Multigerbes

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gerbe

loop spaces

Multigerbes: a new theory of higher gerbes

Chris Kottke Joint work in progress with R. Melrose

New College of Florida

Workshop on Geometric Quantization BIRS, April 2018

Line bundles : $H^2(X; \mathbb{Z})$

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- ▶ A complex line bundle $L \longrightarrow X$ has a Chern class $c_1(L) \in H^2(X; \mathbb{Z})$.
- Naturality:

$$c_1(\underline{\mathbb{C}}) = 0,$$
 $c_1(L \otimes L') = c_1(L) + c_1(L'),$
 $c_1(L^{-1}) = -c_1(L),$ $c_1(f^*L) = f^*c_1(L)$

• $c_1(L) = c_2(L')$ if and only if $L \cong L'$.

Line bundles : $H^2(X; \mathbb{Z})$

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Relation

Relation to loop spaces

- ▶ A complex line bundle $L \longrightarrow X$ has a Chern class $c_1(L) \in H^2(X; \mathbb{Z})$.
- Naturality:

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- $ightharpoonup c_1(L) = c_2(L')$ if and only if $L \cong L'$.
- ▶ Explicit in Čech cohomology: $[L] \in \check{C}^1(X; \mathbb{C}^*)$ satisfies d[L] = 0, unique up to dh, $h \in \check{C}^0(X; \mathbb{C}^*)$, so

$$[L] \in \check{H}^1(X; \mathbb{C}^*) \cong H^2(X; \mathbb{Z}).$$

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▶ Various versions: Giraud, Brylinski, Hitchin and Chattergee, Murray.

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Relation t loop spaces Various versions: Giraud, Brylinski, Hitchin and Chattergee, Murray.

▶ Murray: a bundle gerbe (L, Y, X) is

a locally split map (meaning surjective with local sections)

$$p: Y \longrightarrow X$$
,

a line bundle

$$L \longrightarrow Y^{[2]} = Y \times_X Y = \{(y_1, y_2) : p(y_1) = p(y_2) \in X\}$$

▶ with a product

$$\phi: L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \xrightarrow{\cong} L_{(y_1,y_3)}, \quad (y_1,y_2,y_3) \in Y^{[3]}$$

satisfying associativity:

$$\phi \circ (1 \otimes \phi) = \phi \circ (\phi \otimes 1) : L_{(y_1, y_2)} \otimes L_{(y_2, y_3)} \otimes L_{(y_3, y_4)} \cong L_{(y_1, y_4)},$$
$$(y_1, y_2, y_3, y_4) \in Y^{[4]}$$

▶ (L, Y, X) has a Dixmier Douady class $DD(L, Y, X) \in H^3(X; \mathbb{Z})$.

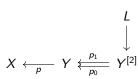
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$$\begin{array}{ccc}
Q & L \\
\downarrow & \downarrow \\
X & \stackrel{p_1}{\longleftarrow} Y^{[2]}
\end{array}$$

▶ Trivialization: an isomorphism $L \cong \delta Q := p_0^* Q \otimes p_1^* Q^{-1}$ for some line bundle $Q \longrightarrow Y$.

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$$X \leftarrow_{p} Y \rightleftharpoons_{p_0}^{p_1} Y^{[2}$$

- ▶ Trivialization: an isomorphism $L \cong \delta Q := p_0^* Q \otimes p_1^* Q^{-1}$ for some line bundle $Q \longrightarrow Y$.
- ► Inverse: $(L, Y, X)^{-1} = (L^{-1}, Y, X)$.
- ▶ Product: $(L, Y, X) \otimes (L', Y', X) = (\pi_1^* L \otimes \pi_2^* L', Y \times_X Y', X)$
- ▶ Pullback: $f^*(L, Y, X) = (f^*L, f^*Y, X')$

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- ▶ Pullback: $f^*(L, Y, X) = (f^*L, f^*Y, X')$
- ▶ Relation with DD class:
 - ▶ DD(L) = 0 if and only if L is trivial.
 - $\triangleright DD(L^{-1}) = -DD(L)$
 - $DD(L \otimes L') = DD(L) + DD(L')$
 - $DD(f^*L) = f^*DD(L).$
 - ▶ DD(L) = DD(L') if and only if L and L' are stably isomorphic, i.e., $L \otimes Q \cong L' \otimes Q'$ for trivial gerbes Q and Q'.
 - ▶ Better: DD(L) = DD(L') if and only if there is a 1-isomorphism (a la Waldorf) $(L, Y, X) \longrightarrow (L', Y', X)$.

