Math 2321 Fall 2015: Quiz 5 Solutions

Problem 1. Consider the double integral $\int_1^{e^2}\int_{\ln x}^2 1\,dy\,dx$. Give an equivalent iterated integral in which you reverse the order of integration from $dy\,dx$ to $dx\,dy$, and evaluate this new integral.

Solution. Sketching the region $R = \{(x,y) : 1 \le x \le e^2,\ \ln x \le y \le 2\}$ of integration, we see that it is bounded by 3 curves: the curve $y = \ln x$, the line $y = 2$ and the line $x = 1$. Reversing the order so that x is integrated first, we can describe the region as

$$R = \{(x,y) : 0 \le y \le 2,\ 1 \le x \le e^y\},$$

thus $\int_1^{e^2}\int_{\ln x}^2 1\,dy\,dx = \int_0^2\int_1^{e^y} 1\,dx\,dy$. We have

$$\begin{aligned}\int_0^2\int_1^{e^y} 1\,dx\,dy &= \int_0^2 x\Big|_{x=1}^{x=e^y}\,dy\\ &= \int_0^2 e^y - 1\,dy\\ &= e^y - y\Big|_{y=0}^2\\ &= (e^2-2)-(e^0-0) = e^2-3.\end{aligned}$$

□

Problem 2. Convert the double integral

$$\int_{-3}^0\int_0^{\sqrt{9-x^2}} xe^{(x^2+y^2)^{3/2}}\,dy\,dx$$

into an integral in polar coordinates. **You do not need to evaluate the integral.**

Solution. Sketching the region of integration $R = \left\{(x,y) : -3 \le x \le 0,\ 0 \le y \le \sqrt{9-x^2}\right\}$, we see that it consists of the quarter disk of radius 3 situated in the quadrant of $\mathbb{R}^2$ where x is negative and y is positive. Converting R into polar coordintes, we have $R = \left\{(r,\theta) : 0 \le r \le 3,\ \frac{\pi}{2} \le \theta \le \pi\right\}$. Converting the integrand into polar coordinates, we have $xe^{(x^2+y^2)^{3/2}} = r\cos\theta e^{r^3}$, and finally $dy\,dx = dA$ becomes $dA = r\,dr\,d\theta$. Thus the integral in polar coordinates is

$$\int_{\frac{\pi}{2}}^{\pi}\int_0^3 (r\cos\theta)e^{r^3}\,r\,dr\,d\theta = \int_{\frac{\pi}{2}}^{\pi}\int_0^3 r^2e^{r^3}\cos\theta\,dr\,d\theta.$$

□

Problem 3. Calculate

$$\iiint_S (x+3z^2)\,dV,$$

where S is the solid region in $\mathbb{R}^3$ given by $0 \le x \le 1$, $0 \le y \le x$, and $0 \le z \le xy$.

Solution. Since the limits in z depend on x and y and the limits in y depend on x, the most convenient order to integrate in is $dV = dz\,dy\,dx$. Then the limits for S are exactly as written and we have

$$\begin{aligned}
\int_0^1 \int_0^x \int_0^{xy} (x+3z^2)\,dz\,dy\,dx &= \int_0^1 \int_0^x (xz+z^3)\Big|_{z=0}^{z=xy}\,dy\,dx \\
&= \int_0^1 \int_0^x (x^2y+x^3y^3)\,dy\,dx \\
&= \int_0^1 (\tfrac{1}{2}x^2y^2+\tfrac{1}{4}x^3y^4)\Big|_{y=0}^{y=x}\,dx \\
&= \int_0^1 (\tfrac{1}{2}x^4+\tfrac{1}{4}x^7)\,dx \\
&= (\tfrac{1}{10}x^5+\tfrac{1}{32}x^8)\Big|_{x=0}^{1} \\
&= \frac{1}{10}+\frac{1}{32}=\frac{42}{320}=\frac{21}{160}.
\end{aligned}$$

$\square$