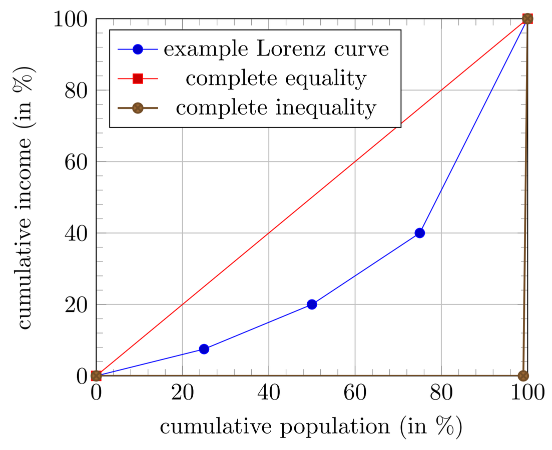
Performance of Empirical Best Predictor in Informative Samples - A Monte Carlo Simulation

Felix Skarke, Enno Tammena, Christian Koopmann
Freie Universität Berlin, Humboldt Universität zu Berlin

Motivation

- Use of direct estimators in domains of interest with insufficient sample size can lead to unreliable results
  $\Rightarrow$ Using small area methods like EBP (Empirical Best Predictor) approach can be preferable
- In model-based inference normally the sampling design is assumed to be uninformative (e.g. simple random sampling):
  $P(s|y) = P(s), \ \forall y \in \mathbb{R}^N, \forall s$
- When complex designs are used for sampling (practical reasons, special interest in small subpopulation), ignoring weights can lead to biased estimators
  $\Rightarrow$ Use of direct estimators like weighted Gini
- Problem: sampling weights cannot be used directly in the EBP approach because of non-linearity of indicator
- The aim of this study is to evaluate the performance of EBP relative to direct estimation under a complex sampling design

The Gini: a measure for inequality



- The Gini coefficient is used to measure inequality of distribution (e.g. income, wealth) in a society
- It is defined as the area between the Lorenz curve and the 45° line (=A) in relation to the area beneath the 45° line (=A+B), where B is the area under the Lorenz curve

Therefore it can be expressed as:

$G = A/(A+B) = 2A = 1 - 2B$

The Gini: unweighted and weighted

The Gini coefficient can be expressed without a direct reference to the Lorenz curve:

unweighted version

$$\hat{G} = \frac{2\sum_{j=1}^{n} jy_j}{n\sum_{j=1}^{n} y_j} - \frac{n+1}{n}$$

weighted version

$$\hat{G} = 100\left[\frac{2\sum_{j=1}^{n}\left(w_j y_j \sum_{l=1}^{j} w_l\right) - \sum_{j=1}^{n} w_j^2 y_j}{\left(\sum_{j=1}^{n} w_j\right)\sum_{j=1}^{n}(w_j y_j)} - 1\right]$$

⚠ Using weights in direct estimators can be important, if a complex sampling design is used.

Empirical Best Predictor

Random Effects model:

$$y_{ij} = x_{ij}\prime\beta + u_i + e_{ij}, \quad j = 1, \dots, n_i, \quad i = 1, \dots, D$$

,where $u_i \sim iid\mathcal{N}(0, \sigma_u^2)$ and $e_{ij} \sim iid\mathcal{N}(0, \sigma_e^2)$

Estimation of model:

1. estimate $\hat{\beta}, \hat{\sigma}_u^2, \hat{\sigma}_e^2, \hat{u}_i, \hat{\gamma}_i = \frac{\hat{\sigma}_u^2}{\hat{\sigma}_u^2 + \frac{\hat{\sigma}_e^2}{n_i}}$ from sample
2. generate $e_{ij}^* \sim \mathcal{N}(0, \hat{\sigma}_e^2)$ and $u_i^* \sim \mathcal{N}(0, \hat{\sigma}_u^2(1 - \hat{\gamma}_i))$ for L pseudo-populations:
   $y_{ij}^{*(l)} = x_{ij}\prime\hat{\beta} + \hat{u}_i + u_i^* + e_{ij}^*$
   $\Rightarrow$ obtain an indicator of interest in each domain for every pseudo-population
3. calculate $\hat{\theta}_i^{EBP} = \frac{1}{L}\sum_{l=1}^{L}\hat{\theta}_i^{(l)}$ for each domain

MSE estimation

Since analytical approximations of the MSE are difficult or even impossible to derive in the case of nonlinear indicators like the Gini. The MSE is approximated using a bootstrap procedure:

