# CALLIAS' INDEX THEOREM AND MONOPOLE DEFORMATION

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Abstract. A generalization of Callias' index theorem for self adjoint Dirac operators with skew adjoint potentials on asymptotically conic manifolds is presented in which the potential term may have constant rank nullspace at infinity. This is subsequently used to compute the formal dimension of the space of SU(2) magnetic monopoles on asymptotically conic 3-manifolds. The dimension is shown to agree with the one computed by Braam for monopoles on certain asymptotically hyperbolic manifolds.

## Introduction

In [Cal78], Callias proved an index theorem on $\mathbb{R}^n$ for operators of the form $D + \Phi$, with $D$ a Dirac type operator and $\Phi$ a skew-adjoint (matrix) potential which is nondegenerate outside a compact set. A primary motivation was the consideration of spinors coupled to a background *magnetic monopole*, a pair consisting of a connection form $A \in \Omega^1(\mathbb{R}^3; \mathfrak{su}(2))$ and a Higgs field $\Phi \in C^\infty(\mathbb{R}^3; \mathfrak{su}(2))$ satisfying a certain PDE (see below). The spin Dirac operator coupled to such a monopole has the required form $D^A_{\mathrm{spin}} + \Phi$ since it turns out that $\Phi$ is necessarily nondegenerate as a $2 \times 2$ matrix outside a sufficiently large ball.

Callias' index theorem was subsequently generalized in [Ang93], [Råd94], [Bun95] and [Kot11] to include the case of arbitrary Riemannian manifolds, certain types of pseudodifferential operators and other settings; however there has remained another connection between Callias' index theorem and monopoles which has lacked an adequate mathematical treatment. Indeed, as noted originally in [Wei79], the linearization of the monopole equation around $(A, \Phi)$ *appears* to be a Callias type operator of the form $D^A_{\mathrm{sig}} + \Phi$ acting on $\mathfrak{su}(2)$ valued forms, whose index should therefore compute the dimension of the monopole moduli space. Indeed, here $D^A_{\mathrm{sig}}$ is a twisted version of the odd signature operator introduced in [APS75]; however $\Phi$ is acting on $\mathfrak{su}(2)$ through the adjoint representation and cannot be nondegenerate outside a compact set (since $[\Phi, \Phi] = 0$), so none of the existing Callias type index theorems apply.

In the present paper we remedy this situation, and present two main results:

(1) a generalization of the Callias index theorem in which the potential may have nullspace of constant rank at infinity, and
(2) a calculation of the formal dimension of the monopole moduli space using this index theorem.

The setting for these results is the class of spaces known variously as *asymptotically conic*, *asymptotically locally Euclidean* or *(exact) scattering manifolds.* These are compact manifolds with boundary equipped with complete interior metrics having

the form
$$g = \frac{dx^2}{x^4} + \frac{h}{x^2}$$
in a product neighborhood of the boundary, where $x$ is a boundary defining function and $h$ is a smooth family (in $x$) of boundary metrics. These include all Euclidean spaces (radially compactified) with $x = 1/r$, and many more spaces besides since the boundary may be essentially arbitrary and $h$ need not be constant in $x$. Our main result regarding monopoles is the following:

**Theorem.** *Let $X$ be an asymptotically conic 3-manifold for which the induced Laplacian on $\partial X$ has smallest nonzero eigenvalue at least that of the standard 2-sphere. Then the formal tangent space $T_{(A,\Phi)}\mathcal{M}_k$ to the moduli space of charge $k$ monopoles at a monopole $(A,\Phi)$ on $X$ has dimension $4k + \frac{1}{2}b^1(\partial X)$ and consists of $(a,\phi)$ smooth on the interior of $X$ having asymptotic expansions at $\partial X$ with components along* $\operatorname{span}\{\Phi|_{\partial X}\} \subset \mathfrak{su}(2)$ *of order*
$$\phi = \mathcal{O}(x^2), \quad a = \begin{cases} \mathcal{O}(x^2) & \text{if } b^1(\partial X) = 0, \\ \mathcal{O}(x) & \text{if } b^1(\partial X) \neq \{0\}. \end{cases}$$
*and with components along* $\operatorname{span}\left\{\Phi^\perp|_{\partial X}\right\}$ *vanishing to infinite order.*

The dimension obtained agrees with the result in [Bra89] for monopoles on certain asymptotically hyperbolic manifolds. We now discuss the results in more detail.

**The index theorem.** To describe the index theorem, it is helpful to consider the following simplified case. Let $X$ be a scattering manifold, and suppose $V = V_0 \oplus V_1$ is a global direct sum of vector bundles on $X$ with respect to which a self-adjoint Dirac operator $D \in \operatorname{Diff}^1_{\mathrm{sc}}(X;V)$ and compatible skew-adjoint potential $\Phi \in C^\infty\big(X;\operatorname{End}(V)\big)$ decompose diagonally as $D = D_0 \oplus D_1$, $\Phi = 0 \oplus \Phi_1$. Here compatibility means that $\Phi$ and $D$ commute to first order at $\partial X$ and that $\Phi_1|_{\partial X}$ is nondegenerate.

In this case $D + \Phi = D_0 \oplus (D_1 + \Phi_1)$ and we can analyze $D_0$ and $D_1 + \Phi_1$ separately. The operator $D_1 + \Phi_1$ is a conventional Callias type operator acting on sections of $V_1$. It has Fredholm extensions on standard Sobolev spaces $H^k_{\mathrm{sc}}(X;V_1)$, where derivatives are taken with respect to vector fields bounded with respect to the metric $g$ (i.e. $\left\{\partial_r, \frac{1}{r}\partial_\theta\right\}$ in the Euclidean case), with nullspace consisting of smooth sections vanishing to all orders at $\partial X$ and index equal to $\operatorname{ind}(\eth^+_+)$ where $\eth^+_+$ is a Dirac operator on $\partial X$ constructed in terms of $D_1$ and $\Phi_1$ — this is the usual Callias index theorem in this setting.

On the other hand $D_0$ has a very different Fredholm theory. It does not have Fredholm extensions with respect to the $H^k_{\mathrm{sc}}$ spaces, but rather with respect to *weighted* Sobolev spaces $x^\alpha H^k_{\mathrm{b}}(X;V_0)$ wherein derivatives are taken with respect to vector fields which are bounded with respect to $x^2 g$ (i.e. $\{r\partial_r, \partial_\theta\}$ in the Euclidean case). More specifically there are Fredholm extensions

$$D_0 : x^{\alpha-1/2}H^k_{\mathrm{b}}(X;V_0) \longrightarrow x^{\alpha+1/2}H^{k-1}_{\mathrm{b}}(X;V_0) \tag{I.1}$$

for all $\alpha \in \mathbb{R}\setminus B$ where $B$ is a discrete set of points for which a family of operators on $\partial X$ induced by $D_0$ and parametrized by $\alpha$ is not invertible, and the index $\operatorname{ind}(D_0,\alpha)$ of the extension depends on the choice of $\alpha$. The difference $\operatorname{ind}(D_0,\alpha) - \operatorname{ind}(D_0,\alpha')$ is the sum of the nullspace dimensions of the operator family on $\partial X$ at $B \cap (\alpha,\alpha')$, and the nullspace of $D_0$ itself consists of sections which are smooth on the interior of $X$, but have asymptotic expansions at $\partial X$ with terms of the form $x^r(\log x)^k$ for $(r,k) \in \mathbb{R}\times\mathbb{N}$ with the $r$ determined by the set $B$.

Putting these together, it follows that

(I.2)
$$D+\Phi : x^{\alpha-1/2}H^k_{\mathrm{b}}(X;V_0)\oplus H^l_{\mathrm{sc}}(X;V_1) \longrightarrow x^{\alpha+1/2}H^{k-1}_{\mathrm{b}}(X;V_0)\oplus H^{l-1}_{\mathrm{sc}}(X;V_1)$$

is Fredholm for $\alpha \notin B$, with index

$$\mathrm{ind}(D+\Phi) = \mathrm{ind}(\partial^+_+) + \mathrm{ind}(D_0,\alpha). \tag{I.3}$$

The second term may be viewed as a kind of *defect index*, and satisfies $\mathrm{ind}(D_0,-\alpha) = -\mathrm{ind}(D_0,\alpha)$ along with the difference formula alluded to above. (In particular $\mathrm{ind}(D_0,0) = 0$ provided $0 \notin B$, which reflects of the fact that the extension (I.1) is self-adjoint when $\alpha = 0$.)

The index theorem we prove says essentially that the above holds as well in the case that $D+\Phi$ does not decompose globally as a product. In general we assume merely that $\Phi$ and $D$ are compatible with $V|_{\partial X} = V_0 \oplus V_1$ decomposing into the nullspace of $\Phi|_{\partial X}$ and a bundle on which it is nondegenerate. After reviewing some background material in §1 we consider Fredholm extensions of $D+\Phi$ in §2, defining a family of *hybrid Sobolev spaces* which behave like the spaces in (I.2) on a neighborhood of $\partial X$ with respect to an extension of the splitting $V|_{\partial X} = V_0 \oplus V_1$.

In §3 we prove Theorem 3.6 which says that the index of the Fredholm extensions of §2 is given by a formula like (I.3), in which $\mathrm{ind}(D_0,\alpha)$ is now replaced by an abstract integer valued 'defect term' $\mathrm{def}(D+\Phi,\alpha)$ with the same properties as $\mathrm{ind}(D_0,\alpha)$ (the latter is no longer meaningful since $D_0$ is not globally defined, though the operators on $\partial X$ whose nullspaces determine $\mathrm{ind}(D_0,\alpha) - \mathrm{ind}(D_0,\alpha')$ *are* well-defined). The proof is by a clever argument suggested by Richard Melrose and Michael Singer, in which $\Phi$ is deformed continuously in a parameter $\tau$ to be nondegenerate at $\partial X$, and then a family of Fredholm parametrices is constructed which is sufficiently uniform in $\tau$ to allow for computation of the index. The parametrix family is constructed using a special *transition calculus* of pseudodifferential operators meant to capture the transition from Fredholm behavior like that of $D_1+\Phi_1$ to behavior like that of $D_0$ as a parameter $\tau \to 0$. Parts of this calculus first appeared in [GH08] where it was used to analyze the low energy limit of the resolvent of the Laplacian on scattering manifolds, and though we don't make use of all of its features, a rather complete development of the calculus has been included in Appendix B in case it may serve as a useful reference.

In fact the geometric setup of the transition calculus leads naturally to an interpretation of the index theorem as an index gluing result, in which a doubly infinite end $\partial X \times \mathbb{R}$ is glued onto $X$ along with a model operator which 'corrects' for the Fredholm behavior of $D_0$, and whose index appears naturally as the defect term.

**Monopoles.** In §4 we apply the index theorem to the deformation theory of monopoles on scattering 3-manifolds. By definition a monopole $(A,\Phi)$ satisfies the *Bogomolny equation* $\star d^A\Phi = F^A$ along with boundary conditions $|\Phi|\,|_{\partial X} = m$ and $H^2(\partial X;\mathbb{Z}) \ni c_1(L) = k$. Here $d^A = d + [A,\cdot]$ and $F^A = dA + \frac{1}{2}[A,A]$ are the covariant derivative and curvature associated to $A$, $\star$ is the Hodge star operator and $L \longrightarrow \partial X$ is the line bundle defined by the $+i$ eigenspace of $\Phi|_{\partial X}$. From the physical interpretation of monopoles, $m \in \mathbb{R}$ is the 'mass' (which we normalize to 1) and $k \in \mathbb{Z}^N$ (here $N$ is the number of connected components of $\partial X$) is the 'charge' of $(A, \Phi)$. The *gauge group*[1] $\mathcal{G} = \{u \in C^\infty(X; \mathrm{SU}(2)) : u|_{\partial X} = 1\}$ acts on monopoles, and of interest are the gauge equivalence classes of monopoles of given charge $k$, forming the *moduli space* $\mathcal{M}_k$. Since the theory of monopoles is only interesting if the underlying manifold is complete and noncompact there are considerable analytical difficulties with the usual geometric approach to deformation theory.

It is a now classical result that for Euclidean space $X = \mathbb{R}^3$, $\mathcal{M}_k$ is a smooth manifold of dimension $4k$. This dimension was computed by Weinberg in [Wei79], essentially by augmenting Callias' original result with some physical arguments to get around the noninvertibility of $\Phi|_{\partial X}$. Actual solutions for general $k$ were subsequently constructed by Jaffe and Taubes in [JT80] by grafting widely separated copies of the explicit charge 1 Bogomolny-Prasad-Sommerfield monopole. In [Ati84] Atiyah computed the dimension $\dim(\mathcal{M}_k) = 4k$ for $X = \mathbb{H}^3$ using a clever argument to avoid the difficulties of working with noncompact spaces; this argument was subsequently utilized by Braam in [Bra89] to consider monopoles on conformally compact $X$ (a type of asymptotically hyperbolic geometry).

Indeed, monopoles on $X$ are in correspondence with $S^1$-equivariant *instantons* on $X \times S^1$; this theory is conformally invariant, so by taking a conformal compactification of $X \times S^1$ (which requires $X$ to have conformally compact geometry) one is reduced to analysis on compact manifolds at the cost of adding equivariance. The equivariant index theorem may be used to compute the formal dimension of $\mathcal{M}_k$, and Braam shows[2] this dimension to be $4k + \frac{1}{2}b^1(\partial X)$. Braam goes on to construct monopoles by gluing in the case that $b^2(X) = 0$, whereas Atiyah uses twistor theory to construct solutions on $\mathbb{H}^3$. Also of note are the posthumously published works [Flo95b] and [Flo95a] of Floer in which he uses gluing arguments to obtain monopoles on manifolds with Euclidean ends.

For $X$ a scattering manifold, we attack the problem directly. We set up a three term elliptic complex whose first map represents infinitesimal gauge transformations and whose second map represents the linearized Bogomolny equation. The Hodge operator of the complex is of Callias type, and a Weitzenbock type result shows the operator to be surjective on a certain range of weighted hybrid Sobolev spaces. Along the way we give a detailed account of the odd signature operator. As is typical for scattering type geometry, the discrete set $B$ controlling the defect index contains points related both to the topology of $\partial X$ as well as the geometry, and by assuming that $\nu_i \geq 2$ where $\nu_i$ are the positive eigenvalues of $\Delta_{\partial X}$, we obtain the result stated above as Theorem 4.8.

The construction of actual monopoles on scattering manifolds along with a compactification of their moduli space will be the subject of a future work by the author along with R. Melrose and M. Singer.

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[1]In fact we consider a more general gauge group consisting of sections smooth on the interior, with polyhomogeneous (singular) asymptotic expansions at $\partial X$.

[2]The actual dimension Braam computes differs from this by $-\#\pi_0(\partial X)$, but this is accounted for by a different convention for the gauge group: Braam allows $\mathcal{G}$ to act by an extra $S^1$ on each component of $\partial X$ whereas we do not.

## 1. Technical Background

1.1. **Differential operators.** Let $X$ be an $n$ dimensional manifold with boundary, with a boundary defining function $x$. By definition this means $x \in C^\infty(X; [0,\infty))$ with $x^{-1}(0) = \partial X$ and $dx|_{\partial X} \neq 0$, and $x$ is unique up to multiplication by a smooth strictly positive function. The *b vector fields* $\mathcal{V}_{\mathrm{b}}(X)$ are those smooth vector fields which are tangent to $\partial X$; they are characterized by the property

$$\mathcal{V}_{\mathrm{b}}(X) \cdot xC^\infty(X) \subset xC^\infty(X). \tag{1.1}$$

The *scattering vector fields* $\mathcal{V}_{\mathrm{sc}}(X)$ are defined by $\mathcal{V}_{\mathrm{sc}}(X) = x\mathcal{V}_{\mathrm{b}}(X)$. While $\mathcal{V}_{\mathrm{b}}(X)$ is naturally associated to $(X, \partial X)$, $\mathcal{V}_{\mathrm{sc}}(X)$ depends on the choice on $x$. As derivations of $C^\infty(X)$ these satisfy

$$[\mathcal{V}_{\mathrm{b}}(X), \mathcal{V}_{\mathrm{b}}(X)] \subset \mathcal{V}_{\mathrm{b}}(X), \tag{1.2}$$

$$[\mathcal{V}_{\mathrm{b}}(X), \mathcal{V}_{\mathrm{sc}}(X)] \subset \mathcal{V}_{\mathrm{sc}}(X) \text{ and} \tag{1.3}$$

$$[\mathcal{V}_{\mathrm{sc}}(X), \mathcal{V}_{\mathrm{sc}}(X)] \subset x\mathcal{V}_{\mathrm{sc}}(X). \tag{1.4}$$

In other words, $\mathcal{V}_{\mathrm{b}}(X)$ and $\mathcal{V}_{\mathrm{sc}}(X)$ are Lie subalgebras of the algebra $\mathcal{V}(X)$ of smooth vector fields and $\mathcal{V}_{\mathrm{sc}}(X)$ is a $\mathcal{V}_{\mathrm{b}}(X)$-module.

Near $\partial X$ these vector fields may be characterized in terms of local coordinates. If $(x, y)$ form coordinates where the $y_i$ are coordinates on $\partial X$ and $x$ is the boundary defining function then

$$\mathcal{V}_{\mathrm{b}}(X) \overset{\mathrm{loc}}{=} \mathrm{span}_{C^\infty(X)} \{x\partial_x, \partial_{y_i}\}$$
$$\mathcal{V}_{\mathrm{sc}}(X) \overset{\mathrm{loc}}{=} \mathrm{span}_{C^\infty(X)} \{x^2\partial_x, x\partial_{y_i}\}$$

Thus, over $C^\infty(X)$, $\mathcal{V}_{\mathrm{b}}(X)$ and $\mathcal{V}_{\mathrm{sc}}(X)$ are locally free sheaves of rank $n = \dim(X)$ and as such may be identified as the sections of well-defined vector bundles, namely the *b tangent* and *scattering tangent* bundles:

$$\mathcal{V}_{\mathrm{b}}(X) \equiv C^\infty(X; {}^{\mathrm{b}}TX) \qquad {}^{\mathrm{b}}TX \overset{\mathrm{loc}}{=} \mathrm{span}_{\mathbb{R}} \{x\partial_x, \partial_{y_i}\}$$
$$\mathcal{V}_{\mathrm{sc}}(X) \equiv C^\infty(X; {}^{\mathrm{sc}}TX) \quad {}^{\mathrm{sc}}TX \overset{\mathrm{loc}}{=} \mathrm{span}_{\mathbb{R}} \left\{x^2\partial_x, x\partial_{y_i}\right\}.$$

The associated cotangent bundles ${}^{\mathrm{b}}T^*X$, ${}^{\mathrm{sc}}T^*X$ are defined by duality; they admit respective local frames $\{dx/x, dy_i\}$ and $\left\{dx/x^2, dy_i/x\right\}$. There are well-defined bundle maps

$${}^{\mathrm{sc}}TX \longrightarrow {}^{\mathrm{b}}TX \longrightarrow TX \tag{1.5}$$

induced by the inclusions $\mathcal{V}_{\mathrm{sc}}(X) \subset \mathcal{V}_{\mathrm{b}}(X) \subset \mathcal{V}(X)$ — indeed, these are just the obvious maps $x^2\partial_x \longmapsto x(x\partial_x) \longmapsto x^2(\partial_x)$ and $x\partial_{y_i} \longmapsto x(\partial_{y_i})$ — which, though they are isomorphisms away from $\partial X$, are neither injective nor surjective at $\partial X$. As operators on $C^\infty(X)$, elements of $\mathcal{V}_{\mathrm{b}}(X)$ (resp. $\mathcal{V}_{\mathrm{sc}}(X)$) may be composed, resulting in *b (resp. scattering) differential operators*

$$\mathrm{Diff}^k_{\mathrm{b}}(X) = \mathrm{Diff}^{k-1}_{\mathrm{b}}(X) + \{V_1 \cdots V_k : V_i \in \mathcal{V}_{\mathrm{b}}(X)\}$$
$$\mathrm{Diff}^k_{\mathrm{sc}}(X) = \mathrm{Diff}^{k-1}_{\mathrm{sc}}(X) + \{V_1 \cdots V_k : V_i \in \mathcal{V}_{\mathrm{sc}}(X)\}$$

where $\mathrm{Diff}^0_{\mathrm{b}}(X) = \mathrm{Diff}^0_{\mathrm{sc}}(X) = C^\infty(X)$. Alternatively, the (filtered) algebra $\mathrm{Diff}^*_{\mathrm{b}}(X)$ may be considered as the universal enveloping algebra of $\mathcal{V}_{\mathrm{b}}(X)$ over the ring $C^\infty(X)$ and likewise for $\mathrm{Diff}^*_{\mathrm{sc}}(X)$. The definition of such operators acting on sections of vector bundles requires the notion of a connection which is discussed next.

1.2. **Connections and Dirac operators.** Let $V \longrightarrow X$ be a vector bundle, with associated principal (frame) bundle $\pi : P \longrightarrow X$. As $P$ is also a manifold with boundary $\partial P = \pi^{-1}(\partial X)$ and defining function $\pi^*(x)$, the b and scattering tangent bundles of $P$ are well-defined, and $d\pi$ extends by continuity from the interior to give short exact sequences

$$
\begin{aligned}
0 \longrightarrow & V_p P \longrightarrow {}^{\mathrm{b}}T_p P \xrightarrow{d\pi} {}^{\mathrm{b}}T_{\pi(p)} X \longrightarrow 0 \\
0 \longrightarrow & V_p P \longrightarrow {}^{\mathrm{sc}}T_p P \xrightarrow{d\pi} {}^{\mathrm{sc}}T_{\pi(p)} X \longrightarrow 0
\end{aligned}
\tag{1.6}
$$

for each $p \in P$, where $VP := \ker d\pi$. As in the ordinary case of closed manifolds, $V_p P$ is isomorphic to the lie algebra $\mathfrak{g}$ of the structure group $G$ of $P$, and a *b (resp. scattering) connection* is defined by one of three equivalent objects:

- An equivariant (with respect to the right action by $G$) choice of splitting ${}^{\mathrm{b}}T_p P \cong {}^{\mathrm{b}}T_{\pi(p)} X \oplus V_p P$ (resp. ${}^{\mathrm{sc}}T_p P \cong {}^{\mathrm{sc}}T_{\pi(p)} X \oplus V_p P$) of the exact sequence above.
- A $\mathfrak{g}$ valued b (resp. scattering) one-form $\omega \in C^\infty(P; {}^{\mathrm{b/sc}}T^* P \otimes \mathfrak{g})$ satisfying $\omega|_{VP \cong \mathfrak{g}} \equiv \mathrm{Id}$ and transforming equivariantly via $\omega_{p \cdot g} = \mathrm{Ad}(g^{-1}) \cdot \omega_p$ for $g \in G$.
- A covariant derivative operator $\nabla : C^\infty(X; V) \longrightarrow C^\infty(X; {}^{\mathrm{b/sc}}T^* X \otimes V)$ satisfying $\nabla f\, s = df\, s + f \nabla s$ for $f \in C^\infty(X)$, $s \in C^\infty(X; V)$. Note that $d : C^\infty(X) \longrightarrow C^\infty({}^{\mathrm{b/sc}}T^* X)$ is well-defined by continuity from the interior of $X$, and $\nabla$ is related to $\omega$ locally via $\nabla = d + \gamma^* \omega$ with respect to a local trivializing section $\gamma : U \subset X \longrightarrow \pi^{-1}(U) \subset P$.

The dual sequence to (1.5) gives rise to maps

$$
C^\infty(P; T^* P) \longrightarrow C^\infty(P; {}^{\mathrm{b}}T^* P) \longrightarrow C^\infty(P; {}^{\mathrm{sc}}T^* P)
$$

and we say that a b (resp. scattering) connection is the *lift of a true (resp. b) connection* if its connection form $\omega \in C^\infty(P; {}^{\mathrm{b/sc}}T^* P \otimes \mathfrak{g})$ — or equivalently each of the local connection forms for $\nabla$ on $X$ — is in the image of the corresponding map. As an example, it will be shown below that the Levi-Civita connection for an exact scattering metric is always the lift of a b-connection. Note that $d$, acting on functions, may be considered as the covariant derivative associated to a b or scattering connection on the trivial line bundle and in this sense is always the lift of a true connection on $X$.

In light of the inclusions $T\partial X \hookrightarrow TX|_{\partial X}$ and $T\partial X \hookrightarrow {}^{\mathrm{b}}TX|_{\partial X}$, true and b connections naturally induce connections on $V|_{\partial X}$ by restriction. In contrast, a scattering connection generally does *not* induce such a connection on $V|_{\partial X}$ unless it is the lift of a b connection.

With a connection on $V$ of the appropriate type, differential operators $\mathrm{Diff}^*_{\mathrm{b}}(X; V)$ (resp. $\mathrm{Diff}^*_{\mathrm{sc}}(X; V)$) may be defined as the universal enveloping algebra of $\mathcal{V}_{\mathrm{b}}(X)$ (resp. $\mathcal{V}_{\mathrm{sc}}(X)$) over $C^\infty(X; \mathrm{End}(V))$:

$$
\begin{aligned}
\mathrm{Diff}^k_{\mathrm{b}}(X; V) &= \mathrm{Diff}^{k-1}_{\mathrm{b}}(X; V) + \left\{ \nabla_{V_1} \cdots \nabla_{V_k} : V_i \in \mathcal{V}_{\mathrm{b}}(X) \right\}, \\
\mathrm{Diff}^k_{\mathrm{sc}}(X; V) &= \mathrm{Diff}^{k-1}_{\mathrm{sc}}(X; V) + \left\{ \nabla_{V_1} \cdots \nabla_{V_k} : V_i \in \mathcal{V}_{\mathrm{sc}}(X) \right\}, \\
\mathrm{Diff}^0_{\mathrm{b}}(X; V) &= \mathrm{Diff}^0_{\mathrm{sc}}(X; V) = C^\infty(X; \mathrm{End}(V)).
\end{aligned}
$$

**Lemma 1.1.** *Suppose $P \in \mathrm{Diff}^k_{\mathrm{sc}}(X; V)$ is defined as above in terms of a scattering connection $\nabla$. If $\nabla$ is the lift of a b-connection, then $P = x^k P'$ where $P' \in \mathrm{Diff}^k_{\mathrm{b}}(X; V)$.*

*Proof.* If $\nabla$ is the lift of a b-connection and $\mathcal{V}_{\mathrm{sc}}(X) \ni V = xV'$ for $V' \in \mathcal{V}_{\mathrm{b}}(X)$ then the identity

$$\nabla_V = \nabla_{xV'} = x\nabla_{V'}$$

holds. The claim then follows by induction, the derivation property of $\nabla$ and the property (1.1). $\square$

Of primary importance are Dirac operators. As in the familiar case of closed manifolds, a (positive definite) metric $g \in C^\infty\big(X; \mathrm{Sym}^2({}^{\mathrm{sc}}T^*X)\big)$ gives rise to the *Clifford bundle*

$$\mathbb{C}\ell_{\mathrm{sc}}(X) \longrightarrow X, \quad \mathbb{C}\ell_{\mathrm{sc}}(X)_p = \mathbb{C}\ell({}^{\mathrm{sc}}T_pX, g).$$

Our convention for the Clifford algebra is that if $\{e_i\}$ is a basis for a vector space $W$, then $\mathbb{C}\ell(W, g)$ is generated by $e_I = e_{i_1} \cdots e_{i_k}$, $I = \{i_1 < \cdots < i_k\}$ subject to canonical anticommutation relations

$$e_ie_j + e_je_i = -2g\,(e_i, e_j)\,.$$

We say $V \longrightarrow X$ is a Clifford module if it admits a fiberwise action $\mathrm{c}\ell : \mathbb{C}\ell_{\mathrm{sc}}(X) \longrightarrow \mathrm{End}(V)$, and given a scattering connection on $V$ with covariant derivative $\nabla$ satisfying $\nabla\big(\mathrm{c}\ell(v)s\big) = \mathrm{c}\ell\big(\nabla^{\mathrm{LC}}v\big)s + \mathrm{c}\ell(v)\nabla s$, there exists a canonical *scattering Dirac operator* defined by

$$D := \mathrm{c}\ell \circ \nabla \in \mathrm{Diff}^1_{\mathrm{sc}}(X; V) \tag{1.7}$$

where we use the identification $g : {}^{\mathrm{sc}}T^*X \cong {}^{\mathrm{sc}}TX$ and the Clifford contraction map $\mathrm{c}\ell : C^\infty(X; {}^{\mathrm{sc}}TX \otimes V) \longrightarrow C^\infty(X; V)$. If $V$ is equipped with a Hermitian structure for which the Clifford action is skew-adjoint, the usual proof (see [LM89], Prop. 5.3) shows that $D$ is formally self-adjoint with respect to the Hilbert space $L^2(X; V)$ defined by the Hermitian structure and $g$.

We restrict attention to *exact scattering metrics*, meaning $g$ has the form

$$g = \frac{dx^2}{x^4} + \frac{h(x, y, dy)}{x^2}, \tag{1.8}$$

in a collar neighborhood of $\partial X$ where the restriction $h|_{\partial X}$ is an ordinary metric on $\partial X$, and therefore by continuity $h(x)$ gives a family of nondegenerate metrics in a neighborhood of $\partial X$. In particular for $g$ to be exact scattering, the vector field $x^2\partial_x$ must have length 1 and there may be no cross terms of the form $\frac{dx}{x^2}\frac{dy}{x}$.

