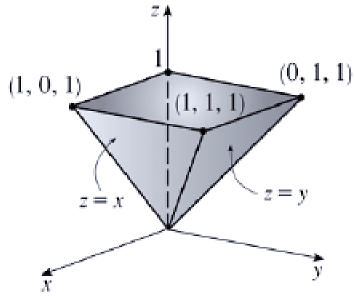
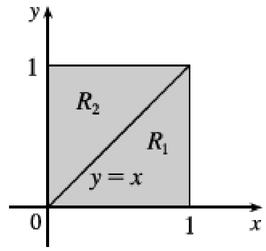
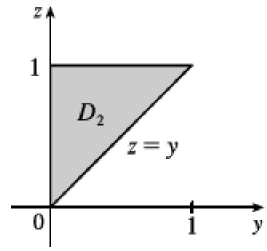
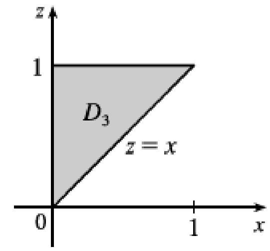
36.









$\int_0^1 \int_y^1 \int_0^z f(x,y,z)\,dx\,dz\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le x \le z, y \le z \le 1, 0 \le y \le 1\}$.

Notice that E is bounded below by two different surfaces, so we must split the projection of E onto the xy-plane into two regions as in the second diagram. If D_1, D_2, and D_3 are the projections of E on the xy-, yz- and xz-planes then

$$\begin{aligned} D_1 = R_1 \cup R_2 &= \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\} \cup \{(x,y) \mid 0 \le x \le 1, x \le y \le 1\} \\ &= \{(x,y) \mid 0 \le y \le 1, y \le x \le 1\} \cup \{(x,y) \mid 0 \le y \le 1, 0 \le x \le y\}, \end{aligned}$$

$D_2 = \{(y,z) \mid 0 \le y \le 1, y \le z \le 1\} = \{(y,z) \mid 0 \le z \le 1, 0 \le y \le z\}$, and

$D_3 = \{(x,z) \mid 0 \le x \le 1, x \le z \le 1\} = \{(x,z) \mid 0 \le z \le 1, 0 \le x \le z\}$.

Thus we also have

$$\begin{aligned} E &= \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, x \le z \le 1\} \cup \{(x,y,z) \mid 0 \le x \le 1, x \le y \le 1, y \le z \le 1\} \\ &= \{(x,y,z) \mid 0 \le y \le 1, y \le x \le 1, x \le z \le 1\} \cup \{(x,y,z) \mid 0 \le y \le 1, 0 \le x \le y, y \le z \le 1\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, 0 \le y \le z, 0 \le x \le z\} = \{(x,y,z) \mid 0 \le x \le 1, x \le z \le 1, 0 \le y \le z\} \\ &= \{(x,y,z) \mid 0 \le z \le 1, 0 \le x \le z, 0 \le y \le z\}. \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 \int_y^1 \int_0^z f(x,y,z)\,dx\,dz\,dy &= \int_0^1 \int_0^x \int_x^1 f(x,y,z)\,dz\,dy\,dx + \int_0^1 \int_x^1 \int_y^1 f(x,y,z)\,dz\,dy\,dx \\ &= \int_0^1 \int_y^1 \int_x^1 f(x,y,z)\,dz\,dx\,dy + \int_0^1 \int_0^y \int_y^1 f(x,y,z)\,dz\,dx\,dy \\ &= \int_0^1 \int_0^z \int_0^z f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_x^1 \int_0^z f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^1 \int_0^z \int_0^z f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

