**1.** The region $R$ is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 0 \le r \le 4, 0 \le \theta \le \frac{3\pi}{2}\right\}$.

Thus $\iint_R f(x,y)\,dA = \int_0^{3\pi/2}\int_0^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

**2.** The region $R$ is more easily described by rectangular coordinates: $R = \left\{(x,y) \mid -1 \le x \le 1, 0 \le y \le 1 - x^2\right\}$.

Thus $\iint_R f(x,y)\,dA = \int_{-1}^1 \int_0^{1-x^2} f(x,y)\,dy\,dx$.

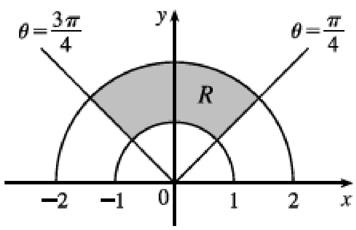
**3.** The region $R$ is more easily described by rectangular coordinates: $R = \left\{(x,y) \mid -1 \le x \le 1, 0 \le y \le \frac{1}{2}x + \frac{1}{2}\right\}$.

Thus $\iint_R f(x,y)\,dA = \int_{-1}^1 \int_0^{(x+1)/2} f(x,y)\,dy\,dx$.

**4.** The region $R$ is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 3 \le r \le 6,\ -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right\}$.

Thus $\iint_R f(x,y)\,dA = \int_{-\pi/2}^{\pi/2} \int_3^6 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

**5.** The integral $\int_{\pi/4}^{3\pi/4}\int_1^2 r\,dr\,d\theta$ represents the area of the region $R = \{(r,\theta) \mid 1 \le r \le 2, \pi/4 \le \theta \le 3\pi/4\}$, the top quarter portion of a ring (annulus).




$$\int_{\pi/4}^{3\pi/4}\int_1^2 r\,dr\,d\theta = \left(\int_{\pi/4}^{3\pi/4} d\theta\right)\left(\int_1^2 r\,dr\right)$$
$$= \left[\theta\right]_{\pi/4}^{3\pi/4}\left[\tfrac{1}{2}r^2\right]_1^2 = \left(\tfrac{3\pi}{4} - \tfrac{\pi}{4}\right)\cdot\tfrac{1}{2}(4-1) = \tfrac{\pi}{2}\cdot\tfrac{3}{2} = \tfrac{3\pi}{4}$$

**7.** The half disk $D$ can be described in polar coordinates as $D = \{(r,\theta) \mid 0 \le r \le 5, 0 \le \theta \le \pi\}$. Then

$$\iint_D x^2 y\,dA = \int_0^\pi \int_0^5 (r\cos\theta)^2(r\sin\theta)\,r\,dr\,d\theta = \left(\int_0^\pi \cos^2\theta\sin\theta\,d\theta\right)\left(\int_0^5 r^4\,dr\right)$$
$$= \left[-\tfrac{1}{3}\cos^3\theta\right]_0^\pi\left[\tfrac{1}{5}r^5\right]_0^5 = -\tfrac{1}{3}(-1-1)\cdot 625 = \tfrac{1250}{3}$$

**10.**
$$\iint_R \frac{y^2}{x^2+y^2}\,dA = \int_0^{2\pi}\int_a^b \frac{(r\sin\theta)^2}{r^2}\,r\,dr\,d\theta = \left(\int_0^{2\pi}\sin^2\theta\,d\theta\right)\left(\int_a^b r\,dr\right)$$
$$= \int_0^{2\pi}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta\int_a^b r\,dr = \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{2}r^2\right]_a^b$$
$$= \tfrac{1}{2}(2\pi - 0 - 0)\left[\tfrac{1}{2}\left(b^2 - a^2\right)\right] = \tfrac{\pi}{2}(b^2 - a^2)$$

**11.**
$$\iint_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2}\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} d\theta\int_0^2 re^{-r^2}\,dr$$
$$= \left[\theta\right]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{2}e^{-r^2}\right]_0^2 = \pi\left(-\tfrac{1}{2}\right)(e^{-4} - e^0) = \tfrac{\pi}{2}(1 - e^{-4})$$

**20.** The paraboloid $z = 18 - 2x^2 - 2y^2$ intersects the $xy$-plane in the circle $x^2 + y^2 = 9$, so

$$V = \iint_{x^2+y^2\le 9} \left(18 - 2x^2 - 2y^2\right) dA = \iint_{x^2+y^2\le 9} \left[18 - 2\left(x^2+y^2\right)\right] dA = \int_0^{2\pi}\int_0^3 \left(18 - 2r^2\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^3 \left(18r - 2r^3\right) dr = \left[\,\theta\,\right]_0^{2\pi} \left[9r^2 - \tfrac{1}{2}r^4\right]_0^3 = (2\pi)\left(81 - \tfrac{81}{2}\right) = 81\pi$$

**23.** By symmetry,

$$V = 2\iint_{x^2+y^2\le a^2} \sqrt{a^2 - x^2 - y^2}\,dA = 2\int_0^{2\pi}\int_0^a \sqrt{a^2 - r^2}\;r\,dr\,d\theta = 2\int_0^{2\pi} d\theta \int_0^a r\sqrt{a^2 - r^2}\,dr$$

$$= 2\left[\,\theta\,\right]_0^{2\pi} \left[-\tfrac{1}{3}(a^2 - r^2)^{3/2}\right]_0^a = 2(2\pi)\left(0 + \tfrac{1}{3}a^3\right) = \tfrac{4\pi}{3}a^3$$