**Lemma 1.2.** *The Levi-Civita connection $\nabla^{\mathrm{LC}}$ with respect to an exact scattering metric $g$ is the lift of a b-connection. In fact, with respect to the local trivializing frame $\big\{e_0 = x^2\partial_x, e_i = x\partial_{y_i}\big\}$ of the frame bundle $P_{SO(n)} \longrightarrow X$ of ${}^{\mathrm{sc}}TX$*

$$\nabla^{\mathrm{LC}} \overset{\mathrm{loc}}{=} d + A^{\mathrm{LC}(h)} + x\sum_{i=1}^{n-1} e_0 \wedge e_i \frac{dy_i}{x} \tag{1.9}$$

*where we use the identification $\Lambda^2\mathbb{R}^n \cong \mathfrak{so}(n)$, and where $A^{\mathrm{LC}(h)} = \sum_i xA_i\frac{dy_i}{x}$ is the image in $C^\infty(X; {}^{\mathrm{sc}}T^*X \otimes \mathfrak{so}(n-1))$ of the local connection form $A^{\mathrm{LC}(h(x))} = \sum_i A_i\,dy_i \in C^\infty(X; {}^{\mathrm{b}}T^*X \otimes \mathfrak{so}(n-1))$ for $\nabla^{\mathrm{LC}(h(x))}$, the family of Levi-Civita connections on $\partial X$ associated to the metrics $h(x)$ in* (1.8) *which are parametrized by $x$.*

*Proof.* The Kozsul formula for (1.8) gives

$$\begin{aligned}
\nabla^{\mathrm{LC}}_{x^2\partial_x} x^2\partial_x &= \nabla^{\mathrm{LC}}_{x^2\partial_x} x\partial_{y_i} = 0 \\
\nabla^{\mathrm{LC}}_{x\partial_{y_i}} x^2\partial_x &= -x(x\partial_{y_i}) \\
\nabla^{\mathrm{LC}}_{x\partial_{y_i}} x^2\partial_{y_j} &= x(x^2\partial_x)\delta_{ij} + x\nabla^{\mathrm{LC}(h)}_{\partial_{y_i}}\partial_{y_j}
\end{aligned}$$

from which the claim follows. □

As a special case of Lemma 1.1 we have the following, which applies in particular to geometric Dirac operators in light of Lemma 1.2.

**Corollary 1.3.** *If $V \longrightarrow X$ is a Clifford module whose Clifford connection $\nabla$ is the lift of a b-connection, then the Dirac operator $D$ of* (1.7) *is equal to $x\widetilde{D}$ for some $\widetilde{D} \in \mathrm{Diff}^1_{\mathrm{b}}(X;V)$.*

In fact we can be more precise. It is a classical fact that if a manifold with boundary $X$ is equipped with an incomplete metric of product type near $\partial X$, then any Dirac operator $D \in \mathrm{Diff}^1(X;V)$ may be written locally near $\partial X$ as

$$D = \mathrm{c}\ell(e_0)\nabla_{e_0} + \sum_{i=1}^{n-1}\mathrm{c}\ell(e_i)\nabla_{e_i} = \mathrm{c}\ell(e_0)\left(\nabla_{e_0} + \underbrace{\sum_{i=1}^{n-1}\mathrm{c}\ell(e_i\, e_0)\nabla_{e_i}}_{D'}\right)$$

where $e_0 \in C^\infty(X; N\partial X)$ is a unit normal section, $\{e_i\}_{i=1}^{n-1}$ is an orthonormal frame for $T\partial X$ and $D' \in \mathrm{Diff}^1(\partial X; V^\pm, V^\mp)$ defines a Dirac operator on $\partial X$ which is $\mathbb{Z}_2$ graded with respect to the splitting $V|_{\partial X} = V^+ \oplus V^-$ according to $\pm 1$ eigenspaces of $i\mathrm{c}\ell(e_0)$.

A similar situation occurs for a scattering Dirac operator $D$ provided the connection is the lift of a b-connection. For in this this case we may write $D$ locally near $\partial X$ as

$$D = \mathrm{c}\ell(x^2\partial_x)\left(\nabla_{x^2\partial_x} + xD'\right) = x\left(\mathrm{c}\ell(x^2\partial_x)\left(\nabla_{x\partial_x} + D'\right)\right)$$

where $D' = D'(x)$ is a family of Dirac operators on $\partial X$ given by

$$D' = D'(x) = \sum_{i=1}^{n-1}\mathrm{c}\ell\big((xe(x)_i)(x^2\partial_x)\big)\nabla_{e(x)_i}$$

in terms of orthonormal frames $\{e(x)_i\}$ for the metrics $h(x)$ of (1.8). We refer to $D'(0) \in \mathrm{Diff}^1(\partial X; V^\pm, V^\mp)$ as the *induced boundary Dirac operator* of $D$ and emphasize that it is entirely determined by

(1) the induced connection $\nabla|_{T\partial X}$, and
(2) the *induced Clifford action* $\mathrm{c}\ell_\partial : \mathbb{C}\ell(T\partial X) \longrightarrow \mathrm{End}_{\mathbb{Z}_2}(V^+ \oplus V^-)$ where

$$\mathrm{c}\ell_\partial(e_i) := \mathrm{c}\ell\big((xe_i)(x^2\partial_x)\big).$$

For a geometric Dirac operator $D_{(X,g)}$ on an exact scattering manifold (the Hodge de Rham operator, the spin Dirac operator etc.), the induced boundary operator is generally *not* the corresponding geometric operator $D_{(\partial X,h)}$[3]. Indeed, according to Lemma 1.2 the induced connection differs from $\nabla^{\mathrm{LC}(h)}$ by the term

[3] Neither is $\widetilde{D} = x^{-1}D$ the corresponding geometric operator with respect to the metric $\widetilde{g} = x^2 g$.

$\sum_i e_0 \wedge e_i \, dy_i$ so that the induced operator $D'$ and geometric operator $D_{(\partial X,h)}$ are related by

$$
\begin{aligned}
D' &= D_{(\partial X,h)} + E,\\
E &= \sum_i \mathrm{c}\ell_\partial(\partial_{y_i}) \circ (e_0 \wedge e_i)\cdot \in C^\infty\big(\partial X, \mathrm{End}_{\mathbb{Z}_2}(V^+ \oplus V^-)\big).
\end{aligned}
\tag{1.10}
$$

We point out however that the index theory of $D'$ and $D_{(\partial X,h)}$ is the same since $E$ is lower order.

1.3. **Half-densities and Sobolev spaces.** Before discussing Fredholm results in the next section, a word must be said about Sobolev spaces. First of all note that a scattering metric such as (1.8) restricts to a complete Riemannian metric on the interior of $X$, and likewise for a b-metric (a positive definite element of $C^\infty\big(X; \mathrm{Sym}^2({}^{\mathrm{b}}T^*X)\big)$). Along with a Hermitian structure on $V$, either of these may be used to define $L^2(X;V)$ spaces; however the most natural and invariant treatment involves half-density bundles.

For any $s \in \mathbb{R}$, the $s$-density bundles $\Omega^s_{\mathrm{b}}(X) \longrightarrow X$ and $\Omega^s_{\mathrm{sc}}(X) \longrightarrow X$ are defined by

$$
\begin{aligned}
\Omega^s_{\mathrm{b}}(X)_p &= \big\{ v : \Lambda^n({}^{\mathrm{b}}T_pX) \setminus 0 \longrightarrow \mathbb{R} \;\big|\; v(tu) = |t|^s \, v(u) \big\}\\
\Omega^s_{\mathrm{sc}}(X)_p &= \big\{ v : \Lambda^n({}^{\mathrm{sc}}T_pX) \setminus 0 \longrightarrow \mathbb{R} \;\big|\; v(tu) = |t|^s \, v(u) \big\}
\end{aligned}
$$

In other words, $\Omega^s_{\mathrm{b}}(X)_p$ behaves essentially as the $s$ power of the absolute value of the maximal exterior product $\Lambda^n({}^{\mathrm{b}}T^*_pX)$, and likewise for $\Omega^s_{\mathrm{sc}}(X)_p$.

For $s = 1$, there are invariantly defined integration maps

$$
\int_X : C^\infty(X; \Omega_{\mathrm{b/sc}}(X)) \longrightarrow \mathbb{R} \cup \pm\infty
$$

which are well-defined regardless of the orientability of $X$ since the density bundles transform in accordance with the change of variables formula for Riemann integration. Likewise, for $s = 1/2$ there is a natural bilinear pairing

$$
\int_X : C^\infty\big(X; \Omega^{1/2}_{\mathrm{b/sc}}(X)\big) \times C^\infty\big(X; \Omega^{1/2}_{\mathrm{b/sc}}(X)\big) \longrightarrow \mathbb{R} \cup \pm\infty
$$

which may be extended to $C^\infty\big(X; V \otimes \Omega^{1/2}_{\mathrm{b/sc}}(X)\big)$ whenever $V$ is equipped with a bilinear form. Taking the completion of sections which are compactly supported in the interior of $X$ leads to the natural Hilbert spaces $L^2(X; V \otimes \Omega^{1/2}_{\mathrm{sc}})$ and $L^2(X; V \otimes \Omega^{1/2}_{\mathrm{b}})$. We may freely pass from one to the other in light of the equivalence

$$
\begin{aligned}
L^2(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) &= x^{n/2} L^2(X; V \otimes \Omega^{1/2}_{\mathrm{b}})\\
&= \Big\{ u = x^{n/2} v : v \in L^2(X; V \otimes \Omega^{1/2}_{\mathrm{b}}) \Big\}
\end{aligned}
\tag{1.11}
$$

which follows from the identification $\Omega^{1/2}_{\mathrm{sc}} \cong x^{-n/2} \Omega^{1/2}_{\mathrm{b}}$.

For $k \in \mathbb{N}$, the *b (resp. scattering) Sobolev spaces* are defined to be the common maximal domains for b (resp. scattering) differential operators of order $k$:

$$
\begin{aligned}
H^k_{\mathrm{b}}(X; V \otimes \Omega^{1/2}_{\mathrm{b/sc}}) &= \Big\{ u \in L^2(X; V \otimes \Omega^{1/2}_{\mathrm{b/sc}}) : Pu \in L^2, \ \forall\, P \in \mathrm{Diff}^k_{\mathrm{b}}(X; V \otimes \Omega^{1/2}_{\mathrm{b/sc}}) \Big\}\\
H^k_{\mathrm{sc}}(X; V \otimes \Omega^{1/2}_{\mathrm{b/sc}}) &= \Big\{ u \in L^2(X; V \otimes \Omega^{1/2}_{\mathrm{b/sc}}) : Pu \in L^2, \ \forall\, P \in \mathrm{Diff}^k_{\mathrm{sc}}(X; V \otimes \Omega^{1/2}_{\mathrm{b/sc}}) \Big\}
\end{aligned}
$$

Spaces of order $s \in \mathbb{R}$ may also be defined using pseudodifferential operators of the appropriate type, though we shall not require them. We will make use of weighted versions $x^\alpha H^k_{\mathrm{b/sc}}$ of these spaces, along with the spaces

$$\begin{aligned} H^{k,l}_{\mathrm{b,sc}}(X; V\otimes\Omega^{1/2}_{\mathrm{b/sc}}) &= \Big\{ u \in L^2 : D\circ Pu \in L^2 \ \forall\ D \in \mathrm{Diff}^k_{\mathrm{b}},\ P\in \mathrm{Diff}^l_{\mathrm{sc}} \Big\} \\ &= \Big\{ u \in L^2 : P\circ Du \in L^2 \ \forall\ D \in \mathrm{Diff}^k_{\mathrm{b}},\ P\in \mathrm{Diff}^l_{\mathrm{sc}} \Big\}. \end{aligned}$$

The equivalence of the two definitions here follows from (1.3).

Finally, if $X$ *is* equipped with a scattering (respectively b) metric $g$, then there is a canonical trivializing section of $\Omega^s_{\mathrm{sc}}(X)$ (resp. $\Omega^s_{\mathrm{b}}(X)$) given by $|d\mathrm{Vol}_g|^s$ which may be used to promote ordinary sections of $V$ to half-densities, and the resulting $L^2$ space is the same as the one described in the beginning of this section. Differential operators may be lifted to act on half-density sections by utilizing a connection with respect to which the canonical section $|d\mathrm{Vol}_g|^s$ is covariant constant.

1.4. **The scattering calculus.** We briefly review the Fredholm theory of scattering operators. We refer the reader to [Mel94] for proofs of the results here.

For $P \in \mathrm{Diff}^k_{\mathrm{sc}}(X;V)$, two kinds of symbols are defined. The first is an version $\sigma_k(P) \in C^\infty({}^{\mathrm{sc}}T^*X; \mathrm{End}(V))$ of the usual principal symbol for differential operators, given locally by

$$\sigma_k : \sum_{j+|\alpha|\leq k} a_{j,\alpha}(x,y)(x^2\partial_x)^j(x\partial_y)^\alpha \longmapsto \sum_{j+|\alpha|=k} a_{j,\alpha}(x,y)(i\xi)^j(i\eta)^\alpha \tag{1.12}$$

The second is the fiberwise Fourier transform with respect to ${}^{\mathrm{sc}}TX|_{\partial X} \longrightarrow \partial X$ of the 'restriction' of $P$ to the boundary in the following sense: It follows from (1.4) that the quotient Lie algebra $\mathcal{V}_{\mathrm{sc}}(X)/x\,\mathcal{V}_{\mathrm{sc}}(X)$ is abelian, and this quotient map along with the restriction $|_{\partial X} : C^\infty(X;\mathrm{End}(V)) \longrightarrow C^\infty(\partial X; \mathrm{End}(V))$ leads to the restriction of $P$ to $\partial X$, denoted by $N_{\mathrm{sc}}(P)$, which may be viewed as a fiberwise differential operator $N_{\mathrm{sc}}(P) \in \mathrm{Diff}^k_{\mathrm{fib},I}({}^{\mathrm{sc}}TX|_{\partial X}, V)$ which is translation invariant with respect to the vector bundle structure (hence the subscript $I$). Locally,

$$N_{\mathrm{sc}} : \sum_{j+|\alpha|\leq k} a_{j,\alpha}(x,y)(x^2\partial_x)^j(x\partial_y)^\alpha \longmapsto \sum_{j+|\alpha|\leq k} a_{j,\alpha}(0,y)(\partial_z)^j(\partial_w)^\alpha$$

where $(z,w)$ are fiber coordinates for ${}^{\mathrm{sc}}TX|_{\partial X}$. Taking the Fourier transform gives the *scattering or boundary symbol* $\sigma_{\mathrm{sc}}(P) \in C^\infty\big({}^{\mathrm{sc}}T^*X|_{\partial X}; \mathrm{End}(V)\big)$ which is a (generally inhomogeneous) polynomial of degree $k$ along the fibers:

$$\sigma_{\mathrm{sc}} : \sum_{j+|\alpha|\leq k} a_{j,\alpha}(x,y)(x^2\partial_x)^j(x\partial_y)^\alpha \longmapsto \sum_{j+|\alpha|\leq k} a_{j,\alpha}(0,y)(i\xi)^j(i\eta)^\alpha. \tag{1.13}$$

Note that (1.12) involves only the highest order part of the operator while (1.13) involves the lower orders as well. They are compatible in that $\sigma_k(P)$ over $\partial X$ captures the leading order asymptotic behavior of $\sigma_{\mathrm{sc}}(P)$ as $|(\xi,\eta)| \longrightarrow \infty$.

The main Fredholm and elliptic regularity results from the theory are summarized by

**Theorem 1.4.** *$P \in \mathrm{Diff}^k_{\mathrm{sc}}(X; V\otimes \Omega^{1/2}_{\mathrm{sc}})$ is said to be* fully elliptic *if $\sigma_k(P)$ is invertible off the zero section and $\sigma_{\mathrm{sc}}(P)$ is invertible everywhere. In this case P admits Fredholm extensions*

$$P : x^\alpha H^m_{\mathrm{sc}}(X; V\otimes\Omega^{1/2}_{\mathrm{sc}}) \longrightarrow x^\alpha H^{m-k}_{\mathrm{sc}}(X; V\otimes\Omega^{1/2}_{\mathrm{sc}})$$

*for all $\alpha, m \in \mathbb{R}$ whose index is independent of $\alpha$ and $m$. The nullspace of $P$ satisfies*

$$\mathrm{Null}(P) \subset x^\infty C^\infty(X; V \otimes \Omega_{\mathrm{sc}}^{1/2})$$

*where $x^\infty C^\infty$ denotes smooth functions vanishing to infinite order at $\partial X$.*

Using (1.3) the result holds as well for the Fredholm extensions

$$P : x^\alpha H^{l,m}_{\mathrm{b,sc}}(V; V \otimes \Omega_{\mathrm{sc}}^{1/2}) \longrightarrow x^\alpha H^{l,m-k}_{\mathrm{b,sc}}(X; V \otimes \Omega_{\mathrm{sc}}^{1/2}).$$

The theorem above is a consequence of a parametrix construction within a calculus of *scattering pseudodifferential operators*

$$\Psi^*_{\mathrm{sc}}(X;V) = \bigcup_{\substack{s\in\mathbb{R}\\ e\in\mathbb{Z}}} \Psi^{s,e}_{\mathrm{sc}}(X;V),$$

which extend the differential operators above. These are defined by their Schwartz kernels on the *scattering double space*, $\beta_{\mathrm{sc}} : X^2_{\mathrm{sc}} \longrightarrow X^2$, a blown-up version of $X^2$. An operator $Q \in \Psi^{s,e}_{\mathrm{sc}}(X;V)$ is conormal of order $s$ with respect to the (lifted) diagonal, has Laurent expansion with leading order $e$ at the unique boundary face (conventionally denoted sc) meeting the diagonal, and the coefficient of the leading order term restricts to a conormal distribution with respect to the zero section of a natural vector bundle structure $\mathring{\mathrm{sc}} \cong {}^{\mathrm{sc}}TX|_{\partial X} \longrightarrow \partial X$, whose fiberwise Fourier transform is therefore a fiberwise (total) symbol $\sigma_{\mathrm{sc}}(Q) \in C^\infty({}^{\mathrm{sc}}T^*X|_{\partial X}; \mathrm{End}(V))$.

Operators compose according to $\Psi^{s,e}_{\mathrm{sc}}(X;V) \circ \Psi^{t,f}_{\mathrm{sc}}(X;V) \subset \Psi^{s+t,e+f}_{\mathrm{sc}}(X;V)$, with symbols composing via

$$\begin{aligned} \sigma_{s+t}(Q\circ P) &= \sigma_s(Q)\sigma_t(P) \\ (\sigma_{\mathrm{sc}})_{e+f}(Q\circ P) &= (\sigma_{\mathrm{sc}})_e(Q)(\sigma_{\mathrm{sc}})_f(P). \end{aligned}$$

$Q \in \Psi^{s,e}_{\mathrm{sc}}(X;V)$ is a compact operator on the $x^\alpha H^k_{\mathrm{sc}}$ spaces provided $s < 0$ and $e > 0$, and is trace-class provided $s < -\dim(X)$ and $e > \dim(X)/2$. Theorem 1.4 follows from the construction of a $Q \in \Psi^{-k,0}_{\mathrm{sc}}(X;V)$ with $\sigma_{-k}(Q) = \sigma_k(P)^{-1}$ and $\sigma_{\mathrm{sc}}(Q) = \sigma_{\mathrm{sc}}(P)^{-1}$, whence $PQ - I, QP - I \in \Psi^{-1,1}_{\mathrm{sc}}(X;V)$.

1.5. **The b calculus.** Finally we summarize the Fredholm theory for b operators, which is somewhat different from the above. Proofs of the results in this section can be found in [Mel93].

Given $P \in \mathrm{Diff}^k_{\mathrm{b}}(X;V)$ the principal symbol $\sigma_k(P) \in C^\infty\big({}^{\mathrm{b}}T^*X; \mathrm{End}(V)\big)$ is well-defined. In local coordinates

$$\sigma_k : \sum_{j+|\alpha|\leq k} a_{j,\alpha}(x,y)(x\partial_x)^j \partial_y^\alpha \longmapsto \sum_{j+|\alpha|=k} a_{j,\alpha}(x,y)(i\xi)^j(i\eta)^\alpha,$$

and $P$ is said to be elliptic if $\sigma_k(P)$ is invertible away from the zero section ${}^{\mathrm{b}}T^*X$.

In contrast to the situation in the previous section, the Lie algebra $\mathcal{V}_{\mathrm{b}}(X)/x\,\mathcal{V}_{\mathrm{b}}(X)$ induced over $\partial X$ is *not* abelian, though the element $x\partial_x$ descends to have trivial bracket with all other elements. The induced restriction of $P$ to $\partial X$, here denoted $I(P)$ to match the usual convention, therefore defines an operator

$$I(P) \in \mathrm{Diff}^k_I({}^{\mathrm{b}}N_+\partial X; V)$$

where ${}^{\mathrm{b}}N_+\partial X \longrightarrow \partial X$ is the $\mathbb{R}_+ = [0,\infty)$ bundle spanned[4] by $x\partial_x$, and the $I$ denotes invariance with respect to the multiplicative structure on ${}^{\mathrm{b}}N_+\partial X \cong \partial X \times \mathbb{R}_+$. Locally,

$$I : \sum_{j+|\alpha|\leq k} a_{j,\alpha}(x,y)(x\partial_x)^j\partial_y^\alpha \longmapsto \sum_{j+|\alpha|\leq k} a_{j,\alpha}(0,y)(s\partial_s)^j\partial_y^\alpha.$$

Here $s \in \mathbb{R}_+$ denotes the fiber variable for ${}^{\mathrm{b}}N_+\partial X$. Conjugation by the Mellin transform in $s$ gives the *indicial family* $I(P,\lambda) = \mathcal{M}^{-1}I(P)\mathcal{M}$, where

$$\mathcal{M}(u) = \int_{\mathbb{R}_+} s^{-i\lambda}u(s)\frac{ds}{s}$$

depends on the choice of $x$ used to trivialize ${}^{\mathrm{b}}N_+\partial X \cong \partial X \times \mathbb{R}_+$. Locally, this just amounts to replacing $x\partial_x$ by $i\lambda$ and evaluating at the boundary:

$$I(\cdot,\lambda) : \sum_{j+|\alpha|\leq k} a_{j,\alpha}(x,y)(x\partial_x)^j\partial_y^\alpha \longmapsto \sum_{j+|\alpha|\leq k} a_{j,\alpha}(0,y)(i\lambda)^j\partial_y^\alpha.$$

$I(P,\lambda)$ is a holomorphic family with respect to $\lambda \in \mathbb{C}$ of differential operators on $\partial X$ which are elliptic if $P$ is elliptic, which we assume from now on. It can be shown that the set

$$\mathrm{spec}_{\mathrm{b}}(P) = \{\lambda \in \mathbb{C} : I(P,\lambda) \text{ not invertible,}\}$$

called the *b-spectrum* of $P$, is discrete, satisfies $|\mathrm{Im}\,\lambda_j| \to \infty$ as $j \to \infty$ and does not depend on the choice of $x$. Equivalently,

$$I(P,\lambda)^{-1} \in \Psi^{-k}(\partial X; V)$$

is a meromorphic family, with (finite order) poles at $\lambda \in \mathrm{spec}_{\mathrm{b}}(P)$.

**Theorem 1.5.** *If $P \in \mathrm{Diff}^k_{\mathrm{b}}(X; V \otimes \Omega_{\mathrm{b}}^{1/2})$ is elliptic, then $P$ admits Fredholm extensions*

$$P_\alpha : x^\alpha H^m_{\mathrm{b}}(X; V\otimes\Omega_{\mathrm{b}}^{1/2}) \longrightarrow x^\alpha H^{m-k}_{\mathrm{b}}(X; V\otimes\Omega_{\mathrm{b}}^{1/2}) \tag{1.14}$$

*for all $\alpha \notin -\mathrm{Im}\ \mathrm{spec}_{\mathrm{b}}(P)$, $m \in \mathbb{R}$. Elements of* $\mathrm{Null}(P_\alpha)$ *are smooth on the interior of $X$, and have asymptotic expansions at $\partial X$ of the form*

$$\mathrm{Null}(P_\alpha) \ni v \sim \sum_{\substack{z\in i\mathrm{spec}_{\mathrm{b}}(P)+\mathbb{N}\\ \mathrm{Re}\,z>\alpha \\ 0\leq l\leq \mathrm{ord}(-iz)}} x^z(\log x)^l v'(y)\left|\frac{dx}{x}\right|^{1/2}, \quad v' \in C^\infty(\partial X; V\otimes\Omega^{1/2}).$$

*Here* $\mathrm{ord}(-iz)$ *denotes the order of the pole of $I(P,\lambda)^{-1}$ at $\lambda = -i(z+n)$ for given $n \in \mathbb{N}$. For terms of leading order (meaning $z \in i\mathrm{spec}_{\mathrm{b}}(P)$), $v' \in \mathrm{Null}\big(I(P,-iz)\big)$.*

In particular, the index of the Fredholm extension does not depend on $m$, but does depend on $\alpha$ in a way we describe below.

The theorem follows by a parametrix construction within the *b calculus* of pseudodifferential operators $\Psi^*_{\mathrm{b}}(X;V) = \bigcup \Psi^{s,\mathcal{E}}_{\mathrm{b}}(X;V)$ with Schwartz kernels on the *b double space* $\beta_{\mathrm{b}} : X^2_{\mathrm{b}} = [X^2; \partial X^2] \longrightarrow X^2$, which are conormal (of order $s$) to the diagonal and have polyhomogeneous expansions (see Appendix A) given by the index sets $\mathcal{E} = (E_{\mathrm{ff}}, E_{\mathrm{lb}}, E_{\mathrm{rb}})$, at the boundary faces of $X^2_{\mathrm{b}}$.

[4]In fact ${}^{\mathrm{b}}N_+\partial X$ is well-defined independent of the choice of $x$ as the span of the inward pointing sections of the kernel of the bundle map ${}^{\mathrm{b}}TX \longrightarrow TX$ over $\partial X$. It is always trivial (being an $\mathbb{R}_+$ bundle, but the trivialization depends on a choice of $x$.

For $Q \in \Psi_{\mathrm{b}}^{s,\mathcal{E}}(X;V)$, the restriction of the leading order term in the expansion of $Q$ at the boundary face ff meeting the diagonal is denoted $I(Q)$. This has an interpretation as a pseudodifferential operator $I(Q) \in \Psi_I^s({}^{\mathrm{b}}N_+\partial X;V)$ of convolution type in the fiber directions, such that with respect to composition $\Psi_{\mathrm{b}}^{s,\mathcal{E}} \times \Psi_{\mathrm{b}}^{t,\mathcal{F}} \ni (Q,P) \longmapsto Q \circ P \in \Psi_{\mathrm{b}}^{s+t,\mathcal{G}}$,

$$\sigma_{s+t}(Q \circ P) = \sigma_s(Q)\sigma_t(P)$$
$$I(Q \circ P) = I(Q)I(P).$$

Here $\mathcal{G} = (G_{\mathrm{ff}}, G_{\mathrm{lb}}, G_{\mathrm{rb}})$ where

$$G_{\mathrm{lb}} = (E_{\mathrm{ff}} + F_{\mathrm{lb}})\overline{\cup}E_{\mathrm{lb}} \qquad G_{\mathrm{rb}} = (E_{\mathrm{rb}} + F_{\mathrm{ff}})\overline{\cup}F_{\mathrm{rb}}$$
$$G_{\mathrm{ff}} = (E_{\mathrm{ff}} + F_{\mathrm{ff}})\overline{\cup}(E_{\mathrm{lb}} + F_{\mathrm{rb}})$$

in terms of the extended union of index sets (see Appendix A), and there is a necessary condition $\operatorname{Re} E_{\mathrm{rb}} + \operatorname{Re} F_{\mathrm{lb}} > 0$ for the composition to be defined (hence the term 'calculus' as opposed to 'algebra' of operators).

$Q \in \Psi_{\mathrm{b}}^{s,\mathcal{E}}(X;V)$ is bounded as an operator $x^\alpha H_{\mathrm{b}}^k \longrightarrow x^\beta H_{\mathrm{b}}^{k-s}$ provided $\operatorname{Re} E_{\mathrm{rb}} > -\alpha$, $\operatorname{Re} E_{\mathrm{lb}} > \beta$. It is compact on $x^\alpha H_{\mathrm{b}}^k$ if $s < 0$ and $\operatorname{Re} E_{\mathrm{ff}} > 0$, $\operatorname{Re} E_{\mathrm{rb}} > -\alpha$, $\operatorname{Re} E_{\mathrm{lb}} > \alpha$, and trace class if in addition $s < -\dim(X)$. Theorem 1.5 is a consequence of constructing a parametrix $Q \in \Psi_{\mathrm{b}}^{-k,\mathcal{E}(\alpha)}(X;V)$ such that $\sigma_{-k}(Q) = \sigma_k(P)^{-1}$ and $I(Q) = \mathcal{M}^{-1}_{\operatorname{Im}\lambda=-\alpha} I(P,\lambda)^{-1}$, for which $PQ - I$ and $QP - I$ are compact. The characterization of the nullspace requires additional analysis we will not describe here.

We next discuss the dependence of the index of (1.14) on $\alpha$. Each pole of $I(P,\lambda)^{-1}$, say at $\lambda = -iz$ for $z \in \mathbb{C}$, is associated with a set of elements of the nullspace of $I(P)$ called the *formal nullspace at* $\lambda$

$$F'(P,\lambda) = \Big\{u = \sum_{0 \le l < \operatorname{ord}(\lambda)} s^z (\log s)^l u'(y) : I(P)u = 0\Big\}. \tag{1.15}$$

which may be identified with $\operatorname{ord}(\lambda)$ copies of $\operatorname{Null}\big(I(P,\lambda)\big) \subset C^\infty(\partial X; V \otimes \Omega^{1/2})$. The leading term in the asymptotic expansion of any $v \in \operatorname{Null}(P_\alpha)$ is in $F'(P,-iz)$ for some $z \in i\operatorname{spec}_{\mathrm{b}}(P)$ with $\operatorname{Re} z > \alpha$. Using the notation

$$F(P,r) = \bigoplus_{\operatorname{Im}\lambda=-r} F'(P,\lambda), \quad r \in \mathbb{R}$$

and

$$\begin{aligned} (1.16)\quad & \operatorname{Null}(P,r) = \\ & \quad \big\{v \sim x^z(\log x)^l v' + o(x^z(\log x)^l) : Pv = 0,\ z \in i\operatorname{spec}_{\mathrm{b}}(P),\ \operatorname{Re}(z) = r\big\} \end{aligned}$$

(so that with this notation $\operatorname{Null}(P_\alpha) = \bigcup_{r>\alpha} \operatorname{Null}(P,r)$), we denote the image of $\operatorname{Null}(P,r)$ in $F(P,r)$ by this leading term map as

$$G(P,r) = \operatorname{Image}\big(\operatorname{Null}(P,r) \longrightarrow F(P,r)\big) \cong \operatorname{Null}(P,r)/\operatorname{Null}(P,r') \tag{1.17}$$

where $r' = \min\{\operatorname{Re} z > r : z \in i\operatorname{spec}_{\mathrm{b}}(P)\}$.

