Multigerbes: a new theory of higher gerbes

Chris Kottke
Joint work in progress with R. Melrose

New College of Florida

Workshop on Geometric Quantization
BIRS, April 2018

Line bundles : $H^2(X;\mathbb{Z})$

- A complex line bundle $L \longrightarrow X$ has a Chern class $c_1(L) \in H^2(X;\mathbb{Z})$.
- Naturality:

$$\begin{aligned} c_1(\underline{\mathbb{C}}) &= 0, & c_1(L \otimes L') &= c_1(L) + c_1(L'), \\ c_1(L^{-1}) &= -c_1(L), & c_1(f^*L) &= f^*c_1(L) \end{aligned}$$

- $c_1(L) = c_2(L')$ if and only if $L \cong L'$.

Line bundles : $H^2(X;\mathbb{Z})$

- A complex line bundle $L \longrightarrow X$ has a Chern class $c_1(L) \in H^2(X;\mathbb{Z})$.
- Naturality:

$$c_1(\underline{\mathbb{C}}) = 0, \qquad c_1(L \otimes L') = c_1(L) + c_1(L'),$$
$$c_1(L^{-1}) = -c_1(L), \qquad c_1(f^* L) = f^* c_1(L)$$

- $c_1(L) = c_2(L')$ if and only if $L \cong L'$.
- Explicit in Čech cohomology: $[L] \in \check{C}^1(X;\mathbb{C}^*)$ satisfies $d[L] = 0$, unique up to dh, $h \in \check{C}^0(X;\mathbb{C}^*)$, so

$$[L] \in \check{H}^1(X;\mathbb{C}^*) \cong H^2(X;\mathbb{Z}).$$

- Various versions: Giraud, Brylinski, Hitchin and Chattergee, Murray.

- Various versions: Giraud, Brylinski, Hitchin and Chattergee, Murray.
- Murray: a *bundle gerbe* (L, Y, X) is
  - a *locally split* map (meaning surjective with local sections)
  $$p : Y \longrightarrow X,$$
  - a line bundle
  $$L \longrightarrow Y^{[2]} = Y \times_X Y = \{(y_1, y_2) : p(y_1) = p(y_2) \in X\}$$
  - with a product
  $$\phi : L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \xrightarrow{\cong} L_{(y_1,y_3)}, \quad (y_1, y_2, y_3) \in Y^{[3]}$$
  - satisfying associativity:
  $$\phi \circ (1 \otimes \phi) = \phi \circ (\phi \otimes 1) : L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \otimes L_{(y_3,y_4)} \cong L_{(y_1,y_4)}, \\ (y_1, y_2, y_3, y_4) \in Y^{[4]}$$
- (L, Y, X) has a *Dixmier Douady class* $DD(L, Y, X) \in H^3(X;\mathbb{Z})$.

Properties of gerbes

$$
\begin{tikzcd}
 & & L \arrow[d] \\
X & Y \arrow[l, "p"'] & Y^{[2]} \arrow[l, shift right, "p_0"'] \arrow[l, shift left, "p_1"]
\end{tikzcd}
$$

Properties of gerbes

$$
\begin{tikzcd}
 & Q \arrow[d] & L \arrow[d] \\
X & Y \arrow[l, "p"'] & Y^{[2]} \arrow[l, shift left, "p_1"'] \arrow[l, shift right, "p_0"]
\end{tikzcd}
$$

- Trivialization: an isomorphism $L \cong \delta Q := p_0^* Q \otimes p_1^* Q^{-1}$ for some line bundle $Q \longrightarrow Y$.

$$\begin{tikzcd} & & L \arrow[d] \\ X & Y \arrow[l, "p"'] & Y^{[2]} \arrow[l, shift right, "p_1"'] \arrow[l, shift left, "p_0"] \end{tikzcd}$$

- Trivialization: an isomorphism $L \cong \delta Q := p_0^* Q \otimes p_1^* Q^{-1}$ for some line bundle $Q \longrightarrow Y$.
- Inverse: $(L, Y, X)^{-1} = (L^{-1}, Y, X)$.
- Product: $(L, Y, X) \otimes (L', Y', X) = (\pi_1^* L \otimes \pi_2^* L', Y \times_X Y', X)$
- Pullback: $f^*(L, Y, X) = (f^* L, f^* Y, X')$

Properties of gerbes

$$\begin{tikzcd} & & L \arrow[d] \\ X & Y \arrow[l, "p"'] & Y^{[2]} \arrow[l, shift right, "p_0"'] \arrow[l, shift left, "p_1"] \end{tikzcd}$$

- Trivialization: an isomorphism $L \cong \delta Q := p_0^* Q \otimes p_1^* Q^{-1}$ for some line bundle $Q \longrightarrow Y$.
- Inverse: $(L, Y, X)^{-1} = (L^{-1}, Y, X)$.
- Product: $(L, Y, X) \otimes (L', Y', X) = (\pi_1^* L \otimes \pi_2^* L', Y \times_X Y', X)$
- Pullback: $f^*(L, Y, X) = (f^* L, f^* Y, X')$
- Relation with DD class:
  - $DD(L) = 0$ if and only if L is trivial.
  - $DD(L^{-1}) = -DD(L)$
  - $DD(L \otimes L') = DD(L) + DD(L')$
  - $DD(f^* L) = f^* DD(L)$.
  - $DD(L) = DD(L')$ if and only if L and L' are *stably isomorphic*, i.e., $L \otimes Q \cong L' \otimes Q'$ for trivial gerbes Q and Q'.
  - Better: $DD(L) = DD(L')$ if and only if there is a 1-isomorphism (a la Waldorf) $(L, Y, X) \longrightarrow (L', Y', X)$.

