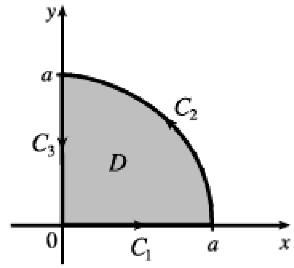
22. By Green's Theorem, $\frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\iint_D 2x\,dA = \frac{1}{A}\iint_D x\,dA = \overline{x}$ and $-\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\iint_D(-2y)\,dA = \frac{1}{A}\iint_D y\,dA = \overline{y}$.

23. We orient the quarter-circular region as shown in the figure.



$A = \frac{1}{4}\pi a^2$ so $\overline{x} = \dfrac{1}{\pi a^2/2}\displaystyle\oint_C x^2\,dy$ and $\overline{y} = -\dfrac{1}{\pi a^2/2}\displaystyle\oint_C y^2\,dx$.

Here $C = C_1 + C_2 + C_3$ where C_1: $x = t,\ \ y = 0,\ \ 0 \le t \le a$;

C_2: $x = a\cos t,\ \ y = a\sin t,\ \ 0 \le t \le \frac{\pi}{2}$; and

C_3: $x = 0, y = a - t, 0 \le t \le a$. Then

$$\oint_C x^2\,dy = \int_{C_1} x^2\,dy + \int_{C_2} x^2\,dy + \int_{C_3} x^2\,dy = \int_0^a 0\,dt + \int_0^{\pi/2}(a\cos t)^2(a\cos t)\,dt + \int_0^a 0\,dt$$
$$= \int_0^{\pi/2} a^3\cos^3 t\,dt = a^3\int_0^{\pi/2}(1-\sin^2 t)\cos t\,dt = a^3\left[\sin t - \tfrac{1}{3}\sin^3 t\right]_0^{\pi/2} = \tfrac{2}{3}a^3$$

so $\overline{x} = \dfrac{1}{\pi a^2/2}\displaystyle\oint_C x^2\,dy = \dfrac{4a}{3\pi}$.

$$\oint_C y^2\,dx = \int_{C_1} y^2\,dx + \int_{C_2} y^2\,dx + \int_{C_3} y^2\,dx = \int_0^a 0\,dt + \int_0^{\pi/2}(a\sin t)^2(-a\sin t)\,dt + \int_0^a 0\,dt$$
$$= \int_0^{\pi/2}(-a^3\sin^3 t)\,dt = -a^3\int_0^{\pi/2}(1-\cos^2 t)\sin t\,dt = -a^3\left[\tfrac{1}{3}\cos^3 t - \cos t\right]_0^{\pi/2} = -\tfrac{2}{3}a^3,$$

so $\overline{y} = -\dfrac{1}{\pi a^2/2}\displaystyle\oint_C y^2\,dx = \dfrac{4a}{3\pi}$. Thus $(\overline{x},\overline{y}) = \left(\dfrac{4a}{3\pi}, \dfrac{4a}{3\pi}\right)$.

20. No. Assume there is such a $\mathbf{G}$. Then $\operatorname{div}(\operatorname{curl}\mathbf{G}) = yz - 2yz + 2yz = yz \neq 0$ which contradicts Theorem 11.

For Exercises 23–29, let $\mathbf{F}(x,y,z) = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G}(x,y,z) = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$.

25. $\operatorname{div}(f\mathbf{F}) = \operatorname{div}(f\langle P_1, Q_1, R_1\rangle) = \operatorname{div}\langle fP_1, fQ_1, fR_1\rangle = \dfrac{\partial(fP_1)}{\partial x} + \dfrac{\partial(fQ_1)}{\partial y} + \dfrac{\partial(fR_1)}{\partial z}$

$$= \left(f\frac{\partial P_1}{\partial x} + P_1\frac{\partial f}{\partial x}\right) + \left(f\frac{\partial Q_1}{\partial y} + Q_1\frac{\partial f}{\partial y}\right) + \left(f\frac{\partial R_1}{\partial z} + R_1\frac{\partial f}{\partial z}\right)$$
$$= f\left(\frac{\partial P_1}{\partial x} + \frac{\partial Q_1}{\partial y} + \frac{\partial R_1}{\partial z}\right) + \langle P_1, Q_1, R_1\rangle\cdot\left\langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right\rangle = f\operatorname{div}\mathbf{F} + \mathbf{F}\cdot\nabla f$$

For Exercises 23−29, let $\mathbf{F}(x, y, z) = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G}(x, y, z) = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$.