It is a fundamental result that there is a nondegenerate bilinear pairing (essentially a generalized Green's formula)

$$\begin{aligned} &B : F(P,r) \times F(P^*,-r) \longrightarrow \mathbb{C} \\ &B(u,v) = \frac{1}{i}\int_{\partial X \times \mathbb{R}_+} \langle I(P)\phi u, \phi v\rangle - \langle \phi u, I(P^*)\phi v\rangle \end{aligned} \tag{1.18}$$

where $\phi \in C^\infty(\mathbb{R}_+)$ is a compactly supported cutoff with $\phi \equiv 1$ in a neighborhood of 0, on which $B$ does not depend. Moreover,

$$G(P,r) = G(P^*,-r)^\perp$$

with respect to $B$, which leads to the so-called *relative index theorem*:

**Theorem 1.6.** *The index* $\operatorname{ind}(P_\alpha)$ *of the extensions* (1.14) *is constant for* $\alpha$ *in connected components of* $\mathbb{R} \setminus -\operatorname{Im}\operatorname{spec}_{\mathrm{b}}(P)$, *and for* $\alpha < \beta$,

$$\operatorname{ind}(P_\alpha) - \operatorname{ind}(P_\beta) = \sum_{\alpha < r < \beta} \dim F(P,r).$$

Of particular interest are self-adjoint operators, for which $\operatorname{spec}_{\mathrm{b}}(P)$ is purely imaginary in light of the fact that $I(P,\lambda) = I(P^*,\overline{\lambda})$. In this case, the identity $\operatorname{ind}(P_\alpha) = -\operatorname{ind}(P_{-\alpha})$ may be used along with explicit computation of the $F(P,r)$ in terms of $\operatorname{Null}\big(I(P,-ir)\big)$ to compute the absolute index of a given extension.

## 2. Fredholm extensions of Callias operators

We now return to the situation described in the introduction. Namely, $X$ is a manifold with boundary equipped with an exact scattering metric and a scattering Dirac operator $D \in \operatorname{Diff}^1_{\mathrm{sc}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}})$ acting on half-density sections of a $\mathbb{C}\ell_{\mathrm{sc}}(X)$ module $V \longrightarrow X$. We assume that the connection $\nabla$ on $V$ used to define $D$ is the lift of a b connection — in particular it induces a connection on $V|_{\partial X}$. We suppose $\Phi \in C^\infty\big(X; \operatorname{End}(V \otimes \Omega^{1/2}_{\mathrm{sc}})\big)$ is skew-Hermitian, that $(\nabla\Phi)|_{\partial X} = 0$, and that $\Phi|_{\partial X}$ commutes with Clifford multiplication and has constant rank over $\partial X$. In particular,

$$V|_{\partial X} = V_0 \oplus V_1, \quad V_0 = \operatorname{Null}(\Phi|_{\partial X})$$

splits into a sum of bundles which are preserved by the Clifford action and connection induced on $\partial X$. We further suppose that the splitting extends to a neighborhood $U \supset \partial X$ for which

$$\Phi = \begin{pmatrix} \Phi_{00} & \Phi_{10} \\ \Phi_{01} & \Phi_{11} \end{pmatrix}, \quad \Phi_i = \mathcal{O}(x^{1+\epsilon}),\ i \in \{00, 01, 10\} \tag{2.1}$$

for some $\epsilon > 0$. This condition is met in the application of interest in §4; indeed in that case the splitting extends to a neighborhood on which $\Phi \equiv \begin{pmatrix} 0 & 0 \\ 0 & \Phi_{11} \end{pmatrix}$.

Given $D$ and $\Phi$ with these properties, we set

$$P := D + \Phi \in \operatorname{Diff}^1_{\mathrm{sc}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}})$$

and refer to it as a (generalized) Callias type operator.

In discussing Fredholm extensions of $P$, we first consider the two extreme cases in which the rank of $\Phi|_{\partial X}$ is maximal or zero.

2.1. **The case of full rank.** When $\Phi|_{\partial X}$ is invertible the following is a consequence of Theorem 1.4 and the index theorem proved in [Kot11].

**Theorem 2.1.** *Under the assumptions above on $D$ and $\Phi$ and with the additional assumption that $\Phi|_{\partial X}$ is nondegenerate, $P$ is fully elliptic with Fredholm extensions*

$$P : x^\gamma H^{l,k}_{\mathrm{b,sc}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) \longrightarrow x^\gamma H^{l,k-1}_{\mathrm{b,sc}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}})$$

*for all $\gamma$, $l, k$, and*

$$\mathrm{ind}(P) = \mathrm{ind}(\partial^+_+)$$

*where $\partial^+_+ \in \mathrm{Diff}^1(\partial X; V^+_+ \otimes \Omega^{1/2}_{\mathrm{sc}}, V^-_+ \otimes \Omega^{1/2}_{\mathrm{sc}})$ is one half of the induced Dirac operator on $\partial X$ obtained from $D$ acting on $V_+|_{\partial X} = V^+_+ \oplus V^-_+$. Here $V_+$ denotes the span of the positive imaginary eigenvectors of $\Phi|_{\partial X}$ and $V^+_+ \oplus V^-_+$ denotes the splitting into $\pm 1$ eigenspaces of $i\mathrm{c}\ell(x^2\partial_x)$. Elements of $\mathrm{Null}(P)$ are smooth and rapidly vanishing at $\partial X$:*

$$\mathrm{Null}(P) \subset x^\infty C^\infty(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}).$$

2.2. **The case of zero rank.** In the case that $\Phi|_{\partial X} \equiv 0$, $P$ fails to be fully elliptic as a scattering operator and Theorem 2.1 does not hold. However since $\Phi = x\Phi'$ in this case, $P$ may be instead considered as a weighted b differential operator using Corollary 1.3.

There is a choice involved in the way $x$ is factored out; indeed $x^{-\alpha} P x^{\alpha-1}$ for $0 \leq \alpha \leq 1$ are essentially equivalent as b-operators, though their b-spectra are shifted relative to one another along the imaginary axis. We opt for the convention $\alpha = -1/2$, which has the virtue of preserving formal self-adjointness of the Dirac operator on the unweighted $L^2$ space. Furthermore, as the standard Fredholm results for b-operators are most naturally stated using b half densities, we also conjugate $P$ by $x^{n/2}$ and make use of the equivalence (1.11).

Thus as a notational convention, for any $Q \in \mathrm{Diff}^1_{\mathrm{sc}}$ we define

$$\widetilde{Q} := x^{-(n+1)/2} Q x^{(n-1)/2} \in \mathrm{Diff}^1_{\mathrm{b}}(X; V \otimes \Omega^{1/2}_{\mathrm{b}}), \tag{2.2}$$

and note that mapping properties (boundedness, Fredholm, etc.) of

$$Q : x^{\gamma-1/2} H^k_{\mathrm{b}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) \longrightarrow x^{\gamma+1/2} H^{k-1}_{\mathrm{b}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}})$$

are equivalent to the corresponding mapping properties of

$$\widetilde{Q} : x^\gamma H^k_{\mathrm{b}}(X; V \otimes \Omega^{1/2}_{\mathrm{b}}) \longrightarrow x^\gamma H^{k-1}_{\mathrm{b}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}).$$

Moreover $\widetilde{Q}$ is formally self-adjoint on $L^2(X; V \otimes \Omega^{1/2}_{\mathrm{b}})$ if and only if $Q$ is formally self-adjoint on $L^2(X; V \otimes \Omega^{1/2}_{\mathrm{sc}})$.

Given the assumption that $\Phi = \mathcal{O}(x^{1+\epsilon})$, it follows that $\widetilde{P} = \widetilde{D} + \widetilde{\Phi}$ where $\widetilde{\Phi} = x^{-1}\Phi = \mathcal{O}(x^\epsilon)$. In particular $\widetilde{P}$ is a compact perturbation of $\widetilde{D}$. From the results in §1.5 we conclude:

**Proposition 2.2.** *$P$ admits a Fredholm extension*

$$P : x^{\gamma-1/2} H^k_{\mathrm{b}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) \longrightarrow x^{\gamma+1/2} H^{k-1}_{\mathrm{b}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) \tag{2.3}$$

*for all $\gamma \notin \mathrm{ispec}_{\mathrm{b}}(\widetilde{D})$ and all $k$, and* $\mathrm{Null}(P)$ *consists of polyhomogeneous sections with expansions*

$$\mathrm{Null}(P) \ni u \sim x^{(n-1)/2} \sum_{\substack{r \in i\mathrm{spec}_{\mathrm{b}}(\widetilde{D}) + \mathbb{N} \\ r > \gamma \\ 0 \leq l < \mathrm{ord}(-ir)}} x^r (\log x)^l u'(y) \left|\tfrac{dx}{x}\right|^{1/2}$$

*where $u' \in C^\infty(\partial X; V \otimes \Omega^{1/2})$. For the leading order terms (corresponding to $r \in i\mathrm{spec}_{\mathrm{b}}(\widetilde{D})$), $u' \in \mathrm{Null}\big(I(\widetilde{D}, -ir)\big)$. The index of the extension* (2.3) *will be written* $\mathrm{ind}(P,\gamma)$, *and satisfies*

$$\begin{aligned} \mathrm{ind}(P, -\gamma) &= -\mathrm{ind}(P,\gamma), \quad \textit{and} \\ \mathrm{ind}(P, \gamma_0 - \epsilon) - \mathrm{ind}(P, \gamma_0 + \epsilon) &= \dim F(\widetilde{D}, \gamma_0) \end{aligned} \tag{2.4}$$

*for $\gamma_0 \in i\mathrm{spec}_{\mathrm{b}}(\widetilde{D})$, for sufficiently small $\epsilon > 0$.*

*Proof.* This is mostly a matter of unwinding the notation and making use of Theorem 1.5.

The properties (2.4) follow from the fact that $\widetilde{P}$ has the same index as $\widetilde{D}$ as an operator $x^\gamma H^k_{\mathrm{b}} \longrightarrow x^\gamma H^{k-1}_{\mathrm{b}}$ which satisfies analogous properties by self-adjointness and Theorem 1.6.

The characterization of $\mathrm{Null}(P)$ comes from the fact that $\mathrm{Null}(P) = x^{(n-1)/2}\mathrm{Null}(\widetilde{P})$ which involves a multiplication by $x^{-1/2}$ along with the natural identification $\Omega^{1/2}_{\mathrm{b}}(X) = x^{n/2}\Omega^{1/2}_{\mathrm{sc}}(X)$. □

2.3. **General constant rank nullspace.** We now consider the general case described at the beginning of this section, wherein $V|_{\partial X} = V_0 \oplus V_1$ according to $V_0 = \mathrm{Null}(\Phi)$, $\Phi|_{V_1} \neq 0$, and the splitting extends to a neighborhood $U$ of $\partial X$ for which (2.1) holds.

To obtain a Fredholm result we are forced to measure regularity near $\partial X$ differently according to the splitting of $V$. This is accomplished by defining families of *hybrid Sobolev spaces*.

**Definition 2.3.** Let $\Pi_0 \in C^\infty(U; \mathrm{End}(V_0 \oplus V_1))$ be the projection onto the $V_0$ subbundle, and denote by $\Pi_1 = (\mathrm{Id} - \Pi_0)$ the other projection. Let $\chi \in C^\infty(X; [0,1])$ be a cutoff supported on $U$ and such that $\chi \equiv 1$ on an open neighborhood of $\partial X$ and let

$$\big[x^\alpha H^{k+l}_{\mathrm{b}} \big| x^\beta H^{k,l}_{\mathrm{b,sc}}\big](X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) \tag{2.5}$$

consist of those $u \in C^{-\infty}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}})$ such that

$$\begin{aligned} \Pi_0 \chi u &\in x^\alpha H^{k+l}_{\mathrm{b}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}), \\ \Pi_1 \chi u &\in x^\beta H^{k,l}_{\mathrm{b,sc}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}), \\ (1-\chi) u &\in H^{k+l}_c(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}). \end{aligned}$$

It follows from the fact that the $H^{k+l}_{\mathrm{b}}$ and $H^{k,l}_{\mathrm{b,sc}}$ norms are equivalent on sections supported away from $\partial X$ that $\big[x^\alpha H^{k+l}_{\mathrm{b}} \big| x^\beta H^{k,l}_{\mathrm{b,sc}}\big]$ is a well-defined, complete Hilbert space with respect to the norm

$$\|u\|^2_{\left[x^\alpha H^{k+l}_{\mathrm{b}} \middle| x^\beta H^{k,l}_{\mathrm{b,sc}}\right]} = \|\Pi_0 \chi u\|^2_{x^\alpha H^{k+l}_{\mathrm{b}}} + \|\Pi_1 \chi u\|^2_{x^\beta H^{k,l}_{\mathrm{sc,b}}} + \|(1-\chi)u\|^2_{H^{k+l}_c}$$

and that it is independent of the choice of $\chi$. The corresponding inner product is obtained by polarization.

We allow the freedom to use different weights $x^\alpha$ and $x^\beta$ along the components, though the next result shows that the relative weight $\beta - \alpha$ is constrained by the necessity to control the off diagonal terms of $P$.

Indeed, from the assumptions that $\Phi|_{\partial X}$ commutes with Clifford multiplication and $(\nabla\Phi)|_{\partial X} = 0$, it follows that $D$ and $\Phi$ commute to leading order at $\partial X$, so with respect to the splitting $P$ has the form

$$P = D + \Phi = \begin{pmatrix} D_0 & R_{01} \\ R_{10} & D_1 \end{pmatrix} + \Phi \tag{2.6}$$

near $\partial X$ with remainder terms $R_i \in x\mathrm{Diff}^1_{\mathrm{sc}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}})$.

**Lemma 2.4.** *$P$ extends to a bounded operator*

$$\begin{aligned} P : \big[x^{\gamma-1/2}H^{k+l}_{\mathrm{b}}\big|x^{\gamma+\beta}H^{k,l}_{\mathrm{b,sc}}\big](X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) \\ \longrightarrow \big[x^{\gamma+1/2}H^{k+l-1}_{\mathrm{b}}\big|x^{\gamma+\beta}H^{k,l-1}_{\mathrm{b,sc}}\big](X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) \end{aligned} \tag{2.7}$$

*for all $k$, $l$, $\gamma$ and $-1/2 \le \beta \le 1/2$. For $-1/2 < \beta < 1/2$ $P$ is homotopic to an operator $P'$ arbitrarily close to $P$ in norm which is of the form*

$$P'|_{U'} = \begin{pmatrix} D_0 & 0 \\ 0 & D_1 + \Phi_{11} \end{pmatrix} \tag{2.8}$$

*on a possibly smaller neighborhood $U'$ of $\partial X$.*

*Proof.* Using (1.3) and the obvious estimates on commutators, we may move past $P$ the factor $x^\gamma$ as well as $k$ b-derivatives and $l$ scattering derivatives, so it suffices to verify the case $\gamma = k = l = 0$. As boundedness is clear on the compactly supported part of $u \in \big[x^{-1/2}H^1_{\mathrm{b}}\big|x^\beta H^{0,1}_{\mathrm{b,sc}}\big]$, we concentrate on the terms $\Pi_0\chi u$ and $\Pi_1\chi u$. Thus is suffices to verify boundedness of

$$\begin{pmatrix} D_0 & R_{01} \\ R_{10} & D_1 \end{pmatrix} + \begin{pmatrix} \Phi_{00} & \Phi_{01} \\ \Phi_{10} & \Phi_{11} \end{pmatrix} : \begin{matrix} x^{-1/2}H^1_{\mathrm{b}} \\ \oplus \\ x^\beta H^1_{\mathrm{sc}} \end{matrix} \longrightarrow \begin{matrix} x^{1/2}L^2 \\ \oplus \\ x^\beta L^2 \end{matrix}$$

with $R_i \in x\mathrm{Diff}^1_{\mathrm{sc}}$, and $\Phi_i = \mathcal{O}(x^{1+\epsilon})$ for $i \neq 11$.

Boundedness of $D_0$ and $D_1$ follows from the analysis of the previous sections, so consider $R_{01}$. As an element of $x\mathrm{Diff}^1_{\mathrm{sc}}$, $R_{01}$ is bounded as an operator

$$R_{01} : x^\beta H^1_{\mathrm{sc}} \longrightarrow x^{\beta+1}L^2,$$

so it will be bounded from $x^\beta H^1_{\mathrm{sc}} \longrightarrow x^{1/2}L^2$ provided $\beta \ge -1/2$.

Now consider $R_{10} \in x\mathrm{Diff}^1_{\mathrm{sc}}$. It may be decomposed into $R_{10} = R^1_{10} + R^2_{10}$, where $R^1_{10} \in x^2\mathcal{V}_{\mathrm{b}}$ and $R^2_{10} \in xC^\infty$. It follows that

$$\begin{aligned} R^1_{10} &: x^{-1/2}H^1_{\mathrm{b}} \longrightarrow x^{3/2}L^2, \quad \text{and} \\ R^2_{10} &: x^{-1/2}H^1_{\mathrm{b}} \longrightarrow x^{1/2}H^1_{\mathrm{b}} \end{aligned}$$

are bounded, and then the inclusions $x^{3/2}L^2, x^{1/2}H^1_{\mathrm{b}} \subset x^\beta L^2$ hold provided $\beta \le 1/2$. The off diagonal terms of $\Phi$ involve a similar computation; they are seen to be bounded provided $-1/2 - \epsilon \le \beta \le 1/2 + \epsilon$. We conclude that $P$ is bounded provided $-1/2 \le \beta \le 1/2$.

If these inequalities are strict, each of the remainder terms may be written as $x^\delta R_i'$, with $R_i'$ bounded. For example, if $\beta \geq -1/2+\delta$, $R_{01}$ can be written as $x^\delta R_{01}'$ with $R_{01}' \in x^{1-\delta}\mathrm{Diff}^1_{\mathrm{sc}}$, and

$$R_{01}' : x^\beta H^1_{\mathrm{sc}} \longrightarrow x^{\beta+1-\delta}L^2 \subset x^{1/2}L^2$$

is bounded. The case of $R_{10}$ is similar.

Thus for any $\varepsilon > 0$ we can choose a neighborhood $U' = \{x < x_0\}$ for some $x_0 \in [0,1)$ and a cutoff $\phi$ supported on $U'$ such that $\|\phi R \phi\| \leq x_0^\delta \|\phi R' \phi\| < \varepsilon$. Making the neighborhood smaller if necessary, we can also arrange that $\|\phi \Phi' \phi\| < \varepsilon$, where $\Phi' = \begin{pmatrix} \Phi_{00} & \Phi_{01} \\ \Phi_{10} & 0 \end{pmatrix}$. It follows that

$$P_t = \begin{pmatrix} D_0 & 0 \\ 0 & D_1 + (\Phi_{11}) \end{pmatrix} + t(1-\phi)(\Phi' + R) + (1-t)(\Phi' + R)$$

is a homotopy between $P = P_0$ and an operator $P_1$ of the required form, and $\|P_t - P_0\| < 2\varepsilon$ uniformly in $t$. $\square$

The term $D_0$ in (2.6) is only defined on the neighborhood $U$ of $\partial X$. Nevertheless, we may consider $\widetilde{D}_0 = x^{-(n+1)/2} D_0 x^{(n-1)/2}$ as in §2.2 and refer to the indicial family $I(\widetilde{D}_0, \lambda) \in \mathrm{Diff}(\partial X; V_0 \otimes \Omega^{1/2})$ and the b-spectrum $\mathrm{spec}_{\mathrm{b}}(\widetilde{D}_0)$ since these only depend on the leading order $I(\widetilde{D}_0)$ of $\widetilde{D}_0$ at $\partial X$.

**Theorem 2.5.** *$P$ has a Fredholm extension as an operator*

$$P : \big[x^{\gamma-1/2}H^{k+l}_{\mathrm{b}} \big| x^{\gamma+\beta}H^{k,l}_{\mathrm{b,sc}}\big](X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) \longrightarrow \big[x^{\gamma+1/2}H^{k+l-1}_{\mathrm{b}} \big| x^{\gamma+\beta}H^{k,l-1}_{\mathrm{b,sc}}\big](X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) \tag{2.9}$$

*for any $\gamma \notin i\mathrm{spec}_{\mathrm{b}}(\widetilde{D}_0)$, $k, l \in \mathbb{N}$, and $-1/2 < \beta < 1/2$.*

*The nullspace of such an extension exhibits the following regularity:*

$$\begin{aligned} \mathrm{Null}(P) \ni u \implies & \Pi_1 \chi u \in x^\infty C^\infty(U; V_1 \otimes \Omega^{1/2}_{\mathrm{sc}}) \\ & \Pi_0 \chi u \sim x^{(n-1)/2} \sum_{\substack{r \in i\mathrm{spec}_{\mathrm{b}}(\widetilde{D}_0)+\mathbb{N} \\ r > \gamma \\ 0 \leq l < \mathrm{ord}(-ir)}} x^r (\log x)^l u'(y) \left|\tfrac{dx}{x}\right|^{1/2} \end{aligned}$$

*where $u' \in C^\infty(\partial X; V_0 \otimes \Omega^{1/2})$ with leading order terms having $u' \in \mathrm{Null}\big(I(\widetilde{D}_0, -ir)\big)$.*

*Proof.* Deforming $P$ by an arbitrarily small norm perturbation (preserving Fredholmness since the set of Fredholm operators is open), we may assume that $P$ has the form (2.8) near $\partial X$.

We will construct a Fredholm parametrix $Q \in C^{-\infty}\big(X^2; \mathrm{End}(V \otimes \Omega^{1/2}_{\mathrm{sc}})\big)$ decomposing as a direct sum of terms near $\partial X^2$ which come from the scattering and b calculi, respectively. Thus let $Q$ be supported in a neighborhood of the diagonal and conormal to it on the interior of $X^2$. Restricted to $U \times U$, let $Q$ have the form

$$Q|_{U^2} = \begin{pmatrix} (\beta_{\mathrm{b}})_*\big(x^{(n-1)/2} Q_0 x'^{-(n+1)/2}\big) & 0 \\ 0 & (\beta_{\mathrm{sc}})_*(Q_1) \end{pmatrix}, \quad \begin{aligned} Q_0 &\in \Psi^{-1,\mathcal{E}}_{\mathrm{b}}(X; V \otimes \Omega^{1/2}_{\mathrm{b}}) \\ Q_1 &\in \Psi^{-1,0}_{\mathrm{sc}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}}) \end{aligned}$$

for $Q_0$ and $Q_1$ yet to be determined, where $x$ and $x'$ denote the lifts of $x$ from the left and right respectively.

Let $\sigma_{-1}(Q_1) = \sigma_1(D_1)^{-1}$, and $\sigma_{-1}(Q_0) = \sigma_1(\widetilde{D}_0)^{-1}$, which is compatible with the condition $\sigma_{-1}(Q) = \sigma_1(P)^{-1}$ on the interior of $X^2$. In addition, let $\sigma_{\mathrm{sc}}(Q_1) = \sigma_{\mathrm{sc}}(D_1 + \Phi_{11})^{-1}$ and $I(Q_0) = \mathcal{M}^{-1}_{\mathrm{Im}\,\lambda=-\gamma} I(\widetilde{D}_0, \gamma)^{-1}$.

By lifting $P$ to $X^2$ from the left or right and composing with $Q$, it follows that that $R_L = I - QP$ and $R_R = I - PQ$ are distributions conormal of order $-1$ with respect to the interior diagonal, and that

$$R_L|_{U^2} = \begin{pmatrix} R^0_L & 0 \\ 0 & R^1_L \end{pmatrix}, \quad R_R|_{U^2} = \begin{pmatrix} R^0_R & 0 \\ 0 & R^1_R \end{pmatrix},$$

where $R^1_{R/L} = (\beta_{\mathrm{sc}})_*(R'_{R/L})$ with $R'_{R/L} \in \Psi^{-1,1}_{\mathrm{sc}}$ and $R^0_{R/L} = (\beta_{\mathrm{b}})_*(x^{(n\pm1)/2} R''_{R/L} x'^{(-n\mp1)/2})$ with $R''_{R/L} \in \Psi^{-1,\mathcal{E}^{\pm}(\gamma)}_{\mathrm{b}}$.

By utilizing the partition of unity $\{\chi, (1-\chi)\}$ with $\chi \in C^\infty_c(U)$, it follows from the compactness criteria discussed in §1.4 and §1.5 that the terms $\chi R_{R/L}\chi$, $(1-\chi)R_{R/L}(1-\chi)$, $(1-\chi)R_{R/L}\chi$ and $\chi R_{R/L}(1-\chi)$ act as compact operators on $\big[x^{\gamma-1/2}H^{k+l}_{\mathrm{b}}\big|x^{\gamma+\beta}H^{k,l}_{\mathrm{b,sc}}\big]$ (in the case of $R_L$) or $\big[x^{\gamma+1/2}H^{k+l-1}_{\mathrm{b}}\big|x^{\gamma+\beta}H^{k,l-1}_{\mathrm{b,sc}}\big]$ (in the case of $R_R$) and hence so do $R_L$ and $R_R$ themselves. □

## 3. The index theorem

To compute the index of the Fredholm extensions of the last section, we employ the following strategy. Consider the family

$$\mathcal{P} := P - i\chi\tau \oplus 0 = D + \Phi - i\chi \begin{pmatrix} \tau & 0 \\ 0 & 0 \end{pmatrix}, \quad 0 \le \tau < 1,$$

with respect to the splitting $V|_U = V_0 \oplus V_1$ and where $\chi \in C^\infty_c(U)$ with $\chi \equiv 1$ at $\partial X$ as before. For $\tau > 0$, $\mathcal{P}$ is of the form described in §2.1, with Fredholm extensions on scattering Sobolev spaces index computed from Theorem 2.1 to be

$$\mathrm{ind}(\mathcal{P}) = \mathrm{ind}(\partial^+_+)$$

since the addition of $-i\tau\mathrm{Id}$ at $\partial X$ moves $V_0$ into the negative imaginary eigenbundle $V_-$. For $\tau = 0$ we consider the Fredholm extensions (2.9). This is cannot be a continuous 1-parameter family of Fredholm operators since the domains are discontinuous at $\tau = 0$. Nevertheless we will be able to treat $\mathcal{P}$ in sufficiently uniform manner as to relate the index at $\tau = 0$ to the index at $\tau > 0$ in a computable way.

### 3.1. Transition calculus.

For simplicity, we first consider the rank zero case in which $\Phi|_{\partial X} = 0$ with $\Phi = \mathcal{O}\big(x^{1+\epsilon}\big)$. Of course for $\tau = 0$ the index of the Fredholm extensions (2.3) may be computed using (2.4), and on the other hand it follows from Theorem 2.1 that for $\tau > 0$ $\mathrm{ind}(P - i\chi\tau) = 0$ since $(\Phi - i\chi\tau)|_{\partial X} = -i\tau\mathrm{Id}$, whence $V^+_+ = \{0\}$. It remains to show how these may be reconciled uniformly in $\tau$.

The basic strategy is to construct a pseudodifferential parametrix $\mathcal{Q}$ for $\mathcal{P}$, which is an approximate Fredholm inverse for $P - i\chi\tau$ on scattering Sobolev spaces when $\tau > 0$ and for $P$ on appropriate b Sobolev spaces when $\tau = 0$. Provided that the *left and right remainder terms*

$$\mathcal{R}_L := \mathrm{Id} - \mathcal{Q}\mathcal{P}, \quad \mathcal{R}_R := \mathrm{Id} - \mathcal{P}\mathcal{Q}$$

are of trace class for each $\tau$, with $\tau \longmapsto \mathrm{Tr}(\mathcal{R}_{R/L})(\tau) \in \mathbb{C}$ continuous, it will follow from the trace formula for the index that

$$\tau \longmapsto \mathrm{ind}(\mathcal{P})(\tau) = \mathrm{Tr}(\mathcal{R}_L - \mathcal{R}_R)(\tau) \in \mathbb{Z}$$

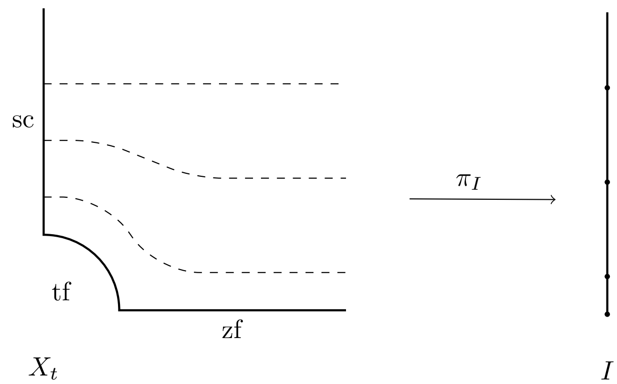


FIGURE 1. The single space $X_t$ and its boundary faces, along with some level sets of $\pi_I$.

is continuous and therefore constant.

To account for the possibility of a nonzero index at $\tau = 0$, which after all depends on the specific choice of weight $\gamma \notin i\mathrm{spec}_{\mathrm{b}}(\widetilde{D})$, the situation is slightly more complicated: while the trace employed is indeed continuous in $\tau$ (and therefore constant), the limit at $\tau = 0$ of $\mathrm{Tr}(\mathcal{R}_L - \mathcal{R}_R)(\tau)$ consists of the sum of the indices of *two* operators, one being the extension (2.3) and the other being a certain model operator on $\partial X \times [0, 1]$, which must therefore have equal and opposite index. In fact this situation arises quite naturally from the geometric methods we employ.

The parametrix for $\mathcal{P}$ is built using the *b-sc transition calculus* of pseudodifferential operators, designed to 'microlocalize' differential operators on $X$ depending on a parameter $\tau \in I := [0, 1)$ which are of scattering type for $\tau > 0$ and of b-type at $\tau = 0$. The calculus is developed in detail in Appendix B, though we briefly recall its main features here. The natural space to consider is a blown-up version of $X \times I$, referred to as the *single space:*

$$X_t := [X \times I; \partial X \times 0].$$

There are three boundary faces of $X_t$ (see Figure 1) which are referred to as the 'scattering face' sc, the 'transition face' tf and the 'zero face' zf. They are diffeomorphic to $\partial X \times I$, $\partial X \times {}_{\mathrm{b}}[0, 1]_{\mathrm{sc}}$ (this notation will be explained shortly) and $X$, respectively.

The projection $\pi_I : X \times I \longrightarrow I$ lifts to a b-fibration (a type of generalized fibration well suited to manifolds with corners — see Appendix A for a definition) $\pi_I : X_t \longrightarrow I$ whose level sets $\pi_I^{-1}(\tau)$, $\tau > 0$ are diffeomorphic to $X$, and whose level set $\pi_I^{-1}(0)$ consists of the union of boundary faces zf and tf.

