Optimization - 2

Agenda :-

1. Differentiation
2. Examples.
3. Check Differentiability Of a function
4. * Commonly used Derivatives
5. Rules of Differentiability
6. Using Derivatives for optimization
   → maxima, minima

Recap :-

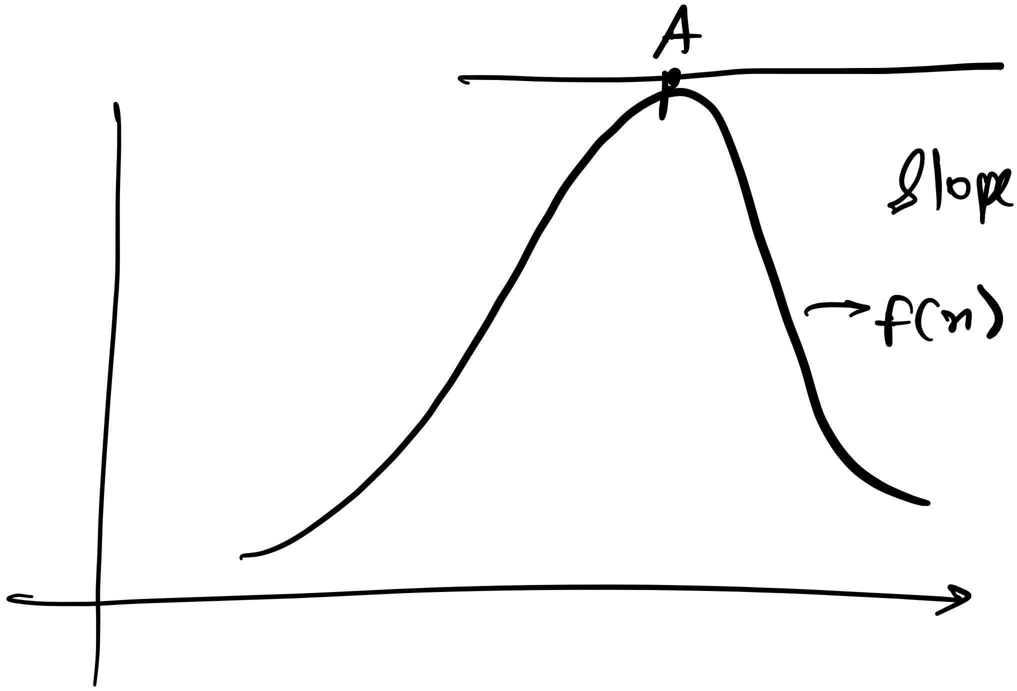
$$\underset{\vec{w}, w_0}{\operatorname{argmax}} \boxed{G(D, \vec{w}, w_0)}$$

