Real Analysis, Fall 2017

September 5, 2017

Chapter 1

The real number line

1.1 Ordered Sets

One basic property of many number systems (natural numbers, integers, rationals, etc) is that they are *ordered*, so we say that "3 *is greater than* 2", and so on.

1.1.1 Definition. A *total order* on a set S is a relation[1] $\leq$ satisfying the following axioms:

(O1) (Reflexivity) For every element a, it always holds that $a \leq a$.

(O2) (Antisymmetry) If $a \leq b$ and $b \leq a$, then it must be that $a = b$.

(O3) (Transitivity) If $a \leq b$ and $b \leq c$, then it holds that $a \leq c$.

(O4) (Totality) For every pair of elements a and b, either $a \leq b$ or $b \leq a$.

We say S is an *ordered set*.

1.1.2 Example. Find some examples of ordered sets.

1.1.3 Example. Find an example of a *partially ordered* set—a set with a relation satisfying axioms (O1)–(O3) but not (O4).

1.1.4 Problem. Suppose S is an ordered set. Formulate a reasonable definition of strict inequality ($a < b$) in terms of the order relation $\leq$. Then write down a definition equivalent to Definition 1.1.1 using strict inequality as the primitive relation; that is, write down a set of axioms that $<$ should satisfy, in terms of which $\leq$ (suitably defined in terms of $<$) has properties (O1)–(O4).

1.1.5 Definition. Let S be an ordered set, and $A \subseteq S$ a subset. An *upper bound* for A is an element $u \in S$ such that $a \leq u$ for every $a \in A$. If such an element exists, we say A is *bounded above*.

Similarly, a *lower bound* for A is an element $l \in S$ such that $l \leq a$ for every $a \in A$. If such a lower bound exists, we say A is *bounded below*.

1.1.6 Definition. A *least upper bound* or *supremum* of a bounded above set A is an element u_0 of S such that

(i) u_0 is an upper bound for A, and

(ii) $u_0 \leq u$ for every other upper bound u.

We denote a supremum for A (if it exists) by $\sup A$.

Similarly, a *greatest lower bound* or *infimum* of a bounded below set A is an element b_0 of S such that

(i) b_0 is a lower bound for A, and

[1] A relation is a comparision operation between two elements which evaluates to either *true* or *false*.

(ii) $b_0 \geq b$ for every other lower bound b.

We denote an infimum for A (if it exists) by $\inf A$.

1.1.7 Proposition. *If a supremum (or infimum) of A exists, then it is unique.*

1.1.8 Proposition. *If A and B are subsets of an ordered set S which are bounded above and below, and if $A \subseteq B$, then*

$$\inf B \leq \inf A \leq \sup A \leq \sup B.$$

1.1.9 Example. Let $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ denote the set of integers, with the usual order. Find some examples of subsets A of $\mathbb{Z}$ such that

(a) A is bounded above and below.

(b) A is bounded above but not below.

(c) A is not bounded above and not bounded below.

Which of these sets have a supremum? Which have an infimum?

1.1.10 Example. Repeat Example 1.1.9 with the set $\mathbb{Q}$ of rational numbers in place of $\mathbb{Z}$. The following Lemma may be of use.

1.1.11 Lemma. *There exists no $q \in \mathbb{Q}$ such that $q^2 = 2$. [Possible hint: write $q = \frac{a}{b}$ in lowest terms and consider the evenness/oddness of a and b.]*

1.1.12 Definition. An ordered set S has the *least upper bound property* if every subset which is bounded above has a supremum. Likewise S has the *greatest lower bound property* if every subset which is bounded below has an infimum.

1.1.13 Example. Does $\mathbb{Z}$ have the least upper bound property? Does $\mathbb{Q}$? Justify your answers with a proof or counterexample.

1.1.14 Theorem. *If S has the least upper bound property, then it has the greatest lower bound property.*

1.2 Fields and ordered fields

Of course the familiar number systems have additional structure. Besides the order, we have addition, subtraction, multiplication and division.

1.2.1 Definition. A *field* is a set $\mathbb{F}$ with two binary operations[2] $+$ and $\cdot$, called *addition* and *multiplication*, respectively, satisfying the following axioms:

(F1) (Associativity of addition) $(a + b) + c = a + (b + c)$ for all a, b, c in $\mathbb{F}$.

