Calc III: Workshop 12, Fall 2017

Problem 1. The electrostatic force (on a positive unit test charge at (x, y, z)) due to a unit point charge at $(0, 0, 0)$ is given by

$$\mathbf{F}(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

(a) Let S_R be the closed sphere of radius R, with $\mathbf{n}$ oriented outward. Show by direct computation that $\iint_{S_R} \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi$, independent of R.

(b) Using the divergence theorem, show that the flux $\iint_{\Sigma} \mathbf{F} \cdot \mathbf{n}\, dS$ of $\mathbf{F}$ across any closed surface containing $(0, 0, 0)$ is 4π.

Problem 2. Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where

$$\mathbf{F}(x, y, z) = z^2 x\mathbf{i} + (\tfrac{1}{3}y^3 + \tan z)\mathbf{j} + (x^2 z + y^2)\mathbf{k}$$

and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$. Note that S is not a closed surface.

Problem 3. Let $\mathbf{v}$ be a constant vector and Σ a closed surface with any orientation. Prove that $\iint_{\Sigma} \mathbf{v} \cdot \mathbf{n}\, dS = 0$.

Problem 4. Let Σ be the closed surface bounding a solid region E, oriented with outward pointing unit normal. Prove that

$$\iint_{\Sigma} (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot \mathbf{n}\, dS = \mathrm{Vol}(E).$$