1. Fit the model to the sample data and therefore obtain estimates for $\beta, \sigma_u^2 and \sigma_e^2$
2. Generate $u_i^* \sim iid\mathcal{N}(0, \hat{\sigma}_u^2)$ and $e_{ij}^* \sim iid\mathcal{N}(0, \hat{\sigma}_e^2)$ independently for every domain and every person in the population
3. Construct a superpopulation using the generated error terms and the population covariates
   $y_{ij}^* = x_{ij}\prime\hat{\beta} + u_i^* + e_{ij}^*$
4. Draw B bootstrap populations from the superpopulation and calculate θ_i^{*b} in every domain for each of the populations
5. Draw a bootstrap sample from every population, implement the EBP and estimate $\hat{\theta}_i^{*b}$
6. Finally estimate the MSE per domain by:
   $$\hat{MSE}(\hat{\theta}_i) = \frac{1}{B}\sum_{b=1}^{B}(\hat{\theta}_i^{*b} - \theta_i^{*b})^2$$

Data and Sampling

Assume the following scenario:

1. Stratified sample (i income groups nested in SMAs) with data on y and X
   - In each SMA, an equal number of observations per income group is sampled such that:
     - Sample size n_{SMA} per SMA is proportional to N_{SMA}
     - Smaller income groups are oversampled
   - Weights differ between income groups and SMAs
     $\rightarrow frequencyweights_{SMA,incomegroup} = \frac{N_{SMA,incomegroup}}{n_{SMA}/i}$
2. Population data on X on SMA level is available
   - approach is generalizable to other forms of sampling, where certain groups are over- or undersampled in a stratified setting
3. Dataset: EUSILC Data as provided by the emdi package
   - to achieve a sufficient population size N, randomly duplicate observations between 1 and 8 times
   - add ε to the dependent variable of duplicated observations, where $\varepsilon \sim N(0, 5000)$

Implementation

- take an initial random sample of size n with same number (g) of observations from each income group
- calculate n_{SMA}
- calculate *frequencyweights*
- for 1:s {
  1. split population by SMA, take a sample of $n_{SMA}/5$ from each income group
  2. estimate $\widehat{Gini}_{direct\_unweighted}, \widehat{Gini}_{direct\_weighted}$ per SMA
  3. estimate $\widehat{Gini}_{EBP\_unweighted}$ based on l pseudo-populations per SMA
  4. estimate $\widehat{MSE}_{Gini_{unweighted\_EBP}}$ based on b bootstraps per SMA
  5. expand the sample by frequency weights
  6. estimate $\widehat{Gini}_{EBP\_weighted}$ based on l pseudo-populations per SMA
  7. save the results per SMA

  } calculate $Gini_{SMA}$ based on population
- calculate quality measures per SMA
  - $MSE_{SMA} = \sum_{i=1}^{s}(\widehat{Gini}_{SMA} - Gini_{SMA})^2$
  - $RelBias_{SMA} = \sum_{i=1}^{s}(\widehat{Gini}_{SMA} - Gini_{SMA})/Gini_{SMA}$
- Parameters: $N = 112644, s = 250, g = 2000, n = 10000, l = 50, b = 10, SMA = district, i = 5$