Example: lifting bundle gerbes

- $E \longrightarrow X$ principal G bundle, where G has a central extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{G} \longrightarrow G \longrightarrow 1$$

- $\widehat{G} \longrightarrow G$ defines an associated line bundle $L = \widehat{G} \times_{\mathrm{U}(1)} \mathbb{C} \longrightarrow G$
- Difference map $u : E^{[2]} \longrightarrow G$, where $u(y_0, y_1) = g$ such that $y_1 = y_0\, g$.
- $(u^* L, E, X)$ is the *lifting bundle gerbe* for E.

$$\begin{tikzcd} u^* L \arrow[d] & L \arrow[d] \\ E^{[2]} \arrow[r, "u"] \arrow[d] & G \\ X & \end{tikzcd}$$

- $DD(u^* L, E, X) \in H^3(X; \mathbb{Z})$ is the obstruction to lifting E to a $\widehat{G}$ bundle $\widehat{E} \longrightarrow X$.

$$X \longleftarrow Y \Lleftarrow Y^{[2]} \Lleftarrow Y^{[3]} \Lleftarrow Y^{[4]} \qquad \cdots$$

- These higher fiber products define a *simplicial space* over X, i.e., a sequence $\left\{ Y_n = Y^{[n+1]} : n \in \mathbb{N}_0 \right\}$ of spaces with *face maps* $p_j : Y_n \longrightarrow Y_{n-1}$, $j = 0, \ldots, n$ and *degeneracy maps* $s_j : Y_{n-1} \longrightarrow Y_n$, $j = 0, \ldots, n-1$ (all commuting with maps to X), satisfying the relations of standard simplices.

$$X \longleftarrow Y \Lleftarrow Y^{[2]} \Lleftarrow Y^{[3]} \Lleftarrow Y^{[4]} \qquad \cdots$$

- These higher fiber products define a *simplicial space* over X, i.e., a sequence $\{Y_n = Y^{[n+1]} : n \in \mathbb{N}_0\}$ of spaces with *face maps* $p_j : Y_n \longrightarrow Y_{n-1}$, $j = 0, \ldots, n$ and *degeneracy maps* $s_j : Y_{n-1} \longrightarrow Y_n$, $j = 0, \ldots, n-1$ (all commuting with maps to X), satisfying the relations of standard simplices.

$$X \longleftarrow Y_0 \Longleftarrow Y_1 \Lleftarrow Y_2 \;\substack{\longleftarrow\\\longleftarrow\\\longleftarrow\\\longleftarrow}\; Y_3 \qquad \cdots$$

- These higher fiber products define a *simplicial space* over X, i.e., a sequence $\{Y_n = Y^{[n+1]} : n \in \mathbb{N}_0\}$ of spaces with *face maps* $p_j : Y_n \longrightarrow Y_{n-1}$, $j = 0, \ldots, n$ and *degeneracy maps* $s_j : Y_{n-1} \longrightarrow Y_n$, $j = 0, \ldots, n-1$ (all commuting with maps to X), satisfying the relations of standard simplices.

Gerbes as simplicial line bundles

$$
\begin{tikzcd}
 & & L \arrow[d] & \delta L \arrow[d] & \delta^2 L \arrow[d] & \\
X & Y_0 \arrow[l] & Y_1 \arrow[l, shift left=1] \arrow[l, shift right=1] & Y_2 \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & Y_3 \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] & \cdots
\end{tikzcd}
$$

- These higher fiber products define a *simplicial space* over X, i.e., a sequence $\left\{Y_n = Y^{[n+1]} : n \in \mathbb{N}_0\right\}$ of spaces with *face maps* $p_j : Y_n \longrightarrow Y_{n-1}$, $j = 0, \dots, n$ and *degeneracy maps* $s_j : Y_{n-1} \longrightarrow Y_n$, $j = 0, \dots, n-1$ (all commuting with maps to X), satisfying the relations of standard simplices.
- [Brylinski-McLaughlin]: A *simplicial line bundle* is a line bundle $L \longrightarrow Y_1$ with a trivialization of $\delta L = p_0^* L \otimes p_1^* L^{-1} \otimes p_2^* L$ pulling back to the canonical trivialization of $\delta^2 L$.

Gerbes as simplicial line bundles

$$\begin{tikzcd}
 & & L \arrow[d] & \delta L \arrow[d] & \delta^2 L \arrow[d] & \\
X & Y_0 \arrow[l] & Y_1 \arrow[l, shift left] \arrow[l, shift right] & Y_2 \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & Y_3 \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] & \cdots
\end{tikzcd}$$

- These higher fiber products define a *simplicial space* over X, i.e., a sequence $\left\{ Y_n = Y^{[n+1]} : n \in \mathbb{N}_0 \right\}$ of spaces with *face maps* $p_j : Y_n \longrightarrow Y_{n-1}$, $j = 0, \ldots, n$ and *degeneracy maps* $s_j : Y_{n-1} \longrightarrow Y_n$, $j = 0, \ldots, n-1$ (all commuting with maps to X), satisfying the relations of standard simplices.
- [Brylinski-McLaughlin]: A *simplicial line bundle* is a line bundle $L \longrightarrow Y_1$ with a trivialization of $\delta L = p_0^* L \otimes p_1^* L^{-1} \otimes p_2^* L$ pulling back to the canonical trivialization of $\delta^2 L$.
- In case $Y_\bullet$ consists of fiber products $Y^{[\bullet-1]}$ of a locally split map $Y \longrightarrow X$, this precisely recovers the definition of a bundle gerbe.

$$\begin{tikzcd} & & L \arrow[d] & \delta L \arrow[d] & \delta^2 L \arrow[d] & \\ X & Y \arrow[l] & Y^{[2]} \arrow[l, shift left] \arrow[l, shift right] & Y^{[3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & Y^{[4]} \arrow[l, shift left=3] \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[l, shift right=3] & \cdots \end{tikzcd}$$

- These higher fiber products define a *simplicial space* over X, i.e., a sequence $\left\{Y_n = Y^{[n+1]} : n \in \mathbb{N}_0\right\}$ of spaces with *face maps* $p_j : Y_n \longrightarrow Y_{n-1}$, $j = 0, \ldots, n$ and *degeneracy maps* $s_j : Y_{n-1} \longrightarrow Y_n$, $j = 0, \ldots, n-1$ (all commuting with maps to X), satisfying the relations of standard simplices.
- [Brylinski-McLaughlin]: A *simplicial line bundle* is a line bundle $L \longrightarrow Y_1$ with a trivialization of $\delta L = p_0^* L \otimes p_1^* L^{-1} \otimes p_2^* L$ pulling back to the canonical trivialization of $\delta^2 L$.
- In case $Y_\bullet$ consists of fiber products $Y^{[\bullet-1]}$ of a locally split map $Y \longrightarrow X$, this precisely recovers the definition of a bundle gerbe.

$$
\begin{array}{ccccccc}
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y;\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(X;\mathbb{C}^*) & \xrightarrow{d} & \cdots
\end{array}
$$

Dixmier-Douady class via Čech-simplicial double complex

$$
\begin{array}{ccccccc}
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y;\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(X;\mathbb{C}^*) & \xrightarrow{d} & \cdots
\end{array}
$$

This double complex is vertically *exact* for $Y \longrightarrow X$ locally split. In particular the total cohomology is isomorphic to $\check{H}^\bullet(X;\mathbb{C}^*)$.

$$
\begin{array}{ccccccc}
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y;\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(X;\mathbb{C}^*) & \xrightarrow{d} & \cdots
\end{array}
$$

This double complex is vertically *exact* for $Y \longrightarrow X$ locally split. In particular the total cohomology is isomorphic to $\check{H}^\bullet(X;\mathbb{C}^*)$.

