Math 2321 Fall 2015: Quiz 7 Solutions

Problem 1. Consider the vector field $\mathbf{F}(x, y, z) = (-3z, 2x, 5y)$.
(a) Compute the divergence, $\nabla \cdot \mathbf{F}$, of $\mathbf{F}$.
(b) Compute the curl, $\nabla \times \mathbf{F}$, of $\mathbf{F}$.

Solution.
(a) For the divergence,

$$\nabla \cdot \mathbf{F} = (\tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z}) \cdot (-3z, 2x, 5y) = \tfrac{\partial}{\partial x}(-3z) + \tfrac{\partial}{\partial y}(2x) + \tfrac{\partial}{\partial z}(5y) = 0.$$

(b) For the curl,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -3z & 2x & 5y \end{vmatrix} = (5, -3, 2).$$

□

Problem 2. Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (y, x + y)$, and C is parameterized by $\mathbf{r}(t) = (2t, 4t^2)$, $0 \leq t \leq 1$.

Solution. We compute

$$\mathbf{r}'(t) = (2, 8t), \quad \mathbf{F}(\mathbf{r}(t)) = (4t^2, 2t + 4t^2)$$

so

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^1 8t^2 + 16t^2 + 32t^3\, dt \\ &= \int_0^1 24t^2 + 32t^3\, dt \\ &= \frac{24}{3} + \frac{32}{8} = 16. \end{aligned}$$

□

Problem 3. Let $\mathbf{F}(x, y, z) = (xz, y + z, x^2)$ be a force field in Newtons, and let C be the oriented curve where $z = \sqrt{x}$ and $y = x^3$, from $(0, 0, 0)$ to $(4, 64, 2)$, where all coordinates are in meters. Compute the work done by $\mathbf{F}$ along C. (You don't need to simplify your numerical answer; you can leave it in terms of powers of 2 or 4.) answer

Solution. We must choose how to parameterize C. One option is to use x as a parameter, writing $\mathbf{r}(x) = (x, x^3, \sqrt{x})$, $0 \leq x \leq 4$, however this leads to a slightly more unpleasant integral, in fractional powers of x. A nicer choice is to use z as a parameter:

$$\begin{aligned} &\mathbf{r}(z) = (z^2, z^6, z), \quad 0 \leq z \leq 2 \\ &\mathbf{r}'(z) = (2z, 6z^5, 1), \quad \mathbf{F}(\mathbf{r}(z)) = (z^3, z^6 + z, z^4). \end{aligned}$$

Then the work is given by

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 \mathbf{F}(\mathbf{r}(z)) \cdot \mathbf{r}'(z)\, dz \\
&= \int_0^2 2z^4 + 6(z^1 1 + z^6) + z^4\, dz \\
&= \frac{3}{5}(2)^5 + \frac{6}{12}(2)^{12} + \frac{6}{7}(2)^7 \text{ joules.}
\end{aligned}$$

$\square$