Higher gerbes, loop spaces, and transgression

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Gerbes

Higher gerbes

Relation to loop spaces

Line bundles : $H^2(X;\mathbb{Z})$

- A complex line bundle $L \longrightarrow X$ has a Chern class $c_1(L) \in H^2(X;\mathbb{Z})$.
- Naturality:

$$c_1(\mathbb{C} \times X) = 0, \qquad c_1(L \otimes L') = c_1(L) + c_1(L'),$$
$$c_1(L^{-1}) = -c_1(L), \qquad c_1(f^*L) = f^*c_1(L)$$

- $c_1(L) = c_2(L')$ if and only if $L \cong L'$.

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- $c_1(L) = c_2(L')$ if and only if $L \cong L'$.
- Seen by Čech cohomology: $[L] \in \check{C}^1(X;\mathbb{C}^*)$ satisfies $d[L] = 1$, unique up to dh, $h \in \check{C}^0(X;\mathbb{C}^*)$, so

$$[L] \in \check{H}^1(X;\mathbb{C}^*) \cong H^2(X;\mathbb{Z}).$$

Gerbes : $H^3(X;\mathbb{Z})$

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- Various versions: Giraud, Brylinski, Hitchin and Chattergee, Murray.
- Murray: a *bundle gerbe* (L, Y, X) is
  - a fiber bundle (more generally: *locally split*, i.e., surjective map admitting local sections)
  $$p : Y \longrightarrow X,$$
  - a line bundle
  $$L \longrightarrow Y^{[2]} = Y \times_X Y = \{(y_1, y_2) : p(y_1) = p(y_2) \in X\}$$
  - with a product
  $$\phi : L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \xrightarrow{\cong} L_{(y_1,y_3)}, \quad (y_1, y_2, y_3) \in Y^{[3]}$$
  - satisfying associativity:
  $$\phi \circ (1 \otimes \phi) = \phi \circ (\phi \otimes 1) : L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \otimes L_{(y_3,y_4)} \cong L_{(y_1,y_4)},$$
  $$(y_1, y_2, y_3, y_4) \in Y^{[4]}$$
- (L, Y, X) has a *Dixmier Douady class* $DD(L, Y, X) \in H^3(X;\mathbb{Z})$.

Properties of gerbes

$$\begin{tikzcd} & & L \arrow[d] \\ X & Y \arrow[l, "p"'] & Y^{[2]} \arrow[l, shift right, "p_0"] \arrow[l, shift left, "p_1"'] \end{tikzcd}$$

Properties of gerbes

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\end{tikzcd}
$$

- Trivialization: an isomorphism $L \cong \delta Q := p_0^* Q \otimes p_1^* Q^{-1}$ for some $Q \longrightarrow Y$.

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- Trivialization: an isomorphism $L \cong \delta Q := p_0^* Q \otimes p_1^* Q^{-1}$ for some $Q \longrightarrow Y$.
- Inverse: $(L, Y, X)^{-1} = (L^{-1}, Y, X)$.
- Product: $(L, Y, X) \otimes (L', Y', X) = (\pi_1^* L \otimes \pi_2^* L', Y \times_X Y', X)$
- Pullback: $f^*(L, Y, X) = (f^* L, f^* Y, X')$

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- Pullback: $f^*(L, Y, X) = (f^* L, f^* Y, X')$
- Relation with DD class:
  - $DD(L) = 0$ if and only if L is trivial.
  - $DD(L^{-1}) = -DD(L)$
  - $DD(L \otimes L') = DD(L) + DD(L')$
  - $DD(f^* L) = f^* DD(L)$.
  - $DD(L) = DD(L')$ if and only if L and L' are *stably isomorphic*, i.e., $L \otimes Q \cong L' \otimes Q'$ for trivial gerbes Q and Q'.

Example: lifting bundle gerbes

- $E \longrightarrow X$ principal G bundle, where G has a central extension

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \widehat{G} \longrightarrow G \longrightarrow 1$$

- $\widehat{G} \longrightarrow G$ defines an associated line bundle $L = \widehat{G} \times_{\mathbb{C}^*} \mathbb{C} \longrightarrow G$
- Difference map $u : E^{[2]} \longrightarrow G$, where "$u(y_0, y_1) = y_0^{-1} y_1$" i.e., $u(y_0, y_1) = g$ such that $y_1 = y_0\, g$.
- $(u^* L, E, X)$ is the *lifting bundle gerbe* for E.

$$\begin{tikzcd} u^* L \arrow[d] & L \arrow[d] \\ E^{[2]} \arrow[r, "u"] \arrow[d] & G \\ X & \end{tikzcd}$$

- $DD(u^* L, E, X) \in H^3(X; \mathbb{Z})$ is the obstruction to lifting E to a $\widehat{G}$ bundle $\widehat{E} \longrightarrow X$.

