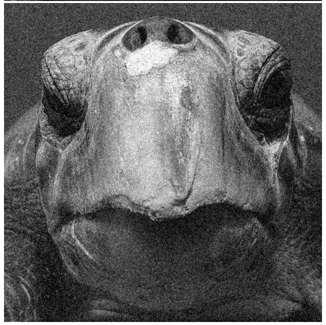

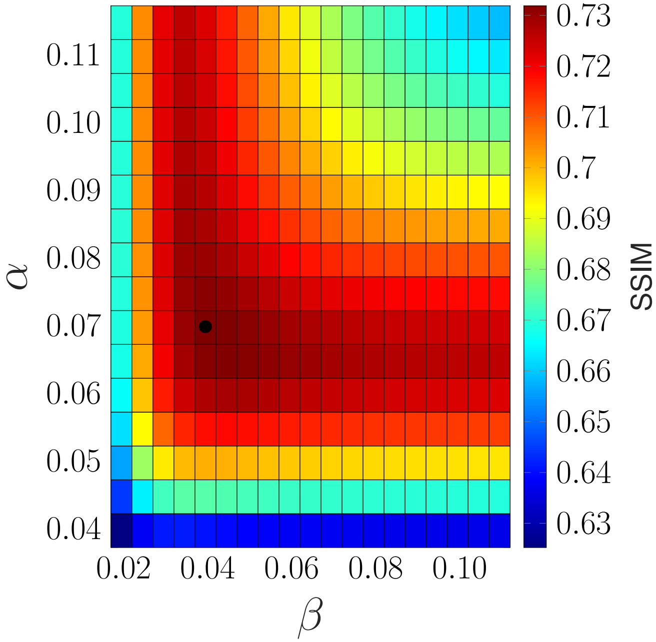

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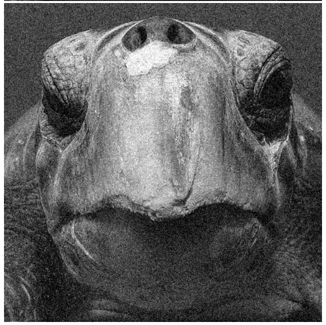

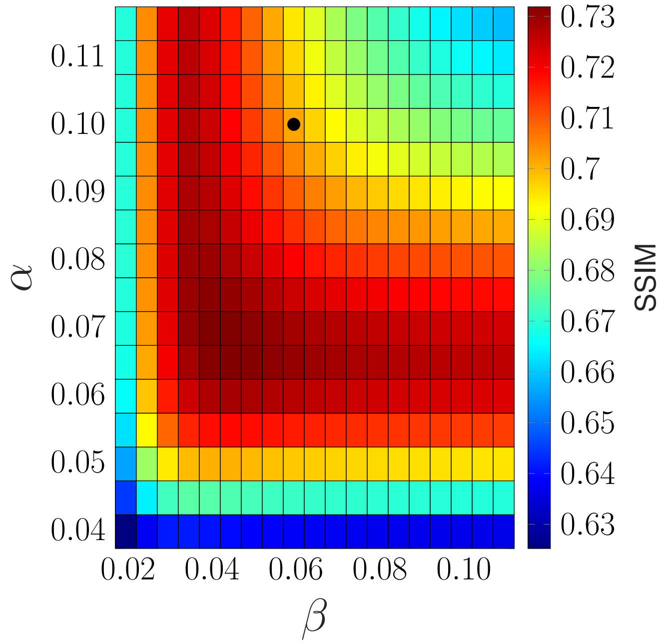

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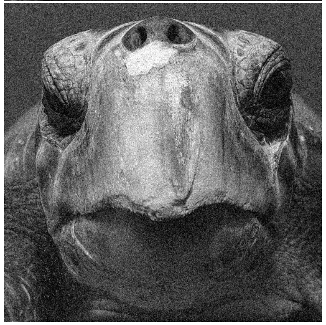

