Calculus III Workshop 10 questions: 11/9/16

Problem 1 (16.3, #20)**.** Show that the line integral $\int_C \sin y\,dx + (x\cos y - \sin y)\,dy$ is independent of path (i.e., the vector field is conservative) and evaluate the integral where C is any path from $(2,0)$ to $(1,\pi)$.

Problem 2 (16.3, #29)**.** Show that if the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is conservative and P, Q, R have continuous first order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Problem 3 (16.3, #30)**.** Use the previous exercise to show that the line integral $\int_C y\,dx + x\,dy + xyz\,dz$ is not independent of path.

Problem 4 (16.3, #35)**.** Let $\mathbf{F}(x,y) = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$.

(a) Show that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

(b) Show that $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ is not independent of path. [Hint: compute $\int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds$ and $\int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds$ where C_1 and C_2 are the upper and lower halves of the unit circle from $(1,0)$ to $(-1,0)$.] Does this contradict Theorem 6 (which says if $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$ on a simply connected region, then $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is path independent)?

Problem 5 (16.4, #9)**.** Use Green's Theorem to evaluate the line integral $\oint_C y^3\,dx - x^3\,dy$ where C is the circle $x^2+y^2=4$ oriented counterclockwise.

Problem 6 (16.4, #22)**.** Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\overline{x},\overline{y})$ (i.e., the center of mass assuming constant density) are

$$\overline{x} = \frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\oint_C 0\,dx + x^2\,dy$$
$$\overline{y} = -\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\oint_C y^2\,dx + 0\,dy$$

where A is the area of D.

Problem 7 (16.4, #23)**.** Use the previous exercise to find the centroid of a quarter-circular region of radius a.