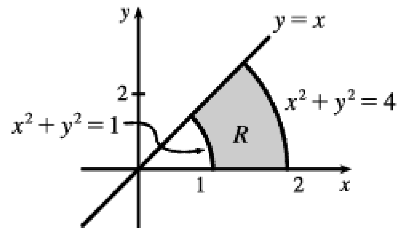
13. R is the region shown in the figure, and can be described by $R = \{(r, \theta) \mid 0 \le \theta \le \pi/4, 1 \le r \le 2\}$. Thus

$\iint_R \arctan(y/x)\, dA = \int_0^{\pi/4} \int_1^2 \arctan(\tan\theta)\, r\, dr\, d\theta$ since $y/x = \tan\theta$.

Also, $\arctan(\tan\theta) = \theta$ for $0 \le \theta \le \pi/4$, so the integral becomes

$\int_0^{\pi/4} \int_1^2 \theta\, r\, dr\, d\theta = \int_0^{\pi/4} \theta\, d\theta \int_1^2 r\, dr = \left[\frac{1}{2}\theta^2\right]_0^{\pi/4} \left[\frac{1}{2}r^2\right]_1^2 = \frac{\pi^2}{32} \cdot \frac{3}{2} = \frac{3}{64}\pi^2.$



38.

The distance from a point (x, y) to the origin is $f(x, y) = \sqrt{x^2 + y^2}$, so the average distance from points in D to the origin is

$$f_{\text{ave}} = \frac{1}{A(D)} \iint_D \sqrt{x^2 + y^2}\, dA = \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a \sqrt{r^2}\; r\, dr\, d\theta$$

$$= \frac{1}{\pi a^2} \int_0^{2\pi} d\theta \int_0^a r^2\, dr = \frac{1}{\pi a^2}\, [\theta]_0^{2\pi}\, \left[\frac{1}{3}r^3\right]_0^a = \frac{1}{\pi a^2} \cdot 2\pi \cdot \frac{1}{3}a^3 = \frac{2}{3}a$$

16. $\rho(x, y) = k/\sqrt{x^2 + y^2} = k/r$.

$$m = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} \frac{k}{r}\, r\, dr\, d\theta = k \int_{\pi/6}^{5\pi/6} [(2\sin\theta) - 1]\, d\theta$$

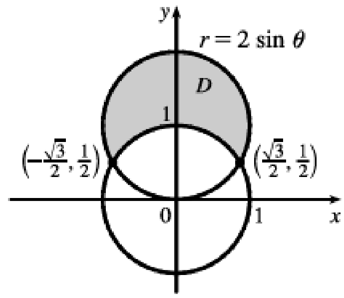
$$= k\big[-2\cos\theta - \theta\big]_{\pi/6}^{5\pi/6} = 2k\left(\sqrt{3} - \tfrac{\pi}{3}\right)$$

By symmetry of D and $f(x) = x$, $M_y = 0$, and

$$M_x = \int_{\pi/6}^{5\pi/6} \int_1^{2\sin\theta} kr\sin\theta\, dr\, d\theta = \tfrac{1}{2}k \int_{\pi/6}^{5\pi/6} (4\sin^3\theta - \sin\theta)\, d\theta$$

$$= \tfrac{1}{2}k\big[-3\cos\theta + \tfrac{4}{3}\cos^3\theta\big]_{\pi/6}^{5\pi/6} = \sqrt{3}\,k$$

Hence $(\overline{x}, \overline{y}) = \left(0, \frac{3\sqrt{3}}{2(3\sqrt{3} - \pi)}\right)$.



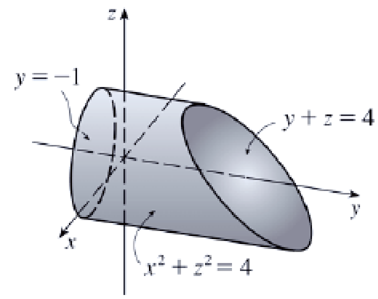
22. Here $E = \left\{(x, y, z) \mid -1 \le y \le 4 - z, x^2 + z^2 \le 4\right\}$, so

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-1}^{4-z} dy\,dz\,dx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4 - z + 1)\,dz\,dx$$

$$= \int_{-2}^{2} \left[5z - \tfrac{1}{2}z^2\right]_{z=-\sqrt{4-x^2}}^{z=\sqrt{4-x^2}} dx = \int_{-2}^{2} 10\sqrt{4-x^2}\,dx$$

$$= 10\left[\tfrac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\tfrac{x}{2}\right)\right]_{-2}^{2} \quad \left[\begin{array}{l}\text{using trigonometric substitution or}\\ \text{Formula 30 in the Table of Integrals}\end{array}\right]$$

$$= 10\left[2\sin^{-1}(1) - 2\sin^{-1}(-1)\right] = 20\left(\tfrac{\pi}{2} - \left(-\tfrac{\pi}{2}\right)\right) = 20\pi$$



Alternatively, use polar coordinates to evaluate the double integral:

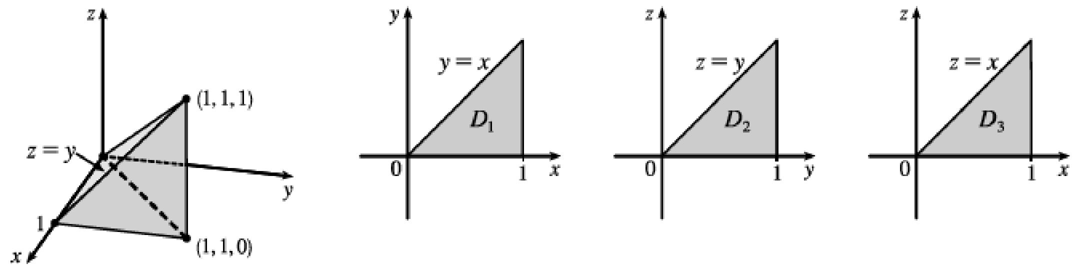
$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (5 - z)\,dz\,dx = \int_{0}^{2\pi} \int_{0}^{2} (5 - r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\tfrac{5}{2}r^2 - \tfrac{1}{3}r^3\sin\theta\right]_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} \left(10 - \tfrac{8}{3}\sin\theta\right) d\theta$$

$$= 10\theta + \tfrac{8}{3}\cos\theta\Big]_0^{2\pi} = 20\pi$$

35.