Example: lifting bundle gerbes

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 $ightharpoonup E \longrightarrow X$ principal G bundle, where G has a central extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{G} \longrightarrow G \longrightarrow 1$$

- $lackbox{}\widehat{G}\longrightarrow G$ defines an associated line bundle $L=\widehat{G} imes_{\mathrm{U}(1)}\mathbb{C}\longrightarrow G$
- ▶ Difference map $u: E^{[2]} \longrightarrow G$, where $u(y_0, y_1) = g$ such that $y_1 = y_0 g$.
- \blacktriangleright (u^*L, E, X) is the *lifting bundle gerbe* for E.

$$\begin{array}{ccc}
u^*L & L \\
\downarrow & & \downarrow \\
E^{[2]} & \xrightarrow{u} & G \\
\downarrow & & X
\end{array}$$

▶ $DD(u^*L, E, X) \in H^3(X; \mathbb{Z})$ is the obstruction to lifting E to a \widehat{G} bundle $\widehat{E} \longrightarrow X$.

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$$X \longleftarrow Y \longleftarrow Y^{[2]} \biguplus Y^{[3]} \biguplus Y^{[4]} \cdots$$

▶ These higher fiber products define a *simplicial space* over X, i.e., a sequence $\{Y_n = Y^{[n+1]} : n \in \mathbb{N}_0\}$ of spaces with *face maps* $p_j : Y_n \longrightarrow Y_{n-1}, j = 0, \ldots, n$ and *degeneracy maps* $s_j : Y_{n-1} \longrightarrow Y_n, j = 0, \ldots, n-1$ (all commuting with maps to X), satisfying the relations of standard simplices.

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Relation

$$X \longleftarrow Y_0 \longleftarrow Y_1 \longleftarrow Y_2 \longleftarrow Y_3 \cdots$$

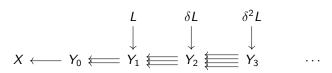
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- ▶ These higher fiber products define a *simplicial space* over X, i.e., a sequence $\{Y_n = Y^{[n+1]} : n \in \mathbb{N}_0\}$ of spaces with *face maps* $p_j : Y_n \longrightarrow Y_{n-1}, j = 0, \dots, n$ and *degeneracy maps* $s_j : Y_{n-1} \longrightarrow Y_n, j = 0, \dots, n-1$ (all commuting with maps to X), satisfying the relations of standard simplices.
- ▶ [Brylinski-McLaughlin]: A simplicial line bundle is a line bundle $L \longrightarrow Y_1$ with a trivialization of $\delta L = p_0^* L \otimes p_1^* L^{-1} \otimes p_2^* L$ pulling back to the canonical trivialization of $\delta^2 L$.

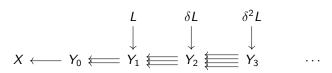
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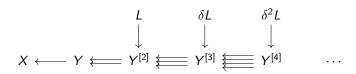
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- ▶ In case Y_{\bullet} consists of fiber products $Y^{[\bullet-1]}$ of a locally split map $Y \longrightarrow X$, this precisely recovers the definition of a bundle gerbe.

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Relation t



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This double complex is vertically *exact* for $Y \longrightarrow X$ locally split. In particular the total cohomology is isomorphic to $\check{H}^{\bullet}(X; \mathbb{C}^*)$.

▶ The Čech chains are with respect to pairs of "admissible covers" of (X, Y) to which $p: Y \longrightarrow X$ and its local sections are adapted, including higher intersections.

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$$\overset{\delta\uparrow}{\check{C}^{0}}(Y^{[2]};\mathbb{C}^{*}) \overset{\delta\uparrow}{\to} \overset{\delta\uparrow}{\check{C}^{1}}(Y^{[2]};\mathbb{C}^{*}) \overset{d}{\to} \check{C}^{2}(Y^{[2]};\mathbb{C}^{*}) \overset{d}{\to} \cdots$$

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- ▶ The Čech chains are with respect to pairs of "admissible covers" of (X,Y) to which $p:Y\longrightarrow X$ and its local sections are adapted, including higher intersections.
- ▶ Fiber products of the pair of covers give covers of $Y^{[k]}$ for k > 1.

