THE STRONG NOVIKOV CONJECTURE AND POSITIVE SCALAR CURVATURE.

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1. Introduction

Let (M, g) be a Riemannian manifold. The scalar curvature of g is a smooth function $\kappa \in C^\infty(M)$ defined as the trace of the Ricci tensor. By [8, Theorem 3.1], for $p \in M$ the value of $\kappa(p)$ is determined by the power series expansion

$$\operatorname{vol} B_r(p, M) \;=\; \operatorname{vol} B_r(0, \mathbb{R}^n) \left(1 - \frac{\kappa(p)}{6(n+2)} r^2 + \dots\right),$$

where $\operatorname{vol} B_r(p, M)$ is the volume of a small geodesic ball of radius r and $\operatorname{vol} B_r(0, \mathbb{R}^n)$ is the volume of a ball in Euclidean space of the same radius r. Positive scalar curvature means that, for r small, geodesics balls of radius r have smaller volume than balls of the same radius in Euclidean space.

An interesting question in differential geometry is the following one.

Question 1.1. *Which manifolds admit a Riemannian metric of positive scalar curvature?*

This question is related with the classification of differential manifolds. In fact, the scalar curvature can be thought of as the weakest curvature invariant one can attach to a differential manifold.

More in general, we could ask which smooth functions can be realized as the scalar curvature functions of some Riemannian metrics on M. It is a remarkable result of Kazdan and Warner (see [11] and [12]) that if M can carry a metric of positive scalar curvature, then every smooth function is the scalar curvature of some Riemannian metric. Therefore, answering the second question boils down to answering Question 1.1. This fact partially explains why we care so much about the existence of a Riemannian metric of *positive* scalar curvature.

In these notes we rather discuss the inverse problem.

Question 1.2. *Which manifolds cannot carry a Riemannian metric of positive scalar curvature?*

The (partial) answer to this question involves a rich theory of obstructions constructed both with analytical and topological methods. The first example is the celebrated theorem of Lichnerowicz establishing that for a closed spin manifold M of dimension $4k$ the $\widehat{A}$-genus is an obstruction to the existence of metrics of positive scalar curvature. The proof of this theorem is a perfect example of the interplay between topology and the global theory of elliptic operators. The first part of these notes is devoted to reviewing this theorem. In Section 2.1, we use techniques of analysis and differential geometry to prove that the index of the spin Dirac operator $\not{D}_M$ must vanish if M admits a metric of positive scalar curvature. In Section 2.4, we use the Atiyah-Singer index theorem to show that the index of $\not{D}_M$ corresponds to the $\widehat{A}$-genus of the manifold M.

In the second part of these notes, we define a family of invariants extending the $\widehat{A}$-genus and discuss the extent to which it is possible to use techniques from index theory to prove that they are actually obstructions to the existence of metrics of positive scalar curvature. This leads us to consider differential operators acting on bundles over the C^*-algebras. The analysis of such operators requires a more K-theoretical formulation of the problem. In this contest, Question 1.2 turns out to be strictly related to a famous conjecture in K-theory, called the 'strong Novikov conjecture'. The connection between this conjecture and the positive scalar curvature was realized for the first time by Rosenberg in [15].

2. $\widehat{A}$-genus and positive scalar curvature

In this section we present a classical result of Lichnerowicz, connecting the positive scalar curvature with the $\widehat{A}$-genus.

2.1. **The spin-Dirac operator.** We start with constructing an analytic obstruction to the existence of a metric of positive scalar curvature on a spin manifold.

