Real Analysis, Fall 2017

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Chapter 1

The real number line

1.1 Ordered Sets

One basic property of many number systems (natural numbers, integers, rationals, etc) is that they are *ordered*, so we say that "3 is greater than 2", and so on.

1.1.1 Definition. A *total order* on a set S is a relation[1] $\leq$ satisfying the following axioms:

(O1) (Reflexivity) For every element a, it always holds that $a \leq a$.

(O2) (Antisymmetry) If $a \leq b$ and $b \leq a$, then it must be that $a = b$.

(O3) (Transitivity) If $a \leq b$ and $b \leq c$, then it holds that $a \leq c$.

(O4) (Totality) For every pair of elements a and b, either $a \leq b$ or $b \leq a$.

We say S is an *ordered set*.

1.1.2 Example. Find some examples of ordered sets.

1.1.3 Example. Find an example of a *partially ordered* set—a set with a relation satisfying axioms (O1)–(O3) but not (O4).

1.1.4 Problem. Suppose S is an ordered set. Formulate a reasonable definition of strict inequality ($a < b$) in terms of the order relation $\leq$. Then write down a definition equivalent to Definition 1.1.1 using strict inequality as the primitive relation; that is, write down a set of axioms that $<$ should satisfy, in terms of which $\leq$ (suitably defined in terms of $<$) has properties (O1)–(O4).

1.1.5 Definition. Let S be an ordered set, and $A \subseteq S$ a subset. An *upper bound* for A is an element $u \in S$ such that $a \leq u$ for every $a \in A$. If such an element exists, we say A is *bounded above*.

Similarly, a *lower bound* for A is an element $l \in S$ such that $l \leq a$ for every $a \in A$. If such a lower bound exists, we say A is *bounded below*.

1.1.6 Definition. A *least upper bound* or *supremum* of a bounded above set A is an element u_0 of S such that

(i) u_0 is an upper bound for A, and

(ii) $u_0 \leq u$ for every other upper bound u.

We denote a supremum for A (if it exists) by $\sup A$.

Similarly, a *greatest lower bound* or *infimum* of a bounded below set A is an element b_0 of S such that

(i) b_0 is a lower bound for A, and

[1] A relation is a comparision operation between two elements which evaluates to either *true* or *false*.

(ii) $b_0 \geq b$ for every other lower bound b.

We denote an infimum for A (if it exists) by $\inf A$.

1.1.7 Proposition. *If a supremum (or infimum) of A exists, then it is unique.*

1.1.8 Proposition. *If A and B are subsets of an ordered set S which both have a supremum and an infimum and satisfy $A \subseteq B$, then*

$$\inf B \leq \inf A \leq \sup A \leq \sup B. \tag{1.1}$$

1.1.9 Example. Let $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ denote the set of integers, with the usual order. Find some examples of subsets A of $\mathbb{Z}$ such that

(i) A is bounded above and below.

(ii) A is bounded above but not below.

(iii) A is not bounded above and not bounded below.

Which of these sets have a supremum? Which have an infimum?

1.1.10 Example. Repeat Example 1.1.9 with the set $\mathbb{Q}$ of rational numbers in place of $\mathbb{Z}$. The following Lemma may be of use.

1.1.11 Lemma. *There exists no $q \in \mathbb{Q}$ such that $q^2 = 2$.*

Proof hint: Write $q = \frac{a}{b}$ in lowest terms and consider the evenness/oddness of a and b. □

1.1.12 Definition. An ordered set S has the *least upper bound property* if every subset which is bounded above has a supremum. Likewise S has the *greatest lower bound property* if every subset which is bounded below has an infimum.

1.1.13 Example. Does $\mathbb{Z}$ have the least upper bound property? Does $\mathbb{Q}$? Justify your answers with a proof or counterexample.

1.1.14 Theorem. *If S has the least upper bound property, then it has the greatest lower bound property.*

1.2 Fields and ordered fields

Of course the familiar number systems have additional structure. Besides the order, we have addition, subtraction, multiplication and division.

1.2.1 Definition. A *field* is a set $\mathbb{F}$ with two binary operations[2] $+$ and $\cdot$, called *addition* and *multiplication*, respectively, satisfying the following axioms:

(F1) (Associativity of addition) $(a + b) + c = a + (b + c)$ for all a, b, c in $\mathbb{F}$.

(F2) (Additive identity) There exists an element $0 \in \mathbb{F}$ such that $0 + a = a + 0 = a$ for all a.

(F3) (Additive inverses) For each a in $\mathbb{F}$ there exists an element $-a$ such that $(-a) + a = a + (-a) = 0$.

(F4) (Commutativity of addition) $a + b = b + a$ for all a, b in $\mathbb{F}$.

(F5) (Associativity of multiplication) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, c in $\mathbb{F}$.

(F6) (Multiplicative identity) There exists an element $1 \in \mathbb{F}$ such that $1 \cdot a = a \cdot 1 = a$ for all a.

(F7) (Multiplicative inverses) For all $a \neq 0$, there exists an element a^{-1} in $\mathbb{F}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

[2] A binary operation is a function/operation taking in two elements of $\mathbb{F}$ and returning a third element of $\mathbb{F}$.

(F8) (Commutativity of multiplication) $a \cdot b = b \cdot a$ for all a, b in $\mathbb{F}$.

(F9) (Distributivity) $a \cdot (b + c) = a \cdot b + a \cdot c$.

(F10) (Nontriviality) $0 \neq 1$.