eg :-

$$\underset{x}{\operatorname{argmax}} \left[- \underline{(x-2)^2} \right]$$

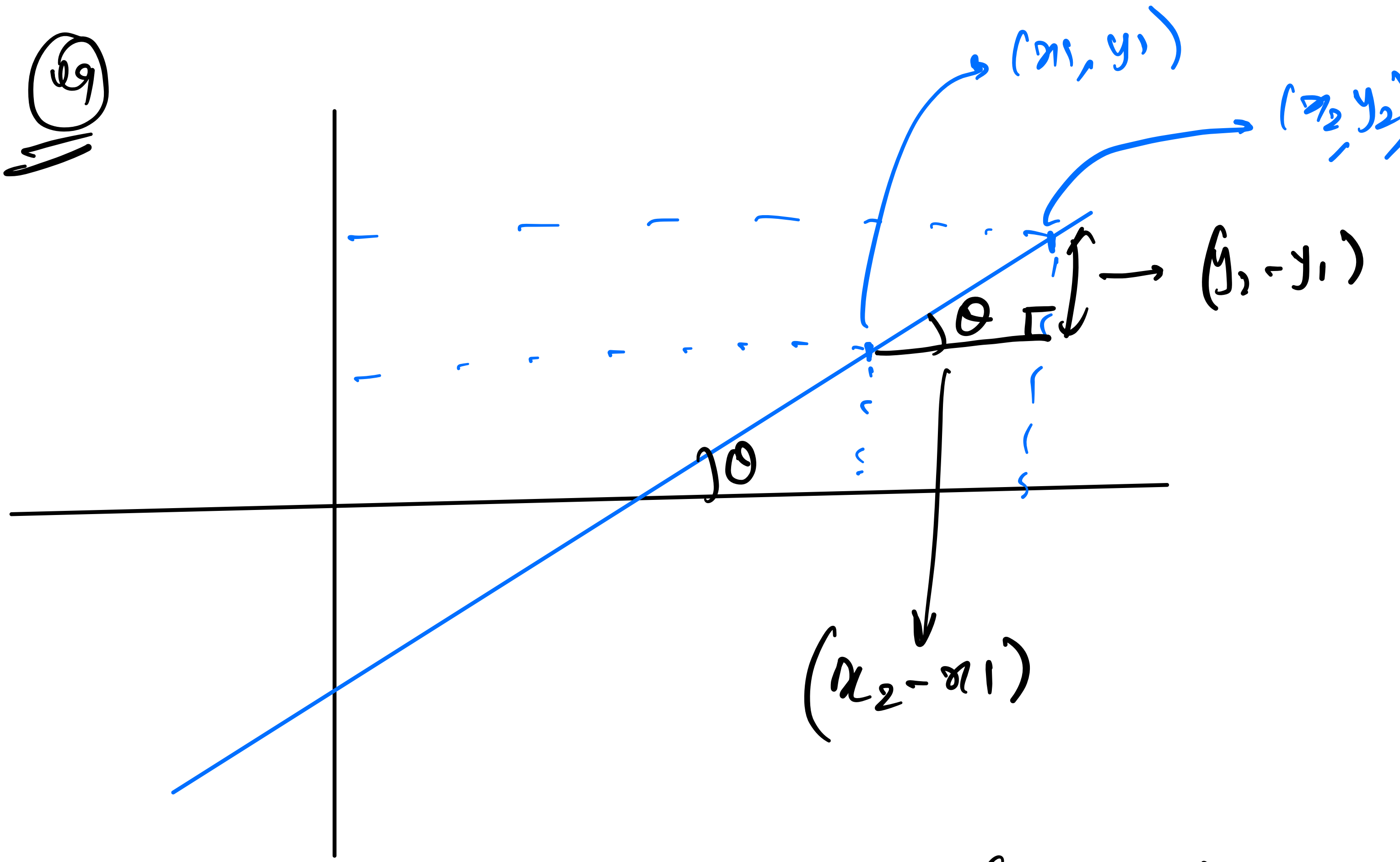
$$\Downarrow$$

$x=2$, the above func is max.



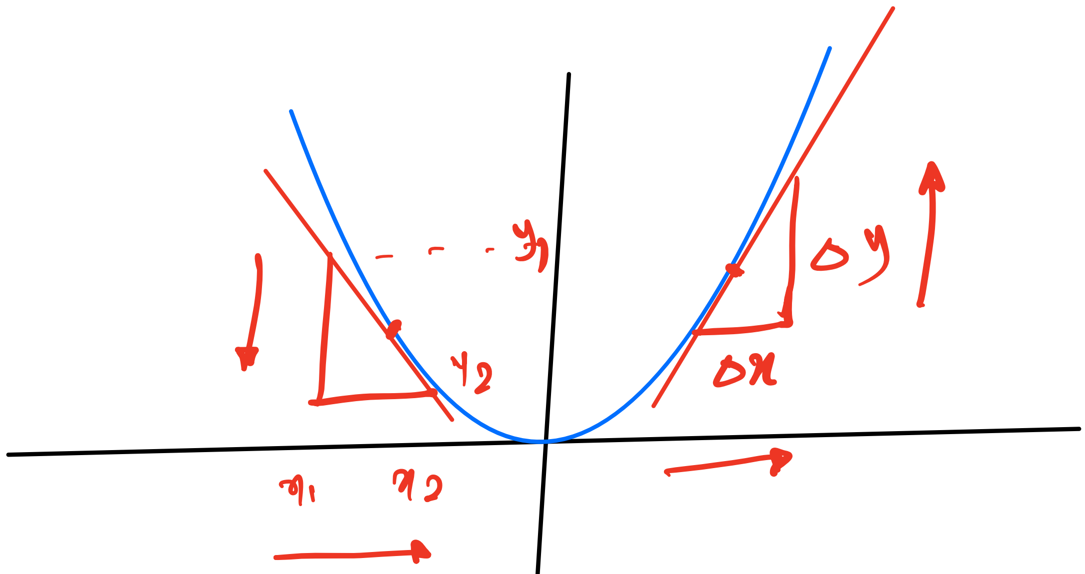
$$slope = \frac{df(x)}{dx} = 0$$

eg



$$\text{Slope} = \tan\theta = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