Suppose M is a closed oriented even dimensional Riemannan spin manifold. We use the Riemannian metric and the spin structure to construct the spin bundle $\mathbb{S}_M$. This is a complex $\mathbb{Z}_2$-graded Clifford bundle $\mathbb{S}_M = \mathbb{S}_M^+ \oplus \mathbb{S}_M^-$ endowed with a Clifford connection

$$\nabla^{\mathbb{S}} : C^\infty(M, \mathbb{S}_M) \longrightarrow C^\infty(M, T^*M \otimes \mathbb{S}_M).$$

The associated Dirac operator is called the *spin Dirac* operator. It is the first-order elliptic odd self-adjoint operator

$$\not{D}_M = \begin{pmatrix} 0 & \not{D}_M^- \\ \not{D}_M^+ & 0 \end{pmatrix},$$

where the operators $\not{D}_M^\pm : C^\infty(M, \mathbb{S}_M^\pm) \to C^\infty(M, \mathbb{S}_M^\mp)$ are given by the compositions

$$C^\infty(M, \mathbb{S}_M^\pm) \xrightarrow{\nabla^{\mathbb{S}}} C^\infty(M, T^*M \otimes \mathbb{S}_M^\pm) \xrightarrow{c} C^\infty(M, \mathbb{S}_M^\mp)$$

and where the bundle map $c : T^*M \otimes \mathbb{S}_M^\pm \to \mathbb{S}_M^\mp$ is the Clifford multiplication (for more details about the spin bundle and the associated Dirac operator, we refer to [13, Chapter II]).

The relationship between scalar curvature and the spin Dirac operator is given by a classical result due to Bochner, Lichnerowicz and Weitzenböck. Before stating this result, we need to construct a second operator out of our geometric data. We use the Clifford connection $\nabla^{\mathbb{S}}$ (that is in particular a differential operator of order one) to define the second order differential operator

$$\Delta_{\mathbb{S}} = \left(\nabla^{\mathbb{S}}\right)^* \circ \left(\nabla^{\mathbb{S}}\right)$$

where $(\nabla^{\mathbb{S}})^* : C^\infty(T^*M \otimes \mathbb{S}) \to C^(\mathbb{S})$ is the formal adjoint of $\nabla^{\mathbb{S}}$. The operator $\Delta_{\mathbb{S}}$ is called the *connection laplacian.* The next proposition compares this operator with the second order operator $\not{D}_M^2$.

Proposition 2.2 (Bochner-Lichnerowicz-Weitzenböck formula)**.** *We have*

$$\not{D}_M^2 = \Delta_{\mathbb{S}} + \frac{1}{4}\kappa, \tag{2.1}$$

where κ is the scalar curvature of the Riemannian metric on M.

For the proof of this proposition, see [13, Theorem 8.8]. Since $\not{D}_M$ is elliptic, $\operatorname{Ker} \not{D}_M^\pm$ are finite dimenisonal vector spaces and we define the index of $\not{D}_M$ to be the integer

$$\operatorname{Ind} \not{D}_M := \dim_{\mathbb{C}} \operatorname{Ker} \not{D}_M^+ - \dim_{\mathbb{C}} \operatorname{Ker} \not{D}_M^-.$$

In the next proposition we show that the index of $\not{D}_M$ is an obstruction to the existence of a Riemannian metric of positive scalar curvature.

Proposition 2.3. *Let M be an even dimensional closed spin manifold. If M admits a Riemannian metric of positive scalar curvature, then* $\operatorname{Ind} \not{D}_M = 0$.

Proof. Suppose there exists a Riemannian metric on M whose scalar curvature κ is a strictly positive function. We will show that both Ker $\not{D}_M^+$ and Ker $\not{D}_M^-$ vanish. From [13, Theorem 5.4], we have

$$\mathrm{Ker}\,\not{D}_M^{\pm} \;=\; \mathrm{Ker}\,\big(\not{D}_M^{\mp}\circ\not{D}_M^{\pm}\big)$$

so that it suffices to show that Ker $\not{D}_M^2 = \{0\}$.