**2.** $Q = \iint_D \sigma(x,y)\,dA = \iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^1 \sqrt{r^2}\,r\,dr\,d\theta$

$= \int_0^{2\pi} d\theta \;\int_0^1 r^2\,dr = [\theta]_0^{2\pi}\;\left[\tfrac{1}{3}r^3\right]_0^1 = 2\pi\cdot\tfrac{1}{3} = \tfrac{2\pi}{3}$ C

**7.** $m = \int_{-1}^1\int_0^{1-x^2} ky\,dy\,dx = k\int_{-1}^1 \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x^2} dx = \tfrac{1}{2}k\int_{-1}^1 (1-x^2)^2\,dx = \tfrac{1}{2}k\int_{-1}^1 (1 - 2x^2 + x^4)\,dx$

$= \tfrac{1}{2}k\left[x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right]_{-1}^1 = \tfrac{1}{2}k\left(1 - \tfrac{2}{3} + \tfrac{1}{5} + 1 - \tfrac{2}{3} + \tfrac{1}{5}\right) = \tfrac{8}{15}k,$

$M_y = \int_{-1}^1\int_0^{1-x^2} kxy\,dy\,dx = k\int_{-1}^1 \left[\tfrac{1}{2}xy^2\right]_{y=0}^{y=1-x^2} dx = \tfrac{1}{2}k\int_{-1}^1 x\,(1-x^2)^2\,dx = \tfrac{1}{2}k\int_{-1}^1 (x - 2x^3 + x^5)\,dx$

$= \tfrac{1}{2}k\left[\tfrac{1}{2}x^2 - \tfrac{1}{2}x^4 + \tfrac{1}{6}x^6\right]_{-1}^1 = \tfrac{1}{2}k\left(\tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{6} - \tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{6}\right) = 0,$

$M_x = \int_{-1}^1\int_0^{1-x^2} ky^2\,dy\,dx = k\int_{-1}^1 \left[\tfrac{1}{3}y^3\right]_{y=0}^{y=1-x^2} dx = \tfrac{1}{3}k\int_{-1}^1 (1-x^2)^3\,dx = \tfrac{1}{3}k\int_{-1}^1 (1 - 3x^2 + 3x^4 - x^6)\,dx$

$= \tfrac{1}{3}k\left[x - x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7\right]_{-1}^1 = \tfrac{1}{3}k\left(1 - 1 + \tfrac{3}{5} - \tfrac{1}{7} + 1 - 1 + \tfrac{3}{5} - \tfrac{1}{7}\right) = \tfrac{32}{105}k.$

Hence $m = \tfrac{8}{15}k$, $(\overline{x}, \overline{y}) = \left(0, \frac{32k/105}{8k/15}\right) = \left(0, \tfrac{4}{7}\right)$.

**15.** Placing the vertex opposite the hypotenuse at $(0,0)$, $\rho(x,y) = k(x^2+y^2)$. Then

$m = \int_0^a\int_0^{a-x} k\left(x^2+y^2\right) dy\,dx = k\int_0^a \left[ax^2 - x^3 + \tfrac{1}{3}\,(a-x)^3\right] dx = k\left[\tfrac{1}{3}ax^3 - \tfrac{1}{4}x^4 - \tfrac{1}{12}\,(a-x)^4\right]_0^a = \tfrac{1}{6}ka^4.$

By symmetry,

$$M_y = M_x = \int_0^a\int_0^{a-x} ky(x^2+y^2)\,dy\,dx = k\int_0^a \left[\tfrac{1}{2}(a-x)^2x^2 + \tfrac{1}{4}(a-x)^4\right] dx$$

$$= k\left[\tfrac{1}{6}a^2x^3 - \tfrac{1}{4}ax^4 + \tfrac{1}{10}x^5 - \tfrac{1}{20}(a-x)^5\right]_0^a = \tfrac{1}{15}ka^5$$

Hence $(\overline{x}, \overline{y}) = \left(\tfrac{2}{5}a, \tfrac{2}{5}a\right)$.

**17.** $I_x = \iint_D y^2 \rho(x,y)dA = \int_{-1}^{1}\int_0^{1-x^2} y^2 \cdot ky\,dy\,dx = k\int_{-1}^{1}\left[\frac{1}{4}y^4\right]_{y=0}^{y=1-x^2} dx = \frac{1}{4}k\int_{-1}^{1}(1-x^2)^4\,dx$

$$= \frac{1}{4}k\int_{-1}^{1}(x^8 - 4x^6 + 6x^4 - 4x^2 + 1)\,dx = \frac{1}{4}k\left[\frac{1}{9}x^9 - \frac{4}{7}x^7 + \frac{6}{5}x^5 - \frac{4}{3}x^3 + x\right]_{-1}^{1} = \frac{64}{315}k,$$