- The Čech chains are with respect to pairs of "admissible covers" of (X, Y) to which $p : Y \longrightarrow X$ and its local sections are adapted, including higher intersections.

$$
\begin{array}{ccccccc}
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y;\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(X;\mathbb{C}^*) & \xrightarrow{d} & \cdots
\end{array}
$$

This double complex is vertically *exact* for $Y \longrightarrow X$ locally split. In particular the total cohomology is isomorphic to $\check{H}^\bullet(X;\mathbb{C}^*)$.

- The Čech chains are with respect to pairs of "admissible covers" of (X, Y) to which $p : Y \longrightarrow X$ and its local sections are adapted, including higher intersections.
- Fiber products of the pair of covers give covers of $Y^{[k]}$ for $k > 1$.

$$
\begin{array}{ccccccc}
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y;\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(X;\mathbb{C}^*) & \xrightarrow{d} & \cdots
\end{array}
$$

This double complex is vertically *exact* for $Y \longrightarrow X$ locally split. In particular the total cohomology is isomorphic to $\check{H}^\bullet(X;\mathbb{C}^*)$.

- The Čech chains are with respect to pairs of "admissible covers" of (X, Y) to which $p : Y \longrightarrow X$ and its local sections are adapted, including higher intersections.
- Fiber products of the pair of covers give covers of $Y^{[k]}$ for $k > 1$.
- The local sections of p induce chain homotopy contractions of each vertical complex.

$$
\begin{array}{ccccccc}
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y;\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(X;\mathbb{C}^*) & \xrightarrow{d} & \cdots
\end{array}
$$

This double complex is vertically *exact* for $Y \longrightarrow X$ locally split. In particular the total cohomology is isomorphic to $\check{H}^\bullet(X;\mathbb{C}^*)$.

- The Čech chains are with respect to pairs of "admissible covers" of (X, Y) to which $p : Y \longrightarrow X$ and its local sections are adapted, including higher intersections.
- Fiber products of the pair of covers give covers of $Y^{[k]}$ for $k > 1$.
- The local sections of p induce chain homotopy contractions of each vertical complex.
- Take the direct limit over all admissible pairs of covers. Over X and Y this is equivalent to the direct limit over all covers.

Dixmier-Douady class via Čech-simplicial double complex

$$
\begin{array}{ccccccc}
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y;\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(X;\mathbb{C}^*) & \xrightarrow{d} & \cdots
\end{array}
$$

The Dixmier-Douady class is the image in $\check{H}^2(X;\mathbb{C}^*)$ of the pure cocycle $-[L] \in \check{C}^1(Y^{[2]};\mathbb{C}^*) \subset \check{C}^\bullet(Y^{[\bullet]};\mathbb{C}^*)$ in $\check{H}^2(X;\mathbb{C}^*)$.

$$
\begin{array}{ccccccc}
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y;\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(X;\mathbb{C}^*) & \xrightarrow{d} & \cdots
\end{array}
$$

The Dixmier-Douady class is the image in $\check{H}^2(X;\mathbb{C}^*)$ of the pure cocycle $-[L] \in \check{C}^1(Y^{[2]};\mathbb{C}^*) \subset \check{C}^\bullet(Y^{[\bullet]};\mathbb{C}^*)$ in $\check{H}^2(X;\mathbb{C}^*)$.

$$
\begin{array}{ccccc}
0 & & & & \\
\uparrow & & & & \\
-[L] & \longrightarrow & 0 & & \\
\uparrow & & \uparrow & & \\
\beta & \longrightarrow & d\beta & \to & 0 \\
 & & \uparrow & & \uparrow \\
 & & \alpha & \longrightarrow & 0
\end{array}
$$

so $DD(L) = [\alpha] \in \check{H}^2(X;\mathbb{C}^*) \cong H^3(X;\mathbb{Z})$.

$$
\begin{array}{ccccccc}
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y^{[2]};\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(Y;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(Y;\mathbb{C}^*) & \xrightarrow{d} & \cdots \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow & & \\
\check{C}^0(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^1(X;\mathbb{C}^*) & \xrightarrow{d} & \check{C}^2(X;\mathbb{C}^*) & \xrightarrow{d} & \cdots
\end{array}
$$

The Dixmier-Douady class is the image in $\check{H}^2(X;\mathbb{C}^*)$ of the pure cocycle $-[L] \in \check{C}^1(Y^{[2]};\mathbb{C}^*) \subset \check{C}^\bullet(Y^{[\bullet]};\mathbb{C}^*)$ in $\check{H}^2(X;\mathbb{C}^*)$.

$$
\begin{array}{ccccc}
0 & & & & \\
\uparrow & & & & \\
-[L] & \longrightarrow & 0 & & \\
\uparrow & & \uparrow & & \\
\beta & \longrightarrow & d\beta & \rightarrow & 0 \\
& & \uparrow & & \uparrow \\
& & \alpha & \longrightarrow & 0
\end{array}
$$

so $DD(L) = [\alpha] \in \check{H}^2(X;\mathbb{C}^*) \cong H^3(X;\mathbb{Z})$. It also follows that Y supports a bundle gerbe with class $[\alpha] \in H^3(X;\mathbb{Z})$ iff $p^*[\alpha] = 0 \in H^3(Y;\mathbb{Z})$.

2-gerbes : $H^4(X;\mathbb{Z})$

- Stevenson: gerbes have pullbacks, trivializations, morphisms, so we can play the same game again.

