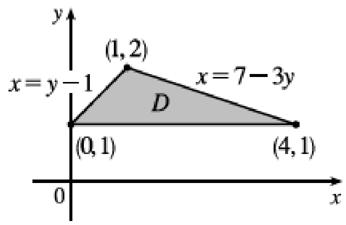
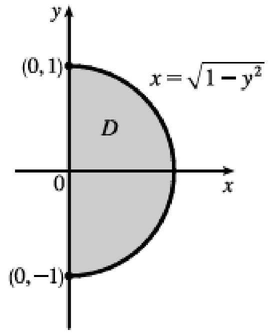
19.



$$\iint_D y^2\,dA = \int_1^2 \int_{y-1}^{7-3y} y^2\,dx\,dy = \int_1^2 \left[xy^2\right]_{x=y-1}^{x=7-3y}\,dy$$
$$= \int_1^2 \left[(7-3y)-(y-1)\right] y^2\,dy = \int_1^2 (8y^2 - 4y^3)\,dy$$
$$= \left[\tfrac{8}{3}y^3 - y^4\right]_1^2 = \tfrac{64}{3} - 16 - \tfrac{8}{3} + 1 = \tfrac{11}{3}$$

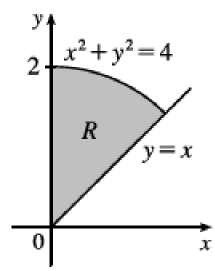
20.



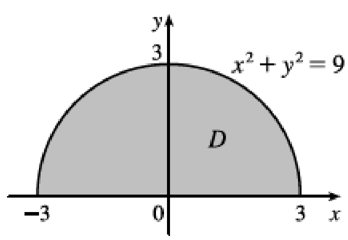
$$\iint_D xy^2\,dA = \int_{-1}^1 \int_0^{\sqrt{1-y^2}} xy^2\,dx\,dy$$
$$= \int_{-1}^1 y^2 \left[\tfrac{1}{2}x^2\right]_{x=0}^{x=\sqrt{1-y^2}}\,dy = \tfrac{1}{2}\int_{-1}^1 y^2(1-y^2)\,dy$$
$$= \tfrac{1}{2}\int_{-1}^1 (y^2 - y^4)\,dy = \tfrac{1}{2}\left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5\right]_{-1}^1$$
$$= \tfrac{1}{2}\left(\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{2}{15}$$

8. The region R is $\frac{1}{8}$ of a disk, as shown in the figure, and can be described by $R = \{(r,\theta) \mid 0 \le r \le 2,\ \pi/4 \le \theta \le \pi/2\}$. Thus

$$\iint_R (2x - y)\,dA = \int_{\pi/4}^{\pi/2} \int_0^2 (2r\cos\theta - r\sin\theta)\,r\,dr\,d\theta$$
$$= \left(\int_{\pi/4}^{\pi/2} (2\cos\theta - \sin\theta)\,d\theta\right)\left(\int_0^2 r^2\,dr\right)$$
$$= \left[2\sin\theta + \cos\theta\right]_{\pi/4}^{\pi/2} \left[\tfrac{1}{3}r^3\right]_0^2$$
$$= \left(2 + 0 - \sqrt{2} - \tfrac{\sqrt{2}}{2}\right)\left(\tfrac{8}{3}\right) = \tfrac{16}{3} - 4\sqrt{2}$$



29.



$$\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx = \int_0^\pi \int_0^3 \sin\left(r^2\right) r\,dr\,d\theta$$
$$= \int_0^\pi d\theta \int_0^3 r\sin\left(r^2\right)dr = [\theta]_0^\pi \left[-\tfrac{1}{2}\cos\left(r^2\right)\right]_0^3$$
$$= \pi\left(-\tfrac{1}{2}\right)(\cos 9 - 1) = \tfrac{\pi}{2}(1 - \cos 9)$$

8. The boundary curves intersect when $x^2 = x + 2 \quad \Leftrightarrow \quad x^2 - x - 2 = 0 \quad \Leftrightarrow \quad x = -1, x = 2$. Thus here

$D = \{(x, y) \mid -1 \le x \le 2, x^2 \le y \le x + 2\}$.

