Real Analysis, Fall 2017

September 16, 2017

Chapter 1

The real number line

1.1 Ordered Sets

One basic property of many number systems (natural numbers, integers, rationals, etc) is that they are *ordered*, so we say that "3 *is greater than* 2", and so on.

1.1.1 Definition. A *total order* on a set S is a relation[1] $\leq$ satisfying the following axioms:

(O1) (Reflexivity) For every element a, it always holds that $a \leq a$.

(O2) (Antisymmetry) If $a \leq b$ and $b \leq a$, then it must be that $a = b$.

(O3) (Transitivity) If $a \leq b$ and $b \leq c$, then it holds that $a \leq c$.

(O4) (Totality) For every pair of elements a and b, either $a \leq b$ or $b \leq a$.

We say S is an *ordered set*.

1.1.2 Example. Find some examples of ordered sets.

1.1.3 Example. Find an example of a *partially ordered* set—a set with a relation satisfying axioms (O1)–(O3) but not (O4).

1.1.4 Problem. Suppose S is an ordered set. Formulate a reasonable definition of strict inequality ($a < b$) in terms of the order relation $\leq$. Then write down a definition equivalent to Definition 1.1.1 using strict inequality as the primitive relation; that is, write down a set of axioms that $<$ should satisfy, in terms of which $\leq$ (suitably defined in terms of $<$) has properties (O1)–(O4).

1.1.5 Definition. Let S be an ordered set, and $A \subseteq S$ a subset. An *upper bound* for A is an element $u \in S$ such that $a \leq u$ for every $a \in A$. If such an element exists, we say A is *bounded above*.

Similarly, a *lower bound* for A is an element $l \in S$ such that $l \leq a$ for every $a \in A$. If such a lower bound exists, we say A is *bounded below*.

1.1.6 Definition. A *least upper bound* or *supremum* of a bounded above set A is an element u_0 of S such that

(i) u_0 is an upper bound for A, and

(ii) $u_0 \leq u$ for every other upper bound u.

We denote a supremum for A (if it exists) by $\sup A$.

Similarly, a *greatest lower bound* or *infimum* of a bounded below set A is an element b_0 of S such that

(i) b_0 is a lower bound for A, and

[1] A relation is a comparision operation between two elements which evaluates to either *true* or *false*.

(ii) $b_0 \geq b$ for every other lower bound b.

We denote an infimum for A (if it exists) by $\inf A$.

1.1.7 Proposition. *If a supremum (or infimum) of A exists, then it is unique.*

1.1.8 Proposition. *If A and B are subsets of an ordered set S which are bounded above and below, and if $A \subseteq B$, then*

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

1.1.9 Example. Let $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ denote the set of integers, with the usual order. Find some examples of subsets A of $\mathbb{Z}$ such that

(a) A is bounded above and below.

(b) A is bounded above but not below.

(c) A is not bounded above and not bounded below.

Which of these sets have a supremum? Which have an infimum?

1.1.10 Example. Repeat Example 1.1.9 with the set $\mathbb{Q}$ of rational numbers in place of $\mathbb{Z}$. The following Lemma may be of use.

1.1.11 Lemma. *There exists no $q \in \mathbb{Q}$ such that $q^2 = 2$. [Possible hint: write $q = \frac{a}{b}$ in lowest terms and consider the evenness/oddness of a and b.]*

1.1.12 Definition. An ordered set S has the *least upper bound property* if every subset which is bounded above has a supremum. Likewise S has the *greatest lower bound property* if every subset which is bounded below has an infimum.

1.1.13 Example. Does $\mathbb{Z}$ have the least upper bound property? Does $\mathbb{Q}$? Justify your answers with a proof or counterexample.

1.1.14 Theorem. *If S has the least upper bound property, then it has the greatest lower bound property.*

1.2 Fields and ordered fields

Of course the familiar number systems have additional structure. Besides the order, we have addition, subtraction, multiplication and division.

1.2.1 Definition. A *field* is a set $\mathbb{F}$ with two binary operations[2] $+$ and $\cdot$, called *addition* and *multiplication*, respectively, satisfying the following axioms:

(F1) (Associativity of addition) $(a + b) + c = a + (b + c)$ for all a, b, c in $\mathbb{F}$.

(F2) (Additive identity) There exists an element $0 \in \mathbb{F}$ such that $0 + a = a + 0 = a$ for all a.