2-gerbes : $H^4(X;\mathbb{Z})$

- Stevenson: gerbes have pullbacks, trivializations, morphisms, so we can play the same game again.
- A *bundle 2-gerbe* (L, Z, Y, X) is a "simpicial bundle gerbe"

$$\begin{tikzcd} & & \mathbb{L} \arrow[d] & \delta\mathbb{L} \arrow[d] & \delta^2\mathbb{L} \arrow[d] & \\ X & Y \arrow[l] & Y^{[2]} \arrow[l, "\times 2"'] & Y^{[3]} \arrow[l, "\times 3"'] & Y^{[4]} \arrow[l, "\times 4"'] & Y^{[5]} \arrow[l, "\times 5"'] \end{tikzcd}$$

  - A locally split map $Y \longrightarrow X$,
  - A gerbe $\mathbb{L} = (L, Z, Y^{[2]})$,
  - A trivialization of $\delta\mathbb{L} = p_0^*\mathbb{L} \otimes p_1^*\mathbb{L}^{-1} \otimes p_2^*\mathbb{L}$ over $Y^{[3]}$,

2-gerbes : $H^4(X;\mathbb{Z})$

- Stevenson: gerbes have pullbacks, trivializations, morphisms, so we can play the same game again.
- A *bundle 2-gerbe* (L, Z, Y, X) is a "simpicial bundle gerbe"

$$\begin{tikzcd} & & \mathbb{L} \arrow[d] & \delta\mathbb{L} \arrow[d] & \delta^2\mathbb{L} \arrow[d] & \\ X & Y \arrow[l] & Y^{[2]} \arrow[l, "\times 2"'] & Y^{[3]} \arrow[l, "\times 3"'] & Y^{[4]} \arrow[l, "\times 4"'] & Y^{[5]} \arrow[l, "\times 5"'] \end{tikzcd}$$

  - A locally split map $Y \longrightarrow X$,
  - A gerbe $\mathbb{L} = (L, Z, Y^{[2]})$,
  - A trivialization of $\delta\mathbb{L} = p_0^*\mathbb{L} \otimes p_1^*\mathbb{L}^{-1} \otimes p_2^*\mathbb{L}$ over $Y^{[3]}$,
  - A 2-morphism (did I mention gerbes have 2-morphisms?) relating the induced trivialization of $\delta^2\mathbb{L}$ to the canonical one,
  - A coherency condition on pulled back 2-morphisms over $Y^{[5]}$.

- Stevenson: gerbes have pullbacks, trivializations, morphisms, so we can play the same game again.
- A *bundle 2-gerbe* (L, Z, Y, X) is a "simpicial bundle gerbe"

$$\begin{array}{ccccccccccc} & & & & \mathbb{L} & & \delta\mathbb{L} & & \delta^2\mathbb{L} & & \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ X & \longleftarrow & Y & \underset{\times 2}{\longleftarrow} & Y^{[2]} & \underset{\times 3}{\longleftarrow} & Y^{[3]} & \underset{\times 4}{\longleftarrow} & Y^{[4]} & \underset{\times 5}{\longleftarrow} & Y^{[5]} \end{array}$$

  - A locally split map $Y \longrightarrow X$,
  - A gerbe $\mathbb{L} = (L, Z, Y^{[2]})$,
  - A trivialization of $\delta\mathbb{L} = p_0^*\mathbb{L} \otimes p_1^*\mathbb{L}^{-1} \otimes p_2^*\mathbb{L}$ over $Y^{[3]}$,
  - A 2-morphism (did I mention gerbes have 2-morphisms?) relating the induced trivialization of $\delta^2\mathbb{L}$ to the canonical one,
  - A coherency condition on pulled back 2-morphisms over $Y^{[5]}$.
- (L, Z, Y, X) has a well-defined characteristic class $C(L, Z, Y, X) \in H^4(X;\mathbb{Z})$.

2-gerbes : $H^4(X;\mathbb{Z})$

- Stevenson: gerbes have pullbacks, trivializations, morphisms, so we can play the same game again.
- A *bundle 2-gerbe* (L, Z, Y, X) is a "simpicial bundle gerbe"

$$\begin{array}{ccccccccccc} & & & & \mathbb{L} & & \delta\mathbb{L} & & \delta^2\mathbb{L} & & \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ X & \longleftarrow & Y & \underset{\times 2}{\longleftarrow} & Y^{[2]} & \underset{\times 3}{\longleftarrow} & Y^{[3]} & \underset{\times 4}{\longleftarrow} & Y^{[4]} & \underset{\times 5}{\longleftarrow} & Y^{[5]} \end{array}$$

  - A locally split map $Y \longrightarrow X$,
  - A gerbe $\mathbb{L} = (L, Z, Y^{[2]})$,
  - A trivialization of $\delta\mathbb{L} = p_0^*\mathbb{L} \otimes p_1^*\mathbb{L}^{-1} \otimes p_2^*\mathbb{L}$ over $Y^{[3]}$,
  - A 2-morphism (did I mention gerbes have 2-morphisms?) relating the induced trivialization of $\delta^2\mathbb{L}$ to the canonical one,
  - A coherency condition on pulled back 2-morphisms over $Y^{[5]}$.
- (L, Z, Y, X) has a well-defined characteristic class $C(L, Z, Y, X) \in H^4(X;\mathbb{Z})$.
- For higher gerbes $(H^{\geq 5}(X;\mathbb{Z}))$, higher and more complicated coherency conditions will appear.
- The roles of Y and Z are very asymmetric.

A new version of 2-gerbes:

$$\begin{tikzcd} Z \arrow[d] & \\ X & Y \arrow[l] \end{tikzcd}$$

- Start with $Y \longrightarrow X$ and $Z \longrightarrow X$ locally split.

A new version of 2-gerbes:

$$\begin{tikzcd} Z & W \arrow[l] \arrow[d] \\ X & Y \arrow[l] \end{tikzcd}$$

with $Z \longrightarrow X$.

- Start with $Y \longrightarrow X$ and $Z \longrightarrow X$ locally split.
- Take $W \longrightarrow Y$, $W \longrightarrow Z$ locally split forming a commutative square. Minimal choice: $W = Y \times_X Z$, but typically W will be larger.