19.
The paraboloid $z = 4 - x^2 - y^2 = 4 - r^2$ intersects the xy-plane in the circle $x^2 + y^2 = 4$ or $r^2 = 4 \quad \Rightarrow \quad r = 2$, so in cylindrical coordinates, E is given by $\left\{(r, \theta, z) \,\middle|\, 0 \le \theta \le \pi/2, 0 \le r \le 2, 0 \le z \le 4 - r^2\right\}$. Thus

$$\iiint_E (x + y + z)\, dV =$$

$$\begin{aligned}
&= \int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)\, r\, dz\, dr\, d\theta = \int_0^{\pi/2} \int_0^2 \left[r^2(\cos\theta + \sin\theta)z + \tfrac{1}{2} r z^2\right]_{z=0}^{z=4-r^2} dr\, d\theta \\
&= \int_0^{\pi/2} \int_0^2 \left[(4r^2 - r^4)(\cos\theta + \sin\theta) + \tfrac{1}{2} r (4 - r^2)^2\right] dr\, d\theta \\
&= \int_0^{\pi/2} \left[\left(\tfrac{4}{3} r^3 - \tfrac{1}{5} r^5\right)(\cos\theta + \sin\theta) - \tfrac{1}{12}(4 - r^2)^3\right]_{r=0}^{r=2} d\theta \\
&= \int_0^{\pi/2} \left[\tfrac{64}{15}(\cos\theta + \sin\theta) + \tfrac{16}{3}\right] d\theta = \left[\tfrac{64}{15}(\sin\theta - \cos\theta) + \tfrac{16}{3}\theta\right]_0^{\pi/2} \\
&= \tfrac{64}{15}(1 - 0) + \tfrac{16}{3} \cdot \tfrac{\pi}{2} - \tfrac{64}{15}(0 - 1) - 0 = \tfrac{8}{3}\pi + \tfrac{128}{15}
\end{aligned}$$

20.
In cylindrical coordinates E is bounded by the planes $z = 0$, $z = r\cos\theta + r\sin\theta + 5$ and the cylinders $r = 2$ and $r = 3$, so E is given by $\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 2 \le r \le 3, 0 \le z \le r\cos\theta + r\sin\theta + 5\}$. Thus

$$\begin{aligned}
\iiint_E x\, dV &= \int_0^{2\pi} \int_2^3 \int_0^{r\cos\theta + r\sin\theta + 5} (r\cos\theta)\, r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_2^3 (r^2\cos\theta)[\,z\,]_{z=0}^{z=r\cos\theta + r\sin\theta + 5} dr\, d\theta \\
&= \int_0^{2\pi} \int_2^3 (r^2\cos\theta)(r\cos\theta + r\sin\theta + 5)\, dr\, d\theta = \int_0^{2\pi} \int_2^3 (r^3(\cos^2\theta + \cos\theta\sin\theta) + 5r^2\cos\theta)\, dr\, d\theta \\
&= \int_0^{2\pi} \left[\tfrac{1}{4} r^4(\cos^2\theta + \cos\theta\sin\theta) + \tfrac{5}{3} r^3\cos\theta\right]_{r=2}^{r=3} d\theta \\
&= \int_0^{2\pi} \left[\left(\tfrac{81}{4} - \tfrac{16}{4}\right)(\cos^2\theta + \cos\theta\sin\theta) + \tfrac{5}{3}(27 - 8)\cos\theta\right] d\theta \\
&= \int_0^{2\pi} \left(\tfrac{65}{4}\left(\tfrac{1}{2}(1 + \cos 2\theta) + \cos\theta\sin\theta\right) + \tfrac{95}{3}\cos\theta\right) d\theta = \left[\tfrac{65}{8}\theta + \tfrac{65}{16}\sin 2\theta + \tfrac{65}{8}\sin^2\theta + \tfrac{95}{3}\sin\theta\right]_0^{2\pi} = \tfrac{65}{4}\pi
\end{aligned}$$

25. (a) The paraboloids intersect when $x^2 + y^2 = 36 - 3x^2 - 3y^2 \quad \Rightarrow \quad x^2 + y^2 = 9$, so the region of integration is $D = \left\{(x, y) \mid x^2 + y^2 \le 9\right\}$. Then, in cylindrical coordinates,