Since M is compact, the scalar curvature κ is uniformly positive, i.e. there exists a constant $c > 0$ such that

$$\kappa(x) \geq c, \qquad\qquad x \in M.$$

Let $u \in$ Ker $\not{D}_M^2$. By elliptic regularity, $u \in C^\infty(M, \mathbb{S}_M)$. By (2.1) we deduce

$$\begin{aligned} 0 &= \Big\langle \not{D}_M^2 u, u\Big\rangle_{L^2(\mathbb{S}_M)} \;=\; \Big\langle\Big(\Delta_{\mathbb{S}} + \frac{1}{4}\kappa\Big)u, u\Big\rangle_{L^2(\mathbb{S}_M)} \\ &= \Big\langle \nabla^{\mathbb{S}}u, \nabla^{\mathbb{S}}u\Big\rangle_{L^2(T^*M\otimes\mathbb{S}_M)} + \Big\langle \frac{1}{4}\kappa u, u\Big\rangle_{L^2(\mathbb{S}_M)} \;\geq\; c\|u\|^2_{L^2(\mathbb{S}_M)}, \end{aligned}$$

from which $u = 0$. Therefore, Ker $\not{D}_M^2 = \{0\}$, from which the thesis follows. ☐

2.4. **The $\widehat{A}$-genus.** We now want to give a cohomological interpretation of the analytic obstruction constructed in the previous section.

Let E be a vector bundle over a compact manifold X. The *total $\widehat{A}$-class* of E is the element $\widehat{\mathbf{A}}(E) \in H^{4*}(X;\mathbb{Q})$ characterized by the following properties:

(1) (naturality) for every map $f : Y \to X$ we have $\widehat{\mathbf{A}}(f^*E) = f^*(\widehat{\mathbf{A}}(E))$;
(2) (exponential property) $\widehat{\mathbf{A}}(E \oplus F) = \widehat{\mathbf{A}}(E) \cup \widehat{\mathbf{A}}(F)$;
(3) (normalization) if L is a complex vector bundle over X, then

$$\widehat{\mathbf{A}}(L) \;=\; \frac{x/2}{\sinh(x/2)},$$

where $x \in H^2(X;\mathbb{Q})$ is the Euler class of L.

In particular, the class $\widehat{\mathbf{A}}(M) := \widehat{\mathbf{A}}(TM)$ is called the total $\widehat{A}$-class of the manifold M. When M is oriented and of dimension $n = 4k$, we define the $\widehat{A}$-genus of M as

$$\widehat{A}(M) \;:=\; \big\langle \widehat{\mathbf{A}}(M), [M]\big\rangle, \tag{2.2}$$

where $[M] \in H_n(M)$ is the fundamental class of M.

The relationship between the $\widehat{A}$-genus and the spin Dirac operator is one of the most famous applications of the index theorem of Atiyah and Singer.

Proposition 2.5 (Atiyah-Singer)**.** *Let M a closed spin manifold of dimension $4k$. Then*

$$\mathrm{Ind}\,\not{D}_M \;=\; \widehat{A}(M).$$

For the proof of this proposition we refer to [13, Theorem 13.10] and to the original paper of Atiyah and Singer [2, Theorem 5.3]. By putting together Propositions 2.3 and 2.5 we finally deduce the result of Lichnerowicz.

Theorem 2.6 (Lichnerowicz)**.** *Let M be a closed spin manifold of dimension $4k$. If M admits a Riemannian metric of positive scalar curvature, then the $\widehat{A}$-genus of M vanishes.*

Remark 2.7. This theorem was proved before the index theorem so the proof that we presented here is not the one used by Lichnerowicz. Nonetheless, this proof is suitable for the generalizations that we will present in the next sections.

3. Higher genera and the Gromov-Lawson conjecture

In this section we use the topological data of our compact manifold M to construct a family of invariants generalizing the $\widehat{A}$-genus.

3.1. **Classifying spaces of discrete groups.** Let $\widetilde{M} \to M$ be a Galois cover with deck transformation group Γ. In particular, Γ is a discrete group and $\widetilde{M}$ is a principal Γ-bundle over M. Let $B\Gamma$ be the *classifying space* of the group Γ and let $E\Gamma \to B\Gamma$ be the *universal principal* Γ-bundle. In particular, $B\Gamma$ is an Eilenberg- MacLane space $K(\Gamma, 1)$ and $E\Gamma$ is simply connected. From the theory of classifying spaces, it follows that there exists a map

$$r : M \to B\Gamma$$

such that $r^*(E\Gamma) = \widetilde{M}$. The map r is called a *classifying map* for $\widetilde{M}$.