A new version of 2-gerbes:

$$
\begin{tikzcd}
Z^{[3]} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & & & \\
Z^{[2]} \arrow[d, shift left=1] \arrow[d, shift right=1] & & & \\
Z \arrow[d] & W \arrow[l] \arrow[d] & & \\
X & Y \arrow[l] & Y^{[2]} \arrow[l, shift left=1] \arrow[l, shift right=1] & Y^{[3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}
$$

- Start with $Y \longrightarrow X$ and $Z \longrightarrow X$ locally split.
- Take $W \longrightarrow Y$, $W \longrightarrow Z$ locally split forming a commutative square. Minimal choice: $W = Y \times_X Z$, but typically W will be larger.
- Fill out the diagram by fiber products.

Bigerbes

A new version of 2-gerbes:

$$
\begin{tikzcd}
Z^{[3]} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & W^{[1,3]} \arrow[l] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & W^{[2,3]} \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & W^{[3,3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
Z^{[2]} \arrow[d, shift left] \arrow[d, shift right] & W^{[1,2]} \arrow[l] \arrow[d, shift left] \arrow[d, shift right] & W^{[2,2]} \arrow[l, shift left] \arrow[l, shift right] \arrow[d, shift left] \arrow[d, shift right] & W^{[2,3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d, shift left] \arrow[d, shift right] \\
Z \arrow[d] & W \arrow[l] \arrow[d] & W^{[2,1]} \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & W^{[3,1]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d] \\
X & Y \arrow[l] & Y^{[2]} \arrow[l, shift left] \arrow[l, shift right] & Y^{[3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}
$$

- Start with $Y \longrightarrow X$ and $Z \longrightarrow X$ locally split.
- Take $W \longrightarrow Y$, $W \longrightarrow Z$ locally split forming a commutative square. Minimal choice: $W = Y \times_X Z$, but typically W will be larger.
- Fill out the diagram by fiber products.

A new version of 2-gerbes:

$$\begin{tikzcd}
Z^{[3]} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & W^{[1,3]} \arrow[l] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & W^{[2,3]} \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] & W^{[3,3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\
Z^{[2]} \arrow[d, shift left=1] \arrow[d, shift right=1] & W^{[1,2]} \arrow[l] \arrow[d, shift left=1] \arrow[d, shift right=1] & W^{[2,2]} \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[d, shift left=1] \arrow[d, shift right=1] & W^{[2,3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d, shift left=1] \arrow[d, shift right=1] \\
Z \arrow[d] & W \arrow[l] \arrow[d] & W^{[2,1]} \arrow[l, shift left=1] \arrow[l, shift right=1] \arrow[d] & W^{[3,1]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d] \\
X & Y \arrow[l] & Y^{[2]} \arrow[l, shift left=1] \arrow[l, shift right=1] & Y^{[3]} \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2]
\end{tikzcd}$$

- Start with $Y \longrightarrow X$ and $Z \longrightarrow X$ locally split.
- Take $W \longrightarrow Y$, $W \longrightarrow Z$ locally split forming a commutative square. Minimal choice: $W = Y \times_X Z$, but typically W will be larger.
- Fill out the diagram by fiber products.
- $W^{[\bullet,\bullet]}$ forms a *bisimplicial space over* X.

$$
\begin{tikzcd}
 & \delta_1^2 L \arrow[dl] & & & & \\
W^{[2,4]} \arrow[dd] & & & & & \\
 & \delta_1 L \arrow[dl] & & \delta_0\delta_1 L \arrow[dl] & & \\
W^{[2,3]} \arrow[dd] & & W^{[3,3]} \arrow[ll] \arrow[dd] & & & \\
 & L \arrow[dl] & & \delta_0 L \arrow[dl] & & \delta_0^2 L \arrow[dl] \\
W^{[2,2]} & & W^{[3,2]} \arrow[ll] & & W^{[4,2]} \arrow[ll] &
\end{tikzcd}
$$

Definition

A bundle bigerbe *is a "bisimplicial line bundle" over $W^{[\bullet,\bullet]}$, i.e., a line bundle L over $W^{[2,2]}$, with trivializations of $\delta_0 L$ and $\delta_1 L$, such that the induced trivializations of $\delta_0\delta_1 L$ agree and which induce the canonical trivializations of $\delta_1^2 L$ and $\delta_0^2 L$.*

Bigerbes

$$
\begin{tikzcd}
 & \delta_1^2 L \arrow[dl] & & & & \\
W^{[2,4]} \arrow[dd] & & & & & \\
 & \delta_1 L \arrow[dl] & & \delta_0\delta_1 L \arrow[dl] & & \\
W^{[2,3]} \arrow[dd] & & W^{[3,3]} \arrow[ll] \arrow[dd] & & & \\
 & L \arrow[dl] & & \delta_0 L \arrow[dl] & & \delta_0^2 L \arrow[dl] \\
W^{[2,2]} & & W^{[3,2]} \arrow[ll] & & W^{[4,2]} \arrow[ll] &
\end{tikzcd}
$$

Definition

A bundle bigerbe *is a "bisimplicial line bundle" over $W^{[\bullet,\bullet]}$, i.e., a line bundle L over $W^{[2,2]}$, with trivializations of $\delta_0 L$ and $\delta_1 L$, such that the induced trivializations of $\delta_0\delta_1 L$ agree and which induce the canonical trivializations of $\delta_1^2 L$ and $\delta_0^2 L$.*

- Products, inverses, pull backs straightforward to define.
- A *trivialization* is an isomorphism $L \cong \delta_1 Q$ (equivalently $L \cong \delta_0 Q'$) for a line bundle Q over $W^{[1,2]}$ (Q' over $W^{[2,1]}$).

Theorem

A bundle bigerbe (L, W, X) has a well-defined characteristic class $C(L) \in H^4(X;\mathbb{Z})$, with

$$\begin{aligned} C(L^{-1}) &= -C(L), \\ C(L \otimes L') &= C(L) + C(L'), \\ C(f^* L) &= f^* C(L). \end{aligned}$$

$C(L) = 0$ if and only if L is trivial. $C(L) = C(L')$ if and only if L and L' are stably isomorphic.

Theorem

A bundle bigerbe (L, W, X) *has a well-defined characteristic class* $C(L) \in H^4(X; \mathbb{Z})$, *with*

$$\begin{aligned} C(L^{-1}) &= -C(L), \\ C(L \otimes L') &= C(L) + C(L'), \\ C(f^* L) &= f^* C(L). \end{aligned}$$

$C(L) = 0$ *if and only if* L *is trivial.* $C(L) = C(L')$ *if and only if* L *and* L' *are stably isomorphic.*

- This generalizes in a straightforward manner to higher degree (Exercise), leading to *bundle multigerbes*.