$$\boxed{\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{df(x)}{dx}}$$



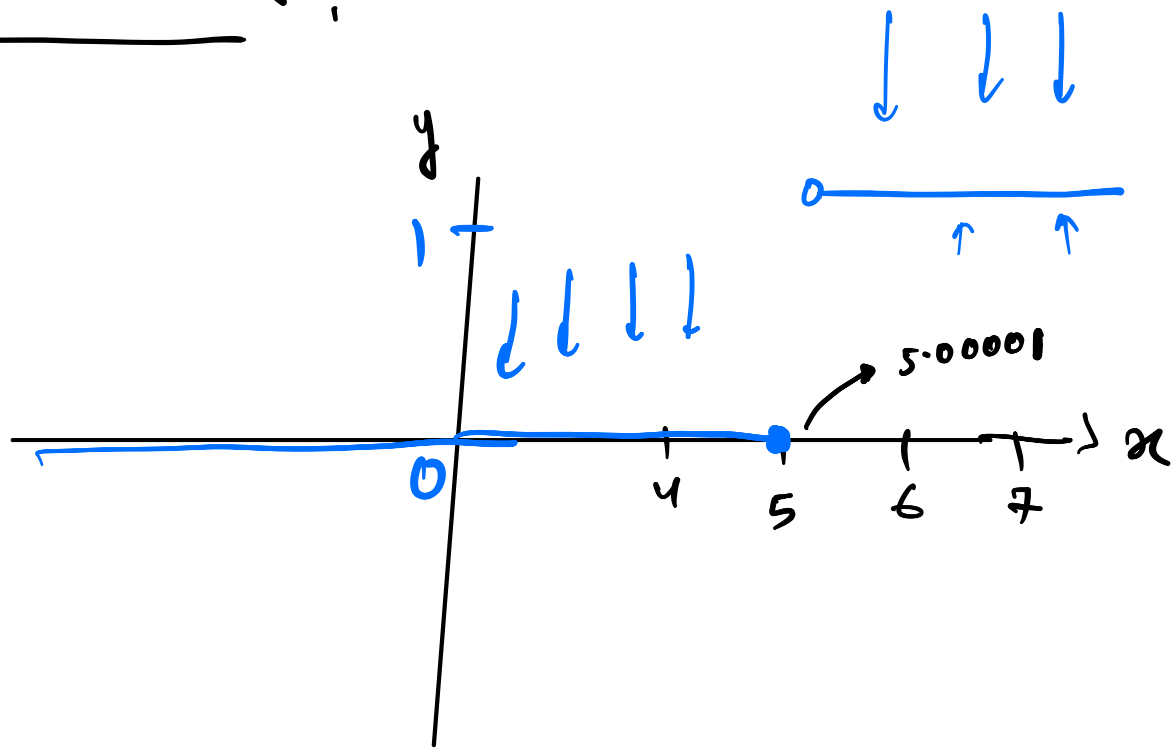
$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$\Rightarrow$ $-ve$

