3.
$\partial(2x-3y)/\partial y = -3 = \partial(-3x+4y-8)/\partial x$ and the domain of $\mathbf{F}$ is $\mathbb{R}^2$ which is open and simply-connected, so by Theorem 6 $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\nabla f = \mathbf{F}$, that is, $f_x(x,y) = 2x-3y$ and $f_y(x,y) = -3x+4y-8$. But $f_x(x,y) = 2x-3y$ implies $f(x,y) = x^2-3xy+g(y)$ and differentiating both sides of this equation with respect to y gives $f_y(x,y) = -3x+g'(y)$. Thus $-3x+4y-8 = -3x+g'(y)$ so $g'(y) = 4y-8$ and $g(y) = 2y^2-8y+K$ where K is a constant. Hence $f(x,y) = x^2-3xy+2y^2-8y+K$ is a potential function for $\mathbf{F}$.

6. $\partial(3x^2-2y^2)/\partial y = -4y$, $\partial(4xy+3)/\partial x = 4y$. Since these are not equal, $\mathbf{F}$ is not conservative.

12. (a) $f_x(x,y) = x^2$ implies $f(x,y) = \frac{1}{3}x^3 + g(y)$ and $f_y(x,y) = 0 + g'(y)$. But $f_y(x,y) = y^2$ so $g'(y) = y^2 \quad \Rightarrow \quad g(y) = \frac{1}{3}y^3 + K$. We can take $K=0$, so $f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3$.

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = f(2,8) - f(-1,2) = \left(\frac{8}{3}+\frac{512}{3}\right) - \left(-\frac{1}{3}+\frac{8}{3}\right) = 171$.

13. (a) $f_x(x,y) = xy^2$ implies $f(x,y) = \frac{1}{2}x^2y^2 + g(y)$ and $f_y(x,y) = x^2y + g'(y)$. But $f_y(x,y) = x^2y$ so $g'(y) = 0 \quad \Rightarrow$ $g(y) = K$, a constant. We can take $K=0$, so $f(x,y) = \frac{1}{2}x^2y^2$.

(b) The initial point of C is $\mathbf{r}(0) = (0,1)$ and the terminal point is $\mathbf{r}(1) = (2,1)$, so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(2,1) - f(0,1) = 2 - 0 = 2$.

15. (a) $f_x(x,y,z) = yz$ implies $f(x,y,z) = xyz + g(y,z)$ and so $f_y(x,y,z) = xz + g_y(y,z)$. But $f_y(x,y,z) = xz$ so $g_y(y,z) = 0 \quad \Rightarrow \quad g(y,z) = h(z)$. Thus $f(x,y,z) = xyz + h(z)$ and $f_z(x,y,z) = xy + h'(z)$. But $f_z(x,y,z) = xy + 2z$, so $h'(z) = 2z \quad \Rightarrow \quad h(z) = z^2 + K$. Hence $f(x,y,z) = xyz + z^2$ (taking $K=0$).

(b) $\int_C \mathbf{F}\cdot d\mathbf{r} = f(4,6,3) - f(1,0,-2) = 81 - 4 = 77$.

17. (a) $f_x(x,y,z) = yze^{xz}$ implies $f(x,y,z) = ye^{xz} + g(y,z)$ and so $f_y(x,y,z) = e^{xz} + g_y(y,z)$. But $f_y(x,y,z) = e^{xz}$ so $g_y(y,z) = 0 \quad \Rightarrow \quad g(y,z) = h(z)$. Thus $f(x,y,z) = ye^{xz} + h(z)$ and $f_z(x,y,z) = xye^{xz} + h'(z)$. But $f_z(x,y,z) = xye^{xz}$, so $h'(z) = 0 \quad \Rightarrow \quad h(z) = K$. Hence $f(x,y,z) = ye^{xz}$ (taking $K=0$).

(b) $\mathbf{r}(0) = \langle 1,-1,0\rangle$, $\mathbf{r}(2) = \langle 5,3,0\rangle$ so $\int_C \mathbf{F}\cdot d\mathbf{r} = f(5,3,0) - f(1,-1,0) = 3e^0 + e^0 = 4$.