RMSE of weighted and unweighted EBP

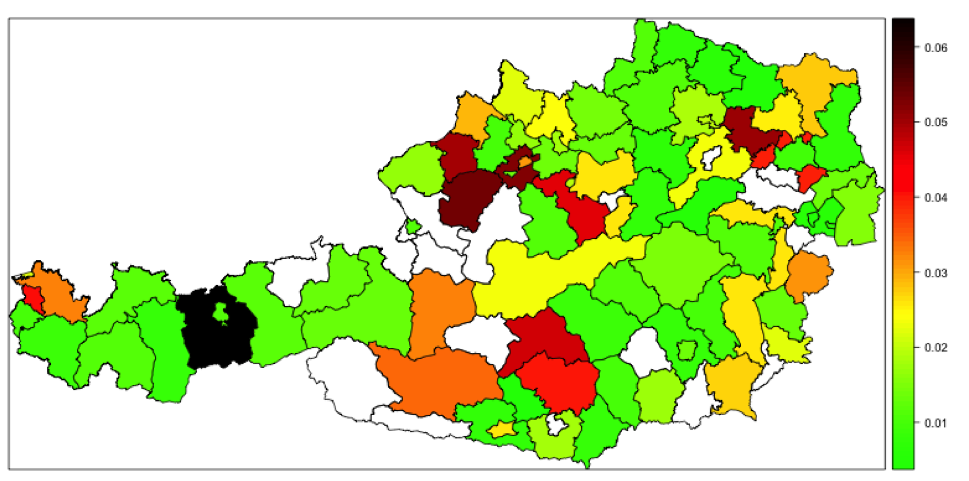


Figure: RMSE of Weighted EBP per Domain

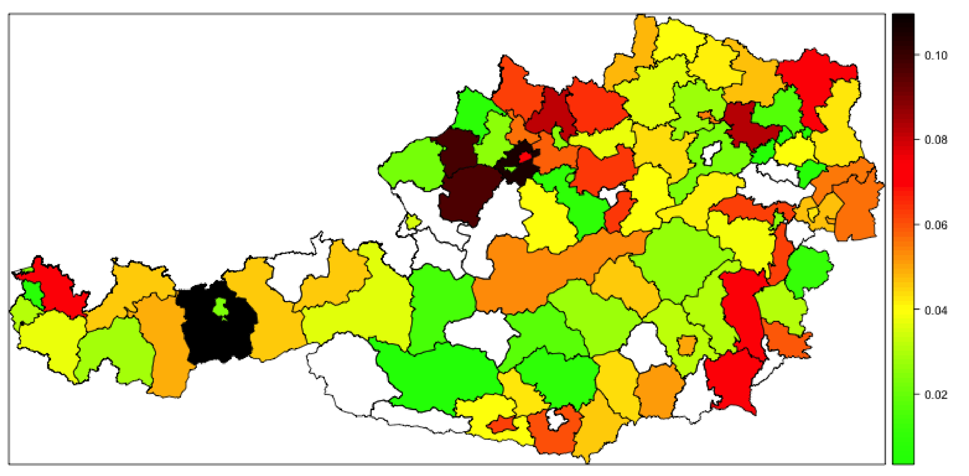


Figure: RMSE of Unweighted EBP per Domain

Domain Level MRE and RMSE of Estimators

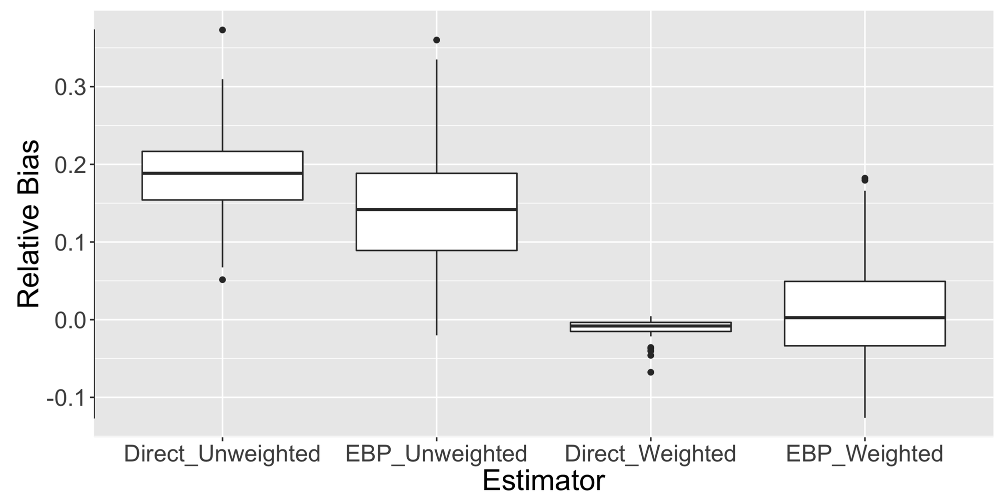


Figure: Boxplot of Relative Bias on Domain Level

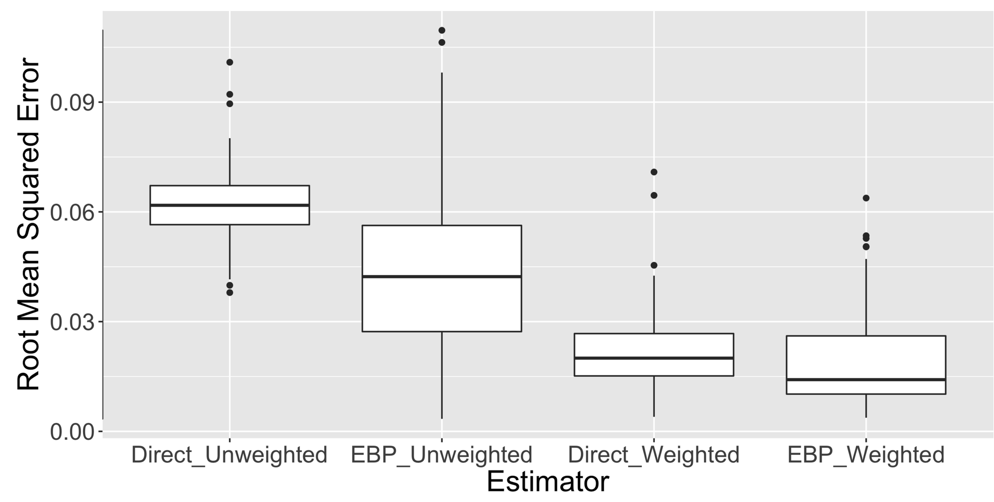


Figure: Boxplot of RMSE on Domain Level

Accuracy of inbuilt MSE estimator

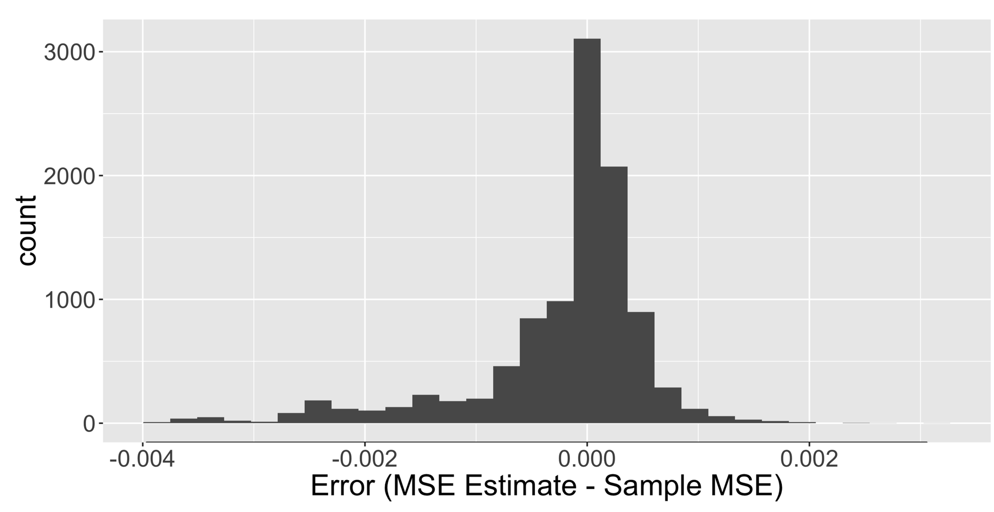


Figure: Histogram of Difference between EBP MSE estimator and the sample MSE across iterations

Observations:

- Bootstrap estimator agrees quite well with sample MSE
- Slightly more likely to underestimate sample MSE

Conclusions

Conclusions:

- Unweighted direct estimator and EBP are both biased unlike their weighted versions.
- Weighting substantially increase accuracy for both estimators.
- Relative accuracy advantage of EBP over direct estimation is much larger in the unweighted case.
- Weights can be included in EBP estimation by expanding the sample using frequency weights.

Future Research:

- Explore alternative ways of including weights into EBP estimation.
- Analyse alternative sampling designs.

Bibliography

- Alfons, A., and Templ, M. (2013) Estimation of Social Exclusion Indicators from Complex Surveys: The R Package laeken. Journal of Statistical Software, 54, 1-25.
- Guadarrama, M., Molina, I., and Rao, J.N.K. (2016) Small area estimation of general parameters under complex sampling designs. UC3M Working Papers, Statistics and Econometrics, 16-05.
- Molina, I., and Rao, J.N.K. (2010) Small area estimation of poverty indicators. Canandian Journal of Statistic, 38, 369-385.
- Verret, F., Rao, J.N.K, and Hidiroglou, M.A. (2015) Model-based small area estimation under informative sampling. Survey Methodology, 41.2, 333-347.

Für weitere Informationen

Kontakt:

Name	Matrikel N.	e-mail
Christian Koopmann	572485	koopmanc@hu-berlin.de
Felix Skarke	573653	f.skarke@fu-berlin.de
Enno Tammena	572575	tammenae@hu-berlin.de