(F2) (Additive identity) There exists an element $0 \in \mathbb{F}$ such that $0 + a = a + 0 = a$ for all a.

(F3) (Additive inverses) For each a in $\mathbb{F}$ there exists an element $-a$ such that $(-a) + a = a + (-a) = 0$.

(F4) (Commutativity of addition) $a + b = b + a$ for all a, b in $\mathbb{F}$.

(F5) (Associativity of multiplication) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, c in $\mathbb{F}$.

(F6) (Multiplicative identity) There exists an element $1 \in \mathbb{F}$ such that $1 \cdot a = a \cdot 1 = a$ for all a.

(F7) (Multiplicative inverses) For all $a \neq 0$, there exists an element a^{-1} in $\mathbb{F}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.

[2] A binary operation is a function/operation taking in two elements of $\mathbb{F}$ and returning a third element of $\mathbb{F}$.

(F8) (Commutativity of multiplication) $a \cdot b = b \cdot a$ for all a, b in $\mathbb{F}$.

(F9) (Distributivity) $a \cdot (b + c) = a \cdot b + a \cdot c$.

(F10) (Nontriviality) $0 \neq 1$.

It is customary to omit the $\cdot$ when writing multiplication; in other words, we usually just write ab instead of $a \cdot b$. Additionally, we usually denote $a + (-b)$ simply by $a - b$, and we may also use the notation $\frac{1}{a}$ in place of a^{-1}. It is important to note that subtraction $-$ and division $\div$ are not really distinct operations; they are just syntactic shorthand for addition (resp. multiplication) by an additive (resp. multiplicative) inverse.

We also use the usual shorthand a^n in place of $\underbrace{a \cdots a}_{n \text{ times}}$ and na in place of $\underbrace{a + \cdots + a}_{n \text{ times}}$.

Remark. Though we shall be entirely concerned with fields in this course, you may be familiar with various mathematical objects satisfying fewer of the above axioms. A set with a single operation satisfying axioms (F1)–(F3) is a *group* which is said to be *commutative* or *abelian* if (F4) also holds.

A *ring* is a set with two operations satisfying all of the above except (F7), (F8) and (F10). A *commutative ring* satisfies (F8). According to some conventions, a ring need not satisfy (F6), though such "rings without identity" are sometimes cutely referred to as '*rng*'s. If (F7) holds but not (F8), then $\mathbb{F}$ is called a *division ring*.

Axiom (F10) might be considered optional for fields, but if we allow $0 = 1$ then $\mathbb{F}$ must be the one element set $\{0\}$ (you can prove this after you prove Proposition 1.2.4 below), which for various reasons is best not regarded as a field.

1.2.2 Example. Come up with some examples of fields, some with infinitely many and some with finitely many elements. Can you construct a field with exactly two elements? Three?

1.2.3 Proposition. *The following properties of addition and multiplication hold in any field. (That is, they follow from the axioms above.)*

(i) *(Uniqueness of identities) If an element b in $\mathbb{F}$ satisfies $b + a = a$ for some a, then $b = 0$. Likewise if b satisfies $ba = a$ for some $a \neq 0$, then $b = 1$.*

(ii) *(Uniqueness of inverses) If b satisfies $a + b = 0$, then $b = -a$. Likewise, if b satisfies $ba = 1$ then $b = a^{-1}$.*

(iii) *(Cancellation) If $a + c = b + c$ then $a = b$. Likewise if $c \neq 0$ and $ac = bc$, then $a = b$.*

(iv) *(Inverse of an inverse) $-(-a) = a$ and $(a^{-1})^{-1} = a$.*

1.2.4 Proposition. *In any field, the following properties hold.*

(i) $0a = 0$ *for all* a.