26.
$$\begin{aligned}
\operatorname{curl}(f\mathbf{F}) &= \left[\frac{\partial(fR_1)}{\partial y} - \frac{\partial(fQ_1)}{\partial z}\right]\mathbf{i} + \left[\frac{\partial(fP_1)}{\partial z} - \frac{\partial(fR_1)}{\partial x}\right]\mathbf{j} + \left[\frac{\partial(fQ_1)}{\partial x} - \frac{\partial(fP_1)}{\partial y}\right]\mathbf{k} \\
&= \left[f\,\frac{\partial R_1}{\partial y} + R_1\,\frac{\partial f}{\partial y} - f\,\frac{\partial Q_1}{\partial z} - Q_1\,\frac{\partial f}{\partial z}\right]\mathbf{i} + \left[f\,\frac{\partial P_1}{\partial z} + P_1\,\frac{\partial f}{\partial z} - f\,\frac{\partial R_1}{\partial x} - R_1\,\frac{\partial f}{\partial x}\right]\mathbf{j} \\
&\qquad + \left[f\,\frac{\partial Q_1}{\partial x} + Q_1\,\frac{\partial f}{\partial x} - f\,\frac{\partial P_1}{\partial y} - P_1\,\frac{\partial f}{\partial y}\right]\mathbf{k} \\
&= f\left[\frac{\partial R_1}{\partial y} - \frac{\partial Q_1}{\partial z}\right]\mathbf{i} + f\left[\frac{\partial P_1}{\partial z} - \frac{\partial R_1}{\partial x}\right]\mathbf{j} + f\left[\frac{\partial Q_1}{\partial x} - \frac{\partial P_1}{\partial y}\right]\mathbf{k} \\
&\qquad + \left[R_1\frac{\partial f}{\partial y} - Q_1\frac{\partial f}{\partial z}\right]\mathbf{i} + \left[P_1\frac{\partial f}{\partial z} - R_1\frac{\partial f}{\partial x}\right]\mathbf{j} + \left[Q_1\frac{\partial f}{\partial x} - P_1\frac{\partial f}{\partial y}\right]\mathbf{k} \\
&= f\operatorname{curl}\mathbf{F} + (\nabla f) \times \mathbf{F}
\end{aligned}$$

For Exercises 23–29, let $\mathbf{F}(x, y, z) = P_1\,\mathbf{i} + Q_1\,\mathbf{j} + R_1\,\mathbf{k}$ and $\mathbf{G}(x, y, z) = P_2\,\mathbf{i} + Q_2\,\mathbf{j} + R_2\,\mathbf{k}$.

29. $\operatorname{curl}(\operatorname{curl}\mathbf{F}) = \nabla \times (\nabla \times \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial R_1/\partial y - \partial Q_1/\partial z & \partial P_1/\partial z - \partial R_1/\partial x & \partial Q_1/\partial x - \partial P_1/\partial y \end{vmatrix}$

$$= \left(\frac{\partial^2 Q_1}{\partial y \partial x} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2} + \frac{\partial^2 R_1}{\partial z \partial x}\right)\mathbf{i} + \left(\frac{\partial^2 R_1}{\partial z \partial y} - \frac{\partial^2 Q_1}{\partial z^2} - \frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial x \partial y}\right)\mathbf{j}$$
$$+ \left(\frac{\partial^2 P_1}{\partial x \partial z} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial y \partial z}\right)\mathbf{k}$$

Now let's consider $\operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F}$ and compare with the above.
(Note that $\nabla^2\mathbf{F}$ is defined on page 1119 [ET 1095].)