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} \Rightarrow +ve$$

* **Differentiability :-**

eg)



$$f(x) = y = \begin{cases} 1, & x > 5 \\ 0, & x <= 5 \end{cases}$$

RHL $\lim_{x \to 5^+} f(x) \longrightarrow 1$

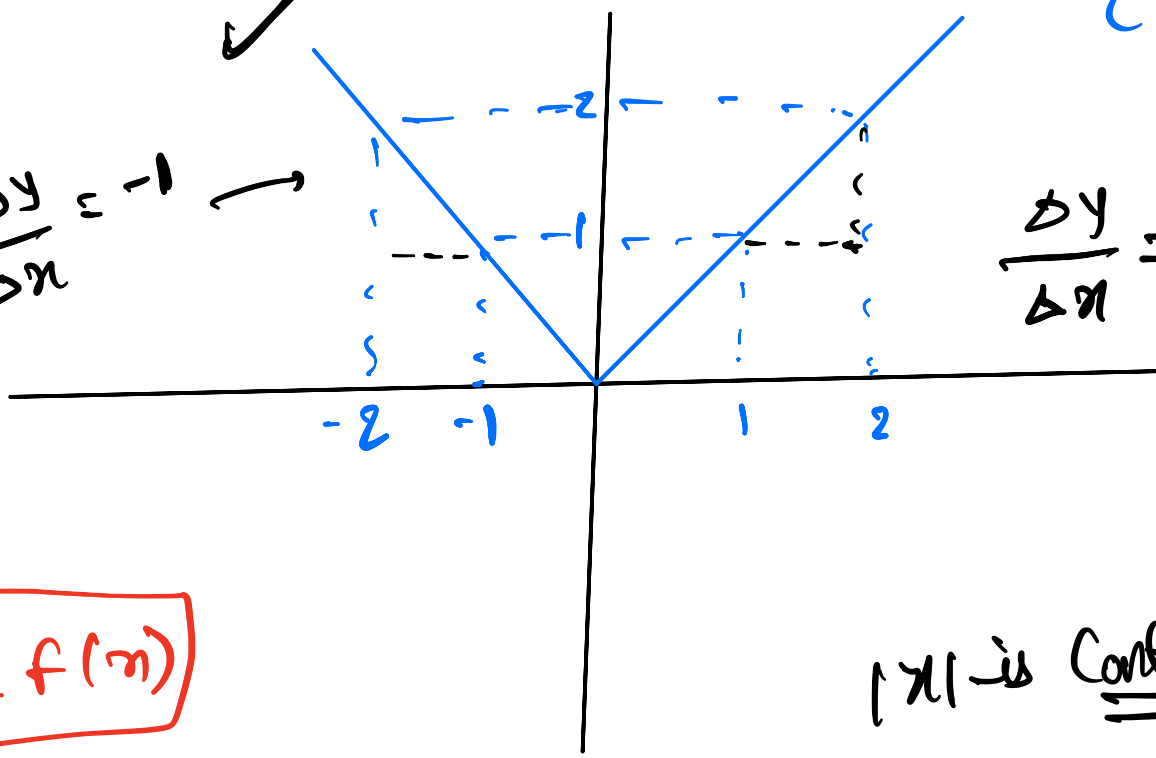
LHL $\lim_{x \to 5^-} f(x) \longrightarrow 0$

eg2 :-

$f(x) = |x|$

$$y = |x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \\ 0, & x = 0 \end{cases}$$

