Calc III: Workshop 12 Solutions, Fall 2017

Problem 1. The electrostatic force (on a positive unit test charge at (x, y, z)) due to a unit point charge at $(0, 0, 0)$ is given by

$$\mathbf{F}(x, y, z) = \frac{1}{(x^2 + y^2 + z^2)^{3/2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

(a) Let S_R be the closed sphere of radius R, with $\mathbf{n}$ oriented outward. Show by direct computation that $\iint_{S_R} \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi$, independent of R.
(b) Using the divergence theorem, show that the flux $\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS$ of $\mathbf{F}$ across any closed surface containing $(0, 0, 0)$ is 4π.

Solution.
(a) We can parameterize the sphere of radius R using spherical coordinates by $\mathbf{r}(\varphi, \theta) = (R \sin\varphi \cos\theta,\ R \sin\varphi \sin\theta,\ R \cos\varphi)$. The normal can be determined either using the cross product formula, or just by geometric reasoning (we know it will be proportional to $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ to be $\mathbf{n} = 1/R(x, y, z)$. Thus $\mathbf{F}\cdot\mathbf{n} = \big(1/R^3(x, y, z)\big)\cdot\big(1/R(x, y, z)\big) = 1/R^2$ is just constant and since $dS = R^2 \sin\varphi\, d\varphi\, d\theta$, we have

$$\iint_{S_R} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_R} \frac{1}{R^2}\, dS = \int_0^{2\pi} \int_0^{\pi} \frac{1}{R^2} R^2 \sin\varphi\, d\varphi\, d\theta = 4\pi,$$

which is independent of R.
(b) Now let Σ be an arbitrary closed surface containing $(0, 0, 0)$. As shown in class, the divergence of $\mathbf{F}$ vanishes where it is defined, namely on $\mathbb{R}^3 \setminus (0, 0, 0)$. We can't use the divergence theorem directly on the region bounded by Σ since it contains $(0, 0, 0)$, but if we consider $S = \Sigma - S_R$ for sufficiently small R, then S is the boundary of a solid region which excludes $(0, 0, 0)$, so

$$\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS - \iint_{S_R} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_E \nabla \cdot \mathbf{F}\, dV = 0$$

or in other words,

$$\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_R} \mathbf{F} \cdot \mathbf{n}\, dS = 4\pi.$$

$\square$

Problem 2. Use the divergence theorem to evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where

$$\mathbf{F}(x, y, z) = z^2x\mathbf{i} + (\tfrac{1}{3}y^3 + \tan z)\mathbf{j} + (x^2z + y^2)\mathbf{k}$$

and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$. Note that S is not a closed surface.

Solution. Since S is not closed, we need to close it, say by adding the unit disk D in the xy-plane. Taking $\mathbf{n}$ to point upward along S and downward along D, we then have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS + \iint_D \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{\partial H} \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_H \nabla \cdot \mathbf{F}\, dV$$

where H is the solid upper hemisphere, and

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(z^2x) + \frac{\partial}{\partial y}(\tfrac{1}{3}y^3 + \tan z) + \frac{\partial}{\partial z}(x^2z + y^2) = x^2 + y^2 + z^2.$$

Thus to compute $\iiint_S \mathbf{F}\cdot\mathbf{n}\,dS$ we need to compute both $\iint_D \mathbf{F}\cdot\mathbf{n}\,dS$ and $\iiint_H (x^2+y^2+z^2)\,dV$. For D, we can see that $\mathbf{n} = -\mathbf{k} = (0,0,-1)$ and $\mathbf{F}(x,y,0) = (*,*,y^2)$ (we don't care about the x or y components since we are going to take the dot product with $-\mathbf{k}$) so

$$\iint_D \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D -y^2\,dA = \int_0^{2\pi}\int_0^1 (r\sin\theta)^2\, r\,dr\,d\theta = -\frac{\pi}{4}.$$

For the triple integral, we can use spherical polar coordinates and write

$$\iiint_H x^2+y^2+z^2\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (\rho^2)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \frac{2\pi}{5}.$$

Putting it all together, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_H \nabla\cdot\mathbf{F}\,dV - \iint_D \mathbf{F}\cdot\mathbf{n}\,dS = \frac{2\pi}{5} + \frac{\pi}{4}.$$

$\square$

Problem 3. Let $\mathbf{v}$ be a constant vector and Σ a closed surface with any orientation. Prove that $\iint_\Sigma \mathbf{v}\cdot\mathbf{n}\,dS = 0$.

Solution. Here $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ is a constant vector, with divergence $\nabla\cdot\mathbf{v} = \frac{\partial}{\partial x}v_1 + \frac{\partial}{\partial y}v_2 + \frac{\partial}{\partial z}v_3 = 0$. Thus taking $\Sigma = \pm\partial E$, where E is a solid region,

$$\iint_\Sigma \mathbf{v}\cdot\mathbf{n}\,dS = \pm\iiint_E 0\,dV = 0.$$

$\square$

Problem 4. Let Σ be the closed surface bounding a solid region E, oriented with outward pointing unit normal. Prove that

$$\iint_\Sigma \tfrac{1}{3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\cdot\mathbf{n}\,dS = \mathrm{Vol}(E).$$

Solution. The divergence of the integrand here is

$$\nabla\cdot\tfrac{(}{x}\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \tfrac{1}{3}\left(\tfrac{\partial x}{\partial x} + \tfrac{\partial y}{\partial y} + \tfrac{\partial z}{\partial z}\right) = 1.$$

Thus by the divergence theorem,

$$\iint_\Sigma \tfrac{1}{3}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\cdot\mathbf{n}\,dS = \iiint_E 1\,dV = \mathrm{Vol}(E).$$

$\square$