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_{-1}^{1}\int_0^{1-x^2} kx^2y\,dy\,dx = k\int_{-1}^{1}\left[\frac{1}{2}x^2y^2\right]_{y=0}^{y=1-x^2} dx = \frac{1}{2}k\int_{-1}^{1} x^2(1-x^2)^2\,dx$

$$= \frac{1}{2}k\int_{-1}^{1}(x^2 - 2x^4 + x^6)\,dx = \frac{1}{2}k\left[\frac{1}{3}x^3 - \frac{2}{5}x^5 + \frac{1}{7}x^7\right]_{-1}^{1} = \frac{8}{105}k,$$

and $I_0 = I_x + I_y = \frac{64}{315}k + \frac{8}{105}k = \frac{88}{315}k.$

**9.** $\iiint_E y\,dV = \int_0^3\int_0^x\int_{x-y}^{x+y} y\,dz\,dy\,dx = \int_0^3\int_0^x \left[yz\right]_{z=x-y}^{z=x+y} dy\,dx = \int_0^3\int_0^x 2y^2\,dy\,dx$

$$= \int_0^3 \left[\frac{2}{3}y^3\right]_{y=0}^{y=x} dx = \int_0^3 \frac{2}{3}x^3\,dx = \left.\frac{1}{6}x^4\right]_0^3 = \frac{81}{6} = \frac{27}{2}$$

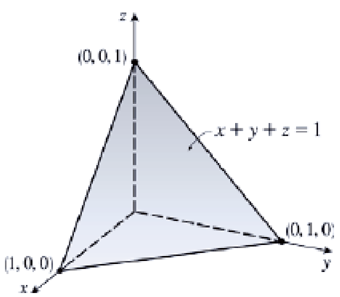
**13.** Here $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le \sqrt{x}, 0 \le z \le 1 + x + y\}$, so

$$\iiint_E 6xy\,dV = \int_0^1\int_0^{\sqrt{x}}\int_0^{1+x+y} 6xy\,dz\,dy\,dx = \int_0^1\int_0^{\sqrt{x}}\left[6xyz\right]_{z=0}^{z=1+x+y} dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{x}} 6xy(1+x+y)\,dy\,dx = \int_0^1 \left[3xy^2 + 3x^2y^2 + 2xy^3\right]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 (3x^2 + 3x^3 + 2x^{5/2})\,dx = \left[x^3 + \frac{3}{4}x^4 + \frac{4}{7}x^{7/2}\right]_0^1 = \frac{65}{28}$$

**15.**




Here $T = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le 1 - x, 0 \le z \le 1 - x - y\}$, so

$$\iiint_T x^2\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x^2\,dz\,dy\,dx = \int_0^1\int_0^{1-x} x^2(1-x-y)\,dy\,dx$$

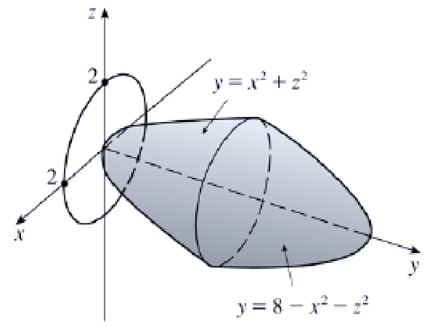
$$= \int_0^1\int_0^{1-x}(x^2 - x^3 - x^2y)\,dy\,dx = \int_0^1 \left[x^2y - x^3y - \frac{1}{2}x^2y^2\right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left[x^2(1-x) - x^3(1-x) - \frac{1}{2}x^2(1-x)^2\right] dx$$

$$= \int_0^1 \left(\frac{1}{2}x^4 - x^3 + \frac{1}{2}x^2\right) dx = \left[\frac{1}{10}x^5 - \frac{1}{4}x^4 + \frac{1}{6}x^3\right]_0^1$$

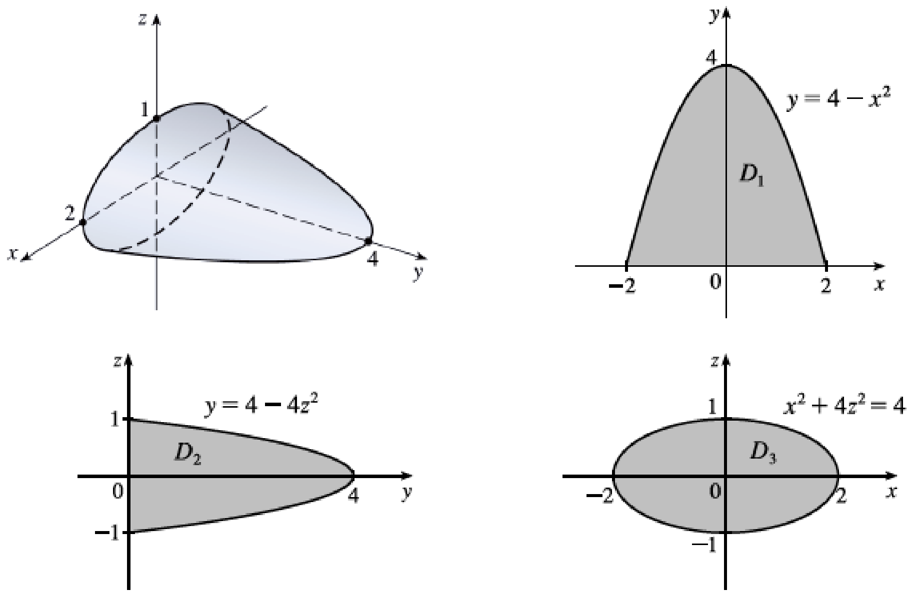
$$= \frac{1}{10} - \frac{1}{4} + \frac{1}{6} = \frac{1}{60}$$

