Braid Groups

Cassandra Kornhiser

June 9, 2020

1 Introduction

Although braiding has been around for tens of thousands of years, its prominence in the world of science and mathematics did not blossom until a few centuries ago. In the early 1900's, mathematicians and scientists became intrigued by the properties of braids and their possible applications, transcending them to a topic of importance by the 1950's [?]. At this time in history, group theory had become a popular and innovative way to analyze collections of elements, providing the building blocks for various concepts. Therefore, for mathematicians, it was the best method to examine and analyze braid formation. To do so, it had to be proven that braid groups exist and it led to its broaden application and another progression in the science and mathematics fields.

2 Introduction to Group Theory

As mentioned above, the best and most powerful way to analyze braids is through group theory. In mathematics, groups are "algebraic structures" that organize a collection of elements closed under a binary operation. [?] To elaborate, a binary operation is an operation on two input elements that produces a single output element, such as addition and subtraction. By definition, a group is a nonempty set G on which there is defined a binary operation $\star$ on $(g_1, g_2) \to g_1 \star g_2$ satisfying various properties [?]. These properties include

1. **Closure:** If elements g_1 and g_2 belong to group G, then $g_1 \star g_2$ is also in G.

2. **Identity:** There exists an identity element $e \in G$ such that $g \star e = e \star g = g$ and for all g in G.

3. **Inverse**: For every element g, there exists its inverse g^{-1} such that $g \star g^{-1} = g^{-1} \star g = e$ for all g, g^{-1} in G

4. **Associativity**: For all $g_1, g_2, g_3 \in G$, $g_1 \star (g_2 \star g_3) = (g_1 \star g_2) \star g_3$

To better understand these properties, let us use them to prove that the set of integers $\mathbb{Z}$ closed under the binary operation addition, is a group.

2.1 Property (1)

Beginning with property (1), an integer added to another integer results in an integer, therefore, the $\mathbb{Z}$ is closed under addition. Let us consider even and odd integers.

1. Let g_1, g_2, g_3 be even integers $\in \mathbb{Z}_{Even}$, where an even number is defined as $2k$ for all $k \in \mathbb{Z}$. Then, $g_1 + g_2 = 2k_1 + 2k_2 = 2k_3 = g_3$, therefore, since adding two even integers results in an even integer, we can say that the set of even integers under addition is *closed.*

2. Let g_1, g_2, g_3 be odd integers $\in \mathbb{Z}_{Odd}$, where an odd number is defined as $2k + 1$ for all $k \in \mathbb{Z}$. Then, $g_1 + g_2 = 2k_1 + 1 + 2k_2 + 1 = 2k_3 = g_3$, which is the definition of an even number. Therefore, the set of odd integers under addition is *not closed.*

2.2 Property (2)

For property (2), there must be an element such that when inputed with another element, it has no effect on it under the chosen binary operation. Continuing with the example using $\mathbb{Z}$ under addition, let the identity element be e and let $g \in \mathbb{Z}$, such that $e + g = g + e = g$. Therefore, e must be equal to 0, where $0 \in \mathbb{Z}$, satisfying this property.

2.3 Property (3)

Property (3) states that every element in a group must have an inverse element such that it is also in the group and as inputs, results in the identity element e. Let us look at examples in $\mathbb{Z}$.

1. If $g = 4$, then $g^{-1} = -4$ such that $g + g^{-1} = 4 + (-4) = 0$, where 0 is the identity element. For $\mathbb{Z}$ under addition, we can define the inverse of an element as the negative value of the element.

2. If we consider $\mathbb{Z}$ closed under multiplication, then the identity element $e = 1$ and so $g * g^{-1} = g^{-1} * g = e$, defining g^{-1} as the reciprocal of g. If $g = 4$, then $g^{-1} = \frac{1}{4}$, since $4 * \frac{1}{4} = 1$. Since $\frac{1}{4}$ is not in $\mathbb{Z}$, then $\mathbb{Z}$ under multiplication is not a group.

2.4 Property (4)

The fourth property defines the associativity property of groups. This property essentially states that the order of elements under an operation is trivial, which holds with $\mathbb{Z}$ under addition. An example of this property not being satisfied is matrix multiplication, as $(AB)C \neq A(BC)$.