$$\operatorname{grad}(\operatorname{div}\mathbf{F}) - \nabla^2\mathbf{F} = \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 R_1}{\partial x \partial z}\right)\mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y \partial x} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial y \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 P_1}{\partial z \partial x} + \frac{\partial^2 Q_1}{\partial z \partial y} + \frac{\partial^2 R_1}{\partial z^2}\right)\mathbf{k}\right]$$
$$- \left[\left(\frac{\partial^2 P_1}{\partial x^2} + \frac{\partial^2 P_1}{\partial y^2} + \frac{\partial^2 P_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 Q_1}{\partial x^2} + \frac{\partial^2 Q_1}{\partial y^2} + \frac{\partial^2 Q_1}{\partial z^2}\right)\mathbf{j}\right.$$
$$\left. + \left(\frac{\partial^2 R_1}{\partial x^2} + \frac{\partial^2 R_1}{\partial y^2} + \frac{\partial^2 R_1}{\partial z^2}\right)\mathbf{k}\right]$$
$$= \left(\frac{\partial^2 Q_1}{\partial x \partial y} + \frac{\partial^2 R_1}{\partial x \partial z} - \frac{\partial^2 P_1}{\partial y^2} - \frac{\partial^2 P_1}{\partial z^2}\right)\mathbf{i} + \left(\frac{\partial^2 P_1}{\partial y \partial x} + \frac{\partial^2 R_1}{\partial y \partial z} - \frac{\partial^2 Q_1}{\partial x^2} - \frac{\partial^2 Q_1}{\partial z^2}\right)\mathbf{j}$$
$$+ \left(\frac{\partial^2 P_1}{\partial z \partial x} + \frac{\partial^2 Q_1}{\partial z \partial y} - \frac{\partial^2 R_1}{\partial x^2} - \frac{\partial^2 R_2}{\partial y^2}\right)\mathbf{k}$$

Then applying Clairaut's Theorem to reverse the order of differentiation in the second partial derivatives as needed and comparing, we have $\operatorname{curl}\operatorname{curl}\mathbf{F} = \operatorname{grad}\operatorname{div}\mathbf{F} - \nabla^2\mathbf{F}$ as desired.

38.

Let $\mathbf{H} = \langle h_1, h_2, h_3 \rangle$ and $\mathbf{E} = \langle E_1, E_2, E_3 \rangle$.

(a) $\nabla \times (\nabla \times \mathbf{E}) = \nabla \times (\operatorname{curl} \mathbf{E}) = \nabla \times \left(-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{1}{c}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial h_1/\partial t & \partial h_2/\partial t & \partial h_3/\partial t \end{vmatrix}$

$$= -\frac{1}{c}\left[\left(\frac{\partial^2 h_3}{\partial y\,\partial t} - \frac{\partial^2 h_2}{\partial z\,\partial t}\right)\mathbf{i} + \left(\frac{\partial^2 h_1}{\partial z\,\partial t} - \frac{\partial^2 h_3}{\partial x\,\partial t}\right)\mathbf{j} + \left(\frac{\partial^2 h_2}{\partial x\,\partial t} - \frac{\partial^2 h_1}{\partial y\,\partial t}\right)\mathbf{k}\right]$$

$$= -\frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial h_3}{\partial y} - \frac{\partial h_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial h_1}{\partial z} - \frac{\partial h_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right)\mathbf{k}\right]$$

[assuming that the partial derivatives are continuous so that the order of differentiation does not matter]

$$= -\frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl}\mathbf{H} = -\frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}$$

(b) $\nabla \times (\nabla \times \mathbf{H}) = \nabla \times (\operatorname{curl} \mathbf{H}) = \nabla \times \left(\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}\right) = \frac{1}{c}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial E_1/\partial t & \partial E_2/\partial t & \partial E_3/\partial t \end{vmatrix}$

$$= \frac{1}{c}\left[\left(\frac{\partial^2 E_3}{\partial y\,\partial t} - \frac{\partial^2 E_2}{\partial z\,\partial t}\right)\mathbf{i} + \left(\frac{\partial^2 E_1}{\partial z\,\partial t} - \frac{\partial^2 E_3}{\partial x\,\partial t}\right)\mathbf{j} + \left(\frac{\partial^2 E_2}{\partial x\,\partial t} - \frac{\partial^2 E_1}{\partial y\,\partial t}\right)\mathbf{k}\right]$$

$$= \frac{1}{c}\frac{\partial}{\partial t}\left[\left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right)\mathbf{k}\right]$$