**20.** The paraboloids intersect when $x^2 + z^2 = 8 - x^2 - z^2 \quad \Leftrightarrow \quad x^2 + z^2 = 4$, thus the intersection is the circle $x^2 + z^2 = 4$, $y = 4$. The projection of $E$ onto the $xz$-plane is the disk $x^2 + z^2 \leq 4$, so $E = \left\{(x, y, z) \mid x^2 + z^2 \leq y \leq 8 - x^2 - z^2, x^2 + z^2 \leq 4\right\}$. Let $D = \left\{(x, z) \mid x^2 + z^2 \leq 4\right\}$. Then using polar coordinates $x = r\cos\theta$ and $z = r\sin\theta$, we have




$$V = \iiint_E dV = \iint_D \left(\int_{x^2+z^2}^{8-x^2-z^2} dy\right) dA = \iint_D (8 - 2x^2 - 2z^2)\, dA$$
$$= \int_0^{2\pi} \int_0^2 (8 - 2r^2)\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 (8r - 2r^3)\, dr$$
$$= \left[\theta\right]_0^{2\pi} \left[4r^2 - \tfrac{1}{2}r^4\right]_0^2 = 2\pi(16 - 8) = 16\pi$$

**29.**




If $D_1$, $D_2$, $D_3$ are the projections of $E$ on the $xy$-, $yz$-, and $xz$-planes, then

$$D_1 = \left\{(x,y) \mid -2 \le x \le 2, 0 \le y \le 4 - x^2\right\} = \left\{(x,y) \mid 0 \le y \le 4,\ -\sqrt{4-y} \le x \le \sqrt{4-y}\right\}$$

$$D_2 = \left\{(y,z) \mid 0 \le y \le 4,\ -\tfrac{1}{2}\sqrt{4-y} \le z \le \tfrac{1}{2}\sqrt{4-y}\right\} = \left\{(y,z) \mid -1 \le z \le 1, 0 \le y \le 4 - 4z^2\right\}$$

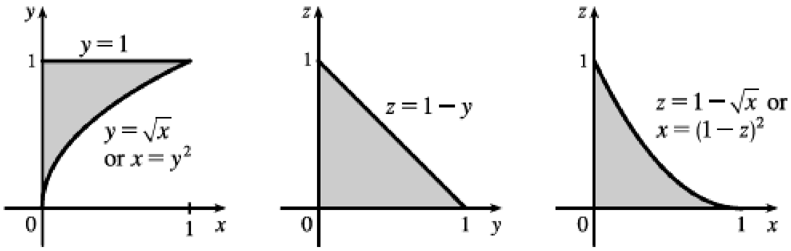
$$D_3 = \left\{(x,z) \mid x^2 + 4z^2 \le 4\right\}$$

Therefore

$$\begin{aligned}
E &= \left\{(x,y,z) \mid -2 \le x \le 2, 0 \le y \le 4 - x^2,\ -\tfrac{1}{2}\sqrt{4 - x^2 - y} \le z \le \tfrac{1}{2}\sqrt{4 - x^2 - y}\right\} \\
&= \left\{(x,y,z) \mid 0 \le y \le 4,\ -\sqrt{4-y} \le x \le \sqrt{4-y},\ -\tfrac{1}{2}\sqrt{4 - x^2 - y} \le z \le \tfrac{1}{2}\sqrt{4 - x^2 - y}\right\} \\
&= \left\{(x,y,z) \mid -1 \le z \le 1, 0 \le y \le 4 - 4z^2,\ -\sqrt{4 - y - 4z^2} \le x \le \sqrt{4 - y - 4z^2}\right\} \\
&= \left\{(x,y,z) \mid 0 \le y \le 4,\ -\tfrac{1}{2}\sqrt{4-y} \le z \le \tfrac{1}{2}\sqrt{4-y},\ -\sqrt{4 - y - 4z^2} \le x \le \sqrt{4 - y - 4z^2}\right\} \\
&= \left\{(x,y,z) \mid -2 \le x \le 2,\ -\tfrac{1}{2}\sqrt{4 - x^2} \le z \le \tfrac{1}{2}\sqrt{4 - x^2}, 0 \le y \le 4 - x^2 - 4z^2\right\} \\
&= \left\{(x,y,z) \mid -1 \le z \le 1,\ -\sqrt{4 - 4z^2} \le x \le \sqrt{4 - 4z^2}, 0 \le y \le 4 - x^2 - 4z^2\right\}
\end{aligned}$$

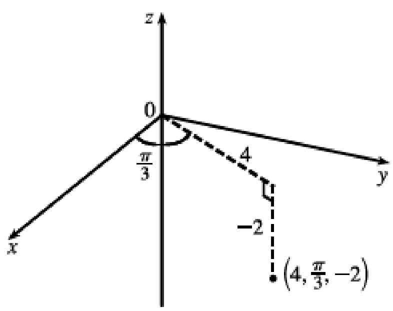
Then

$$\begin{aligned}
\iiint_E f(x,y,z)\,dV &= \int_{-2}^{2}\int_{0}^{4-x^2}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x,y,z)\,dz\,dy\,dx = \int_{0}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_{-\sqrt{4-x^2-y}/2}^{\sqrt{4-x^2-y}/2} f(x,y,z)\,dz\,dx\,dy \\
&= \int_{-1}^{1}\int_{0}^{4-4z^2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\,dx\,dy\,dz = \int_{0}^{4}\int_{-\sqrt{4-y}/2}^{\sqrt{4-y}/2}\int_{-\sqrt{4-y-4z^2}}^{\sqrt{4-y-4z^2}} f(x,y,z)\,dx\,dz\,dy \\
&= \int_{-2}^{2}\int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2}\int_{0}^{4-x^2-4z^2} f(x,y,z)\,dy\,dz\,dx = \int_{-1}^{1}\int_{-\sqrt{4-4z^2}}^{\sqrt{4-4z^2}}\int_{0}^{4-x^2-4z^2} f(x,y,z)\,dy\,dx\,dz
\end{aligned}$$