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- ▶ Fiber products of the pair of covers give covers of $Y^{[k]}$ for k > 1.
- ▶ The local sections of *p* induce chain homotopy contractions of each vertical complex.

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- ▶ The Čech chains are with respect to pairs of "admissible covers" of (X, Y) to which $p: Y \longrightarrow X$ and its local sections are adapted, including higher intersections.
- ▶ Fiber products of the pair of covers give covers of $Y^{[k]}$ for k > 1.
- ▶ The local sections of *p* induce chain homotopy contractions of each vertical complex.
- ► Take the direct limit over all admissible pairs of covers. Over X and Y this is equivalent to the direct limit over all covers.

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$$\overset{\delta\uparrow}{\check{C}^{0}}(Y^{[2]};\mathbb{C}^{*}) \overset{\delta}{\to} \overset{\delta\uparrow}{\check{C}^{1}}(Y^{[2]};\mathbb{C}^{*}) \overset{d}{\to} \overset{\delta\uparrow}{\check{C}^{2}}(Y^{[2]};\mathbb{C}^{*}) \overset{d}{\to} \cdots$$

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The Dixmier-Douady class is the image in $\check{H}^2(X;\mathbb{C}^*)$ of the pure cocycle $-[L] \in \check{C}^1(Y^{[2]};\mathbb{C}^*) \subset \check{C}^{\bullet}(Y^{[\bullet]};\mathbb{C}^*)$ in $\check{H}^2(X;\mathbb{C}^*)$.

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The Dixmier-Douady class is the image in $\check{H}^2(X; \mathbb{C}^*)$ of the pure cocycle $-[L] \in \check{C}^1(Y^{[2]}; \mathbb{C}^*) \subset \check{C}^{\bullet}(Y^{[\bullet]}; \mathbb{C}^*)$ in $\check{H}^2(X; \mathbb{C}^*)$.

$$\begin{array}{c}
0\\
\uparrow\\
-[L] \longrightarrow 0\\
\uparrow\\
\beta \longrightarrow d\beta \rightarrow 0\\
\uparrow\\
\alpha \longrightarrow 0
\end{array}$$

so
$$DD(L) = [\alpha] \in \check{H}^2(X; \mathbb{C}^*) \cong H^3(X; \mathbb{Z}).$$

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The Dixmier-Douady class is the image in $\check{H}^2(X; \mathbb{C}^*)$ of the pure cocycle $-[L] \in \check{C}^1(Y^{[2]}; \mathbb{C}^*) \subset \check{C}^{\bullet}(Y^{[\bullet]}; \mathbb{C}^*)$ in $\check{H}^2(X; \mathbb{C}^*)$.

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so $DD(L) = [\alpha] \in \check{H}^2(X; \mathbb{C}^*) \cong H^3(X; \mathbb{Z})$. It also follows that Y supports a bundle gerbe with class $[\alpha] \in H^3(X; \mathbb{Z})$ iff $p^*[\alpha] = 0 \in H^3(Y; \mathbb{Z})$.

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Higher gerbes

Relation t loop spaces ► Stevenson: gerbes have pullbacks, trivializations, morphisms, so we can play the same game again.

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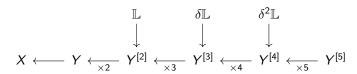
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loop spaces Stevenson: gerbes have pullbacks, trivializations, morphisms, so we can play the same game again.

▶ A bundle 2-gerbe (L, Z, Y, X) is a "simpicial bundle gerbe"



- ▶ A locally split map $Y \longrightarrow X$,
- ightharpoonup A gerbe $\mathbb{L} = (L, Z, Y^{[2]}),$
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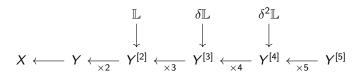
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- ► A 2-morphism (did I mention gerbes have 2-morphisms?) relating the induced trivialization of $\delta^2 \mathbb{L}$ to the canonical one,
- A coherency condition on pulled back 2-morphisms over Y^[5].

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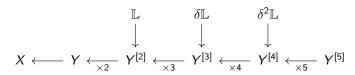
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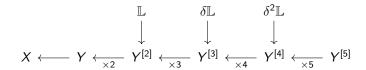
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gerbes Relation Stevenson: gerbes have pullbacks, trivializations, morphisms, so we can play the same game again.