11.
$\mathbf{F}(x,y) = \langle y\cos x - xy\sin x, xy + x\cos x\rangle$ and the region D enclosed by C is given by

$\{(x,y) \mid 0 \le x \le 2, 0 \le y \le 4-2x\}$. C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_{-C}(y\cos x - xy\sin x)\,dx + (xy + x\cos x)\,dy = -\iint_D \left[\tfrac{\partial}{\partial x}(xy + x\cos x) - \tfrac{\partial}{\partial y}(y\cos x - xy\sin x)\right] dA$$
$$= -\iint_D (y - x\sin x + \cos x - \cos x + x\sin x)\,dA = -\int_0^2 \int_0^{4-2x} y\,dy\,dx$$
$$= -\int_0^2 \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x} dx = -\int_0^2 \tfrac{1}{2}(4-2x)^2\,dx = -\int_0^2 (8 - 8x + 2x^2)\,dx = -\left[8x - 4x^2 + \tfrac{2}{3}x^3\right]_0^2$$
$$= -\left(16 - 16 + \tfrac{16}{3} - 0\right) = -\tfrac{16}{3}$$

12. $\mathbf{F}(x,y) = \left\langle e^{-x} + y^2, e^{-y} + x^2\right\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid -\pi/2 \le x \le \pi/2, 0 \le y \le \cos x\}$.

C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_{-C}\left(e^{-x} + y^2\right)dx + \left(e^{-y} + x^2\right)dy = -\iint_D \left[\tfrac{\partial}{\partial x}\left(e^{-y} + x^2\right) - \tfrac{\partial}{\partial y}\left(e^{-x} + y^2\right)\right] dA$$
$$= -\int_{-\pi/2}^{\pi/2}\int_0^{\cos x}(2x - 2y)\,dy\,dx = -\int_{-\pi/2}^{\pi/2}\left[2xy - y^2\right]_{y=0}^{y=\cos x} dx$$
$$= -\int_{-\pi/2}^{\pi/2}(2x\cos x - \cos^2 x)\,dx = -\int_{-\pi/2}^{\pi/2}\left(2x\cos x - \tfrac{1}{2}(1 + \cos 2x)\right)dx$$
$$= -\left[2x\sin x + 2\cos x - \tfrac{1}{2}\left(x + \tfrac{1}{2}\sin 2x\right)\right]_{-\pi/2}^{\pi/2} \qquad \text{[integrate by parts in the first term]}$$
$$= -\left(\pi - \tfrac{1}{4}\pi - \pi - \tfrac{1}{4}\pi\right) = \tfrac{1}{2}\pi$$

13. $\mathbf{F}(x,y) = \langle y - \cos y, x\sin y\rangle$ and the region D enclosed by C is the disk with radius 2 centered at $(3,-4)$.

C is traversed clockwise, so $-C$ gives the positive orientation.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\int_{-C}(y - \cos y)\,dx + (x\sin y)\,dy = -\iint_D \left[\tfrac{\partial}{\partial x}(x\sin y) - \tfrac{\partial}{\partial y}(y - \cos y)\right] dA$$
$$= -\iint_D (\sin y - 1 - \sin y)\,dA = \iint_D dA = \text{area of } D = \pi(2)^2 = 4\pi$$

14. $\mathbf{F}(x,y) = \left\langle \sqrt{x^2+1}, \tan^{-1} x\right\rangle$ and the region D enclosed by C is given by $\{(x,y) \mid 0 \le x \le 1, x \le y \le 1\}$.

C is oriented positively, so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \sqrt{x^2+1}\,dx + \tan^{-1} x\,dy = \iint_D \left[\frac{\partial}{\partial x}\left(\tan^{-1} x\right) - \frac{\partial}{\partial y}\left(\sqrt{x^2+1}\right)\right] dA$$
$$= \int_0^1 \int_x^1 \left(\frac{1}{1+x^2} - 0\right) dy\,dx = \int_0^1 \frac{1}{1+x^2}\,[y]_{y=x}^{y=1}\,dx = \int_0^1 \frac{1}{1+x^2}(1-x)\,dx$$
$$= \int_0^1 \left(\frac{1}{1+x^2} - \frac{x}{1+x^2}\right) dx = \left[\tan^{-1} x - \frac{1}{2}\ln(1+x^2)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

18.

By Green's Theorem, $W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C x\,dx + (x^3 + 3xy^2)\,dy = \iint_D (3x^2 + 3y^2 - 0)\,dA$, where D is the semicircular region bounded by C. Converting to polar coordinates, we have $W = 3\int_0^2 \int_0^\pi r^2 \cdot r\,d\theta\,dr = 3\pi\left[\frac{1}{4}r^4\right]_0^2 = 12\pi$.

1. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x + yz & y + xz & z + xy \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(z + xy) - \frac{\partial}{\partial z}(y + xz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(z + xy) - \frac{\partial}{\partial z}(x + yz)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(y + xz) - \frac{\partial}{\partial y}(x + yz)\right]\mathbf{k}$$

$$= (x - x)\,\mathbf{i} - (y - y)\,\mathbf{j} + (z - z)\,\mathbf{k} = \mathbf{0}$$

(b) $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(x + yz) + \dfrac{\partial}{\partial y}(y + xz) + \dfrac{\partial}{\partial z}(z + xy) = 1 + 1 + 1 = 3$

3. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xye^z & 0 & yze^x \end{vmatrix} = (ze^x - 0)\,\mathbf{i} - (yze^x - xye^z)\,\mathbf{j} + (0 - xe^z)\,\mathbf{k}$

$$= ze^x\,\mathbf{i} + (xye^z - yze^x)\,\mathbf{j} - xe^z\,\mathbf{k}$$

(b) $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(xye^z) + \dfrac{\partial}{\partial y}(0) + \dfrac{\partial}{\partial z}(yze^x) = ye^z + 0 + ye^x = y(e^z + e^x)$

5. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \dfrac{x}{\sqrt{x^2 + y^2 + z^2}} & \dfrac{y}{\sqrt{x^2 + y^2 + z^2}} & \dfrac{z}{\sqrt{x^2 + y^2 + z^2}} \end{vmatrix}$

$$= \frac{1}{(x^2 + y^2 + z^2)^{3/2}}\left[(-yz + yz)\,\mathbf{i} - (-xz + xz)\,\mathbf{j} + (-xy + xy)\,\mathbf{k}\right] = \mathbf{0}$$

(b) $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}\left(\dfrac{x}{\sqrt{x^2 + y^2 + z^2}}\right) + \dfrac{\partial}{\partial y}\left(\dfrac{y}{\sqrt{x^2 + y^2 + z^2}}\right) + \dfrac{\partial}{\partial z}\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$

$$= \frac{x^2 + y^2 + z^2 - x^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{x^2 + y^2 + z^2 - y^2}{(x^2 + y^2 + z^2)^{3/2}} + \frac{x^2 + y^2 + z^2 - z^2}{(x^2 + y^2 + z^2)^{3/2}} = \frac{2x^2 + 2y^2 + 2z^2}{(x^2 + y^2 + z^2)^{3/2}} = \frac{2}{\sqrt{x^2 + y^2 + z^2}}$$

7. (a) $\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x \sin y & e^y \sin z & e^z \sin x \end{vmatrix} = (0 - e^y \cos z)\,\mathbf{i} - (e^z \cos x - 0)\,\mathbf{j} + (0 - e^x \cos y)\,\mathbf{k}$

$$= \langle -e^y \cos z, -e^z \cos x, -e^x \cos y \rangle$$

(b) $\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \dfrac{\partial}{\partial x}(e^x \sin y) + \dfrac{\partial}{\partial y}(e^y \sin z) + \dfrac{\partial}{\partial z}(e^z \sin x) = e^x \sin y + e^y \sin z + e^z \sin x$

12. (a) $\operatorname{curl} f = \nabla \times f$ is meaningless because f is a scalar field.

(b) $\operatorname{grad} f$ is a vector field.

(c) $\operatorname{div} \mathbf{F}$ is a scalar field.

(d) $\operatorname{curl}(\operatorname{grad} f)$ is a vector field.

(e) $\operatorname{grad} \mathbf{F}$ is meaningless because $\mathbf{F}$ is not a scalar field.

(f) $\operatorname{grad}(\operatorname{div} \mathbf{F})$ is a vector field.

(g) $\operatorname{div}(\operatorname{grad} f)$ is a scalar field.

(h) $\operatorname{grad}(\operatorname{div} f)$ is meaningless because f is a scalar field.

(i) $\operatorname{curl}(\operatorname{curl} \mathbf{F})$ is a vector field.

(j) $\operatorname{div}(\operatorname{div} \mathbf{F})$ is meaningless because $\operatorname{div} \mathbf{F}$ is a scalar field.

(k) $(\operatorname{grad} f) \times (\operatorname{div} \mathbf{F})$ is meaningless because $\operatorname{div} \mathbf{F}$ is a scalar field.

(l) $\operatorname{div}(\operatorname{curl}(\operatorname{grad} f))$ is a scalar field.

13. $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = (6xyz^2 - 6xyz^2)\,\mathbf{i} - (3y^2z^2 - 3y^2z^2)\,\mathbf{j} + (2yz^3 - 2yz^3)\,\mathbf{k} = \mathbf{0}$

and $\mathbf{F}$ is defined on all of $\mathbb{R}^3$ with component functions which have continuous partial derivatives, so by Theorem 4, $\mathbf{F}$ is conservative. Thus, there exists a function f such that $\mathbf{F} = \nabla f$. Then $f_x(x, y, z) = y^2z^3$ implies $f(x, y, z) = xy^2z^3 + g(y, z)$ and $f_y(x, y, z) = 2xyz^3 + g_y(y, z)$. But $f_y(x, y, z) = 2xyz^3$, so $g(y, z) = h(z)$ and $f(x, y, z) = xy^2z^3 + h(z)$. Thus $f_z(x, y, z) = 3xy^2z^2 + h'(z)$ but $f_z(x, y, z) = 3xy^2z^2$ so $h(z) = K$, a constant. Hence a potential function for $\mathbf{F}$ is $f(x, y, z) = xy^2z^3 + K$.