$m = \int_{-1}^{2} \int_{x^2}^{x+2} kx \, dy \, dx = k \int_{-1}^{2} x \, [y]_{y=x^2}^{y=x+2} \, dx = k \int_{-1}^{2} (x^2 + 2x - x^3) \, dx = k \left[\frac{1}{3}x^3 + x^2 - \frac{1}{4}x^4\right]_{-1}^{2} = k \left(\frac{8}{3} - \frac{5}{12}\right) = \frac{9}{4}k,$

$M_y = \int_{-1}^{2} \int_{x^2}^{x+2} kx^2 \, dy \, dx = k \int_{-1}^{2} x^2 \, [y]_{y=x^2}^{y=x+2} \, dx = k \int_{-1}^{2} (x^3 + 2x^2 - x^4) \, dx = k \left[\frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{1}{5}x^5\right]_{-1}^{2} = \frac{63}{20}k,$

$M_x = \int_{-1}^{2} \int_{x^2}^{x+2} kxy \, dy \, dx = k \int_{-1}^{2} x \left[\frac{1}{2}y^2\right]_{y=x^2}^{y=x+2} dx = \frac{1}{2}k \int_{-1}^{2} x \, (x^2 + 4x + 4 - x^4) \, dx$

$\quad = \frac{1}{2}k \int_{-1}^{2} (x^3 + 4x^2 + 4x - x^5) \, dx = \frac{1}{2}k \left[\frac{1}{4}x^4 + \frac{4}{3}x^3 + 2x^2 - \frac{1}{6}x^6\right]_{-1}^{2} = \frac{45}{8}k.$

Hence $m = \frac{9}{4}k$, $(\overline{x}, \overline{y}) = \left(\frac{63k/20}{9k/4}, \frac{45k/8}{9k/4}\right) = \left(\frac{7}{5}, \frac{5}{2}\right)$.

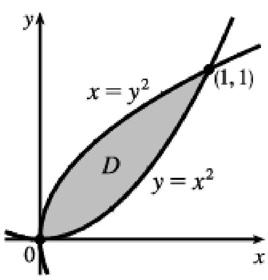
11. $\rho(x, y) = ky = kr \sin\theta$, $m = \int_0^{\pi/2} \int_0^1 kr^2 \sin\theta \, dr \, d\theta = \frac{1}{3}k \int_0^{\pi/2} \sin\theta \, d\theta = \frac{1}{3}k \left[-\cos\theta\right]_0^{\pi/2} = \frac{1}{3}k,$

$M_y = \int_0^{\pi/2} \int_0^1 kr^3 \sin\theta \cos\theta \, dr \, d\theta = \frac{1}{4}k \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta = \frac{1}{8}k \left[-\cos 2\theta\right]_0^{\pi/2} = \frac{1}{8}k,$

$M_x = \int_0^{\pi/2} \int_0^1 kr^3 \sin^2\theta \, dr \, d\theta = \frac{1}{4}k \int_0^{\pi/2} \sin^2\theta \, d\theta = \frac{1}{8}k \left[\theta + \sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{16}k.$

Hence $(\overline{x}, \overline{y}) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)$.

14.



E is the solid above the region shown in the xy-plane and below the plane $z = x + y$.

Thus,

$$\begin{aligned}
\iiint_E xy \, dV &= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy \, dz \, dy \, dx = \int_0^1 \int_{x^2}^{\sqrt{x}} xy(x + y) \, dy \, dx \\
&= \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2 y + xy^2) \, dy \, dx = \int_0^1 \left[\tfrac{1}{2}x^2 y^2 + \tfrac{1}{3}xy^3\right]_{y=x^2}^{y=\sqrt{x}} dx \\
&= \int_0^1 \left(\tfrac{1}{2}x^3 + \tfrac{1}{3}x^{5/2} - \tfrac{1}{2}x^6 - \tfrac{1}{3}x^7\right) dx \\
&= \left[\tfrac{1}{8}x^4 + \tfrac{2}{21}x^{7/2} - \tfrac{1}{14}x^7 - \tfrac{1}{24}x^8\right]_0^1 = \tfrac{1}{8} + \tfrac{2}{21} - \tfrac{1}{14} - \tfrac{1}{24} = \tfrac{3}{28}
\end{aligned}$$

41. $m = \int_0^a \int_0^a \int_0^a (x^2 + y^2 + z^2) \, dx \, dy \, dz = \int_0^a \int_0^a \left[\frac{1}{3}x^3 + xy^2 + xz^2\right]_{x=0}^{x=a} dy \, dz = \int_0^a \int_0^a \left(\frac{1}{3}a^3 + ay^2 + az^2\right) dy \, dz$

