**Calc III: Workshop 5 Solutions, Fall 2018**

**Problem 1.** The temperature at a point $(x, y, z)$ is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2}$$

where $T$ is measured in Celcius and $x, y, z$ are in meters.

(a) Find the rate of change of the temperature at the point $P(2, -1, 2)$ in the direction from $P$ toward the point $Q(3, -3, 3)$.

(b) In what direction does the temperature increase fastest at $P$?

(c) Find the maximum rate of increase at $P$.

*Solution.*

(a) We want to compute a directional derivative. The direction is given by the difference vector $\mathbf{u} = \overrightarrow{PQ} = \langle 1, -2, 1\rangle$. Next we need to normalize this vector to get the unit vector

$$\mathbf{u} = \tfrac{1}{\sqrt{6}}\langle 1, -2, 1\rangle .$$

The gradient of $T$ is given by

$$\nabla T(x, y, z) = 200e^{-x^2-3y^2-9z^2}\langle -2x, -6y, -18z\rangle$$

The rate of change is given by the directional derivative

$$D_{\mathbf{u}}T(2, -1, 2) = \nabla T(2, -1, 2) \cdot \mathbf{v} = 200e^{-(4+3+36)}\langle -4, 6, -36\rangle \cdot \tfrac{1}{\sqrt{6}}\langle 1, -2, 1\rangle$$

$$= \frac{200e^{-43}}{\sqrt{6}}(-4 - 12 - 36) = -\frac{200(52)e^{-43}}{\sqrt{6}} \approx 9 \times 10^{-16}{}^\circ C/m.$$

(b) The rate of increase is fastest in the direction of the gradient, which is the unit vector

$$\frac{\nabla T(2, -1, 2)}{|\nabla T(2, -1, 2)|} = \frac{1}{\sqrt{337}}\langle -2, 3, -18\rangle .$$

(c) The rate of increase in this direction is the magnitude of the gradient:

$$|\nabla T(2, -1, 2)| = \left|200e^{-43}\langle -4, 6, -36\rangle\right| = 200e^{-43}2\sqrt{337} \approx 1.5 \times 10^{-15}{}^\circ C/m.$$

□

**Problem 2.** Show that every plane tangent to the cone $z^2 = x^2 + y^2$ passes through the origin.

*Solution.* The cone is a level surface of the function $g(x, y, z) = x^2 + y^2 - z^2$. The gradient of $g$ is

$$\nabla g = \langle 2x, 2y, -2z\rangle .$$

This is always normal to the surface, so the tangent plane to the cone through the point $(x_0, y_0, z_0)$ is given by

$$\langle 2x_0, 2y_0, -2z_0\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle = 0 \quad\Longleftrightarrow\quad x_0(x-x_0)+y_0(y-y_0)-z_0(z-z_0) = 0$$

To see if this plane contains the origin, we plug in $(x, y, z) = (0, 0, 0)$ and see if it solves the equation:

$$x_0(0 - x_0) + y_0(0 - y_0) - z_0(0 - z_0) = x_0^2 + y_0^2 - z_0^2 = 0$$

which holds since $(x_0, y_0, z_0)$ is a point on the cone! □

**Problem 3.** At what point on the ellipsoid $x^2 + y^2 + 2z^2 = 1$ is the tangent plane parallel to the plane $x + 2y + z = 1$?

*Solution.* Equivalently, we want to know at what point a normal vector to the ellipsoid is parallel to the normal vector of the plane, which is the coefficient vector $\langle 1, 2, 1\rangle$. The normal to the ellipsoid is given by the gradient of the function $f(x, y, z) = x^2 + y^2 + 2z^2$:

$$\nabla f(x, y, z) = \langle 2x, 2y, 4z\rangle .$$

So we want to know is for what point $x, y, z$ is there a solution to the equations

$$\nabla f(x, y, z) = \langle 2x, 2y, 4z\rangle = \lambda \langle 1, 2, 1\rangle , \quad x^2 + y^2 + 2z^2 = 1$$

for some $\lambda$, which boils down to the system of equations

$$\begin{aligned} 2x &= \lambda \\ y &= \lambda \\ 4z &= \lambda \\ x^2 + y^2 + 2z^2 &= 1. \end{aligned}$$