It is customary to omit the $\cdot$ when writing multiplication; in other words, we usually just write ab instead of $a \cdot b$. Additionally, we usually denote $a + (-b)$ simply by $a - b$, and we may also use the notation $\frac{1}{a}$ in place of a^{-1}. It is important to note that subtraction $-$ and division $\div$ are not really distinct operations; they are just syntactic shorthand for addition (resp. multiplication) by an additive (resp. multiplicative) inverse.

We also use the usual shorthand a^n in place of $\underbrace{a \cdots a}_{n \text{ times}}$ and na in place of $\underbrace{a + \cdots + a}_{n \text{ times}}$.

Remark. Though we shall be entirely concerned with fields in this course, you may be familiar with various mathematical objects satisfying fewer of the above axioms. A set with a single operation satisfying axioms (F1)–(F3) is a *group* which is said to be *commutative* or *abelian* if (F4) also holds.

A *ring* is a set with two operations satisfying all of the above except (F7), (F8) and (F10). A *commutative ring* satisfies (F8). According to some conventions, a ring need not satisfy (F6), though such "rings without identity" are sometimes cutely referred to as '*rng*'s. If (F7) holds but not (F8), then $\mathbb{F}$ is called a *division ring*.

Axiom (F10) might be considered optional for fields, but if we allow $0 = 1$ then $\mathbb{F}$ must be the one element set $\{0\}$ (you can prove this after you prove Proposition 1.2.4 below), which for various reasons is best not regarded as a field.

1.2.2 Example. Come up with some examples of fields, some with infinitely many and some with finitely many elements. Can you construct a field with exactly two elements? Three?

1.2.3 Proposition. *The following properties of addition and multiplication hold in any field. (That is, they follow from the axioms above.)*

(i) *(Uniqueness of identities) If an element b in $\mathbb{F}$ satisfies $b + a = a$ for some a, then $b = 0$. Likewise if b satisfies $ba = a$ for some $a \neq 0$, then $b = 1$.*

(ii) *(Uniqueness of inverses) If b satisfies $a + b = 0$, then $b = -a$. Likewise, if b satisfies $ba = 1$ then $b = a^{-1}$.*

(iii) *(Cancellation) If $a + c = b + c$ then $a = b$. Likewise if $c \neq 0$ and $ac = bc$, then $a = b$.*

(iv) *(Inverse of an inverse) $-(-a) = a$ and $(a^{-1})^{-1} = a$.*

1.2.4 Proposition. *In any field, the following properties hold.*

(i) $0a = 0$ *for all a.*

(ii) *If $ab = 0$, then either $a = 0$ or $b = 0$. (We say $\mathbb{F}$ "has no divisors of zero".)*

(iii) $(-a)b = a(-b) = -(ab)$ *for all a and b. In particular $-a = (-1)a$.*

(iv) $(-a)(-b) = ab$ *for all a and b.*

1.2.5 Problem. In a field, show that if $b \neq 0$ and $d \neq 0$ then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

1.2.6 Definition. An *ordered field* is a field $\mathbb{F}$ equipped with a total order, so a set with a relation $\leq$ and two operations $+$ and $\cdot$ satisfying axioms (O1)–(O4) and (F1)–(F10), which is additionally required to satisfy the following axioms:

(OF1) (Compatibility of order and addition) If $a \leq b$ then $a + c \leq b + c$ for any c.

(OF2) (Compatibility of order and multiplication) If $a \leq b$ and $0 \leq c$, then $ac \leq bc$.

1.2.7 Example. Which examples from Example 1.2.2 are ordered fields? In case there is not an obvious order, is there any order at all satisfying (OF1) and (OF2)?

1.2.8 Proposition. *The following properties always hold in an ordered field.*

(i) *If* $0 \leq a$ *then* $-a \leq 0$.

(ii) *If* $0 \leq a$ *and* $0 \leq b$ *then* $0 \leq ab$. *(In fact, this is equivalent to (OF2) and is often used in place of it as the other ordered field axiom).*

(iii) *If* $a \leq 0$ *and* $0 \leq b$*, then* $ab \leq 0$.

(iv) $0 \leq a^2$ *for any* a. *In particular* $0 < 1$.

(v) *If* $0 < a \leq b$ *then* $0 < b^{-1} \leq a^{-1}$.

In light of Proposition 1.2.4.(ii) *the above identities hold with strict inequality* $<$ *used in place of inequality* $\leq$.

1.2.9 Problem. Let $\mathbb{F}$ be an ordered field and consider the subset $Z \subset \mathbb{F}$ generated by taking 0, 1, $1+1$, $1+1+1$, etc. along with -1, $-1-1$, $-1-1-1$, etc. Show that this set is in bijection with the set of integers $\mathbb{Z}$.

Likewise, let $Q \subset \mathbb{F}$ be the subset generated by taking the multiplicative inverses of the nonzero elements in Z along with their integer multiples. Show that this set is in bijection with $\mathbb{Q}$.

Thus every ordered field contains a copy of $\mathbb{Q}$, which may be regarded as the "smallest" possible ordered field.

1.2.10 Definition. Let $\mathbb{F}$ be an ordered field. The *absolute value* or *magnitude* of a number $a \in \mathbb{F}$ is defined by

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

1.2.11 Proposition*. *The absolute value satisfies the following properties. For all* a *and* b *in* $\mathbb{F}$*:*

(i) $|a| \geq 0$.