$\quad = \int_0^a \left[\frac{1}{3}a^3 y + \frac{1}{3}ay^3 + ayz^2\right]_{y=0}^{y=a} dz = \int_0^a \left(\frac{2}{3}a^4 + a^2 z^2\right) dz = \left[\frac{2}{3}a^4 z + \frac{1}{3}a^2 z^3\right]_0^a = \frac{2}{3}a^5 + \frac{1}{3}a^5 = a^5$

$M_{yz} = \int_0^a \int_0^a \int_0^a \left[x^3 + x(y^2 + z^2)\right] dx \, dy \, dz = \int_0^a \int_0^a \left[\frac{1}{4}a^4 + \frac{1}{2}a^2(y^2 + z^2)\right] dy \, dz$

$\quad = \int_0^a \left(\frac{1}{4}a^5 + \frac{1}{6}a^5 + \frac{1}{2}a^3 z^2\right) dz = \frac{1}{4}a^6 + \frac{1}{3}a^6 = \frac{7}{12}a^6 = M_{xz} = M_{xy}$ by symmetry of E and $\rho(x, y, z)$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{7}{12}a, \frac{7}{12}a, \frac{7}{12}a\right)$.

19.
The paraboloid $z = 4 - x^2 - y^2 = 4 - r^2$ intersects the xy-plane in the circle $x^2 + y^2 = 4$ or $r^2 = 4 \quad \Rightarrow \quad r = 2$, so in cylindrical coordinates, E is given by $\left\{(r, \theta, z) \,\middle|\, 0 \le \theta \le \pi/2, 0 \le r \le 2, 0 \le z \le 4 - r^2\right\}$. Thus

$$\iiint_E (x + y + z)\, dV =$$

$$\begin{aligned}
&= \int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)\, r\, dz\, dr\, d\theta = \int_0^{\pi/2} \int_0^2 \left[r^2(\cos\theta + \sin\theta)z + \tfrac{1}{2} r z^2\right]_{z=0}^{z=4-r^2} dr\, d\theta \\
&= \int_0^{\pi/2} \int_0^2 \left[(4r^2 - r^4)(\cos\theta + \sin\theta) + \tfrac{1}{2} r(4 - r^2)^2\right] dr\, d\theta \\
&= \int_0^{\pi/2} \left[\left(\tfrac{4}{3}r^3 - \tfrac{1}{5}r^5\right)(\cos\theta + \sin\theta) - \tfrac{1}{12}(4 - r^2)^3\right]_{r=0}^{r=2} d\theta \\
&= \int_0^{\pi/2} \left[\tfrac{64}{15}(\cos\theta + \sin\theta) + \tfrac{16}{3}\right] d\theta = \left[\tfrac{64}{15}(\sin\theta - \cos\theta) + \tfrac{16}{3}\theta\right]_0^{\pi/2} \\
&= \tfrac{64}{15}(1 - 0) + \tfrac{16}{3} \cdot \tfrac{\pi}{2} - \tfrac{64}{15}(0 - 1) - 0 = \tfrac{8}{3}\pi + \tfrac{128}{15}
\end{aligned}$$

24. In cylindrical coordinates, E is bounded below by the paraboloid $z = r^2$ and above by the sphere $r^2 + z^2 = 2$ or $z = \sqrt{2 - r^2}$. The paraboloid and the sphere intersect when $r^2 + r^4 = 2 \quad \Rightarrow \quad (r^2 + 2)(r^2 - 1) = 0 \quad \Rightarrow \quad r = 1$, so $E = \left\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, r^2 \le z \le \sqrt{2 - r^2}\right\}$ and the volume is

$$\begin{aligned}
\iiint_E dV &= \int_0^{2\pi} \int_0^1 \int_{r^2}^{\sqrt{2-r^2}} r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^1 \left[rz\right]_{z=r^2}^{z=\sqrt{2-r^2}} dr\, d\theta = \int_0^{2\pi} \int_0^1 \left(r\sqrt{2 - r^2} - r^3\right) dr\, d\theta \\
&= \int_0^{2\pi} d\theta \int_0^1 \left(r\sqrt{2 - r^2} - r^3\right) dr = 2\pi \left[-\tfrac{1}{3}(2 - r^2)^{3/2} - \tfrac{1}{4}r^4\right]_0^1 \\
&= 2\pi\left(-\tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{3} \cdot 2^{3/2} - 0\right) = 2\pi\left(-\tfrac{7}{12} + \tfrac{2}{3}\sqrt{2}\right) = \left(-\tfrac{7}{6} + \tfrac{4}{3}\sqrt{2}\right)\pi
\end{aligned}$$