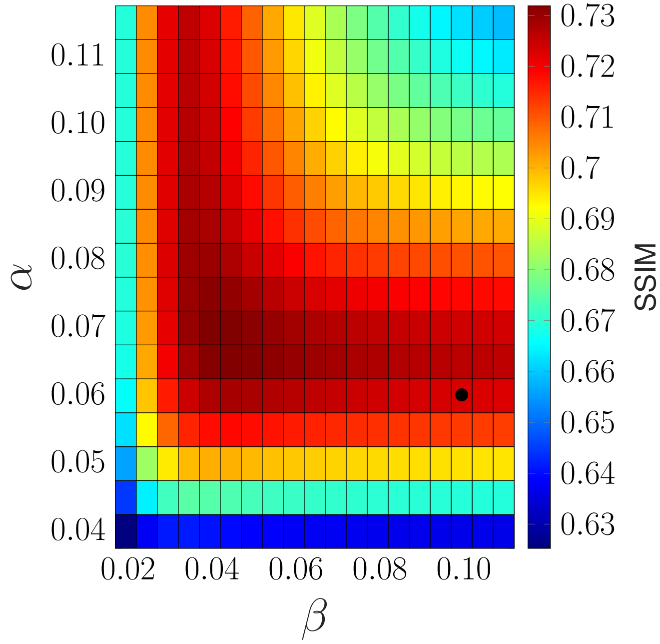

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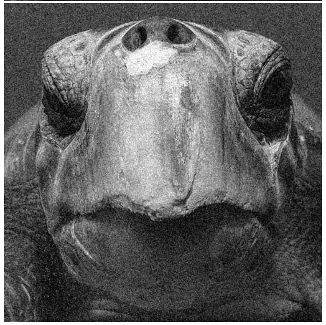
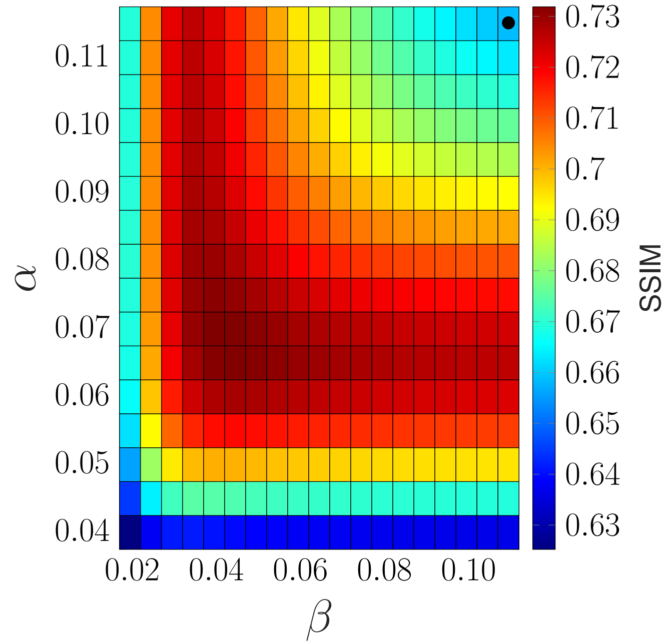




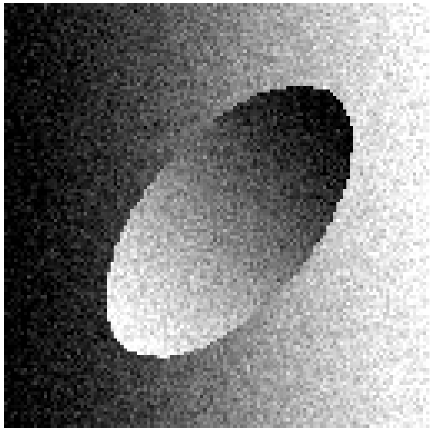
When TGV becomes TV-like?