[assuming that the partial derivatives are continuous so that the order of differentiation does not matter]

$$= \frac{1}{c}\frac{\partial}{\partial t}\operatorname{curl}\mathbf{E} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$$

(c) Using Exercise 29, we have that $\operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{grad}\operatorname{div}\mathbf{E} - \nabla^2\mathbf{E} \quad \Rightarrow$

$$\nabla^2\mathbf{E} = \operatorname{grad}\operatorname{div}\mathbf{E} - \operatorname{curl}\operatorname{curl}\mathbf{E} = \operatorname{grad}\mathbf{0} + \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \quad \text{[from part (a)]} \quad = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

(d) As in part (c), $\nabla^2\mathbf{H} = \operatorname{grad}\operatorname{div}\mathbf{H} - \operatorname{curl}\operatorname{curl}\mathbf{H} = \operatorname{grad}\mathbf{0} + \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$ [using part (b)] $= \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}$.

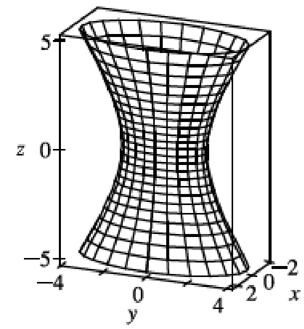
60.

(a) $x = a\cosh u\,\cos v,\quad y = b\cosh u\,\sin v,\quad z = c\sinh u \quad\Rightarrow$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \cosh^2 u\,\cos^2 v + \cosh^2 u\,\sin^2 v - \sinh^2 u$$
$$= \cosh^2 u - \sinh^2 u = 1$$

and the parametric equations represent a hyperboloid of one sheet.

(b)



(c) $\mathbf{r}_u = \sinh u\,\cos v\,\mathbf{i} + 2\sinh u\,\sin v\,\mathbf{j} + 3\cosh u\,\mathbf{k}$ and

$\mathbf{r}_v = -\cosh u\,\sin v\,\mathbf{i} + 2\cosh u\,\cos v\,\mathbf{j}$, so $\mathbf{r}_u \times \mathbf{r}_v = -6\cosh^2 u\,\cos v\,\mathbf{i} - 3\cosh^2 u\,\sin v\,\mathbf{j} + 2\cosh u\,\sinh u\,\mathbf{k}$.

We integrate between $u = \sinh^{-1}(-1) = -\ln\left(1+\sqrt{2}\right)$ and $u = \sinh^{-1} 1 = \ln\left(1+\sqrt{2}\right)$, since then z varies between -3 and 3, as desired. So the surface area is

$$A(S) = \int_0^{2\pi}\int_{-\ln\left(1+\sqrt{2}\right)}^{\ln\left(1+\sqrt{2}\right)} |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv$$
$$= \int_0^{2\pi}\int_{-\ln\left(1+\sqrt{2}\right)}^{\ln\left(1+\sqrt{2}\right)} \sqrt{36\cosh^4 u\,\cos^2 v + 9\cosh^4 u\,\sin^2 v + 4\cosh^2 u\,\sinh^2 u}\,du\,dv$$

64. (a) Here $z = a\sin\alpha$, $y = |AB|$, and $x = |OA|$. But

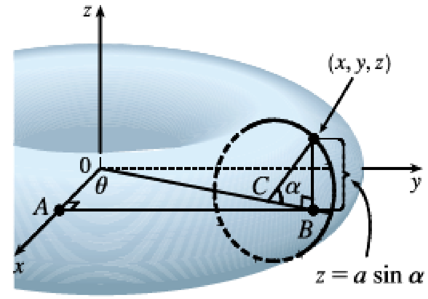
$|OB| = |OC| + |CB| = b + a\cos\alpha$ and $\sin\theta = \dfrac{|AB|}{|OB|}$ so that

$y = |OB|\sin\theta = (b + a\cos\alpha)\sin\theta$. Similarly $\cos\theta = \dfrac{|OA|}{|OB|}$ so