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- ▶ (L, Z, Y, X) has a well-defined characteristic class $C(L, Z, Y, X) \in H^4(X; \mathbb{Z})$.
- ▶ For higher gerbes $(H^{\geq 5}(X; \mathbb{Z}))$, higher and more complicated coherency conditions will appear.
- ▶ The roles of *Y* and *Z* are very asymmetric.

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Higher gerbes

loop spaces A new version of 2-gerbes:



▶ Start with $Y \longrightarrow X$ and $Z \longrightarrow X$ locally split.

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- ▶ Start with $Y \longrightarrow X$ and $Z \longrightarrow X$ locally split.
- ▶ Take $W \longrightarrow Y$, $W \longrightarrow Z$ locally split forming a commutative square. Minimal choice: $W = Y \times_X Z$, but typically W will be larger.

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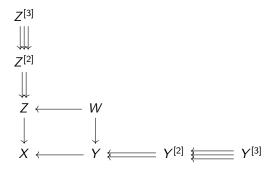
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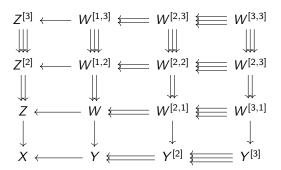
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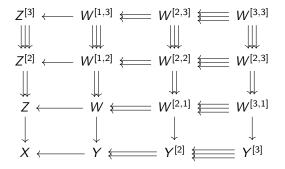
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- ▶ Fill out the diagram by fiber products.
- $\blacktriangleright W^{[\bullet,\bullet]}$ forms a bisimplicial space over X.

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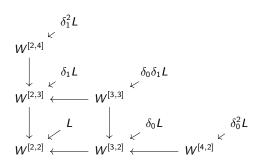
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Definition

A bundle bigerbe is a "bisimplicial line bundle" over $W^{[\bullet,\bullet]}$, i.e., a line bundle L over $W^{[2,2]}$, with trivializations of $\delta_0 L$ and $\delta_1 L$, such that the induced trivializations of $\delta_0 \delta_1 L$ agree and which induce the canonical trivializations of $\delta_1^2 L$ and $\delta_0^2 L$.

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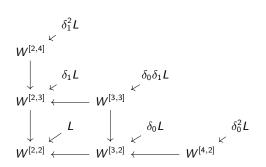
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Definition

A bundle bigerbe is a "bisimplicial line bundle" over $W^{[\bullet,\bullet]}$, i.e., a line bundle L over $W^{[2,2]}$, with trivializations of $\delta_0 L$ and $\delta_1 L$, such that the induced trivializations of $\delta_0 \delta_1 L$ agree and which induce the canonical trivializations of $\delta_1^2 L$ and $\delta_0^2 L$.

- ▶ Products, inverses, pull backs straightforward to define.
- ▶ A trivialization is an isomorphism $L \cong \delta_1 Q$ (equivalently $L \cong \delta_0 Q'$) for a line bundle Q over $W^{[1,2]}$ (Q' over $W^{[2,1]}$).

Theorem

A bundle bigerbe (L, W, X) has a well-defined characteristic class $C(L) \in H^4(X; \mathbb{Z})$, with

$$C(L^{-1}) = -C(L),$$

$$C(L \otimes L') = C(L) + C(L'),$$

$$C(f^*L) = f^*C(L).$$

C(L) = 0 if and only if L is trivial. C(L) = C(L') if and only if L and L' are stably isomorphic.

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► This generalizes in a straightforward manner to higher degree (Exercise), leading to *bundle multigerbes*.

Theorem

A bundle multigerbe L of degree n has a well-defined characteristic class $C(L) \in H^{2+n}(X; \mathbb{Z})$, with

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Bigerbes and Čech cohomology

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Gerbes Higher

gerbes Relatio Taking Čech cochains with respect to a certain class of covers, we obtain a triple complex $(\check{C}^{\bullet}(W^{[\bullet,\bullet]};\mathbb{C}^*),d,\delta_0,\delta_1)$.

▶ The simplicial complexes $(\check{C}^p(W^{[\bullet,q]};\mathbb{C}^*),\delta_0)$ and $(\check{C}^p(W^{[q,\bullet]};\mathbb{C}^*),\delta_1)$ are exact; in fact they admit chain homotopy contractions (commuting with each other, but not with the Čech differential).

