Calc III: Workshop 8 Solutions, Fall 2017

Problem 1. Use spherical coordinates to evaluate $\iiint_E y\, dV$, where E is the solid hemisphere inside $x^2 + y^2 + z^2 = 9$ where $y \geq 0$.

Solution. E is parameterized in spherical coordinates by $0 \leq \rho \leq 3$, $0 \leq \theta \leq \pi$ (only half way around), and $0 \leq \varphi \leq \pi$. Then

$$\begin{aligned}
\iiint_E y^2\, dV &= \int_0^\pi \int_0^\pi \int_0^3 (\rho \sin\varphi \sin\theta)\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \\
&= \int_0^\pi \sin\theta\, d\theta \int_0^\pi \sin^2\varphi\, d\varphi \int_0^3 \rho^3\, d\rho \\
&= (2)(\pi/2)(\tfrac{3^4}{4}) \\
&= \frac{81\pi}{4}
\end{aligned}$$

□

Problem 2. Suppose a hemisphere H has constant density. Find its center of mass.

Solution. We can arrange the hemisphere however we like, say the upper hemisphere of radius a, so inside $x^2 + y^2 + z^2 = a^2$ where $z \geq 0$. Then H would be parameterized by spherical coordinates $0 \leq \rho \leq a$, $0 \leq \theta \leq 2\pi$, and $0 \leq \varphi \leq \pi/2$ (halfway around). By symmetry, the center of mass is along the axis of the hemisphere, so has coordinates $(0, 0, \overline{z})$, where

$$\overline{z} = \frac{\iiint_H z(\delta)\, dV}{\iiint_H \delta\, dV} = \frac{\iiint_H z\, dV}{\iiint_H\, dV}$$

where the constant denisty δ cancels from the numerator and denominator (so we may as well take $\delta = 1$).

Alternatively, we can utilize the previous problem and parameterize the hemisphere where $y \geq 0$, in which case the center of mass is $(0, \overline{y}, 0)$, with

$$\overline{y} = \frac{\iiint_H y\, dV}{\iiint_H\, dV}.$$

The numerator was computed above (in the case that the radius $a = 3$), while the denominator is just half the volume of a sphere (of radius 3), so

$$\overline{y} = \frac{3^4\pi}{4} \frac{6}{4\pi 3^3} = \frac{3^2}{8},$$

or 3/8 of the way along the axis from the base to the outer shell. It follows that for a sphere of radius a, the center of mass will be at $\frac{3a}{8}$ along the axis. □

Problem 3. Use cylindrical coordinates to evaluate $\iiint_E z\, dV$, where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

Solution. E is parameterized in cylindrical coordinates by $r^2 \leq z \leq 4$, $0 \leq r \leq 2$ (the upper limit is given by setting $z = x^2 + y^2 = r^2$ equal to $z = 4$), and $0 \leq \theta \leq 2\pi$. Thus

$$\begin{aligned}
\iiint_E z\,dV &= \int_0^{2\pi} \int_0^2 \int_{r^2} 4z\,r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^2 \tfrac{r}{2}(z^2)\Big|_{z=r^2} 4\,dr\,d\theta \\
&= 2\pi \int_0^2 8r - \tfrac{1}{2}r^3\,dr \\
&= 2\pi(4r^2 + \tfrac{1}{8}r^4)\Big|_{r=0}^2 \\
&= 28\pi.
\end{aligned}$$

□

Problem 4. Find the center of mass of the solid S bounded by the parabolid $z = 4x^2 + 4y^2$ and the plane $z = a$ (where $a > 0$) if S has constant density.

Solution. As above, the constant density will cancel out of the center of mass computation, so we may as well assume that the density is 1, so mass equals volume. By symmetry, the center of mass will have coordinates $(0, 0, \overline{z})$, where

$$\overline{z} = \frac{\iiint_S z\,dV}{\iiint_S dV}.$$

It is convenient to set $a = b^2$ for some b and substitute back at the end. Then S is parameterized by $4r^2 \leq z \leq b^2$, $0 \leq r \leq b/2$ (solve $z = 4r^2 = b^2$ for r), and $0 \leq \theta \leq 2\pi$. The denominator is given by

$$\begin{aligned}
\iiint_S dV &= \int_0^{2\pi} \int_0^{b/2} \int_{4r^2}^{b^2} r\,dz\,dr\,d\theta \\
&= 2\pi \int_0^{b/2} r(b^2 - 4r^2)\,dr \\
&= 2\pi(\tfrac{b^2}{2}r^2 - r^4)\Big|_{r=0}^{b/2} \\
&= 2\pi(\tfrac{b^4}{2^3} - \tfrac{b^4}{2^4}) \\
&= \frac{b^4\pi}{8}.
\end{aligned}$$

Then the numerator is given by

$$\begin{aligned}
\iiint_S z\,dV &= \int_0^{2\pi} \int_0^{b/2} \int_{4r^2}^{b^2} zr\,dz\,dr\,d\theta \\
&= 2\pi\tfrac{1}{2} \int_0^{b/2} (b^4 - 16r^4)r\,dr \\
&= \pi(\tfrac{b^4}{2}r^2 - \tfrac{16}{6}r^6)\Big|_{r=0}^{b/2} \\
&= \frac{b^6\pi}{12}
\end{aligned}$$

Evaluating the quotient gives

$$\overline{z} = \frac{2b^2}{3} = \frac{2a}{3}.$$

□

Problem 5. Evaluate $\int_C (2+xy^2)\,ds$, where C is the upper half of the unit circle $x^2+y^2=1$.

Solution. We can parameterize C by $(x,y) = (\cos t, \sin t)$ where $0 \le t \le \pi$. Then

$$ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt = \sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = dt,$$

and

$$\begin{aligned}\int_C (2+xy^2)\,ds &= \int_0^\pi 2 + \cos t\,\sin^2 t\,dt \\ &= 4\pi + \int_0^\pi \cos t\,\sin^2 t\,dt \\ &= 4\pi.\end{aligned}$$

□

Problem 6. Let C be a curve in the plane, parameterized by $\big(x(t), y(t)\big)$ for $a \le t \le b$.

(a) Note that $\big(x(a+b-t), y(a+b-t)\big)$, where $a \le t \le b$ parameterizes the same curve, but in the opposite direction. Denote this "opposite direction curve" by $-C$.

(b) Show that $\int_C f(x,y)\,ds = \int_{-C} f(x,y)\,ds$.

Solution.

(a) If $x(t)$ and $y(t)$ satisfy the relevant equation for C (for any t), then so do $x(a+b-t)$ and $y(a+b-t)$. When $t=a$, then $a+b-t=b$ and when $t=b$, then $a+b-t=a$, so this traverses the same curve, but in the opposite direction.

(b) Choose a parameterization of C, with respect to which

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt.$$

Then use the change of variables $u = a+b-t$, $du = -dt$, to obtain

$$\begin{aligned}\int_C f(x,y)\,ds &= -\int_b^a f(x(a+b-u), y(a+b-u))\sqrt{x'(a+b-u)^2 + y'(a+b-u)^2}\,du \\ &= \int_a^b f(x(a+b-u), y(a+b-u))\sqrt{x'(a+b-u)^2 + y'(a+b-u)^2}\,du.\end{aligned}$$

But as noted above, this latter expression is a parameterization for $\int_{-C} f(x,y)\,ds$.

□