The vector fields (and by extension differential operators) considered are those tangent to the fibers of $\pi_I$ which are of scattering type on $\pi_I^{-1}(\tau)$, $\tau > 0$, b-type on zf, and b-sc-type on tf, meaning that they are of b-type near $\mathrm{tf} \cap \mathrm{zf} \cong \partial X \times 0$ and of scattering type near $\mathrm{tf} \cap \mathrm{sc} \cong \partial X \times 1$ (hence the notation $\mathrm{tf} \cong \partial X \times {}_{\mathrm{b}}[0, 1]_{\mathrm{sc}}$).

It follows that the indicial families of the restrictions of such a vector field $V$ to tf and zf are related by $I(N_{\mathrm{tf}}(V), \lambda) = I(N_{\mathrm{zf}}(V), -\lambda)$, where $N_*(V)$ denotes the restriction of $V$ to a boundary face (see Proposition B.10). This relation is ultimately the source of the 'equal and opposite' index phenomenon in the trace formula at $\tau = 0$ alluded to above.

*Remark.* In fact, $X_t$ may be alternatively thought of as a 'surgery' or 'gluing' space in which a scattering end is glued onto the b end of a manifold with boundary by attaching the cylindrical end $\partial X \times {}_{\mathrm{b}}[0,1]_{\mathrm{sc}}$, and the index theorem below may be seen as a gluing formula for the index. The indicial family relation imposes some matching conditions across the gluing hypersurface.

Restricting differential operators to various submanifolds results in the following homomorphisms[5] of differential operator algebras:

$$
\begin{aligned}
N_\tau &: \mathrm{Diff}^k_t(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \longrightarrow \mathrm{Diff}^k_{\mathrm{sc}}(X; V \otimes \Omega_{\mathrm{b}}^{1/2}),\ \tau > 0 \\
N_{\mathrm{tf}} &: \mathrm{Diff}^k_t(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \longrightarrow \mathrm{Diff}^k_{\mathrm{b,sc}}(\partial X \times {}_{\mathrm{b}}[0,1]_{\mathrm{sc}}; V \otimes \Omega_{\mathrm{b}}^{1/2}) \\
N_{\mathrm{zf}} &: \mathrm{Diff}^k_t(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \longrightarrow \mathrm{Diff}^k_{\mathrm{b}}(X; V \otimes \Omega_{\mathrm{b}}^{1/2}).
\end{aligned}
\tag{3.1}
$$

Here $N_\tau$ denotes restriction to the submanifold $\pi_I^{-1}(\tau)$, $\tau > 0$. The symbols and indicial operators of these objects are compatible in the sense that

$$
\begin{aligned}
\sigma_{\mathrm{sc}}(N_{\mathrm{tf}}(P)) &= \lim_{\tau \to 0} \sigma_{\mathrm{sc}}(N_\tau(P)), \\
I(N_{\mathrm{tf}}(P), \lambda) &= I(N_{\mathrm{zf}}(P), -\lambda).
\end{aligned}
\tag{3.2}
$$

We refer to (3.1) as the 'normal operator' homomorphisms. Restriction to the boundary face sc gives fiberwise translation invariant differential operators on the bundle $\pi_X^* {}^{\mathrm{sc}}TX|_{\mathrm{sc}} \longrightarrow \mathrm{sc} \cong \partial X \times I$, and taking the Fourier transform with respect to the vector bundle structure gives a family of the scattering symbols which we denote $\sigma_{\mathrm{sc}}(P)$ and which satisfy

$$
\begin{aligned}
\sigma_{\mathrm{sc}}(P)(\tau) &= \sigma_{\mathrm{sc}}(N_\tau(P)), \quad \tau > 0, \\
\sigma_{\mathrm{sc}}(P)(0) &= \sigma_{\mathrm{sc}}(P)|_{\mathrm{sc} \cap \mathrm{tf}} = \sigma_{\mathrm{sc}}(N_{\mathrm{tf}}(P)).
\end{aligned}
\tag{3.3}
$$

Pseudodifferential operators $\Psi^*_t(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$ are defined in terms of their Schwartz kernels on the *double space* $X_t^2$, a blown-up version of $X^2 \times I$, and there are analogous normal operator homomorphisms at the pseudodifferential level:

$$
\begin{aligned}
N_\tau &: \Psi^*_t(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \longrightarrow \Psi^*_{\mathrm{sc}}(X; V \otimes \Omega_{\mathrm{b}}^{1/2}),\ \tau > 0 \\
N_{\mathrm{tf}} &: \Psi^*_t(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \longrightarrow \Psi^*_{\mathrm{b,sc}}(\partial X \times {}_{\mathrm{b}}[0,1]_{\mathrm{sc}}; V \otimes \Omega_{\mathrm{b}}^{1/2}) \\
N_{\mathrm{zf}} &: \Psi^*_t(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \longrightarrow \Psi^*_{\mathrm{b}}(X; V \otimes \Omega_{\mathrm{b}}^{1/2}) \\
\sigma_{\mathrm{sc}} &: \Psi^*_t(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \longrightarrow C^\infty\big({}^{\mathrm{sc}}T^*_{\partial X}X \times I; \mathrm{End}(V \otimes \Omega_{\mathrm{b}}^{1/2})\big)
\end{aligned}
$$

These satisfy the same compatibility relations as in (3.2) and (3.3) as a consequence of the geometry of $X_t^2$.

Finally, for appropriately well-behaved operators there is a trace

$$
\mathrm{Tr} : \Psi^*_t(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \longrightarrow \mathcal{A}^*_{\mathrm{phg}}(I)
$$

where the range space denotes functions which are smooth on the interior of $I$ with polyhomogeneous expansions (meaning in terms of $\tau^z (\log \tau)^k$, $z \in \mathbb{C}$, $k \in \mathbb{N}$) at

[5]Meaning the restriction of a composition is the composition of restrictions.

$\tau = 0$. As a function of $\tau$, this trace is continuous and

$$\begin{aligned} \operatorname{Tr}(P)(\tau) &= \operatorname{Tr}(N_\tau(P)), \quad \tau > 0 \\ \operatorname{Tr}(P)(0) &= \operatorname{Tr}(N_{\mathrm{tf}}(P)) + \operatorname{Tr}(N_{\mathrm{zf}}(P)), \end{aligned} \tag{3.4}$$

provided all the operators on the right hand side are trace-class (see Proposition B.12).

3.2. **The case of zero rank.** We will next consider the operator $\mathcal{P} = D + \Phi - i\chi\tau$ in the zero rank case ($\Phi|_{\partial X} = 0$) within the framework of the transition calculus. Before doing so however we note two things: first, the transition calculus is most naturally defined using b half densities rather than scattering half densities, so we will need to conjugate $\mathcal{P}$ by $x^{n/2}$ in order to make it act on sections of $\Omega_{\mathrm{b}}^{1/2}$. Second, we observe

**Lemma 3.1.** *The lift of $\mathcal{P}$ to $X_t$ vanishes to first order at the boundary face* $\mathrm{tf} \subset X_t$.

*Proof.* This is easily checked in local coordinates: if $(x, y, \tau)$ are coordinates for $X \times I$ near $\partial X \times 0$, then natural local coordinates for $X_t$ are given by

$$\begin{aligned} (\xi, y, \tau) &:= (x/\tau, y, \tau) \quad \text{near } \mathrm{tf} \cap \mathrm{sc}, \text{ and} \\ (x, y, \eta) &:= (x, y, \tau/x) \quad \text{near } \mathrm{tf} \cap \mathrm{zf}. \end{aligned}$$

(These coordinates are related to the global variable $\sigma \in {}_{\mathrm{b}}[0,1]_{\mathrm{sc}}$ via $\eta = 1/\xi = \sigma/(1-\sigma)$.)

The relevant vector fields (of which $D$ is a linear combination) and coordinates lift in the first case according to $\{x^2\partial_x, x\partial_y\} \longmapsto \{\tau\xi^2\partial_\xi, \tau\xi\partial_y\}$ and $\tau \longmapsto \tau$ is locally a boundary defining function. In the second case they lift according to $\{x^2\partial_x, x\partial_y\} \longmapsto \{x^2\partial_x - x\eta\partial_\eta, x\partial_y\}$ and $\tau \longmapsto x\eta$, where $x$ is locally boundary defining for tf (and $\eta$ is boundary defining for zf).

Since we are assuming $\Phi = \mathcal{O}(x^{1+\epsilon}) = \mathcal{O}((\tau\xi)^{1+\epsilon})$ it follows that, as a transition differential operator, $\mathcal{P}$ has an overall factor of $\tau$ near $\mathrm{tf} \cap \mathrm{sc}$ and $x$ near $\mathrm{tf} \cap \mathrm{zf}$ so in fact $\mathcal{P}|_{\mathrm{tf}} = 0$. □

Thus in order to consider $\mathcal{P}$ as an operator in the calculus, we must also factor out a power of the boundary defining function $\rho_{\mathrm{tf}}$ for tf. The choice of such a $\rho_{\mathrm{tf}}$ matters here, as does the manner in which it is factored out (compare the discussion in §2.2). We will use a function satisfying

$$\rho_{\mathrm{tf}} = \begin{cases} x & \text{near } \mathrm{tf} \cap \mathrm{zf}, \text{ and} \\ \tau & \text{near } \mathrm{tf} \cap \mathrm{sc}. \end{cases} \tag{3.5}$$

and take

$$\widetilde{\mathcal{P}} = x^{-n/2} \rho_{\mathrm{tf}}^{-1/2} \mathcal{P} \rho_{\mathrm{tf}}^{-1/2} x^{n/2} \tag{3.6}$$

(compare (2.2)). With this convention $N_{\mathrm{zf}}(\widetilde{\mathcal{P}})$ agrees with the operator $\widetilde{P}$ in §2.2 and again has the virtue of preserving formal self-adjointness of the Dirac operator.

*Remark.* Note that the convention (2.2) can also be understood as taking the boundary defining function $\rho_{\mathrm{tf}}$ on the double space to coincide with $(x\, x')^{1/2}$ near $\rho_{\mathrm{tf}} \cap \rho_{\mathrm{zf}}$, where $x$ and $x'$ are pulled back from the left and right factors of $X$, respectively.

With these conventions in place, we may prove

**Lemma 3.2.** *$\widetilde{\mathcal{P}} \in \mathrm{Diff}^1_t(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$ and*

(a) *$N_{\mathrm{zf}}(\widetilde{\mathcal{P}}) = \widetilde{P}$, where $\widetilde{P}$ is the operator in §2.2.*

(b) *$N_{\mathrm{tf}}(\widetilde{\mathcal{P}}) = \mathsf{D} - i\phi$, where $\mathsf{D}$ is a formally self-adjoint first order operator and $\phi$ is a non-negative scalar function, strictly positive on the interior, such that $\phi|_{\partial X \times 0} = 0$ and $\phi|_{\partial X \times 1} = 1$. The indicial families are related by*

$$I(N_{\mathrm{tf}}(\widetilde{\mathcal{P}}), \lambda) = I(\mathsf{D}, \lambda) = I(\widetilde{D}, -\lambda) = I(N_{\mathrm{zf}}(\widetilde{\mathcal{P}}), -\lambda) \tag{3.7}$$

*Proof.* The first claim has been discussed already and follows from the fact that the normalization convention (3.6) agrees with (2.2).

For the second claim, observe that the assumption $\Phi = \mathcal{O}(x^{1+\epsilon})$ means that $\rho_{\mathrm{tf}}^{-1/2}\Phi\rho_{\mathrm{tf}}^{-1/2} = \mathcal{O}(\rho_{\mathrm{tf}}^{\epsilon})$ still vanishes at tf, so $N_{\mathrm{tf}}(\widetilde{\mathcal{P}})$ is given by the restriction of $x^{-n/2}\rho_{\mathrm{tf}}^{-1/2}(D - i\chi\tau)\rho_{\mathrm{tf}}^{-1/2}x^{n/2}$.

At the b end of tf (i.e. tf $\cap$ zf), the normalization convention and the vector field computations in the proof of Lemma 3.1 show that $\mathsf{D}$ agrees with $\widetilde{D} = x^{-(n+1)/2}Dx^{(n-1)/2}$ apart from the replacement of $x\partial_x$ by $-\eta\partial_\eta$ (giving a direct verification of the indicial relation $I(\mathsf{D}, \lambda) = I(\widetilde{D}, -\lambda)$). Near the scattering end, they show that $\mathsf{D}$ agrees with $x^{-n/2}Dx^{n/2}$ apart from the replacement of $x$ by $\xi$.

Finally, it follows from (3.5) that $\phi$, which is the restriction of $x^{-n/2}\rho_{\mathrm{tf}}^{-1/2}\tau\chi\rho_{\mathrm{tf}}^{-1/2}x^{n/2} = \rho_{\mathrm{tf}}^{-1/2}\tau\chi\rho_{\mathrm{tf}}^{-1/2}$ agrees locally with boundary defining function $\eta$ near the b end and with 1 near the scattering end. □

Along with the evident invertibility of the scattering symbol of $N_{\mathrm{tf}}(\widetilde{\mathcal{P}})$, given explicitly by $\sigma_{\mathrm{sc}}(x^{-n/2}Dx^{n/2}) - i\mathrm{Id} = \sigma_{\mathrm{sc}}(D) - i\mathrm{Id}$, the indicial family relations imply

**Corollary 3.3.** *$N_{\mathrm{tf}}(\widetilde{\mathcal{P}})$ admits Fredholm extensions*

$$\begin{aligned} N_{\mathrm{tf}}(\widetilde{\mathcal{P}}) : \rho_{\mathrm{b}}^{\gamma}\rho_{\mathrm{sc}}^{\eta}H^k_{\mathrm{b,sc}}(\partial X \times {}_{\mathrm{b}}[0,1]_{\mathrm{sc}}; V \otimes \Omega_{\mathrm{b}}^{1/2}) \\ \longrightarrow \rho_{\mathrm{b}}^{\gamma}\rho_{\mathrm{sc}}^{\eta}H^{k-1}_{\mathrm{b,sc}}(\partial X \times {}_{\mathrm{b}}[0,1]_{\mathrm{sc}}; V \otimes \Omega_{\mathrm{b}}^{1/2}) \end{aligned} \tag{3.8}$$

*for all $k$, $\eta$ and $\gamma \notin i\mathrm{spec}_{\mathrm{b}}(\mathsf{D}) = -i\mathrm{spec}_{\mathrm{b}}(\widetilde{D})$.*

*Remark.* Here it should be noted that we're using the notation $H^*_{\mathrm{b,sc}}$ in a different way than used previously, to mean regularity with respect to derivatives which are of b type near $\partial X \times 0$ and scattering type near $\partial X \times 1$.

From the indicial relations it is *almost* obvious, and no doubt possible to show directly, that the index of (3.8) satisfies

$$\mathrm{ind}(N_{\mathrm{tf}}(\widetilde{\mathcal{P}}), \gamma) = \mathrm{ind}(N_{\mathrm{zf}}(\widetilde{\mathcal{P}}), -\gamma) = -\mathrm{ind}(N_{\mathrm{zf}}(\widetilde{\mathcal{P}}), \gamma). \tag{3.9}$$

Indeed, loosely speaking $N_{\mathrm{tf}}(\widetilde{\mathcal{P}}) = \mathsf{D} + i\phi$ should be thought of as a compact perturbation of self-adjoint operator $\mathsf{D}$ near the b end, with index computed by the relative index theorem. Of course it is not possible to entirely deform away the $i\phi$ term as its presence at the scattering end is necessary for $N_{\mathrm{tf}}(\widetilde{\mathcal{P}})$ to be Fredholm, although there is ultimately no index contribution from that end.

In any case, (3.9) will be obtained as a consequence of the parametrix construction for $\widetilde{\mathcal{P}}$ which we address next.

**Proposition 3.4.** *Under the assumption that $\Phi = \mathcal{O}(x^{1+\epsilon})$, for any choice of $\gamma \notin i\mathrm{spec}_{\mathrm{b}}(\widetilde{D})$ there exists a parametrix $\mathcal{Q} \in \Psi_t^{-1,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$, such that*

$$\mathcal{R}_L = \mathrm{Id} - \mathcal{Q}\widetilde{\mathcal{P}} \in \Psi_t^{-\infty,\mathcal{F}^L}, \mathcal{R}_R = \mathrm{Id} - \widetilde{\mathcal{P}}\mathcal{Q} \in \Psi_t^{-\infty,\mathcal{F}^R}$$

*are trace class, with continuous trace in $\tau$. Here*

$$\begin{aligned}\mathcal{E} &= (0_{\mathrm{sc}}, 0_{\mathrm{tf}}, E^+(\gamma)_{\mathrm{lb}_0}, E^-(\gamma)_{\mathrm{rb}_0}),\\ \mathcal{F}^L &= (\infty_{\mathrm{sc}}, 0_{\mathrm{tf}}, E^+(\gamma)_{\mathrm{lb}_0}, E^-(\gamma)_{\mathrm{rb}_0})\\ \mathcal{F}^R &= (\infty_{\mathrm{sc}}, 0_{\mathrm{tf}}, E^+(\gamma)_{\mathrm{lb}_0}, E^-(\gamma)_{\mathrm{rb}_0})\end{aligned}$$

*where* $E^\pm(\gamma) = \left\{(\pm z, l) : z \in i\mathrm{spec}_{\mathrm{b}}(\widetilde{D}) + \mathbb{N},\ 0 \leq l \leq \mathrm{ord}(-iz),\ z \gtrless \tau\right\}.$

*In particular, $N_{\mathrm{zf}}(\mathcal{Q})$ is a Fredholm inverse for the extension*

$$\widetilde{P} = N_{\mathrm{zf}}(\widetilde{\mathcal{P}}) : x^\gamma H_{\mathrm{b}}^k(X; V \otimes \Omega_{\mathrm{b}}^{1/2}) \longrightarrow x^\gamma H_{\mathrm{b}}^{k-1}(X; V \otimes \Omega_{\mathrm{b}}^{1/2}), \tag{3.10}$$

*and $N_{\mathrm{tf}}(\mathcal{Q})$ is a Fredholm inverse for the extension*

$$\begin{aligned} N_{\mathrm{tf}}(\widetilde{\mathcal{P}}) : \rho_{\mathrm{b}}^{-\gamma} H_{\mathrm{b,sc}}^{k,l}(\partial X \times {}_{\mathrm{b}}[0,1]_{\mathrm{sc}}; V \otimes \Omega_{\mathrm{b}}^{1/2}) \\ \longrightarrow \rho_{\mathrm{b}}^{-\gamma} H_{\mathrm{b,sc}}^{k-1,l-1}(\partial X \times {}_{\mathrm{b}}[0,1]_{\mathrm{sc}}; V \otimes \Omega_{\mathrm{b}}^{1/2}). \end{aligned} \tag{3.11}$$

*Finally,*

$$\mathrm{Tr}(\mathcal{R}_L - \mathcal{R}_R)(0) = \mathrm{ind}(N_{\mathrm{zf}}(\widetilde{\mathcal{P}}), \gamma) + \mathrm{ind}(N_{\mathrm{tf}}(\widetilde{\mathcal{P}}), -\gamma) = 0. \tag{3.12}$$

To say that the remainder terms are trace class here means that $N_\tau(\mathcal{R}_{R/L})$, $N_{\mathrm{tf}}(\mathcal{R}_{R/L})$ and $N_{\mathrm{zf}}(\mathcal{R}_{R/L})$ are each trace class, so in particular (3.4) holds.

*Proof.* As $\widetilde{\mathcal{P}}$ is elliptic, its principal symbol $\sigma_1(\widetilde{\mathcal{P}})$ is invertible, so we may initially choose $\mathcal{Q}_0 \in \Psi_t^{-1,\mathcal{E}}$ such that $\sigma_{-1}(\mathcal{Q}_0) = \sigma_1(\widetilde{\mathcal{P}})^{-1}$, and likewise $\sigma_{\mathrm{sc}}(\mathcal{Q}_0) = \sigma_{\mathrm{sc}}(\widetilde{\mathcal{P}})^{-1}$.

Since $\sigma_{\mathrm{sc}}(\widetilde{\mathcal{P}})|_{\mathrm{sc}\cap\mathrm{tf}} = \sigma_{\mathrm{sc}}\big(N_{\mathrm{tf}}(\widetilde{\mathcal{P}})\big)$, this is compatible with a choice of $\mathcal{Q}_0$ such that $N_{\mathrm{tf}}(\mathcal{Q}_0)$ is a Fredholm parametrix for $N_{\mathrm{tf}}(\widetilde{\mathcal{P}})$ as an operator (3.11) at the scattering end of $\mathrm{tf} \cong \partial X \times_{\mathrm{b}} [0,1]_{\mathrm{sc}}$. At the b-end, this requires taking $I\big(N_{\mathrm{tf}}(\mathcal{Q}_0)\big) = \mathcal{M}^{-1}_{\mathrm{Im}\,\lambda=\gamma} I(N_{\mathrm{tf}}(\widetilde{\mathcal{P}}), \gamma)^{-1}$.

This is in turn compatible with the requirement that $N_{\mathrm{zf}}(\mathcal{Q}_0)$ be a Fredholm parametrix for (3.10), since

$$I\big(N_{\mathrm{zf}}(\mathcal{Q}_0)\big) = \mathcal{M}^{-1}_{\mathrm{Im}\,\lambda=-\gamma} I(N_{\mathrm{zf}}(\widetilde{\mathcal{P}}), \gamma) = \mathcal{M}^{-1}_{\mathrm{Im}\,\lambda=-\gamma} I(N_{\mathrm{tf}}(\widetilde{\mathcal{P}}), -\gamma) = I\big(N_{\mathrm{tf}}(\mathcal{Q}_0)\big).$$

The order $\mathcal{E} = (0_{\mathrm{sc}}, 0_{\mathrm{tf}}, E^+(\gamma)_{\mathrm{lb}_0}, E^-(\gamma)_{\mathrm{rb}_0})$ is a standard consequence of taking this inverse Mellin transform along $\mathrm{Im}\,\lambda = -\gamma$.

$\mathcal{Q}_0$ is not yet a sufficiently good parametrix for the remainders to be trace-class; this requires inverting $\widetilde{\mathcal{P}}$ to higher order along the diagonal and at sc. Indeed, from Propositions B.5 and B.7 it follows that $\mathcal{R}_0 := I - \widetilde{\mathcal{P}}\mathcal{Q}_0 \in \Psi_t^{-1,\mathcal{G}_0}$ where $\mathcal{G}_0 = (-1_{\mathrm{sc}}, 0_{\mathrm{tf}}, E^+(\gamma)_{\mathrm{lb}_0}, E^-(\gamma)_{\mathrm{rb}_0}) = \mathcal{E} - 1_{\mathrm{sc}}$. We will proceed by induction on $i \in \mathbb{N}$, constructing additional terms $\mathcal{Q}_i$ as follows. For the inductive step, we choose a $\mathcal{Q}_i \in \Psi_t^{-(i+1),\mathcal{E}'}$, $\mathcal{E}' = (i_{\mathrm{sc}}, 0_{\mathrm{tf}}, 0_{\mathrm{lb}_0}, 0_{\mathrm{rb}_0})$ satisfying

$$\begin{aligned}\sigma_{-(i+1)}(\mathcal{Q}_i) &= \sigma_1(\widetilde{\mathcal{P}})^{-1}\sigma_{-i}(\mathcal{R}_{i-1})\\ \sigma_{\mathrm{sc}}(\mathcal{Q}_i) &= \sigma_{\mathrm{sc}}(\widetilde{\mathcal{P}})^{-1}\sigma_{\mathrm{sc}}(\mathcal{R}_{i-1})\end{aligned}$$

and vanishing on tf and zf except near $\mathrm{tf} \cap \mathrm{sc}$ (for we want this to improve the parametrix for $N_{\mathrm{tf}}(\widetilde{\mathcal{P}})$ near the scattering end as well). It then follows from composition that $\widetilde{\mathcal{P}}(\mathcal{Q}_0 + \mathcal{Q}_1 + \cdots + \mathcal{Q}_i) = I - \mathcal{R}_i$ with $\mathcal{R}_i \in \Psi_t^{-i,\mathcal{G}_i}$ with $\mathcal{G}_i = \mathcal{G}_{i-1} - 1_{\mathrm{sc}}$.

Asymptotically summing the resulting infinite series, we obtain $\mathcal{Q} \in \Psi_t^{-1,\mathcal{E}}(X; V \otimes \Omega_{\mathrm{b}}^{1/2})$ with the remainder terms $\mathcal{R}_L$ and $\mathcal{R}_R$ as claimed. It follows that the remainders

$$N_{\mathrm{tf}}(\mathcal{R}_L) = I - N_{\mathrm{tf}}(\mathcal{Q})N_{\mathrm{tf}}(\widetilde{\mathcal{P}}), \quad N_{\mathrm{tf}}(\mathcal{R}_R) = I - N_{\mathrm{tf}}(\widetilde{\mathcal{P}})N_{\mathrm{tf}}(\mathcal{Q})$$
$$N_{\mathrm{zf}}(\mathcal{R}_L) = I - N_{\mathrm{zf}}(\mathcal{Q})N_{\mathrm{zf}}(\widetilde{\mathcal{P}}) \quad N_{\mathrm{zf}}(\mathcal{R}_R) = I - N_{\mathrm{zf}}(\widetilde{\mathcal{P}})N_{\mathrm{zf}}(\mathcal{Q})$$

are trace class. (3.12) follows from Proposition B.12 (which shows that (3.4) holds) and continuity in $\tau$ since $\mathrm{ind}(N_\tau(\widetilde{\mathcal{P}})) = 0$ for $\tau > 0$. □

From Proposition 2.2 and (3.12) we obtain

**Corollary 3.5.** *The indices of the extensions* (3.11) *of* $N_{\mathrm{tf}}(\widetilde{\mathcal{P}})$ *satisfy*

$$\begin{aligned} \mathrm{ind}(N_{\mathrm{tf}}(\widetilde{\mathcal{P}}), -\gamma) &= -\mathrm{ind}(N_{\mathrm{tf}}(\widetilde{\mathcal{P}}), \gamma), \\ \mathrm{ind}(N_{\mathrm{tf}}(\widetilde{\mathcal{P}}), \gamma_0 - \epsilon) - \mathrm{ind}(N_{\mathrm{tf}}(\widetilde{\mathcal{P}}), \gamma_0 + \epsilon) &= -\dim F(\widetilde{D}, \gamma_0) \end{aligned} \tag{3.13}$$

*for* $\gamma_0 \in i\mathrm{spec}_{\mathrm{b}}(\widetilde{D}_0)$ *and* $\epsilon > 0$ *sufficiently small.*

3.3. **The case of general rank.** We may now compute the index of the Fredholm extensions (2.9).

**Theorem 3.6.** *The Fredholm extension* (2.9) *has index*

$$\mathrm{ind}(P) = \mathrm{ind}(\eth_+^+) + \mathrm{def}(P, \gamma) \tag{3.14}$$

*where* $\eth_+^+ \in \mathrm{Diff}^1(\partial X; V_+^+ \otimes \Omega_{\mathrm{sc}}^{1/2}, V_+^- \otimes \Omega_{\mathrm{sc}}^{1/2})$ *is one half of the induced Dirac operator on* $\partial X$ *obtained from* $D$ *acting on* $V_+|_{\partial X} = V_+^+ \oplus V_+^-$. *Here* $V_+$ *denotes the span of the positive imaginary eigenspace of* $\Phi|_{\partial X}$ *and* $V_+^+ \oplus V_+^-$ *denotes the splitting into* $\pm 1$ *eigenspaces of* $i\mathrm{c}\ell(x^2\partial_x)$.

*The defect term* $\mathrm{def}(P, \gamma) \in \mathbb{Z}$ *is constant for* $\gamma \notin i\mathrm{spec}_{\mathrm{b}}(\widetilde{D}_0)$ *and satisfies*

$$\begin{aligned} \mathrm{def}(P, -\gamma) &= -\mathrm{def}(P, \gamma), \\ \mathrm{def}(P, \gamma_0 - \epsilon) - \mathrm{def}(P, \gamma_0 + \epsilon) &= \dim F(\widetilde{D}_0, \gamma_0) \end{aligned} \tag{3.15}$$

*for* $\gamma_0 \in i\mathrm{spec}_{\mathrm{b}}(\widetilde{D}_0)$ *and sufficiently small* $\epsilon > 0$.

*Proof.* The strategy is to give a parameterized version of the parametrix construction in the proof of Theorem 2.5.

As in that proof we may assume that $P$ is diagonal with respect to $V = V_0 \oplus V_1$ in a neighborhood $U$ of $\partial X$. Let $\mathcal{P} = D + \Phi - i\chi\tau \oplus 0$ as in the beginning of this section so that

$$\mathcal{P}|_U = \begin{pmatrix} D_0 - i\chi\tau & 0 \\ 0 & D_1 + \Phi \end{pmatrix},$$

and fix $\gamma \notin i\mathrm{spec}_{\mathrm{b}}(\widetilde{D}_0)$.

We construct a parametrix $\mathcal{Q} \in C^{-\infty}(X^2 \times I; \mathrm{End}(V) \otimes \Omega_{\mathrm{sc}}^{1/2})$ for $\mathcal{P}$ as a distribution supported near and conormal to the fiber diagonal in $X^2 \times I$, such that

$$\mathcal{Q}|_{U^2 \times I} = \begin{pmatrix} \mathcal{Q}_0 & 0 \\ 0 & \mathcal{Q}_1 \end{pmatrix}.$$

We suppose that $\mathcal{Q}_1 = \pi^*_{X^2}\big(\beta^*_{\mathrm{sc}}(Q'_1)\big)$; that is, $\mathcal{Q}_1$ is the pushforward from $X^2_{\mathrm{sc}}$ of a distribution $Q'_1$, pulled back to be constant along $I$, We further suppose

$$\mathcal{Q}_0 = (\beta_t)_*\big(x^{n/2}\rho_{\mathrm{tf}}^{-1/2}\widetilde{\mathcal{Q}}_0{\rho'}_{\mathrm{tf}}^{-1/2}{x'}^{-n/2}\big)$$

for a distribution $\widetilde{\mathcal{Q}}_0$ on $X^2_t$, with $x, \rho_{\mathrm{tf}}$ and $x', \rho'_{\mathrm{tf}}$ denoting the lifts from the left and right, respectively.