Example 3.2. When $\Gamma = \mathbb{Z}^k$, then $E\Gamma$ is the k-dimensional torus $T^k = (S^1)^k$ and $E\Gamma$ is its universal cover $\mathbb{R}^k$ (see [10, Example 1B.5]).

Example 3.3. Let π_g be the fundamental group of the oriented Riemann surface S_g of genus $g \geq 2$. In this case, S_g is a classifying space of its own fundamental group, i.e. $B\pi_g = S_g$ (see [10, Example 1B.2]). The universal principal π_g-bundle coincides with the universal cover of S_g, i.e. $E\pi_g$ is the upper half-plane $\mathbb{H} = \{(x_1, x_2) \in \mathbb{R}^2 | x_1 > 0\}$.

Example 3.4. If $\Gamma = \mathbb{Z}_2$, then a classifying space is given by the infinite real projective space, i.e. $B\mathbb{Z}_2 = \mathbb{RP}^\infty$. The double cover of this space is the infinite sphere $\mathbb{S}^\infty$, that is contractible (see [10, Example 1B.3]). Therefore, $E\mathbb{Z}_2 = \mathbb{S}^\infty$. Notice that the space $\mathbb{RP}^\infty$ is not compact. Let us show this fact (according to Jose'). For a fixed positive integer i, consider the open set $U_i = \{[x_1 : x_2 : \ldots] | x_i \neq 0\}$. Then $\{U_i\}_{i=0}^\infty$ is an open cover of $\mathbb{RP}^\infty$ that doesn't admit a finite subcover. This example illustrates that the classifying space of a discrete group might be not compact.

3.5. **Higher $\widehat{A}$-genera.** Let $\widetilde{M} \to M$ be a Galois cover with classifying map $r : M \to B\Gamma$. We want to use these data to construct new obstructions to the existence of a positive scalar curvature metric on the manifolds M.

Definition 3.6. For every cohomology class $[u] \in H^*(B\Gamma, \mathbb{Q})$, the *higher $\widehat{A}$-genus associated to* $[u]$ is the rational number

$$\widehat{A}(M; [u]) := \Big\langle r^*([u]) \cup \widehat{\mathbf{A}}(M), [M] \Big\rangle, \tag{3.1}$$

where $[M]$ is the fundamental homology clas of the oriented manifold M and where $\widehat{\mathbf{A}}(M)$ is the total $\widehat{A}$-class of the manifold M (see Section 2.4).

Gromov and Lawson were able to prove in many circunstances that if the manifold M carries a positive scalar curvature metric, then the higher $\widehat{A}$-genera must vanish (see [9] and also [13, Section VI.5]). Moreover, there are no known examples of manifolds admitting a metric of positive scalar curvature such that one of the higher $\widehat{A}$-genera doesn't vanish. This leads to the following

Conjecture 3.7 (Gromov-Lawson)**.** *For any compact spin manifold with positive scalar curvature, all the higher $\widehat{A}$-genera must vanish.*

This conjecture is still open. Notice that it can be regarded as a 'higher version' of Theorem 2.6. Therefore, it is natural to try to adapt the arguments we used in Section 2 to prove (at least some cases of) this conjecture. The remaining part of these notes is devoted to explain this approach. In particular, in Section 4 we construct an 'higher version' of the spin Dirac operator and prove the analogous of Proposition 2.3 for such operators. Finally, in Section 5 we construct a map connecting these higher analytic obstructions to the higher $\widehat{A}$-genera. Finally, we will formulate a sufficient condition for using this map to prove the analogous of Proposition 2.5.

4. Higher Dirac obstructions

In this section we construct an analytic counterpart of the higher $\widehat{A}$-genera defined in the previous section.