14. $\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz^2 & x^2yz^2 & x^2y^2z \end{vmatrix} = (2x^2yz - 2x^2yz)\,\mathbf{i} - (2xy^2z - 2xyz)\,\mathbf{j} + (2xyz^2 - xz^2)\,\mathbf{k} \neq \mathbf{0}$,

so $\mathbf{F}$ is not conservative.

17. $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^{yz} & xze^{yz} & xye^{yz} \end{vmatrix}$

$$= [xyze^{yz} + xe^{yz} - (xyze^{yz} + xe^{yz})]\,\mathbf{i} - (ye^{yz} - ye^{yz})\,\mathbf{j} + (ze^{yz} - ze^{yz})\,\mathbf{k} = \mathbf{0}$$

$\mathbf{F}$ is defined on all of $\mathbb{R}^3$, and the partial derivatives of the component functions are continuous, so $\mathbf{F}$ is conservative. Thus there exists a function f such that $\nabla f = \mathbf{F}$. Then $f_x(x,y,z) = e^{yz}$ implies $f(x,y,z) = xe^{yz} + g(y,z) \quad\Rightarrow$ $f_y(x,y,z) = xze^{yz} + g_y(y,z)$. But $f_y(x,y,z) = xze^{yz}$, so $g(y,z) = h(z)$ and $f(x,y,z) = xe^{yz} + h(z)$. Thus $f_z(x,y,z) = xye^{yz} + h'(z)$ but $f_z(x,y,z) = xye^{yz}$ so $h(z) = K$ and a potential function for $\mathbf{F}$ is $f(x,y,z) = xe^{yz} + K$.

3. $\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + (3-v)\,\mathbf{j} + (1+4u+5v)\,\mathbf{k} = \langle 0,3,1\rangle + u\,\langle 1,0,4\rangle + v\,\langle 1,-1,5\rangle$. From Example 3, we recognize this as a vector equation of a plane through the point $(0,3,1)$ and containing vectors $\mathbf{a} = \langle 1,0,4\rangle$ and $\mathbf{b} = \langle 1,-1,5\rangle$. If we wish to find a more conventional equation for the plane, a normal vector to the plane is $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 4 \\ 1 & -1 & 5 \end{vmatrix} = 4\,\mathbf{i} - \mathbf{j} - \mathbf{k}$ and an equation of the plane is $4(x-0) - (y-3) - (z-1) = 0$ or $4x - y - z = -4$.

5. $\mathbf{r}(s,t) = \langle s, t, t^2 - s^2\rangle$, so the corresponding parametric equations for the surface are $x = s,\ \ y = t,\ z = t^2 - s^2$. For any point (x,y,z) on the surface, we have $z = y^2 - x^2$. With no restrictions on the parameters, the surface is $z = y^2 - x^2$, which we recognize as a hyperbolic paraboloid.

20. From Example 3, parametric equations for the plane through the point $(0,-1,5)$ that contains the vectors $\mathbf{a} = \langle 2,1,4\rangle$ and $\mathbf{b} = \langle -3,2,5\rangle$ are $x = 0 + u(2) + v(-3) = 2u - 3v,\ y = -1 + u(1) + v(2) = -1 + u + 2v$, $z = 5 + u(4) + v(5) = 5 + 4u + 5v$.

21.
Solving the equation for x gives $x^2 = 1 + y^2 + \frac{1}{4}z^2 \quad\Rightarrow\quad x = \sqrt{1 + y^2 + \frac{1}{4}z^2}$. (We choose the positive root since we want the part of the hyperboloid that corresponds to $x \geq 0$.) If we let y and z be the parameters, parametric equations are $y = y$, $z = z,\ x = \sqrt{1 + y^2 + \frac{1}{4}z^2}$.

23. Since the cone intersects the sphere in the circle $x^2 + y^2 = 2$, $z = \sqrt{2}$ and we want the portion of the sphere above this, we can parametrize the surface as $x = x$, $y = y$, $z = \sqrt{4 - x^2 - y^2}$ where $x^2 + y^2 \le 2$.

Alternate solution: Using spherical coordinates, $x = 2\sin\phi\cos\theta$, $y = 2\sin\phi\sin\theta$, $z = 2\cos\phi$ where $0 \le \phi \le \frac{\pi}{4}$ and $0 \le \theta \le 2\pi$.