Theorem

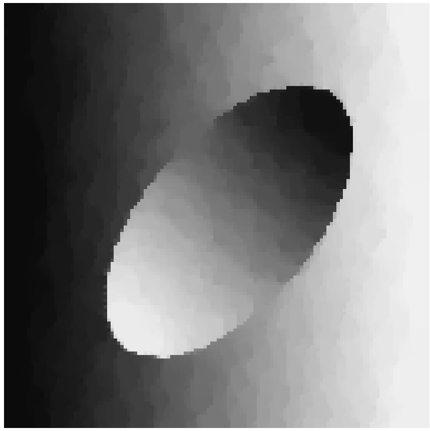
There exists a constant C (depending only on Ω) such that if $\boldsymbol{\beta/\alpha > C}$ then

$$\mathrm{TGV}(u) = \alpha \int_{\Omega} d|Du - g|, \quad \textit{for all } u \in \mathrm{BV}(\Omega),$$

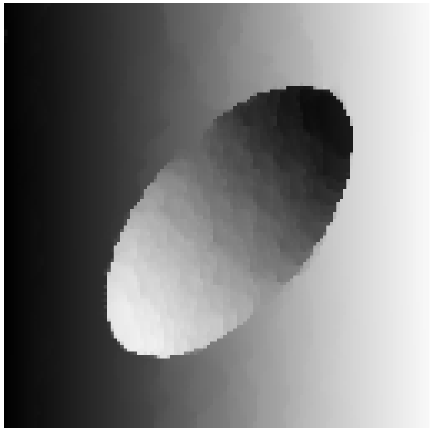
where g is an affine function (also depending on u).



Noisy



TV, $\alpha = 0.1$



TGV, $\alpha = 0.1, \beta = 100$

References: Papafitsoros, Valkonen (2015)], [Papafitsoros, Bredies (2015)

Minimisation algorithms for TV/TGV minimization

A variety of *primal*, *dual* or *primal-dual* approaches:

- Chambolle projection's algorithm
- Augmented Langrangian methods
- Alternating direction method of multipliers (ADMM)
- Primal-dual hybrid methods (Chambolle-Pock)
- Bregman iteration methods
- Smoothing approaches, e.g. Huber TV
- Newton methods for the dual problem or prima-dual optimality conditions
- . . .

Brief look on spatial dependent parameters

$$\mathrm{TV}_\alpha(u) \simeq \int_\Omega \alpha(x)|\nabla u|\,dx, \quad \mathrm{TGV}_{\beta,\alpha}(u) \simeq \min_w \int_\Omega \alpha(x)|\nabla u - w|\,dx + \int_\Omega \beta(x)|Ew|\,dx$$

scalar TV, PSNR=27.57, SSIM=0.7597
spatial TV, PSNR=27.55, SSIM=0.7750

scalar TGV, PSNR=27.71, SSIM=0.8024
spatial TGV, PSNR=**28.01**, SSIM=**0.8037**

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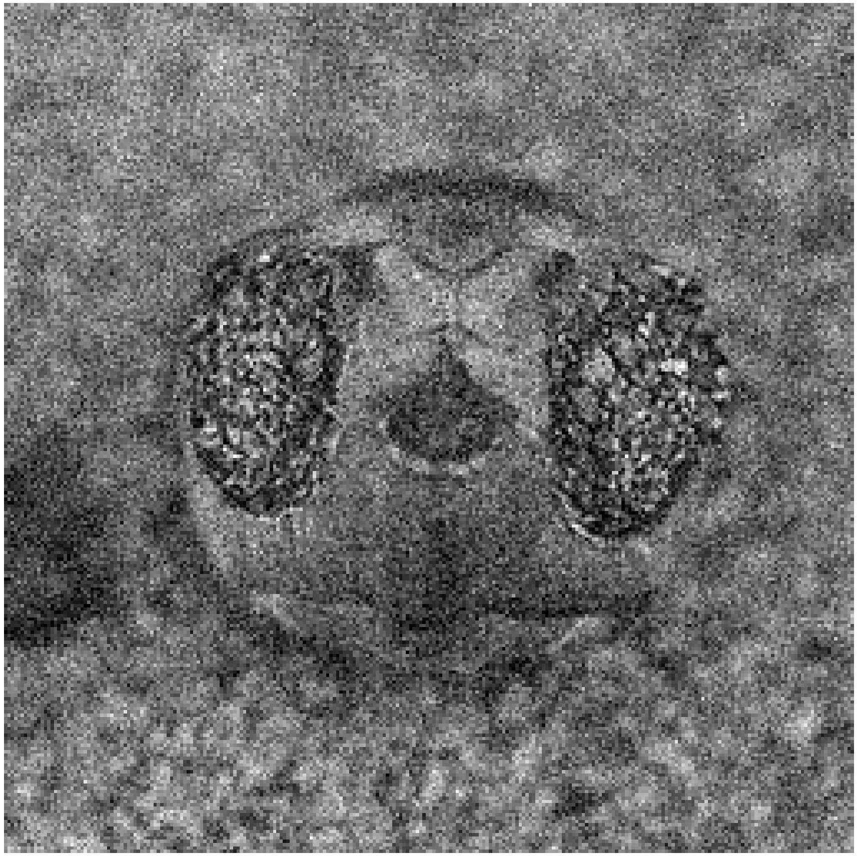