4.1. **The algebra $C_r^*\Gamma$.** Let Γ be a discrete group and denote with $\mathbb{C}\Gamma$ the group algebra of Γ. It consists of finite complex combinations of elemnts in Γ. Consider the inner product

$$\Big\langle \sum_{g\in\Gamma} a_g g \Big| \sum_{h\in\Gamma}^{N} b_h h \Big\rangle := \sum_{g\in\Gamma} \overline{a_g}\, b_g, \qquad\qquad \sum_{g\in\Gamma} a_g g,\ \sum_{g\in\Gamma} b_g g \in \mathbb{C}\Gamma.$$

Let $l^2(\Gamma)$ be the Hilbert space obtained as the completion of $\mathbb{C}\Gamma$ with respect to the norm induced by this inner product. We regard any element $f \in l^2(\Gamma)$ as a square-integrable function on Γ. For $g \in \Gamma$, we define the *left regular representation* through

$$L_g(f)(h) = f(g^{-1}h), \qquad\qquad h \in \Gamma. \tag{4.1}$$

Notice that L_g is a bounded operator on $l^2(\Gamma)$. Therefore, the left regular representation induces an action of $\mathbb{C}\Gamma$ on $l^2(\Gamma)$ through bounded operators. This allows us to identify $\mathbb{C}\Gamma$ with a subalgebra of $\mathcal{B}(l^2(\Gamma))$. The closure of $\mathbb{C}\Gamma$ in $\mathcal{B}(l^2(\Gamma))$ with respect to the operator norm is called the *reduced group C^*-algebra* of Γ and is denoted with $C_r^*\Gamma$.

Remark 4.2. Observe that $1 = L_e$, where e is the neutral element of Γ, and that $L_g^* = L_{g^{-1}}$. It follows that the group algebra $\mathbb{C}\Gamma$ is a unital subalgebra of $\mathcal{B}(l^2(\Gamma))$ closed under the $*$-operation. Therefore, $C_r^*\Gamma$ is a unital C^*-algebra.

Example 4.3. When $\Gamma = \mathbb{Z}^k$, we use Fourier transform to identify $l^2(\mathbb{Z}) \cong L^2(T^k)$, where $T^k = (S^1)^k$ is the k-dimensional torus. Under this identification, $C_r^*(\mathbb{Z}^k)$ is isomorphic as a C^*-algebra to $C^0(T^k)$.

4.4. **K-theory.** Let A be a unital C^*-algebra. Let $V(A)$ be the set of the isomorphism classes of finitely generated projective modules over A. Notice that $V(A)$ is an abelian semigroup, where the operation is given by the direct sum of modules and where the neutral element is the class $[0]$. The group $K_0(A)$ is defined as the Grothendieck group of $V(A)$. Its elements are formal differences of isomorphism classes of finitely generated projective modules over A. Notice that a morphism of unital C^*-algebras $\phi : A \to B$ induces a homomorphism of groups $\phi_* : K_0(A) \to K_0(B)$.

When A is not unital, we consider the unital algebra A^+ obtained by 'adding a unit' to A. More precisely, set $A^+ := A \oplus \mathbb{C}$. A^+ is a unital algebra with unit $0\oplus 1$. We consider the map $\phi : A^+ \to \mathbb{C}$ defined by $\phi(a \oplus \lambda) = \lambda$. Define $K_0(A)$ to be the kernel of the induced homomorphism of K-groups $\phi_* : K_0(A^+) \to K_0(\mathbb{C})$. We say that the C^*-algebra $SA := A \otimes C_0(\mathbb{R})$ is the *suspension* of A and define $K_1(A)$ to be $K_0(SA)$. Notice $SA \cong C_0(\mathbb{R}, A)$ is not unital so that the definition of the K_1-group requires to define the K_0-group of a nonunital C^*-algebra. We can iterate this construction and define the *higher K-groups* $K_n(A) := K_0(S^nA)$, where S^nA is defined recursively as the suspension of $S^{n-1}A$. The *Bott periodicity theorem* also holds for the K-theory of C^* algebra. More precisely, there exists an isomorphism $\beta_A : K_0(A) \to K_1(SA)$, where β_A is called the *Bott map* (see [17, Section 9.1]).

