Calc III: Workshop 4, Fall 2017

Problem 1.

(a) Show that $f(x, y) = \sin(x + cy) + \cos(x - cy)$ satisfies the 1-dimensional wave equation

$$\frac{\partial^2 f}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial y^2} = 0. \tag{1}$$

(b) Let $u(t)$ and $v(t)$ be twice differentiable functions of a single variable. Show that $f(x, y) = u(x + cy) + v(x - cy)$ is a solution of (**??**).

Problem 2.

(a) Find the tangent plane to the surface $x^2 + y^2 - z^2 = 0$ at the point $P = (3, 4, 5)$.
(b) Find the tangent plane to the surface $x^2 + y^2 = 4$ at the point $P = (\sqrt{3}, 1, 0)$.

Problem 3. Prove the product and quotient rules for gradients:

$$\nabla(f\,g) = f\nabla g + g\nabla f, \quad \nabla(f/g) = \frac{g\nabla f - f\nabla g}{g^2}, \quad g(x, y) \neq 0.$$

Problem 4. The function $r(x, y) = \sqrt{x^2 + y^2}$ is the length of the position vector $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j}$ for each point $(x, y) \in \mathbb{R}^2$. Show that $\nabla r = \frac{1}{r}\mathbf{r}$ when $(x, y) \neq (0, 0)$, and that $\nabla(r^2) = 2\mathbf{r}$.

Problem 5. Recall that the *linear approximation* to a function $f(x, y)$ of two variables, at a point (x_0, y_0) is the linear function

$$L(x, y) = a(x - x_0) + b(y - y_0) + c$$

whose graph $z = L(x, y)$ is the tangent plane to the graph $z = f(x, y)$ of f at (x_0, y_0). (Hint: what are a, b and c in terms of $f(x_0, y_0)$ and the partial derivatives of f at (x_0, y_0)?)

(a) Given that f is a differentiable function with $f(2, 5) = 6$, $f_x(2, 5) = 1$, and $f_y(2, 5) = -1$, use the linear approximation to estimate $f(2.2, 4.9)$.
(b) Generalize the formula for linear approximations to functions of three variables, find the linear approximation to the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at $(3, 2, 6)$ and use it to approximate the number $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2}$.