Gerbes as simplicial line bundles

$$X \longleftarrow Y \leftleftarrows Y^{[2]} \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} Y^{[3]} \overset{\longleftarrow}{\underset{\longleftarrow}{\leftleftarrows}} Y^{[4]} \qquad \cdots$$

Gerbes as simplicial line bundles

$$X \longleftarrow Y \Lleftarrow Y^{[2]} \Lleftarrow Y^{[3]} \Lleftarrow Y^{[4]} \qquad \cdots$$

- A *simplicial space* (over X) is a sequence $\{Y_n : n \in \mathbb{N}_0\}$ of spaces with *face maps* $p_j : Y_n \longrightarrow Y_{n-1}$, $j = 0, \dots, n$ and *degeneracy maps* $s_j : Y_{n-1} \longrightarrow Y_n$, $j = 0, \dots, n-1$ (commuting with maps to X), satisfying relations derived from those of standard simplices.

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X & Y_0 \arrow[l] & Y_1 \arrow[l, shift left] \arrow[l, shift right] & Y_2 \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & Y_3 \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] & \cdots
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- [Brylinski-McLaughlin]: A *simplicial line bundle* is a line bundle $L \longrightarrow Y_1$ with a trivialization of $\delta L = p_0^* L \otimes p_1^* L^{-1} \otimes p_2^* L$ pulling back to the canonical trivialization of $\delta\delta L$.

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& & & & \downarrow & & \downarrow & & \downarrow \\
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- In case $Y_\bullet$ consists of fiber products $Y^{[\bullet-1]}$ of a locally split map $Y \longrightarrow X$, this recovers the definition of a bundle gerbe.

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Dixmier-Douady class via Čech theory

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$Y \longrightarrow X$ locally split induces a double complex

$$
\begin{array}{ccccccc}
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
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which is vertically *exact*.

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which is vertically *exact*. The Dixmier-Douady class is just a diagram chase:

$$\begin{array}{ccccc}
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[L] & \longrightarrow & 1 & & \\
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Then $DD(L) = [\alpha] \in \check{H}^2(X;\mathbb{C}^*) \cong H^3(X;\mathbb{Z})$.

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Then $DD(L) = [\alpha] \in \check{H}^2(X;\mathbb{C}^*) \cong H^3(X;\mathbb{Z})$. Also Y supports a bundle gerbe with class $[\alpha] \in H^3(X;\mathbb{Z})$ iff $\delta[\alpha] = 0 \in H^3(Y;\mathbb{Z})$.

2-gerbes : $H^4(X; \mathbb{Z})$

- Stevenson: gerbes have pullbacks, trivializations, morphisms, so we can play the same game again.

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- A *bundle 2-gerbe* (L, Z, Y, X) is a "simpicial bundle gerbe"

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  - A coherency condition on pulled back 2-morphisms over $Y^{[5]}$.

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- (L, Z, Y, X) has a well-defined class $DD(L, Z, Y, X) \in H^4(X;\mathbb{Z})$.
- For higher gerbes $(H^{\geq 5}(X;\mathbb{Z}))$, higher and more complicated coherency conditions will appear.
- Y and Z are not on equal footing.

Bigerbes

A nicer version of 2-gerbes:

$$\begin{tikzcd} Z & W_{00} \arrow[l] \arrow[d] \\ X \arrow[from=u] & Y \arrow[l] \end{tikzcd}$$

- Start with $Y \longrightarrow X$ and $Z \longrightarrow X$ fiber bundles (or locally split).
- Take $W \longrightarrow Y$, $W \longrightarrow Z$ fiber bundles forming a commutative square. Minimal choice: $W = Y \times_X Z$, but typically W will be larger. Set $W_{00} = W$.

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- Fill out the diagram by fiber products.

