Calc III: Workshop 10, Fall 2018

Problem 1. Calculate $\oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = \langle x^2 + y, 3x - y^2 \rangle$ and C is the positively oriented boundary curve of a region D that has area 6.

Problem 2. Let D be a region bounded by a simple closed curve C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid (i.e., center of mass assuming uniform density) $(\overline{x}, \overline{y})$ of D are

$$\overline{x} = \frac{1}{2A} \oint_C x^2\, dy = \frac{1}{2A} \oint_C 0\, dx + x^2\, dy,$$
$$\overline{y} = \frac{1}{2A} \oint_C y^2\, dx = \frac{1}{2A} \oint_C y^2\, dx + 0\, dy,$$

where A is the area of D.

Problem 3. Use the previous exercise to show that the centroid of a quarter-circular region of radius a.

Problem 4. Use Green's Theorem to evaluate the line integral $\int_C \langle y + 1, x \rangle \cdot d\mathbf{r}$, where C is the upper half of the unit circle starting at $(1, 0)$ and ending at $(-1, 0)$. Note that C is not closed!

Problem 5.

(a) Show that for any vector field $\mathbf{F}(x, y, z)$ with twice continuously differentiable components, that

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

(b) Is there a vector field $\mathbf{G}$ such that $\nabla \times \mathbf{G} = \langle x \sin y, \cos y, z - xy \rangle$? Explain.

Problem 6.

(a) Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(x)\mathbf{i} + g(y)\mathbf{j} + h(z)\mathbf{k}$$

is irrotational ($\nabla \times \mathbf{F} = \mathbf{0}$).

(b) Show that any vector field of the form

$$\mathbf{F}(x, y, z) = f(y, z)\mathbf{i} + g(x, z)\mathbf{j} + h(x, y)\mathbf{k}$$

is incompressible ($\nabla \cdot \mathbf{F} = 0$).

Problem 7. Find a parametric representation for the following surfaces:

(a) The plane that passes through the point $(0, -1, 5)$ and contains the vectors $\langle 2, 1, 4 \rangle$ and $\langle -3, 2, 5 \rangle$.

(b) The part of the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ that lies to the left of the xz-plane.