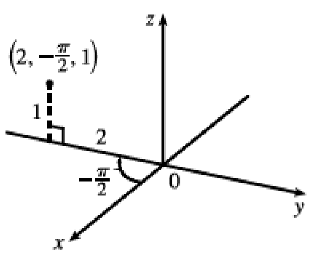
**33.** 


The diagrams show the projections of $E$ on the $xy$-, $yz$-, and $xz$-planes. Therefore

$$\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$$
$$= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy = \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$$
$$= \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$$

**1.** (a) 


From Equations 1, $x = r\cos\theta = 4\cos\frac{\pi}{3} = 4\cdot\frac{1}{2} = 2$, $y = r\sin\theta = 4\sin\frac{\pi}{3} = 4\cdot\frac{\sqrt{3}}{2} = 2\sqrt{3}$, $z = -2$, so the point is $\left(2, 2\sqrt{3}, -2\right)$ in rectangular coordinates.

(b) 


$x = 2\cos\left(-\frac{\pi}{2}\right) = 0$, $y = 2\sin\left(-\frac{\pi}{2}\right) = -2$, and $z = 1$, so the point is $(0, -2, 1)$ in rectangular coordinates.

**3.** (a) From Equations 2 we have $r^2 = (-1)^2 + 1^2 = 2$ so $r = \sqrt{2}$; $\tan\theta = \frac{1}{-1} = -1$ and the point $(-1, 1)$ is in the second quadrant of the $xy$-plane, so $\theta = \frac{3\pi}{4} + 2n\pi$; $z = 1$. Thus, one set of cylindrical coordinates is $\left(\sqrt{2}, \frac{3\pi}{4}, 1\right)$.

(b) $r^2 = (-2)^2 + (2\sqrt{3})^2 = 16$ so $r = 4$; $\tan\theta = \frac{2\sqrt{3}}{-2} = -\sqrt{3}$ and the point $\left(-2, 2\sqrt{3}\right)$ is in the second quadrant of the $xy$-plane, so $\theta = \frac{2\pi}{3} + 2n\pi$; $z = 3$. Thus, one set of cylindrical coordinates is $\left(4, \frac{2\pi}{3}, 3\right)$.

**7.** $z = 4 - r^2 = 4 - (x^2 + y^2)$ or $4 - x^2 - y^2$, so the surface is a circular paraboloid with vertex $(0, 0, 4)$, axis the $z$-axis, and opening downward.

**8.** Since $2r^2 + z^2 = 1$ and $r^2 = x^2 + y^2$, we have $2(x^2 + y^2) + z^2 = 1$ or $2x^2 + 2y^2 + z^2 = 1$, an ellipsoid centered at the origin with intercepts $x = \pm\frac{1}{\sqrt{2}}$, $y = \pm\frac{1}{\sqrt{2}}$, $z = \pm 1$.

**9.** (a) Substituting $x^2 + y^2 = r^2$ and $x = r\cos\theta$, the equation $x^2 - x + y^2 + z^2 = 1$ becomes $r^2 - r\cos\theta + z^2 = 1$ or $z^2 = 1 + r\cos\theta - r^2$.

(b) Substituting $x = r\cos\theta$ and $y = r\sin\theta$, the equation $z = x^2 - y^2$ becomes $z = (r\cos\theta)^2 - (r\sin\theta)^2 = r^2(\cos^2\theta - \sin^2\theta)$ or $z = r^2\cos 2\theta$.

**17.** In cylindrical coordinates, $E$ is given by $\{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4, -5 \le z \le 4\}$. So

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^4\int_{-5}^4 \sqrt{r^2}\,r\,dz\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^4 r^2\,dr \int_{-5}^4 dz$$

$$= \big[\,\theta\,\big]_0^{2\pi}\,\big[\tfrac{1}{3}r^3\big]_0^4\,\big[\,z\,\big]_{-5}^4 = (2\pi)\big(\tfrac{64}{3}\big)(9) = 384\pi$$

**22.**
In cylindrical coordinates $E$ is the solid region within the cylinder $r = 1$ bounded above and below by the sphere $r^2 + z^2 = 4$,

so $E = \left\{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, -\sqrt{4-r^2} \le z \le \sqrt{4-r^2}\right\}$. Thus the volume is

$$\iiint_E dV = \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r\sqrt{4-r^2}\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 2r\sqrt{4-r^2}\,dr = 2\pi\Big[-\tfrac{2}{3}(4-r^2)^{3/2}\Big]_0^1 = \tfrac{4}{3}\pi(8 - 3^{3/2})$$

**23.**
In cylindrical coordinates, $E$ is bounded below by the cone $z = r$ and above by the sphere $r^2 + z^2 = 2$ or $z = \sqrt{2-r^2}$. The

cone and the sphere intersect when $2r^2 = 2 \quad \Rightarrow \quad r = 1$, so $E = \left\{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, r \le z \le \sqrt{2-r^2}\right\}$