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\begin{tikzcd}
Z^{[3]} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & & & \\
Z^{[2]} \arrow[d, shift left] \arrow[d, shift right] & & & \\
Z \arrow[d] & W_{00} \arrow[l] \arrow[d] & & \\
X & Y \arrow[l] & Y^{[2]} \arrow[l, shift left] \arrow[l, shift right] & Y^{[3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
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\begin{tikzcd}
Z^{[3]} \arrow[d, shift left=1.5] \arrow[d] \arrow[d, shift right=1.5] & W_{02} \arrow[l] \arrow[d, shift left=1.5] \arrow[d] \arrow[d, shift right=1.5] & W_{12} \arrow[l, shift left=0.75] \arrow[l, shift right=0.75] \arrow[d, shift left=1.5] \arrow[d] \arrow[d, shift right=1.5] & W_{22} \arrow[l, shift left=1.5] \arrow[l] \arrow[l, shift right=1.5] \arrow[d, shift left=1.5] \arrow[d] \arrow[d, shift right=1.5] \\
Z^{[2]} \arrow[d, shift left=0.75] \arrow[d, shift right=0.75] & W_{01} \arrow[l] \arrow[d, shift left=0.75] \arrow[d, shift right=0.75] & W_{11} \arrow[l, shift left=0.75] \arrow[l, shift right=0.75] \arrow[d, shift left=0.75] \arrow[d, shift right=0.75] & W_{12} \arrow[l, shift left=1.5] \arrow[l] \arrow[l, shift right=1.5] \arrow[d, shift left=0.75] \arrow[d, shift right=0.75] \\
Z \arrow[d] & W_{00} \arrow[l] \arrow[d] & W_{10} \arrow[l, shift left=0.75] \arrow[l, shift right=0.75] \arrow[d] & W_{20} \arrow[l, shift left=1.5] \arrow[l] \arrow[l, shift right=1.5] \arrow[d] \\
X & Y \arrow[l] & Y^{[2]} \arrow[l, shift left=0.75] \arrow[l, shift right=0.75] & Y^{[3]} \arrow[l, shift left=1.5] \arrow[l] \arrow[l, shift right=1.5]
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Z^{[3]} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & W_{02} \arrow[l] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & W_{12} \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & W_{22} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
Z^{[2]} \arrow[d, shift left] \arrow[d, shift right] & W_{01} \arrow[l] \arrow[d, shift left] \arrow[d, shift right] & W_{11} \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift left] \arrow[d, shift right] & W_{12} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d, shift left] \arrow[d, shift right] \\
Z \arrow[d] & W_{00} \arrow[l] \arrow[d] & W_{10} \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & W_{20} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d] \\
X & Y \arrow[l] & Y^{[2]} \arrow[l, shift left] \arrow[l, shift right] & Y^{[3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}
$$

- Start with $Y \longrightarrow X$ and $Z \longrightarrow X$ fiber bundles (or locally split).
- Take $W \longrightarrow Y$, $W \longrightarrow Z$ fiber bundles forming a commutative square. Minimal choice: $W = Y \times_X Z$, but typically W will be larger. Set $W_{00} = W$.
- Fill out the diagram by fiber products.
- $W_{\bullet\bullet}$ forms a *bisimplicial space over* X.

Bigerbes

$$
\begin{tikzcd}
 & \delta_1^2 L \arrow[dl] & & & & \\
W_{13} \arrow[dd] & & & & & \\
 & \delta_1 L \arrow[dl] & & \delta_0\delta_1 L \arrow[dl] & & \\
W_{12} \arrow[dd] & & W_{22} \arrow[ll] \arrow[dd] & & & \\
 & L \arrow[dl] & & \delta_0 L \arrow[dl] & & \delta_0^2 L \arrow[dl] \\
W_{11} & & W_{21} \arrow[ll] & & W_{31} \arrow[ll] &
\end{tikzcd}
$$

Definition

A bundle bigerbe *is a “bisimplicial line bundle” over $W_{\bullet\bullet}$, i.e., a line bundle L over W_{11}, with trivializations of $\delta_0 L$ and $\delta_1 L$, such that the induced trivializations of $\delta_0\delta_1 L$ agree and which induce the canonical trivializations of $\delta_1^2 L$ and $\delta_0^2 L$.*

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- Products, inverses, pull backs straightforward to define.
- A *trivialization* is an isomorphism $L \cong \delta_0\delta_1 Q$ for a line bundle Q over W_{00}.