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- ► The tototal cohomology of $(\check{C}^{\bullet}(W^{[\bullet,\bullet]};\mathbb{C}^*),d,\delta_0,\delta_1)$ is isomorphic to $\check{H}^{\bullet}(X;\mathbb{C}^*)$.

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 $[L] \in \check{C}^1(W^{[2,2]}; \mathbb{C}^*).$

a triple complex $(\check{C}^{\bullet}(W^{[\bullet,\bullet]};\mathbb{C}^*),d,\delta_0,\delta_1)$.

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$\check{C}^p(Z^{[3]}) \stackrel{\delta_0}{\rightarrow} \check{C}^p(W^{[1,3]}) \stackrel{\delta_0}{\rightarrow} \check{C}^p(W^{[2,3]}) \stackrel{\delta_0}{\rightarrow} \check{C}^p(W^{[3,3]})$ $\check{C}^p(Z^{[2]}) \xrightarrow{\delta_0} \check{C}^p(W^{[1,2]}) \xrightarrow{\delta_0} \check{C}^p(W^{[2,2]}) \xrightarrow{\delta_0} \check{C}^p(W^{[3,2]})$ $\begin{array}{ccc} \delta_1 \uparrow & \delta_1 \uparrow & \delta_1 \uparrow & \delta_1 \uparrow \\ \check{C}^p(Z) \xrightarrow{\delta_0} \check{C}^p(W^{[1,1]}) \xrightarrow{\delta_0} \check{C}^p(W^{[2,1]}) \xrightarrow{\delta_0} \check{C}^p(W^{[3,1]}) \end{array}$ $\check{C}^p(X) \xrightarrow{\delta_0} \check{C}^p(Y) \xrightarrow{\delta_0} \check{C}^p(Y^{[2]}) \xrightarrow{\delta_0} \check{C}^p(Y^{[3]})$ ▶ The simplicial complexes $(\check{C}^p(W^{[\bullet,q]};\mathbb{C}^*),\delta_0)$ and $(\check{C}^p(W^{[q,\bullet]};\mathbb{C}^*),\delta_1)$ are exact; in fact they admit chain homotopy contractions (commuting with each other, but not with the Čech differential). ▶ The tototal cohomology of $(\check{C}^{\bullet}(W^{[\bullet,\bullet]};\mathbb{C}^*),d,\delta_0,\delta_1)$ is isomorphic

 $ightharpoonup C(L) \in \check{H}^3(X; \mathbb{C}^*) \cong H^4(X; \mathbb{Z})$ is the image of the pure cocycle

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Relation to spaces

- ▶ Suppose *X* is connected, and take $Y = \mathcal{P}_*X$, the based path space.
- ▶ Then $Y^{[2]} = \mathcal{P}_*^{[2]} X \cong \Omega X$, the based loop space.
- Every class in $H^3(X; \mathbb{Z})$ is represented by a bundle gerbe (L, \mathcal{P}_*X, X) , i.e., a simplicial line bundle L on ΩX .

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- Likewise, if X is simply connected, with $Y=Z=\mathcal{P}_*X$ and $W=\mathcal{P}_*\mathcal{P}_*X$, Then $W^{[2,2]}=\Omega^2X$, the double based loop space of X.
- Every class in $H^4(X;\mathbb{Z})$ is represented by a bundle bigerbe $(L,\mathcal{P}_*X,\mathcal{P}_*\mathcal{P}_*X,X)$, equivalently a "doubly fusion" line bundle $L\longrightarrow \Omega^2X$. (c.f. Carey Johnson Murray Stevenson Wang)

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Proposition

If X is k-connected, then every class in $H^{3+k}(X; \mathbb{Z})$ is represented by a multigerbe on $\Omega^{2+k}X$, (aka a 2+k-fold fusion line bundle).

Existence: free loop spaces

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Relation to loop spaces

- ▶ Alternatively, take $Y = \mathcal{P}X$, the free path space, fibering over X^2 .
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- ▶ Every class in $H^3(X; \mathbb{Z})$ is represented by a bundle gerbe $L = (L, \mathcal{P}X, X^2)$ on $\mathcal{L}X$, with the additional condition of a trivialization of the alternating product of pullbacks to the "figure-of-eight" loop space [K-Melrose, 2013].
- ► Figure-of-eight is yet another simplicial condition "over" the simplicial space

$$X \longleftarrow X^2 \longleftarrow X^3$$

guaranteeing that the class in $H^3(X^2; \mathbb{Z})$ comes from $H^3(X; \mathbb{Z})$.