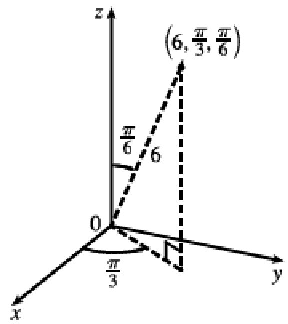
and the volume is

$$\iiint_E dV = \int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 [rz]_{z=r}^{z=\sqrt{2-r^2}}\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \left(r\sqrt{2-r^2} - r^2\right)dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 \left(r\sqrt{2-r^2} - r^2\right)dr = 2\pi\left[-\tfrac{1}{3}(2-r^2)^{3/2} - \tfrac{1}{3}r^3\right]_0^1$$

$$= 2\pi\left(-\tfrac{1}{3}\right)\left(1 + 1 - 2^{3/2}\right) = -\tfrac{2}{3}\pi\left(2 - 2\sqrt{2}\right) = \tfrac{4}{3}\pi\left(\sqrt{2} - 1\right)$$

**1.** (a)

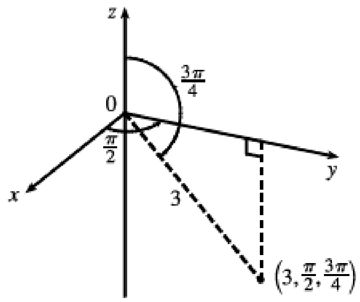



From Equations 1, $x = \rho \sin\phi \cos\theta = 6 \sin\frac{\pi}{6} \cos\frac{\pi}{3} = 6 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{2}$,

$y = \rho \sin\phi \sin\theta = 6 \sin\frac{\pi}{6} \sin\frac{\pi}{3} = 6 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$, and

$z = \rho \cos\phi = 6 \cos\frac{\pi}{6} = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$, so the point is $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}, 3\sqrt{3}\right)$ in rectangular coordinates.

(b)




$x = 3 \sin\frac{3\pi}{4} \cos\frac{\pi}{2} = 3 \cdot \frac{\sqrt{2}}{2} \cdot 0 = 0$,

$y = 3 \sin\frac{3\pi}{4} \sin\frac{\pi}{2} = 3 \cdot \frac{\sqrt{2}}{2} \cdot 1 = \frac{3\sqrt{2}}{2}$, and

$z = 3 \cos\frac{3\pi}{4} = 3\left(-\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{2}}{2}$, so the point is $\left(0, \frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$ in rectangular coordinates.

**3.** (a) From Equations 1 and 2, $\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{0^2 + (-2)^2 + 0^2} = 2$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{0}{2} = 0 \quad\Rightarrow\quad \phi = \dfrac{\pi}{2}$, and

$$\cos\theta = \frac{x}{\rho\sin\phi} = \frac{0}{2\sin(\pi/2)} = 0 \quad\Rightarrow\quad \theta = \frac{3\pi}{2} \quad [\text{since } y < 0]. \text{ Thus spherical coordinates are } \left(2, \frac{3\pi}{2}, \frac{\pi}{2}\right).$$

(b) $\rho = \sqrt{1+1+2} = 2$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{-\sqrt{2}}{2} \quad\Rightarrow\quad \phi = \dfrac{3\pi}{4}$, and

$$\cos\theta = \frac{x}{\rho\sin\phi} = \frac{-1}{2\sin(3\pi/4)} = \frac{-1}{2\left(\sqrt{2}/2\right)} = -\frac{1}{\sqrt{2}} \quad\Rightarrow\quad \theta = \frac{3\pi}{4} \quad [\text{since } y > 0]. \text{ Thus spherical coordinates}$$

are $\left(2, \dfrac{3\pi}{4}, \dfrac{3\pi}{4}\right)$.

**9.** (a) $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, so the equation $z^2 = x^2 + y^2$ becomes $(\rho\cos\phi)^2 = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$ or $\rho^2\cos^2\phi = \rho^2\sin^2\phi$. If $\rho \neq 0$, this becomes $\cos^2\phi = \sin^2\phi$. ($\rho = 0$ corresponds to the origin which is included in the surface.) There are many equivalent equations in spherical coordinates, such as $\tan^2\phi = 1$, $2\cos^2\phi = 1$, $\cos 2\phi = 0$, or even $\phi = \frac{\pi}{4}$, $\phi = \frac{3\pi}{4}$.

(b) $x^2 + z^2 = 9 \quad\Leftrightarrow\quad (\rho\sin\phi\cos\theta)^2 + (\rho\cos\phi)^2 = 9 \quad\Leftrightarrow\quad \rho^2\sin^2\phi\cos^2\theta + \rho^2\cos^2\phi = 9$ or $\rho^2\left(\sin^2\phi\cos^2\theta + \cos^2\phi\right) = 9$.

