

Calculus III Workshop 10 questions: 11/9/16

Problem 1 (16.3, #20). Show that the line integral $\int_C \sin y \, dx + (x \cos y - \sin y) \, dy$ is independent of path (i.e., the vector field is conservative) and evaluate the integral where C is any path from $(2, 0)$ to $(1, \pi)$.

Problem 2 (16.3, #29). Show that if the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is conservative and P, Q, R have continuous first order partial derivatives, then

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Problem 3 (16.3, #30). Use the previous exercise to show that the line integral $\int_C y \, dx + x \, dy + xyz \, dz$ is not independent of path.

Problem 4 (16.3, #35). Let $\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$.

- (a) Show that $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$.
(b) Show that $\int_C \mathbf{F} \cdot \mathbf{T} \, ds$ is not independent of path. [Hint: compute $\int_{C_1} \mathbf{F} \cdot \mathbf{T} \, ds$ and $\int_{C_2} \mathbf{F} \cdot \mathbf{T} \, ds$ where C_1 and C_2 are the upper and lower halves of the unit circle from $(1, 0)$ to $(-1, 0)$.] Does this contradict Theorem 6 (which says if $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$ on a simply connected region, then $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is path independent)?

Problem 5 (16.4, #9). Use Green's Theorem to evaluate the line integral $\oint_C y^3 \, dx - x^3 \, dy$ where C is the circle $x^2 + y^2 = 4$ oriented counterclockwise.

Problem 6 (16.4, #22). Let D be a region bounded by a simple closed path C in the xy -plane. Use Green's Theorem to prove that the coordinates of the centroid (\bar{x}, \bar{y}) (i.e., the center of mass assuming constant density) are

$$\begin{aligned}\bar{x} &= \frac{1}{2A} \oint_C x^2 \, dy = \frac{1}{2A} \oint_C 0 \, dx + x^2 \, dy \\ \bar{y} &= -\frac{1}{2A} \oint_C y^2 \, dx = -\frac{1}{2A} \oint_C y^2 \, dx + 0 \, dy\end{aligned}$$

where A is the area of D .

Problem 7 (16.4, #23). Use the previous exercise to find the centroid of a quarter-circular region of radius a .