Math 2321 Fall 2015: Quiz 6 Solutions

Problem 1. Let S be the solid region between the half cones $z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2}$ and $z = \sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2=1$. Compute the volume of S *using spherical coordinates.*

Solution. The cones $z = \sqrt{x^2+y^2} = r$ and $z = \frac{1}{\sqrt{3}}\sqrt{x^2+y^2} = \frac{1}{\sqrt{3}}r$ are given in spherical coordinates by the surfaces $\varphi = \frac{\pi}{4}$ and $\varphi = \frac{\pi}{3}$, respectively. The spherical surface $x^2+y^2+z^2=1$ is given by $\rho = 1$. Thus, S is parameterized in spherical coordinates by

$$S = \left\{(\rho,\varphi,\theta) : 0 \le \theta \le 2\pi,\ \tfrac{\pi}{4} \le \varphi \le \tfrac{\pi}{3},\ 0 \le \rho \le 1\right\},$$

and the volume is computed by the integral

$$\begin{aligned}
\mathrm{Vol}(S) = \iiint_S dV &= \int_0^{2\pi}\int_{\pi/4}^{\pi/3}\int_0^1 \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta \\
&= 2\pi \int_{\pi/4}^{\pi/3} \frac{1}{3}\sin\varphi\, d\varphi \\
&= \frac{2\pi}{3}(-\cos\varphi)\Big|_{\pi/3}^{\pi/4} \\
&= \frac{2\pi}{3}\left(-\frac{1}{\sqrt{2}} + \frac{1}{2}\right).
\end{aligned}$$

□

Problem 2. Let S be the solid region outside the cylinder $x^2+y^2=1$ and inside the sphere $x^2+y^2+z^2=4$, with mass density given by $\delta(x,y,z) = x^2+y^2$. Write an integral which computes the mass of S *in cylindrical coordinates.* **You do not have to evaluate the integral.**

Solution. The cylinder is given by the equation $r = 1$ in cylindrical coordinates, while the sphere is given by $r^2+z^2=4$. Solving for z in terms of r, the latter becomes $z = \pm\sqrt{4-r^2}$. Alternatively, solving for r in terms of z, we have $r = \sqrt{4-z^2}$. Simultaneously solving $r=1$ and $z = \pm\sqrt{4-r^2}$ shows that the sphere and cylinder intersect in the planes $z = \pm\sqrt{3}$. We can write S in one of two ways:

$$\begin{aligned}
S &= \left\{(r,\theta,z) : 0 \le \theta \le 2\pi,\ 1 \le r \le 2,\ -\sqrt{4-r^2} \le z \le \sqrt{4-r^2}\right\} \quad \text{or} \\
S &= \left\{(r,\theta,z) : 0 \le \theta \le 2\pi,\ -\sqrt{3} \le z \le \sqrt{3},\ 1 \le r \le \sqrt{4-z^2}\right\}.
\end{aligned}$$

The density $\delta(x,y,z) = x^2+y^2$ is given in cylindrical coordinates by $\delta = r^2$, and $dV = r\,dz\,dr\,d\theta$ (or $dV = r\,dr\,dz\,d\theta$), so we have

$$\begin{aligned}
\mathrm{Mass}(S) = \iiint_S \delta\, dV &= \int_0^{2\pi}\int_1^2\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^3\, dz\, dr\, d\theta, \quad \text{or} \\
&= \int_0^{2\pi}\int_{-\sqrt{3}}^{\sqrt{3}}\int_1^{\sqrt{4-z^2}} r^3\, dr\, dz\, d\theta.
\end{aligned}$$

□

Problem 3. Let E be the surface given by the portion of the graph $z = x^2 - y^2$ which lies over the disk of radius 5 in the x-y plane (centered at the origin). Write down a double integral which computes the surface area of E. **You do not have to evaluate the integral**.

Solution. We use the parameterization $\mathbf{r}(x, y) = (x, y, x^2 - y^2)$. Either from the formula $dS = |\mathbf{r}_x \times \mathbf{r}_y|\, dx\, dy$ or from the formula $dS = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$ for a graph $z = f(x, y)$, we find

$$dS = \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy.$$

Thus the surface area is given by an integral

$$\text{Area}(E) = \iint_E dS = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy$$

where R is the disk of radius 5. We can either write this as

$$\iint_E dS = \int_{-5}^{5} \int_{-\sqrt{25-y^2}}^{\sqrt{25-y^2}} \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy,$$

or convert to polar coordinates, giving

$$\iint_E dS = \int_0^{2\pi} \int_0^5 \sqrt{4r^2 + 1}\, r\, dr\, d\theta.$$

□