**21.** In spherical coordinates, $B$ is represented by $\{(\rho,\theta,\phi)\,|\,0\le\rho\le 5, 0\le\theta\le 2\pi, 0\le\phi\le\pi\}$. Thus

$$\begin{aligned}\iiint_B (x^2+y^2+z^2)^2\,dV &= \int_0^\pi\int_0^{2\pi}\int_0^5 (\rho^2)^2\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^\pi \sin\phi\,d\phi\,\int_0^{2\pi}d\theta\,\int_0^5\rho^6\,d\rho \\ &= \big[-\cos\phi\big]_0^\pi\,\big[\,\theta\,\big]_0^{2\pi}\,\big[\tfrac17\rho^7\big]_0^5 = (2)(2\pi)\big(\tfrac{78,125}{7}\big) \\ &= \tfrac{312,500}{7}\pi \approx 140,249.7\end{aligned}$$

**22.** In spherical coordinates, $H$ is represented by $\left\{(\rho,\theta,\phi)\,\middle|\,0\le\rho\le 3, 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{2}\right\}$. Thus

$$\begin{aligned}\iiint_H (9-x^2-y^2)\,dV &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^3\left[9-(\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta)\right]\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \int_0^{\pi/2}\int_0^{2\pi}\int_0^3 (9-\rho^2\sin^2\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \int_0^{\pi/2}\int_0^{2\pi}\left[3\rho^3-\tfrac15\rho^5\sin^2\phi\right]_{\rho=0}^{\rho=3}\sin\phi\,d\rho\,d\theta\,d\phi \\ &= \int_0^{\pi/2}\int_0^{2\pi}\left(81\sin\phi-\tfrac{243}{5}\sin^3\phi\right)d\theta\,d\phi \\ &= \int_0^{2\pi}d\theta\int_0^{\pi/2}\left[81\sin\phi-\tfrac{243}{5}(1-\cos^2\phi)\sin\phi\right]d\phi \\ &= 2\pi\left[-81\cos\phi-\tfrac{243}{5}\left(\tfrac13\cos^3\phi-\cos\phi\right)\right]_0^{\pi/2} \\ &= 2\pi\left[0+81+\tfrac{243}{5}\left(-\tfrac23\right)\right] = \tfrac{486}{5}\pi\end{aligned}$$

**24.** In spherical coordinates, $E$ is represented by $\{(\rho,\theta,\phi)\,|\,0\le\rho\le 3, 0\le\theta\le\pi, 0\le\phi\le\pi\}$. Thus

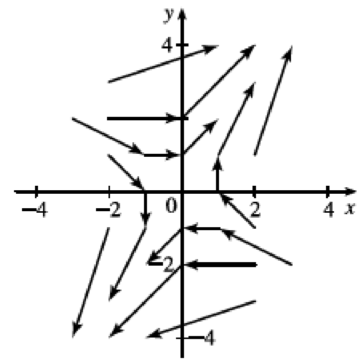
$$\begin{aligned}\iiint_E y^2\,dV &= \int_0^\pi\int_0^\pi\int_0^3(\rho\sin\phi\sin\theta)^2\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^\pi\sin^3\phi\,d\phi\,\int_0^\pi\sin^2\theta\,d\theta\,\int_0^3\rho^4\,d\rho \\ &= \int_0^\pi(1-\cos^2\phi)\,\sin\phi\,d\phi\,\int_0^\pi\tfrac12(1-\cos2\theta)\,d\theta\,\int_0^3\rho^4\,d\rho \\ &= \left[-\cos\phi+\tfrac13\cos^3\phi\right]_0^\pi\,\left[\tfrac12\left(\theta-\tfrac12\sin2\theta\right)\right]_0^\pi\,\left[\tfrac15\rho^5\right]_0^3 \\ &= \left(\tfrac23+\tfrac23\right)\left(\tfrac12\pi\right)\left(\tfrac15(243)\right) = \left(\tfrac43\right)\left(\tfrac{\pi}{2}\right)\left(\tfrac{243}{5}\right) = \tfrac{162\pi}{5}\end{aligned}$$

**27.** The solid region is given by $E=\left\{(\rho,\theta,\phi)\mid 0\le\rho\le a, 0\le\theta\le2\pi, \frac{\pi}{6}\le\phi\le\frac{\pi}{3}\right\}$ and its volume is

$$\begin{aligned}V &= \iiint_E dV = \int_{\pi/6}^{\pi/3}\int_0^{2\pi}\int_0^a\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/6}^{\pi/3}\sin\phi\,d\phi\,\int_0^{2\pi}d\theta\int_0^a\rho^2\,d\rho \\ &= [-\cos\phi]_{\pi/6}^{\pi/3}\,[\theta]_0^{2\pi}\,\left[\tfrac13\rho^3\right]_0^a = \left(-\tfrac12+\tfrac{\sqrt3}{2}\right)(2\pi)\left(\tfrac13a^3\right) = \tfrac{\sqrt3-1}{3}\pi a^3\end{aligned}$$

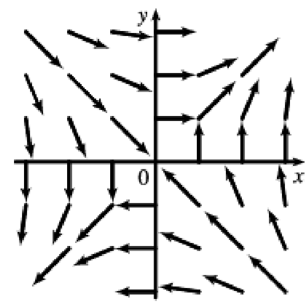
**4.** $\mathbf{F}(x,y) = y\,\mathbf{i} + (x+y)\,\mathbf{j}$

The length of the vector $y\,\mathbf{i} + (x+y)\,\mathbf{j}$ is $\sqrt{y^2 + (x+y)^2}$. Vectors along the $x$-axis are vertical, and vectors along the line $y = -x$ are horizontal with length $|y|$.



**5.** $\mathbf{F}(x,y) = \dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2+y^2}}$

The length of the vector $\dfrac{y\,\mathbf{i} + x\,\mathbf{j}}{\sqrt{x^2+y^2}}$ is 1.



