Calc III Fall 2017: Exam 1 Solutions

Problem 1. For the curve $\mathbf{f}(t) = \big(\cos^2(t), 3\sin(t), 2\sqrt{t}\big)$, compute the derivative $\mathbf{f}'(t)$ and find an equation for the tangent line to the curve at $t = \pi$.

Solution. The derivative is

$$\mathbf{f}'(t) = \big(-2\cos(t)\sin(t), 3\cos(t), t^{-1/2}\big).$$

The tangent line to the curve is given by

$$\mathbf{l}(s) = \mathbf{f}(\pi) + s\mathbf{f}'(\pi) = \big(1, -3s, 2\sqrt{\pi} + \tfrac{1}{\sqrt{\pi}}s\big).$$

$\square$

Problem 2. An ant  is located on the xy-plane at the point $(1, 0)$. The density of ant pheromone (which signals that a DELICIOUS FOOD SOURCE is nearby) is given by

$$f(x, y) = ye^{xy}.$$

(a) What is the rate of change (directional derivative) of pheromone percieved by the ant if it is heading in the $(-1, 1)$ direction?

(b) If the ant wants to head toward the DELICIOUS FOOD SOURCE as quickly as possible by maximizing the rate of change of pheromone, in which direction should it initially move?

Solution.

(a)

$$\nabla f(x, y) = \big(y^2e^{xy},\, e^{xy} + xye^{xy}\big) \quad \nabla f(1, 0) = (0, 1)\,.$$

The directional derivative in the direction of $\mathbf{u} = (-1, 1)$, after normalizing to $\mathbf{v} = \mathbf{u}/\|\mathbf{u}\| = \frac{1}{\sqrt{2}}(-1, 1)$ is

$$\nabla f(1, 0) \cdot \mathbf{v} = (0, 1) \cdot \frac{1}{\sqrt{2}}(-1, 1) = \frac{1}{\sqrt{2}}.$$

(b) To get to the food source as quickly as possible, the ant should head in the direction of $\nabla f(1, 0)$, namely $(0, 1)$, for in this direction the rate of change is $\|\nabla f(1, 0)\| = 1$.

$\square$

Problem 3. Find an equation for the tangent plane to the hyperboloid $z^2 - x^2 - y^2 = 2$ at the point $(1, 1, 2)$.

Solution. The hyperboloid is a level surface of the function $g(x, y, z) = z^2 - x^2 - y^2$, the gradient of which is

$$\nabla g(x, y, z) = (-2x, -2y, 2z).$$

The gradient is normal to the surface at each point, so a normal vector to the tangent plane is given by $\mathbf{n} = \nabla g(1, 1, 2) = (-2, -2, 4)$. An equation for the plane is then

$$\mathbf{n} \cdot (x - 1, y - 1, z - 2) = -2(x - 1) - 2(y - 1) + 4(z - 2) = 0.$$

Alternatively, we could solve for z (locally) as a function of x and y near the point $(1, 1, 2)$, namely $z = \sqrt{2 + x^2 + y^2} = f(x, y)$. Then the tangent plane is given by the formula

$$f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) - z + 2 = 0$$

where $f_x(x,y) = \frac{x}{\sqrt{2+x^2+y^2}}$ and $f_y(x,y) = \frac{y}{\sqrt{2+x^2+y^2}}$, so $f_x(1,1) = f_y(1,1) = \frac{1}{2}$, giving

$$\frac{1}{2}(x-1) + \frac{1}{2}(y-1) - z + 2 = 0.$$

$\square$

Problem 4. Compute the arc length of the curve $\mathbf{f}(t) = (\frac{1}{3}t^3, \frac{1}{3}t^3+1, t^2+2)$, where $0 \leq t \leq 4$.