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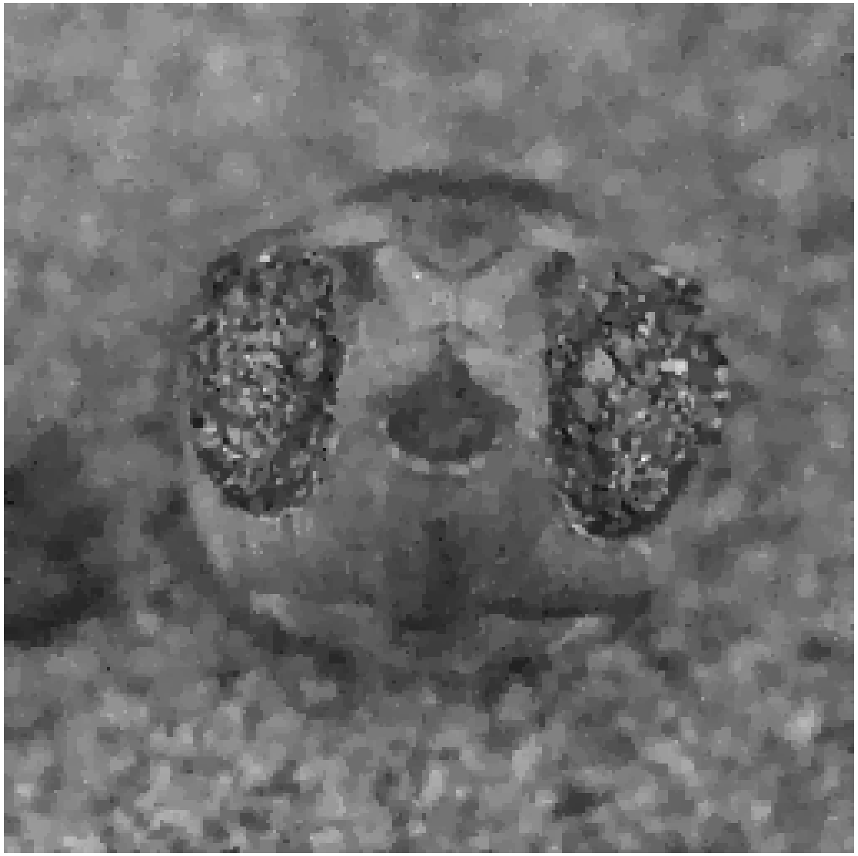