Example 4.5. Let X be a locally compact Hausdorff space. Suppose X is noncompact and consider the C^*-algebra $C_0(X)$ of continuous functions on X vanishing at infinity. Denote by X^+ the one-point compactification of X and let $C(X^+)$ be the unital algebra of continous functions on X^+. Then the unital algebras $C_0(X)^+$ and $C(X^+)$ coincide. This illustrates the fact that the operation of 'adding a unit' can be regarded as the noncommutative counterpart of the one-point compactification of a topological space (cf [17, Section 1]).

Example 4.6. We recall that the K-theory group $K^0(X)$ is defined as the set of formal differences of isomorphism classes of vector bundles over X. The Swann's theorem states that $K^0(X)$ is isomorphic to $K_0(C(X))$. From Example 4.3, we deduce that $K_0(\mathbb{Z}^k) = K_0(C(T^k)) = K^0(T^k)$.

4.7. **The Miščenko-Fomenko line bundle.** Fix a discrete group Γ. Let $B\Gamma$ be the classifying space of Γ and $E\Gamma \to B\Gamma$ be the universal principal Γ-bundle. The group Γ acts in a natural way on $C_r^*(\Gamma)$ by right translations. It also acts on $E\Gamma$ by deck transformations. The associated bundle

$$\mathcal{V}(\Gamma) := E\Gamma \times_\Gamma C_r^*(\Gamma)$$

is called the *Miščenko-Fomenko line bundle.* Notice that $\mathcal{V}(\Gamma)$ is not a vector bundle, since its typical fiber is the algebra $C_r^*(\Gamma)$, that in general is infinite dimensional over $\mathbb{C}$. The exterior derivative endows the product bundle $E\Gamma \times C_r^*\Gamma \to E\Gamma$ with a flat cooection, called the *trivial connection.* Since Γ is discrete, the trivial connection induces a (non trivial) flat connection on the bundle $\mathcal{V}(\Gamma)$ (for a precise definition of connections on bundle over C^*-algebras, we refer to [16, Section 3]).

Let M be a closed even dimensional spin manifold and let $\mathbb{S}_M$ be the spin Clifford bundle over M with associated spin Dirac operator $\not{D}_M$ (see Section 2.1). Suppose $\widetilde{M} \to M$ is a Galois cover with deck transformation group Γ and classifying map $r : M \to B\Gamma$ (see Section 3.1). Since Γ is discrete, the pull-back bundle $r^*(\mathcal{V}(\Gamma))$ over M is endowed with a flat connection $\nabla^{\mathcal{V}}$. We consider the bundle $\mathbb{S}_M \otimes r^*(\mathcal{V}(\Gamma))$ endowed with the connection $\nabla^{\mathbb{S}} \otimes 1 + 1 \otimes \nabla^{\mathcal{V}}$. The spin Dirac operator $\not{D}_M$ twisted with the bundle $r^*(\mathcal{V}(\Gamma))$ is the odd self-adjoint elliptic differential operator

$$\not{D}_{(M,r)} = \begin{pmatrix} 0 & \not{D}^-_{(M,r)} \\ \not{D}^+_{(M,r)} & 0 \end{pmatrix},$$

where the operators $\not{D}^\pm_{(M,r)} : C_c^\infty(M, \mathbb{S}_M^\pm \otimes r^*(\mathcal{V}(\Gamma))) \to C_c^\infty(M, \mathbb{S}_M^\mp \otimes r^*(\mathcal{V}(\Gamma)))$ are defined through the compositions

$$C_c^\infty(M, \mathbb{S}_M^\pm \otimes r^*(\mathcal{V}(\Gamma))) \xrightarrow{\nabla^{\mathbb{S}}\otimes 1 + 1\otimes\nabla^{\mathcal{V}}} C_c^\infty(M, T^*M \otimes \mathbb{S}_M^\pm \otimes r^*(\mathcal{V}(\Gamma))) \xrightarrow{c\otimes 1} C_c^\infty(M, \mathbb{S}_M^\mp \otimes r^*(\mathcal{V}(\Gamma)))$$

and where c is the Clifford multiplication on $\mathbb{S}_M$.

