Calc III Fall 2016: Exam 1 Solutions

Problem 1. Find an equation for the tangent line of the parameterized curve

$$\mathbf{r}(t) = \left\langle \cos^2(t),\, 2\sin(t),\, t^3 \right\rangle$$

at the point $(1, 0, \pi^3)$.

Solution. Note that $(1, 0, \pi^3) = \mathbf{r}(t)$ for $t = \pi$. We compute the derivative

$$\mathbf{r}'(t) = \left\langle -2\cos(t)\sin(t), 2\cos(t), 3t^2 \right\rangle$$

and evaluate it at $t = \pi$ to get the tangent vector

$$\mathbf{r}'(\pi) = \left\langle 0, -2, 3\pi^2 \right\rangle.$$

Then the tangent line is given as a parameterized line by

$$\mathbf{l}(s) = (1, 0, \pi^3) + s(0, -2, 3\pi^2).$$

Alternatively, we could write the line in non-parametric form by

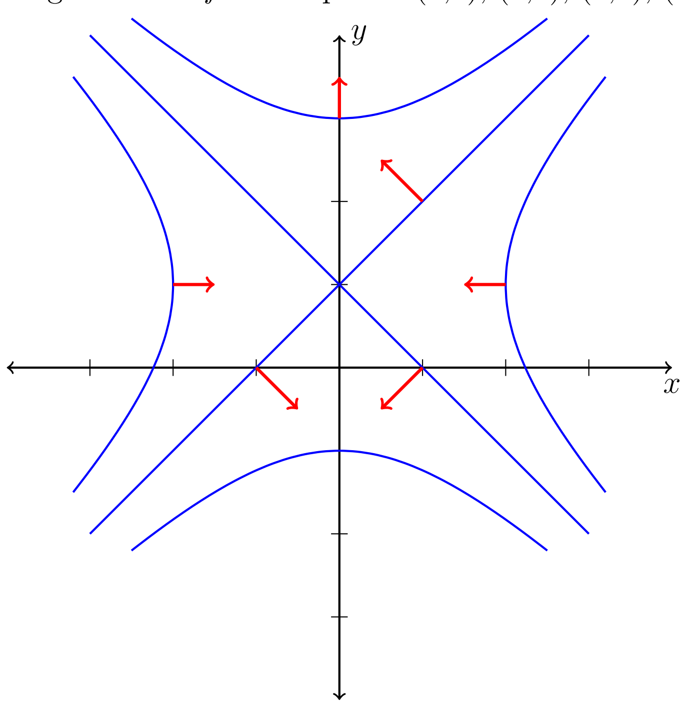
$$-\frac{y}{2} = \frac{z - \pi^3}{3\pi^2}, \quad (x - 1) = 0.$$

□

Problem 2. For the function $f(x, y) = (y - 1)^2 - x^2 + 1$, plot the following:

(a) the level curves where $f = -3, 1$, and 5. (Hint: think about what the function would be if $(y - 1)$ were replaced by y and the $+1$ at the end was removed.)

(b) the direction of the gradient of f at the points (1,0), (2,1), (1,2), (0,3), (-2,1) and (-1,0).



Problem 3. For the function

$$f(x, y, z) = x^2 + xz + y^3 - 2xy + z$$

(a) Compute the linear approximation, $L(x, y, z)$, of f at the point $(1, 1, 0)$.
(b) Use part (a) to approximate the value of $f(1.01, 0.08, 0.02)$. (Do **not** compute the exact value.)

Solution.
(a) The partial derivatives are

$$f_x = 2x + z - 2y, \quad f_y = 3y^2 - 2x, \quad f_z = x + 1$$

which, evaluated at $(1, 1, 0)$ give

$$f_x(1, 1, 0) = 0, \quad f_y(1, 1, 0) = 1, \quad f_z(1, 1, 0) = 2,$$

and $f(1, 1, 0) = 0$. The linear approximation is

$$L(x, y, z) = f(1, 1, 0) + f_x(1, 1, 0)(x - 1) + f_y(1, 1, 0)(y - 1) + f_z(1, 1, 0)(z - 0) = y - 1 + 2z.$$

(b) The linear approximation gives

$$f(1, 01, 0.08, 0.02) \approx L(1.01, 0.08, 0.02) = -0.92 + 2(0.02) = -0.88$$

$\square$

Problem 4. Let S be the surface defined by the equation

$$\ln(x^2y^3) + z = 3.$$

Find an equation for the tangent plane to S at the point $(1, 1, 3)$.