$\frac{\Delta y}{\Delta x} = -1$



$\frac{\Delta y}{\Delta x} = \underline{\underline{1}}$

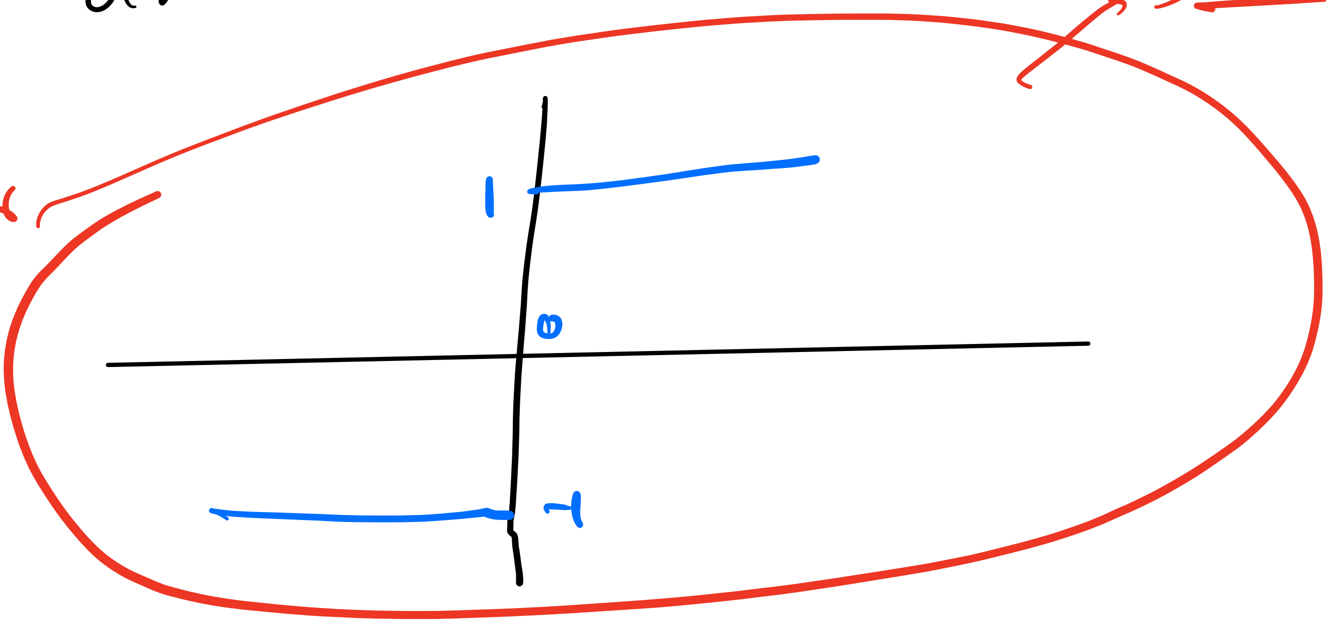
$\boxed{y = f(x)}$

$|x|$ is $\underline{\underline{\text{Continuous}}}$

$$f'(x) = \frac{df(x)}{dx} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

$\underline{\text{Discontinuous}}$

Plot for $f'(x)$



* 2 Conditions for Differentiability

(1) $f(x)$ is Continuous.

and (2) $f'(x)$ is also Continuous.

* $\boxed{|x| \text{ is not Differentiable}}$

eg

$f(x) = x^2$

$f'(x) = 2x$

$y = f(x) = x^2$

$$f'(x) = \frac{df(x)}{dx} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$x_2 \rightarrow x + \Delta x$

$\Delta x \rightarrow 0.0001$

$$= \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$f(x) = x^2$

$$= \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$

$(a+b)^2 = a^2 + 2ab + b^2$

$$= \frac{\cancel{x^2} + 2x \cdot \Delta x + (\Delta x)^2 - \cancel{x^2}}{\Delta x}$$

$$= \frac{2x \cdot \cancel{\Delta x} + (\Delta x)^2}{\cancel{\Delta x}}$$

$$\frac{df(x)}{dx} = 2x + \cancel{\Delta x}^{\to 0}$$

$\Delta x \to 0$

$$\boxed{\frac{df(x)}{dx} = \frac{d(x^2)}{dx} = 2x}$$

* Commonly used Derivatives

(1) $\frac{d}{dx}(\text{Const}) = 0$

(2) $\frac{d}{dx}(x^n) = \boxed{n * x^{n-1}}$

$x^2 \Rightarrow 2 * x^{2-1} = 2x$

$x^3 \Rightarrow 3 * x^{3-1} = 3 * x^2$

(3) $\frac{d}{dx}(\sin x) = \cos x$

(4) $\frac{d}{dx}(\cos x) = -\sin x$

(5) $\frac{d}{dx}(\tan x) = \sec^2(x)$

(6) $\boxed{\frac{d}{dx}(\log x) = \frac{1}{x}}$

(7) $\frac{d}{dx}(e^x) = e^x$

(*) **Rules** :-

(i) Sum Rule / Linearity Rule

$$\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

$$= f'(x) + g'(x)$$

eg :-

$f(x) = x^2 + 7x + 12$

$f'(x) = ?$

$$f'(x) = \frac{d}{dx}(x^2) + \frac{d}{dx}(7x) + \frac{d}{dx}(12)$$

$= 2x + 7 + 0$

$= \boxed{2x+7}$ ✓

(2) Product Rule :-

$$\frac{d}{dx}\left(f(x) * g(x)\right)$$

$$= f(x) * \frac{d}{dx}\left(g(x)\right) + g(x) * \frac{d}{dx}\left(f(x)\right)$$