(ii) $|a| = 0$ *if and only if* $a = 0$.

(iii) $|ab| = |a||b|$.

(iv) *(Triangle inequality)* $|a+b| \leq |a| + |b|$.

(v) *(Reverse triangle inequality)* $\big||a| - |b|\big| \leq |a - b|$.

Remark. Combining (iv) and (v) of the last proposition gives the useful strings of inequalities:

$$|a| - |b| \leq \big||a| - |b|\big| \leq |a+b| \leq |a| + |b|, \quad \text{and} \quad |a| - |b| \leq \big||a| - |b|\big| \leq |a-b| \leq |a| + |b|. \tag{1.2}$$

1.2.12 Definition. The *distance* between numbers a and b in an ordered field $\mathbb{F}$ is the quantity

$$d(a,b) = |a - b|.$$

1.2.13 Proposition*. *The distance satisfies the following properties. For all* a*,* b*, and* c *in* $\mathbb{F}$*:*

(i) $d(a,b) \geq 0$.

(ii) $d(a,b) = 0$ *if and only if* $a = b$.

(iii) *(Symmetry)* $d(a, b) = d(b, a)$.

(iv) *(Triangle inequality)* $d(a, c) \leq d(a, b) + d(b, c)$.

1.2.14 Lemma (Suprema/infima in an ordered field)**.** *Let A be a bounded above subset of an ordered field. Then $u = \sup A$ if and only if*

(i) *$a \leq u$ for all $a \in A$ (i.e., u is an upper bound), and*

(ii) *for every $\varepsilon > 0$, there exists $a \in A$ such that $u - \varepsilon < a$ (i.e., $u - \varepsilon$ fails to be an upper bound).*

Similarly, if A is bounded below, then $b = \inf A$ if and only if

(i) *$b \leq a$ for all $a \in A$, and*

(ii) *for every $\varepsilon > 0$, there exists $a \in A$ such that $a < b + \varepsilon$.*

1.2.15 Definition ($\pm\infty$ notation)**.** As a notation convention, it is useful to introduce the symbols $+\infty$ and $-\infty$ when speaking of suprema and infima in an ordered field. We write $\sup A = +\infty$ if A is not bounded above, and $\inf A = -\infty$ if A is not bounded below. With these conventions $\sup A$ and $\inf A$ are always defined for a nonempty set A, and (1.1) holds identically whenever $A \subseteq B$.

A more formal way to do this is to embed $\mathbb{F}$ into a larger ordered set $\overline{\mathbb{F}} = \mathbb{F} \cup \{+\infty, -\infty\}$ with the order defined so that $-\infty < a < +\infty$ for all $a \in \mathbb{F}$. Note that $\overline{\mathbb{F}}$ is *not* a field, though we may observe the following notation conventions: if $a > 0 \in \mathbb{F}$, then

$$a + (+\infty) = +\infty, \quad a + (-\infty) = -\infty, \quad a(+\infty) = +\infty, \quad a(-\infty) = -\infty,$$
$$(-a)(+\infty) = -\infty, \quad (-a)(-\infty) = +\infty, \quad \frac{\pm a}{\pm\infty} = 0.$$

Expressions such as $+\infty - \infty$ and $\pm\infty / \pm\infty$ are not defined.

1.3 Completeness and the real number field

1.3.1 Definition. An ordered field $\mathbb{F}$ is *complete* if it satisfies the least upper bound property (c.f. Definition 1.1.12), in other words, if for every bounded above subset $A \subset \mathbb{F}$, the supremum (least upper bound) $\sup A$ exists in $\mathbb{F}$.

1.3.2 Theorem† (Characterization/definition of $\mathbb{R}$)**.** *There exists a unique*[3] *complete ordered field called the* real numbers *and denoted by* $\mathbb{R}$.

Remark. We omit the proof of Theorem 1.3.2 for now; we may come back to it later on. However, it is worth mentioning one construction which is possible at this point: define a *Dedekind cut* to be a subset $A \subset \mathbb{Q}$ of the rationals with the properties that

(i) A is neither empty nor all of $\mathbb{Q}$,

(ii) if $q \in A$ and $p < q$, then $p \in A$,

(iii) if $q \in A$ then $q < r$ for some $r \in A$.

[3] Here "uniqueness" means the following: given two complete ordered fields F_1 and F_2, there exists an *isomorphism* (a bijection compatible with the order and field operations) $\phi : F_1 \longrightarrow F_2$. Moreover ϕ is unique. Using ϕ we can regard F_1 and F_2 as being "the same" field.

In other words, a cut is essentially a half infinite open interval in $\mathbb{Q}$; take as an example $\{q \in \mathbb{Q} : q < 2\} = (-\infty, 2)$. It is tempting to want to write $\{q \in \mathbb{Q} : q < \sqrt{2}\}$ as another example, but this is ill-specified since we do not have such a number as $\sqrt{2}$ at this point. The equivalent set may be specified as $\{q \in \mathbb{Q} : q < 0 \text{ or } q^2 < 2\}$. The idea here is that real numbers are represented by the "upper endpoints" of the cuts, though since these are not well-defined, the whole cut stands in as a replacement.

It is then possible to define an order, addition, and multiplication on the set of Dedekind cuts (order and addition are straightforward; multiplication is a little tricky) and verify that they satisfy all the axioms of an ordered field along with completeness, with subfield $\mathbb{Q}$ identified with those cuts of the form $\{q \in \mathbb{Q} : q < p\}$ for $p \in \mathbb{Q}$.