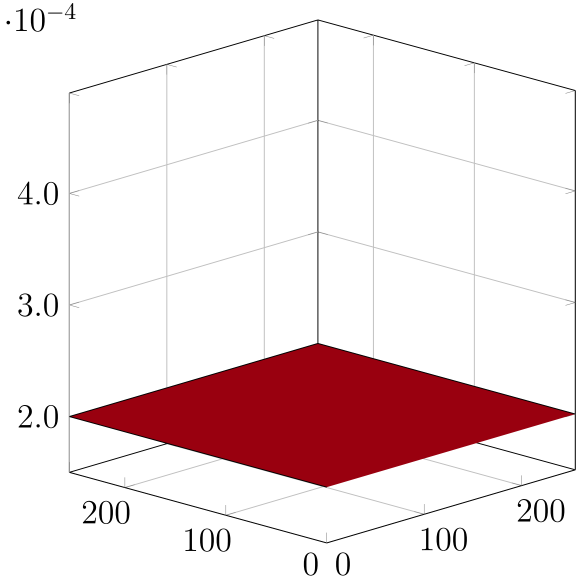

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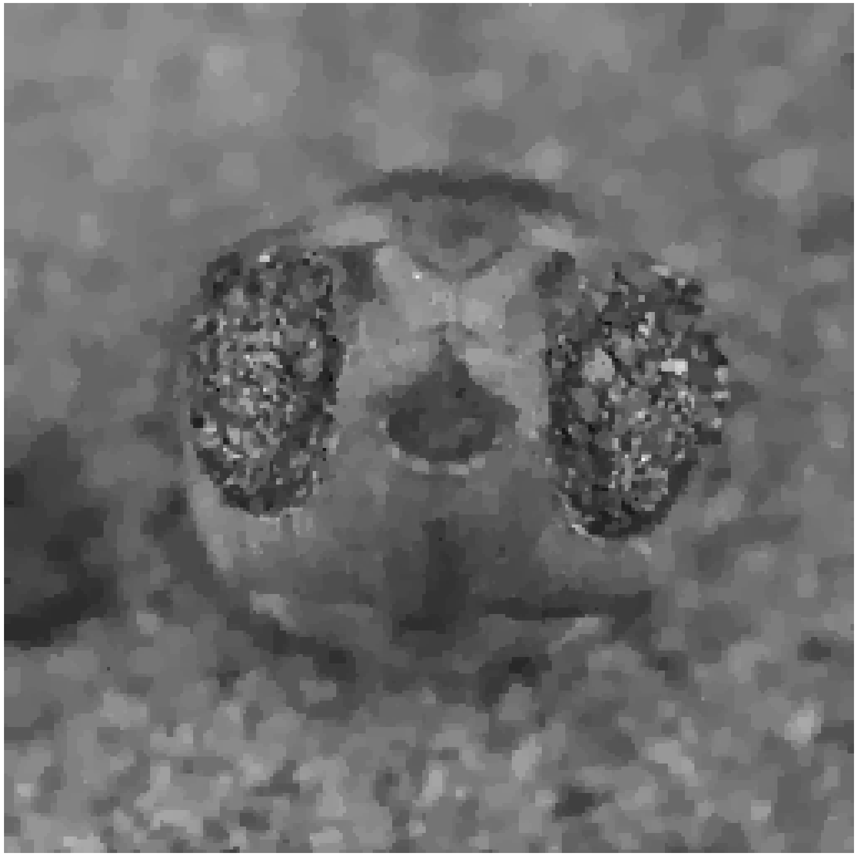
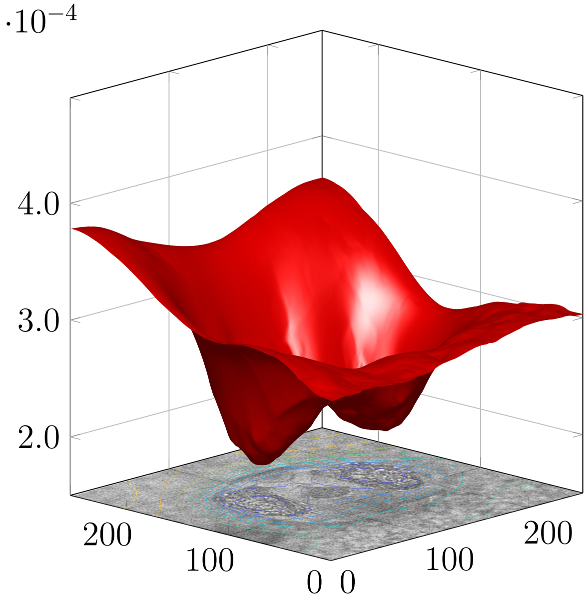
$$\mathrm{TV}_{\alpha}(u) \simeq \int_{\Omega} \alpha(x)|\nabla u|\,dx, \quad \mathrm{TGV}_{\beta,\alpha}(u) \simeq \min_{w} \int_{\Omega} \alpha(x)|\nabla u - w|\,dx + \int_{\Omega} \beta(x)|Ew|\,dx$$