Let $\sigma_{-1}(Q'_1) = \sigma_1(D_1)^{-1}$ and $\sigma_{-1}(\widetilde{\mathcal{Q}}_0) = \sigma_1(\widetilde{D}_0)^{-1}$, which are compatible with an overall choice such that $\sigma_{-1}(\mathcal{Q}) = \sigma_1(\mathcal{P})^{-1}$ in the interior. We furthermore let $\sigma_{\mathrm{sc}}(Q'_1) = \sigma_{\mathrm{sc}}(D_1 + \Phi)^{-1}$. For $\widetilde{\mathcal{Q}}_0$, let $\sigma_{\mathrm{sc}}(\widetilde{\mathcal{Q}}_0) = \sigma_{\mathrm{sc}}(\widetilde{\mathcal{P}}_0)^{-1}$, and take $N_{\mathrm{tf}}(\widetilde{\mathcal{Q}}_0)$ and $N_{\mathrm{zf}}(\widetilde{\mathcal{Q}}_0)$ to be suitable Fredholm parametricies for $N_{\mathrm{tf}}(\widetilde{\mathcal{P}}_0)$ and $N_{\mathrm{zf}}(\widetilde{\mathcal{P}}_0)$ as operators (3.11) and (3.10), respectively.

Finally, we improve $\mathcal{Q}^0 := \mathcal{Q}$ by an iterative procedure (here the superscripts are indices not powers), taking

$$\begin{aligned}
\sigma_{-(i+1)}(\mathcal{Q}^i) &= \sigma_1(\mathcal{P})^{-1}\sigma_{-i}(\mathcal{R}^{i-1}),\\
\sigma_{\mathrm{sc}}(\mathcal{Q}^i_1) &= \sigma_{\mathrm{sc}}(\mathcal{P}_1)^{-1}\sigma_{\mathrm{sc}}(\mathcal{R}^{i-1}), \quad \text{and}\\
\sigma_{\mathrm{sc}}(\widetilde{\mathcal{Q}}^i_0) &= \sigma_{\mathrm{sc}}(\widetilde{\mathcal{P}}_0)^{-1}\sigma_{\mathrm{sc}}(\widetilde{\mathcal{R}}^{i-1}_0),
\end{aligned}$$

where $\mathcal{R}^i = I - \mathcal{P}\mathcal{Q}^i$, and asymptotically summing the resulting series, denoting the result again by $\mathcal{Q}$.

The remainder terms $\mathcal{R}_L = I - \mathcal{Q}\mathcal{P}$ and $\mathcal{R}_R = I - \mathcal{P}\mathcal{Q}$ are smooth on the interior of $X^2 \times I$ and are diagonal with respect to $V_0 \oplus V_1$ near $\partial X^2 \times I$. For fixed $\tau > 0$ these are trace class operators on any $x^\alpha H^k_{\mathrm{sc}}(X; V \otimes \Omega^{1/2}_{\mathrm{sc}})$, and the families trace — given by restriction to the fiber diagonal $X \times I \subset X^2 \times I$ followed by push forward to $I$ — of $\mathcal{R}_L$ and $\mathcal{R}_R$ is continuous by the considerations of the previous section.

At $\tau = 0$, $\mathrm{Tr}\,(\mathcal{R}_L)$ has the form[6]

$$\begin{aligned}
(\mathrm{Tr}\mathcal{R}_L)(0) = &\begin{pmatrix} (\beta_t)_*\,\mathrm{Tr}\big(I_{\mathrm{tf}} - N_{\mathrm{tf}}(\widetilde{\mathcal{Q}}_0)N_{\mathrm{tf}}(\widetilde{\mathcal{P}}_0)\big) & 0 \\ 0 & 0\end{pmatrix}\\
&+ \begin{pmatrix} (\beta_t)_*\,\mathrm{Tr}\big(I_{\mathrm{zf}} - N_{\mathrm{zf}}(\widetilde{\mathcal{Q}}_0)N_{\mathrm{zf}}(\widetilde{\mathcal{P}}_0)\big) & 0 \\ 0 & (\beta_{\mathrm{sc}})_*\big(I - Q'_1(D_1 + \Phi)\big)\end{pmatrix}
\end{aligned}$$

and similarly for $\mathrm{Tr}\mathcal{R}_R(0)$, where $\widetilde{\mathcal{P}}_0 = x^{n/2}\rho_{\mathrm{tf}}^{-1/2}(D_0 + i\chi\tau){\rho'}_{\mathrm{tf}}^{-1/2}{x'}^{n/2}$. We observe that the rightmost term above is the error for a Fredholm parametrix for the extension (2.9), since roughly speaking[7] $\begin{pmatrix} N_{\mathrm{zf}}(\mathcal{P}_0) & 0 \\ 0 & D_1 + \Phi\end{pmatrix}$ agrees with $P$ and the distribution having the form

$$\begin{pmatrix} N_{\mathrm{zf}}(\mathcal{Q}_0) & 0 \\ 0 & Q'_1\end{pmatrix}$$

near $\partial X^2$ is a parametrix as in the proof of Theorem 2.5.

From the trace formula for the index it follows that

$$\mathrm{Tr}\big(\mathcal{R}_L - \mathcal{R}_R\big)(\tau) = \mathrm{ind}(\mathcal{P}) = \mathrm{ind}(\partial^+_+), \quad \tau > 0$$

---

[6]Evaluation at the diagonal has the effect of removing terms of the form $\rho_{\mathrm{tf}}^{-1/2}{\rho'}_{\mathrm{tf}}^{1/2}$ and $x^{n/2}{x'}^{-n/2}$ which would otherwise be present

[7]Meaning we are omitting the normalizing factors of $x$, $x'$, $\rho_{\mathrm{tf}}$ and $\rho'_{\mathrm{tf}}$.

using Theorem 2.1 and the fact that the positive imaginary eigenbundle $V_+$ of $\Phi - i\chi\tau \oplus 0$ coincides with that of $\Phi$. Then from the constancy of the trace of $\mathcal{R}_{L/R}$, it follows that

$$\mathrm{ind}(\partial_+^+) = \mathrm{Tr}\big(\mathcal{R}_L - \mathcal{R}_R\big)(0) = \mathrm{ind}(N_{\mathrm{tf}}(\widetilde{\mathcal{P}}_0), -\gamma) + \mathrm{ind}(P)$$

where $\mathrm{ind}(P)$ denotes the index of the extension (2.9) which we are trying to compute. The proof is completed by defining

$$\mathrm{def}(P,\gamma) := -\mathrm{ind}(N_{\mathrm{tf}}(\widetilde{\mathcal{P}}_0), -\gamma) = \mathrm{ind}(N_{\mathrm{tf}}(\widetilde{\mathcal{P}}_0), \gamma)$$

which satisfies (3.15) as a consequence of Corollary 3.5. □

*Remark.* Whereas $D_0$ is not a globally defined operator, the operator $N_{\mathrm{tf}}(\widetilde{\mathcal{P}}_0) = N_{\mathrm{tf}}(\widetilde{D_0 - i\chi\tau}) \in \mathrm{Diff}^1_{\mathrm{b,sc}}(\partial X \times_{\mathrm{b}} [0,1]_{\mathrm{sc}}; V \otimes \Omega_{\mathrm{b}}^{1/2})$ whose index gives rise to $\mathrm{def}(P,\gamma)$ is nevertheless well defined, and depends only on the expansion of $D_0$ near $\partial X$. In particular it is independent of $\chi$ and $\tau$.

## 4. Deformation theory of monopoles

4.1. **Monopoles.** We consider SU(2) 'magnetic monopoles' on a 3-dimensional exact scattering manifold $(X, g)$. A monopole consists of a principal SU(2) bundle $P \longrightarrow X$ (necessarily trivial since $SU(2)$ is 2-connected), a connection $A$ on $P$, and a section $\Phi \in \Gamma(X; \mathrm{ad}P)$ satisfying the *Bogomolny equation*

$$F^A = \star d^A \Phi \tag{4.1}$$

where $F^A$ is the curvature of $A$ and $d^A$ is the covariant derivative (here written as an exterior covariant derivative) defined by $A$. Monopoles are minimizers of the Yang-Mills-Higgs action

$$(A,\Phi) \longmapsto \big\|F^A\big\|^2_{L^2} + \big\|d^A\Phi\big\|^2_{L^2}$$

within connected components of the *configuration space* $\mathcal{C}$ of pairs $(A,\Phi) \in \Gamma(X; \Lambda^1 \otimes \mathrm{ad}P) \oplus \Gamma(X; \mathrm{ad}P)$ which are bounded up to $\partial X$ having finite action. Here $\mathrm{ad}P = X \times \mathfrak{su}(2)$ and is equipped with the Hermitian inner product given by negative of the Killing form, and as a matter of notation we use $\Gamma(X;V)$ to denote sections of $V \longrightarrow X$ which are bounded and conormal at $\partial X$ with polyhomogeneous expansions. (For a general scattering manifold, monopoles are not expected to be $C^\infty$ up to $\partial X$ as we will see below, but they are expected to be polyhomogeneous.) We also use the notational shorthand $\Lambda^k$ to refer to the bundle $\Lambda^k({}^{\mathrm{sc}}T^*X)$.

Components of this configuration space are labeled by an integral *charge parameter* $k \in \mathbb{Z}^N$ (here $N$ is the number of components of $\partial X$) defined by the first Chern class

$$k = c_1(L) \in H^2(\partial X; \mathbb{Z}) \cong \mathbb{Z}^N$$

of a line bundle $L \longrightarrow \partial X$ consisting of the positive imaginary eigenspace of $\Phi|_{\partial X}$ as an endomorphism of the trivial $\mathbb{C}^2$ bundle over $\partial X$ associated to the standard representation of SU(2) — in other words, viewing $\Phi|_{\partial X}$ as valued in skew-adjoint $2\times 2$ matrices. Indeed, finite action implies that $|\Phi|\,|_{\partial X}$ is a constant, conventionally normalized to 1, so that $\partial X \times \mathbb{C}^2$ splits as a bundle over $\partial X$ into $L \oplus L^*$ according to $\pm i$ eigenspaces of $\Phi|_{\partial X}$.

Solutions of (4.1) are preserved by the action of the gauge group $\mathcal{G} = \Gamma(X; \mathrm{Ad}P)$, here taken by convention to be bounded sections $g$ with $g|_{\partial X} = \mathrm{Id}$. This convention restricts the gauge group from acting at $\partial X$, so the boundary data $(\Phi, A)|_{\partial X}$ are thought of as being prescribed. One is then interested in the moduli space $\mathcal{M}_k$ of solutions to (4.1) modulo $\mathcal{G}$, with charge parameter $k$ and prescribed boundary data $(\Phi, A)|_{\partial X}$.

4.2. **Deformation complex.** The problem of computing the formal dimension of $\mathcal{M}_k$ is an infinitesimal one, and may be recast in the form of an elliptic complex.

Indeed, at a pair $(A, \Phi)$ the tangent space $T_{(A,\Phi)}\mathcal{C}$ to the configuration space is $\Gamma(X; \Lambda^1 \otimes \mathrm{ad}P) \oplus \Gamma(X; \mathrm{ad}P)$ and the Lie algebra of the gauge group is given by $\Gamma(X; \mathrm{ad}P)$. The vector field on $\mathcal{C}$ induced from the action by $\gamma \in \Gamma(X; \mathrm{ad}P)$ is

$$\widetilde{\gamma}_{(A,\Phi)} = (-d^A\gamma, -[\Phi, \gamma]) \in T_{(A,\Phi)}\mathcal{C} \tag{4.2}$$

On the other hand, the linearization of the Bogomolny map

$$\mathcal{B} : \mathcal{C} \ni (A, \Phi) \longmapsto F^A - \star d^A\Phi \in \Gamma(X; \Lambda^2 \otimes \mathrm{ad}P)$$

is

$$d\mathcal{B} : T_{(A,\Phi)}\mathcal{C} \ni (a, \phi) \longmapsto d^A a - \star d^A\phi + \star[\Phi, a] \in \Gamma(X; \Lambda^2 \otimes \mathrm{ad}P). \tag{4.3}$$

It is convenient at this point to make use of the isomorphism $\star : \Gamma(X; \mathrm{ad}P) \cong \Gamma(X; \Lambda^3 \otimes \mathrm{ad}P)$, after which we may arrange (4.2) and (4.3) into a sequence

$$\begin{aligned}0 \longrightarrow \Gamma(X; \mathrm{ad}P) \xrightarrow{\mathsf{D}_0} \Gamma(X; \Lambda^1 \otimes \mathrm{ad}P) &\oplus \Gamma(X; \Lambda^3 \otimes \mathrm{ad}P)\\ &\xrightarrow{\mathsf{D}_1} \Gamma(X; \Lambda^2 \otimes \mathrm{ad}P) \longrightarrow 0\end{aligned} \tag{4.4}$$

where

$$\mathsf{D}_0 : \gamma \longmapsto (-d^A\gamma, - \star [\Phi, \gamma]),$$

represents the infinitesimal gauge action and

$$\mathsf{D}_1 : (a, \phi) \longmapsto d^A a + \star [\Phi, a] + \delta^A\phi$$

represents the linearization of the Bogomolny equation. Here $\delta^A = (d^A)^* = -\star d^A\star$ is the formal adjoint of $d^A$ on 3-forms with respect to the $L^2$ pairing defined by the metric and the inner product on $\mathrm{ad}P$. (Note that $\star[\Phi, \cdot]$ is skew-adjoint since $[\Phi, \cdot]$ is skew-adjoint with respect to the Killing form and $\star^* = \star^{-1} = \star$ in odd dimensions.)

**Proposition 4.1.** *If* $(A, \Phi)$ *satisfy the Bogomolny equation* (4.1)*, then the sequence* (4.4) *is an elliptic chain complex.*

*Proof.* One computes

$$\begin{aligned}\mathsf{D}_1\,\mathsf{D}_0 &= \begin{bmatrix} d^A + \star[\Phi, \cdot] & \delta^A \end{bmatrix} \begin{bmatrix} -d^A \\ - \star [\Phi, \cdot] \end{bmatrix}\\ &= -[F^A, \cdot] - \star[\Phi, d^A\cdot] + \star d^A ([\Phi, \cdot])\\ &= -[F^A, \cdot] - \star[\Phi, d^A\cdot] + \star[d^A\Phi, \cdot] + \star[\Phi, d^A\cdot]\\ &= [\star d^A\Phi - F^A, \cdot]\end{aligned}$$

which is 0 if $F^A = \star d^A\Phi$.

On the other hand, at the principal symbolic level, $\sigma(\mathsf{D}_0)(\xi) = \begin{bmatrix} -i\xi\wedge\cdot & 0 \end{bmatrix}^\dagger$ and $\sigma(\mathsf{D}_1)(\xi) = \begin{bmatrix} i\xi\wedge\cdot & -i\xi\lrcorner\cdot \end{bmatrix}$ which is seen to be exact. (Note that $i\xi\lrcorner\cdot$ is injective on forms of maximal dimension 3.) □

From now on we assume from now on that $(A,\Phi)$ satisfies (4.1). Observe that the formal dimension of $\mathcal{M}_k$ is given by the dimension of the degree 1 cohomology space of (4.4):

$$\dim(\mathcal{M}_k) = \dim \mathscr{H}^1 := \dim\left(\ker \mathsf{D}_1/\mathrm{Im}\,\mathsf{D}_0\right),$$

while the index of $\mathsf{D}_1+\mathsf{D}_0^*$ (assuming a Fredholm extension has been chosen) is

$$\mathrm{ind}\,(\mathsf{D}_1+\mathsf{D}_0^*) = \dim \mathscr{H}^1 - \left(\dim \mathscr{H}^0 + \dim \mathscr{H}^2\right).$$

Here

$$\mathsf{D}_1+\mathsf{D}_0^* : \Gamma\big(X;(\Lambda^1\oplus\Lambda^3)\otimes \mathrm{ad}P\big) \longrightarrow \Gamma\big(X;(\Lambda^0\oplus\Lambda^2)\otimes\mathrm{ad}P\big)$$

is constructed using the formal $L^2$ adjoint of $\mathsf{D}_0$. The condition

$$\mathsf{D}_0^*(a,\star\phi)=0 \iff -\delta^A a + [\Phi,\phi]=0$$

is known classically as the *Coulomb gauge condition.*

We will prove the following:

(1) $\mathsf{D}_1+\mathsf{D}_0^*$ is a Callias type operator with respect to the boundary splitting $\mathrm{ad}P|_{\partial X} = \mathrm{ad}P_0\oplus\mathrm{ad}P_1 := \mathbb{C}\Phi\oplus\Phi^\perp$.
(2) For a range of weighted hybrid Sobolev spaces, $\mathsf{D}_1+\mathsf{D}_0^*$ is surjective (so in particular $\mathscr{H}^0=\mathscr{H}^2=\{0\}$).
(3) At the low end of this range, $\mathsf{D}_1+\mathsf{D}_0^*$ is Fredholm with index $4k+1/2b^1(\partial X)$, where $b^1(\partial X)=\dim H^1(\partial X;\mathbb{R})$.

4.3. **Sobolev spaces and surjectivity.** The splitting of $\mathrm{ad}P|_{\partial X}$ (leading to the hybrid Sobolev spaces used below) is determined by the potential term $[\Phi,\cdot]\in\Gamma\big(X;\mathrm{End}(\mathrm{ad}P)\big)$. Note that $\Phi\neq 0$ on a neighborhood of $\partial X$ by continuity and the hypothesis that $|\Phi|\,|_{\partial X}=1$. Then from the basic properties of $\mathfrak{su}(2)$ it follows that $[\Phi,\cdot]$ is nondegenerate on $\Phi^\perp\subset\mathrm{ad}P|_U$ with nullspace bundle the span of $\Phi|_U$. Indeed, as $\mathfrak{su}(2)$ is a simple Lie algebra we may take $\Phi$ to be the Cartan element at each point of $\partial X$ and then the decomposition

$$\mathrm{ad}P|_U = \mathrm{ad}P_0\oplus\mathrm{ad}P_1 = \mathrm{ad}P_0\oplus\mathrm{ad}P_+\oplus\mathrm{ad}P_-, \tag{4.5}$$

where $\mathrm{ad}P_\pm$ denote the positive/negative imaginary eigenspaces of $\Phi$, coincides with the root space decomposition

$$\mathfrak{su}(2) = \mathfrak{h}\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha}.$$

Later on we will make use of the following relationship between $\mathrm{ad}P|_U$ and the line bundle $L\longrightarrow\partial X$ defining the charge, though it seems appropriate to give the proof here.

**Lemma 4.2.** *The complex line bundles* $\mathrm{ad}P_+$ *and* $L\otimes L$ *(respectively* $\mathrm{ad}P_-$ *and* $L^*\otimes L^*$*) are isomorphic over* $U$*. Thus,*

$$\mathrm{ad}P|_U = \mathrm{ad}P_0\oplus\mathrm{ad}P_+\oplus\mathrm{ad}P_- \cong \underline{\mathbb{C}}\oplus L^{\otimes 2}\oplus(L^*)^{\otimes 2}$$

*where* $\underline{\mathbb{C}}$ *denotes the trivial bundle.*

*Proof.* From the representation theory of $\mathfrak{su}(2)$, the 4-dimensional representations $\pi_3\oplus\pi_1$ and $\pi_2\otimes\pi_2$ are necessarily isomorphic, where $\pi_1$ is the trivial representation, $\pi_2$ the standard representation and $\pi_3$ the adjoint representation; in particular the isomorphism identifies the subspaces of highest weight.

Taking $\Phi|_U$ to be the Cartan element, this leads to the vector bundle isomorphism

$$\mathrm{ad}P\oplus\underline{\mathbb{C}} \cong (L\oplus L^*)^{\otimes 2} = L^{\otimes 2}\oplus(L^*)^{\otimes 2}\oplus\underline{\mathbb{C}^2}$$

in which highest weight subbundles $\mathrm{ad}P_+$ and $L \otimes L$ are identified. The lowest weight subbundles $\mathrm{ad}P_-$ and $L^* \otimes L^*$ are likewise isomorphic, and $\mathrm{ad}P_0 \oplus \underline{\mathbb{C}} \cong \underline{\mathbb{C}^2}$. (of course $\mathrm{ad}P_0|_U$ is has already been seen to be trivial since it has non-vanishing section $\Phi$.) □

Consider next the connection $A$ on $\mathrm{ad}P$. Finite action of $(\Phi, A)$ implies that $A$ is the lift of a b-connection, inducing a connection on $\partial X$ which respects the splitting $\mathrm{ad}P|_{\partial X} = \mathrm{ad}P_0 \oplus \mathrm{ad}P_+ \oplus \mathrm{ad}P_-$ since $d^A\Phi$ must vanish there. In fact since $\mathrm{ad}P_0$ is the span of $\Phi$ and $d^A\Phi|_{\partial X} = 0$ it follows that the induced connection on $\mathrm{ad}P_0$ is actually flat.

It is then convenient to make use of the gauge freedom for $(\Phi, A)$ and take $A$ to be in *radial gauge* near $\partial X$. Thus we may assume that $A$ is of product type with respect to $U \cong \partial X \times [0, x_0)$, with normal component $\nabla^A_{x^2\partial_x} \equiv x^2\partial_x$ and tangential component pulled back from $\partial X$, in particular respecting the decomposition (4.5) and restricting to a flat connection on $\mathrm{ad}P_0$.

With these considerations in place, it follows that the operators $d^A$, $\delta^A$ and $[\Phi, \cdot]$, and therefore also $\mathsf{D}_0$, $\mathsf{D}_1$ and their formal adjoints, are bounded as operators

$$\mathsf{D}_i, \mathsf{D}_i^* : \big[x^\alpha H_{\mathrm{b}}^{k+l} \big| x^\beta H_{\mathrm{b,sc}}^{k,l}\big](X; \Lambda^*\otimes \mathrm{ad}P) \longrightarrow \big[x^{\alpha+1} H_{\mathrm{b}}^{k+l-1} \big| x^{\beta+1} H_{\mathrm{b,sc}}^{k,l-1}\big](X; \Lambda^*\otimes \mathrm{ad}P).$$

where the hybrid spaces are defined with respect to $\mathrm{ad}P|_U = \mathrm{ad}P_0 \oplus \mathrm{ad}P_1$. For $k \geq 0$ and $l \geq 1$ these may alternately be considered as domains for $\mathsf{D}_i, \mathsf{D}_i^*$ as unbounded operators on the weighted $L^2$ space $x^\eta L^2(X; \Lambda^* \otimes \mathrm{ad}P)$ for $\eta = \min(\alpha, \beta)$.

A lower bound for $\eta$ is determined by the convention that monopole boundary data and limit of the gauge group at $\partial X$ are fixed, and the deformation spaces should therefore consist of sections with some degree of vanishing at $\partial X$. This requires taking $\eta \geq -3/2$ since $x^{-3/2-\epsilon}L^2$ contains constant sections for all $\epsilon > 0$.

An upper bound on the allowed weights is determined using the following Weitzenbock type vanishing results.

**Proposition 4.3.** *The operators* $\mathsf{D}_0^*\mathsf{D}_0$ *and* $\mathsf{D}_1\mathsf{D}_1^*$ *are positive semidefinite operators given by*

$$\begin{aligned} \mathsf{D}_0^*\mathsf{D}_0 &= (\nabla^A)^*\nabla^A - [\Phi, [\Phi, \cdot]], \quad \text{on } \Gamma(X; \Lambda^0 \otimes \mathrm{ad}P), \text{ and} \\ \mathsf{D}_1\mathsf{D}_1^* &= (\nabla^A)^*\nabla^A - [\Phi, [\Phi, \cdot]], \quad \text{on } \Gamma(X; \Lambda^2 \otimes \mathrm{ad}P). \end{aligned}$$

*Proof.* The result is immediate for $\mathsf{D}_0^*\mathsf{D}_0$, using that $d^A = \nabla^A$ on 0-forms, $\star^2 = 1$ (since $\dim(X)$ is odd), and $\star[\Phi, \cdot]$ is skew-adjoint.

For $\mathsf{D}_1\mathsf{D}_1^*$ we compute[8]

$$\begin{aligned} \mathsf{D}_1\mathsf{D}_1^* &= \begin{bmatrix} d^A + \star[\Phi, \cdot] & \delta^A \end{bmatrix} \begin{bmatrix} \delta^A - \star[\Phi, \cdot] \\ d^A \end{bmatrix} \\ &= \Delta^A + [\Phi, d^A \star \cdot] - d^A\left([\Phi, \star\cdot]\right) - [\Phi, [\Phi, \cdot]] \\ &= \Delta^A - [d^A\Phi, \star\cdot] - [\Phi, [\Phi, \cdot]] \\ &= \Delta^A - [\star F^A, \star\cdot] - [\Phi, [\Phi, \cdot]], \end{aligned}$$

[8]One must be a bit careful with the bracket operation on forms of odd degree; in terms of the associative matrix product, $[A, B]$ on 1-forms is really $A \wedge B + B \wedge A$.

using $F^A = \star d^A\Phi$ on the last line. To complete the proof, a straightforward computation reveals that $[\star F^A, \star\cdot]$ is equivalent to the curvature term in the Weitzenbock formula — that is, $\Delta^A = (\nabla^A)^*\nabla^A + [\star F^A, \star\cdot]$ on $\Gamma(X; \Lambda^2 \otimes \mathrm{ad}P)$. □

**Corollary 4.4.** *If $\alpha \le 0$ and $\beta \le 3/2$, then $\mathsf{D}_1 + \mathsf{D}_0^*$ is surjective as an operator*

$$\mathsf{D}_1 + \mathsf{D}_0^* : \left[x^\alpha H_{\mathrm{b}}^{k+l} \middle| x^\beta H_{\mathrm{b,sc}}^{k,l}\right](X; \Lambda^{\mathrm{odd}} \otimes \mathrm{ad}P) \longrightarrow \left[x^{\alpha+1} H_{\mathrm{b}}^{k+l-1} \middle| x^\beta H_{\mathrm{b,sc}}^{k,l-1}\right](X; \Lambda^{\mathrm{even}} \otimes \mathrm{ad}P)$$

*for all $k, l \in \mathbb{Z}$.*

*Proof.* Let $P = \mathsf{D}_0 + \mathsf{D}_1^*$ be the adjoint of $\mathsf{D}_1 + \mathsf{D}_0^*$; then the result is equivalent to injectivity of

$$P : \left[x^{-\alpha-1} H_{\mathrm{b}}^{1-k-l} \middle| x^{-\beta} H_{\mathrm{b,sc}}^{-k,1-l}\right] \longrightarrow \left[x^{-\alpha} H_{\mathrm{b}}^{-k-l} \middle| x^{-\beta} H_{\mathrm{b,sc}}^{-k,-l}\right].$$

Suppose then that $u \in \mathrm{Null}(P) \cap \left[x^{-\alpha-1} H_{\mathrm{b}}^{1-k-l} \middle| x^{-\beta} H_{\mathrm{b,sc}}^{-k,1-l}\right]$ and consider the pairing $\langle \nabla^* \nabla u, u \rangle_{L^2}$. Note that this pairing only makes sense provided that $\nabla^*\nabla u \in \left[x^{-\alpha+1} H_{\mathrm{b}}^{-k-l-1} \middle| x^{-\beta} H_{\mathrm{b,sc}}^{-k,-l-1}\right]$ lies also inside the dual space $\left[x^{\alpha+1} H_{\mathrm{b}}^{k+l-1} \middle| x^{-\beta} H_{\mathrm{b,sc}}^{k,l-1}\right]$ of $u$, which requires $\alpha \le 0$.

We may then integrate by parts, noting that there are no boundary terms since $\nabla u \in \left[x^{-\alpha} H_{\mathrm{b}}^{-k-l} \middle| x^{-\beta} H_{\mathrm{b,sc}}^{-k,-l}\right]$ decays at $\partial X$ (using that $\beta \le 3/2$), to obtain

$$\|\nabla u\|_{L^2}^2 \le \langle \nabla^* \nabla u, u \rangle + \|[\Phi, u]\|^2 = \langle P^* P u, u \rangle = 0.$$

It follows that $u = 0$ since there are no covariant constant $L^2$ sections of any bundle on an infinite volume manifold. □

4.4. **The Callias operator.** By inspection, the operator $\mathsf{D}_1 + \mathsf{D}_0^*$ is given by

$$\mathsf{D}_1 + \mathsf{D}_0^* = \left(d^A - \delta^A\right)\tau + \star[\Phi, \cdot] : \Gamma(X; \Lambda^{\mathrm{odd}} \otimes \mathrm{ad}P) \longrightarrow \Gamma(X; \Lambda^{\mathrm{even}} \otimes \mathrm{ad}P)$$

where $\tau = (-1)^{k(k-1)/2+k+1}$ (the reason for writing $\tau$ in this apparently complicated manner will become clear below) on $\Lambda^k$; in particular $\tau = 1$ on $\Lambda^1$ and $\tau = -1$ on $\Lambda^3$. Equivalently,

$$\mathsf{D}_1 + \mathsf{D}_0^* = \tau\left(d^A + \delta^A\right) + \star[\Phi, \cdot],$$

where now $\tau$ evaluates to 1 on $\Lambda^0$ and $-1$ on $\Lambda^2$.

To proceed we will consider the operator $\star(\mathsf{D}_1 + \mathsf{D}_0^*)$ which has the same index and nullspace as $\mathsf{D}_1 + \mathsf{D}_0^*$ but has the same domain and range bundle, namely $\Lambda^{\mathrm{odd}} \otimes \mathrm{ad}P$. Thus

$$\star(\mathsf{D}_1 + \mathsf{D}_0^*) = \star\tau\left(d^A + \delta^A\right) + [\Phi, \cdot].$$

The first term is a twisted (by $\mathrm{ad}P$) version of a Dirac operator known as the *odd signature operator* $\not{D}_{\mathrm{odd}} = \star\tau(d + \delta)$ first introduced in [APS75]. Since both $\nabla^{\mathrm{LC}}$ and $\nabla^A$ are lifted from b-connections and $[\Phi, \cdot]$ is skew-Hermitian it follows that $\star(\mathsf{D}_1 + \mathsf{D}_0^*)$ is indeed of Callias type. Moreover, in light of the fact that $\mathrm{ad}P_0$ admits a trivialization over $U$ in which $A$ is flat, it follows that $(\star(\mathsf{D}_1 + \mathsf{D}_0^*))_0$ — the restriction of the Callias operator to $\mathrm{ad}P_0$ over $U$ — may be identified with $\not{D}_{\mathrm{odd}}$ itself. To better understand this operator, some discussion is in order.