1.3.3 Definition. An ordered field $\mathbb{F}$ is *Archimedean* if for every $a \in \mathbb{F}$, there exists an integer[4] N such that $a \leq N$.

1.3.4 Example*. Show that $\mathbb{Q}$ is Archimedean.

1.3.5 Example (Research Allowed)**.** Find an example of a non-Archimedean field.

1.3.6 Theorem. *As a complete ordered field, $\mathbb{R}$ is Archimedean.*

1.3.7 Proposition. *A field is Archimedean if and only if, for every $a > 0$, there exists a positive integer N such that*

$$0 < \frac{1}{N} < a.$$

Remark. The Archimedean property says that a field has no "infinitely large" elements, and via Proposition 1.3.7, it implies that there are no "infinitely small" elements. The next result gives a technically useful if strange seeming characterization of the zero element.

1.3.8 Corollary. *In an Archimedean field, if $0 \leq a$ and $a < \varepsilon$ for every $0 < \varepsilon$, then $a = 0$.*

1.3.9 Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *Let a and b be real numbers with $a < b$. Then there exists a rational number q such that*

$$a < q < b.$$

We say $\mathbb{Q}$ is dense *in $\mathbb{R}$.*

Remark. This may be a surprising result, especially when juxtaposed with the following one. Recall that an infinite set is said to be *countable* if it is in bijection with the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of natural numbers.

1.3.10 Theorem.

(i) $\mathbb{Q}$ *is countable.*

(ii) $\mathbb{R}$ *is uncountable.*

One more result at this point will be useful later on, though the proof is rather technical and tricky, so you may go ahead and take it as given rather than trying to prove it.

1.3.11 Theorem† (Positive nth roots)**.** *For every $y > 0$ in $\mathbb{R}$ and $n \in \mathbb{N}$, there exists a unique $x > 0$ in $\mathbb{R}$ such that $x^n = y$.*

The proof is obtained from the following two results, the first of which is more or less straightforward while the second is the tricky one.

1.3.12 Lemma. *For fixed $y > 0$ and $n \in \mathbb{N}$, the set $E = \{t \in \mathbb{R} : 0 < t,\ t^n < y\}$ is nonempty and bounded above.*

1.3.13 Lemma†. *The element $x = \sup E$ satisfies $x^n = y$.*

[4] Here we are identifying a subset of $\mathbb{F}$ with the integers as in Problem 1.2.9.

1.4 Sequences of real numbers

While least upper bounds give an expedient way to express the completeness of $\mathbb{R}$, *sequences* play a much more ubiquitous role in analysis.

1.4.1 Definition. A *sequence* of real numbers is a function[5] from $\mathbb{N}$ into $\mathbb{R}$. As a matter of notation, if $x : \mathbb{N} \longrightarrow \mathbb{R}$ is a sequence, we prefer to write x_n instead of $x(n)$, and denote the sequence by

$$(x_1, x_2, x_3, \ldots), \quad \text{or} \quad (x_n)_{n=1}^{\infty}, \quad \text{or just} \quad (x_n).$$

It is permissible and often convenient to index a sequence starting from 0 instead of 1, or starting from a number greater than 1.

1.4.2 Definition. A sequence (x_n) in $\mathbb{R}$ is said to be

(i) *bounded* if there exists some $B > 0$ such that $|x_n| \leq B$ for all n.

(ii) *increasing* if $x_n \leq x_{n+1}$ for all n. It is *strictly increasing* if $x_n < x_{n+1}$ for all n.

(iii) *decreasing* if $x_n \geq x_{n+1}$ for all n. It is *strictly decreasing* if $x_n > x_{n+1}$ for all n.

(iv) *monotone* if it is either increasing or decreasing.

(v) *convergent* if there exists some $L \in \mathbb{R}$ with the following property: for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\text{for all } n \geq N, \quad |x_n - L| < \varepsilon.$$

Equivalently, for every $\varepsilon > 0$, the interval $(L - \varepsilon, L + \varepsilon) \subset \mathbb{R}$ contains all but finitely many terms of the sequence. In this case we say L is the *limit* of the sequence (x_n) and write $\lim_{n\to\infty} x_n = L$ or $x_n \to L$.

1.4.3 Exposition*. Explain in plain English what is meant by the limit of a sequence. Address the order of the quantifiers (i.e., "for all" or "there exists"): if the order or type of the quantifiers is changed, why is this a bad definition of limit?

1.4.4 Proposition. *The limit of a sequence, if it exists, is unique.*

1.4.5 Example*.

(i) Show that the constant sequence $x_n = c$ for all n converges and $\lim x_n = c$.

(ii) Show that $x_n = \frac{1+3n}{1+5n}$ has limit $\frac{3}{5}$.

1.4.6 Example.

(i) Show that $x_n = \frac{1}{n} \to 0$.

(ii) If $0 \leq p < 1$, show that $x_n = p^n \to 0$. [Hint: such p can be written as $p = \frac{1}{1+a}$ for $a > 0$. The binomial estimate $(1 + a)^n \geq 1 + na$ for $a \geq 0$ is also useful here.]

1.4.7 Proposition. *If a sequence converges, then it is bounded.*

1.4.8 Theorem. *In $\mathbb{R}$, every bounded monotone sequence converges.*

Remark. The previous property of $\mathbb{R}$ is referred to as the *monotone sequence property.* In fact, it is equivalent to completeness: it can be proved that an ordered field in which every bounded monotone sequence converges has the least upper bound property.