(ii) *If $ab = 0$, then either $a = 0$ or $b = 0$. (We say $\mathbb{F}$ "has no divisors of zero".)*

(iii) $(-a)b = a(-b) = -(ab)$ *for all a and b. In particular $-a = (-1)a$.*

(iv) $(-a)(-b) = ab$ *for all a and b.*

1.2.5 Problem. In a field, show that if $b \neq 0$ and $d \neq 0$ then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

1.2.6 Definition. An *ordered field* is a field $\mathbb{F}$ equipped with a total order, so a set with a relation $\leq$ and two operations $+$ and $\cdot$ satisfying axioms (O1)–(O4) and (F1)–(F10), which is additionally required to satisfy the following axioms:

(OF1) (Compatibility of order and addition) If $a \leq b$ then $a + c \leq b + c$ for any c.

(OF2) (Compatibility of order and multiplication) If $a \le b$ and $0 \le c$, then $ac \le bc$.

1.2.7 Example. Which examples from Example 1.2.2 are ordered fields? In case there is not an obvious order, is there any order at all satisfying (OF1) and (OF2)?

1.2.8 Proposition. *The following properties always hold in an ordered field.*

(a) *If* $0 \le a$ *then* $-a \le 0$.

(b) *If* $0 \le a$ *and* $0 \le b$ *then* $0 \le ab$. *(In fact, this is equivalent to (OF2) and is often used in place of it as the other ordered field axiom).*

(c) *If* $a \le 0$ *and* $0 \le b$, *then* $ab \le 0$.

(d) $0 \le a^2$ *for any* a. *In particular* $0 < 1$.

(e) *If* $0 < a \le b$ *then* $0 < b^{-1} \le a^{-1}$.

In light of Proposition 1.2.4.(ii) *the above identities hold with strict inequality* $<$ *used in place of inequality* $\le$.

1.2.9 Problem. Let $\mathbb{F}$ be an ordered field and consider the subset $Z \subset \mathbb{F}$ generated by taking 0, 1, $1+1$, $1+1+1$, etc. along with -1, $-1-1$, $-1-1-1$, etc. Show that this set is in bijection with the set of integers $\mathbb{Z}$.

Likewise, let $Q \subset \mathbb{F}$ be the subset generated by taking the multiplicative inverses of the nonzero elements in Z along with their integer multiples. Show that this set is in bijection with $\mathbb{Q}$.

Thus every ordered field contains a copy of $\mathbb{Q}$, which may be regarded as the "smallest" possible ordered field.

1.2.10 Definition. Let $\mathbb{F}$ be an ordered field. The *absolute value* or *magnitude* of a number $a \in \mathbb{F}$ is defined by

$$|a| = \begin{cases} a & \text{if } a \ge 0, \\ -a & \text{if } a < 0. \end{cases}$$

1.2.11 Proposition*. *The absolute value satisfies the following properties. For all* a *and* b *in* $\mathbb{F}$*:*

(i) $|a| \ge 0$.

(ii) $|a| = 0$ *if and only if* $a = 0$.

(iii) $|ab| = |a||b|$.

(iv) *(Triangle inequality)* $|a+b| \le |a| + |b|$.

(v) *(Reverse triangle inequality)* $\big||a| - |b|\big| \le |a-b|$.

Remark. Combining (iv) and (v) of the last proposition gives the useful strings of inequalities:

$$|a| - |b| \le \big||a| - |b|\big| \le |a+b| \le |a| + |b|, \quad \text{and} \quad |a| - |b| \le \big||a| - |b|\big| \le |a-b| \le |a| + |b|. \tag{1.1}$$

1.2.12 Definition. The *distance* between numbers a and b in an ordered field $\mathbb{F}$ is the quantity

$$d(a, b) = |a - b|.$$

1.2.13 Proposition*. *The distance satisfies the following properties. For all* a, b, *and* c *in* $\mathbb{F}$*:*

(i) $d(a, b) \ge 0$.

(ii) $d(a, b) = 0$ *if and only if* $a = b$.

(iii) *(Symmetry)* $d(a,b) = d(b,a)$.

(iv) *(Triangle inequality)* $d(a,c) \leq d(a,b) + d(b,c)$.