(F3) (Additive inverses) For each a in $\mathbb{F}$ there exists an element $-a$ such that $(-a) + a = a + (-a) = 0$.

(F4) (Commutativity of addition) $a + b = b + a$ for all a, b in $\mathbb{F}$.

(F5) (Associativity of multiplication) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, c in $\mathbb{F}$.

(F6) (Multiplicative identity) There exists an element $1 \in \mathbb{F}$ such that $1 \cdot a = a \cdot 1 = a$ for all a.

(F7) (Multiplicative inverses) For all $a \neq 0$, there exists an element a^{-1} in $\mathbb{F}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

[2] A binary operation is a function/operation taking in two elements of $\mathbb{F}$ and returning a third element of $\mathbb{F}$.

(F8) (Commutativity of multiplication) $a \cdot b = b \cdot a$ for all a, b in $\mathbb{F}$.

(F9) (Distributivity) $a \cdot (b + c) = a \cdot b + a \cdot c$.

(F10) (Nontriviality) $0 \neq 1$.

It is customary to omit the $\cdot$ when writing multiplication; in other words, we usually just write ab instead of $a \cdot b$. Additionally, we usually denote $a + (-b)$ simply by $a - b$, and we may also use the notation $\frac{1}{a}$ in place of a^{-1}. It is important to note that subtraction $-$ and division $\div$ are not really distinct operations; they are just syntactic shorthand for addition (resp. multiplication) by an additive (resp. multiplicative) inverse.

We also use the usual shorthand a^n in place of $\underbrace{a \cdots a}_{n \text{ times}}$ and na in place of $\underbrace{a + \cdots + a}_{n \text{ times}}$.

Remark. Though we shall be entirely concerned with fields in this course, you may be familiar with various mathematical objects satisfying fewer of the above axioms. A set with a single operation satisfying axioms (F1)–(F3) is a *group* which is said to be *commutative* or *abelian* if (F4) also holds.

A *ring* is a set with two operations satisfying all of the above except (F7), (F8) and (F10). A *commutative ring* satisfies (F8). According to some conventions, a ring need not satisfy (F6), though such "rings without identity" are sometimes cutely referred to as '*rng*'s. If (F7) holds but not (F8), then $\mathbb{F}$ is called a *division ring*.

Axiom (F10) might be considered optional for fields, but if we allow $0 = 1$ then $\mathbb{F}$ must be the one element set $\{0\}$ (you can prove this after you prove Proposition 1.2.4 below), which for various reasons is best not regarded as a field.

1.2.2 Example. Come up with some examples of fields, some with infinitely many and some with finitely many elements. Can you construct a field with exactly two elements? Three?

1.2.3 Proposition. *The following properties of addition and multiplication hold in any field. (That is, they follow from the axioms above.)*

(i) *(Uniqueness of identities) If an element b in $\mathbb{F}$ satisfies $b + a = a$ for some a, then $b = 0$. Likewise if b satisfies $ba = a$ for some $a \neq 0$, then $b = 1$.*

(ii) *(Uniqueness of inverses) If b satisfies $a + b = 0$, then $b = -a$. Likewise, if b satisfies $ba = 1$ then $b = a^{-1}$.*

(iii) *(Cancellation) If $a + c = b + c$ then $a = b$. Likewise if $c \neq 0$ and $ac = bc$, then $a = b$.*

(iv) *(Inverse of an inverse) $-(-a) = a$ and $(a^{-1})^{-1} = a$.*

1.2.4 Proposition. *In any field, the following properties hold.*

(i) $0a = 0$ *for all a.*

(ii) *If $ab = 0$, then either $a = 0$ or $b = 0$. (We say $\mathbb{F}$ "has no divisors of zero".)*

(iii) $(-a)b = a(-b) = -(ab)$ *for all a and b. In particular $-a = (-1)a$.*

(iv) $(-a)(-b) = ab$ *for all a and b.*

1.2.5 Problem. In a field, show that if $b \neq 0$ and $d \neq 0$ then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

1.2.6 Definition. An *ordered field* is a field $\mathbb{F}$ equipped with a total order, so a set with a relation $\leq$ and two operations $+$ and $\cdot$ satisfying axioms (O1)–(O4) and (F1)–(F10), which is additionally required to satisfy the following axioms:

(OF1) (Compatibility of order and addition) If $a \leq b$ then $a + c \leq b + c$ for any c.