$\int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le z \le y,\, y \le x \le 1,\, 0 \le y \le 1\}$.

If D_1, D_2, and D_3 are the projections of E on the xy-, yz- and xz-planes then

$$D_1 = \{(x,y) \mid 0 \le y \le 1,\, y \le x \le 1\} = \{(x,y) \mid 0 \le x \le 1,\, 0 \le y \le x\},$$

$$D_2 = \{(y,z) \mid 0 \le y \le 1,\, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1,\, z \le y \le 1\}, \text{ and}$$

$$D_3 = \{(x,z) \mid 0 \le x \le 1,\, 0 \le z \le x\} = \{(x,z) \mid 0 \le z \le 1,\, z \le x \le 1\}.$$

Thus we also have

$$\begin{aligned} E &= \{(x,y,z) \mid 0 \le x \le 1,\, 0 \le y \le x,\, 0 \le z \le y\} = \{(x,y,z) \mid 0 \le y \le 1,\, 0 \le z \le y,\, y \le x \le 1\} \\ &= \{(x,y,z) \mid 0 \le z \le 1,\, z \le y \le 1,\, y \le x \le 1\} = \{(x,y,z) \mid 0 \le x \le 1,\, 0 \le z \le x,\, z \le y \le x\} \\ &= \{(x,y,z) \mid 0 \le z \le 1,\, z \le x \le 1,\, z \le y \le x\}. \end{aligned}$$

Then

$$\begin{aligned} \int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy &= \int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_0^y \int_y^1 f(x,y,z)\,dx\,dz\,dy \\ &= \int_0^1 \int_z^1 \int_y^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_0^x \int_z^x f(x,y,z)\,dy\,dz\,dx \\ &= \int_0^1 \int_z^1 \int_z^x f(x,y,z)\,dy\,dx\,dz \end{aligned}$$

37. The region C is the solid bounded by a circular cylinder of radius 2 with axis the z-axis for $-2 \le z \le 2$. We can write $\iiint_C (4 + 5x^2yz^2)\,dV = \iiint_C 4\,dV + \iiint_C 5x^2yz^2\,dV$, but $f(x,y,z) = 5x^2yz^2$ is an odd function with respect to y. Since C is symmetrical about the xz-plane, we have $\iiint_C 5x^2yz^2\,dV = 0$. Thus $\iiint_C (4 + 5x^2yz^2)\,dV = \iiint_C 4\,dV = 4 \cdot V(E) = 4 \cdot \pi(2)^2(4) = 64\pi$.

40. $m = \int_{-1}^{1} \int_0^{1-y^2} \int_0^{1-z} 4\,dx\,dz\,dy = 4\int_{-1}^{1} \int_0^{1-y^2} (1-z)\,dz\,dy = 4\int_{-1}^{1} \left[z - \frac{1}{2}z^2\right]_{z=0}^{z=1-y^2} dy = 2\int_{-1}^{1} (1-y^4)\,dy = \frac{16}{5}$,

$$M_{yz} = \int_{-1}^{1} \int_0^{1-y^2} \int_0^{1-z} 4x\,dx\,dz\,dy = 2\int_{-1}^{1} \int_0^{1-y^2} (1-z)^2\,dz\,dy = 2\int_{-1}^{1} \left[-\tfrac{1}{3}(1-z)^3\right]_{z=0}^{z=1-y^2} dy$$
$$= \tfrac{2}{3}\int_{-1}^{1} \left(1-y^6\right) dy = \left(\tfrac{4}{3}\right)\left(\tfrac{6}{7}\right) = \tfrac{24}{21}$$

$$M_{xz} = \int_{-1}^{1} \int_0^{1-y^2} \int_0^{1-z} 4y\,dx\,dz\,dy = \int_{-1}^{1} \int_0^{1-y^2} 4y(1-z)\,dz\,dy$$
$$= \int_{-1}^{1} \left[4y(1-y^2) - 2y(1-y^2)^2\right] dy = \int_{-1}^{1} (2y - 2y^5)\,dy = 0 \quad \text{[the integrand is odd]}$$

$$M_{xy} = \int_{-1}^{1} \int_0^{1-y^2} \int_0^{1-z} 4z\,dx\,dz\,dy = \int_{-1}^{1} \int_0^{1-y^2} (4z - 4z^2)\,dz\,dy = 2\int_{-1}^{1} \left[(1-y^2)^2 - \tfrac{2}{3}(1-y^2)^3\right] dy$$
$$= 2\int_{-1}^{1} \left[\tfrac{1}{3} - y^4 + \tfrac{2}{3}y^6\right] dy = \left[\tfrac{4}{3}y - \tfrac{4}{5}y^5 + \tfrac{8}{21}y^7\right]_0^1 = \tfrac{96}{105} = \tfrac{32}{35}$$

Thus, $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{5}{14}, 0, \frac{2}{7}\right)$