Solving the first three equations for $x$, $y$, and $z$ and plugging them into the fourth gives

$$\lambda^2(\tfrac{1}{4}1 + \tfrac{1}{16}) = 1 \implies \lambda = \pm\frac{4}{\sqrt{21}}.$$

This gives the pair of points $\pm(2/\sqrt{21}, 4/\sqrt{21}, 1/\sqrt{21})$, at which the tangent plane to the ellipsoid is parallel to $x + 2y + z = 1$. □

**Problem 4.** Find all local maxima, minima, and saddle points of the function $f(x, y) = 2 - x^4 + 2x^2 - y^2$.

*Solution.* The critical points are determined from the equation

$$\langle 0, 0\rangle = \nabla f(x, y) = \left\langle -4x^3 + 4x, -2y\right\rangle$$

which amounts to the pair of equations

$$x(x^2 - 1) = 0 \quad y = 0.$$

From the second equation $y$ must vanish, and from the first equation we get $x = 0$, $1$, or $-1$. Thus there are three critical points

$$(0, 0), \quad (1, 0), \quad \text{and} \quad (-1, 0).$$

The discriminant is

$$Df = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} -12x^2 + 4 & 0 \\ 0 & -2 \end{vmatrix} = (-12x^2 + 4)(-2).$$

We have

$$Df(0, 0) = (4)(-2) < 0, \quad Df(\pm 1, 0) = (-12 + 4)(-2) > 0,$$

moreover $f_x x(\pm 1, 0) = -12 + 4 < 0$, so we conclude that $(0, 0)$ is a saddle point, and $(\pm 1, 0)$ are local maxima. □

**Problem 5.** Find the shortest distance from the point $(2, 0, -3)$ to the plane $x + y + z = 1$. [Hint: instead of minimizing the function $\sqrt{(x-2)^2 + (y-0)^2 + (z+3)^2}$, it is computationally much easier to minimize its square.]

*Solution.* We want to minimize the function $(x-2)^2+y^2+(z-3)^2$ subject to the constraint $x+y+z=1$. Solving for $y$ (say) in terms of $x$ and $z$ reduces the problem to minimizing the function

$$f(x,z)=(x-2)^2+(1-x-z)^2+(z-3)^2.$$

The critical points are given by the solutions to the pair of equations

$$f_x=2(x-2)-2(1-x-z)=0, \quad \text{and} \quad f_y=-2(1-x-z)+2(z-3)=0$$

which simply to the system of two equations

$$2x+z+1=0 \quad x+2z+2=0.$$

Subtracting twice the second equation from the first gives $x=0$, and from the second equation we then get $z=-1$. Plugging back in for $y=1-x-z$ gives $y=2$. Thus the point on the plane which minimizes the distance is the point $(0,2,-1)$, and the minimizing distance is

$$\sqrt{(2-0)^2+(0-2)^2+(-3+1)^2}=2\sqrt{3}.$$

□

**Problem 6.** The base of an aquarium of given volume $V$ is made of slate and its four sides are made of glass. If slate costs five times as much as glass (per unit area), find the dimensions of the aquarium that minimize the cost of the materials.

*Solution.* We want to minimize the cost

$$f(x,y,z)=2xz+2yz+5xy$$

subject to the constraint $xyz=V$. Solving for $z=V/(xy)$ and plugging this into $f$ reduces the problem to minimizing the function

$$g(x,y)=f(x,y,V/xy)=\frac{2V}{y}+\frac{2V}{x}+5xy$$

Setting $\nabla g=\mathbf{0}$ gives the equations

$$5y=\frac{2V}{x^2}, \quad \text{and} \quad 5x=\frac{2V}{y^2}.$$

plugging the first into the second results in the equation

$$x=\frac{5x^4}{2V}, \quad \text{or} \quad x(\tfrac{5}{2V}x^3-1)=0.$$

The solutions to this equation are $x=0$ (which is not acceptable since it does not satisfy the constraint $V=xyz$), and $x=\pm\,(2V/5)^{1/3}$, the negative one of which we do not accept since the dimensions must be positive. Thus $x=(2V/5)^{1/3}$ and plugging this back into $y$ gives

$$y=(2V/5)(2V/5)^{-2/3}=(2V/5)^{1/3}$$

also. The cost minimizing dimensions are

$$x=y=(2V/5)^{1/3}, \quad z=V^{1/3}(5/2)^{2/3}.$$

□