1.2.14 Lemma (Suprema/infima in an ordered field)**.** *Let A be a bounded above subset of an ordered field. Then $u = \sup A$ if and only if*

(i) *$a \leq u$ for all $a \in A$ (u is an upper bound), and*

(ii) *for every $\varepsilon > 0$, there exists $a \in A$ such that $u - \varepsilon < a$ ($u - \varepsilon$ fails to be an upper bound).*

Similarly, if A is bounded below, then $b = \inf A$ if and only if

(i) *$b \leq a$ for all $a \in A$, and*

(ii) *for every $\varepsilon > 0$, there exists $a \in A$ such that $a < b + \varepsilon$.*

1.2.15 Definition ($\pm\infty$ notation)**.** As a notation convention, it is useful to introduce the symbols $+\infty$ and $-\infty$ when speaking of suprema and infima in an ordered field. We write $\sup A = +\infty$ if A is not bounded above, and $\inf A = -\infty$ if A is not bounded below. With these conventions $\sup A$ and $\inf A$ are always defined for a nonempty set A.

A more formal way to do this is to embed $\mathbb{F}$ into a larger ordered set $\overline{\mathbb{F}} = \mathbb{F} \cup \{+\infty, -\infty\}$ with the order defined so that $-\infty < a < +\infty$ for all $a \in \mathbb{F}$. Note that $\overline{\mathbb{F}}$ is *not* a field, though we may observe the following notation conventions: if $a > 0 \in \mathbb{F}$, then

$$a + (+\infty) = +\infty, \quad a + (-\infty) = -\infty, \quad a(+\infty) = +\infty, \quad a(-\infty) = -\infty,$$
$$(-a)(+\infty) = -\infty, \quad (-a)(-\infty) = +\infty, \quad \frac{\pm a}{\pm\infty} = 0.$$

Expressions such as $+\infty - \infty$ and $\pm\infty / \pm\infty$ are not defined.

1.3 Completeness and the real number field

1.3.1 Definition. An ordered field $\mathbb{F}$ is *complete* if it satisfies the least upper bound property (c.f. Definition 1.1.12), in other words, if for every bounded above subset $A \subset \mathbb{F}$, the supremum (least upper bound) $\sup A$ exists in $\mathbb{F}$.

1.3.2 Theorem (Characterization/definition of $\mathbb{R}$)**.** *There exists a unique*[3] *complete ordered field called the* real numbers *and denoted by* $\mathbb{R}$.

Remark. We omit the proof of Theorem 1.3.2 for now; we may come back to it later on.

1.3.3 Definition. An ordered field F is *Archimedean* if for every $a \in F$, there exists an integer[4] N such that $a \leq N$.

1.3.4 Example*. Show that $\mathbb{Q}$ is Archimedean.

1.3.5 Example (Research Allowed)**.** Find an example of a non-Archimedean field.

1.3.6 Theorem. *As a complete ordered field, $\mathbb{R}$ is Archimedean.*

1.3.7 Proposition. *If $0 < a$ in an Archimedean field such as $\mathbb{R}$, then there exists a positive integer N such that*

$$0 < \frac{1}{N} < a.$$

[3] Here "uniqueness" means the following: given two complete ordered fields F_1 and F_2, there exists an *isomorphism* (a bijection compatible with the order and field operations) $\phi : F_1 \longrightarrow F_2$. Moreover ϕ is unique. Using ϕ we can regard F_1 and F_2 as being "the same" field.

[4] Here we are identifying a subset of F with the integers as in Problem 1.2.9.

Remark. The Archimedean property says that a field has no "infinitely large" elements, and via Proposition 1.3.7, it implies that there are no "infinitely small" elements. The next result gives a technically useful if strange seeming characterization of the zero element.

1.3.8 Corollary. *In an Archimedean field, if* $0 \leq a$ *and* $a < \varepsilon$ *for every* $0 < \varepsilon$*, then* $a = 0$.

1.3.9 Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *Let* a *and* b *be real numbers with* $a < b$*. Then there exists a rational number* q *such that*

$$a < q < b.$$

We say $\mathbb{Q}$ *is* dense *in* $\mathbb{R}$.

Remark. This may be a surprising result, especially when juxtaposed with the following one. Recall that an infinite set is said to be *countable* if it is in bijection with the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of natural numbers.

1.3.10 Theorem.

(i) $\mathbb{Q}$ *is countable.*

(ii) $\mathbb{R}$ *is uncountable.*