33.
$\mathbf{r}(u,v) = (u+v)\,\mathbf{i} + 3u^2\,\mathbf{j} + (u-v)\,\mathbf{k}$.

$\mathbf{r}_u = \mathbf{i} + 6u\,\mathbf{j} + \mathbf{k}$ and $\mathbf{r}_v = \mathbf{i} - \mathbf{k}$, so $\mathbf{r}_u \times \mathbf{r}_v = -6u\,\mathbf{i} + 2\,\mathbf{j} - 6u\,\mathbf{k}$. Since the point $(2,3,0)$ corresponds to $u = 1$, $v = 1$, a normal vector to the surface at $(2,3,0)$ is $-6\,\mathbf{i} + 2\,\mathbf{j} - 6\,\mathbf{k}$, and an equation of the tangent plane is $-6x + 2y - 6z = -6$ or $3x - y + 3z = 3$.

41. Here we can write $z = f(x,y) = \frac{1}{3} - \frac{1}{3}x - \frac{2}{3}y$ and D is the disk $x^2 + y^2 \le 3$, so by Formula 9 the area of the surface is

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint_D \sqrt{1 + \left(-\tfrac{1}{3}\right)^2 + \left(-\tfrac{2}{3}\right)^2}\,dA = \tfrac{\sqrt{14}}{3}\iint_D dA$$
$$= \tfrac{\sqrt{14}}{3}\,A(D) = \tfrac{\sqrt{14}}{3}\cdot\pi\left(\sqrt{3}\right)^2 = \sqrt{14}\,\pi$$

44. $z = f(x,y) = 1 + 3x + 2y^2$ with $0 \le x \le 2y$, $0 \le y \le 1$. Thus, by Formula 9,

$$A(S) = \iint_D \sqrt{1 + 3^2 + (4y)^2}\,dA = \int_0^1\int_0^{2y}\sqrt{10 + 16y^2}\,dx\,dy = \int_0^1 2y\sqrt{10 + 16y^2}\,dy$$
$$= \tfrac{1}{16}\cdot\tfrac{2}{3}(10 + 16y^2)^{3/2}\Big]_0^1 = \tfrac{1}{24}(26^{3/2} - 10^{3/2})$$

6. $\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\,\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi/2$ and

$\mathbf{r}_u \times \mathbf{r}_v = (\cos v\,\mathbf{i} + \sin v\,\mathbf{j} + \mathbf{k}) \times (-u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j}) = -u\cos v\,\mathbf{i} - u\sin v\,\mathbf{j} + u\,\mathbf{k} \quad \Rightarrow$

$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2u^2} = \sqrt{2}\,u$ [since $u \ge 0$]. Then by Formula 2,

$$\iint_S xyz\,dS = \iint_D (u\cos v)(u\sin v)(u)\ |\mathbf{r}_u \times \mathbf{r}_v|\ dA = \int_0^1\int_0^{\pi/2}(u^3\sin v\cos v)\cdot\sqrt{2}\,u\,dv\,du$$
$$= \sqrt{2}\int_0^1 u^4\,du\ \int_0^{\pi/2}\sin v\cos v\,dv = \sqrt{2}\left[\tfrac{1}{5}u^5\right]_0^1\ \left[\tfrac{1}{2}\sin^2 v\right]_0^{\pi/2} = \sqrt{2}\cdot\tfrac{1}{5}\cdot\tfrac{1}{2} = \tfrac{1}{10}\sqrt{2}$$

7. $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \pi$ and

$\mathbf{r}_u \times \mathbf{r}_v = \langle\cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 1\rangle = \langle \sin v, -\cos v, u\rangle \quad \Rightarrow$

$|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{u^2 + 1}$. Then

$$\iint_S y\,dS = \iint_D (u\sin v)\ |\mathbf{r}_u \times \mathbf{r}_v|\ dA = \int_0^1\int_0^\pi (u\sin v)\cdot\sqrt{u^2+1}\,dv\,du = \int_0^1 u\sqrt{u^2+1}\,du\ \int_0^\pi \sin v\,dv$$
$$= \left[\tfrac{1}{3}(u^2+1)^{3/2}\right]_0^1\ \left[-\cos v\right]_0^\pi = \tfrac{1}{3}(2^{3/2} - 1)\cdot 2 = \tfrac{2}{3}(2\sqrt{2} - 1)$$

9. $z = 1 + 2x + 3y$ so $\dfrac{\partial z}{\partial x} = 2$ and $\dfrac{\partial z}{\partial y} = 3$. Then by Formula 4,

$$\iint_S x^2yz\,dS = \iint_D x^2yz\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA = \int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{4+9+1}\,dy\,dx$$
$$= \sqrt{14}\int_0^3\int_0^2(x^2y + 2x^3y + 3x^2y^2)\,dy\,dx = \sqrt{14}\int_0^3\left[\tfrac{1}{2}x^2y^2 + x^3y^2 + x^2y^3\right]_{y=0}^{y=2}\,dx$$
$$= \sqrt{14}\int_0^3(10x^2 + 4x^3)\,dx = \sqrt{14}\left[\tfrac{10}{3}x^3 + x^4\right]_0^3 = 171\sqrt{14}$$