1.4.9 Problem. Let $x_n = \sqrt{n^2 + 1} - n$. Show (x_n) converges and compute its limit.

[5] Recall that a *function* $f : A \longrightarrow B$ is an assignment to every element a of the *domain* A an element $b = f(a)$ of the *target* B.

1.4.10 Theorem. *Let (x_n) and (y_n) be convergent sequences with limits x and y, respectively. Then*

(i) $x_n + y_n \to x + y$.

(ii) $-x_n \to -x$.

(iii) $x_n y_n \to xy$.

(iv) *If $x_n \neq 0$ for all n and $x \neq 0$, then $x_n^{-1} \to x^{-1}$.*

(v) *If $x_n \leq y_n$ for all n, then $x \leq y$.*

Remark. Combining the above, it follows that $x_n - y_n \to x - y$ and $x_n/y_n \to x/y$ (provided $y_n \neq 0$ and $y \neq 0$).

1.4.11 Problem. Let $p(t) = a_0 + \cdots + a_l t^l$ and $q(t) = b_0 + \cdots + b_m t^m$ be polynomials with real coefficients, where $a_l \neq 0$ and $b_m \neq 0$. If $l \leq m$, prove that

$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \begin{cases} 0, & \text{if } l < m, \text{ and} \\ a_l/b_m, & \text{if } l = m. \end{cases}$$

1.4.12 Problem. Define a sequence inductively by setting $x_1 = \sqrt{2}$ and for $n \geq 2$, $x_n = \sqrt{2 + x_{n-1}}$. Prove that (x_n) converges and find its limit.

1.4.13 Example*. Show that strict inequality cannot be obtained in Theorem 1.4.10.(v), by producing an example where $x_n < y_n$ for all n but $x = y$.

1.4.14 Example* (Harmonic series/sequence)**.** Define a sequence by $x_1 = 1$, $x_2 = 1 + \frac{1}{2}$, and $x_n = 1 + \cdots + \frac{1}{n}$. Show (x_n) is monotone increasing, but unbounded above, hence does not converge. [Possible hint: compare to the sequence $y_1 = 1$, $y_2 = 1 + \frac{1}{2}$, $y_3 = 1 + \frac{1}{2} + \frac{1}{4}$, $y_4 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4}$, $y_5 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8}$ etc., where y_n has each term from the corresponding x_n replaced by the largest power of 2^{-1} which is less than or equal to it.]

The following limits are useful to know but tricky to prove at this point[6], so you can take them as given rather than trying to prove them.

1.4.15 Proposition†.

(i) *For any $a > 0$, the sequence $a^{1/n} \to 1$.*

(ii) *The sequence $n^{1/n} \to 1$.*

In addition to bounded monotone sequences, another very useful class of sequences which "ought to converge" are the *Cauchy sequences*.

1.4.16 Definition. A sequence (x_n) is *Cauchy* if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\text{for every } n, m \geq N, \quad |x_n - x_m| < \varepsilon. \tag{1.3}$$

Remark. Intuitively, a Cauchy sequence is one in which the *tails* of the sequence (i.e., the sequences $(x_n)_{n=N}^{\infty}$ for various N) become arbitrarily "bunched up".

1.4.17 Proposition. *Every Cauchy sequence is bounded.*

1.4.18 Proposition. *Every convergent sequence is Cauchy.*

[6] In fact they become quite easy to prove once we have developed the logarithm, but we do not have this yet.

The converse is not true in general; however one of the distiguishing features of $\mathbb{R}$ over $\mathbb{Q}$ as a complete ordered field is the following main result.

1.4.19 Theorem. *In $\mathbb{R}$, every Cauchy sequence converges.*

Proof hint. For each k let $a_k = \sup\{x_n : n \geq k\}$. Then (a_k) is a bounded decreasing sequence. Show that $x_n \to a$, where $a = \lim_k a_k$. □

Remark. This property is known as the *Cauchy completeness* of $\mathbb{R}$. It is possible to show that it is equivalent to both the least upper bound property and to the monotone sequence property.

1.4.20 Example. Show that (1.3) cannot be replaced by "for every $n \geq N$, $|x_n - x_{n+1}| < \varepsilon$", by finding an example of a divergent sequence in $\mathbb{R}$ with the latter property.

Remark. The previous exercise shows that it is generally not enough for pairs of adjacent elements in the sequence to be getting close together; rather, we need the distance between any pair of not-necessarily-adjacent elements in some tail of the sequence to be getting close. On the other hand, if adjacent pairs become close fast enough, then the sequence actually is Cauchy, as the next result shows.

1.4.21 Lemma. *Let (x_n) be a sequence in $\mathbb{R}$ such that for all $n \in \mathbb{N}$,*

$$|x_n - x_{n+1}| \leq \frac{a}{2^n}, \quad \textit{for some } a > 0.$$

Then (x_n) is Cauchy, and therefore convergent. (In fact $a/2^n$ can be replaced by a/b^n for any $b > 1$.)

Remark. Having discussed sequences, we can now mention two more methods of constructing $\mathbb{R}$ from $\mathbb{Q}$. In both cases we consider as elements equivalence classes of sequences (x_n) in $\mathbb{Q}$, with the equivalence relation $(x_n) \sim (y_n)$ if $x_n - y_n \to 0$.