(OF2) (Compatibility of order and multiplication) If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

1.2.7 Example. Which examples from Example 1.2.2 are ordered fields? In case there is not an obvious order, is there any order at all satisfying (OF1) and (OF2)?

1.2.8 Proposition. *The following properties always hold in an ordered field.*

(a) *If* $0 \leq a$ *then* $-a \leq 0$.

(b) *If* $0 \leq a$ *and* $0 \leq b$ *then* $0 \leq ab$. *(In fact, this is equivalent to (OF2) and is often used in place of it as the other ordered field axiom).*

(c) *If* $a \leq 0$ *and* $0 \leq b$, *then* $ab \leq 0$.

(d) $0 \leq a^2$ *for any* a. *In particular* $0 < 1$.

(e) *If* $0 < a \leq b$ *then* $0 < b^{-1} \leq a^{-1}$.

In light of Proposition 1.2.4.(ii) *the above identities hold with strict inequality* $<$ *used in place of inequality* $\leq$.

1.2.9 Problem. Let $\mathbb{F}$ be an ordered field and consider the subset $Z \subset \mathbb{F}$ generated by taking 0, 1, $1+1$, $1+1+1$, etc. along with -1, $-1-1$, $-1-1-1$, etc. Show that this set is in bijection with the set of integers $\mathbb{Z}$.

Likewise, let $Q \subset \mathbb{F}$ be the subset generated by taking the multiplicative inverses of the nonzero elements in Z along with their integer multiples. Show that this set is in bijection with $\mathbb{Q}$.

Thus every ordered field contains a copy of $\mathbb{Q}$, which may be regarded as the "smallest" possible ordered field.

1.2.10 Definition. Let $\mathbb{F}$ be an ordered field. The *absolute value* or *magnitude* of a number $a \in \mathbb{F}$ is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

1.2.11 Proposition*. *The absolute value satisfies the following properties. For all* a *and* b *in* $\mathbb{F}$*:*

(i) $|a| \geq 0$.

(ii) $|a| = 0$ *if and only if* $a = 0$.

(iii) $|ab| = |a||b|$.

(iv) *(Triangle inequality)* $|a + b| \leq |a| + |b|$.

(v) *(Reverse triangle inequality)* $\big||a| - |b|\big| \leq |a - b|$.

Remark. Combining (iv) and (v) of the last proposition gives the useful strings of inequalities:

$$|a| - |b| \leq \big||a| - |b|\big| \leq |a + b| \leq |a| + |b|, \quad \text{and} \quad |a| - |b| \leq \big||a| - |b|\big| \leq |a - b| \leq |a| + |b|. \tag{1.1}$$

1.2.12 Definition. The *distance* between numbers a and b in an ordered field $\mathbb{F}$ is the quantity

$$d(a, b) = |a - b|.$$

1.2.13 Proposition*. *The distance satisfies the following properties. For all* a, b, *and* c *in* $\mathbb{F}$*:*

(i) $d(a, b) \geq 0$.

(ii) $d(a, b) = 0$ *if and only if* $a = b$.

(iii) *(Symmetry)* $d(a,b) = d(b,a)$.

(iv) *(Triangle inequality)* $d(a,c) \le d(a,b) + d(b,c)$.

1.2.14 Lemma (Suprema/infima in an ordered field)**.** *Let A be a bounded above subset of an ordered field. Then $u = \sup A$ if and only if*

(i) *$a \le u$ for all $a \in A$ (i.e., u is an upper bound), and*

(ii) *for every $\varepsilon > 0$, there exists $a \in A$ such that $u - \varepsilon < a$ (i.e., $u - \varepsilon$ fails to be an upper bound).*

Similarly, if A is bounded below, then $b = \inf A$ if and only if

(i) *$b \le a$ for all $a \in A$, and*

(ii) *for every $\varepsilon > 0$, there exists $a \in A$ such that $a < b + \varepsilon$.*

1.2.15 Definition ($\pm\infty$ notation)**.** As a notation convention, it is useful to introduce the symbols $+\infty$ and $-\infty$ when speaking of suprema and infima in an ordered field. We write $\sup A = +\infty$ if A is not bounded above, and $\inf A = -\infty$ if A is not bounded below. With these conventions $\sup A$ and $\inf A$ are always defined for a nonempty set A.

