Calc III: Workshop 10 Solutions, Fall 2017

Problem 1. Let $R = \{(x, y) : x^2 + y^2 \le a^2, y \ge 0\}$ be the upper half disk of radius a, with mass density $\delta(x, y) = x^2 + y^2$.
(a) Compute the mass of R.
(b) Compute the center of mass $(\overline{x}, \overline{y})$ of R.

Solution.
(a) The mass is given by $\iint_R \delta(x, y)\, dA$. It is easiest to parameterize R in polar coordinates, by $0 \le r \le a$ and $0 \le \theta \le \pi$. Then since $\delta(x, y) = x^2 + y^2 = r^2$, we have

$$\begin{aligned} M &= \iint_R \delta(x, y)\, dA \\ &= \int_0^\pi \int_0^a (r^2)\, r\, dr\, d\theta \\ &= \frac{\pi a^4}{4} \end{aligned}$$

(b) By symmetry of both R and δ with respect to reflection about the y axis, it follows that $\overline{x} = 0$. To compute $\overline{y}$, we have

$$\begin{aligned} \overline{y} &= \frac{1}{M} \iint_R y\, \delta(x, y)\, dA \\ &= \frac{1}{M} \int_0^\pi \int_0^a r \sin\theta (r^2)\, r\, dr\, d\theta \\ &= \frac{1}{M} \int_0^\pi \sin\theta\, d\theta \int_0^a r^4\, dr \\ &= \frac{1}{M} \frac{2a^5}{5} \\ &= \frac{4}{\pi a^4} \frac{2a^5}{5} = \frac{8a}{5\pi}. \end{aligned}$$

□

Problem 2. Use Green's Theorem to compute the line integral $\oint_C \mathbf{F} \cdot \mathbf{T}\, ds$, where $\mathbf{F}(x, y) = (x^2+y^2)\mathbf{i} + (2xy+x)\mathbf{j}$ and C is the closed triangular path consisting of straight line segments from $(0, 0)$ to $(1, 1)$, then to $(1, 0)$ and back to $(0, 0)$.

Solution. Here $P(x, y) = x^2 + y^2$ and $Q(x, y) = (2xy + x)$, so

$$Q_x - P_y = (2y + 1) - (2y) = 1.$$

The closed curve C traverses the boundary ∂R of the solid triangle, but in the opposite direction, i.e., $C = -\partial R$. Thus by Green's Theorem,

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = -\oint_{\partial R} \mathbf{F} \cdot \mathbf{T}\, ds = -\iint_R Q_x - P_y\, dA = -\iint_R dA = -\text{Area}(R) = -\frac{1}{2}.$$

□

Problem 3. Compute the volume of the solid region between the surfaces $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$

Solution. Cylindrical coordinates are best for this; then the surfaces have the form $z = r^2$ and $z = 8 - r^2$, respectively. The region is given by the limits $r^2 \leq z \leq 8 - r^2$, $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$ (the upper limit in r is given by the intersection of the two paraboloids: set $z = r^2 = 8 - r^2$ and solve for $r = 2$). Thus the volume is

$$\begin{aligned}
\text{Vol} &= \iiint_E dV \\
&= \int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} r \, dz \, dr \, d\theta \\
&= 2\pi \int_0^2 r\big((8 - r^2) - (r^2)\big) \, dr \\
&= 2\pi \int_0^2 8r - 2r^3 \, dr \\
&= 2\pi \left(\frac{8(2^2)}{2} - \frac{2(2^4)}{4} \right) \\
&= 16\pi.
\end{aligned}$$

□

Problem 4. Set up, but do not evaluate, the triple integral $\iiint_E xy \, dV$, where E is the region bounded below by the cone $z = \sqrt{3(x^2 + y^2)}$ and above by the sphere $x^2 + y^2 + z^2 = 4$, using:

(a) Cartesian coordinates (x, y, z),
(b) Cylindrical coordinates (z, r, θ), and
(c) Spherical coordinates (ρ, φ, θ).

Solution. The two surfaces meet where $z = \sqrt{3(x^2 + y^2)} = \sqrt{4 - x^2 - y^2}$, which gives the circle $x^2 + y^2 = 1$ lying in the plane $z = \sqrt{3}$.

(a) In cartesian coordinates, if we do z first, the integral becomes

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{\sqrt{3(x^2+y^2)}}^{\sqrt{4-x^2-y^2}} xy \, dz \, dy \, dx$$

(b) In cylindrical coordinates, it becomes

$$\int_0^{2\pi} \int_0^1 \int_{\sqrt{3}r}^{\sqrt{4-r^2}} (r\cos\theta)(r\sin\theta) \, r \, dz \, dr \, d\theta.$$

(c) Finally, in spherical coordinates we note that the cone $z = \sqrt{3(x^2 + y^2)} = (\sqrt{3})r$ is described by the constant angle $\varphi = \pi/6$ (there is a 30-60-90 triangle in z and r). The other limits are $0 \leq \rho \leq 2$ and $0 \leq \theta \leq 2\pi$, so we have

$$\int_0^{2\pi} \int_0^{\pi/6} \int_0^2 (\rho\sin\varphi\cos\theta)(\rho\sin\varphi\sin\theta)\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$\square$

Problem 5.

(a) Verify that the vector field $\mathbf{F}(x,y) = (2xy + ye^x)\mathbf{i} + (x^2 + e^x)\mathbf{j}$ is conservative, and find a potential function $f(x,y)$.

(b) Compute the line integral $\int_C \mathbf{F}(x,y) \cdot \mathbf{T}\, ds$, where C is the curve $y = 1 + x^2$ from $(0,1)$ to $(1,2)$.

Solution.

(a) We compute

$$Q_x - P_y = \tfrac{\partial}{\partial x}(x^2 + e^x) - \tfrac{\partial}{\partial y}(2xy + ye^x) = (2x + e^x) - (2x + e^x) = 0$$

which vanishes on all of $\mathbb{R}^2$, so $\mathbf{F}$ must be conservative. To find a potential function we need to solve $\nabla f = \mathbf{F}$, or

$$f_x(x,y) = 2xy + ye^x$$
$$f_y(x,y) = x^2 + e^x.$$

Integrating the first equation with respect to x gives $f(x,y) = x^2y + ye^x + c(y)$, where $c(y)$ is an unknown function of y. Plugging this into the second equation gives

$$f_y(x,y) = x^2 + e^x + c'(y) = x^2 + e^x$$

so $c'(y) = 0$ and therefore $c(y)$ is a constant, which we can take to be 0. Thus a potential function is given by $f(x,y) = x^2y + ye^x$.

(b) By the fundamental theorem for line integrals,

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \nabla f \cdot \mathbf{T}\, ds = f(1,2) - f(0,1) = (2 + 2e) - (0 + 1) = 1 + 2e.$$

$\square$