Math 2321 Fall 2015: Quiz 4 Solutions

Problem 1. Consider the parameterization $\mathbf{r}(u, v) = (u, u\cos v, u \sin v)$ of a surface in $\mathbb{R}^3$. Give a parametric description of the tangent plane of $\mathbf{r}$ at the point $(u_0, v_0) = (1, 0)$.

Solution. The derivatives with respect to the parameters are

$$\begin{aligned} \mathbf{r}_u &= (1, \cos v, \sin v), & \mathbf{r}_u(1,0) &= (1,1,0) \\ \mathbf{r}_v &= (0, -u\sin v, u\cos v) & \mathbf{r}_v(1,0) &= (0,0,1), \end{aligned}$$

so the tangent plane to the surface at $\mathbf{r}(1, 0) = (1, 1, 0)$ is given by

$$\Gamma(s,t) = (1,1,0) + s(1,1,0) + t(0,0,1) = (1+s, 1+s, t) \qquad \square$$

Problem 2. Find a parameterization of the surface $x^3 - y^2 - z^2 = 1$ which is regular at the point $(1, 0, 0)$.

Solution. One way is to parameterize the surface as a graph. However, if we try solving for z in terms of y and x, or for y in terms of x and z, we find the parameterization is not regular. For example:

$$\mathbf{r}(x,y) = (x, y, \sqrt{x^3 - y^2 - 1}) \implies \mathbf{r}_x = (1, 0, \tfrac{3x^2}{2\sqrt{x^3-y^2-1}}),$$

and $(1, 0, 0) = \mathbf{r}(1, 0)$, but $\mathbf{r}_x(1, 0)$ doesn't exist since the denominator in the third component is 0.

So we can consider the surface as a "sideways graph" where $x = f(y, z) = (1+y^2+z^2)^{1/3}$. The parameterization given by

$$\begin{aligned} \mathbf{r}(y,z) &= \big((1+y^2+z^2)^{1/3}, y, z\big), & \mathbf{r}(0,1) &= (1,0,0) \\ \mathbf{r}_y(y,z) &= (\tfrac{2y}{3(1+y^2+z^2)^{2/3}}, 1, 0) & \mathbf{r}_y(0,1) &= (0,1,0) \\ \mathbf{r}_z(y,z) &= (\tfrac{2z}{3(1+y^2+z^2)^{2/3}}, 0, 1) & \mathbf{r}_y(0,1) &= (\tfrac{2}{3(2)^{2/3}}, 0, 1) = (\tfrac{2^{1/3}}{3}, 0, 1) \end{aligned}$$

is regular, since $\mathbf{r}_y(0, 1)$ and $\mathbf{r}_z(0, 1)$ exist, are non-zero and are linearly independent. (The last assertion follows from the fact that $\mathbf{r}_y(0, 1) \neq a\mathbf{r}_z(0, 1)$ for any $a \in \mathbb{R}$; indeed, this is impossible by inspecting the y and z components of the vectors). $\square$

Problem 3. Find all the critical points of the function $f(x, y) = x^3 + 3xy - y^3 + 2$ and classify them as local minima, local maxima, saddle points, or degenerate critical points.

Solution. Setting $\nabla f = (0, 0)$, we have

$$\nabla f(x,y) = (3x^2 + 3y, 3x - 3y^2) = (0,0) \iff \begin{cases} -x^2 = y \\ y^2 = x \end{cases}$$

Plugging the first equation into the second yeilds $x = x^4$, which has solutions $x = 0$ and $x = 1$. Plugging these solutions into the first equation, we find $x = 0 \implies y = 0$, and $x = 1 \implies y = -1$, so we have the two critical points $(0, 0)$ and $(1, -1)$. Next we use the second derivative test, evaluating the discriminant

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = (6x)(-6y) - 9$$

at the critical points. At $(0,0)$ we have

$$D(0,0) = -9 < 0$$

so $(0,0)$ is a **saddle**. At $(1,-1)$ we have

$$D(1,-1) = 6^2 - 9 > 0, \quad f_{xx}(1,-1) = 6 > 0$$

so $(1,-1)$ is a **local minimum**. $\square$