Calc III: Workshop 2 Solutions, Fall 2017

Problem 1. Find the point at which the line $x = 3 - t$, $y = 2 + t$, $z = 5t$ intersects the plane $x - y + 2z = 9$.

Solution. Plugging in for x, y, and z in terms of t, we have the equation

$$(3 - t) - (2 + t) + 2(5t) = 9,$$

which may be simplified to $8t = 8$, or $t = 1$. This is the point $(2, 3, 5)$. $\square$

Problem 2. Find the line of intersection of the planes

$$x + 3y + 2z - 6 = 0, \quad 2x - y + z + 2 = 0.$$

Solution. The planes have normal vectors $\mathbf{n}_1 = (1, 3, 2)$ and $\mathbf{n}_2 = (2, -1, 1)$, respectively. Since these are not parallel, the two planes must intersect, and the resulting line will be parallel to $\mathbf{n}_1 \times \mathbf{n}_2 = (5, 3, -7)$. It remains to find any single point in their intersection. Requiring both equations above to hold, we can set $x = 0$ (for instance), to get the system of equations

$$\begin{aligned} 3y + 2z &= 6, \\ y &= z + 2. \end{aligned}$$

The second is easily substituted into the first to get $z = 0$, from which we then have $y = 2$. Thus $(0, 2, 0)$ is a point on the line, and we can write a parameterized equation for the line as

$$\begin{aligned} x &= 0 + 5t, \\ y &= 2 + 3t, \\ z &= 0 - 7t. \end{aligned}$$

$\square$

Problem 3. Find the point of intersection (if any) of the line $\frac{x-6}{4} = y + 3 = z$ with the plane $x + 3y + 2z - 6 = 0$.

Solution. Plugging the equations for the line into the equation for the plane to eliminate y and z, we have

$$x + 3\left(\frac{x+6}{4} - 3\right) + 2\left(\frac{x+6}{4}\right) - 6 = 0$$

which simplifies to $x = 4$. Plugging this into the equation for the line gives the point $(4, -7/2, -1/2)$. $\square$

Problem 4. In general, any four non-coplanar points determine a unique sphere. Find the equation for the sphere determined by the points $(0, 0, 0)$, $(0, 0, 2)$, $(1, -4, 3)$, and $(0, -1, 3)$.

Solution. Plug these into the general form $x^2 + y^2 + z^2 + ax + by + cz + d = 0$ to get the system of equations

$$\begin{aligned} d &= 0, \\ 2c + d &= -4, \\ a - 4b + 3c + d &= -26, \\ -b + 3c + d &= -10 \end{aligned}$$

These can be solved by substitution to get $a = -4$, $b = 4$, $c = -2$ and $d = 0$. Completing the square and rewriting the equation in the form $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$, we find that the center of the sphere is $(x_0, y_0, z_0) = (-a/2, -b/2, -c/2) = (2, -2, 1)$ and the radius is $r = \sqrt{\frac{1}{4}(a^2 + b^2 + c^2) - d} = 3$. □

Problem 5. Let S be the sphere with radius 1 centered at $(0, 0, 1)$, and let S^* be S without the "north pole" at the point $(0, 0, 2)$. Let (a, b, c) be an arbitrary point on S^*. Then the line passing through $(0, 0, 2)$ and (a, b, c) intersects the xy-plane at a unique point $(x, y, 0)$. Find the equation for this point $(x, y, 0)$ in terms of (a, b, c). See Figure 1.6.10 in the book.

Remark. This sets up a one-to-one correspondence between points in the plane and points on the sphere with the north pole removed. This is known as *stereographic projection*.

Solution. The (vector) equation for the line through $(0, 0, 2)$ and (a, b, c) is

$$(x, y, z) = (0, 0, 2) + t(a, b, c - 2).$$

Solving for when $z = 0$ gives $t = \frac{2}{2-c}$, and then $(x, y, 0) = (\frac{2a}{2-c}, \frac{2b}{2-c}, 0)$. □