Calc III Fall 2016: Exam 2 Solutions

Problem 1. Evaluate $\iint_E (x^2+y^2)^{3/2}\,dA$, where E is the region between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution. In polar coordinates,

$$\begin{aligned}\iint_E (x^2+y^2)^{3/2}\,dA &= \int_0^{2\pi}\int_1^2 (r^3)r\,dr\,d\theta \\ &= 2\pi\left(\frac{2^5}{5}-\frac{1}{5}\right).\end{aligned}$$

$\square$

Problem 2. Compute the volume of the region bounded between the surfaces $z=0$ and $z=x^2+y^2$ where $0\le x\le 1$, $0\le y\le x$.

Solution.

$$\begin{aligned}\int_0^1\int_0^x\int_0^{x^2+y^2} dz\,dy\,dx &= \int_0^1\int_0^x (x^2+y^2)\,dy\,dx \\ &= \int_0^1\int_0^x (x^2+y^2)\,dy\,dx \\ &= \int_0^1 x^2y+\frac{y^3}{3}\Big|_0^x \\ &= \int_0^1 x^3+\frac{x^3}{3} \\ &= \frac{4}{3}\frac{1}{4}x^4\Big|_0^1 \\ &= \frac{1}{3}\end{aligned}$$

$\square$

Problem 3. Convert the following triple integral to cylindrical coordinates and evaluate it:

$$\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{4-x^2-y^2} dz\,dy\,dx$$

Solution.

$$\begin{aligned}\int_0^{\pi/2}\int_0^2\int_0^{4-r^2} r\,dz\,dr\,d\theta &= \frac{\pi}{2}\int_0^2 r(4-r^2)\,dr \\ &= \frac{\pi}{2}\left(\frac{r^2}{2}-\frac{r^4}{4}\right)\Big|_0^2 \\ &= 2\pi.\end{aligned}$$

$\square$

Problem 4. A solid S lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2+y^2+z^2 = 1$, with density at each point equal to the distance to the z-axis. Find the mass of S, using spherical coordinates.

Solution. The density is $\delta = \sqrt{x^2 + y^2} = \rho \sin\varphi$. So the mass is given by

$$\begin{aligned}
\iiint_S \sqrt{x^2 + y^2}\, dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 (\rho\sin\varphi)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin^2\varphi \, d\varphi \int_0^1 \rho^3 \, d\rho \\
&= 2\pi \int_0^{\pi/4} \tfrac{1}{2}(1 - \cos 2\varphi)\, d\varphi (\tfrac{1}{4}) \\
&= \frac{\pi}{4}\left(\frac{\pi}{4} - \frac{1}{2}\right)
\end{aligned}$$

□

Problem 5. Evaluate the line integral $\int_C x\, ds$, where C is the curve along $y = x^2$ from $(0, 0)$ to $(2, 4)$.

Solution. We may parameterize the curve by $\mathbf{r}(t) = \langle t, t^2\rangle$, where $0 \le t \le 2$. Then $\mathbf{r}'(t) = \langle 1, 2t\rangle$, so the arc length element is given by $ds = |\mathbf{r}'(t)|\, dt = \sqrt{4t^2 + 1}\, dt$. The line integral is

$$\begin{aligned}
\int_C x\, ds &= \int_0^2 t\sqrt{4t^2 + 1}\, dt \\
&= \frac{1}{8}\int_0^{17} \sqrt{u}\, du \\
&= \frac{(17)^{3/2}}{12}.
\end{aligned}$$

□