Math 2321 Fall 2015: Exam 1 Solutions

Problem 1. Consider the function $T(x, y, z) = ze^{-x^2-y^2}$.

(a) Compute the gradient $\nabla T(x, y, z)$.

Solution. $\nabla T(x, y, z) = \big((-2x)ze^{-x^2-y^2}, (-2y)ze^{-x^2-y^2}, e^{-x^2-y^2}\big)$. $\square$

(b) Compute the directional derivative of T at the point $(1, 0, 1)$ in the direction $\frac{1}{\sqrt{3}}(1, 1, 1)$.

Solution. We have $\nabla T(1, 0, 1) = (-2e^{-1}, 0, e^{-1})$, and $\mathbf{u} = \frac{1}{\sqrt{3}}(1, 1, 1)$. Then

$$D_{\mathbf{u}}T(1, 0, 1) = \nabla T(1, 0, 1) \cdots \mathbf{u} = \frac{1}{\sqrt{3}}(-2e^{-1} + 0 + e^{-1}) = -\frac{1}{e\sqrt{3}}.$$ $\square$

Problem 2. The problem concerns the surface $x + y^2 + z^2 = 4$ in $\mathbb{R}^3$.

(a) Determine a parameterization for this surface which is regular.

Solution. Rearranging the equation, we can write $x = 4 - y^2 - z^2$, so this is a "sideways graph" of a function $x = f(y, z)$. We can therefore use y and z as parameters, to get

$$\mathbf{r}(y, z) = (4 - y^2 - z^2, y, z).$$

The derivatives of this parameterization are

$$\mathbf{r}_y(y, z) = (-2y, 1, 0), \quad \mathbf{r}_z(y, z) = (-2z, 0, 1),$$

which always exist, are nonzero and are never parallel. Hence $\mathbf{r}$ is regular. $\square$

(b) Give a description (as an equation or by parameterization) of the tangent plane to the surface at the point $(2, 1, 1)$.

Solution. In terms of the parameterization $\mathbf{r}$ above, we can parameterize the tangent plane by

$$\Gamma(s, t) = (2, 1, 1)+s\mathbf{r}_y(1, 1)+t\mathbf{r}_z(1, 1) = (2, 1, 1)+s(-2, 1, 0)+t(-2, 0, 1) = (2-2s-2t, 1+s, 1+t).$$

(Here we used that $(2, 1, 1) = \mathbf{r}(1, 1)$.)

Alternatively, we can think of the surface as the level set $f(x, y, z) = x + y^2 + z^2 = 4$ and use the fact that $\nabla f(2, 1, 1)$ is normal to the surface there, to write the tangent plane as

$$0 = \nabla f(2, 1, 1)\cdot(x-2, y-1, z-1) = (1, 2, 2)\cdot(x-2, y-1, z-1) = (x-2)+2(y-1)+2(z-1).$$ $\square$

Problem 3. In $\mathbb{R}^2$, consider the level curve $f(x, y) = (x - 1)^2 - (y - 2)^2 = 3$. Find an equation for the tangent line to this level curve at the point $(3, 3)$.

Solution. The gradient is $\nabla f(x, y) = \big(2(x - 1), -2(y - 2)\big)$, and $\nabla f(3, 3) = (4, -2)$. The tangent line is given by

$$0 = \nabla f(3, 3) \cdot (x - 3, y - 3) = (4, -2) \cdot (x - 3, y - 3) = 4(x - 3) - 2(y - 3).$$

$\square$

Problem 4. Find all critical points of $f(x, y) = 2x^2y - 2x^2 - y^2$. and classify then as local maxima, local minima, saddle points, or degenerate critical points.

Solution. Computing the gradient, we find

$$\nabla f(x, y) = (4xy - 4x, 2x^2 - 2y) = (0, 0) \iff \begin{cases} x(y-1) = 0 \\ y = x^2 \end{cases}$$

From the first equation, either $x = 0$ or $y = 1$. If $x = 0$, then from the second equation $y = 0$, so we have a critical point at $(0, 0)$. If $y = 1$, then from the second equation we have $x = \pm 1$, so we have another two critical points at $(-1, 1)$ and $(1, 1)$.

The Hessian matrix of f is

$$Hf(x, y) = \begin{pmatrix} 4y - 4 & 4x \\ 4x & -2 \end{pmatrix}$$

which has determinant

$$D(x, y) = \det Hf(x, y) = (4y - 4)(-2) - (4x^2) = 8 - 8y - 16x^2.$$

- At the critical point $(0, 0)$, we have $D(0, 0) = 8 > 0$, and $f_{xx}(0, 0) = -4 < 0$, so this is a *local max.*
- At $(\pm 1, 1)$ we have $D(\pm 1, 1) = -16 < 0$, so these are *saddle points.*

□

Problem 5. Find the global maximum and minimum values of the function $f(x, y) = 3x - 4y$ on the disk $\{(x, y) : x^2 + y^2 \leq 4\}$ and the points where these values occur.

Solution. The gradient $\nabla f(x, y) = (3, -4)$ never vanishes, so there are no critical points inside the disk. Passing to the boundary, we can either parameterize it as $\mathbf{r}(\theta) = (2\cos\theta, 2\sin\theta)$ or use Lagrange multipliers, with the constraint $g(x, y) = x^2 + y^2 = 4$.

Using Lagrange multipliers, we solve $\nabla f = \lambda \nabla g$ along with $g = 4$, giving the system of equations

$$\begin{aligned} 3 &= 2\lambda x \\ -4 &= 2\lambda y \\ x^2 + y^2 &= 4. \end{aligned}$$

Sovling the first and second equations for x and y in terms of λ gives $x = \frac{3}{2\lambda}$ and $y = -\frac{4}{2\lambda}$. Plugging these into the third equation gives

$$4 = \frac{9}{4\lambda^2} + \frac{16}{4\lambda^2} = \frac{25}{4\lambda^2} \implies \lambda = \pm\frac{5}{4}.$$

Plugging these back in and solving for x and y gives the pair of solutions

$$(\tfrac{6}{5}, -\tfrac{8}{5}), \quad (-\tfrac{6}{5}, \tfrac{8}{5}).$$

Evaluating f at these points, we have

$$f(\tfrac{6}{5}, -\tfrac{8}{5}) = 10, \quad f(-\tfrac{6}{5}, \tfrac{8}{5}) = -10,$$

which are the global max and min, respectively. □