Calculus III Workshop questions: 8/24/16

Problem 1. Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = -2\mathbf{i} - \mathbf{j} + 5\mathbf{k}$. Compute the following:
(a) $\mathbf{a} + \mathbf{b}$
(b) $2\mathbf{a} + 3\mathbf{b}$
(c) $|\mathbf{a}|$
(d) $|\mathbf{a} - \mathbf{b}|$

Solution. (a) $\mathbf{a} + \mathbf{b} = \langle 1-2, 2-1, -3+5\rangle = \langle -1, 1, 2\rangle$.
(b) $2\langle 1, 2, -3\rangle + 3\langle -2, -1, 5\rangle = \langle -4, 1, 9\rangle$.
(c) $|\mathbf{a}| = \sqrt{1 + 2^2 + (-3)^2} = \sqrt{14}$.
(d) $|\mathbf{a} - \mathbf{b}| = \sqrt{(1+2)^2 + (2+1)^2 + (-3-5)^2} = \sqrt{9 + 9 + 64} = \sqrt{82}$.

□

Problem 2. A vector $\mathbf{v}$ in $\mathbb{R}^2$ lies in the positive quandrant (i.e., where $x \geq 0$ and $y \geq 0$), makes an angle of $\pi/3$ with the positive x-axis, and satisfies $|\mathbf{v}| = 4$. Write $\mathbf{v}$ in component form.

Solution. We're given that $\cos\theta = \pi/3$, where θ is the angle off the x-axis. The x component, v_1 of $\mathbf{v}$ is then given by $v_1 = \mathbf{v} \cdot \mathbf{i} = |v|\cos\theta = 4(\frac{1}{2}) = 2$. The y-component v_2 satisfies

$$\begin{aligned} 4 = |\mathbf{v}| &= \sqrt{v_1^2 + v_2^2} = \sqrt{4 + v_2^2} \\ \implies v_2 &= \sqrt{12} = 2\sqrt{3} \end{aligned}$$

Thus $\mathbf{v} = \langle 2, 2\sqrt{3}\rangle$. □

Problem 3. Let $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ and $\mathbf{b} = \mathbf{j} + \frac{1}{2}\mathbf{k}$. Find the vector and scalar projections of $\mathbf{a}$ onto $\mathbf{b}$.

Solution. Relevant quantities are

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= \langle 2, -1, 4\rangle \cdot \langle 0, 1, \tfrac{1}{2}\rangle = 3, \\ |\mathbf{b}| &= \sqrt{\tfrac{5}{4}} = \tfrac{\sqrt{5}}{2}. \end{aligned}$$

Then the scalar projection is

$$\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{6}{\sqrt{5}}$$

and the vector projection is

$$\frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{b}}{|\mathbf{b}|^2} = \tfrac{12}{5}\langle 0, 1, \tfrac{1}{2}\rangle = \langle 0, 1, \tfrac{6}{5}\rangle.$$

□

Problem 4. Find two unit vectors which are orthogonal to both of the specified vectors:
(a) $\mathbf{i} + \mathbf{j}$ and $\mathbf{i} + \mathbf{k}$
(b) $\langle 3, 2, 1\rangle$ and $\langle -1, 1, 0\rangle$.

Solution. (a) This can be done easily enough by hand: if $\mathbf{v} = \langle a, b, c\rangle$ is orthogonal to the first vector, it must be that $0 = \mathbf{v} \cdot (\mathbf{i} + \mathbf{j}) = a + b$, which implies $a = -b$. Likewise, to be orthogonal to the second vector requires $0 = \mathbf{v} \cdot (\mathbf{i} + \mathbf{k}) = a + c$, so that $a = -c$. Then $b = c$, so two possibilities are $\langle 1, -1, -1\rangle$ and $\langle -1, 1, 1\rangle$, which after normalization (i.e., dividing by the length to form unit vectors) become

$$\pm \left\langle \tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}} \right\rangle .$$

Alternatively, you can use the cross product method.

(b) For this we use the cross product: calling the vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, we have

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \mathbf{i}\big(2(0) - 1(1)\big) - \mathbf{j}\big(3(0) - 1(-1)\big) + \mathbf{k}\big(3(1) - (-1)2\big) = \langle -1, 1, 5\rangle .$$

This vector is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ by a property of the cross product, but is not a unit vector: $|\mathbf{a} \times \mathbf{b}| = \sqrt{1 + 1 + 25} = \sqrt{27}$. The two unit vectors are therefore

$$\pm \tfrac{1}{\sqrt{27}} \langle -1, 1, 5\rangle .$$

□

Problem 5. Let $\mathbf{a} = \langle 1, 0, 1\rangle$, $\mathbf{b} = \langle 2, 1, -1\rangle$ and $\mathbf{c} = \langle 0, 1, 3\rangle$. Show that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

Solution. Omitting some work, you can compute that

$$\mathbf{a} \times \mathbf{b} = \langle -1, 3, 1\rangle , \quad \mathbf{b} \times \mathbf{c} = \langle 4, -6, 2\rangle ,$$
$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \langle 8, 3, -1\rangle \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \langle 6, 2, -6\rangle .$$

□

Problem 6. Let A, B and C be the vertices of a triangle in $\mathbb{R}^2$. Compute the vector $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

Solution. The sum is the zero vector. To see this, think in terms of displacement vectors. $\overrightarrow{AB}$ is the vector which, when added to the point A, gives the point B, etc. Thus the sum represents travelling from A, then to B, then to C, and back to A, which is a net displacement of 0. □

Problem 7. Find all vectors $\mathbf{v}$ such that $\langle 1, 2, 1\rangle \times \mathbf{v} = \langle 3, 1, -5\rangle$.

Solution. Let $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ have variable components. Writing the cross product, we have

$$\langle 1, 2, 1\rangle \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ v_1 & v_2 & v_3 \end{pmatrix} = \langle 2v_3 - v_2, v_1 - v_3, v_2 - 2v_1\rangle .$$

Setting this equal to $\langle 3, 1, -5\rangle$ gives the equations

$$\begin{aligned} 2v_3 - v_2 &= 3 \\ v_1 - v_3 &= 1 \\ v_2 - 2v_1 &= -5 \end{aligned}$$

which can be solved to get $\langle v_1, v_2, v_3\rangle = \langle 1, -3, 0\rangle$. □