$$= f(x) * g'(x) + g(x) * f'(x)$$

eg:- $H(x) = \overset{f}{(x+3)} * \overset{g}{(x+4)}$

$H'(x) = ?$

$H'(x) = f * g' + g * f'$

$= (x+3) \times \frac{d}{dx}(x+4)$

$+$

$(x+4) * \frac{d}{dx}(x+3)$

$= (x+3) + (x+4)$

$= 2x+7$

App2 :-

$$H(x) = (x+3) * (x+4)$$

$$= x(x+4) + 3(x+4)$$

$$= x^2 + 4x + 3x + 12$$

$$H(x) = x^2 + 7x + 12$$

$$H'(x) = \underline{\underline{2x+7}}$$

eg3 :- $y = \overset{f}{x} * \overset{g}{\log(x)}$

find $y' = ?$

$$y' = x * \frac{d}{dx}(\log(x)) + \log(x) * \frac{d}{dx}(x)$$

$$= \cancel{x} * \frac{1}{\cancel{x}} + \log x * 1$$

$$= \boxed{1 + \log(x)}$$

Rule ③ Quotient Rule :-

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g * f' - f * g'}{(g)^2}$$

eg:- $y = \frac{\log x}{x}$

$y' = ?$

$$\Rightarrow y' = \frac{x * \frac{d}{dx}(\log x) - \log x * \frac{d}{dx}(x)}{(x)^2}$$

$$= \frac{\cancel{x} * \frac{1}{\cancel{x}} - \log x * 1}{x^2}$$

$$\boxed{y' = \frac{(1 - \log x)}{x^2}}$$

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* **Chain Rule :-**

$$\frac{d}{dx}\left(f(g(x))\right) = f'(g(x)) * g'(x)$$

eg :- $y = e^{(5x^2+2)} \rightarrow e^{x}$

$$y' = e^{(5x^2+2)} * \frac{d}{dx}(5x^2+2)$$

$$\boxed{y' = e^{(5x^2+2)} * (10x)}$$

$\underline{\underline{eg}}$:-

$$\boxed{f(x) = \frac{1}{1+e^{-x}}} \quad \text{(sigmoid)}$$

$$f'(x) = ?$$

$$f'(x) = \frac{(1+e^{-x}) * \frac{d}{dx}(1) - 1 * \frac{d}{dx}(1+e^{-x})}{(1+e^{-x})^2} =$$