Theorem

A bundle bigerbe (L, W, X) *has a well-defined Dixmier-Douady class* $DD(L) \in H^4(X; \mathbb{Z})$*, with*

$$\begin{aligned} DD(L^{-1}) &= -DD(L), \\ DD(L \otimes L') &= DD(L) + DD(L'), \\ DD(f^* L) &= f^* DD(L). \end{aligned}$$

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- Definition generalizes in a straightforward manner to higher degree (Exercise).

Theorem

A bundle multigerbe L of degree n has a well-defined Dixmier-Douady class $DD(L) \in H^{2+n}(X;\mathbb{Z})$*, with*

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Existence: based loop spaces

- Suppose X is connected, and take $Y = \mathcal{P}_* X$, the based path space.
- Then $Y^{[2]} = \mathcal{P}_*^{[2]} X \cong \Omega X$, the based loop space.
- Every class in $H^3(X;\mathbb{Z})$ is represented by a bundle gerbe $(L, \mathcal{P}_* X, X)$, i.e., a simplicial line bundle L on ΩX.

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- Likewise, if X is simply connected, with $Y = Z = \mathcal{P}_*X$ and $W_{00} = \mathcal{P}_*\mathcal{P}_*X$, Then $W_{11} = \Omega^2X$, the double based loop space of X.
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Proposition

If X is k-connected, then every class in $H^{3+k}(X; \mathbb{Z})$ is represented by a multisimplicial line bundle on $\Omega^{2+k} X$.

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- Alternatively, take $Y = \mathcal{P}X$, the free path space, fibering over X^2.
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- Figure-of-eight is yet another simplicial condition "over" the simplicial space

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Transgression and loop-fusion cohomology

Transgression and loop-fusion cohomology

- Take $\alpha \in H^3(X;\mathbb{Z})$ and $L \longrightarrow \mathcal{L}X$ with $DD(L, \mathcal{P}X, X^2) = \alpha$.
- $c_1(L) \in H^2(\mathcal{L}X;\mathbb{Z})$ is the *transgression* of α:

$$\begin{tikzcd} H^k(X;\mathbb{Z}) \arrow[r, "\mathrm{ev}^*"] \arrow[dr, "T"'] & H^k(\mathbb{S}^1 \times \mathcal{L}X;\mathbb{Z}) \arrow[d, "\int_{\mathbb{S}^1}"] \\ & H^{k-1}(\mathcal{L}X;\mathbb{Z}) \end{tikzcd}$$

- Loses information since it forgets the simplicial properties of L.

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Theorem

On $\mathcal{L}^\ell X$ there is a well-defined loop-fusion cohomology $\check{H}^\bullet_{\mathrm{lf}}(\mathcal{L}^\ell X;\mathbb{Z})$ through which iterated transgression factors as an isomorphism:

$$H^k_{\mathrm{lf}}(\mathcal{L}^\ell X;\mathbb{Z}) \xrightarrow{\cong} H^{k-n}_{\mathrm{lf}}(\mathcal{L}^{\ell+n} X;\mathbb{Z}).$$

Spin structures on loop space

- Let X be a spin manifold and $E \longrightarrow X$ the principal $G = \mathrm{Spin}_n$ bundle.
- Then $\mathcal{L}E \longrightarrow \mathcal{L}X$ is a $\mathcal{L}G$ bundle, and $\mathcal{L}G$ has a central extension

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Proposition

The lifting bundle gerbe $(u^*\widehat{\mathcal{L}G}, \mathcal{L}E, \mathcal{L}X)$ is a bundle bigerbe associated to the bisimplicial space generated by

$$\begin{tikzcd} E^2 \arrow[d] & \mathcal{P}E \arrow[l] \arrow[d] \\ X^2 & \mathcal{P}X \arrow[l] \end{tikzcd}$$

with Dixmier-Douady class $\frac{1}{2}p_1(X) \in H^4(X;\mathbb{Z})$.

- c.f. McLaughlin, Redden, Waldorf, K.-Melrose.

Questions and future directions

- Connection structures, representations of differential cohomology.
- On $\mathcal{L}X$ (and generally $\mathcal{L}^k X$), equivariance of L with respect to action of $\mathrm{Diffeo}^+(\mathbb{S}^1)$ (and its central extension) [c.f. Brylinski]. This may have an important role to play in elliptic cohomology theories.
- Loop-fusion K-theory of $\mathcal{L}X$ and $\mathcal{L}^\ell X$.