22. $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \pi$ and

$\mathbf{r}_u \times \mathbf{r}_v = \langle \cos v, \sin v, 0\rangle \times \langle -u\sin v, u\cos v, 1\rangle = \langle \sin v, -\cos v, u\rangle$. Here $\mathbf{F}(\mathbf{r}(u,v)) = v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\cos v\,\mathbf{k}$ and, by Formula 9,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA = \int_0^1\int_0^\pi(v\sin v - u\sin v\cos v + u^2\cos v)\,dv\,du$$
$$= \int_0^1\left[\sin v - v\cos v - \tfrac{1}{2}u\sin^2 v + u^2\sin v\right]_{v=0}^{v=\pi}\,du = \int_0^1 \pi\,du = \pi u\big]_0^1 = \pi$$

23. $\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$, $z = g(x,y) = 4 - x^2 - y^2$, and D is the square $[0,1]\times[0,1]$, so by Equation 10

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D[-xy(-2x) - yz(-2y) + zx]\,dA = \int_0^1\int_0^1[2x^2y + 2y^2(4-x^2-y^2) + x(4-x^2-y^2)]\,dy\,dx$$
$$= \int_0^1\left(\tfrac{1}{3}x^2 + \tfrac{11}{3}x - x^3 + \tfrac{34}{15}\right)dx = \tfrac{713}{180}$$

27.

Let S_1 be the paraboloid $y = x^2 + z^2$, $0 \le y \le 1$ and S_2 the disk $x^2 + z^2 \le 1$, $y = 1$. Since S is a closed surface, we use the outward orientation.

On S_1: $\mathbf{F}(\mathbf{r}(x,z)) = (x^2+z^2)\,\mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_x \times \mathbf{r}_z = 2x\,\mathbf{i} - \mathbf{j} + 2z\,\mathbf{k}$ (since the $\mathbf{j}$-component must be negative on S_1). Then

$$\iint_{S_1}\mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1}[-(x^2+z^2) - 2z^2]\,dA = -\int_0^{2\pi}\int_0^1(r^2 + 2r^2\sin^2\theta)\,r\,dr\,d\theta$$
$$= -\int_0^{2\pi}\int_0^1 r^3(1+2\sin^2\theta)\,dr\,d\theta = -\int_0^{2\pi}(1+1-\cos 2\theta)\,d\theta\int_0^1 r^3\,dr$$
$$= -\left[2\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^1 = -4\pi\cdot\tfrac{1}{4} = -\pi$$

On S_2: $\mathbf{F}(\mathbf{r}(x,z)) = \mathbf{j} - z\,\mathbf{k}$ and $\mathbf{r}_z \times \mathbf{r}_x = \mathbf{j}$. Then $\iint_{S_2}\mathbf{F}\cdot d\mathbf{S} = \iint_{x^2+z^2\le 1}(1)\,dA = \pi$.

Hence $\iint_S \mathbf{F}\cdot d\mathbf{S} = -\pi + \pi = 0$.

2. The boundary curve C is the circle $x^2 + y^2 = 9$, $z = 0$ oriented in the counterclockwise direction when viewed from above. A vector equation of C is $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$, $0 \le t \le 2\pi$, so $\mathbf{r}'(t) = -3\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j}$ and

$\mathbf{F}(\mathbf{r}(t)) = 2(3\sin t)(\cos 0)\,\mathbf{i} + e^{3\cos t}(\sin 0)\,\mathbf{j} + (3\cos t)e^{3\sin t}\,\mathbf{k} = 6\sin t\,\mathbf{i} + (3\cos t)e^{3\sin t}\,\mathbf{k}$. Then, by Stokes' Theorem,

$\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^{2\pi}(-18\sin^2 t + 0 + 0)\,dt = -18\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]_0^{2\pi} = -18\pi.$

5. C is the square in the plane $z = -1$. Rather than evaluating a line integral around C we can use Equation 3:

$\iint_{S_1} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_{S_2}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ where S_1 is the original cube without the bottom and S_2 is the bottom face of the cube. $\operatorname{curl}\mathbf{F} = x^2z\,\mathbf{i} + (xy - 2xyz)\,\mathbf{j} + (y - xz)\,\mathbf{k}$. For S_2, we choose $\mathbf{n} = \mathbf{k}$ so that C has the same orientation for both surfaces. Then $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = y - xz = x + y$ on S_2, where $z = -1$. Thus $\iint_{S_2}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \int_{-1}^{1}\int_{-1}^{1}(x+y)\,dx\,dy = 0$ so $\iint_{S_1}\operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 0$.