3 Braid Groups

Now that we have defined what a group is and the properties it must satisfy, let us show that braid groups are groups. Before we begin proving that each property is satisfied, we must understand some concepts about braids and braid groups. A braid has n number of strings of any length with fixed endpoints, usually vertically conceptualized with a downward path between the two sets of endpoints. Another important thing to know is that there are very special sets of braids that are used to build all others are known as generators. For general understanding, generators make up braids and braids can be broken down into generators, similar to the notion that numbers are simply combinations of ones and twos. For the sake of simplicity, we will be exploring the properties of braid groups using the braids of the three string braid group, $\mathcal{B}_3$, with elements β_n .

In the above section, we were primarily looking at addition as the binary operation; for braids, concatenation is the only binary operator. Concatenation is considered the linking of two things, so in the case of braids, it is the the attachment of one braid to the end of another. For the purpose of this paper, concatenation will be defined as $\star$.

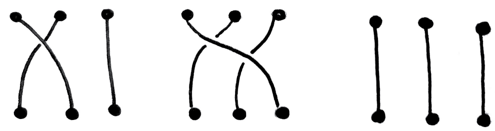

Figure 1: Examples of three string braids.

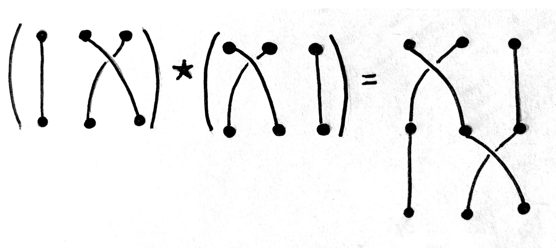

Figure 2: The concatenation of two braids.

3.1 Satisfying Property (1)

Beginning with property (1) of a group, braid groups are closed due to the fact that a braid connected to another braid results in the continuation of the first braid, and therefore, is a braid, as seen in Figure (1).

$$\beta_1 \star \beta_2 = \beta_3$$

3.2 Satisfying Property (2)

The identity element of braid groups is a braid with straight lines, such that parallel strings are directly connected between their endpoints, without any crossings with the other strings [?]. The addition of the element braid to β does not alter β as seem in Figure (3).

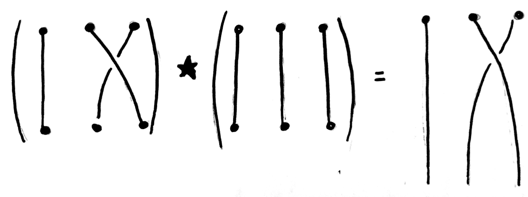

Figure 3: $\beta \star \ e = \beta$

3.3 Satisfying Property (3)

The inverse of a braid β^{-1} is the horizontal reflection of β. Essentially, β^{-1} represents the "undoing" of braid β, where it's first crossing of strings is the last crossing β. Another related property of braid groups is known as

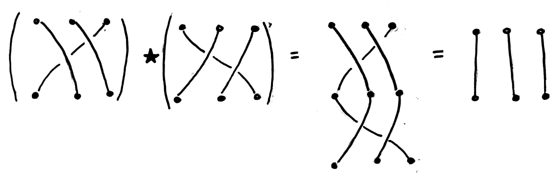

Figure 4: $\beta \star \beta^{-1} = e$

the "shoes socks" property where $(g \star h)^{-1} = h^{-1} \star g^{-}1$. The "shoes socks" theorem is based on the real life concept of putting on shoes and socks and then removing them: we first put on socks, then shoes, but take our shoes off before taking off our socks. The same goes for the inverse of the concatenation between two braids.

3.4 Satisfying Property (4)

Associativity of braid groups can be proven by showing that $(\beta_1 \star \beta_2) \star \beta_3 = \beta_1 \star (\beta_2 \star \beta_3)$. Since it is a biconditional statement, both sides if the equation must be shown true.