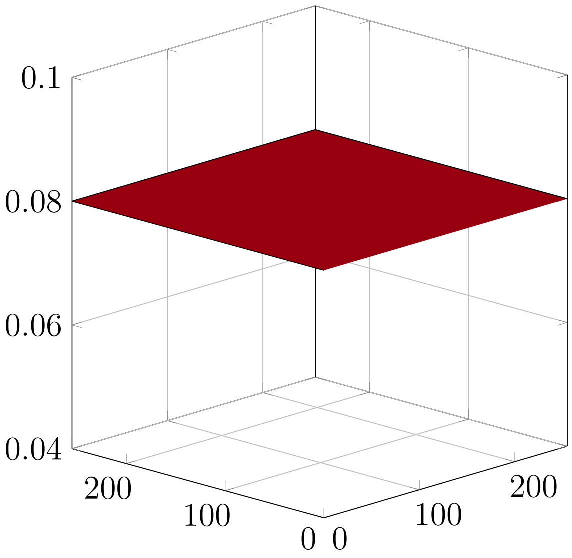


scalar TV, PSNR=27.57, SSIM=0.7597
spatial TV, PSNR=27.55, SSIM=0.7750

scalar TGV, PSNR=27.71, SSIM=0.8024
spatial TGV, PSNR=**28.01**, SSIM=**0.8037**

Brief look on spatial dependent parameters

$$\mathrm{TV}_\alpha(u) \simeq \int_\Omega \alpha(x)|\nabla u|\,dx, \quad \mathrm{TGV}_{\beta,\alpha}(u) \simeq \min_w \int_\Omega \alpha(x)|\nabla u - w|\,dx + \int_\Omega \beta(x)|Ew|\,dx$$