Solution. The arc length is given by the integral of the magnitude of the derivative:

$$\begin{aligned}
s &= \int_0^4 \|\mathbf{f}'(t)\|\ dt \\
&= \int_0^4 \left\|(t^2, t^2, 2t)\right\|\ dt \\
&= \int_0^4 \sqrt{t^4 + t^4 + 4t^2}\, dt \\
&= \int_0^4 t\sqrt{2t^2+4}\, dt \\
&= \frac{1}{4}\int_4^{36} \sqrt{u}\, du \\
&= \frac{1}{6}(36^{3/2} - 4^{3/2}) \\
&= \frac{104}{3}.
\end{aligned}$$

$\square$

Problem 5. Find the critical points of the function

$$f(x,y) = x^2y - 2xy^2 + 6xy + 4$$

and classify them as local minima, local maxima, saddle points, or undetermined.

Solution. We set the gradient of f equal to zero:

$$\nabla f(x,y) = (2xy - 2y^2 + 6y, x^2 - 4xy + 6x) = (0,0)$$

which yields the pair of equations

$$2y(x-y+3) = 0 \quad x(x-4y+6) = 0.$$

There are four possibilities:
(1) $y=0$, $x=0$, giving the critical point $(0,0)$,
(2) $y=0$, $(x+6)=0$, giving the critical point $(-6,0)$,
(3) $x=0$, $(-y+3)=0$, giving the critical point $(0,3)$, and
(4) $(x-y+3)=0$, $(x-4y+6)=0$, giving the critical point $(-2,1)$.

We then apply the second derivative test, with $f_{xx} = 2y$, $f_{yy} = -4x$, $f_{xy} = 2x-4y+6$, so

$$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (2y)(-4x) - (2x-4y+6)^2.$$

(1) $D(0,0) = -(6)^2 < 0$, so $(0,0)$ is a *saddle point.*

(2) $D(-6,0) = -(-12+6)^2 < 0$, so $(-6,0)$ is a *saddle point*.
(3) $D(0,3) = -(-12+6)^2 < 0$, so $(0,3)$ is a *saddle point*.
(4) $D(-2,1) = (2)(8) - (-4-4+6)^2 = 16 - (-2)^2 = 12 > 0$, while $f_{xx}(-2,1) = 2 > 0$, so $(-2,1)$ is a *local minimum*.

□

Problem 6. Find the maximum and minimum values of the function $f(x,y) = xy$ on the ellipse $8x^2 + 2y^2 = 1$, and the points at which these values occur.

Solution. We use Lagrange multipliers, setting $\nabla f = \lambda \nabla g$, where $g(x,y) = 8x^2 + 2y^2$, to get the system of equations

$$y = 16\lambda x$$
$$x = 4\lambda y$$
$$8x^2 + 2y^2 = 1$$

Solving for λ in the first two equations and setting them equal leads to $\frac{y}{16x} = \frac{x}{4y}$, or $y^2 = 4x^2$. This has two solutions $y = 2x$ and $y = -2x$. Plugging either of these into the third equation gives $16x^2 = 1$, or $x = \pm\frac{1}{4}$. Plugging back into y, we have the four solutions $(x,y) = (\frac{1}{4}, \frac{1}{8})$, $(\frac{1}{4}, -\frac{1}{8})$, $(-\frac{1}{4}, \frac{1}{8})$, and $(-\frac{1}{4}, -\frac{1}{8})$.

Evaluating f at these four points gives

$$f\left(\tfrac{1}{4}, \tfrac{1}{8}\right) = \frac{1}{32} \qquad f\left(\tfrac{1}{4}, -\tfrac{1}{8}\right) = -\frac{1}{32}$$
$$f\left(-\tfrac{1}{4}, \tfrac{1}{8}\right) = -\frac{1}{32} \qquad f\left(-\tfrac{1}{4}, -\tfrac{1}{8}\right) = \frac{1}{32}$$

So the minumum of f is $-\frac{1}{32}$, achieved at the two points $(\frac{1}{4}, -\frac{1}{8})$ and $(-\frac{1}{4}, \frac{1}{8})$, while the maximum is $\frac{1}{32}$, achieved at the two points $(\frac{1}{4}, \frac{1}{8})$ and $(-\frac{1}{4}, -\frac{1}{8})$. □