Calculus III Workshop 11 questions: 11/16/16

Two good problems that you might not have done from last time:

Problem 1 (16.4, #22). Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's Theorem to prove that the coordinates of the centroid $(\overline{x}, \overline{y})$ (i.e., the center of mass assuming constant density) are

$$\overline{x} = \frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\oint_C 0\,dx + x^2\,dy$$
$$\overline{y} = -\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\oint_C y^2\,dx + 0\,dy$$

where A is the area of D.

Problem 2 (16.4, #23). Use the previous exercise to find the centroid of a quarter-circular region of radius a.

New problems:

Problem 3 (16.5, #20). Is there a vector field $\mathbf{G}$ on $\mathbb{R}^3$ such that $\operatorname{curl}\mathbf{G} = \nabla\times\mathbf{G} = \langle xyz, -y^2z, yz^2\rangle$? Explain.

Problem 4 (16.5, #25, 26). Prove the following identities, assuming the appropriate partial derivatives exist. Assume f is a scalar function and $\mathbf{F}$ is a vector field.

(a) $\operatorname{div}(f\mathbf{F}) = f\operatorname{div}\mathbf{F} + \nabla f\cdot\mathbf{F}$

(b) $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + \nabla f\times\mathbf{F}$

Problem 5 (16.6, #60.(a)). Show that the parametric equations $x = a\cosh u\cos v$, $y = b\cosh u\sin v$, $z = c\sinh u$ represent a hyperboloid of one sheet. [Hint: recall that $\cosh^2 x - \sinh^2 x = 1$.]

Problem 6 (16.6, #64.(a)). Find a parametric representation for the torus obtained by rotating about the z-axis the circle in the xz-plane with center $(b, 0, 0)$ and radius $a < b$. [Hint: the torus can be viewed as a "little circle" rotated around a "big circle". Use the angles around these two circles as parameters.]

Bonus cool problems:

Problem 7 (16.5, #29). If $\mathbf{F}$ and $\mathbf{G}$ are vector fields, prove the identity $\nabla\times(\nabla\times\mathbf{F}) = \nabla(\nabla\cdot\mathbf{F}) - \nabla^2\mathbf{F}$, where ∇^2 is the *laplacian*

$$\nabla^2 f = \nabla\cdot\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2},$$
$$\nabla^2\mathbf{F} = \nabla^2(P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) = \nabla^2 P\mathbf{i} + \nabla^2 Q\mathbf{j} + \nabla^2 R\mathbf{k}.$$

Problem 8 (16.5, #38). Maxwell's equations relating the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ as they very with time in a region with no charge and no current can be stated as follows:

$$\operatorname{div}\mathbf{E} = 0 \qquad\qquad \operatorname{div}\mathbf{H} = 0$$
$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t} \qquad\qquad \operatorname{curl}\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}$$

where c is the speed of light. Use these equations, and the identity in the previous problem to prove the following

(a) $\nabla \times (\nabla \times \mathbf{E}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$

(b) $\nabla \times (\nabla \times \mathbf{H}) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$

(c) $\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$

(d) $\nabla^2 \mathbf{H} = \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$

The last two equations are known as the *wave equation*, solutions of which represent electromagnetic waves travelling through space.