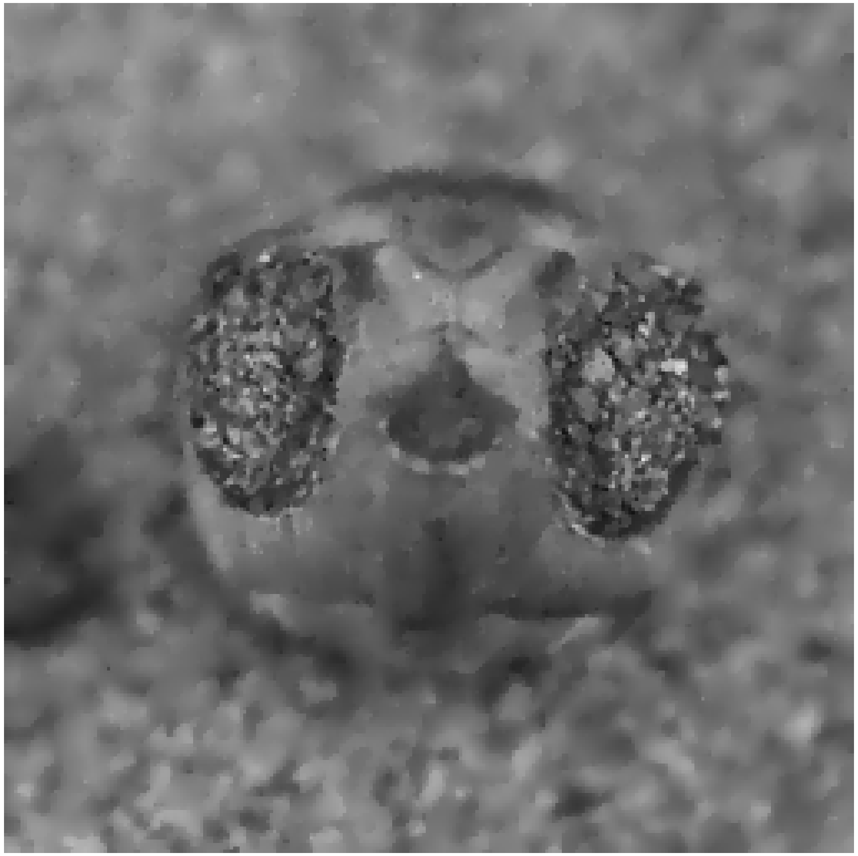


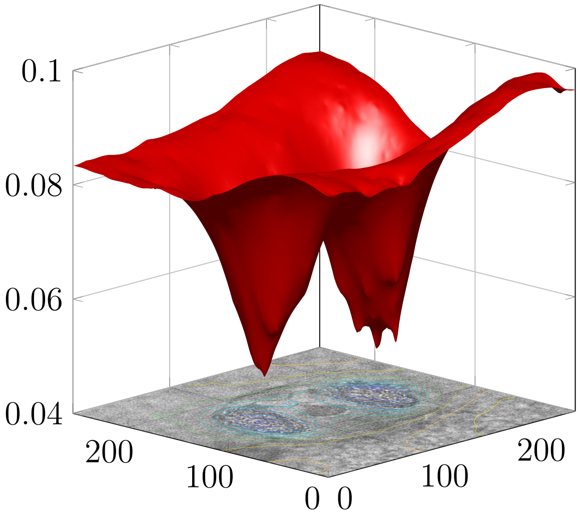
scalar TV, PSNR=27.57, SSIM=0.7597
spatial TV, PSNR=27.55, SSIM=0.7750

scalar TGV, PSNR=27.71, SSIM=0.8024
spatial TGV, PSNR=**28.01**, SSIM=**0.8037**

Brief look on spatial dependent parameters

$$\mathrm{TV}_\alpha(u) \simeq \int_\Omega \alpha(x)|\nabla u|\,dx, \quad \mathrm{TGV}_{\beta,\alpha}(u) \simeq \min_w \int_\Omega \alpha(x)|\nabla u - w|\,dx + \int_\Omega \beta(x)|Ew|\,dx$$

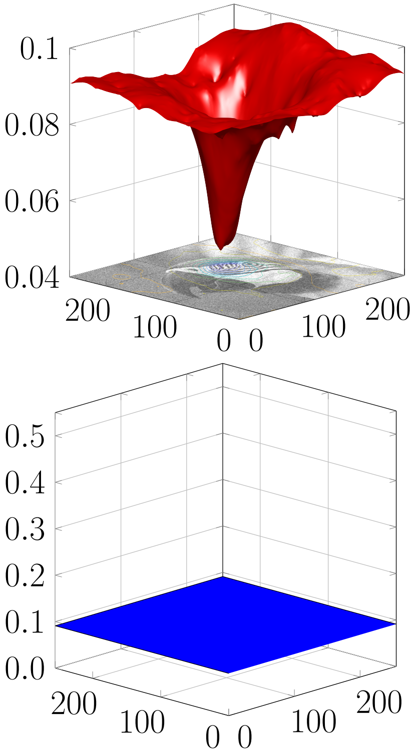




scalar TV, PSNR=27.57, SSIM=0.7597
spatial TV, PSNR=27.55, SSIM=0.7750

scalar TGV, PSNR=27.71, SSIM=0.8024
spatial TGV, PSNR=**28.01**, SSIM=**0.8037**

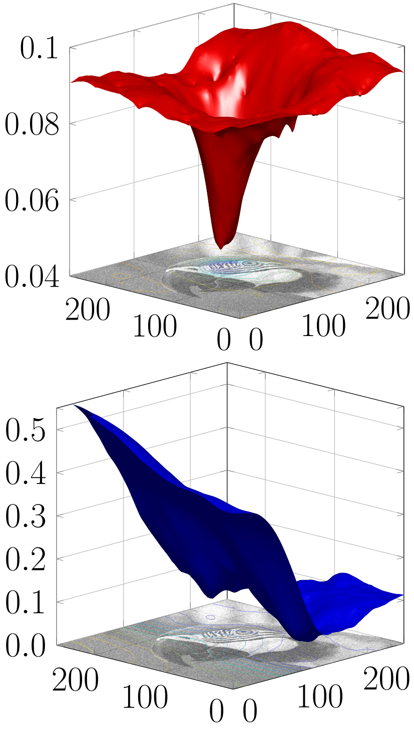
Brief look on spatial dependent parameters



TGV: spatial α, scalar β, PSNR=29.47, SSIM=0.8628

TGV: spatial α, spatial β, PSNR=**29.50**, SSIM=**0.8640**

Brief look on spatial dependent parameters



TGV: spatial α, scalar β, PSNR=29.47, SSIM=0.8628

TGV: spatial α, spatial β, PSNR=**29.50**, SSIM=**0.8640**