4.5. **The odd signature operator.** The $\tau$ introduced above is a special case of the general sign operator

$$\tau = i^{k(k-1)+2nk+\left[\frac{n+1}{2}\right]}, \quad n = \dim(X)$$

defined so that the *normalized Clifford volume element* $\omega_{\mathbb{C}} \in \mathbb{C}\ell(X)$ acts via

$$\omega_{\mathbb{C}} \cdot = \star\tau \cdot$$

under the identification of $\mathbb{C}\ell(X)$ and $\Lambda^* X$ (as vector spaces). Here

$$\omega_{\mathbb{C}} := i^{\left[\frac{n+1}{2}\right]} e_1 \cdots e_n$$

is an involution which is well-defined in terms of any orthonormal frame $\{e_i\}$. In the more familiar case that $n = 2l$ is even, $\tau = i^{k(k-1)+l}$ and the *signature splitting* $\Lambda^* X = \Lambda^+ X \oplus \Lambda^- X$ is defined in terms of the $\pm 1$ eigenspaces of $\omega_{\mathbb{C}} = \star\tau$.

The odd signature operator originally appeared in [APS75] as the boundary Dirac operator induced by the signature operator on a manifold with boundary of dimension $4m$. Here we reverse the procedure and show the following, first supposing that $X$ has an incomplete product-type metric near $\partial X$.

**Proposition 4.5.** *For any manifold $X$ with product type metric near $\partial X$ and dimension $\dim(X) = n$ odd, the induced boundary operator of $\not{D}_{\mathrm{odd}} \in \mathrm{Diff}^1(X; \Lambda^{\mathrm{odd}} X)$ is the (even) signature operator*

$$d + \delta \in \mathrm{Diff}^1(\partial X; \Lambda^{\pm}\partial X, \Lambda^{\mp}\partial X).$$

*Proof.* The connection induced on $\partial X$ is the Levi-Civita connection with respect to the metric on $\partial X$, so the operator is geometric and determined by the induced Clifford action.

The identification $\Lambda^* X \cong \mathbb{C}\ell(X)$ makes this computation easy. Indeed, $\Lambda^{\mathrm{odd}} X|_{\partial X} \cong \Lambda^* \partial X$ corresponds to identifying $\mathbb{C}\ell^1(\mathbb{R}^n) \ni e_0 e_I \cong e_I \in \mathbb{C}\ell^0(\mathbb{R}^{n-1})$ and $\mathbb{C}\ell^1(\mathbb{R}^n) \ni e_J \cong e_J \in \mathbb{C}\ell(\mathbb{R}^{n-1})$ where $e_0$ is the unit normal to $\partial X$ and $I$ and $J$ are multi-indices in $\{1, \ldots, n-1\}$.

By inspection the Clifford action giving rise to $\not{D}_{\mathrm{odd}}$ is given by

$$\mathrm{c}\ell_{\mathrm{odd}}(e) = \star\tau(e \wedge \cdot - e \lrcorner \cdot) \cong \omega_{\mathbb{C}}^n e \cdot$$

where we have introduced the notation $\omega^n = e_0 e_1 \cdots e_{n-1}$ to highlight the dimension. (Note that this is a *self-adjoint* Clifford action, satisfying $\{\mathrm{c}\ell_{\mathrm{odd}}(e), \mathrm{c}\ell_{\mathrm{odd}}(f)\} = 2\langle e, f\rangle$ rather than $-2\langle e, f\rangle$ so in particular $\not{D}_{\mathrm{odd}}^2 = -\Delta$.) Now $\omega_{\mathbb{C}}^n = i e_0 \omega_{\mathbb{C}}^{n-1}$ so it follows that

$$\mathrm{c}\ell_{\mathrm{odd}}(e_0) = \omega_{\mathbb{C}}^n e_0 = i e_0 \omega_{\mathbb{C}}^{n-1} e_0 = -i\omega_{\mathbb{C}}^{n-1}$$

and so the $\pm 1$ eigenspaces of $i\mathrm{c}\ell_{\mathrm{odd}}(e_0)$ are $\Lambda^{\pm}\partial X$. The induced $\mathbb{C}\ell(\partial X)$ action may be written

$$\mathrm{c}\ell_{\partial}(e_j) = \mathrm{c}\ell_{\mathrm{odd}}(e_j e_0) = \omega_{\mathbb{C}}^n e_j \omega_{\mathbb{C}}^n e_0 = -(\omega_{\mathbb{C}}^n)^2 e_j e_0 = -e_j e_0.$$

This acts by

$$\begin{aligned} \mathrm{c}\ell_{\partial}(e_j) e_0 e_I &= -e_j e_0 \cdot e_0 e_I = e_j \cdot e_I \\ \mathrm{c}\ell_{\partial}(e_j) e_J &= -e_j e_0 \cdot e_J = e_0 (e_j \cdot e_J), \end{aligned}$$

so in particular under the identification $\Lambda^{\mathrm{odd}} X|_{\partial X} \cong \Lambda^* X$ above $\mathrm{c}\ell_{\partial}(\cdot)$ is the standard Clifford action on forms. $\square$

When $X$ is an exact scattering manifold, there is an additional lower order term coming from the induced connection (see §1.2).

**Proposition 4.6.** *For an exact scattering manifold $X$ of odd dimension $n$, the induced boundary operator of $\not{D}_{\mathrm{odd}} \in \mathrm{Diff}^1_{\mathrm{sc}}(X; \Lambda^{\mathrm{odd}} X)$ is*

$$d + \delta - N \in \mathrm{Diff}^1(\partial X; \Lambda^+ \oplus \Lambda^-), \quad N = \begin{cases} k & \text{on } \Lambda^k,\ k \text{ odd} \\ m - k & \text{on } \Lambda^k,\ k \text{ even} \end{cases}$$

*where $m = n - 1 = \dim(\partial X)$. $N$ has (graded) degree 0 while $d + \delta$ has degree 1 with respect to the grading $\Lambda^+ \partial X \oplus \Lambda^- \partial X$.*

*Proof.* The Clifford algebra computation giving rise to $d + \delta$ and the signature splitting $\Lambda^+ \oplus \Lambda^-$ is the same as for Proposition 4.5. It remains only to compute the difference term $E = \sum_{1 \le i \le n-1} \mathrm{c}\ell_\partial(e_i) \circ (e_0 \wedge e_i)$ of (1.10).

Here $e_0 \wedge e_i \in \mathfrak{so}(n)$ is the skew-adjoint matrix mapping $e_i \longmapsto e_0$ and $e_0 \longmapsto -e_i$ and which is 0 otherwise; we denote it henceforth by $E_{0i}$ to avoid confusion with forms. $E_{0i}$ is given by the same matrix in the contragredient representation (i.e. on ${}^{\mathrm{sc}}T^*X$) by skew-adjointness, and acts on $\Lambda^* X$ as an (ungraded) derivation. Thus

$$E_{0i} e_J = e_{J(i,0)}, \quad E_{0i} e_0 \wedge e_J = -e_i \wedge e_J$$

where $e_J = e_{j_1} \wedge \cdots \wedge e_{j_k}$ and $e_{J(i,0)}$ is the form obtained by replacing $e_i$ by $e_0$ in $e_J$ if it occurs and which is 0 otherwise.

Composing this with the Clifford action $\mathrm{c}\ell_\partial(e_i) = \mathrm{c}\ell(e_i\, e_0)$ it follows that

$$(\mathrm{c}\ell_\partial(e_i) \circ E_{0i})\, e_J = \begin{cases} -e_J & i \in J \\ 0 & i \notin J \end{cases}, \quad \text{and} \quad (\mathrm{c}\ell_\partial(e_i) \circ E_{0i})\, e_0 \wedge e_J = \begin{cases} 0 & i \in J \\ -e_0 \wedge e_J & i \notin J \end{cases}$$

and the claim then follows from the identification of the $e_J$ with the odd forms on $\partial X$ and the $e_0 \wedge e_J$ with the even ones. □

It remains to compute the indicial roots of $\widetilde{\not{D}}_{\mathrm{odd}}$, and we will continue to consider arbitrary odd $n = \dim(X)$ since it is not any more difficult than taking $n = 3$.

**Proposition 4.7.** *Let $X$ be an exact scattering manifold of odd dimension $n = m + 1$. The indicial roots of $\widetilde{\not{D}}_{\mathrm{odd}} = x^{-(n+1)/2} \not{D}_{\mathrm{odd}} x^{(n-1)/2} \in \mathrm{Diff}^1_{\mathrm{b}}(X; \Lambda^{\mathrm{odd}})$ are given by*

$$i\mathrm{spec}_{\mathrm{b}}(\widetilde{\not{D}}_{\mathrm{odd}}) = \left\{(-1)^k \left(\tfrac{m}{2} - k\right) : k = 0, 1, \ldots, m\right\}$$
$$\bigcup \pm \left\{ \tfrac{1}{2} \pm \sqrt{\left(\tfrac{m-1}{2} - k\right)^2 + \nu} : 0 \neq \nu \in \mathrm{spec}(\Delta_{\partial X}),\ 0 \le k \le m - 1. \right\}$$

*The formal nullspace $F(\widetilde{\not{D}}_{\mathrm{odd}}, r)$ corresponding to the root $r = \big((-1)^k(\frac{m}{2} - k)\big)$ may be identified with harmonic $k$ forms:*

$$F(\widetilde{\not{D}}_{\mathrm{odd}}, r) = \mathcal{H}^k(\partial X) \cong H^k_{dR}(\partial X), \quad r = (-1)^k(\tfrac{m}{2} - k).$$

*In particular, $F(\widetilde{\not{D}}_{\mathrm{odd}}, 0) \cong H^{m/2}_{dR}(\partial X)$, and so $0 \in \mathrm{spec}_{\mathrm{b}}(\widetilde{D})$ if and only if $H^{m/2}_{dR}(\partial X) \neq \{0\}$.*

*Proof.* From the preceding results, it follows that near $\partial X$ we may write

$$\not{D}_{\mathrm{odd}} = x\left(-i\mathrm{c}\ell(\omega^{n-1}_{\mathbb{C}})(x\partial_x - N + (d + \delta)_{\partial X})\right)$$

where we have identified ${}^{\mathrm{sc}}\Lambda^{\mathrm{odd}} X \cong \Lambda^* \partial X$.

Since $-ic\ell(\omega_{\mathbb{C}}^{n-1})$ is an isomorphism, it suffices to consider $D = x(x\partial_x - N + (d+\delta)_{\partial X})$ which will have the same indicial roots and formal nullspace. Since taking $\widetilde{D} = x^{-(n+1)/2} D x^{(n-1)/2}$ amounts to removing the leftmost factor of $x$ and replacing $x\partial_x$ by $x\partial_x + (n-1)/2 = x\partial_x + m/2$, and then $I(\widetilde{\not{D}}_{\mathrm{odd}}, \lambda)$ amounts to replacing $x\partial_x$ by $i\lambda$, it follows that

$$I(\widetilde{D}, -ir) = \begin{bmatrix} r+M & d+\delta \\ d+\delta & r-M \end{bmatrix} \tag{4.6}$$

with respect to the splitting $\Lambda^*\partial X = \Lambda^{\mathrm{odd}}\partial X \oplus \Lambda^{\mathrm{even}}\partial X$, where

$$M = \tfrac{m}{2} - k, \quad m = \dim(\partial X).$$

To analyze the roots of $I(\widetilde{D}, -ir)$, we utilize the Hodge decomposition on $C^\infty(\partial X; \Lambda^*)$. On harmonic forms $\mathcal{H}^*(\partial X)$, (4.6) reduces to

$$r + (-1)^{k+1}(m/2 - k) \quad \text{on } \mathcal{H}^k(\partial X)$$

giving the first part of the b-spectrum and identifying the formal nullspace

$$F\big(\widetilde{\not{D}}_{\mathrm{odd}}, r\big) := \mathrm{Null}\big(I(\widetilde{\not{D}}_{\mathrm{odd}}, -ir)\big) = \mathcal{H}^k(\partial X), \quad r = (-1)^k(m/2-k).$$

Off of the harmonic forms, we observe that $I(\widetilde{D}, -ir)$ preserves eigenspaces of $\Delta_{\partial X}$ and that the only coupling occurs between closed and co-closed forms. Thus it suffices to consider the action of $I(\widetilde{D}, -ir)$ first on pairs $(\phi_\nu, \psi_\nu) \in C^\infty(\partial X; \Lambda^k) \times C^\infty(\partial X; \Lambda^{k+1})$ such that $d\phi_\nu = \sqrt{\nu}\psi_\nu$, $\delta\psi_\nu = \sqrt{\nu}\phi_\nu$, for which

$$I(\widetilde{D}, -ir) = \begin{bmatrix} r + (m/2-k) & \sqrt{\nu} \\ \sqrt{\nu} & r - (m/2-k) + 1 \end{bmatrix}, \quad 0 \le k \le m-1$$

and then likewise on pairs $(\gamma_\nu, \eta_\nu) \in C^\infty(\partial X; \Lambda^{m-k}) \times C^\infty(\partial X; \Lambda^{m-(k+1)})$ such that $\delta\gamma_\nu = \sqrt{\nu}\eta_\nu$, $d\eta_\nu = \sqrt{\nu}\gamma_\nu$, for which

$$I(\widetilde{D}, -ir) = \begin{bmatrix} r - (m/2-k) & \sqrt{\nu} \\ \sqrt{\nu} & r + (m/2-k) - 1 \end{bmatrix}.$$

These are respectively invertible unless

$$r = -\tfrac{1}{2} \pm \sqrt{\left(\tfrac{m}{2} - k - \tfrac{1}{2}\right)^2 + \nu}, \quad \text{resp.} \quad r = \tfrac{1}{2} \pm \sqrt{\left(\tfrac{m}{2} - k - \tfrac{1}{2}\right)^2 + \nu}. \qquad \square$$

*Remark.* From this computation of $\mathrm{spec}_{\mathrm{b}}(\widetilde{\not{D}}_{\mathrm{odd}})$ it can be seen that, not only is the b-spectrum symmetric about 0 (as indeed it must be by self-adjointness), but that the formal nullspaces at $r$ and $-r$ are explicitly isomorphic by the Hodge star on $\partial X$. Indeed, $\star_{\partial X}$ gives an isomorphism between eigenforms of $\Delta_{\partial X}$ of form degree $k$ and eigenvalue $\nu$ and those of form degree $m-k$ and eigenvalue $\nu$, (including harmonic forms as the special case $\nu = 0$), and the formal nullspace at each point of the b-spectrum consists precisely of such eigenforms.

4.6. **Moduli dimension.** The set $i\mathrm{spec}_{\mathrm{b}}(\widetilde{\not{D}}_{\mathrm{odd}})$ for $n = 3$ is depicted in Figure 2. In particular, $\pm 1 \in i\mathrm{spec}_{\mathrm{b}}(\widetilde{\not{D}}_{\mathrm{odd}})$ always occurs, with formal nullspaces corresponding to $H^0(\partial X; \mathbb{C})$ and $H^2(\partial X; \mathbb{C})$, respectively. $0 \in i\mathrm{spec}_{\mathrm{b}}(\widetilde{\not{D}}_{\mathrm{odd}})$ occurs with formal nullspace $H^1(\partial X; \mathbb{C})$ only if the latter is nonempty. Then there are points corresponding to positive eigenvalues of $\Delta_{\partial X}$ which are therefore sensitive to the metric on $\partial X$.

We say $\partial X$ is *sufficiently small* provided that the first nonzero eigenvalue of $\Delta_{\partial X}$ satisfies $\nu_1 \ge 2$. (Recall that under the metric scaling $h \longmapsto \lambda h$, the eigenvalues

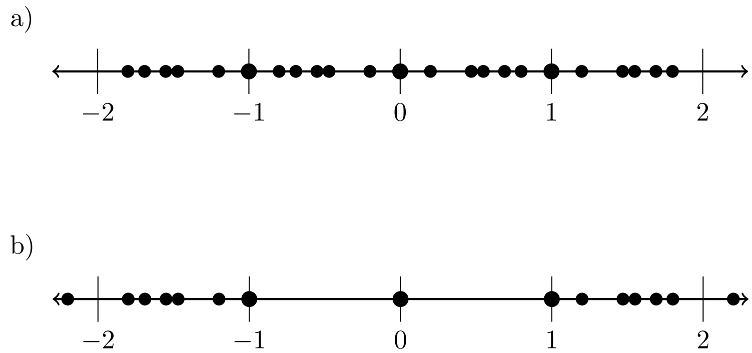


FIGURE 2. The b-spectrum of $\widetilde{\not{D}}_{\mathrm{odd}}$ in case a) $\partial X$ is not sufficiently small, and b) $\partial X$ is sufficiently small.

of $\partial X$ scale by $\nu \longmapsto \nu/\lambda$ and $\mathrm{Vol}(\partial X) \longmapsto \lambda^{m/2}\mathrm{Vol}(\partial X)$.) An example is the round sphere for which $\nu_1 = 2$ exactly, and for which $i\mathrm{spec}_{\mathrm{b}}(\widetilde{\not{D}}_{\mathrm{odd}}) \subset \mathbb{Z}$. For $\partial X$ sufficiently small, it follows that $i\mathrm{spec}_{\mathrm{b}}(\widetilde{\not{D}}_{\mathrm{odd}}) \cap [-1,1] = \{-1,0,1\}$, since

$$\sqrt{\left(\tfrac{m-1}{2} - k\right)^2 + \nu_1} \geq \sqrt{\left(\tfrac{m-1}{2}\right)^2 + 2} \geq \sqrt{\tfrac{9}{4}} \geq \tfrac{3}{2},$$

for here $m = 1$ and $k \in \{0,1\}$.

From Theorem 2.5 (using the volume form to trivialize the bundle $\Omega^{1/2}_{\mathrm{sc}}(X)$), $\mathsf{D}_1 + \mathsf{D}_0^*$ is Fredholm as an operator

$$\mathsf{D}_1 + \mathsf{D}_0^* : \left[x^{\gamma-1/2}H^{k+l}_{\mathrm{b}} \middle| x^{\gamma}H^{k,l}_{\mathrm{b,sc}}\right] \longrightarrow \left[x^{\gamma+1/2}H^{k+l-1}_{\mathrm{b}} \middle| x^{\gamma}H^{k,l-1}_{\mathrm{b,sc}}\right]$$

for $\gamma \notin i\mathrm{spec}_{\mathrm{b}}(\widetilde{\not{D}}_{\mathrm{odd}})$, and from the considerations of §4.3 we are specifically interested in $\gamma \in [-1, 1/2]$, for which the operator is surjective and the domain consists of sections with some decay. From the geometric perspective, we want as large a domain as possible, and we may finally conclude:

**Theorem 4.8.** *For an exact scattering manifold $(X, g)$ with sufficiently small boundary $\partial X$, the formal tangent space $T_{(\Phi,A)}\mathcal{M}_k$ is given by the nullspace of*

$$\begin{aligned}\mathsf{D}_1 + \mathsf{D}_0^* : \left[x^{\gamma-1/2}H^{k+l}_{\mathrm{b}} \middle| x^{\gamma}H^{k,l}_{\mathrm{b,sc}}\right]&(X; \Lambda^{\mathrm{odd}} \otimes \mathrm{ad}P) \\ &\longrightarrow \left[x^{\gamma+1/2}H^{k+l-1}_{\mathrm{b}} \middle| x^{\gamma}H^{k,l-1}_{\mathrm{b,sc}}\right](X; \Lambda^{\mathrm{even}} \otimes \mathrm{ad}P)\end{aligned}$$

*for any $k, l \in \mathbb{N}$ and $\gamma \in (-1, 0)$. This space has dimension*

$$\dim\left(T_{(\Phi,A)}\mathcal{M}_k\right) = 4\underline{k} + \tfrac{1}{2}b^1(\partial X)$$

*and consists of polyhomogeneous sections. Here $\underline{k} = \int_{\partial X} c_1(L) = k_1 + \cdots + k_N$ is a sum over components of $\partial X$. Near $\partial X$, the $\mathrm{ad}P_1 = \Phi^{\perp}$ components of $(a, \phi) \in \mathrm{Null}(\mathsf{D}_1 + \mathsf{D}_0^*)$ are rapidly vanishing, and the $\mathrm{ad}P_0 = \mathbb{C}\Phi$ components have leading order*

$$\phi = \mathcal{O}(x^2), \quad a = \begin{cases} \mathcal{O}(x^2) & \text{if } H^1(\partial X; \mathbb{C}) = \{0\}, \\ \mathcal{O}(x) & \text{if } H^1(\partial X; \mathbb{C}) \neq \{0\}. \end{cases}$$

*Proof.* From Theorem 3.6 the index of $\mathsf{D}_1 + \mathsf{D}_0^*$ is given by

$$\operatorname{ind}(\mathsf{D}_1 + \mathsf{D}_0^*, \gamma) = \operatorname{ind}(\partial_+^+) + \operatorname{def}(\mathsf{D}_1 + \mathsf{D}_0^*, \gamma) = \operatorname{ind}\big((\partial_{\mathrm{sig}})^+_{\mathrm{ad}P_+}\big) + \operatorname{def}(\mathsf{D}_1 + \mathsf{D}_0^*, \gamma)$$

where we have used Proposition 4.6, and where

$$\operatorname{def}(\mathsf{D}_1 + \mathsf{D}_0^*, \gamma) = \operatorname{ind}(\widetilde{\not{D}}_{\mathrm{odd}}, \gamma)$$

From the standard index formula (see [LM89], Thm. 13.9),

$$\operatorname{ind}\big((\partial_{\mathrm{sig}})^+_{\mathrm{ad}P_+}\big) = \int_{\partial X} \mathrm{ch}_2(\mathrm{ad}P_+)\hat{\mathbf{L}}(\partial X) = \int_{\partial X} 2c_1(\mathrm{ad}P_+) = \int_{\partial X} 4c_1(L) = 4\underline{k}$$

where $\mathrm{ch}_2(E) = \sum_k 2^k \mathrm{ch}^k(E)$ and $\mathrm{ch}^k(E)$ denotes the $H^{2k}(\partial X;\mathbb{R})$ component of the Chern character $\mathrm{ch}(E)$, and we have made use of the isomorphism $\mathrm{ad}P_+ \cong L \otimes L$ of Lemma 4.2.

On the other hand, under the assumption that $\partial X$ is sufficiently small, it follows from

$$\operatorname{ind}(\widetilde{\not{D}}_{\mathrm{odd}}, -\epsilon) = -\operatorname{ind}(\widetilde{\not{D}}_{\mathrm{odd}}, \epsilon)$$
$$\operatorname{ind}(\widetilde{\not{D}}_{\mathrm{odd}}, -\epsilon) - \operatorname{ind}(\widetilde{\not{D}}_{\mathrm{odd}}, \epsilon) = \dim H^1(\partial X;\mathbb{C})$$

for $\epsilon \in (0,1)$ that

$$\operatorname{ind}(\widetilde{\not{D}}_{\mathrm{odd}}, \gamma) = \tfrac{1}{2}\dim H^1(\partial X;\mathbb{C}), \quad \gamma \in (-1,0).$$

The asymptotics follow from Theorem 2.5, and we note that the asymptotic order $x^1 = x^{(n-1)/2+0}$ corresponding to $0 \in i\mathrm{spec}_{\mathrm{b}}(\widetilde{\not{D}}_{\mathrm{odd}})$ only occurs if $H^1(\partial X;\mathbb{C})$ is non-trivial, and then only shows up in the 1 form component, namely $a$. The next order asymptotic which occurs is $x^2 = x^{(n-1)/2+1}$ corresponding to $1 \in i\mathrm{spec}_{\mathrm{b}}(\widetilde{\not{D}}_{\mathrm{odd}})$. $\square$

As a special case, when $X = \overline{\mathbb{R}^3}$ is the radial compactification of $\mathbb{R}^3$ with the Euclidean metric

$$g_0 = \frac{dx^2}{x^4} + \frac{d\theta^2}{x^2}, \quad x = 1/r$$

where $d\theta^2$ is the standard round metric on $S^2$, we obtain the standard result

$$\dim\big(T_{(\Phi,A)}\mathcal{M}_k\big) = 4k$$

with $T_{(\Phi,A)}\mathcal{M}_k$ consisting of *smooth* functions, since $i\mathrm{spec}_{\mathrm{b}}(\widetilde{\not{D}}_{\mathrm{odd}}) \subset \mathbb{Z}$ in this case.

## Appendix A. Polyhomogeneity

Here we summarize some standard results regarding polyhomogeneous conormal distributions. References for the material this section include [Mel93] and [Mel92]. We say $E \subset \mathbb{C} \times \mathbb{N}$ (here $\mathbb{N} = \{0, 1, \cdots\}$) is an *index set* if it is discrete and satisfies $\operatorname{Re}(z_j) \longrightarrow \infty$ when $|(z_j, k_j)| \longrightarrow \infty$ for $(z_j, k_j) \in E$.

Let $X$ be a manifold with boundary and boundary defining function $x$. We say $u \in C^\infty(X \setminus \partial X)$ has *polyhomogeneous expansion in* $x$ with index set $E$ if $u$ is asymptotic to the sum

$$u \sim \sum_{(z,k)\in E} x^z (\log x)^k u_{z,k}$$

at $\partial X$, with $u_{z,k} \in C^\infty(\partial X)$. The condition that $\operatorname{Re} z_j \longrightarrow \infty$ guarantees that such sums are Borel summable. Such an expansion is dependent on the choice of $x$, but if we require that $E$ is a *smooth index set*, meaning in addition

$$(z,k) \in E \implies (z+n, l) \in E, \quad \forall n \in \mathbb{N},\ 0 \le l \le k,$$

then the notion that $u$ has a polyhomogeneous expansion with index set $E$ is independent of the choice of $x$, and we let $\mathcal{A}^E_{\mathrm{phg}}(X)$ denote the set of such functions. Likewise if $V \longrightarrow X$ is a vector bundle, the set $\mathcal{A}^E_{\mathrm{phg}}(X;V)$ is well-defined for smooth $E$ and consists of sections with expansions in terms of some local frame.

The presence or absence of a term associated to a particular $(z,k) \in E$ in the expansion of $u$ generally depends on the choice of boundary defining function, but there are certain terms for which the vanishing or non-vanishing of the corresponding coefficient makes coordinate-invariant sense. We say $(z,k) \in E$ is *high order* if $z = \min\{z+n : n \in \mathbb{Z},\ (z,l) \in E\}$ and $k = \max\{l : (z,l) \in E\}$. It follows that if $u_{(z,k)} \neq 0$ for a high order term with respect to one choice of $x$, then the same coefficient will be nonzero in the expansion with respect to any other choice, and $u_{(z,k)} = 0$ is equivalent to $u \in \mathcal{A}^{E\setminus(z,k)}_{\mathrm{phg}}(X)$.

For a fixed $x$, polyhomogeneity with index set $E$ is equivalent to the *Mellin transform*

$$\mathcal{M}(\phi u)(\lambda) = \int_{\mathbb{R}_+} x^{-i\lambda}\phi(x)u(x)\frac{dx}{x}$$

being a meromorphic function (section of $V|_{\partial X}$) with respect to $\lambda \in \mathbb{C}$ with poles at $\lambda = -iz$ of order $k+1$ where $k = \max\{l : (z,l) \in E\}$, and with rapid decay uniformly in strips $|\mathrm{Im}\,\lambda| \leq c$ as $|\mathrm{Re}\,\lambda| \longrightarrow \infty$. Here $\phi \equiv 1$ on a neighborhood of $\partial X$ with support in a larger neighborhood, and $\mathcal{M}(\phi u) - \mathcal{M}(\phi' u)$ is holomorphic for any other such $\phi'$.

We make use of the following notation for index sets. We identify $e \in \mathbb{Z}$ with the index set $\{(e+\mathbb{N}, 0)\}$, and write $\infty$ for the empty index set (since it corresponds to functions vanishing to infinite order). We also write

$$\mathrm{Re}\,E := \inf\{\mathrm{Re}\ (z) : (z,k) \in E\}, \quad \mathrm{Im}\,E := \inf\{\mathrm{Im}\ (z) : (z,k) \in E\}.$$

Now suppose $X$ is a manifold with corners and let $\mathcal{M}_l(X)$ be the set of its boundary faces of codimension $l$; in particular $\mathcal{M}_1(X)$ is the set of boundary hypersurfaces. We use the notation $\mathcal{E} = \{E_H : H \in \mathcal{M}_1(X)\}$ for a multi-index of smooth index sets, and the space $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X;V)$ is defined recursively by

$$u \in \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X;V) \iff u \sim \sum_{(z,k)\in E_H} \rho_H^z (\log \rho_H)^k u_{z,k}, \quad u_{z,k} \in \mathcal{A}^{\mathcal{E}(H)}_{\mathrm{phg}}(H;V)$$

where $\mathcal{E}(H) = \{F_G : G \in \mathcal{M}_1(H)\}$ and $F_G = E_{H'}$ for the unique $H' \in \mathcal{M}_1(X)$ such that $H \cap H' = G$. For a closed manifold $Y$, $\mathcal{A}^*_{\mathrm{phg}}(Y) \equiv C^\infty(Y)$, so that the recursion eventually terminates and $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}(X;V)$ is well-defined.

For any boundary face $F \in \mathcal{M}_n(M)$ and a normal neighborhood $U \cong F \times [0,1)^n$ defined in terms of fixed boundary defining functions $(x_1,\ldots,x_n)$ for hypersurfaces $H_1,\ldots,H_n$ such that $F \subset H_1 \cap \cdots \cap H_n$, the *multi-Mellin transform* of $\phi\, u|_U$ (where $\phi$ is a compactly supported cutoff function in $U$ nowhere vanishing on $F$) is defined by

$$\mathcal{M}_F(\phi\, u)(z,\lambda_1,\ldots,\lambda_n) = \int_{\mathbb{R}^n_+} x_1^{-i\lambda_1}\cdots x_n^{-i\lambda_n}\,\phi\, u, \frac{dx_1}{x_1}\ldots\frac{dx_n}{x_n},$$

where $(z,\lambda_1,\ldots,\lambda_n) \in F \times \mathbb{C}^n$. Then $u$ is polyhomogeneous with multi-index $\mathcal{E}$ if and only if, for each such boundary face $F$, $\mathcal{M}_F(\phi\, u)$ is a product of meromorphic functions valued in $\mathcal{A}^*_{\mathrm{phg}}(F;V)$, with factors having poles of order $k_i+1$ at $\{\lambda_i = -iz_i\}$ only if $(z_i,k_i) \in E_{H_i}$, and decaying rapidly and uniformly in strips $|\mathrm{Im}\,\lambda_i| \leq c$. For a fixed choice of defining functions $\rho_i$, a particular pole may or may not occur according to whether the corresponding term appears in the expansion in terms of the $\rho_i$. However the poles corresponding to high order terms are fundamental, and if such a pole does not occur then $u$ is polyhomogeneous for the multi-index in which the associated element has been removed.