8. $\operatorname{curl}\mathbf{F} = (x - y)\,\mathbf{i} - y\,\mathbf{j} + \mathbf{k}$ and S is the portion of the plane $3x + 2y + z = 1$ over $D = \left\{(x,y) \mid 0 \le x \le \frac{1}{3}, 0 \le y \le \frac{1}{2}(1 - 3x)\right\}$. We orient S upward and use Equation 16.7.10 with $z = g(x,y) = 1 - 3x - 2y$:

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} &= \iint_D\left[-(x-y)(-3) - (-y)(-2) + 1\right]dA = \int_0^{1/3}\int_0^{(1-3x)/2}(1 + 3x - 5y)\,dy\,dx\\ &= \int_0^{1/3}\left[(1+3x)y - \tfrac{5}{2}y^2\right]_{y=0}^{y=(1-3x)/2}dx = \int_0^{1/3}\left[\tfrac{1}{2}(1+3x)(1-3x) - \tfrac{5}{2}\cdot\tfrac{1}{4}(1-3x)^2\right]dx\\ &= \int_0^{1/3}\left(-\tfrac{81}{8}x^2 + \tfrac{15}{4}x - \tfrac{1}{8}\right)dx = \left[-\tfrac{27}{8}x^3 + \tfrac{15}{8}x^2 - \tfrac{1}{8}x\right]_0^{1/3} = -\tfrac{1}{8} + \tfrac{5}{24} - \tfrac{1}{24} = \tfrac{1}{24}\end{aligned}$$

9. $\operatorname{curl}\mathbf{F} = (xe^{xy} - 2x)\,\mathbf{i} - (ye^{xy} - y)\,\mathbf{j} + (2z - z)\,\mathbf{k}$ and we take S to be the disk $x^2 + y^2 \le 16$, $z = 5$. Since C is oriented counterclockwise (from above), we orient S upward. Then $\mathbf{n} = \mathbf{k}$ and $\operatorname{curl}\mathbf{F}\cdot\mathbf{n} = 2z - z$ on S, where $z = 5$. Thus

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_S(2z - z)\;dS = \iint_S(10 - 5)\,dS = 5(\text{area of } S) = 5(\pi\cdot 4^2) = 80\pi$$

14.

The paraboloid intersects the plane $z = 1$ when $1 = 5 - x^2 - y^2 \quad \Leftrightarrow \quad x^2 + y^2 = 4$, so the boundary curve C is the circle $x^2 + y^2 = 4$, $z = 1$ oriented in the counterclockwise direction as viewed from above. We can parametrize C by $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + \mathbf{k}$, $0 \le t \le 2\pi$, and then $\mathbf{r}'(t) = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$. Thus $\mathbf{F}(\mathbf{r}(t)) = -4\sin t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 6\cos t\,\mathbf{k}$, $\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 8\sin^2 t + 4\sin t\cos t$, and

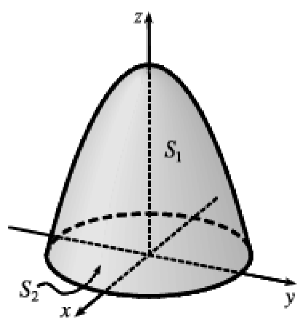
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (8\sin^2 t + 4\sin t\cos t)\,dt = 8\left(\tfrac{1}{2}t - \tfrac{1}{4}\sin 2t\right) + 2\sin^2 t\Big]_0^{2\pi} = 8\pi$$

Now $\operatorname{curl}\mathbf{F} = (-3 - 2y)\,\mathbf{j} + 2z\,\mathbf{k}$, and the projection D of S on the xy-plane is the disk $x^2 + y^2 \le 4$, so by Equation 16.7.10 with $z = g(x, y) = 5 - x^2 - y^2$ we have

$$\begin{aligned}\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} &= \iint_D [-0 - (-3 - 2y)(-2y) + 2z]\,dA = \iint_D [-6y - 4y^2 + 2(5 - x^2 - y^2)]\,dA \\ &= \int_0^{2\pi}\int_0^2 \left[-6r\sin\theta - 4r^2\sin^2\theta + 2(5 - r^2)\right] r\,dr\,d\theta = \int_0^{2\pi}\left[-2r^3\sin\theta - r^4\sin^2\theta + 5r^2 - \tfrac{1}{2}r^4\right]_{r=0}^{r=2} d\theta \\ &= \int_0^{2\pi}\left(-16\sin\theta - 16\sin^2\theta + 20 - 8\right) d\theta = 16\cos\theta - 16\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right) + 12\theta\Big]_0^{2\pi} = 8\pi\end{aligned}$$