Theorem

A bundle multigerbe L of degree n has a well-defined characteristic class $C(L) \in H^{2+n}(X; \mathbb{Z})$, with

$$\begin{aligned} C(L^{-1}) &= -C(L), \\ C(L \otimes L') &= C(L) + C(L'), \\ C(f^* L) &= f^* C(L). \end{aligned}$$

$C(L) = 0$ if and only if L is trivial. $C(L) = C(L')$ if and only if L and L' are stably isomorphic.

- This generalizes in a straightforward manner to higher degree (Exercise), leading to *bundle multigerbes*.

Taking Čech cochains with respect to a certain class of covers, we obtain a triple complex $\big(\check{C}^\bullet(W^{[\bullet,\bullet]};\mathbb{C}^*), d, \delta_0, \delta_1\big)$.

$$\begin{tikzcd}
\check{C}^p(Z^{[3]}) \arrow[r, "\delta_0"] & \check{C}^p(W^{[1,3]}) \arrow[r, "\delta_0"] & \check{C}^p(W^{[2,3]}) \arrow[r, "\delta_0"] & \check{C}^p(W^{[3,3]}) \\
\check{C}^p(Z^{[2]}) \arrow[r, "\delta_0"] \arrow[u, "\delta_1"] & \check{C}^p(W^{[1,2]}) \arrow[r, "\delta_0"] \arrow[u, "\delta_1"] & \check{C}^p(W^{[2,2]}) \arrow[r, "\delta_0"] \arrow[u, "\delta_1"] & \check{C}^p(W^{[3,2]}) \arrow[u, "\delta_1"] \\
\check{C}^p(Z) \arrow[r, "\delta_0"] \arrow[u, "\delta_1"] & \check{C}^p(W^{[1,1]}) \arrow[r, "\delta_0"] \arrow[u, "\delta_1"] & \check{C}^p(W^{[2,1]}) \arrow[r, "\delta_0"] \arrow[u, "\delta_1"] & \check{C}^p(W^{[3,1]}) \arrow[u, "\delta_1"] \\
\check{C}^p(X) \arrow[r, "\delta_0"] \arrow[u, "\delta_1"] & \check{C}^p(Y) \arrow[r, "\delta_0"] \arrow[u, "\delta_1"] & \check{C}^p(Y^{[2]}) \arrow[r, "\delta_0"] \arrow[u, "\delta_1"] & \check{C}^p(Y^{[3]}) \arrow[u, "\delta_1"]
\end{tikzcd}$$

- The simplicial complexes $\big(\check{C}^p(W^{[\bullet,q]};\mathbb{C}^*), \delta_0\big)$ and $\big(\check{C}^p(W^{[q,\bullet]};\mathbb{C}^*), \delta_1\big)$ are exact; in fact they admit chain homotopy contractions (commuting with each other, but not with the Čech differential).

Taking Čech cochains with respect to a certain class of covers, we obtain a triple complex $\big(\check{C}^\bullet(W^{[\bullet,\bullet]};\mathbb{C}^*), d, \delta_0, \delta_1\big)$.

$$\begin{array}{ccccccc}
\check{C}^p(Z^{[3]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[1,3]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[2,3]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[3,3]}) \\
\delta_1\uparrow & & \delta_1\uparrow & & \delta_1\uparrow & & \delta_1\uparrow \\
\check{C}^p(Z^{[2]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[1,2]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[2,2]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[3,2]}) \\
\delta_1\uparrow & & \delta_1\uparrow & & \delta_1\uparrow & & \delta_1\uparrow \\
\check{C}^p(Z) & \xrightarrow{\delta_0} & \check{C}^p(W^{[1,1]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[2,1]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[3,1]}) \\
\delta_1\uparrow & & \delta_1\uparrow & & \delta_1\uparrow & & \delta_1\uparrow \\
\check{C}^p(X) & \xrightarrow{\delta_0} & \check{C}^p(Y) & \xrightarrow{\delta_0} & \check{C}^p(Y^{[2]}) & \xrightarrow{\delta_0} & \check{C}^p(Y^{[3]})
\end{array}$$

- The simplicial complexes $\big(\check{C}^p(W^{[\bullet,q]};\mathbb{C}^*), \delta_0\big)$ and $\big(\check{C}^p(W^{[q,\bullet]};\mathbb{C}^*), \delta_1\big)$ are exact; in fact they admit chain homotopy contractions (commuting with each other, but not with the Čech differential).
- The tototal cohomology of $\big(\check{C}^\bullet(W^{[\bullet,\bullet]};\mathbb{C}^*), d, \delta_0, \delta_1\big)$ is isomorphic to $\check{H}^\bullet(X;\mathbb{C}^*)$.

Taking Čech cochains with respect to a certain class of covers, we obtain a triple complex $\left(\check{C}^\bullet(W^{[\bullet,\bullet]};\mathbb{C}^*), d, \delta_0, \delta_1\right)$.

$$\begin{array}{ccccccc}
\check{C}^p(Z^{[3]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[1,3]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[2,3]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[3,3]}) \\
\delta_1\uparrow & & \delta_1\uparrow & & \delta_1\uparrow & & \delta_1\uparrow \\
\check{C}^p(Z^{[2]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[1,2]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[2,2]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[3,2]}) \\
\delta_1\uparrow & & \delta_1\uparrow & & \delta_1\uparrow & & \delta_1\uparrow \\
\check{C}^p(Z) & \xrightarrow{\delta_0} & \check{C}^p(W^{[1,1]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[2,1]}) & \xrightarrow{\delta_0} & \check{C}^p(W^{[3,1]}) \\
\delta_1\uparrow & & \delta_1\uparrow & & \delta_1\uparrow & & \delta_1\uparrow \\
\check{C}^p(X) & \xrightarrow{\delta_0} & \check{C}^p(Y) & \xrightarrow{\delta_0} & \check{C}^p(Y^{[2]}) & \xrightarrow{\delta_0} & \check{C}^p(Y^{[3]})
\end{array}$$

- The simplicial complexes $\left(\check{C}^p(W^{[\bullet,q]};\mathbb{C}^*), \delta_0\right)$ and $\left(\check{C}^p(W^{[q,\bullet]};\mathbb{C}^*), \delta_1\right)$ are exact; in fact they admit chain homotopy contractions (commuting with each other, but not with the Čech differential).
- The tototal cohomology of $\left(\check{C}^\bullet(W^{[\bullet,\bullet]};\mathbb{C}^*), d, \delta_0, \delta_1\right)$ is isomorphic to $\check{H}^\bullet(X;\mathbb{C}^*)$.
- $C(L) \in \check{H}^3(X;\mathbb{C}^*) \cong H^4(X;\mathbb{Z})$ is the image of the pure cocycle $[L] \in \check{C}^1(W^{[2,2]};\mathbb{C}^*)$.