23. In spherical coordinates, E is represented by $\{(\rho, \theta, \phi) \mid 2 \le \rho \le 3, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$ and $x^2 + y^2 = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta = \rho^2 \sin^2\phi \left(\cos^2\theta + \sin^2\theta\right) = \rho^2 \sin^2\phi$. Thus

$$\begin{aligned}
\iiint_E (x^2 + y^2)\, dV &= \int_0^{\pi} \int_0^{2\pi} \int_2^3 (\rho^2 \sin^2\phi)\, \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \int_0^{\pi} \sin^3\phi\, d\phi \int_0^{2\pi} d\theta \int_2^3 \rho^4\, d\rho \\
&= \int_0^{\pi} (1 - \cos^2\phi)\, \sin\phi\, d\phi \,\left[\,\theta\,\right]_0^{2\pi} \left[\tfrac{1}{5}\rho^5\right]_2^3 = \left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi} (2\pi) \cdot \tfrac{1}{5}(243 - 32) \\
&= \left(1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}\right)(2\pi)\left(\tfrac{211}{5}\right) = \tfrac{1688\pi}{15}
\end{aligned}$$

34. Place the center of the base at $(0, 0, 0)$, then the density is $\rho(x, y, z) = Kz$, K a constant. Then

$m = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a (K\rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 2\pi K \int_0^{\pi/2} \cos\phi \sin\phi \cdot \frac{1}{4}a^4\, d\phi = \frac{1}{2}\pi K a^4 \left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/2} = \frac{\pi}{4} K a^4.$

By the symmetry of the problem $M_{xz} = M_{yz} = 0$, and

$M_{xy} = \int_0^{2\pi} \int_0^{\pi/2} \int_0^a K\rho^4 \cos^2\phi \sin\phi\, d\rho\, d\phi\, d\theta = \frac{2}{5}\pi K a^5 \int_0^{\pi/2} \cos^2\phi \sin\phi\, d\phi = \frac{2}{5}\pi K a^5 \left[-\frac{1}{3}\cos^3\theta\right]_0^{\pi/2} = \frac{2}{15}\pi K a^5.$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \frac{8}{15}a\right)$.

4. Parametric equations for C are $x = 4t, \;\; y = 3 + 3t, \;\; 0 \le t \le 1$. Then

$$\int_C x \sin y \, ds = \int_0^1 (4t)\sin(3+3t)\sqrt{4^2+3^2}\,dt = 20\int_0^1 t\sin(3+3t)dt$$

Integrating by parts with $u = t \;\; \Rightarrow \;\; du = dt$, $dv = \sin(3+3t)dt \;\; \Rightarrow \;\; v = -\frac{1}{3}\cos(3+3t)$ gives

$$\int_C x \sin y \, ds = 20\left[-\tfrac{1}{3}t\cos(3+3t) + \tfrac{1}{9}\sin(3+3t)\right]_0^1 = 20\left[-\tfrac{1}{3}\cos 6 + \tfrac{1}{9}\sin 6 + 0 - \tfrac{1}{9}\sin 3\right]$$
$$= \tfrac{20}{9}(\sin 6 - 3\cos 6 - \sin 3)$$

20. $\mathbf{F}(\mathbf{r}(t)) = (t^2+t^3)\,\mathbf{i} + (t^3-t^2)\,\mathbf{j} + (t^2)^2\mathbf{k} = (t^2+t^3)\,\mathbf{i} + (t^3-t^2)\,\mathbf{j} + t^4\,\mathbf{k}$, $\mathbf{r}'(t) = 2t\,\mathbf{i} + 3t^2\,\mathbf{j} + 2t\,\mathbf{k}$. Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_0^1 (2t^3 + 2t^4 + 3t^5 - 3t^4 + 2t^5)\,dt = \int_0^1 (5t^5 - t^4 + 2t^3)\,dt$$
$$= \left[\tfrac{5}{6}t^6 - \tfrac{1}{5}t^5 + \tfrac{1}{2}t^4\right]_0^1 = \tfrac{5}{6} - \tfrac{1}{5} + \tfrac{1}{2} = \tfrac{17}{15}.$$