$$\frac{d}{dx}(e^{-x}) = -e^{-x}$$

$$= \frac{-\left(0 + e^{-x} * (-1)\right)}{(1+e^{-x})^2}$$

$$= \frac{e^{-x}}{(1+e^{-x})^2}$$

$$= \frac{1}{1+e^{-x}} * \boxed{\frac{e^{-x}}{(1+e^{-x})}}$$

$$= f(x) * \left(1 - \frac{1}{1+e^{-x}} \right)$$

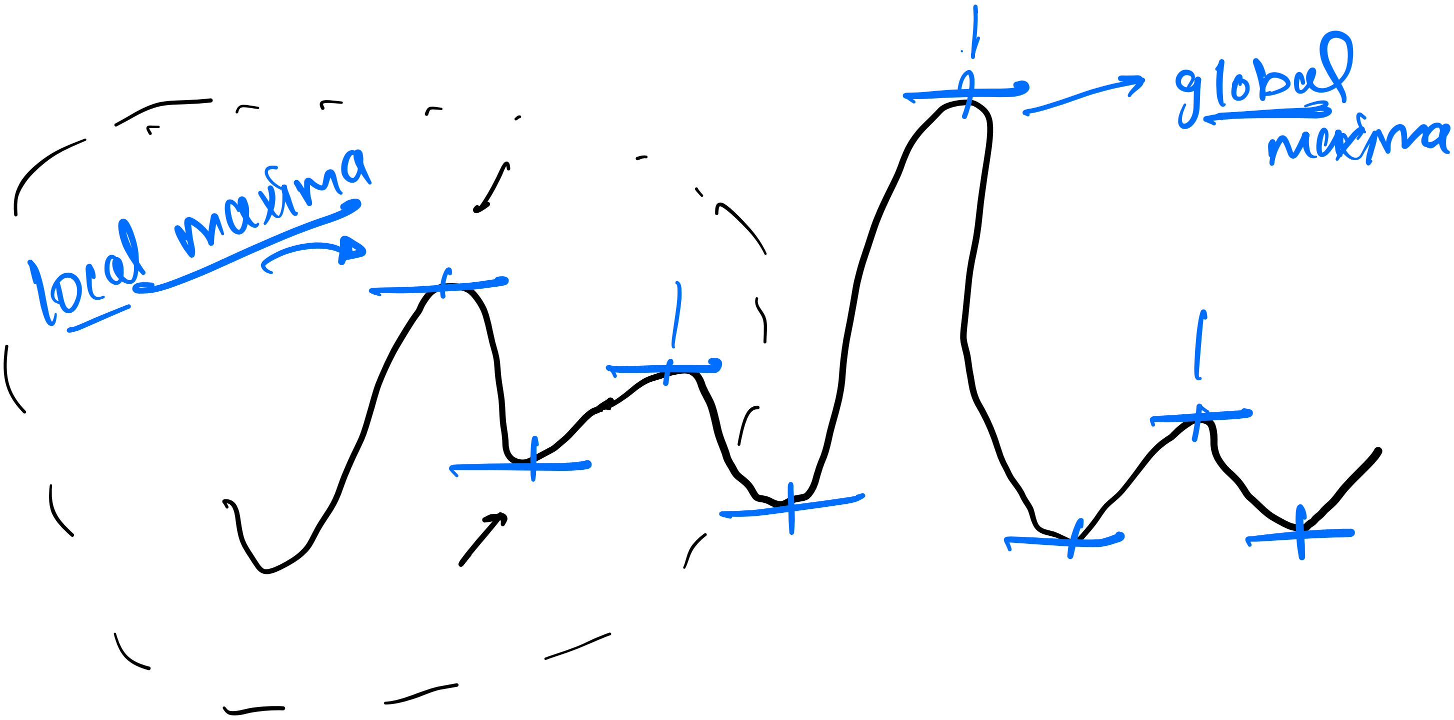
$$\boxed{\underline{f'(x)} = \underline{f(x)} * (\underline{1 - f(x)})}$$

$$1 - \frac{1}{1+e^{-x}}$$

$$\frac{\cancel{1} + e^{-x} - \cancel{1}}{1+e^{-x}} = \frac{e^{-x}}{1+e^{-x}}$$

⊛ Using Derivatives for optimization

maxima, minima.



$$y = f(x)$$

$$\boxed{f'(x) = 0}$$ → Critical points

$f''(x) > 0$ → minima

$f''(x) < 0$ → maxima

eg :-

$$f(x) = x^2$$

$$f'(x) = 2x$$

1st derivative

$$f''(x) = \frac{d}{dx}\left(f'(x)\right) = 2$$

2nd derivative

eg :- $f(x) = x^2 - x + 2$

$f'(x) = 0$

$\Rightarrow f(x) = x^2 - x + 2$

$f'(x) = 2x - 1$

$f'(x) = 0$

$2x - 1 = 0$

$\boxed{x = 1/2} \longrightarrow$ CP

$f''(x) = \frac{d}{dx}(2x - 1)$

$\boxed{f''(x) = 2}$

$f''(1/2) = 2 > 0 \longrightarrow$ Minima exists at $x = 1/2$

② Minimum value of the function.

$f(x) = x^2 - x + 2$

$f(1/2) = (1/2)^2 - 1/2 + 2$

$= 1/4 - 1/2 + 2$

$= \boxed{7/4}$

eq2:

$$P(x) = 41 - 72x - 18x^2$$

Max P ?

