Calc III: Quiz 5 Solutions, Fall 2017

Problem 1. Evaluate the triple integral $\int_0^3 \int_0^2 \int_0^1 xyz\,dx\,dy\,dz$.

Solution. Because of the constant limits and the fact that the integrand is a product of single variable functions, we can evaluate the integral as

$$\begin{aligned}\int_0^3 \int_0^2 \int_0^1 xyz\,dx\,dy\,dz &= \int_0^3 z\,dz \int_0^2 y\,dy \int_0^1 x\,dx \\ &= \frac{z^2}{2}\Big|_{z=0}^3 \frac{y^2}{2}\Big|_{y=0}^2 \frac{x^2}{2}\Big|_{x=0}^2 \\ &= \frac{9}{2}.\end{aligned}$$

□

Problem 2. Find the volume inside both the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 1$, using cylindrical coordinates.

Solution. The cylinder admits a difficult description in spherical coordinates, whereas both surfaces are more or less easily described in cylindrical coordinates, where the sphere is given by $r^2 + z^2 = 4$ and the cylinder by $r = 1$. The solid region contained in both shapes is therefore

$$E = \left\{(r, \theta, z) : -\sqrt{4 - r^2} \leq z \leq \sqrt{4 - r^2},\ 0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi\right\},$$

and the volume is therefore (remembering that $dV = r\,dz\,dr\,d\theta$ in cylindrical coordinates)

$$\begin{aligned}\mathrm{Vol}(E) &= \int_0^{2\pi} \int_0^1 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta \\ &= 2\pi \int_0^1 2r\sqrt{4 - r^2}\,dr \\ &= \frac{4\pi}{3}(4^{3/2} - 3^{3/2}).\end{aligned}$$

□

Problem 3. Find the volume inside both the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$, using spherical coordinates.

Solution. The sphere is given by $\rho = 1$ in spherical coordinates, while the cone is given by $\varphi = \pi/4$. The region of integration is

$$E = \{(\rho, \varphi, \theta) : 0 \leq \rho \leq 1,\ 0 \leq \varphi \leq \pi/4,\ 0 \leq \theta \leq 2\pi\},$$

and the volume is given by (remembering that $dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$)

$$\begin{aligned}
\mathrm{Vol}(E) &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho \sin\varphi \, d\rho \, d\varphi \, d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi \, d\varphi \int_0^1 \rho^2 \, d\rho \\
&= 2\pi \big(\cos(0) - \cos(\pi/4) \big) (1/3) \\
&= \frac{\pi}{3} (2 - \sqrt{2})
\end{aligned}$$

$\square$