Calc III: Workshop 9 Solutions, Fall 2017

Problem 1. Let

$$\mathbf{F}(x,y,z) = (2xy+1)z\mathbf{i} + x^2z\mathbf{j} + (x^2y + x + 2z)\mathbf{k}.$$

Compute the line integral $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ where C is the line segment from $(0,0,0)$ to $(1,2,3)$.

Solution. We can parameterize C by $\gamma(t) = (0,0,0) + t(1-0, 2-0, 3-0) = (t, 2t, 3t)$, where $0 \le t \le 1$. Then $\gamma'(t) = (1,2,3)$ adn

$$\mathbf{F}\big(\gamma(t)\big) = \big((2(t)(2t)+1)3t,\; t^2(3t),\; (t^2(2t) + t + 2(3t))\big) = \big(12t^3 + 3t,\; 3t^3,\; 2t^3 + 7t\big)$$

so

$$\begin{aligned}
\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_0^1 \mathbf{F}(\gamma(t))\cdot\gamma'(t)\,dt \\
&= \int_0^1 12t^3 + 3t + 2(3t^3) + 3(2t^3 + 7t)\,dt \\
&= \int_0^1 24t^3 + 24t\,dt \\
&= \frac{24}{4} + \frac{24}{2} = 18.
\end{aligned}$$

□

Problem 2. In fact the vector field of problem 1 is conservative. Find a potential function (i.e., $f(x,y,z)$ such that $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$) and re-evaluate the line integral using the FTCLI.

Solution. We want to find $f(x,y,z)$ such that

(i) $f_x(x,y,z) = (2xy+1)z$

(ii) $f_y(x,y,z) = x^2z$

(iii) $f_z(x,y,z) = x^2y + x + 2z$

Integrating the first equation in x gives

$$f(x,y,z) = x^2yz + xz + c(y,z)$$

where $c(y,z)$ is an arbitrary function of y and z yet to be determined. Plugging this into the second equation, we obtain

$$f_y(x,y,z) = x^2z + c_y(y,z) = x^2z$$

so $c_y(y,z) = 0$, which means that $c(y,z) = c(z)$ is actually just a pure function of z. Plugging this into the third equation, we obtain

$$f_z(x,y,z) = x^2y + x + c'(z) = x^2y + x + 2z \qquad \Longrightarrow \qquad c'(z) = 2z$$

so $c(z) = z^2$. Thus $f(x,y,z) = x^2yz + xz + z^2$ is a potential function.
Using the fundamental theorem,

$$\int_C \nabla f\cdot\mathbf{T}\,ds = f(1,2,3) - f(0,0,0) = 18.$$

□

Problem 3. Let

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$$

be a vector field defined on $\mathbb{R}^2 \setminus \{(0,0)\}$, i.e., the whole plane minus the origin, where $\mathbf{F}(x,y)$ is undefined. Let C be the unit circle, oriented counterclockwise and compute

$$\oint_C \mathbf{F}(x,y) \cdot \mathbf{T}\, ds$$

Solution. We parameterize C by $\gamma(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$. Then $\gamma'(t) = (-\sin t, \cos t)$ and

$$\mathbf{F}(\gamma(t)) = -\frac{\sin t}{\cos^2 t + \sin^2 t}\mathbf{i} + \frac{\cos t}{\cos^2 t + \sin^2 t}\mathbf{j} = \big(-\sin t,\ \cos t\big).$$

Thus

$$\begin{aligned} \oint_C \mathbf{F} \cdot \mathbf{T}\, ds &= \int_0^{2\pi} \underbrace{\big(-\sin t, \cos t\big)}_{\mathbf{F}(\gamma(t))} \cdot \underbrace{\big(-\sin t, \cos t\big)}_{\gamma'(t)}\, dt \\ &= \int_0^{2\pi} \sin^2 t + \cos^2 t\, dt = 2\pi. \end{aligned}$$

□

Problem 4. Show that the vector field from the previous problem satisfies $Q_x - P_y = 0$ on $\mathbb{R}^2 \setminus \{(0,0)\}$. Is $\mathbf{F}$ conservative? Is there a potential function?

Solution. We have

$$Q_x = \frac{\partial}{\partial x}\frac{x}{x^2+y^2} = \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2}, \quad P_y = \frac{\partial}{\partial y}\frac{-y}{x^2+y^2} = \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2},$$

so

$$Q_x - P_y = \frac{x^2+y^2-2x^2+(x^2+y^2)-2y^2}{(x^2+y^2)^2} = \frac{2(x^2+y^2)-2(x^2+y^2)}{(x^2+y^2)^2} = 0.$$

However, from the previous problem the line integral over the closed curve consisting of the unit circle is nonvanishing, so $\mathbf{F}$ can't be conservative. The test in which $Q_x - P_y = 0$ implies that a vector field is conservative only applies if $Q_x - P_y$ vanishes on all of $\mathbb{R}^2$ (here it is not defined at $(0,0)$), or more generally if it vanishes on a *simply connected region*, meaning a region having no "holes".

In fact, $\mathbf{F}(x,y) = \nabla \tan^{-1}(y/x)$, where $\tan^{-1}(y/x) = \theta$. However, this does not count as a potential function since θ is not a valid differentiable function as it is discontinuous across the positive x axis. □