The first method uses equivalence classes of *bounded increasing sequences*, while the second method uses equivalence classes of *Cauchy sequences*. Each element $q \in \mathbb{Q}$ is represented by the equivalence class of the constant sequence $x_n = q$ for all n.

In either method, one has to define a notion of addition, multiplication, and order on sequences, show these are well-defined on equivalence classes, and prove that the set of equivalence classes satisfies all the axioms for an ordered field, along with some version of completeness (usually the monotone sequence property if you are using monotone sequences, and the Cauchy sequence property if you are using Cauchy sequences).

1.4.22 Theorem (Squeeze theorem)**.** *Let (x_n), (y_n) and (z_n) be sequences in $\mathbb{R}$ such that $x_n \leq y_n \leq z_n$ for all n, and suppose that $x_n \to l$ and $z_n \to l$. Then (y_n) also converges and* $\lim_n y_n = l$.

1.4.23 Exposition*. Explain in plain English what is the significance of the completeness of $\mathbb{R}$. Why is this a useful property to have?

1.5 Subsequences

1.5.1 Definition. Let (x_n) be a sequence in $\mathbb{R}$. A *subsequence* of (x_n) is a sequence $(x_{n_k})_{k=1}^{\infty}$ where $n_1 < n_2 < n_3 < \cdots$ form a strictly increasing sequence (n_k) of natural numbers.

If $x_{n_k} \to l$ for some subsequence (x_{n_k}) of (x_n), we say l is a *subsequential limit* of (x_n).

1.5.2 Example. Find examples of a non-convergent sequence (x_n) with

(i) No subsequential limits.

(ii) Exactly one subsequential limit.

(iii) Exactly two subsequential limits.

(iv) Exactly three subsequential limits.

(v) Infinitely many subsequential limits.

1.5.3 Proposition. *A number $l \in \mathbb{R}$ is a subsequential limit of a sequence (x_n) if and only if, for every $\varepsilon > 0$,*

$$|x_n - l| < \varepsilon$$

for infinitely many $n \in \mathbb{N}$.

1.5.4 Proposition. *If $x_n \to x$, then every subsequence of (x_n) converges to x.*

1.5.5 Proposition. *If (x_n) is a Cauchy sequence (not necessarily in $\mathbb{R}$, perhaps in $\mathbb{Q}$), and $x_{n_k} \to x$ for some subsequence (x_{n_k}), then $x_n \to x$.*

1.5.6 Theorem. *Every sequence in $\mathbb{R}$ or $\mathbb{Q}$ has a monotone subsequence (either increasing or decreasing).*

1.5.7 Corollary. *If (x_n) is a sequence in $[a, b] \subset \mathbb{R}$, for some $a \leq b$, then (x_n) has a convergent subsequence in $\mathbb{R}$.*

Prior to discussing limit superior and inferior, it is convenient to make the following definition.

1.5.8 Definition. Given a sequence, we say $x_n \to +\infty$ if, for every $M > 0$, there exists $N \in \mathbb{N}$ such that

$$M < x_n \quad \text{for all } n \geq N.$$

Likewise, we say $x_n \to -\infty$ if for every $M < 0$, there exists $N \in \mathbb{N}$ such that

$$x_n < M \quad \text{for all } n \geq N.$$

In neither case do we regard (x_n) as a convergent sequence in $\mathbb{R}$; however, it is possible to regard it as convergent in the *extended real numbers* $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, regarded as an ordered set (but not a field) as in Definition 1.2.15.

1.5.9 Proposition*. *If (x_n) is a monotone sequence, then $x_n \to l$ for some $l \in \mathbb{R} \cup \{\pm\infty\}$.*

1.5.10 Definition. Let (x_n) be a sequence in $\mathbb{R}$ and let $L = \{l \in \mathbb{R} \cup \{\pm\infty\} : x_{n_k} \to l, \text{ for some subsequence } (x_{n_k})\}$ be the set of its subsequential limits, including possibly $+\infty$ and $-\infty$. The *limit superior* of (x_n) is the supremum

$$\limsup x_n = \sup L,$$

and the *limit inferior* of (x_n) is the infimum

$$\liminf x_n = \inf L.$$

1.5.11 Proposition. *For every sequence (x_n) in $\mathbb{R}$, there exists a subsequence (x_{n_k}) such that*

$$x_{n_k} \to \limsup x_n.$$

Likewise, there exists a subsequence (x_{n_k}) such that $x_{n_k} \to \liminf x_n$.

In other words, in the definition of limit superior, $\limsup x_n = \sup L$ and

The next result justifies the names "limit inferior" and "limit superior".

1.5.12 Proposition. *An equivalent characterization of limit superior and limit inferior are as follows. Let (x_n) be a real sequence and for each $m \in \mathbb{N}$, let $a_m = \inf\{x_n : n \geq m\}$ and $b_m = \sup\{x_n : n \geq m\}$. Then $a = \liminf x_n$ if and only if*

$$a = \lim a_m = \lim_m \inf\{x_n : n \geq m\} = \sup\{\inf\{x_n : n \geq m\}\}.$$

Likewise, $b = \limsup x_n$ if and only if

$$b = \lim b_m = \lim_m \sup\{x_n : n \geq m\} = \inf\{\sup\{x_n : n \geq m\}\}.$$

1.5.13 Proposition. *For a sequence (x_n),*

$$\liminf x_n \leq \limsup x_n$$

with equality if and only if (x_n) converges to this value.