$\rightarrow$ $P'(x) = 0$

$$-72 - 36x = 0$$

$$\boxed{x = -2} \rightarrow \text{CP}$$

$$P''(x) = \frac{d}{dx}(-72 - 36x)$$

$$P''(x) = -36$$

$$P''(-2) = -36 \rightarrow < 0$$

✓

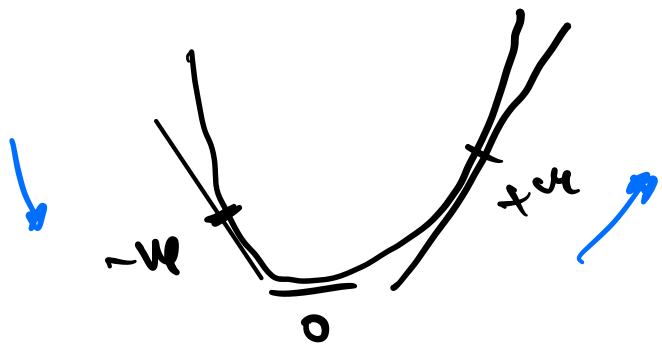
(maxima exists at $x = -2$)

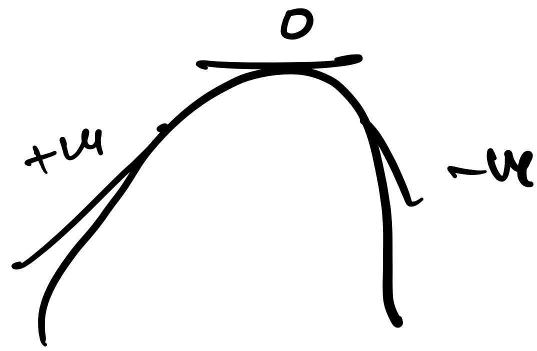
$$P''(-2) = 41 - 72(x) - 18x^2$$

$$= 41 - 72(-2) - 18(-2)^2$$

$$= \boxed{113} \rightarrow \underline{\underline{\text{Max}}}$$

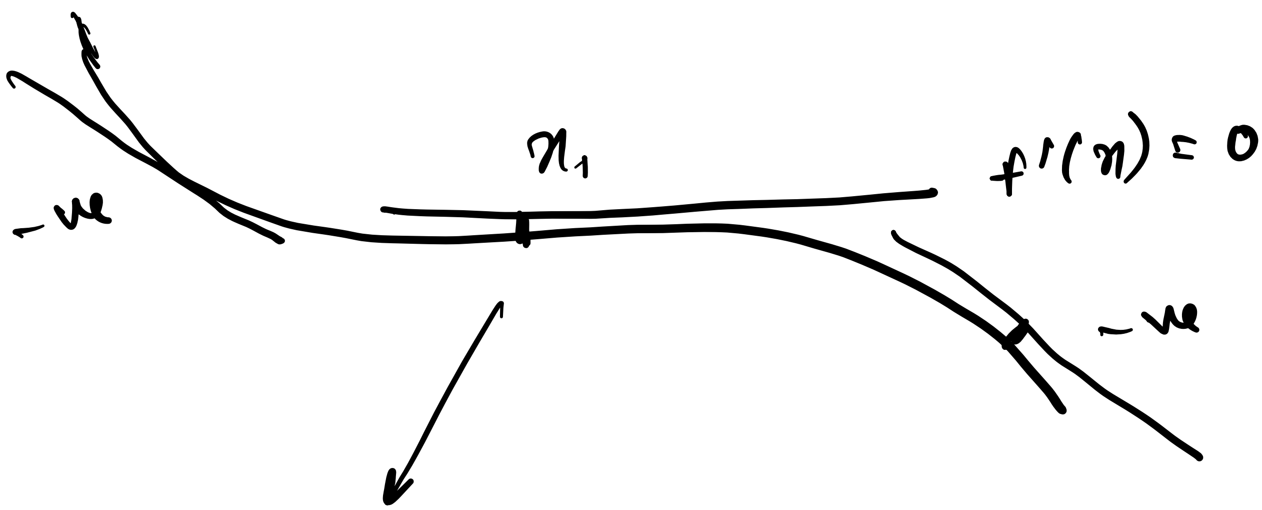
* Saddle Point





minima

(-ve → +ve)



Saddle point

(Q)

$$h(x) = f(g(x))$$

H.W

given: $g(-1) = 2$

$g'(-1) = 3$

$f'(2) = -4$

$$\boxed{h'(-1) = ?}$$

f & g are differentiation.

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