Calc III: Workshop 4 Solutions, Fall 2017

Problem 1.

(a) Show that $f(x,y) = \sin(x+cy) + \cos(x-cy)$ satisfies the 1-dimensional wave equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial y^2} = 0. \tag{1}$$

(b) Let $u(t)$ and $v(t)$ be twice differentiable functions of a single variable. Show that $f(x,y) = u(x+cy) + v(x-cy)$ is a solution of (1).

Solution.

(a) We compute the first two partial derivatives of f with respect to x:

$$\frac{\partial f}{\partial x} = \cos(x+cy) - \sin(x+cy), \quad \frac{\partial^2 f}{\partial x^2} = -\sin(x+cy) - \cos(x+cy),$$

and with respect to y:

$$\frac{\partial f}{\partial y} = c\cos(x+cy) - c\sin(x+cy), \quad \frac{\partial^2 f}{\partial y^2} = -c^2\sin(x+cy) - c^2\cos(x+cy).$$

Then subtracting $1/c^2$ times $\frac{\partial^2 f}{\partial y^2}$ from $\frac{\partial^2 f}{\partial x^2}$ gives 0.

(b) Proceding as above,

$$\frac{\partial f}{\partial x} = u'(x+cy) + v'(x+cy), \quad \frac{\partial^2 f}{\partial x^2} = u''(x+cy) + v''(x+cy),$$
$$\frac{\partial f}{\partial y} = c\left(u'(x+cy) + v'(x+cy)\right), \quad \frac{\partial^2 f}{\partial y^2} = c^2\left(u''(x+cy) + v''(x+cy)\right).$$

Note that u' and u'', etc., denote the ordinary derivatives of u and v as one variable functions, which we then evaluate at $x+cy$. In any case, subtracting as in part (a) verifies that f solves the wave equation.

□

Problem 2.

(a) Find the tangent plane to the surface $x^2+y^2-z^2 = 0$ at the point $P = (3,4,5)$.

(b) Find the tangent plane to the surface $x^2+y^2 = 4$ at the point $P = (\sqrt{3},1,0)$.

Solution.

(a) To use the formula for a graph $z = f(x,y)$, we need to solve for z as a function of x and y. Since $z > 0$ in the point P, the surface near P is given by the positive square root: $z = \sqrt{x^2+y^2}$. Then $\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}$ and $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$. Then the tangent plane at $P = (3,4,5)$ is given by

$$\frac{3}{5}(x-3) + \frac{4}{5}(y-4) - z + 5 = 0.$$

Alternatively, you can use the formula that the tangent plane for a surface of the form $F(x,y,z)=0$ is given by

$$\frac{\partial F(x_0,y_0,z_0)}{\partial x}(x-x_0)+\frac{\partial F(x_0,y_0,z_0)}{\partial y}(y-y_0)+\frac{\partial F(x_0,y_0,z_0)}{\partial z}(z-z_0)=0.$$

In this case, the partial derivatives are $\frac{\partial F}{\partial x}=2x$, $\frac{\partial F}{\partial y}=2y$, and $\frac{\partial F}{\partial z}=-2z$, giving

$$6(x-3)+8(y-4)+(-10)(z-1)=0,$$

which is the same plane (after multiplying through by 1/10).

(b) Since we cannot solve for z as a function of x and y, we use the second way. Here, $F(x,y,z)=x^2+y^2-4$, and $\frac{\partial F}{\partial x}=2x$, $\frac{\partial F}{\partial y}=2y$, and $\frac{\partial F}{\partial z}=0$. The formula for the tangent plane is thus

$$2\sqrt{3}(x-\sqrt{3})+2(1)(y-1)=0.$$

□

Problem 3. Prove the product and quotient rules for gradients:

$$\nabla(f\,g)=f\nabla g+g\nabla f,\quad \nabla(f/g)=\frac{g\nabla f-f\nabla g}{g^2},\quad g(x,y)\neq 0.$$