A more formal way to do this is to embed $\mathbb{F}$ into a larger ordered set $\overline{\mathbb{F}} = \mathbb{F} \cup \{+\infty, -\infty\}$ with the order defined so that $-\infty < a < +\infty$ for all $a \in \mathbb{F}$. Note that $\overline{\mathbb{F}}$ is *not* a field, though we may observe the following notation conventions: if $a > 0 \in \mathbb{F}$, then

$$
\begin{aligned}
&a + (+\infty) = +\infty, \quad a + (-\infty) = -\infty, \quad a(+\infty) = +\infty, \quad a(-\infty) = -\infty, \\
&(-a)(+\infty) = -\infty, \quad (-a)(-\infty) = +\infty, \quad \frac{\pm a}{\pm\infty} = 0.
\end{aligned}
$$

Expressions such as $+\infty - \infty$ and $\pm\infty / \pm\infty$ are not defined.

1.3 Completeness and the real number field

1.3.1 Definition. An ordered field $\mathbb{F}$ is *complete* if it satisfies the least upper bound property (c.f. Definition 1.1.12), in other words, if for every bounded above subset $A \subset \mathbb{F}$, the supremum (least upper bound) $\sup A$ exists in $\mathbb{F}$.

1.3.2 Theorem† (Characterization/definition of $\mathbb{R}$)**.** *There exists a unique*[3] *complete ordered field called the* real numbers *and denoted by* $\mathbb{R}$.

Remark. We omit the proof of Theorem 1.3.2 for now; we may come back to it later on. However, it is worth mentioning one construction which is possible at this point: define a *Dedekind cut* to be a subset $A \subset \mathbb{Q}$ of the rationals with the properties that

(i) A is neither empty nor all of $\mathbb{Q}$,

(ii) if $q \in A$ and $p < q$, then $p \in A$,

(iii) if $q \in A$ then $q < r$ for some $r \in A$.

[3] Here "uniqueness" means the following: given two complete ordered fields F_1 and F_2, there exists an *isomorphism* (a bijection compatible with the order and field operations) $\phi : F_1 \longrightarrow F_2$. Moreover ϕ is unique. Using ϕ we can regard F_1 and F_2 as being "the same" field.

In other words, a cut is essentially a half infinite open interval in $\mathbb{Q}$; take as an example $\{q \in \mathbb{Q} : q < 2\} = (-\infty, 2)$. It is tempting to want to write $\left\{q \in \mathbb{Q} : q < \sqrt{2}\right\}$ as another example, but this is ill-specified since we do not have such a number as $\sqrt{2}$ at this point. The equivalent set may be specified as $\left\{q \in \mathbb{Q} : q < 0 \text{ or } q^2 < 2\right\}$. The idea here is that real numbers are represented by the "upper endpoints" of the cuts, though since these are not well-defined, the whole cut stands in as a replacement.

It is then possible to define an order, addition, and multiplication on the set of Dedekind cuts (order and addition are straightforward; multiplication is a little tricky) and verify that they satisfy all the axioms of an ordered field along with completeness, with subfield $\mathbb{Q}$ identified with those cuts of the form $\{q \in \mathbb{Q} : q < p\}$ for $p \in \mathbb{Q}$.

1.3.3 Definition. An ordered field $\mathbb{F}$ is *Archimedean* if for every $a \in \mathbb{F}$, there exists an integer[4] N such that $a \leq N$.

1.3.4 Example*. Show that $\mathbb{Q}$ is Archimedean.

1.3.5 Example (Research Allowed)**.** Find an example of a non-Archimedean field.

1.3.6 Theorem. *As a complete ordered field, $\mathbb{R}$ is Archimedean.*

1.3.7 Proposition. *If $0 < a$ in an Archimedean field such as $\mathbb{R}$, then there exists a positive integer N such that*

$$0 < \frac{1}{N} < a.$$

Remark. The Archimedean property says that a field has no "infinitely large" elements, and via Proposition 1.3.7, it implies that there are no "infinitely small" elements. The next result gives a technically useful if strange seeming characterization of the zero element.

1.3.8 Corollary. *In an Archimedean field, if $0 \leq a$ and $a < \varepsilon$ for every $0 < \varepsilon$, then $a = 0$.*

1.3.9 Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *Let a and b be real numbers with $a < b$. Then there exists a rational number q such that*

$$a < q < b.$$

We say $\mathbb{Q}$ is dense *in $\mathbb{R}$.*

Remark. This may be a surprising result, especially when juxtaposed with the following one. Recall that an infinite set is said to be *countable* if it is in bijection with the set $\mathbb{N} = \{1, 2, 3, \dots\}$ of natural numbers.

