Calc III: Workshop 10 Solutions, Fall 2017

Problem 1. Find the surface area of the part of the plane $x + 2y + 3z = 1$ which lies inside the cylinder $x^2 + y^2 = 3$.

Solution. Solving for z in the equation for the plane, we have $z = \frac{1}{3}(1 - x - 2y)$. We can use x and y as parameters, with parameterization

$$\mathbf{r}(x, y) = \big(x, y, \tfrac{1}{3}(1 - x - 2y)\big)$$

where (x, y) vary in the disk R of radius 3. Then

$$\mathbf{r}_x(x, y) = \big(1, 0, -\tfrac{1}{3}\big), \quad \mathbf{r}_y(x, y) = (0, 1, -\tfrac{2}{3})$$

and

$$dS = \|\mathbf{r}_x \times \mathbf{r}_y\|\, dx\, dy = \sqrt{(\tfrac{1}{3})^2 + (\tfrac{2}{3})^2 + 1}\, dx\, dy = \frac{\sqrt{14}}{3}\, dx\, dy.$$

Thus the area is

$$\iint_S 1\, dS = \iint_R \frac{\sqrt{14}}{3}\, dx\, dy = \frac{\sqrt{14}}{3}\text{Area}(R) = \frac{\sqrt{14}}{3}(3\pi) = \pi\sqrt{14},$$

since R is the disk of radius $\sqrt{3}$. □

Problem 2. Find the surface area of the part of the cone $z = \sqrt{x^2 + y^2}$ between $z = 0$ and $z = H$.

Solution. We can use x and y as paramters, with $\mathbf{r}(x, y) = (x, y, \sqrt{x^2 + y^2})$ and (x, y) varying in the disk of radius H, or we can use polar/cylindrical coordinates directly, with parameterization

$$\mathbf{r}(r, \theta) = (r\cos\theta, r\sin\theta, r), \quad 0 \le \theta \le 2\pi,\ 0 \le r \le H,$$

using the fact that $z = r$ on the cone. Using the latter parameterization, we find

$$\mathbf{r}_r(r, \theta) = (\cos\theta, \sin\theta, 1), \quad \mathbf{r}_\theta(r, \theta) = (-r\sin\theta, r\cos\theta, 0), \quad \mathbf{r}_r \times \mathbf{r}_\theta = (-r\cos\theta, -r\sin\theta, r),$$

so

$$dS = \|\mathbf{r}_r \times \mathbf{r}_\theta\|\ dr\, d\theta = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2}\, dr\, d\theta = \sqrt{2}\, r\, dr\, d\theta.$$

The surface area is given by

$$\text{Area}(S) = \iint_S dS = \int_0^{2\pi} \int_0^H \sqrt{2}\, r\, dr\, d\theta = (\sqrt{2})(2\pi)(\tfrac{H^2}{2}) = \sqrt{2}\pi H^2.$$

□

Problem 3. Find the flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ of the vector field $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ across the unit sphere $x^2 + y^2 + z^2 = 1$, oriented with outward facing unit normal vector.

Solution. We may use the spherical coordinate parameterization

$$\begin{aligned} \mathbf{r}(\varphi, \theta) &= (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi), \\ \mathbf{r}_\varphi &= (\cos\varphi\cos\theta, \cos\varphi\sin\theta, -\sin\varphi), \\ \mathbf{r}_\theta &= (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0) \end{aligned}$$

so that

$$\mathbf{n}\, dS = \pm\mathbf{r}_\varphi \times \mathbf{r}_\theta\, d\varphi\, d\theta = \pm(\sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \cos\varphi\sin\varphi)d\varphi\, d\theta \tag{1}$$

To figure the sign out, we can test at a point, say $(1,0,0) = \mathbf{r}(\pi/2, 0)$ where we know the outward unit normal vector will be $\mathbf{n} = (1,0,0)$, in which case (1) evaluates to $\pm(1,0,0)$, so we take the $+$ sign.
Evaluating $\mathbf{F}\big(\mathbf{r}(\varphi,\theta)\big)$ gives

$$\mathbf{F}\big(\mathbf{r}(\varphi,\theta)\big) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)$$

and finally

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \int_0^{2\pi}\int_0^{\pi} (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi)\cdot(\sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \cos\varphi\sin\varphi)\,d\varphi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} 2\sin^3\varphi\cos^2\theta + 2\sin^3\varphi\sin^2\theta + 2\cos^2\varphi\sin\varphi\,d\varphi\,d\theta \\
&= 2\int_0^{2\pi}\int_0^{\pi} \sin^3\varphi + \cos^2\varphi\sin\varphi\,d\varphi\,d\theta \\
&= 2\int_0^{2\pi}\int_0^{\pi} \sin\varphi\,d\varphi\,d\theta \\
&= 8\pi.
\end{aligned}$$

$\square$

Problem 4. Compute the flux $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ of the vector field $\mathbf{F}(x,y,z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ lying over the square $0 \le x \le 1$, $0 \le y \le 1$ and has upward orientation.

Solution. Given the limits $0 \le x \le 1$ and $0 \le y \le 1$, it is best to parameterize S by x and y here, so

$$\mathbf{r}(x,y) = (x,y,4-x^2-y^2), \quad \mathbf{r}_x(x,y) = (1,0,-2x), \quad \mathbf{r}_y(x,y) = (0,1,-2y), \quad \mathbf{r}_x\times\mathbf{r}_y = (2x,2y,1).$$

Then

$$\mathbf{n}\,dS = \pm\mathbf{r}_x\times\mathbf{r}_y\,dx\,dy$$

with the $\pm$ sign determined by the orientation. Since we want $\mathbf{n}$ to point "upward" and $\mathbf{r}_x\times\mathbf{r}_y$ has positive z component, we take the $+$ sign. So

$$\begin{aligned}
\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \int_0^1\int_0^1 (xy, y(4-x^2-y^2), (4-x^2-y^2)x)\cdot(2x,2y,1)\,dx\,dy \\
&= \int_0^1\int_0^1 2x^2y + 2y^2(4-x^2-y^2) + (4-x^2-y^2)x\,dx\,dy \\
&= \int_0^1\int_0^1 2x^2y + 8y^2 - 2y^2x^2 - 2y^4 + 4x - x^3 - xy^2\,dx\,dy \\
&= \int_0^1 \frac{2}{3}y + 8y^2 - \frac{2}{3}y^2 - 2y^4 + 2 - \frac{1}{4} - \frac{y^2}{2}\,dy \\
&= \frac{2}{6} + \frac{8}{3} - \frac{2}{9} - \frac{2}{5} + 2 - \frac{1}{4} - \frac{1}{6} \\
&= \frac{713}{180}
\end{aligned}$$

$\square$