Calc III: Quiz 2 Solutions, Fall 2017

Problem 1. Write the normal form for the plane containing the points $P = (1, 0, 3)$, $Q = (1, 2, -1)$, and $R = (6, 1, 6)$.

Solution. The vectors $\overrightarrow{PQ} = (0, 2, -4)$ and $\overrightarrow{PR} = (5, 1, 3)$ are tangent to the plane in question, so their cross product,

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = (10, -20, -10),$$

is a normal vector. Any nonzero multiple of $\mathbf{n}$ is still a normal vector, so we could replace $\mathbf{n}$ by $(1, -2, -1)$ if we want. Then a normal form for the plane is given by $\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$, where $\mathbf{x}_0 = \overrightarrow{OP} = (1, 0, 3)$, and $\mathbf{x} = (x, y, z)$, so we obtain

$$10(x - 1) - 20(y - 0) - 10(z - 3) = 0, \quad \text{or} \quad x - 2y - z + 2 = 0.$$

$\square$

Problem 2. Calculate the derivative $\mathbf{f}'(t)$ of the curve $\mathbf{f}(t) = (t + 1, t^2 + 1, t^3 + 1)$ and find the tangent line to this curve at $\mathbf{f}(0)$.

Solution. The derivative is taken component-wise, so $\mathbf{f}'(t) = (1, 2t, 3t^2)$. The tangent line is given parametrically by

$$\mathbf{l}(s) = \mathbf{f}(0) + s\mathbf{f}'(0) = (1, 1, 1) + s(1, 0, 0),$$

or equivalently,

$$x(s) = 1 + s, \quad y(s) = z(s) = 1.$$

$\square$

Problem 3. Calculate the arc length of the curve $\mathbf{f}(t) = (2\cos 3t, 2\sin 3t, 2t^{3/2})$ over the interval $0 \leq t \leq 1$.

Solution. First we compute $\mathbf{f}'(t) = (-6\sin 3t, 6\cos 3t, 3t^{1/2})$, and then the arc length is given by

$$\begin{aligned} s &= \int_0^1 \|\mathbf{f}'(t)\|\, dt \\ &= \int_0^1 \sqrt{(-6\sin 3t)^2 + (6\cos 3t)^2 + (3t^{1/2})^2}\, dt \\ &= \int_0^1 \sqrt{36 + 9t}\, dt \\ &= \int_0^1 3\sqrt{4 + t}\, dt \\ &= 2(4 + t)^{3/2}\Big|_{t=0}^{1} = 2(5^{3/2} - 4^{3/2}). \end{aligned}$$

$\square$