Solution. The components of $\nabla(fg)$ are $\frac{\partial}{\partial x}(fg)$, $\frac{\partial}{\partial y}(fg)$, and $\frac{\partial}{\partial z}(fg)$, respectively. For each of these, a product rule holds: for example

$$\frac{\partial fg}{\partial x}=g\frac{\partial f}{\partial x}+f\frac{\partial g}{\partial x},$$

and so on. Thus

$$\nabla(fg)=\left(g\tfrac{\partial f}{\partial x}+f\tfrac{\partial g}{\partial x},g\tfrac{\partial f}{\partial y}+f\tfrac{\partial g}{\partial y},g\tfrac{\partial f}{\partial z}+f\tfrac{\partial g}{\partial z}\right)=g\nabla f+f\nabla g.$$

The quotient rule follows similarly from the quotient rules for partial derivatives. □

Problem 4. The function $r(x,y)=\sqrt{x^2+y^2}$ is the length of the position vector $\mathbf{r}(x,y)=x\mathbf{i}+y\mathbf{j}$ for each point $(x,y)\in\mathbb{R}^2$. Show that $\nabla r=\frac{1}{r}\mathbf{r}$ when $(x,y)\neq(0,0)$, and that $\nabla(r^2)=2\mathbf{r}$.

Solution. Computing the partial derivatives,

$$\frac{\partial r}{\partial x}=\frac{x}{\sqrt{x^2+y^2}},\quad\text{and}\quad\frac{\partial r}{\partial y}=\frac{y}{\sqrt{x^2+y^2}},$$

so

$$\nabla(r)=\left(\frac{x}{\sqrt{x^2+y^2}},\frac{y}{\sqrt{x^2+y^2}}\right)=\frac{1}{\sqrt{x^2+y^2}}(x,y)=\frac{\mathbf{r}}{r}.$$

Likewise,

$$\frac{\partial r^2}{\partial x}=2x,\quad\text{and}\quad\frac{\partial r^2}{\partial y}=2y,$$

so

$$\nabla(r^2)=(2x,2y)=2\mathbf{r}.$$

□

Problem 5. Recall that the *linear approximation* to a function $f(x,y)$ of two variables, at a point (x_0, y_0) is the linear function

$$L(x,y) = a(x - x_0) + b(y - y_0) + c$$

whose graph $z = L(x,y)$ is the tangent plane to the graph $z = f(x,y)$ of f at (x_0, y_0). (Hint: what are a, b and c in terms of $f(x_0, y_0)$ and the partial derivatives of f at (x_0, y_0)?)

(a) Given that f is a differentiable function with $f(2,5) = 6$, $f_x(2,5) = 1$, and $f_y(2,5) = -1$, use the linear approximation to estimate $f(2.2, 4.9)$.

(b) Generalize the formula for linear approximations to functions of three variables, find the linear approximation to the function $f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$ at $(3,2,6)$ and use it to approximate the number $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$.

Solution. The a, b, and c are given by $f_x(x_0, y_0)$, $f_y(x_0, y_0)$, and $f(x_0, y_0)$, respectively, so

$$L(x,y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

(a) The linear approximation here is

$$L(x,y) = 6 + 1(x - 2) - (y - 5),$$

so

$$f(2.2, 4.9) \approx L(2.2, 4.9) = 6 + (0.2) - (-0.1) = 6.3.$$

(b) The generalization to three variables is

$$L(x,y,z) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0).$$

In this case the partial derivatives are given by

$$f_x = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \quad f_y = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \quad f_z = \frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

Evaluating at $(3,2,6)$ and computing $L(x,y,z)$ gives

$$L(x,y,z) = 7 + \tfrac{3}{7}(x - 3) + \tfrac{2}{7}(y - 2) + \tfrac{6}{7}(z - 6).$$

Then

$$f(3.02, 1.97, 5.99) \approx L(3.02, 1.97, 5.99) = 7 + \tfrac{3}{7}(0.02) + \tfrac{2}{7}(-0.03) + \tfrac{6}{7}(-0.01) = 7 - \tfrac{6}{700}.$$

$\square$