7. $f(x,y) = \sin(2x+3y)$

(a) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\,\mathbf{i} + \dfrac{\partial f}{\partial y}\,\mathbf{j} = [\cos(2x+3y)\cdot 2]\,\mathbf{i} + [\cos(2x+3y)\cdot 3]\,\mathbf{j} = 2\cos(2x+3y)\,\mathbf{i} + 3\cos(2x+3y)\,\mathbf{j}$

(b) $\nabla f(-6,4) = (2\cos 0)\,\mathbf{i} + (3\cos 0)\,\mathbf{j} = 2\,\mathbf{i} + 3\,\mathbf{j}$

(c) By Equation 9, $D_{\mathbf{u}}\,f(-6,4) = \nabla f(-6,4)\cdot\mathbf{u} = (2\,\mathbf{i}+3\,\mathbf{j})\,\cdot\,\frac{1}{2}\left(\sqrt{3}\,\mathbf{i}-\mathbf{j}\right) = \frac{1}{2}\left(2\sqrt{3}-3\right) = \sqrt{3}-\frac{3}{2}$.

9. $f(x,y,z) = x^2yz - xyz^3$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \left\langle 2xyz - yz^3, x^2z - xz^3, x^2y - 3xyz^2\right\rangle$

(b) $\nabla f(2,-1,1) = \langle -4+1, 4-2, -4+6\rangle = \langle -3,2,2\rangle$

(c) By Equation 14, $D_{\mathbf{u}}f(2,-1,1) = \nabla f(2,-1,1)\cdot\mathbf{u} = \langle -3,2,2\rangle\cdot\left\langle 0, \frac{4}{5}, -\frac{3}{5}\right\rangle = 0 + \frac{8}{5} - \frac{6}{5} = \frac{2}{5}$.

11. $f(x,y) = e^x \sin y \quad\Rightarrow\quad \nabla f(x,y) = \langle e^x\sin y, e^x\cos y\rangle$, $\nabla f(0,\pi/3) = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle$, and a

unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{(-6)^2+8^2}}\langle -6,8\rangle = \frac{1}{10}\langle -6,8\rangle = \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle$, so

$D_{\mathbf{u}}\,f(0,\pi/3) = \nabla f(0,\pi/3)\cdot\mathbf{u} = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}\right\rangle\cdot\left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle = -\frac{3\sqrt{3}}{10} + \frac{4}{10} = \frac{4-3\sqrt{3}}{10}$.

21. $f(x,y) = 4y\sqrt{x} \quad\Rightarrow\quad \nabla f(x,y) = \left\langle 4y\cdot\frac{1}{2}x^{-1/2}, 4\sqrt{x}\right\rangle = \langle 2y/\sqrt{x}, 4\sqrt{x}\,\rangle$.

$\nabla f(4,1) = \langle 1,8\rangle$ is the direction of maximum rate of change, and the maximum rate is $|\nabla f(4,1)| = \sqrt{1+64} = \sqrt{65}$.

32.

$\nabla T = -400e^{-x^2-3y^2-9z^2}\langle x, 3y, 9z\rangle$

(a) $\mathbf{u} = \frac{1}{\sqrt{6}}\langle 1,-2,1\rangle$, $\nabla T(2,-1,2) = -400e^{-43}\langle 2,-3,18\rangle$ and

$D_{\mathbf{u}}\,T(2,-1,2) = \left(-\dfrac{400e^{-43}}{\sqrt{6}}\right)(26) = -\dfrac{5200\sqrt{6}}{3e^{43}}\ {}^\circ\mathrm{C/m}$.

(b) $\nabla T(2,-1,2) = 400e^{-43}\langle -2,3,-18\rangle$ or equivalently $\langle -2,3,-18\rangle$.

(c) $|\nabla T| = 400e^{-x^2-3y^2-9z^2}\sqrt{x^2+9y^2+81z^2}\ {}^\circ\mathrm{C/m}$ is the maximum rate of increase. At $(2,-1,2)$ the maximum rate of increase is $400e^{-43}\sqrt{337}\ {}^\circ\mathrm{C/m}$.

33. $\nabla V(x,y,z) = \langle 10x-3y+yz, xz-3x, xy\rangle$, $\nabla V(3,4,5) = \langle 38,6,12\rangle$

(a) $D_{\mathbf{u}}\,V(3,4,5) = \langle 38,6,12\rangle\cdot\frac{1}{\sqrt{3}}\langle 1,1,-1\rangle = \frac{32}{\sqrt{3}}$

(b) $\nabla V(3,4,5) = \langle 38,6,12\rangle$, or equivalently, $\langle 19,3,6\rangle$.