Existence: based loop spaces

- Suppose X is connected, and take $Y = \mathcal{P}_* X$, the based path space.
- Then $Y^{[2]} = \mathcal{P}_*^{[2]} X \cong \Omega X$, the based loop space.
- Every class in $H^3(X;\mathbb{Z})$ is represented by a bundle gerbe $(L, \mathcal{P}_* X, X)$, i.e., a simplicial line bundle L on ΩX.

- Suppose X is connected, and take $Y = \mathcal{P}_* X$, the based path space.
- Then $Y^{[2]} = \mathcal{P}_*^{[2]} X \cong \Omega X$, the based loop space.
- Every class in $H^3(X; \mathbb{Z})$ is represented by a bundle gerbe $(L, \mathcal{P}_* X, X)$, i.e., a simplicial line bundle L on ΩX.
- In this case, the gerbe product is also known as the "fusion product" (Stolz-Teichner, Waldorf), and L is a "fusion line bundle".

- Suppose X is connected, and take $Y = \mathcal{P}_* X$, the based path space.
- Then $Y^{[2]} = \mathcal{P}_*^{[2]} X \cong \Omega X$, the based loop space.
- Every class in $H^3(X; \mathbb{Z})$ is represented by a bundle gerbe $(L, \mathcal{P}_* X, X)$, i.e., a simplicial line bundle L on ΩX.
- In this case, the gerbe product is also known as the "fusion product" (Stolz-Teichner, Waldorf), and L is a "fusion line bundle".
- Likewise, if X is simply connected, with $Y = Z = \mathcal{P}_* X$ and $W = \mathcal{P}_* \mathcal{P}_* X$, Then $W^{[2,2]} = \Omega^2 X$, the double based loop space of X.
- Every class in $H^4(X; \mathbb{Z})$ is represented by a bundle bigerbe $(L, \mathcal{P}_* X, \mathcal{P}_* \mathcal{P}_* X, X)$, equivalently a "doubly fusion" line bundle $L \longrightarrow \Omega^2 X$. (c.f. Carey Johnson Murray Stevenson Wang)

- Suppose X is connected, and take $Y = \mathcal{P}_* X$, the based path space.
- Then $Y^{[2]} = \mathcal{P}_*^{[2]} X \cong \Omega X$, the based loop space.
- Every class in $H^3(X; \mathbb{Z})$ is represented by a bundle gerbe $(L, \mathcal{P}_* X, X)$, i.e., a simplicial line bundle L on ΩX.
- In this case, the gerbe product is also known as the "fusion product" (Stolz-Teichner, Waldorf), and L is a "fusion line bundle".
- Likewise, if X is simply connected, with $Y = Z = \mathcal{P}_* X$ and $W = \mathcal{P}_* \mathcal{P}_* X$, Then $W^{[2,2]} = \Omega^2 X$, the double based loop space of X.
- Every class in $H^4(X; \mathbb{Z})$ is represented by a bundle bigerbe $(L, \mathcal{P}_* X, \mathcal{P}_* \mathcal{P}_* X, X)$, equivalently a "doubly fusion" line bundle $L \longrightarrow \Omega^2 X$. (c.f. Carey Johnson Murray Stevenson Wang)

Proposition

If X is k-connected, then every class in $H^{3+k}(X; \mathbb{Z})$ is represented by a multigerbe on $\Omega^{2+k} X$, (aka a $2+k$-fold fusion line bundle).

Existence: free loop spaces

- Alternatively, take $Y = \mathcal{P}X$, the free path space, fibering over X^2.
- Then $Y^{[2]} = \mathcal{P}^{[2]}X \cong \mathcal{L}X$, the free loop space.

- Alternatively, take $Y = \mathcal{P}X$, the free path space, fibering over X^2.
- Then $Y^{[2]} = \mathcal{P}^{[2]}X \cong \mathcal{L}X$, the free loop space.
- Every class in $H^3(X;\mathbb{Z})$ is represented by a bundle gerbe $L = (L, \mathcal{P}X, X^2)$ on $\mathcal{L}X$, with the additional condition of a trivialization of the alternating product of pullbacks to the "figure-of-eight" loop space [K-Melrose, 2013].
- Figure-of-eight is yet another simplicial condition "over" the simplicial space

$$X \;\substack{\longleftarrow \\ \longleftarrow}\; X^2 \;\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}\; X^3 \;\cdots\cdots\cdots$$

guaranteeing that the class in $H^3(X^2;\mathbb{Z})$ comes from $H^3(X;\mathbb{Z})$.

- Alternatively, take $Y = \mathcal{P}X$, the free path space, fibering over X^2.
- Then $Y^{[2]} = \mathcal{P}^{[2]}X \cong \mathcal{L}X$, the free loop space.
- Every class in $H^3(X;\mathbb{Z})$ is represented by a bundle gerbe $L = (L, \mathcal{P}X, X^2)$ on $\mathcal{L}X$, with the additional condition of a trivialization of the alternating product of pullbacks to the "figure-of-eight" loop space [K-Melrose, 2013].
- Figure-of-eight is yet another simplicial condition "over" the simplicial space

$$X \leftleftarrows X^2 \Lleftarrow X^3 \;\cdots\cdots\cdots$$

guaranteeing that the class in $H^3(X^2;\mathbb{Z})$ comes from $H^3(X;\mathbb{Z})$.

Proposition

Every class in $H^{3+k}(X;\mathbb{Z})$ is represented by a multisimplicial (and multi figure-of-eight) line bundle on $\mathcal{L}^{2+k}X$.