Chapter 2

Topology of metric spaces

2.1 Metric spaces

The study of sequences, convergence, and of continuous functions in $\mathbb{R}$ really depend only on certain properties of the distance $d(x, y) = |x - y|$. It is useful therefore to work in a more abstract setting, where the results obtained may then apply more broadly. We will work in the setting of *metric spaces*.

2.1.1 Definition. A *metric space* (M, d) is a set M equipped with a real-valued function

$$\begin{aligned} d : M \times M &\longrightarrow \mathbb{R}, \\ (x, y) &\longmapsto d(x, y) \end{aligned}$$

satisfying the following properties:

(M1) (Positivity) $d(x, y) \geq 0$ for all $x, y \in M$.

(M2) (Nondegeneracy) $d(x, y) = 0$ if and only if $x = y$.

(M3) (Symmetry) $d(x, y) = d(y, x)$ for all $x, y \in M$.

(M4) (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in M$.

We say d is a *metric on* M.

If $S \subset M$ is any subset, then (S, d) is a metric space in its own right, which we refer to as a *subspace* of M.

2.1.2 Example. In light of Proposition 1.2.13, $\mathbb{R}$ is a metric space with respect to the metric $d(x, y) = |x - y|$.

2.1.3 Example*. Any set S may be equipped with the *discrete metric* δ, defined by

$$\delta(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Show that (S, δ) is a metric space.

2.1.4 Definition. We define n-dimensional *Euclidean space* to be the set $\mathbb{R}^n = \{x = (x_1, \ldots, x_n) : x_i \in \mathbb{R},\ i = 1, \ldots, n\}$ of ordered n-tuples of real numbers. From linear algebra (or otherwise), we know that $\mathbb{R}^n$ is a *vector space*, meaning it has an *vector addition* operation

$$x + y = (x_1 + y_1, \ldots, x_n + y_n), \quad x, y \in \mathbb{R}^n$$

which is associative, commutative, with identity $0 = (0, \ldots, 0)$, and inverses $-x = (-x_1, \ldots, -x_n)$ (compare axioms (F1)–(F4) of Definition 1.2.1), and a *scalar multiplication* operation

$$ax = (ax_1, \ldots, ax_n), \quad a \in \mathbb{R},\ x \in \mathbb{R}^n,$$

which is appropriately associative and distributive and satisfies $1x = x$. $\mathbb{R}^n$ also has a natural *Euclidean inner product*

$$\langle x, y\rangle = x_1y_1 + \cdots + x_ny_n \in \mathbb{R}$$

which satisfies $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$, $\langle x, y\rangle = \langle y, x\rangle$ and $a\langle x, y\rangle = \langle ax, y\rangle$ for $x, y, z \in \mathbb{R}^n$, $a \in \mathbb{R}$. We define the *Euclidean norm* by

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + \cdots + x_n^2}.$$

2.1.5 Lemma (Cauchy-Schwartz inequality)**.** *For all $x, y \in \mathbb{R}^n$,*

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

[Hint: consider $\langle x - \alpha y, x - \alpha y\rangle$ for a particularly well chosen value of α. Note that using calculus to find a minimizing α is not out of the question here, since the calculus need not enter the actual proof!

2.1.6 Example. Show that each of the following are metrics on $\mathbb{R}^n$:

(i) The *Euclidean (aka ℓ^2) metric*

$$d_2(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

(ii) The ℓ^1 metric

$$d_1(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|.$$

(iii) The ℓ^∞ metric

$$d_\infty(x, y) = \max_{i\in\{1,\ldots,n\}} |x_i - y_i|.$$

Unless otherwise specified, we usually consider $\mathbb{R}^n$ with the Euclidean metric.

Remark. The previous example shows that the same set may have the structure of a metric space with respect to several different metrics (we could even consider $\mathbb{R}^n$ with the discrete metric). This is why a particular metric must be specified (or genreally understood) when we assert that a set is a metric space.

The following estimates are routinely useful in $\mathbb{R}^n$, and can be interpreted as saying that the metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$ are comparable to one another (though not generally equal).

2.1.7 Lemma. *For any $z = (z_1, \ldots, z_n) \in \mathbb{R}^n$,*

$$\frac{1}{\sqrt{n}}(|z_1| + \cdots + |z_n|) \leq \sqrt{|z_1|^2 + \cdots + |z_n|^2} \leq (|z_1| + \cdots + |z_n|)$$

and

$$\max_{i\in\{1,\ldots,n\}} |z_i| \leq \sqrt{|z_1|^2 + \cdots + |z_n|^2} \leq \sqrt{n}\,(\max_i |z_i|)$$

In particular, for any $x, y \in \mathbb{R}^n$,

$$\frac{1}{\sqrt{n}} d_1(x, y) \leq d_2(x, y) \leq d_1(x, y), \quad d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{n}\, d_\infty(x, y).$$

2.2 Sequences

Apart from monotonicity, and any statements relating to order, most of the definitions and results about sequences in $\mathbb{R}$ carry over to the general setting of metric spaces.

2.2.1 Definition. Let M be a metric space and $S \subseteq M$. Then S is said to be *bounded* if for some point $p \in M$ there exists a constant $B \geq 0$ such that $d(q,p) \leq B$ for all $q \in S$.

Remark. Using the triangle inequality, it follows that S is bounded if and only if, for *every* point $p \in M$, there exists $B \geq 0$ such that $d(q,p) \leq B$ for all $q \in S$.