(c) $|\nabla V(3,4,5)| = \sqrt{38^2+6^2+12^2} = \sqrt{1624} = 2\sqrt{406}$

41. Let $F(x,y,z) = 2(x-2)^2 + (y-1)^2 + (z-3)^2$. Then $2(x-2)^2 + (y-1)^2 + (z-3)^2 = 10$ is a level surface of F.

$F_x(x,y,z) = 4(x-2) \quad \Rightarrow \quad F_x(3,3,5) = 4$, $F_y(x,y,z) = 2(y-1) \quad \Rightarrow \quad F_y(3,3,5) = 4$, and

$F_z(x,y,z) = 2(z-3) \quad \Rightarrow \quad F_z(3,3,5) = 4$.

(a) Equation 19 gives an equation of the tangent plane at $(3,3,5)$ as $4(x-3) + 4(y-3) + 4(z-5) = 0 \quad \Leftrightarrow$ $4x + 4y + 4z = 44$ or equivalently $x + y + z = 11$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x-3}{4} = \dfrac{y-3}{4} = \dfrac{z-5}{4}$ or equivalently $x - 3 = y - 3 = z - 5$. Corresponding parametric equations are $x = 3 + t$, $y = 3 + t$, $z = 5 + t$.

43. Let $F(x,y,z) = xyz^2$. Then $xyz^2 = 6$ is a level surface of F and $\nabla F(x,y,z) = \langle yz^2, xz^2, 2xyz \rangle$.

(a) $\nabla F(3,2,1) = \langle 2, 3, 12 \rangle$ is a normal vector for the tangent plane at $(3,2,1)$, so an equation of the tangent plane is $2(x-3) + 3(y-2) + 12(z-1) = 0$ or $2x + 3y + 12z = 24$.

(b) The normal line has direction $\langle 2, 3, 12 \rangle$, so parametric equations are $x = 3 + 2t,\ y = 2 + 3t,\ z = 1 + 12t$, and symmetric equations are $\dfrac{x-3}{2} = \dfrac{y-2}{3} = \dfrac{z-1}{12}$.

1. (a) First we compute $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (1)^2 = 7$. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$ by the Second Derivatives Test.

(b) $D(1,1) = f_{xx}(1,1)\, f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (3)^2 = -1$. Since $D(1,1) < 0$, f has a saddle point at $(1,1)$ by the Second Derivatives Test.

2. (a) $D = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(1) - (6)^2 = -37$. Since $D < 0$, g has a saddle point at $(0,2)$ by the Second Derivatives Test.

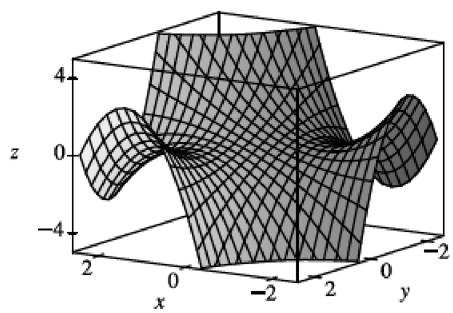
(b) $D = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(-8) - (2)^2 = 4$. Since $D > 0$ and $g_{xx}(0,2) < 0$, g has a local maximum at $(0,2)$ by the Second Derivatives Test.

(c) $D = g_{xx}(0,2)\, g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (4)(9) - (6)^2 = 0$. In this case the Second Derivatives Test gives no information about g at the point $(0,2)$.

3. In the figure, a point at approximately $(1,1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1,1)$. The level curves near $(0,0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

To verify our predictions, we have $f(x,y) = 4 + x^3 + y^3 - 3xy \quad\Rightarrow\quad f_x(x,y) = 3x^2 - 3y$, $f_y(x,y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$, $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3(x^2)^2 - 3x = 0 \quad\Rightarrow\quad 3x(x^3 - 1) = 0 \quad\Rightarrow\quad x = 0$ or $x = 1$. Then we have two critical points, $(0,0)$ and $(1,1)$. The second partial derivatives are $f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = -3$, and $f_{yy}(x,y) = 6y$, so $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then $D(0,0) = 36(0)(0) - 9 = -9$, and $D(1,1) = 36(1)(1) - 9 = 27$. Since $D(0,0) < 0$, f has a saddle point at $(0,0)$ by the Second Derivatives Test. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$.