1. First, let us construct the braid created from $(\beta_1 \star \beta_2)$.

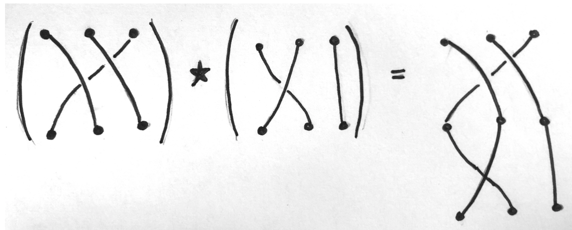

Figure 5: $(\beta_1 \star \beta_2)$

Now, let us attach β_3.

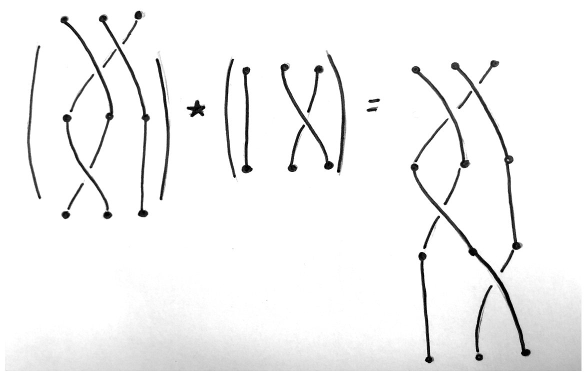

Figure 6: $(\beta_1 \star \beta_2) \star \beta_3$

2. For the right side of the equation, we proceed in the same manner, beginning with the construction of the $(\beta_2 \star \beta_3)$ term.

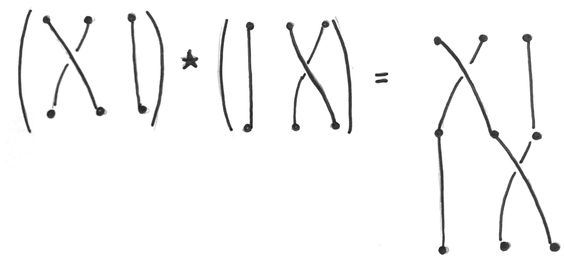

Figure 7: $\beta_2 \star \beta_3$

Then, connecting β_1, we result with a braid equivalent to the output of the left sight of the equality.

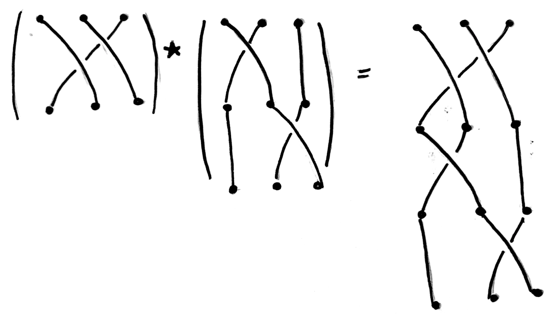

Figure 8: $\beta_1 \star (\beta_2 \star \beta_3)$

Since Figures (6) and (8) are the same braid, the equality holds and therefore, the associativity property is satisfied.

3.5 Braid Group Consensus

From the above fulfillment of the group properties, it can be understood that braid groups are indeed groups. Due to this discovery, major leaps have been made in the science and mathematics fields with many more to come.

4 Conclusion

Braid groups are a beautiful mathematic phenomena that have been used to model and understand a plethora of concepts, from flows to the construction of physical things. Braid groups are another mathematical understanding that can be used to further intertwine the physical and analytical worlds in a naturally occurring way, attributing them a dynamic essence. There are so many ways in which one can study and analyze braids and braid groups and their limitless potential provides a never ending offering of explorations.

5 Bibliography

References

[1] Dale, R. (2007). *Tutorial on the braid groups*. Pages 2-14
Available electronically at:
https://arxiv.org/pdf/1010.4051.pdf
Originally presented to the summer school and workshop at the Institute for Mathematical Sciences, National University of Singapore, June 2007

[2] Jackson, N. (2004). *Notes on braid groups*. Pages 1-6.
Available electronically at:
https://homepages.warwick.ac.uk/ maseay/doc/braids.pdf

[3] *Chapter 1: Group Fundamentals*. Pages 1-13
Available electronically at:
https://faculty.math.illinois.edu/ r-ash/Algebra/Chapter1.pdf