2.2.2 Definition. A sequence (x_n) in a metric space M is said to be

(i) *bounded* if the set $\{x_n : n \in \mathbb{N}\} \subset M$ is bounded,

(ii) *convergent*, with *limit* $l \in M$, if there exists $l \in M$ with the property that, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, l) < \varepsilon \quad \text{for all } n \geq N.$$

We write $x_n \to l$ or $l = \lim_{n\to\infty} x_n$ as usual.

(iii) *Cauchy* if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(x_n, x_m) < \varepsilon \quad \text{for all } n, m \geq N.$$

We define a *subsequence* (x_{n_k}) in the usual way, via a strictly increasing sequence $n_1 < n_2 < \cdots$ of indices, and we say $l \in M$ is a *subsequential limit* if $x_{n_k} \to l$ for a subsequence (x_{n_k}).

The proofs of Propositions 1.4.4, 1.4.17, 1.4.18, 1.5.3, 1.5.4, and 1.5.5 rely fundamentally only on the properties (M1)–(M4) of the metric $d(x,y) = |x-y|$ on $\mathbb{R}$. Hence their proofs carry over at once to prove the following.

2.2.3 Theorem. *The following hold for sequences in a metric space M:*

(i) *The limit of a sequence, if it exists, is unique.*

(ii) *Every convergent sequence is Cauchy.*

(iii) *Every Cauchy sequence (and hence by (ii) every convergent sequence) is bounded.*

(iv) *$x_n \to l$ if and only if every subsequence of (x_n) converges to l.*

(v) *If (x_n) is Cauchy and some subsequence converges to l, then $x_n \to l$.*

(vi) *$l \in M$ is a subsequential limit of (x_n) if and only if, for every $\varepsilon > 0$, $d(x_n, l) < \varepsilon$ for infinitely many n.*

(vii) *$l \in M$ is the limit of (x_n) if and only if, for every $\varepsilon > 0$, $d(x_n, l) < \varepsilon$ for all but finitely many n.*

The following notion is fundamentally useful in the study of metric spaces.

2.2.4 Definition. For $\varepsilon > 0$, we define the *ε-ball centered at $p \in M$* to be the set

$$D_\varepsilon(p) = \{q \in M : d(p,q) < \varepsilon\}.$$

The name "ball" is justified by the example $M = \mathbb{R}^n$, with the Euclidean metric.

Remark. In terms of ε-balls, we can reinterpret the last two items in Theorem 2.2.3 as follows: l is a subsequential limit (respectively, the limit) of (x_n) if and only if for every $\varepsilon > 0$, $D_\varepsilon(l)$ contains infinitely many (resp. all but finitely many) elements of the sequence.

2.2.5 Example. Let S be a metric space with the discrete metric δ (c.f. Example 2.1.3). What do the ε-balls in S look like for various ε? Which sequences are Cauchy in S? Which sequences are convergent?

2.2.6 Definition. Let (x_k) be a sequence in $\mathbb{R}^n$. Temporarily writing superscripts for components, i.e., $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$, we can write each term in the sequence as

$$x_k = (x_k^1, \ldots, x_k^n) \in \mathbb{R}^n.$$

Fixing the superscript i and letting k vary, we get sequences $(x_k^i)_{k=1}^\infty$ in $\mathbb{R}$, for $i = 1, \ldots, n$, which we call the *component sequences* of (x_k).

2.2.7 Proposition. *Let (x_k) be a sequence in $\mathbb{R}^n$.*

(i) *(x_k) converges to $x = (x^1, \ldots, x^n)$ if and only if each component sequence converges:*

$$x_k^i \to x^i, \quad i = 1, \ldots, n.$$

(ii) *(x_k) is Cauchy if and only if each component sequence $(x_k^i)_{k=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$.*

2.2.8 Definition. We say a metric space (M, d) is *complete* if every Cauchy sequence converges in M.

2.2.9 Theorem*.

(i) *Every Euclidean space $\mathbb{R}^n$, $n \in \mathbb{N}$, is a complete metric space.*

(ii) *Any discrete metric space (S, δ) is complete.*

2.2.10 Example. What are some examples of incomplete metric spaces?

The process by which we obtain $\mathbb{R}$ from $\mathbb{Q}$ is an example of a more general operation in the setting of metric spaces, called the *metric completion*. In this general setting, we say a subspace $S \subset M$ is *dense in M* if for every $p \in M$ and for every $\varepsilon > 0$, there is some $q \in S$ such that $d(p, q) < \varepsilon$. In other words, for every $\varepsilon > 0$ and $p \in M$, the ε-ball $D_\varepsilon(p)$ contains some point of S.

2.2.11 Theorem†. *For any metric space (M, d), there exists a complete metric space (M', d') in which M is dense. More precisely, M is in bijection $i : M \longrightarrow S'$ with a dense subspace $S' \subset M'$ under which the metrics agree: $d'(i(p), i(q)) = d(p, q)$.*

Proof sketch. Define M' to be the set of equivalence classes of all Cauchy sequences (x_n) in M, where (x_n) and (y_n) are equivalent if $d(x_n, y_n) \to 0$. In fact one can show that for any two Cauchy sequences (x_n) and (y_n) in M, the sequence $d(x_n, y_n)$ is Cauchy (hence convergent) in $\mathbb{R}$; the metric on M' is then defined by $d'\big((x_n), (y_n)\big) = \lim_n d(x_n, y_n)$. For $p \in M$, take $i(p)$ to be the constant sequence $(p, p, p, \ldots)$. $\square$