Calc III: Workshop 7 Solutions, Fall 2017

Problem 1.

(a) Find the appropriate description of the triangular region R in polar coordinates, where R has vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$. (There will be variable limits somewhere!)

(b) Compute the integral

$$\iint_R \frac{y}{\sqrt{x^2+y^2}}\, dA$$

using polar coordinates. (You can check your answer by computing the integral in cartesian coordinates as well.)

Solution.

(a) The three boundaries of the triangle are given in cartesian coordinates by $x = 0$, $y = x$ and $x = 1$. The first and second of these are the lines $\theta = 0$ and $\theta = \pi/4$, respectively, while $x = 1$ becomes the equation $r \cos\theta = 1$, or $r = \sec\theta$. The innermost radius is $r = 0$. So

$$R = \{(r, \theta) : 0 \le r \le \sec\theta,\ 0 \le \theta \le \pi/4\}.$$

(b) To compute the integral, we change to polar coordinates, where the integrand becomes $f\big(x(r,\theta), y(r,\theta)\big) = \frac{r\sin\theta}{r} = \sin\theta$, and $dV = r\, dr\, d\theta$, so

$$\begin{aligned}
\iint_R \frac{y}{\sqrt{x^2+y^2}}\, dA &= \int_0^{\pi/4} \int_0^{\sec\theta} r \sin\theta\, dr\, d\theta \\
&= \frac{1}{2} \int_0^{\pi/4} r^2 \sin\theta \Big|_{r=0}^{\sec\theta} d\theta \\
&= \frac{1}{2} \int_0^{\pi/4} \frac{\sin\theta}{\cos^2\theta}\, d\theta \\
&= \frac{1}{2} \frac{1}{\cos\theta} \Big|_{\theta=0}^{\pi/4} \\
&= \frac{1}{2}(\sqrt{2} - 1).
\end{aligned}$$

□

Problem 2. Find the volume of the solid region bounded by the paraboloid $z = x^2 + y^2$ and the cone $z^2 = 4(x^2 + y^2)$.

Solution. The region is best expressed in cylindrical coordinates, with lower boundary $z = r^2$ and upper boundary $z = 2r$, lying over the disk $0 \le r \le 2$ in the xy-plane (the two surfaces intersect where $z = r^2 = 2r$, which has solutions $r = 0$ and $r = 2$). Thus

$$\begin{aligned}
\text{Vol} = \iiint_E 1\, dV &= \int_0^{2\pi} \int_0^2 \int_{r^2}^{2r} r\, dz\, dr\, d\theta \\
&= 2\pi \int_0^2 r(2r - r^2)\, dr \\
&= 2\pi \int_0^2 2r^2 - r^3\, dr \\
&= \frac{8\pi}{3}.
\end{aligned}$$

□

Problem 3. Compute the volume of a sphere of radius R using *cylindrical* (instead of spherical) coordinates (r, θ, z).

Solution. By writing the equation for the sphere $x^2 + y^2 + z^2 = R^2$ as $r^2 + z^2 = R^2$ and solving for z, we can write the sphere of radius R in cylindrical coordinates as the region where $-\sqrt{R^2 - r^2} \leq z \leq \sqrt{R^2 - r^2}$, $0 \leq r \leq R$ and $0 \leq \theta \leq 2\pi$. Thus the volume of the sphere is given by

$$\begin{aligned}
\text{Vol} = \iiint_S 1\, dV &= \int_0^{2\pi} \int_0^R \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r\, dz\, dr\, d\theta \\
&= 2\pi \int_0^R 2r\sqrt{R^2 - r^2}\, dr \\
&= -2\pi \frac{2}{3}(R^2 - r^2)^{3/2}\Big|_{r=0}^{R} \\
&= \frac{4\pi}{3} R^3.
\end{aligned}$$

□

Problem 4. Find the volume inside the elliptic cylinder $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $0 \leq z \leq 1$, by using the change of variables $x = ar\cos\theta$, $y = br\sin\theta$, $z = z$.

Solution. In the new variables, the cylinder is given by the limits $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$, and $0 \leq z \leq 2$. Computing the Jacobian determinant, we have

$$\begin{aligned}
\det \frac{\partial(x, y, z)}{\partial(r, \theta, z)} &= \det \begin{pmatrix} a\cos\theta & -ar\sin\theta & 0 \\ b\sin\theta & br\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= abr\cos^2\theta + abr\sin^2\theta \\
&= abr.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{Vol} = \iiint_C dV &= \int_0^{2\pi} \int_0^1 \int_0^2 abr\, dz\, dr\, d\theta \\
&= ab \int_0^{2\pi} d\theta \int_0^1 dr \int_0^2 dz \\
&= ab(2\pi)(1/2)(2) \\
&= 2\pi ab.
\end{aligned}$$

□

Problem 5. Find the volume inside the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, using a change of variables $x = ax'$, $y = by'$, $z = cz'$. (You may use the fact, proved in class, that the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.)

Solution. In the new coordinates (x', y', z'), the ellipsoid becomes the unit sphere $(x')^2 + (y')^2 + (z')^2 = 1$. The Jacobian determinant is

$$\det \frac{\partial(x, y, z)}{\partial(x', y', z')} = \det \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = abc.$$

Thus the volume $\iiint_E 1\, dV$ becomes $\iiint_S abc\, dx'\, dy'\, dz' = abc \iiint_S dx'\, dy'\, dz'$ where S is the unit sphere, or $abc(4\pi/3)$. □