Transgression and loop-fusion cohomology

- Take $\alpha \in H^3(X;\mathbb{Z})$ and $L \longrightarrow \mathcal{L}X$ with $DD(L, \mathcal{P}X, X^2) = \alpha$.
- $c_1(L) \in H^2(\mathcal{L}X;\mathbb{Z})$ is the *transgression* of α:

$$\begin{tikzcd} H^k(X;\mathbb{Z}) \arrow[r, "\mathrm{ev}^*"] \arrow[dr, "T"'] & H^k(\mathbb{S}^1 \times \mathcal{L}X;\mathbb{Z}) \arrow[d, "\int_{\mathbb{S}^1}"] \\ & H^{k-1}(\mathcal{L}X;\mathbb{Z}) \end{tikzcd}$$

- Loses information since it forgets the simplicial properties of L.

Transgression and loop-fusion cohomology

- Take $\alpha \in H^3(X;\mathbb{Z})$ and $L \longrightarrow \mathcal{L}X$ with $DD(L, \mathcal{P}X, X^2) = \alpha$.
- $c_1(L) \in H^2(\mathcal{L}X;\mathbb{Z})$ is the *transgression* of α:

$$\begin{tikzcd} H^k(X;\mathbb{Z}) \arrow[r, "\mathrm{ev}^*"] \arrow[dr, "T"'] & H^k(\mathbb{S}^1 \times \mathcal{L}X;\mathbb{Z}) \arrow[d, "\int_{\mathbb{S}^1}"] \\ & H^{k-1}(\mathcal{L}X;\mathbb{Z}) \end{tikzcd}$$

- Loses information since it forgets the simplicial properties of L.
- [K.-Melrose, 2015]: "Loop-fusion" Čech cohomology $H^\bullet_{\mathrm{lf}}(\mathcal{L}X;\mathbb{Z})$ such that transgression factors through an isomorphism

$$H^k(X;\mathbb{Z}) \xrightarrow{\cong} H^{k-1}_{\mathrm{lf}}(\mathcal{L}X;\mathbb{Z}) \longrightarrow H^{k-1}(\mathcal{L}X;\mathbb{Z}).$$

- Take $\alpha \in H^3(X;\mathbb{Z})$ and $L \longrightarrow \mathcal{L}X$ with $DD(L, \mathcal{P}X, X^2) = \alpha$.
- $c_1(L) \in H^2(\mathcal{L}X;\mathbb{Z})$ is the *transgression* of α:

$$\begin{tikzcd} H^k(X;\mathbb{Z}) \arrow[r, "\mathrm{ev}^*"] \arrow[dr, "T"'] & H^k(\mathbb{S}^1 \times \mathcal{L}X;\mathbb{Z}) \arrow[d, "\int_{\mathbb{S}^1}"] \\ & H^{k-1}(\mathcal{L}X;\mathbb{Z}) \end{tikzcd}$$

- Loses information since it forgets the simplicial properties of L.
- [K.-Melrose, 2015]: “Loop-fusion” Čech cohomology $H^\bullet_{\mathrm{lf}}(\mathcal{L}X;\mathbb{Z})$ such that transgression factors through an isomorphism

$$H^k(X;\mathbb{Z}) \xrightarrow{\cong} H^{k-1}_{\mathrm{lf}}(\mathcal{L}X;\mathbb{Z}) \longrightarrow H^{k-1}(\mathcal{L}X;\mathbb{Z}).$$

Theorem

On $\mathcal{L}^\ell X$ there is a well-defined loop-fusion cohomology $\check{H}^\bullet_{\mathrm{lf}}(\mathcal{L}^\ell X;\mathbb{Z})$ through which iterated transgression factors as an isomorphism:

$$H^k_{\mathrm{lf}}(\mathcal{L}^\ell X;\mathbb{Z}) \xrightarrow{\cong} H^{k-n}_{\mathrm{lf}}(\mathcal{L}^{\ell+n} X;\mathbb{Z}).$$

- Let X be a manifold and $E \longrightarrow X$ a principal G bundle for a simple, simply connected Lie group G (e.g. $G = \mathrm{Spin}$).
- Then $\mathcal{L}E \longrightarrow \mathcal{L}X$ is a $\mathcal{L}G$ bundle, and $\mathcal{L}G$ has a central extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\mathcal{L}G} \longrightarrow \mathcal{L}G \longrightarrow 1$$

- Let X be a manifold and $E \longrightarrow X$ a principal G bundle for a simple, simply connected Lie group G (e.g. $G = \mathrm{Spin}$).
- Then $\mathcal{L}E \longrightarrow \mathcal{L}X$ is a $\mathcal{L}G$ bundle, and $\mathcal{L}G$ has a central extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\mathcal{L}G} \longrightarrow \mathcal{L}G \longrightarrow 1$$

- A lift of $\mathcal{L}E$ to a $\widehat{\mathcal{L}G}$-bundle may be thought of as a "spin structure" on loop space [Atiyah, Witten].

- Let X be a manifold and $E \longrightarrow X$ a principal G bundle for a simple, simply connected Lie group G (e.g. $G = \mathrm{Spin}$).
- Then $\mathcal{L}E \longrightarrow \mathcal{L}X$ is a $\mathcal{L}G$ bundle, and $\mathcal{L}G$ has a central extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\mathcal{L}G} \longrightarrow \mathcal{L}G \longrightarrow 1$$

- A lift of $\mathcal{L}E$ to a $\widehat{\mathcal{L}G}$-bundle may be thought of as a "spin structure" on loop space [Atiyah, Witten].

Proposition

The lifting bundle gerbe $(u^*\widehat{\mathcal{L}G}, \mathcal{L}E, \mathcal{L}X)$ is a bundle bigerbe associated to the bisimplicial space generated by

$$\begin{tikzcd} E^2 \arrow[d] & \mathcal{P}E \arrow[l] \arrow[d] \\ X^2 & \mathcal{P}X \arrow[l] \end{tikzcd}$$

with Dixmier-Douady class $\frac{1}{2}p_1(E) \in H^4(X;\mathbb{Z})$.

- c.f. McLaughlin, Redden, CJMSW, Waldorf, K.-Melrose.

- Connection structures, representations of differential cohomology when X, Y, Z, W are manifolds.
- Satisfactory notion of morphisms for multigerbes.
- On $\mathcal{L}X$ (and generally $\mathcal{L}^k X$), equivariance of L with respect to action of $\mathrm{Diffeo}^+(\mathbb{S}^1)$ (and its central extension) [c.f. Brylinski].
- Loop-fusion K-theory of $\mathcal{L}X$ and $\mathcal{L}^\ell X$.