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Proposition

Every class in $H^{3+k}(X;\mathbb{Z})$ is represented by a multisimplicial (and multi figure-of-eight) line bundle on $\mathcal{L}^{2+k}X$.

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Relation to loop spaces

Multigerbes

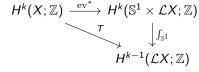
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Highe gerbe

Relation to loop spaces

- ▶ Take $\alpha \in H^3(X; \mathbb{Z})$ and $L \longrightarrow \mathcal{L}X$ with $DD(L, \mathcal{P}X, X^2) = \alpha$.
- ▶ $c_1(L) \in H^2(\mathcal{L}X; \mathbb{Z})$ is the *transgression* of α :



▶ Loses information since it forgets the simplicial properties of *L*.

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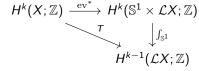
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- ► [K.-Melrose, 2015]: "Loop-fusion" Čech cohomology $H^{\bullet}_{lf}(\mathcal{L}X;\mathbb{Z})$ such that transgression factors through an isomorphism

$$H^k(X;\mathbb{Z}) \stackrel{\cong}{\longrightarrow} H^{k-1}_{\mathrm{lf}}(\mathcal{L}X;\mathbb{Z}) \longrightarrow H^{k-1}(\mathcal{L}X;\mathbb{Z}).$$

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$$H^{k}(X;\mathbb{Z}) \xrightarrow{\operatorname{ev}^{*}} H^{k}(\mathbb{S}^{1} \times \mathcal{L}X;\mathbb{Z})$$

$$\downarrow^{\int_{\mathbb{S}^{1}}} H^{k-1}(\mathcal{L}X;\mathbb{Z})$$

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Theorem

On $\mathcal{L}^{\ell}X$ there is a well-defined loop-fusion cohomology $\check{H}^{\bullet}_{lf}(\mathcal{L}^{\ell}X;\mathbb{Z})$ through which iterated transgression factors as an isomorphism:

$$H^k_{\mathrm{lf}}(\mathcal{L}^{\ell}X;\mathbb{Z}) \stackrel{\cong}{\longrightarrow} H^{k-n}_{\mathrm{lf}}(\mathcal{L}^{\ell+n}X;\mathbb{Z}).$$

Chern-Simons 2-gerbe as a bigerbe

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gerbe

Relation to loop spaces ▶ Let X be a manifold and $E \longrightarrow X$ a principal G bundle for a simple, simply connected Lie group G (e.g. $G = \operatorname{Spin}$).

▶ Then $\mathcal{L}E \longrightarrow \mathcal{L}X$ is a $\mathcal{L}G$ bundle, and $\mathcal{L}G$ has a central extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\mathcal{L}G} \longrightarrow \mathcal{L}G \longrightarrow 1$$

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Proposition

The lifting bundle gerbe $(u^*\widehat{\mathcal{LG}},\mathcal{LE},\mathcal{LX})$ is a bundle bigerbe associated to the bisimplicial space generated by

$$E^2 \longleftarrow \mathcal{P}E$$

$$\downarrow \qquad \qquad \downarrow$$

$$X^2 \longleftarrow \mathcal{P}X$$

with Dixmier-Douady class $\frac{1}{2}p_1(E) \in H^4(X; \mathbb{Z})$.

c.f. McLaughlin, Redden, CJMSW, Waldorf, K.-Melrose.

Questions and future directions

Multigerbes

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High

Relation to loop spaces

- ► Connection structures, representations of differential cohomology when *X*, *Y*, *Z*, *W* are manifolds.
- ▶ Satisfactory notion of morphisms for multigerbes.
- ▶ On $\mathcal{L}X$ (and generally \mathcal{L}^kX), equivariance of L with respect to action of $\operatorname{Diffeo}^+(\mathbb{S}^1)$ (and its central extension) [c.f. Brylinski].
- ▶ Loop-fusion K-theory of $\mathcal{L}X$ and $\mathcal{L}^{\ell}X$.