A.1. **Pullback and pushforward.** Fundamental to the use of polyhomogeneous distributions on manifolds with corners are two results dictating their behavior with respect to pullback and pushforward operations. We say a map $f : X \longrightarrow Y$ between manifolds with corners is a *b-map* provided that for each boundary defining function $\rho_G$ for $G \in \mathcal{M}_1(Y)$,

$$f^*(\rho_G) = a \prod_{H \in \mathcal{M}_1(X)} \rho_H^{e(H,G)}, \quad e(H,G) \in \mathbb{N},\ 0 < a \in C^\infty(X),$$

and b-maps give a well-defined set of morphisms with respect to which compact manifolds with corners form a category. The *boundary exponents* $e(H,G) \in \mathbb{N}$ do not depend on the choice of the $\rho_G$ or $\rho_H$. A b-map $f$ is said to be a *b-fibration* if for each $H \in \mathcal{M}_1(X)$, $e(H,G) \neq 0$ for at most one $G \in \mathcal{M}_1(Y)$. Such a map restricts to a fibration between the interiors of $X$ and $Y$, and restricted to any boundary face of $X$, $f$ is again a b-fibration.

Suppose $f : X \longrightarrow Y$ is a b-map with boundary exponents $e(H,G)$. For a collection $\mathcal{F} = \{F_G\}_{G \in \mathcal{M}_1(Y)}$ of smooth index sets for $Y$, we define the following pullback operation on index sets:

$$\begin{aligned} f^\# \mathcal{F} &= \{E_H\}_{H \in \mathcal{M}_1(X)}, \quad \text{where} \\ E_H &= \left\{ \textstyle\sum_{e(H,G) \neq 0} (e(H,G) z_i, k_i) : (z_i, k_i) \in F_G \right\}. \end{aligned}$$

If $f$ is a b-fibration, we define a pushforward operation by

$$\begin{aligned} f_\#(\mathcal{E}) &= \{F_G\}_{G \in \mathcal{M}_1(Y)}, \quad \text{where} \\ F_G &= \overline{\textstyle\bigcup}_{H \subset f^{-1}(G)} \{(z/e(H,G), k) : (z,k) \in E_H\}, \end{aligned}$$

and where the *extended union operation* is defined on index sets as

$$E \overline{\cup} F = E \cup F \cup \{(z, k+l+1) : (z,k) \in E \text{ and } (z,l) \in F\}.$$

The following results can be found in [Mel92].

**Theorem A.1.** *Let $f : X \longrightarrow Y$ be a b-map. Then the pullback $f^* : \dot{C}^\infty(Y;V) \longrightarrow \dot{C}^\infty(X; f^*V)$ (here $\dot{C}^\infty$ denotes smooth functions vanishing to infinite order at all boundary faces) extends to a map*

$$f^* : \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(Y;V) \longrightarrow \mathcal{A}^{f^\#\mathcal{F}}_{\mathrm{phg}}(X; f^*V).$$

The corresponding pushforward result requires the use of densities.

**Theorem A.2.** *Let $f : X \longrightarrow Y$ be a b-fibration. If $\mathrm{Re}\,(E_H) > 0$ for all $H$ such that $f(H) \cap \mathring{Y} \neq \emptyset$, then the pushforward map $f_* : \dot{C}^\infty(X; \Omega_{\mathrm{b}}(X)) \longrightarrow \dot{C}^\infty(Y; \Omega_{\mathrm{b}}(Y))$ extends to a map*

$$f_* : \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}}\big(X; \Omega_{\mathrm{b}}(X)\big) \longrightarrow \mathcal{A}^{f_\#\mathcal{E}}_{\mathrm{phg}}\big(Y; \Omega_{\mathrm{b}}(Y)\big).$$

*Remark.* In terms of the multi-Mellin characterization of polyhomogeneity, the extended union corresponds roughly to multiplication of meromorphic functions and the addition of the degrees of their poles where they align. (See the proof of Proposition B.12.)

## Appendix B. b-sc transition calculus

We include here a self-contained summary of the b-sc transition calculus of pseudodifferential operators. This idea is due to Melrose and Sa Barretto, and was used by Guillarmou and Hassell in [GH08].

Let $X$ be a compact manifold with boundary, and let $\mathcal{V}_{\mathrm{b}}(X)$, $\mathcal{V}_{\mathrm{sc}}(X)$ denote the Lie algebras of b vector fields and scattering vector fields, respectively. Let $I = [0, 1)$ be a half open interval. The calculus is meant to microlocalize families of differential operators, parametrized by $\tau \in I$, which fail to be fully elliptic in the scattering sense precisely as $\tau \to 0$, where they are treated as weighted b type operators.

### B.1. **Single, Double, and Triple Spaces.**

The operators are constructed as Schwartz kernels on $X^2 \times I$, acting on functions on $X \times I$, with composition occurring on $X^3 \times I$ all via pullback, multiplication and pushforward. The operators and functions considered will be those resolved to have polyhomogeneous expansions by particular blow-ups of these spaces.

The *single space* is defined by

$$X_t = [X \times I; \partial X \times \{0\}].$$

The boundary faces of $X_t$ are denoted as follows.

$$\begin{aligned} \mathrm{sc} &= \text{lift of } \partial X \times I \\ \mathrm{tf} &= \text{lift of } \partial X \times \{0\} \\ \mathrm{zf} &= \text{lift of } X \times \{0\} \end{aligned}$$

See Figure 1.

The double space is defined in two steps. Let $C_n$ denote the union of boundary faces of codimension $n$ of $X^2 \times I$. Thus $C_3 = \partial X \times \partial X \times \{0\}$, while $C_2$ is a union of the faces $\partial X \times \partial X \times I$, $\partial X \times X \times \{0\}$ and $X \times \partial X \times \{0\}$. The *b blowup* or *total boundary blowup* is well-defined for any manifold with corners to be the blow up of all boundary faces in order of decreasing codimension. in this case,

$$(X^2 \times I)_{\mathrm{b}} = [X^2 \times I; C_3, C_2].$$

Now denote by $C_V$ and $\Delta$ the lifts of $\partial X \times \partial X \times I$ and the fiber diagonal $\Delta \times I$, respectively. These intersect transversally in $(X^2 \times I)_{\mathrm{b}}$ and the *double space* is defined by

$$X_t^2 = [(X^2 \times I)_{\mathrm{b}}; C_V \cap \Delta]$$

We denote by $\beta_2 : X_t^2 \longrightarrow X^2 \times I$ the composite blow down map, and denote the boundary faces of $X_t^2$ by

$$\begin{aligned} \mathrm{sc} &= \text{lift of } C_V \cap \Delta & \mathrm{bf} &= \text{lift of } \partial X \times \partial X \times I \\ \mathrm{tf} &= \text{lift of } C_3 & \mathrm{lb}_0 &= \text{lift of } X \times \partial X \times \{0\} \\ \mathrm{rb}_0 &= \text{lift of } \partial X \times X \times \{0\} & \mathrm{lb} &= \text{lift of } X \times \partial X \times I \\ \mathrm{rb} &= \text{lift of } \partial X \times X \times I & \mathrm{zf} &= \text{lift of } X \times X \times \partial I \end{aligned}$$

See Figure 3. The reuse of the names sc, tf and zf should not cause any confusion as it should be clear from context which space is being referred to. Observe that these three faces of $X_t^2$ coincide with those of $X_t$ upon identifying $X_t$ with the lifted diagonal.

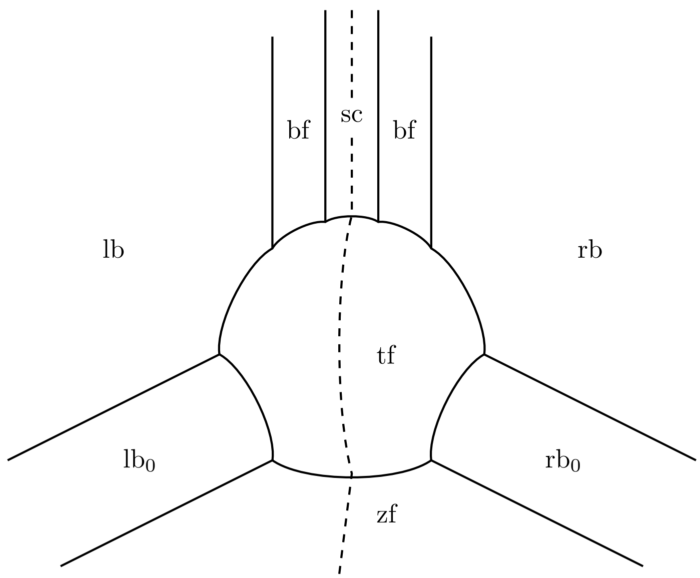


FIGURE 3. The double space $X_t^2$ and its boundary faces

The triple space is similarly defined in two steps. Again letting $C_n$ denote the union of boundary faces of codimension $n$, the b-blowup is given by

$$(X^3 \times I)_{\mathrm{b}} = [M^3; C_4, C_3, C_2].$$

Now consider the b-fibrations $\pi_{L/R/C} : (X^3 \times I)_{\mathrm{b}} \longrightarrow (X^2 \times I)_b$. The double space $X_t^2$ was obtained from $(X^2 \times I)_{\mathrm{b}}$ by blowing up the intersection $C_V \cap \Delta$, which has preimage under each of $\pi_L$, $\pi_R$ and $\Pi_C$ lying in two separate boundary faces of $(X^3 \times I)_{\mathrm{b}}$. Let $G_L$ denote the preimage of $C_V \cap \Delta$ with respect to $\pi_L$ intersecting the preimage of $(\partial X \times X^2 \times I)$ in $(X^3 \times I)_{\mathrm{b}}$, and let $J_L$ denote the preimage of $C_V \cap \Delta$ intersecting the preimage of $(\mathring{X} \times X^2 \times I)$. Define $G_{R/C}$ and $J_{R/C}$ similarly. A moment's consideration reveals that $G_L \cap G_R \cap G_C = K$ is a nonempty submanifold, while the $J_*$ only intersect the corresponding $G_*$.

Finally, define the triple space by

$$X_t^3 = [(X^3 \times I)_{\mathrm{b}}; K, G_L, G_R, G_C, J_L, J_R, J_C].$$

It is well-defined since the $G_*$ are separated after blowing up $K$.

The important features of these spaces is that they lift the obvious projections to b-fibrations. A proof of the following theorem can be found in [GH08].

**Theorem B.1.** *There are b-fibrations $\pi_*$ making the following diagram commute*

(B.1)
$$
\begin{tikzcd}
X_t^3 \arrow[r, shift left=2, "\pi_{L,R,C}"] \arrow[r] \arrow[r, shift right=2] \arrow[d, "\beta_3"] & X_t^2 \arrow[r, shift left=1, "\pi_{L,R}"] \arrow[r, shift right=1] \arrow[d, "\beta_2"] & X_t \arrow[r, "\pi_X"] \arrow[dr, "\pi_I"] \arrow[d, "\beta_1"] & X \\
X^3 \times I \arrow[r, shift left=2, "\pi_{L,R,C}"] \arrow[r] \arrow[r, shift right=2] & X^2 \times I \arrow[r, shift left=1, "\pi_{L,R}"] \arrow[r, shift right=1] & X \times I \arrow[r] \arrow[ur] & I
\end{tikzcd}
$$

*Remark.* We 'overload' the notation for projections, so that for instance, $\pi_I : X_t^3 \longrightarrow I$ should mean the unique b-fibration lifting $X^3 \times I \longrightarrow I$, which by the above theorem is given by any appropriate composition.

The following identifications of various submanifolds and boundary faces of $X_t^2$ are fundamental to the calculus, and are easily verified in local coordinates.

$$
\begin{aligned}
\mathrm{sc} &\cong \overline{{}^{\mathrm{sc}}T_{\partial X} X} \times I \longrightarrow \partial X \times I \\
\mathrm{tf} &\cong (\partial X \times [0,1])^2_{\mathrm{b,sc}}, \\
\mathrm{zf} &\cong X^2_{\mathrm{b}}, \\
\pi_I^{-1}(\tau) &\cong X^2_{\mathrm{sc}}, \quad \tau > 0, \\
\Delta &\cong X_t,
\end{aligned}
$$

where $\Delta \subset X_t^2$ denotes the lifted fiber diagonal.

B.2. **Densities.** We will make use primarily b half densities for operator kernels and functions, in order to facilitate the invocation of the pushforward theorem for polyhomogeneous conormal distributions. Observe on the unresolved spaces $X^n \times I$, $n = 1, 2, 3$ there are canonical identifications

(B.2)
$$
\begin{aligned}
\Omega_{\mathrm{b}}^{1/2}(X \times I) &\cong \pi_X^*(\Omega_{\mathrm{b}}^{1/2}(X)) \otimes \pi_I^*(\Omega_{\mathrm{b}}^{1/2}(I)) \\
\Omega_{\mathrm{b}}^{1/2}(X^2 \times I) &\cong \pi_{X,L}^*(\Omega_{\mathrm{b}}^{1/2}(X)) \otimes \pi_{X,R}^*(\Omega_{\mathrm{b}}^{1/2}(X)) \otimes \pi_I^*(\Omega_{\mathrm{b}}^{1/2}(I)) \\
\Omega_{\mathrm{b}}^{1/2}(X^3 \times I) &\cong \pi_{X,L}^*(\Omega_{\mathrm{b}}^{1/2}(X)) \otimes \pi_{X,C}^*(\Omega_{\mathrm{b}}^{1/2}(X)) \otimes \pi_{X,R}^*(\Omega_{\mathrm{b}}^{1/2}(X)) \otimes \pi_I^*(\Omega_{\mathrm{b}}^{1/2}(I))
\end{aligned}
$$

where $\pi_{X,R}$, etc. are shorthand for $\pi_X \circ \pi_R$ and so on.

**Lemma B.2.** *On $X_t^2$ and $X_t^3$, respectively, there are canonical identifications*

$$
\pi_R^*(\Omega_{\mathrm{b}}^{1/2}(X_t)) \otimes \pi_L^*(\Omega_{\mathrm{b}}^{1/2}(X_t)) \cong \rho_{\mathrm{sc}}^{n/2}\Omega_{\mathrm{b}}^{1/2}(X_t^2) \otimes \pi_I^*(\Omega_{\mathrm{b}}^{1/2}(I)), \quad \textit{and}
$$
$$
\pi_R^*(\Omega_{\mathrm{b}}^{1/2}(X_t^2)) \otimes \pi_C^*(\Omega_{\mathrm{b}}^{1/2}(X_t^2)) \otimes \pi_L^*(\Omega_{\mathrm{b}}^{1/2}(X_t^2)) \cong (\sigma_{\mathrm{sc}}^{n/2}\Omega_{\mathrm{b}}(X_t^3) \otimes \pi_I^*(\Omega_{\mathrm{b}}^2(I))
$$

*where $\sigma_{\mathrm{sc}}$ is a product of all the boundary defining functions $\rho_{G,L/R/C}$ and $\rho_{J,L/R/C}$ for the faces obtained by blowing up $G_{R/L/C}$ and $J_{R/L/C}$ in the process of obtaining $X_t^3$.*

*Proof.* The proof follows from (B.2) and the standard fact that

$$
\beta^*(\Omega^{1/2}(X)) = \rho_{\mathrm{ff}}^{(\mathrm{codim}(Y)-1)/2}\Omega^{1/2}([X;Y])
$$

with respect to a blow-down map $\beta : [X;Y] \longrightarrow X$, where $\rho_{\mathrm{ff}}$ denotes a boundary defining function for the front face of the blow-up. $\square$

We will make use of the canonical trivializing section $\nu^s := \left|\frac{d\tau}{\tau}\right|^s \in C^\infty(I; \Omega^s)$ and its pullback to various spaces.

We denote the *kernel density bundle* $\omega_{\mathrm{kd}} \longrightarrow X_t^2$ by

$$\omega_{\mathrm{kd}} = \rho_{\mathrm{sc}}^{-n/2}\Omega_{\mathrm{b}}^{1/2}(X_t^2).$$

This convention normalizes the densities so that the kernel of the identity operator on b half densities has smooth asymptotic expansion of order 0 at all boundary faces meeting the lifted diagonal.

**Lemma B.3.** *The restriction of the kernel density bundle gives the following identifications*

$$\begin{aligned}
(\omega_{\mathrm{kd}})|_{\mathrm{zf}} &\cong \Omega_{\mathrm{b}}^{1/2}(X_b^2)\\
(\omega_{\mathrm{kd}})|_{\mathrm{tf}} &\cong \rho_{\mathrm{sc}}^{-n/2}\Omega_{\mathrm{b}}^{1/2}\big((\partial X \times [0,1])^2_{\mathrm{b,sc}}\big)\\
(\omega_{\mathrm{kd}})|_{\pi_I^{-1}(\tau)} &\cong \rho_{\mathrm{sc}}^{-n/2}\Omega_{\mathrm{b}}^{1/2}(X^2_{\mathrm{sc}}) \cong \Omega_{\mathrm{sc}}^{1/2}(X^2_{\mathrm{sc}}), \quad \tau > 0\\
(\omega_{\mathrm{kd}})|_{\Delta} &\cong \rho_{\mathrm{sc}}^{-n}\Omega_{\mathrm{b}}((X_t)_{\mathrm{fib}}) \otimes \Omega_{\mathrm{b}}^{1/2}(I)
\end{aligned}$$

*where* $(X_t)_{\mathrm{fib}}$ *denotes the (generalized) fiber of the b-fibration* $\pi_I : X_t \longrightarrow I$*.*

*Proof.* The restriction of b half densities to boundary faces is well-defined, corresponding locally to the cancellation of a boundary defining factor, from which the first and second identifications follow.

Along the submanifold $\pi_I^{-1}(\tau), \tau > 0$,

$$\Omega_{\mathrm{b}}^{1/2}(X_t^2)|_{\pi_I^{-1}(\tau)} \cong \Omega_{\mathrm{b}}^{1/2}(\pi_I^{-1}(\tau)) \otimes \Omega_{\mathrm{b}}^{1/2}(I) \cong \Omega_{\mathrm{b}}^{1/2}(\pi_I^{-1}(\tau))$$

using the trivializing section $\nu^{1/2}$ of $\Omega_{\mathrm{b}}^{1/2}(I)$.

For the last claim, we note that, near $\mathrm{zf} \cap \Delta$,

$$\begin{aligned}
\Omega_{\mathrm{b}}^{1/2}(X_t^2) &\cong \Omega_{\mathrm{b}}^{1/2}(X) \otimes \Omega_{\mathrm{b}}^{1/2}(N\Delta) \otimes \Omega_{\mathrm{b}}^{1/2}(I)\\
&\cong \Omega_{\mathrm{b}}(X) \otimes \Omega_{\mathrm{b}}^{1/2}(I).
\end{aligned}$$

Similarly, near $\mathrm{sc} \cap \Delta$, we have

$$\begin{aligned}
\omega_{\mathrm{kd}} &= \rho_{\mathrm{sc}}^{-n/2}\Omega_{\mathrm{b}}^{1/2}(X) \otimes \Omega_{\mathrm{b}}^{1/2}(N\Delta) \otimes \Omega_{\mathrm{b}}^{1/2}(I)\\
&= \rho_{\mathrm{sc}}^{-n}\Omega_{\mathrm{b}}(X) \otimes \Omega_{\mathrm{b}}^{1/2}(I).
\end{aligned}$$

□

B.3. **The Calculus.** Fix a vector bundle $V \longrightarrow X$, and denote also by $V \longrightarrow X_t^i$ $i \in \{1,2,3\}$ the pullback of $V$ to the single, double and triple spaces. The *b-sc transition pseudodifferential operators* are defined by

$$\begin{aligned}
\Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) &= \mathcal{A}^{\mathcal{E}}_{\mathrm{phg}} I^m(X_t^2, \Delta; \mathrm{End}(V) \otimes \omega_{\mathrm{kd}}),\\
\mathcal{E} &= (E_{\mathrm{zf}}, E_{\mathrm{tf}}, E_{\mathrm{sc}}, E_{\mathrm{lb}_0}, E_{\mathrm{rb}_0}, \infty_{\mathrm{lb}}, \infty_{\mathrm{rb}}, \infty_{\mathrm{bf}})
\end{aligned}$$

where $I^m(X_t^2, \Delta)$ denotes the space of distributions conormal to $\Delta$ in the sense of Hörmander, with symbol order $m$. In particular, these distributions vanish to infinite order at $\mathrm{lb}, \mathrm{rb}$, and $\mathrm{bf}$. For notational convenience we are denoting the bundle $\mathrm{Hom}(\pi_R^* V, \pi_L^* V) \longrightarrow X_t^2$ simply as $\mathrm{End}(V)$.

A distinguished subclass of these operators form the *small calculus*

$$\begin{aligned}
\Psi_t^{m,(e_{\mathrm{zf}}, e_{\mathrm{tf}}, e_{\mathrm{sc}})}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) &= \Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}),\\
&\text{with } \mathcal{E} = (e_{\mathrm{zf}}, e_{\mathrm{tf}}, e_{\mathrm{sc}}, \infty_{\mathrm{lb}_0}, \infty_{\mathrm{rb}_0}, \infty_{\mathrm{lb}}, \infty_{\mathrm{rb}}, \infty_{\mathrm{bf}})
\end{aligned}$$

where $e_i \in \mathbb{Z}$ are identified with the smooth index sets $\{(e_i + \mathbb{N}, 0)\}$.

The action of $P \in \Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$ on $u \in \dot{C}^\infty(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$ (here $\dot{C}^\infty$ denotes smooth sections vanishing to infinite order at all boundary faces) is given by

$$Pu = (\pi_L)_* \left(\kappa_P \cdot \pi_R^*(u)\, \pi_I^*(\nu^{-1/2})\right),$$

where $\kappa_P$ is the Schwartz kernel of $P$.

**Proposition B.4.** *The action of $\Psi_t^*$ on $\dot{C}^\infty$ extends to an operation*

$$\Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \cdot \mathcal{A}_{\mathrm{phg}}^{\mathcal{F}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}) \subset \mathcal{A}_{\mathrm{phg}}^{\mathcal{G}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$$

*where*

$$\begin{aligned} G_{\mathrm{sc}} &= F_{\mathrm{sc}} + E_{\mathrm{sc}} \\ G_{\mathrm{tf}} &= (F_{\mathrm{tf}} + E_{\mathrm{tf}})\,\overline{\cup}\,(F_{\mathrm{zf}} + E_{\mathrm{rb}_0}) \\ G_{\mathrm{zf}} &= (F_{\mathrm{zf}} + E_{\mathrm{zf}})\,\overline{\cup}\,(F_{\mathrm{tf}} + E_{\mathrm{lb}_0}) \end{aligned}$$

*Remark.* In particular, the small calculus maps $\mathcal{A}_{\mathrm{phg}}^{\mathcal{F}}$ to $\mathcal{A}_{\mathrm{phg}}^{\mathcal{G}}$ with

$$\begin{aligned} G_{\mathrm{sc}} &= F_{\mathrm{sc}} + e_{\mathrm{sc}} \\ G_{\mathrm{tf}} &= F_{\mathrm{tf}} + e_{\mathrm{tf}} \\ G_{\mathrm{zf}} &= F_{\mathrm{zf}} + e_{\mathrm{zf}} \end{aligned}$$

*Proof.* This is a direct consequence of Theorems A.1 and A.2. Indeed, taking $u \in \mathcal{A}_{\mathrm{phg}}^{\mathcal{F}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$, it follows that

$$\kappa_P \cdot \pi_R^*(u)\, \pi_I^*(\nu^{-1/2}) \in \mathcal{A}_{\mathrm{phg}}^{\mathcal{H}} I^m(X_t^2, \Delta; V \otimes \rho_{\mathrm{sc}}^{-n/2} \Omega_{\mathrm{b}}^{1/2}(X_t^2) \otimes \pi_R^* \Omega_{\mathrm{b}}^{1/2}(X_t) \otimes \pi_I^*(\Omega_{\mathrm{b}}^{-1/2}(I)))$$

where

$$\begin{aligned} H_{\mathrm{sc}} &= E_{\mathrm{sc}} + F_{\mathrm{sc}} & H_{\mathrm{bf}} &= E_{\mathrm{bf}} + F_{\mathrm{sc}} = \infty + F_{\mathrm{sc}} = \infty \\ H_{\mathrm{lb}} &= E_{\mathrm{lb}} + F_{\mathrm{sc}} = \infty + F_{\mathrm{sc}} = \infty & H_{\mathrm{tf}} &= E_{\mathrm{tf}} + F_{\mathrm{tf}} \\ H_{\mathrm{lb}_0} &= E_{\mathrm{lb}_0} + F_{\mathrm{tf}} & H_{\mathrm{rb}_0} &= E_{\mathrm{rb}_0} + F_{\mathrm{zf}} \\ H_{\mathrm{zf}} &= E_{\mathrm{zf}} + F_{\mathrm{zf}}. \end{aligned}$$

The index sets $G_i$ are obtained from the pushforward theorem, since all boundary exponents of $\pi_L$ are either 0 or 1 and

$$\begin{aligned} \pi_L^{-1}(\mathrm{sc}) &= \mathrm{bf} \cup \mathrm{sc} \cup \mathrm{rb} \\ \pi_L^{-1}(\mathrm{tf}) &= \mathrm{tf} \cup \mathrm{rb}_0 \\ \pi_L^{-1}(\mathrm{zf}) &= \mathrm{lb}_0 \cup \mathrm{zf}. \end{aligned}$$

The interior conormal singularity is killed since $\pi_L$ is transversal to $\Delta$.

It remains to verify that the density bundles behave as expected. The claim is that the pushforward under $\pi_L$ of $\kappa_P\, \pi_R^*(u)\pi_I^*(\nu^{-1/2})$ can be identified with a section of $\Omega_{\mathrm{b}}^{1/2}(X_t) \otimes V$, or equivalently that it pairs with $\Omega_{\mathrm{b}}^{1/2}(X_t)$ to produce an element of $\Omega_{\mathrm{b}}(X_t) \otimes V$. Thus let $\gamma \in C^\infty(X_t; \Omega_{\mathrm{b}}^{1/2}(X_t))$ and consider

$$\gamma\, (\pi_L)_* \left(\kappa_P \cdot \pi_R^*(u)\pi_I^*(\nu^{-1/2})\right) = (\pi_L)_* \left(\pi_L^*(\gamma)\, \kappa_P \cdot \pi_R^*(u)\pi_I^*(\nu^{-1/2})\right).$$

The element in parentheses on the right hand side is a section of

$$\pi_L^*(\Omega_{\mathrm{b}}^{1/2}(X_t)) \otimes \pi_R^*(\Omega_{\mathrm{b}}^{1/2}(X_t)) \otimes \rho_{\mathrm{sc}}^{-n/2} \Omega_{\mathrm{b}}^{1/2}(X_t^2) \otimes \pi_I^*(\Omega_{\mathrm{b}}^{-1/2}(I)) \otimes V,$$

By Lemma B.2, we can identify this with the bundle

$$\rho_{\mathrm{sc}}^{n/2}\Omega_{\mathrm{b}}^{1/2}(X_t^2)\otimes\pi_I^*(\Omega_{\mathrm{b}}^{1/2}(I))\otimes\rho_{\mathrm{sc}}^{-n/2}\Omega_{\mathrm{b}}^{1/2}(X_t^2)\otimes\pi_I^*(\Omega_{\mathrm{b}}^{-1/2}(I))\otimes V\cong\Omega_{\mathrm{b}}(X_t^2)\otimes V,$$

and $(\pi_L)_*$ maps this into $\Omega_{\mathrm{b}}(X_t)\otimes V$ as claimed. □

Composition of the operators is defined by pulling back to the triple space, multiplying and pushing forward:

$$\kappa_{P\circ Q}=(\pi_C)_*\left(\pi_L^*(\kappa_P)\cdot\pi_R^*(\kappa_Q)\,\pi_I^*(\nu^{-2})\right)$$

**Proposition B.5.** *The composition of operators is well-defined, with*

$$\Psi_t^{m,\mathcal{E}}(X_t;V\otimes\Omega_{\mathrm{b}}^{1/2})\circ\Psi_t^{m',\mathcal{F}}(X_t;V\otimes\Omega_{\mathrm{b}}^{1/2})\subset\Psi_t^{m+m',\mathcal{G}}(X_t;V\otimes\Omega_{\mathrm{b}}^{1/2})$$

*where*

$$\begin{aligned}
G_{\mathrm{sc}}&=E_{\mathrm{sc}}+F_{\mathrm{sc}}\\
G_{\mathrm{zf}}&=(E_{\mathrm{zf}}+F_{\mathrm{zf}})\,\overline{\cup}\,(E_{\mathrm{rb}_0}+F_{\mathrm{lb}_0})\\
G_{\mathrm{tf}}&=(E_{\mathrm{tf}}+F_{\mathrm{tf}})\,\overline{\cup}\,(E_{\mathrm{lb}_0}+F_{\mathrm{rb}_0})\\
G_{\mathrm{lb}_0}&=(E_{\mathrm{lb}_0}+F_{\mathrm{zf}})\,\overline{\cup}\,(E_{\mathrm{bf}_0}+F_{\mathrm{lb}_0})\\
G_{\mathrm{rb}_0}&=(E_{\mathrm{zf}}+F_{\mathrm{rb}_0})\,\overline{\cup}\,(E_{\mathrm{rb}_0}+F_{\mathrm{bf}_0})
\end{aligned}$$

*Remark.* In particular, the small calculus composes as

$$\Psi_t^{m,(e_{\mathrm{zf}},e_{\mathrm{tf}},e_{\mathrm{sc}})}\circ\Psi_t^{m',(f_{\mathrm{zf}},f_{\mathrm{tf}},f_{\mathrm{sc}})}\subset\Psi_t^{m+m',(e_{\mathrm{zf}}+f_{\mathrm{zf}},e_{\mathrm{tf}}+f_{\mathrm{tf}},e_{\mathrm{sc}}+f_{\mathrm{sc}})}.$$

*Proof.* First we consider how the densities behave. From Lemma B.2 it follows that $\pi_L^*(\kappa_P)\pi_R^*(\kappa_Q)\pi_I^*(\nu^{-2})$ is a section of $(\rho_{J,C}\rho_{G,C})^{-n/2}\,\Omega_{\mathrm{b}}(X_t^3)\otimes\pi_C^*\Omega_{\mathrm{b}}^{-1/2}(X_t^2)$. This may subsequently be identified with $\Omega_{\mathrm{fib}}(X_t^3)\otimes\pi_C^*(\omega_{\mathrm{kd}})$ where $\Omega_{\mathrm{fib}}$ denotes fiber densities with respect to $\pi_C$.

