Cancer Tumor Growth Dynamics

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1 Introduction

Cancer is a phenomenon that affects everyone these days, whether they themselves or a family member or close friend. Some cancers respond well to treatment, allowing their victims to go into remission, while others are extremely aggressive and cannot be slowed down from killing their prey. Due to their dynamic and varying nature, tumors are a fascinating topic of study and analysis, which was one of the reasons we chose this subject matter to model.

2 Construction of Tumor Growth Model

Using a general logarithmic population model as a basis, we chose to create two models to study. Our first model represents the rate of cell population growth, denoted by $\frac{dN}{dt}$.

$$\frac{dN}{dt} = rN(1 - \frac{N}{K})$$

where N is the net cell population, r is the ratio of cell proliferation, a, and cell death, b, and K is the carrying capacity. An assumption that we employ is that $a, b > 0$, and therefore, $r > 0$, which biologically makes sense.
The second model we constructed modeled the rate of carrying capacity, K, a rate that changes along with the growth of a tumor.

$$\frac{dK}{dt} = \phi N - \psi K N^{\frac{2}{3}}$$

where ϕ is the value of the angiogenesis factor and ψ is the value of the angiogenesis inhibitor. For the purpose of this model, we have chosen ϕ and ψ to equal 5.85 and .00873, respectively.

3 Equilibria, Jacobian, Eigenvalues

3.1 Equilibria

Setting $\frac{dN}{dt} = 0$ we get that $N = K$. Then, we set $\frac{dK}{dt} = 0$ and using the concept that $N = K$, we found that three equilibria of our system are as follows:

$$(N^*, K^*) = (0, 0)$$
$$(N^*, K^*) = ((\tfrac{\phi}{\psi})^{\frac{3}{2}}, (\tfrac{\phi}{\psi})^{\frac{3}{2}})$$
$$(N^*, K^*) = (-(\tfrac{\phi}{\psi})^{\frac{3}{2}}, -(\tfrac{\phi}{\psi})^{\frac{3}{2}})$$

When analyzing our model, we did not include $(N^*, K^*) = (-(\frac{\phi}{\psi})^{\frac{3}{2}}, -(\frac{\phi}{\psi})^{\frac{3}{2}})$ due to its completely negative graphical nature as well as unrealistic quality.

3.2 Jacocian

With $N = K$, our resulting Jacobian Matrix is

$$J = \begin{pmatrix} -r & r \\ \phi - \frac{2}{3}\psi N^{\frac{2}{3}} & -\psi N^{\frac{2}{3}} \end{pmatrix}$$

Evaluating the Jacobian at $N = 0$ and $N = (\frac{\phi}{\psi})^{\frac{3}{2}}$, we got the following matrices. For (0,0)

$$J = \begin{pmatrix} -r & r \\ \phi & 0 \end{pmatrix}$$

and for $((\frac{\phi}{\psi})^{\frac{3}{2}}, (\frac{\phi}{\psi})^{\frac{3}{2}})$

$$J = \begin{pmatrix} -r & r \\ -\frac{1}{3}\phi & -\phi \end{pmatrix}$$

Using the above matrices, we computed the associated Eigen values and used them to analyze the stability of the equilibrium points.

3.3 Eigen Values and Stability

It follows that the characteristic equations for the Jacobian Matrices are,

$N = 0$

$$\lambda^2 + r\lambda - \phi r = 0$$

And the resulting eigenvalues.

$$\lambda_{1,2} = \frac{-r \pm \sqrt{r^2 + 4r\phi}}{2}$$

$N = (\frac{\phi}{\psi})^{\frac{3}{2}}$

$$\lambda^2 + \lambda(r + \phi) + \frac{4}{3}\phi r = 0$$

And the corresponding eigenvalues are given by,

$$\lambda_{1,2} = \frac{-(r + \phi) \pm \sqrt{(r + \phi)^2 - \frac{8}{3}\phi r}}{2}$$

We analyzed the Eigenvalues and concluded that $(0, 0)$ is a saddle point and $((\frac{\phi}{\psi})^{\frac{3}{2}}, (\frac{\phi}{\psi})^{\frac{3}{2}})$ is a stable node.

3.4 Nullclines and Graphical Analysis

To graphically represent our results, we computed the nullclines associated with our model. To find the nullclines of this system, we set $\frac{dN}{dt}$ and $\frac{dK}{dt}$ equal to zero.

$$\frac{dN}{dt} = rN(1 - \frac{N}{K})$$
$$rN(1 - \frac{N}{K}) = 0$$

$$\boxed{N = K} \tag{1}$$

$$\frac{dK}{dt} = \phi N - \psi K N^{\frac{2}{3}}$$
$$\phi N - \psi K N^{\frac{2}{3}} = 0$$

$$\boxed{K = \frac{\phi}{\psi} N^{\frac{1}{3}}} \tag{2}$$

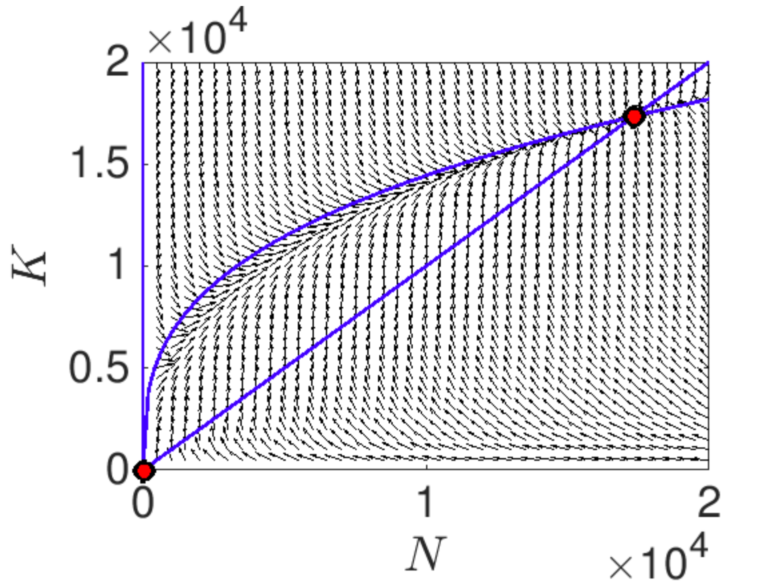

Figure 1: $a = b$

Then, using a predator-prey population model as a guide, we used Matlab to plot the equilibria, nullclines, and resulting direction fields, playing around with the value of r. From these graphical results, it is concluded that the closer to $r = 1$, the closer the direction field hugs the nullcline $K = \frac{\phi}{\psi} N^{\frac{1}{3}}$. For $a > b$, the closer the direction field follows the nullcline $N = K$. From these results, it is clear to understand that the growth of a tumor is increased the less dominant the proliferate rate a is compared to the death rate b.

4 Conclusion

Curating and exploring our model led to the understanding of how tumor growth can be modeled and its relationship between net cell population and carrying capacity. It would be quite interesting to introduce a treatment variable with changing assumptions to see how this model is affected and understand which treatment is the most affective as a method for limiting tumor growth.