2. $\operatorname{div}\mathbf{F} = 2x + x + 1 = 3x + 1$ so

$$\begin{aligned}\iiint_E \operatorname{div}\mathbf{F}\,dV &= \iiint_E (3x + 1)\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} (3r\cos\theta + 1)\,r\,dz\,dr\,d\theta \\ &= \int_0^2\int_0^{2\pi} r(3r\cos\theta + 1)(4 - r^2)\,d\theta\,dr \\ &= \int_0^{2\pi} r(4 - r^2)\big[3r\sin\theta + \theta\big]_{\theta=0}^{\theta=2\pi}\,dr \\ &= 2\pi\int_0^2 (4r - r^3)\,dr = 2\pi\left[2r^2 - \tfrac{1}{4}r^4\right]_0^2 \\ &= 2\pi(8 - 4) = 8\pi\end{aligned}$$



On S_1: The surface is $z = 4 - x^2 - y^2$, $x^2 + y^2 \le 4$, with upward orientation, and $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j} + (4 - x^2 - y^2)\,\mathbf{k}$. Then

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_D [-(x^2)(-2x) - (xy)(-2y) + (4 - x^2 - y^2)]\,dA \\ &= \iint_D \left[2x(x^2 + y^2) + 4 - (x^2 + y^2)\right]\,dA = \int_0^{2\pi}\int_0^2 (2r\cos\theta \cdot r^2 + 4 - r^2)\,r\,dr\,d\theta \\ &= \int_0^{2\pi}\left[\tfrac{2}{5}r^5\cos\theta + 2r^2 - \tfrac{1}{4}r^4\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi}\left(\tfrac{64}{5}\cos\theta + 4\right) d\theta = \left[\tfrac{64}{5}\sin\theta + 4\theta\right]_0^{2\pi} = 8\pi\end{aligned}$$

On S_2: The surface is $z = 0$ with downward orientation, so $\mathbf{F} = x^2\,\mathbf{i} + xy\,\mathbf{j}$, $\mathbf{n} = -\mathbf{k}$ and $\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\,dS = \iint_{S_2} 0\,dS = 0$.

Thus $\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = 8\pi$.

5. $\operatorname{div}\mathbf{F} = \frac{\partial}{\partial x}(xye^z) + \frac{\partial}{\partial y}(xy^2z^3) + \frac{\partial}{\partial z}(-ye^z) = ye^z + 2xyz^3 - ye^z = 2xyz^3$, so by the Divergence Theorem,

$$\begin{aligned}\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^3\int_0^2\int_0^1 2xyz^3\,dz\,dy\,dx = 2\int_0^3 x\,dx\int_0^2 y\,dy\int_0^1 z^3\,dz \\ &= 2\left[\tfrac{1}{2}x^2\right]_0^3\left[\tfrac{1}{2}y^2\right]_0^2\left[\tfrac{1}{4}z^4\right]_0^1 = 2\left(\tfrac{9}{2}\right)(2)\left(\tfrac{1}{4}\right) = \tfrac{9}{2}\end{aligned}$$

7. $\operatorname{div}\mathbf{F} = 3y^2 + 0 + 3z^2$, so using cylindrical coordinates with $y = r\cos\theta$, $z = r\sin\theta$, $x = x$ we have

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E (3y^2 + 3z^2)\,dV = \int_0^{2\pi}\int_0^1\int_{-1}^2 (3r^2\cos^2\theta + 3r^2\sin^2\theta)\,r\,dx\,dr\,d\theta$$
$$= 3\int_0^{2\pi} d\theta\,\int_0^1 r^3\,dr\,\int_{-1}^2 dx = 3(2\pi)\left(\tfrac{1}{4}\right)(3) = \tfrac{9\pi}{2}$$

8. $\operatorname{div}\mathbf{F} = 3x^2 + 3y^2 + 3z^2$, so by the Divergence Theorem,

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E 3(x^2 + y^2 + z^2)\,dV = \int_0^{\pi}\int_0^{2\pi}\int_0^2 3\rho^2\cdot\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = 3\int_0^{\pi}\sin\phi\,d\phi\,\int_0^{2\pi} d\theta\,\int_0^2 \rho^4\,d\rho$$
$$= 3\left[-\cos\phi\right]_0^{\pi}\,\left[\theta\right]_0^{2\pi}\,\left[\tfrac{1}{5}\rho^5\right]_0^2 = 3\,(2)\,(2\pi)\left(\tfrac{32}{5}\right) = \tfrac{384}{5}\pi$$