7. $f(x,y) = (x-y)(1-xy) = x - y - x^2y + xy^2 \quad\Rightarrow\quad f_x = 1 - 2xy + y^2,\ f_y = -1 - x^2 + 2xy,\ f_{xx} = -2y,$ $f_{xy} = -2x + 2y,\ f_{yy} = 2x$. Then $f_x = 0$ implies $1 - 2xy + y^2 = 0$ and $f_y = 0$ implies $-1 - x^2 + 2xy = 0$. Adding the two equations gives $1 + y^2 - 1 - x^2 = 0 \quad\Rightarrow\quad y^2 = x^2 \quad\Rightarrow\quad y = \pm x$, but if $y = -x$ then $f_x = 0$ implies $1 + 2x^2 + x^2 = 0 \quad\Rightarrow\quad 3x^2 = -1$ which has no real solution. If $y = x$ then substitution into $f_x = 0$ gives $1 - 2x^2 + x^2 = 0 \quad\Rightarrow\quad x^2 = 1 \quad\Rightarrow\quad x = \pm 1$, so the critical points are $(1,1)$ and $(-1,-1)$. Now $D(1,1) = (-2)(2) - 0^2 = -4 < 0$ and $D(-1,-1) = (2)(-2) - 0^2 = -4 < 0$, so $(1,1)$ and $(-1,-1)$ are saddle points.



8. $f(x,y) = xe^{-2x^2-2y^2} \quad \Rightarrow \quad f_x = (1-4x^2)e^{-2x^2-2y^2},\ f_y = -4xye^{-2x^2-2y^2},\ f_{xx} = (16x^2-12)xe^{-2x^2-2y^2},$

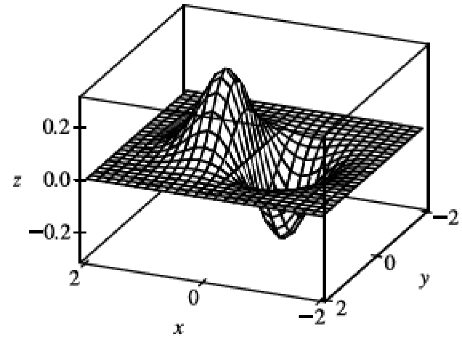
$f_{xy} = (16x^2-4)ye^{-2x^2-2y^2},\ f_{yy} = (16y^2-4)xe^{-2x^2-2y^2}$. Then $f_x = 0$ implies $1-4x^2 = 0 \quad \Rightarrow \quad x = \pm\frac{1}{2}$, and substitution into $f_y = 0 \quad \Rightarrow \quad -4xy = 0$ gives $y = 0$, so the critical points are $\left(\pm\frac{1}{2}, 0\right)$. Now

$D\left(\frac{1}{2}, 0\right) = (-4e^{-1/2})(-2e^{-1/2}) - 0^2 = 8e^{-1} > 0$ and

$f_{xx}\left(\frac{1}{2}, 0\right) = -4e^{-1/2} < 0$, so $f\left(\frac{1}{2}, 0\right) = \frac{1}{2}e^{-1/2}$ is a local maximum.

$D\left(-\frac{1}{2}, 0\right) = (4e^{-1/2})(2e^{-1/2}) - 0^2 = 8e^{-1} > 0$ and

$f_{xx}\left(-\frac{1}{2}, 0\right) = 4e^{-1/2} > 0$, so $f\left(-\frac{1}{2}, 0\right) = -\frac{1}{2}e^{-1/2}$ is a local minimum.



19.
$f(x,y) = x^2 + 4y^2 - 4xy + 2 \quad \Rightarrow \quad f_x = 2x - 4y$, $f_y = 8y - 4x$, $f_{xx} = 2$, $f_{xy} = -4$, $f_{yy} = 8$. Then $f_x = 0$ and $f_y = 0$ each implies $y = \frac{1}{2}x$, so all points of the form $\left(x_0, \frac{1}{2}x_0\right)$ are critical points and for each of these we have $D\left(x_0, \frac{1}{2}x_0\right) = (2)(8) - (-4)^2 = 0$. The Second Derivatives Test gives no information, but $f(x,y) = x^2 + 4y^2 - 4xy + 2 = (x-2y)^2 + 2 \geq 2$ with equality if and only if $y = \frac{1}{2}x$. Thus $f\left(x_0, \frac{1}{2}x_0\right) = 2$ are all local (and absolute) minima.