**11.** $\mathbf{F}(x,y) = \langle x, -y \rangle$ corresponds to graph IV. In the first quadrant all the vectors have positive $x$-components and negative $y$-components, in the second quadrant all vectors have negative $x$- and $y$-components, in the third quadrant all vectors have negative $x$-components and positive $y$-components, and in the fourth quadrant all vectors have positive $x$- and $y$-components. In addition, the vectors get shorter as we approach the origin.

**12.** $\mathbf{F}(x,y) = \langle y, x - y \rangle$ corresponds to graph III. All vectors in quadrants I and II have positive $x$-components while all vectors in quadrants III and IV have negative $x$-components. In addition, vectors along the line $y = x$ are horizontal, and vectors get shorter as we approach the origin.

**13.** $\mathbf{F}(x,y) = \langle y, y + 2 \rangle$ corresponds to graph I. As in Exercise 12, all vectors in quadrants I and II have positive $x$-components while all vectors in quadrants III and IV have negative $x$-components.Vectors along the line $y = -2$ are horizontal, and the vectors are independent of $x$ (vectors along horizontal lines are identical).

**14.** $\mathbf{F}(x,y) = \langle \cos(x+y), x \rangle$ corresponds to graph II. All vectors in quadrants I and IV have positive $y$-components while all vectors in quadrants II and III have negative $y$-components. Also, the $y$-components of vectors along any vertical line remain constant while the $x$-component oscillates.

**15.** $\mathbf{F}(x,y,z) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$ corresponds to graph IV, since all vectors have identical length and direction.

**16.** $\mathbf{F}(x,y,z) = \mathbf{i} + 2\,\mathbf{j} + z\,\mathbf{k}$ corresponds to graph I, since the horizontal vector components remain constant, but the vectors above the $xy$-plane point generally upward while the vectors below the $xy$-plane point generally downward.

**17.**
$\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$ corresponds to graph III; the projection of each vector onto the $xy$-plane is $x\,\mathbf{i} + y\,\mathbf{j}$, which points away from the origin, and the vectors point generally upward because their $z$-components are all $3$.

**18.** $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ corresponds to graph II; each vector $\mathbf{F}(x,y,z)$ has the same length and direction as the position vector of the point $(x,y,z)$, and therefore the vectors all point directly away from the origin.

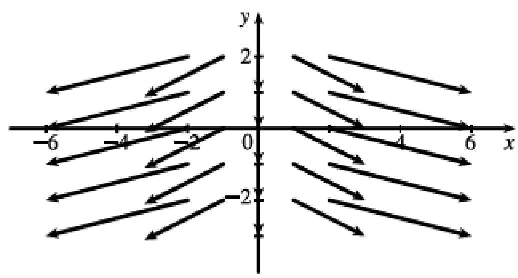
**21.** $f(x,y) = xe^{xy} \quad \Rightarrow$

$\nabla f(x,y) = f_x(x,y)\,\mathbf{i} + f_y\,(x,y)\;\mathbf{j} = (xe^{xy}\cdot y + e^{xy})\,\mathbf{i} + (xe^{xy}\cdot x)\,\mathbf{j} = (xy+1)e^{xy}\,\mathbf{i} + x^2e^{xy}\,\mathbf{j}$

**23.** $\nabla f(x,y,z) = f_x(x,y,z)\,\mathbf{i} + f_y(x,y,z)\,\mathbf{j} + f_z(x,y,z)\,\mathbf{k} = \dfrac{x}{\sqrt{x^2+y^2+z^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2+z^2}}\,\mathbf{j} + \dfrac{z}{\sqrt{x^2+y^2+z^2}}\,\mathbf{k}$

**25.** $f(x,y) = x^2 - y \quad \Rightarrow \quad \nabla f(x,y) = 2x\,\mathbf{i} - \mathbf{j}$.

The length of $\nabla f(x,y)$ is $\sqrt{4x^2+1}$. When $x \neq 0$, the vectors point away from the $y$-axis in a slightly downward direction with length that increases as the distance from the $y$-axis increases.




**1.** $x = t^3$ and $y = t$, $0 \le t \le 2$, so by Formula 3

$$\int_C y^3\,ds = \int_0^2 t^3\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^2 t^3\sqrt{(3t^2)^2+(1)^2}\,dt = \int_0^2 t^3\sqrt{9t^4+1}\,dt$$

$$= \tfrac{1}{36}\cdot\tfrac{2}{3}\left(9t^4+1\right)^{3/2}\Big]_0^2 = \tfrac{1}{54}(145^{3/2}-1) \text{ or } \tfrac{1}{54}\left(145\sqrt{145}-1\right)$$

**3.** Parametric equations for $C$ are $x = 4\cos t,\;\; y = 4\sin t,\; -\frac{\pi}{2} \le t \le \frac{\pi}{2}$. Then

$$\int_C xy^4\,ds = \int_{-\pi/2}^{\pi/2}(4\cos t)(4\sin t)^4\sqrt{(-4\sin t)^2+(4\cos t)^2}\,dt = \int_{-\pi/2}^{\pi/2} 4^5\cos t\,\sin^4 t\,\sqrt{16(\sin^2 t+\cos^2 t)}\,dt$$

$$= 4^5\int_{-\pi/2}^{\pi/2}(\sin^4 t\cos t)(4)\,dt = (4)^6\left[\tfrac{1}{5}\sin^5 t\right]_{-\pi/2}^{\pi/2} = \tfrac{2\cdot 4^6}{5} = 1638.4$$