1.3.10 Theorem.

(i) $\mathbb{Q}$ *is countable.*

(ii) $\mathbb{R}$ *is uncountable.*

One more result at this point will be useful later on, though the proof is rather technical and tricky, so you may go ahead and take it as given rather than trying to prove it.

1.3.11 Theorem† (Positive nth roots)**.** *For every $y > 0$ in $\mathbb{R}$ and $n \in \mathbb{N}$, there exists a unique $x > 0$ in $\mathbb{R}$ such that $x^n = y$.*

The proof is obtained from the following two results, the first of which is more or less straightforward while the second is the tricky one.

1.3.12 Lemma. *For fixed $y > 0$ and $n \in \mathbb{N}$, the set $E = \{t \in \mathbb{R} : 0 < t,\ t^n < y\}$ is nonempty and bounded above.*

1.3.13 Lemma†. *The element $x = \sup E$ satisfies $x^n = y$.*

[4] Here we are identifying a subset of $\mathbb{F}$ with the integers as in Problem 1.2.9.

1.4 Sequences of real numbers

While least upper bounds give an expedient way to express the completeness of $\mathbb{R}$, *sequences* play a much more ubiquitous role in analysis.

1.4.1 Definition. A *sequence* of real numbers is a function[5] from $\mathbb{N}$ into $\mathbb{R}$. As a matter of notation, if $x : \mathbb{N} \longrightarrow \mathbb{R}$ is a sequence, we prefer to write x_n instead of $x(n)$, and denote the sequence by

$$(x_1, x_2, x_3, \ldots), \quad \text{or} \quad (x_n)_{n=1}^{\infty}, \quad \text{or just} \quad (x_n).$$

It is permissible and often convenient to index a sequence starting from 0 instead of 1, or starting from a number greater than 1.

1.4.2 Definition. A sequence (x_n) in $\mathbb{R}$ is said to be

(i) *bounded* if there exists some $B > 0$ such that $|x_n| \leq B$ for all n.

(ii) *increasing* if $x_n \leq x_{n+1}$ for all n. It is *strictly increasing* if $x_n < x_{n+1}$ for all n.

(iii) *decreasing* if $x_n \geq x_{n+1}$ for all n. It is *strictly decreasing* if $x_n > x_{n+1}$ for all n.

(iv) *monotone* if it is either increasing or decreasing.

(v) *convergent* if there exists some $L \in \mathbb{R}$ with the following property: for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that
$$\text{for all } n \geq N, \quad |x_n - L| < \varepsilon.$$
Equivalently, for every $\varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon) \subset \mathbb{R}$ contains all but finitely many terms of the sequence. In this case we say L is the *limit* of the sequence (x_n) and write $L = \lim_{n\to\infty} x_n$ or $x_n \to L$.

1.4.3 Exposition*. Explain in plain English what is meant by the limit of a sequence. Address the order of the quantifiers (i.e., "for all" or "there exists"): if the order or type of the quantifiers is changed, why is this a bad definition of limit?

1.4.4 Proposition. *The limit of a sequence, if it exists, is unique.*

1.4.5 Example*.

(i) Show that the constant sequence $x_n = c$ for all n converges and $\lim x_n = c$.

(ii) Show that $x_n = \frac{1+3n}{1+5n}$ has limit $\frac{3}{5}$.

1.4.6 Example.

(i) Show that $x_n = \frac{1}{n} \to 0$.

(ii) If $0 \leq p < 1$, show that $x_n = p^n \to 0$. [Hint: such p can be written as $p = \frac{1}{1+a}$ for $a > 0$. The binomial estimate $(1+a)^n \leq 1 + na$ for $a \geq 0$ is also useful here.]

1.4.7 Proposition. *If a sequence converges, then it is bounded.*

1.4.8 Theorem. *In $\mathbb{R}$, every bounded monotone sequence converges.*

Remark. The previous property of $\mathbb{R}$ is referred to as the *monotone sequence property.* In fact, it is equivalent to completeness: it can be proved that an ordered field in which every bounded monotone sequence converges has the least upper bound property.

[5] Recall that a *function* $f : A \longrightarrow B$ is an assignment to every element a of the *domain* A an element $b = f(a)$ of the *target* B.