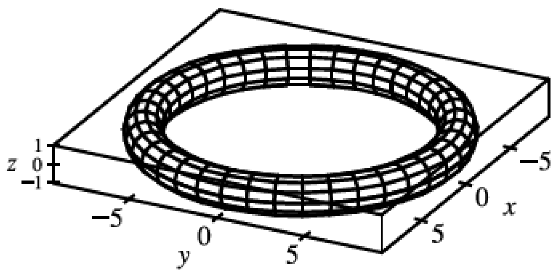
$x = (b + a\cos\alpha)\cos\theta$. Hence a parametric representation for the

torus is $x = b\cos\theta + a\cos\alpha\,\cos\theta$, $y = b\sin\theta + a\cos\alpha\,\sin\theta$,

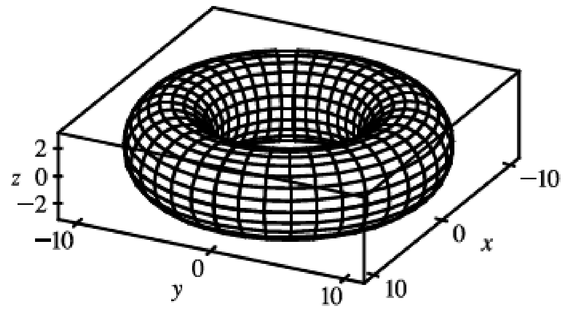
$z = a\sin\alpha$, where $0 \le \alpha \le 2\pi$, $0 \le \theta \le 2\pi$.



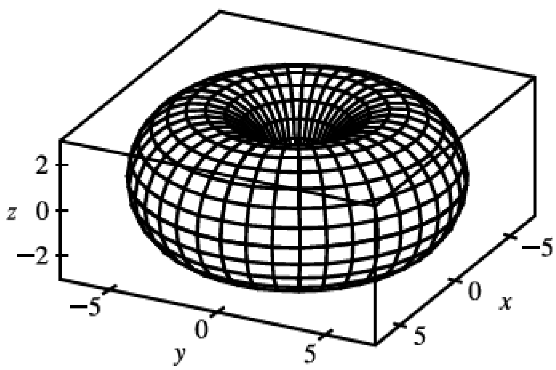
(b)



$a = 1, b = 8$



$a = 3, b = 8$



$a = 3, b = 4$

(c) $x = b\cos\theta + a\cos\alpha\,\cos\theta$, $y = b\sin\theta + a\cos\alpha\,\sin\theta$, $z = a\sin\alpha$, so $\mathbf{r}_\alpha = \langle -a\sin\alpha\,\cos\theta, -a\sin\alpha\,\sin\theta, a\cos\alpha\rangle$,

$\mathbf{r}_\theta = \langle -(b + a\cos\alpha)\sin\theta, (b + a\cos\alpha)\cos\theta, 0\rangle$ and

$$\begin{aligned}\mathbf{r}_\alpha \times \mathbf{r}_\theta &= \left(-ab\cos\alpha\,\cos\theta - a^2\cos\alpha\,\cos^2\theta\right)\mathbf{i} + \left(-ab\sin\alpha\,\cos\theta - a^2\sin\alpha\,\cos^2\theta\right)\mathbf{j}\\ &\quad + \left(-ab\cos^2\alpha\,\sin\theta - a^2\cos^2\alpha\,\sin\theta\,\cos\theta - ab\sin^2\alpha\,\sin\theta - a^2\sin^2\alpha\,\sin\theta\,\cos\theta\right)\mathbf{k}\\ &= -a\,(b + a\cos\alpha)\,[(\cos\theta\,\cos\alpha)\,\mathbf{i} + (\sin\theta\,\cos\alpha)\,\mathbf{j} + (\sin\alpha)\,\mathbf{k}]\end{aligned}$$

Then $|\mathbf{r}_\alpha \times \mathbf{r}_\theta| = a(b + a\cos\alpha)\sqrt{\cos^2\theta\,\cos^2\alpha + \sin^2\theta\,\cos^2\alpha + \sin^2\alpha} = a(b + a\cos\alpha)$.

Note: $b > a$, $-1 \le \cos\alpha \le 1$ so $|b + a\cos\alpha| = b + a\cos\alpha$. Hence

$A\,(S) = \int_0^{2\pi}\int_0^{2\pi} a(b + a\cos\alpha)\,d\alpha\,d\theta = 2\pi\left[ab\alpha + a^2\sin\alpha\right]_0^{2\pi} = 4\pi^2 ab$.