The interior conormal singularities of $\kappa_P$ and $\kappa_Q$ compose transversally as in the classical case, and the only contribution to survive the pushforward comes from $\pi_C^{-1}(\Delta)$, since the conormal singularity everywhere else is transversal to $\pi_C$.

Finally, the index sets are determined by the pushforward theorem, once we identify the relationships between the inverse images under $\pi_{L/R/C}$ of the boundary hypersurfaces of $X_t^2$, in $X_t^3$. In fact, as all boundary faces in question are the lifts of boundary faces of the product spaces $X^2\times I$ and $X^3\times I$ under blowups, it suffices to consider the maps $\pi_{L/R/C}:X^3\times I\longrightarrow X^2\times I$, using commutativity of (B.1).

For instance, $\mathrm{tf}\subset X_t^2$ is the lift under $\beta$ of the face $\partial X\times\partial X\times\{0\}\subset X^2\times I$. So consider $\pi_C^{-1}(\partial X\times\partial X\times\{0\})\subset X^3\times I$. This consists of two boundary faces[9], namely

$$\partial X\times\partial X\times\partial X\times\{0\}\quad\text{and}\quad\partial X\times X\times\partial X\times\{0\}.$$

The first face projects down to $\partial X\times\partial X\times\{0\}$ under both $\pi_L$ and $\pi_R$, corresponding to the face $\mathrm{tf}\subset X_t^2$, while the second face projects to $\partial X\times X\times\{0\}$ under $\pi_L$ (corresponding to $\mathrm{lb}_0\subset X_t^2$) and to $X\times\partial X\times\{0\}$ under $\pi_R$ (corresponding to $\mathrm{rb}_0\subset X_t^2$). From this we conclude that

$$G_{\mathrm{tf}}=(E_{\mathrm{tf}}+F_{\mathrm{tf}})\,\overline{\cup}\,(E_{\mathrm{lb}_0}+F_{\mathrm{rb}_0}).$$

The index sets for the other faces are obtained similarly. □

[9]Though one face is included in the other in $X^3\times I$, this inclusion relationship is not preserved under blowup to $X_t^3$, so we consider them separately.

B.4. **Normal operators and symbols.** The leading order term in the asymptotic expansion of operator kernels at the boundary faces intersecting the lifted diagonal play a special role.

**Definition B.6.** Given $P \in \Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$ with kernel $\kappa_P$, let

- $N_{\mathrm{tf}}(P)$ be the restriction of $\kappa_P$ to tf,
- $\sigma_{\mathrm{sc}}(P)$ be the fiberwise Fourier transform of the restriction of $\kappa_P$ to sc with respect to the vector bundle structure on sc.
- $N_\tau(P)$ be the restriction of $\kappa_P$ to $\pi_I^{-1}(\tau), \tau > 0$, and
- $N_{\mathrm{zf}}(P)$ be the restriction of $\kappa_P$ to zf.

Here restriction means as a section of the kernel density bundle, and it follows that the distributions $N_*(P)$ are well-defined by transversality of the various faces and the lifted diagonal. The composition theorem allows these distributions to be interpreted as various model operators.

**Proposition B.7.** *With $P \in \Psi_t^{m,\mathcal{E}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2}), Q \in \Psi_t^{m',\mathcal{F}}(X_t; V \otimes \Omega_{\mathrm{b}}^{1/2})$, there are identifications*

$$\begin{aligned}
&\sigma_{\mathrm{sc}}(P) \in C^\infty({}^{\mathrm{sc}}T^*_{\partial X} X \times I; \mathrm{End}(V) \otimes \Omega^{1/2}) \\
&N_{\mathrm{tf}}(P) \in \Psi_{\mathrm{b,sc}}^{m,(E_{\mathrm{zf}},E_{\mathrm{lb}_0},E_{\mathrm{rb}_0}),E_{\mathrm{sc}}}(\partial X \times {}_{\mathrm{b}}[0,1]_{\mathrm{sc}}; V \otimes \Omega_{\mathrm{b}}^{1/2}) \\
&N_{\mathrm{zf}}(P) \in \Psi_{\mathrm{b}}^{m,(E_{\mathrm{bf}_0},E_{\mathrm{lb}_0},E_{\mathrm{rb}_0})}(X; V \otimes \Omega_{\mathrm{b}}^{1/2}) \\
&N_\tau(P) \in \Psi_{\mathrm{sc}}^{m,E_{\mathrm{sc}}}(X; V \otimes \Omega_{\mathrm{b}}^{1/2}) \quad \tau > 0.
\end{aligned}$$

*With respect to composition, these satisfy*

$$\begin{aligned}
\sigma_{\mathrm{sc}}(P \circ Q) &= \sigma_{\mathrm{sc}}(P)\sigma_{\mathrm{sc}}(Q) \\
N_{\mathrm{tf}}(P \circ Q) &= N_{\mathrm{tf}}(P) \circ N_{\mathrm{tf}}(Q) \\
N_{\mathrm{zf}}(P \circ Q) &= N_{\mathrm{zf}}(P) \circ N_{\mathrm{zf}}(Q) \\
N_\tau(P \circ Q) &= N_\tau(P) \circ N_\tau(Q)
\end{aligned}$$

*provided* $\mathrm{Re}\,(E_{\mathrm{rb}_0} + F_{\mathrm{lb}_0}) > \mathrm{Re}\,(E_{\mathrm{zf}} + F_{\mathrm{zf}})$ *in the case of* $N_{\mathrm{zf}}(P \circ Q)$*, and provided* $\mathrm{Re}\,(E_{\mathrm{lb}_0} + F_{\mathrm{rb}_0}) > \mathrm{Re}\,(E_{\mathrm{bf}_0} + F_{\mathrm{bf}_0})$ *in the case of* $N_{\mathrm{tf}}(P \circ Q)$.

*Remark.* $N_{\mathrm{bf}_0}(P)$ is an operator on $\partial X \times {}_{\mathrm{b}}[0,1]_{\mathrm{sc}}$ which is of b type near 0 and scattering type near 1. It has index sets $(E_{\mathrm{zf}}, E_{\mathrm{lb}_0}, E_{\mathrm{rb}_0})$ as a b operator, and the index set $E_{\mathrm{sc}}$ as a scattering operator. $\sigma_{\mathrm{sc}}(P)$ is just a family of scattering symbols, parametrized smoothly by $\tau/x \in [0,1)$.

*Proof.* Observe that in the triple space $\pi_I^{-1}(\tau) \cong X^3_{\mathrm{sc}}$ for $\tau > 0$, so for fixed $\tau > 0$ composition coincides with the composition of scattering operators. From this the composition formulae for $\sigma_{\mathrm{sc}}$ and $N_\tau$ follow.

Next consider $N_{\mathrm{zf}}(P)$. From the identification $\mathrm{zf} \cong X^2_{\mathrm{b}}$, this may be viewed as the Schwartz kernel of a b operator on $X$, it must be shown that $N_{\mathrm{zf}}(P \circ Q) = N_{\mathrm{zf}}(P) \circ N_{\mathrm{zf}}(Q)$ under appropriate conditions. As $\mathrm{zf} \subset X^2_t$ is the lift of $X \times X \times \{0\} \subset X^2 \times I$, consider the inverse image of the this boundary face under $\pi_C$ in $X^3 \times I$:

$$\pi_C^{-1}(X \times X \times \{0\}) = X \times X \times X \times \{0\} \cup X \times \partial X \times X \times \{0\}\,.$$

The first face projects to $X^2 \times \{0\} \subset X^2 \times I$ under both $\pi_L$ and $\pi_R$, while the second projects to $X \times \partial X \times \{0\}$ under $\pi_L$ and to $\partial X \times X \times \{0\}$ under $\pi_R$. (This is the reason for the index set behavior $G_{\mathrm{zf}} = (E_{\mathrm{zf}} + F_{\mathrm{zf}})\,\overline{\cup}\,(E_{\mathrm{rb}_0} + F_{\mathrm{lb}_0})$ with respect to composition.) Provided $\mathrm{Re}\,(E_{\mathrm{rb}_0}+F_{\mathrm{lb}_0}) > \mathrm{Re}\,(E_{\mathrm{zf}}+F_{\mathrm{zf}})$, the contribution of the former vanishes upon restriction to zf, so under this condition the value $(\kappa_{P\circ Q})|_{\mathrm{zf}}$ is determined by the composition on the lift of $X \times X \times X \times \{0\}$ in $X^3_t$.

A straightforward coordinate computation shows that the lift of this face is isomorphic to the b triple space $X^3_{\mathrm{b}}$, with $\pi_{L/R/C}$ restricting to the analogous b-fibrations $X^3_{\mathrm{b}} \longrightarrow X^2_{\mathrm{b}}$, and $N_{\mathrm{zf}}(P\circ Q) = N_{\mathrm{zf}}(P) \circ N_{\mathrm{zf}}(Q)$ follows at once.

Similar considerations apply to $N_{\mathrm{tf}}(P)$. From the composition formula, provided $\mathrm{Re}\,(E_{\mathrm{lb}_0} + F_{\mathrm{rb}_0}) > \mathrm{Re}\,(E_{\mathrm{tf}} + F_{\mathrm{tf}})$, the contribution to $(\kappa_{P\circ Q})|_{\mathrm{tf}}$ consists of the composition of $P$ and $Q$ on the boundary hypersurface in $X^3_t$ which is the lift of $\partial X \times \partial X \times \partial X \times \{0\}$ in $X^3 \times I$. A coordinate computation again shows that this face is diffeomorphic to the triple space $(\partial X \times [0,1])^3_{\mathrm{b,sc}}$ and that $\pi_{L/R/C}$ induce the appropriate b-fibrations $(\partial X \times [0,1])^2_{\mathrm{b,sc}}$. □

**Proposition B.8.** *The symbols and normal operators are related by*

$$\begin{aligned} \sigma_{\mathrm{sc}}(P)|_{\pi_I^{-1}(\tau)} &= \sigma_{\mathrm{sc}}(N_\tau(P)) \\ \sigma_{\mathrm{sc}}(P)|_{\mathrm{sc}\cap\mathrm{tf}} &= \sigma_{\mathrm{sc}}(N_{\mathrm{tf}}(P)) \\ I(N_{\mathrm{tf}}(P),\lambda) &= I(N_{\mathrm{zf}}(P),-\lambda) \end{aligned} \tag{B.3}$$

*Proof.* The first two equations are a straightforward consequence of the geometry, the symbols in question being given by the fiberwise Fourier transform along $\pi_I^{-1}(\tau)\cap\mathrm{sc}$ and $\mathrm{sc}\cap\mathrm{tf}$, respectively, using the vector bundle structure induced from the identification $\mathrm{sc} \cong \overline{{}^{\mathrm{sc}}T_{\partial X}X} \times I \longrightarrow \partial X \times I$.

The third equation follows similarly from the geometry, the indicial operators of $N_{\mathrm{tf}}(P)$ and $N_{\mathrm{zf}}(P)$ being given by the fiberwise Mellin transform of the restriction of $\kappa_P$ to $\mathrm{tf}\cap\mathrm{zf} \cong [0,\infty] \times \partial X^2$. The difference in the sign of $\lambda$ is explained by the fact that the fiber variable $s \in [0,\infty]$ (with respect to which the Mellin transform is computed) differs by $s \longmapsto 1/s$ between the two operators.

To see this, suppose $(x,y,\tau)$ and $(x',y',\tau)$ are two copies of coordinates on $X \times I$ near $\partial X \times \{0\}$, so that $(x,y,x',y',\tau)$ form coordinates on $X^2 \times I$ near the corner. After blow-up, coordinates near $\mathrm{zf} \cong X^2_{\mathrm{b}}$ are given by $(x,y,s = x'/x, y',\tau)$ (where $\mathrm{zf} = \{\tau = 0\}$) and the action of the indicial operator $I(N_{\mathrm{zf}}(P))$ is given by integration in $y'$ and (multiplicative) convolution in $s \in [0,\infty)$ with fiber density $ds/s$. (This follows for instance from the fact that $\beta^*\,(x'\partial_{x'})\,|_{\mathrm{tf}\cap\mathrm{zf}} = s\partial_s$.) On the other hand, coordinates for $\mathrm{tf} \cong (\partial X \times [0,1])^2_{\mathrm{b,sc}}$ near the b front face $\mathrm{tf}\cap\mathrm{zf}$ are given by

$$(t,y,s',y') = (\tau/x, y, t'/t = (\tau/x')(x/\tau) = 1/s, y')$$

and $I(N_{\mathrm{tf}}(P))$ acts by convolution in $s' = 1/s$ with fiber density $ds'/s'$. □

B.5. **Symbols of Differential operators.** Next we discuss the Lie algebra giving rise to the transition calculus, and give an alternate definition of the normal operators for differential operators.

Consider the b fibration $\pi_I : X_t \longrightarrow I$. Let $\mathcal{V}_{\mathrm{b,f}}(X_t) \subset \mathcal{V}_{\mathrm{b}}(X_t)$ denote the subset of the b vector fields on $X_t$ (so those tangent to the boundary hypersurfaces), which are tangent to the fibers of $\pi_I$. Tangency is preserved under Lie bracket, so $\mathcal{V}_{\mathrm{b,f}}(X_t)$ is a Lie subalgebra. Then the *b-sc transition vector fields* are defined to be those vector fields which additionally vanish at sc:

$$\mathcal{V}_t(X_t) = \rho_{\mathrm{sc}}\mathcal{V}_{\mathrm{b,f}}(X_t).$$

**Lemma B.9.** (a) $[\mathcal{V}_t(X_t), \mathcal{V}_t(X_t)] \subset \rho_{\mathrm{sc}}\mathcal{V}_t(X_t)$, *so* $\mathcal{V}_t(X_t)$ *is a well-defined Lie subalgebra.*
(b) *Evaluation of* $\mathcal{V}_t(X_t)$ *at* sc *takes values in the abelian Lie algebra* $\mathcal{V}_t(X_t)/\rho_{\mathrm{sc}}\mathcal{V}_t(X_t)$.
(c) *Evaluation at* tf *takes values in* $\mathcal{V}_t(X_t)/\rho_{\mathrm{tf}}\mathcal{V}_t(X_t) \cong \mathcal{V}_{\mathrm{b,sc}}(\partial X \times {}_{\mathrm{b}}[0,1]_{\mathrm{sc}})$.
(d) *Evaluation at* zf *takes values in* $\mathcal{V}_t(X_t)/\rho_{\mathrm{zf}}\mathcal{V}_t(X_t) \cong \mathcal{V}_{\mathrm{b}}(X)$.

*Proof.* (a) follows exactly as in the proof of the identity $[\mathcal{V}_{\mathrm{sc}}(X), \mathcal{V}_{\mathrm{sc}}(X)] \subset x\mathcal{V}_{\mathrm{sc}}(X)$, using the fact that $\mathcal{V}_{\mathrm{b,f}}(X_t) \cdot \rho_{\mathrm{sc}}C^\infty(X_t) \subset \rho_{\mathrm{sc}}C^\infty(X_t)$. Then (b) follows from (a) and the fact that $\rho_{\mathrm{sc}}\mathcal{V}_t(X_t)$ is an ideal.

(c) follows from the fact that, since tf is a (generalized) fiber of $\pi_I$, $\mathcal{V}_t(X_t)/\rho_{\mathrm{tf}}\mathcal{V}_t(X_t)$ consists of vector fields on tf which are tangent to its boundaries $\mathrm{tf} \cap \mathrm{sc}$ and $\mathrm{tf} \cap \mathrm{zf}$, with an additional vanishing factor at $\mathrm{tf} \cap \mathrm{sc}$. A similar consideration results in (d). □

The *transition differential operators* are the enveloping algebra of $\mathcal{V}_t(X_t)$; equivalently, they are defined by iterated composition of $\mathcal{V}_t(X_t)$ as operators on $C^\infty(X_t; V)$.

$$\mathrm{Diff}_t^k(X_t; V) = \left\{ \sum_{j \le k} a_j\, V_1 \cdots V_j \ ;\ V_i \in \mathcal{V}_t(X_t), a_j \in C^\infty(X_t; \mathrm{End}(V)) \right\}$$

The evaluation maps above then extend to normal operator homomorphisms

$$\begin{aligned}
N_{\mathrm{sc}} &: \mathrm{Diff}_t^k(X_t; V) \longrightarrow \mathrm{Diff}_{\mathrm{I,fib}}^k(({}^{\mathrm{sc}}T_{\partial X}X) \times I; V) \\
N_{\mathrm{tf}} &: \mathrm{Diff}_t^k(X_t; V) \longrightarrow \mathrm{Diff}_{\mathrm{b,sc}}^k(\partial X \times I; V) \\
N_{\tau} &: \mathrm{Diff}_t^k(X_t; V) \longrightarrow \mathrm{Diff}_{\mathrm{sc}}^k(X; V) \\
N_{\mathrm{zf}} &: \mathrm{Diff}_t^k(X_t; V) \longrightarrow \mathrm{Diff}_{\mathrm{b}}^k(X; V),
\end{aligned}$$

where $\mathrm{Diff}_{\mathrm{I,fib}}^k(({}^{\mathrm{sc}}T_{\partial X}X) \times I; V)$ denotes translation invariant differential operators along the fibers of ${}^{\mathrm{sc}}T_{\partial X}X \times I \longrightarrow \partial X \times I$ which are smoothly parametrized by the base. Fiberwise Fourier transform of the first of these gives the scattering symbol

$$\sigma_{\mathrm{sc}} : \mathrm{Diff}_t^k(X_t; V) \longrightarrow C^\infty({}^{\mathrm{sc}}T^*_{\partial X}X \times I; \mathrm{End}(V)).$$

**Proposition B.10.** *For any* $P \in \mathrm{Diff}_t^k(X_t; V)$, *the indicial families of* $N_{\mathrm{tf}}(P)$ *and* $N_{\mathrm{zf}}(P)$ *are related by*

$$I(N_{\mathrm{tf}}(P), \lambda) = I(N_{\mathrm{zf}}(P), -\lambda).$$

*Proof.* It suffices to verify this in the case of a vector field $W \in \mathcal{V}_t(X_t)$. By definition $W$ is tangent to the fibers of $\pi_I : X_t \longrightarrow I$, which has the local coordinate form

$$\pi_I : (x, y, t) \longrightarrow \tau = xt$$

near $\mathrm{zf} \cap \mathrm{tf}$ where $t = \tau/x$ is a blow-up coordinate, and $y \in \partial X$. In these coordinates, $\mathcal{V}_t(X_t)$ is span over $C^\infty(X_t)$ of the vector fields $\partial_{y_i}$ and $x\partial_x - t\partial_t$, so

$$W \overset{\mathrm{loc}}{=} a_0(x, y, t)(x\partial_x - t\partial_t) + \sum_i a_i(x, y, t)\, \partial_{y_i}$$

from which it follows that

$$\begin{aligned}
I(N_{\mathrm{zf}}(W), \lambda) &= a_0(0, y, 0)(i\lambda) + \textstyle\sum_i a_i(0, y, 0)\, \partial_{y_i} \\
I(N_{\mathrm{tf}}(W), \lambda) &= a_0(0, y, 0)(-i\lambda) + \textstyle\sum_i a_i(0, y, 0)\, \partial_{y_i}. \quad \square
\end{aligned}$$

By lifting vector fields to $X_t^2$ from the left or the right and applying them to the kernel of the identity, it follows from the analogous results for the b and scattering calculi that these definitions of $N_*(P)$ and $\sigma_{\mathrm{sc}}(P)$ agree with those defined for pseudodifferential operators on $X_t^2$.

B.6. **The Trace.** For operators in $\Psi_{\mathrm{b}}^*(X)$ or $\Psi_{\mathrm{sc}}^*(X)$ which extend to be trace class, the trace may be computed by restricting to the diagonal and integrating (taking the trace fiberwise if operating on sections of a vector bundle). Here sufficient conditions for $A \in \Psi_{\mathrm{b}}^{m,\mathcal{E}}(X;V)$ to be trace class are that $m < -\dim(X)$ and $\operatorname{Re} E_{\mathrm{ff}} > 0$, and for $B \in \Psi_{\mathrm{sc}}^{m,e_{\mathrm{sc}}}(X;V)$ that $m < -\dim(X)$ and $e_{\mathrm{sc}} > n$.

We define an analogous 'families trace' for sufficiently well-behaved $P \in \Psi_t^*(X_t;V)$.

**Definition B.11.** Let $P \in \Psi_t^{m,\mathcal{E}}(X_t;V)$ with $m < -\dim(X)$, Re $E_{\mathrm{sc}} > n$. Then $\operatorname{Tr}(P) \in \mathcal{A}^F_{\mathrm{phg}}(I)$ is defined by

$$\operatorname{Tr}(P) = (\pi_I)_*\big(\operatorname{tr}(\kappa_P|_\Delta)\big)\nu^{-1/2} \in \mathcal{A}^F_{\mathrm{phg}}(I)$$

where $F = E_{\mathrm{tf}}\overline{\cup}E_{\mathrm{zf}}$, $\operatorname{tr}(\cdot)$ denotes the trace in the endomorphism bundle $\operatorname{End}(V)$, $\pi_I : X_t \longrightarrow I$ is the lift of the projection $X \times I \longrightarrow I$, and $\nu^s = \left|\frac{d\tau}{\tau}\right|^s$ is the canonical trivializing section of $\Omega_{\mathrm{b}}^s(I)$.

To see that this operation is well-defined, observe that $m < -\dim(X)$ implies that $\kappa_P$ is continuous so that the restriction to the diagonal is well-defined, and by Lemma B.2 results in a section $\kappa_P|_\Delta \in \mathcal{A}^{\mathcal{F}}\big(X_t;\rho_{\mathrm{sc}}^{-n}\Omega_{\mathrm{b,fib}} \otimes \Omega_{\mathrm{b}}^{1/2}(I)\big)$, where $\Omega_{\mathrm{b,fib}}$ denotes fiber b-densities with respect to $\pi_I : X_t \longrightarrow I$. The condition Re $E_{\mathrm{sc}} > n$ implies that, viewed as a (fiber) b-density, $\kappa_P|_\Delta$ has index set at sc with strictly positive real part so that the pushforward by $\pi_I$ exists, giving a section of $\mathcal{A}^F_{\mathrm{phg}}(I;\Omega_{\mathrm{b}}^{1/2}(I))$ after which the density factor is removed by multiplication by $\nu^{-1/2}$. The index set $F = E_{\mathrm{tf}}\overline{\cup}E_{\mathrm{zf}}$ is a consequence of Theorem A.2.

It is clear that for $\tau > 0$, the value of $\operatorname{Tr}(P)$ at $\tau$ coincides with the trace of $N_\tau(P)$ as a scattering operator:

$$\operatorname{Tr}(P)(\tau) = \operatorname{Tr}(N_\tau(P))$$

More subtle is the fact that $\operatorname{Tr}(P)(0)$ coincides with $\operatorname{Tr}(N_{\mathrm{tf}}(P)) + \operatorname{Tr}(N_{\mathrm{zf}}(P))$, provided these exist.

**Proposition B.12.** *Provided $P \in \Psi_t^{m,\mathcal{E}}(X_t;V)$ satisfies $m < -\dim(X)$, Re $E_{\mathrm{sc}} > \dim(X)$, $E_{\mathrm{tf}} = E_{\mathrm{zf}} = 0$, with $N_{\mathrm{zf}}(P) \in \Psi_{\mathrm{b,sc}}^*$ and $N_{\mathrm{tf}}(P) \in \Psi_{\mathrm{b}}^*$ trace class, then $\operatorname{Tr}(P)$ is a continuous function of $\tau$ down to $\tau = 0$ and*

$$\operatorname{Tr}(P)(0) = \operatorname{Tr}(N_{\mathrm{tf}}(P)) + \operatorname{Tr}(N_{\mathrm{zf}}(P)) \tag{B.4}$$

*Proof.* According to the pushforward theorem, $\operatorname{Tr}(P) \in \mathcal{A}^F_{\mathrm{phg}}(I)$ where $F = E_{\mathrm{zf}}\overline{\cup}E_{\mathrm{tf}}$. In particular the leading order index is $(0,0)\overline{\cup}(0,0) = (0,1)$, meaning a priori $\operatorname{Tr}(P)(\tau)$ has leading order asymptotic $\log(\tau)$ at $\tau = 0$. To prove the first part it suffices to show this leading order behavior does not occur, since the next order term is $\mathcal{O}(\tau^0) = \mathcal{O}(1)$ (corresponding to index $(0,0)$), implying boundedness and hence continuity.

We claim that the offending log term does not occur provided that the restriction $(\kappa_P|_\Delta)|_{\mathrm{tf}\cap\mathrm{zf}}$ to the corner $\mathrm{tf} \cap \mathrm{zf}$ (which is canonically a density on $\mathrm{tf} \cap \mathrm{zf} \cong \partial X$) integrates to 0. From this the result follows, for then the assumption that $N_{\mathrm{zf}}(P)$

and $N_{\mathrm{tf}}(P)$ are trace class means they have vanishing Schwartz kernels at $\mathrm{tf} \cap \mathrm{zf} \subset X_t^2$, so in particular $(\kappa_P|_\Delta)|_{\mathrm{tf}\cap\mathrm{zf}}$ vanishes identically.

Thus suppose $u \in \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(X_t; \Omega_{\mathrm{b}})$ is a bounded continuous density with leading order index $(0,0) \in F_{\mathrm{zf}},\ F_{\mathrm{tf}}$, and that $u|_{\mathrm{tf}\cap\mathrm{zf}} \in C^\infty(\partial X; \Omega)$ has vanishing integral. The b-fibration $X_t \longrightarrow I$ is given locally near $\mathrm{tf} \cap \mathrm{zf}$ by

$$\partial X \times [0,1)^2 \ni (y,t,x) \longmapsto \tau = tx \in I.$$

using coordinates $y \in \partial X$, the boundary defining function $x$ on $X$, and blow-up coordinate $t = \tau/x$. Let $\phi \in C^\infty_c([0,1)^2)$ be a smooth cutoff which is identically equal to 1 on a small neighborhood of $\{0,0\}$ .

Recall that polyhomogeneity of $\phi u \in \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}(\partial X \times [0,1)^2, \Omega_{\mathrm{b}})$ is equivalent to the multi-Mellin transform[10]

$$\mathcal{M}_2(\phi\, u)(\eta,\xi) = \int_{[0,1)^2} t^{-i\eta} x^{-i\xi} \phi(t,x)\, u(y,t,x)\, \frac{dt}{t}\, \frac{dx}{x}$$

being a product-type meromorphic function on $\mathbb{C}^2$ valued in $C^\infty(\partial X; \Omega)$ with poles of order $p+1$ and $q+1$ along $\{\eta = -iz_1\}$ and $\{\xi = -iz_2\}$ respectively, for each $(z_1, p) \in F_{\mathrm{zf}}$ and $(z_2, q) \in F_{\mathrm{tf}}$, along with a uniform decay condition in strips $|\operatorname{Im}\xi| \leq C$ and $|\operatorname{Im}\eta| \leq C'$.

Similarly, the index set $F$ of $v \in \mathcal{A}^F_{\mathrm{phg}}(I; \Omega_{\mathrm{b}})$ is determined by the orders and locations of the poles of

$$\mathcal{M}_1(\phi v)(\lambda) = \int_I \tau^{-i\lambda} \phi(\tau) v(\tau)\, \frac{d\tau}{\tau}$$

where a pole of order $p+1$ at $\lambda = -iz$ corresponds to $(z,p) \in F$.

Thus to show $f_*\phi\, u$ has no log term corresponding to $(0,1) \in F$, it suffices to show that $\mathcal{M}_1(\, f_*\phi u)(\lambda)$ has only a simple pole at $\lambda = 0$. We compute

$$\begin{aligned}
(i\lambda)^2 \mathcal{M}_1(\phi\, f_* u)(\lambda) &= (i\lambda)^2 \int_I \tau^{-i\lambda}\, (f_*\phi\, u)(\tau)\, \frac{d\tau}{\tau} \\
&= (i\lambda)^2 \int_{\partial X \times [0,1)^2} (t\, x)^{-i\lambda} \phi(t,x)\, u(z,t,x)\, \frac{dt}{t}\frac{dx}{x}\, dV_{\partial X} \\
&= \int_{\partial X} \int_{[0,1)^2} (t\, x)^{-i\lambda} \partial_t\, \partial_x \big(\phi(t,x)\, u(z,t,x)\big)\, dx\, dt\, dV_{\partial X}
\end{aligned}$$

using $f^*(\tau^{-i\lambda}) = (t\, x)^{-i\lambda}$, the identities $i\lambda(t\, x)^{-i\lambda} = -x\partial_x(t\, x)^{-i\lambda} = -t\partial_t(t\, x)^{-i\lambda}$, integration by parts (which valid in the region $\operatorname{Im}\lambda >> 0$) and analytic continuation. It follows that

$$\lim_{\lambda\to 0} (i\lambda)^2 \mathcal{M}_1(f_*\phi\, u)(\lambda) = \int_{\partial X} u(z,0,0)\, dV_{\partial X}$$

so that vanishing of $\int_{\partial X} u|_{\partial X \times \{0,0\}}$ implies simplicity of the pole at $\lambda = 0$.

By considering localizations at other boundary faces away from $\mathrm{tf} \cap \mathrm{zf}$, (where the b-fibration is modeled on a simple projection), it is easily seen that the only extended union (i.e. additional log term) contribution can come from the corner $\mathrm{tf} \cap \mathrm{zf}$. Thus, vanishing of the integral of $u|_{\mathrm{tf}\cap\mathrm{zf}}$ is sufficient (and indeed necessary from the equivalence of the Mellin characterization of polyhomogeneity) for $f_*u$ to have leading index $(0,0) \in F$.

[10] Observe that the density element $dt/t\, dx/x$ is part of $u$.

Once it is established that $\mathrm{Tr}(P) \in \mathcal{A}^{(0,0)}_{\mathrm{phg}}(I) \subset C^0(I)$, it is immediate $\mathrm{Tr}(P)(0)$ consists of the pushforward from $\pi_I^{-1}(0) = \mathrm{tf} \cup \mathrm{zf} \subset X_t$ and is given by the integral of the induced b-densities thereon, from which (B.4) follows. $\square$

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