$E = \left\{(r, \theta, z) \mid r^2 \le z \le 36 - 3r^2, 0 \le r \le 3, 0 \le \theta \le 2\pi\right\}$ and

$$V = \int_0^{2\pi} \int_0^3 \int_{r^2}^{36-3r^2} r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^3 \left(36r - 4r^3\right) dr\, d\theta = \int_0^{2\pi} \left[18r^2 - r^4\right]_{r=0}^{r=3} d\theta = \int_0^{2\pi} 81\, d\theta = 162\pi.$$

(b) For constant density K, $m = KV = 162\pi K$ from part (a). Since the region is homogeneous and symmetric, $M_{yz} = M_{xz} = 0$ and

$$\begin{aligned}
M_{xy} &= \int_0^{2\pi} \int_0^3 \int_{r^2}^{36-3r^2} (zK)\, r\, dz\, dr\, d\theta = K \int_0^{2\pi} \int_0^3 r\left[\tfrac{1}{2} z^2\right]_{z=r^2}^{z=36-3r^2} dr\, d\theta \\
&= \tfrac{K}{2} \int_0^{2\pi} \int_0^3 r\left((36 - 3r^2)^2 - r^4\right) dr\, d\theta = \tfrac{K}{2} \int_0^{2\pi} d\theta \int_0^3 (8r^5 - 216r^3 + 1296r)\, dr \\
&= \tfrac{K}{2}(2\pi)\left[\tfrac{8}{6} r^6 - \tfrac{216}{4} r^4 + \tfrac{1296}{2} r^2\right]_0^3 = \pi K(2430) = 2430\pi K
\end{aligned}$$

Thus $(\overline{x}, \overline{y}, \overline{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right) = \left(0, 0, \frac{2430\pi K}{162\pi K}\right) = (0, 0, 15).$

29. The region of integration is the region above the cone $z = \sqrt{x^2 + y^2}$, or $z = r$, and below the plane $z = 2$. Also, we have $-2 \le y \le 2$ with $-\sqrt{4 - y^2} \le x \le \sqrt{4 - y^2}$ which describes a circle of radius 2 in the xy-plane centered at $(0, 0)$. Thus,

$$\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} xz\, dz\, dx\, dy = \int_0^{2\pi} \int_0^2 \int_r^2 (r \cos \theta)\, z\, r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^2 \int_r^2 r^2 (\cos \theta)\, z\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 r^2 (\cos \theta) \left[\tfrac{1}{2} z^2\right]_{z=r}^{z=2} dr\, d\theta = \tfrac{1}{2} \int_0^{2\pi} \int_0^2 r^2 (\cos \theta) \left(4 - r^2\right)\, dr\, d\theta$$

$$= \tfrac{1}{2} \int_0^{2\pi} \cos \theta\, d\theta \int_0^2 \left(4r^2 - r^4\right)\, dr = \tfrac{1}{2} \left[\sin \theta\right]_0^{2\pi} \left[\tfrac{4}{3} r^3 - \tfrac{1}{5} r^5\right]_0^2 = 0$$

30. The region of integration is the region above the plane $z = 0$ and below the paraboloid $z = 9 - x^2 - y^2$. Also, we have $-3 \le x \le 3$ with $0 \le y \le \sqrt{9 - x^2}$ which describes the upper half of a circle of radius 3 in the xy-plane centered at $(0, 0)$. Thus,

$$\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2 + y^2}\, dz\, dy\, dx = \int_0^{\pi} \int_0^3 \int_0^{9-r^2} \sqrt{r^2}\, r\, dz\, dr\, d\theta = \int_0^{\pi} \int_0^3 \int_0^{9-r^2} r^2\, dz\, dr\, d\theta$$

$$= \int_0^{\pi} \int_0^3 r^2 \left(9 - r^2\right) dr\, d\theta = \int_0^{\pi} d\theta \int_0^3 \left(9r^2 - r^4\right) dr$$

$$= \left[\theta\right]_0^{\pi} \left[3r^3 - \tfrac{1}{5} r^5\right]_0^3 = \pi \left(81 - \tfrac{243}{5}\right) = \tfrac{162}{5} \pi$$