Calc III: Workshop 6 Solutions, Fall 2018

Problem 1. In evaluating a double integral over a region D, a sum of iterated integrals was obtained as follows:

$$\iint_D f(x,y)\,dA = \int_0^1 \int_0^{2y} f(x,y)\,dx\,dy + \int_1^3 \int_0^{3-y} f(x,y)\,dx\,dy.$$

Sketch the region D and reverse the order of integration.

Solution. The region is a triangle with vertices at $(0,0)$, $(2,1)$, and $(0,3)$. The reversed order integral is

$$\int_0^2 \int_{x/2}^{3-x} f(x,y)\,dy\,dx.$$

$\square$

Problem 2. Evaluate the integral

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$$

by reversing the order of integration.

Solution. The region is a triangle with vertices at $(0,0)$, $(3,0)$, and $(3,1)$. The reversed order integral is

$$\begin{aligned}\int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx &= \int_0^3 ye^{x^2}\Big|_{y=0}^{x/3}\,dx \\ &= \frac{1}{3}\int_0^3 xe^{x^2}\,dx \\ &= \frac{1}{6}e^{x^2}\Big|_{x=0}^{3} \\ &= \frac{e^9-1}{6}.\end{aligned}$$

$\square$

Problem 3. Find the volume of the tetrahedron enclosed by the coordinate planes and the plane $2x+y+z=4$.

Solution. The given plane meets the coordinate axes at $x=2$, $y=4$ and $z=4$. In particular, we want the volume under the surface $z=4-2x-y$ over the triangle D in the xy-plane with vertices $(0,0)$, $(2,0)$ and $(0,4)$. This is the double integral

$$\begin{aligned}
\text{Vol} &= \iint_D (4-2x-y)\,dA \\
&= \int_0^2 \int_0^{4-2x} (4-2x-y)\,dy\,dx \\
&= \int_0^2 \tfrac{1}{2}(4-2x)^2\,dx \\
&= -\tfrac{1}{12}(4-2x)^3\Big|_{x=0}^{2} \\
&= \frac{16}{3}.
\end{aligned}$$

□

Problem 4. Use polar coordinates to find the volume of the region bounded between the paraboloids $z = 6 - x^2 - y^2$ and $z = 2x^2 + 2y^2$.

Solution. The two surfaces in polar coordinates take the form $z = 6 - r^2$ and $z = 2r^2$. The desired volume between the two surfaces lies over the disk whose radius that of their circle of intersection, i.e., where $6 - r^2 = 2r^2$, or $r = \sqrt{2}$. To find the volume *between* the surfaces, we subtract the volume under the lower surface from the volume under the upper surface. Thus

$$\begin{aligned}
\text{Vol} &= \int_0^{2\pi} \int_0^{\sqrt{2}} (6-r^2)\,r\,dr\,d\theta - \int_0^{2\pi} \int_0^{\sqrt{2}} (2r^2)\,r\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} (6-r^2-2r^2)\,r\,dr\,d\theta \\
&= 6\pi.
\end{aligned}$$

□

Problem 5. A cylindrical drill with radius r_1 is used to bore a hole through the center of a sphere of radius $r_2 > r_1$. Find the volume of the ring-shaped solid that remains.

Solution. If the sphere is centered at the origin, its upper and lower hemispheres are given by $z = \sqrt{r_2^2 - x^2 - y^2} = \sqrt{r_2^2 - r^2}$ and $z = -\sqrt{r_2^2 - r^2}$, respectively. We may find the volume under just the northern hemisphere and double it. We imagine that the hole is drilled along the z-axis, so the volume is under the north hemisphere and over the annulus where $r_1 \le r \le r_2$, which is given by

$$\begin{aligned}
\tfrac{1}{2}\text{Vol} &= \int_0^{2\pi} \int_{r_1}^{r_2} \left(\sqrt{r_2^2 - r^2}\right) r\,dr\,d\theta \\
&= \int_0^{2\pi} -\tfrac{1}{3}(r_2^2 - r^2)^{3/2}\Big|_{r=r_1}^{r_2}\,d\theta \\
&= \int_0^{2\pi} \tfrac{1}{3}(r_2^2 - r_1^2)^{3/2}\,d\theta \\
&= \tfrac{2\pi}{3}(r_2^2 - r_1^2)^{3/2}.
\end{aligned}$$

So the volume is $\frac{4\pi}{3}(r_2^2 - r_1^2)^{3/2}$. □

Problem 6.

(a) Find the appropriate description of the triangular region R in polar coordinates, where R has vertices $(0,0)$, $(1,0)$ and $(1,1)$. (There will be variable limits somewhere!)

(b) Compute the integral

$$\iint_R \frac{y}{\sqrt{x^2+y^2}}\,dA$$

using polar coordinates. (You can check your answer by computing the integral in cartesian coordinates as well.)

Solution.

(a) The three boundaries of the triangle are given in cartesian coordinates by $x=0$, $y=x$ and $x=1$. The first and second of these are the lines $\theta=0$ and $\theta=\pi/4$, respectively, while $x=1$ becomes the equation $r\cos\theta=1$, or $r=\sec\theta$. The innermost radius is $r=0$. So

$$R=\{(r,\theta): 0\le r\le \sec\theta,\ 0\le\theta\le\pi/4\}\,.$$

(b) To compute the integral, we change to polar coordinates, where the integrand becomes $f\big(x(r,\theta),y(r,\theta)\big)=\frac{r\sin\theta}{r}=\sin\theta$, and $dV=r\,dr\,d\theta$, so

$$\begin{aligned}
\iint_R \frac{y}{\sqrt{x^2+y^2}}\,dA &= \int_0^{\pi/4}\int_0^{\sec\theta} r\sin\theta\,dr\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/4} r^2\sin\theta\Big|_{r=0}^{\sec\theta}\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/4}\frac{\sin\theta}{\cos^2\theta}\,d\theta \\
&= \frac{1}{2}\frac{1}{\cos\theta}\Big|_{\theta=0}^{\pi/4} \\
&= \frac{1}{2}(\sqrt{2}-1).
\end{aligned}$$

□