Solution. We may view this as a level surface $f(x, y, z) = 3$ where $f(x, y, z) = \ln(x^2y^3) + z$, with gradient

$$\nabla f(x, y, z) = \left(2x/(x^2y^3), 3y^2/x^2y^3, 1\right) = \left(2/xy^3, 3/x^2y, 1\right).$$

Evaluated at $\mathbf{p} = (x, y, z) = (1, 1, 3) = \mathbf{p}_0$, this gives the vector

$$\nabla f(1, 1, 3) = (2, 3, 1),$$

which is normal to S. Then the tangent plane is given by

$$0 = \nabla f(1, 1, 3) \cdot (\mathbf{p} - \mathbf{p}_0) = (2, 3, 1) \cdot (x - 1, y - 1, z - 3) = 2(x - 1) + 3(y - 1) + 1(z - 3).$$

Alternatively, we can view this as a graph of $z = g(x, y)$, where $g(x, y) = 3 - \ln(x^2y^3)$, with $g(1, 1) = 3$, $g_x(1, 1) = -2$, $g_y(1, 1) = -3$. Then the tangent plane is the graph of the linear approximation to $g(x, y)$ at $(1, 1)$:

$$z = g(1, 1) + g_x(1, 1)(x - 1) + g_y(1, 1)(y - 1) = 3 - 2(x - 1) - 3(y - 1).$$

$\square$

Problem 5. Suppose

$$T(x, y, z) = 100e^{-(x^2+y^2)/100} - z$$

represents the temperature in degrees Fahrenheit at a point (x, y, z) in space above the x-y plane, with (x, y, z) measured in fathoms.
(a) What is the rate of increase/decrease of temperature at the location $(10, 0, 3)$ in the direction $\mathbf{v} = \frac{1}{5}(3\mathbf{j} + 4\mathbf{k})$?
(b) In what direction is the temperature increasing fastest at the location $(10, 0, 3)$, and what is the rate of increase in this direction?

Solution.

(a) The gradient of T is

$$\nabla T(x,y,z) = \left\langle -2xe^{-(x^2+y^2)/100}, -2ye^{-(x^2+y^2)/100}, -1 \right\rangle,$$

which evaluates at $(10,0,3)$ to

$$\nabla T(10,0,3) = \left\langle -20e^{-1}, 0, -1 \right\rangle.$$

To compute the directional derivative along $\mathbf{v}$ (which is already a unit vector), we take the dot product

$$\nabla T(10,0,3)\cdot \mathbf{v} = \left\langle -20e^{-1}, 0, -1 \right\rangle \cdot \langle 0, 3/5, 4/5\rangle = -4/5 \ ^\circ \text{ F/fathom}.$$

(b) This is the direction of maximum increase, and the rate of change in that direction is given by

$$|\nabla T(10,0,3)| = \sqrt{400e^{-2}+1} \ ^\circ \text{ F/fathom}.$$

□

Problem 6. Find and classify (as local minima, maxima, saddle points, etc) the critical points of the function $f(x,y) = 2x^2y - 2x^2 - y^2$.

Solution. Solving $\nabla f(x,y) = \mathbf{0}$ gives the system of equations

$$(4xy - 4x, 2x^2 - 2y) = (0,0) \iff \begin{cases} x(y-1) = 0 \\ x^2 = y \end{cases}.$$

Plugging the second equation into the first, we have $x(x^2-1) = 0$ so $x = 0, \pm 1$. Plugging these values into the second equation gives the three critical points

$$(0,0), \quad (-1,1), \quad (1,1).$$

Next we compute the discriminant

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 4(y-1)(-2) - (4x)^2 = -8(y-1) - 16x^2$$

Evaluated at the critical points, we have

$$\begin{aligned} D(0,0) = 8, \quad f_{xx}(0,0) = -4 &\implies (0,0) \text{ is a maximum,} \\ D(-1,1) = -16, &\implies (-1,1) \text{ is a saddle,} \\ D(1,1) = -16, &\implies (1,1) \text{ is a saddle.} \end{aligned}$$

□