We want to use the index of this operator as an analytic obstruction to the existence of a positive scalar caurvature metric on M. The first problem is how to associate an index to this operator. In fact, the vector spaces $\operatorname{Ker} \not{D}^\pm_{(M,r)}$ are not finite dimensional over $\mathbb{C}$. Even worse, they are not even finitely generated as modules over the algebra $C_r^*\Gamma$.

In order to define the index of $\not{D}_{(M,r)}$, we need to modify this operator. In [7], Miščenko and Fomenko defined a pseudodifferential calculus for operators acting on the bundle $\mathbb{S}_M \otimes r^*(\mathcal{V}(\Gamma))$. In particular, they defined the class $\Psi^{-\infty}(M, \mathbb{S}_M^\pm \otimes r^*(\mathcal{V}(\Gamma)))$ of infinitely smoothing operators acting on $\mathbb{S}_M^\pm \otimes r^*(\mathcal{V}(\Gamma))$ and showed that there exist infinitely smoothing operators $R^\pm$ such that $\operatorname{Ker}(\not{D}^+_{(M,r)} + R^+)$ and $\operatorname{Ker}(\not{D}^-_{(M,r)} - R^-)$are finitely generetad projective modules over $C_r^*\Gamma$. Let $[\operatorname{Ker}(\not{D}^\pm_{(M,r)} \pm R^\pm)]$ denote the isomorphism classes of these modules and define the *index class* of $\not{D}_{(M,r)}$ as

$$\operatorname{Ind} \not{D}_{(M,r)} := [\operatorname{Ker}(\not{D}^+_{(M,r)} + R^+)] - [\operatorname{Ker}(\not{D}^-_{(M,r)} - R^-)] \in K_0(C_r^*\Gamma). \tag{4.2}$$

In [7], it is also proved that the class (4.2) is independent of the choice of smoothing operators $R^\pm$ such that $\operatorname{Ker}(\not{D}^\pm_{(M,r)} \pm R^\pm)$ are finitely generetad projective modules over $C_r^*\Gamma$.

4.8. **Higher analytic obstructions.** We want to show that the class $\operatorname{Ind} \not{D}_{(M,r)}$ is an obstruction to the existence of a metric of positive scalar curvature on M. We consider the *twisted connection laplacian*

$$\Delta_{\mathbb{S}_M\otimes\mathcal{V}} := \left(\nabla^{\mathbb{S}\otimes\mathcal{V}}\right)^* \circ \left(\nabla^{\mathbb{S}\otimes\mathcal{V}}\right) \tag{4.3}$$

associated with the **flat** connection $\nabla^{\mathbb{S}\otimes\mathcal{V}} := \nabla^{\mathbb{S}} \otimes 1 + 1 \otimes \nabla^{\mathcal{V}}$.

Proposition 4.9 (Twisted Bochner-Lichnerowicz formula)**.** *We have*

$$\not{D}^2_{(M,r)} \;=\; \Delta_{\mathbb{S}_M\otimes\mathcal{V}} + \frac{1}{4}\kappa,$$

where κ is the scalar curvature of the Riemannian metric on M.

Remark 4.10. This proposition holds more in general any time we twist the operator $\not{D}_{(M,r)}$ with a **flat** bundle W. For the proof of the case when W is a vector bundle, we refer to [13, Theorem 8.17]. When W is a bundle over a C^* algebra (as in our case), see for example [18, Section 3]. When the bundle W is not flat, on the right hand side of Equation 4.3 it appears a remainder term depending on the curvature of the connection ∇^W (cf. [13, Theorem 8.17]).

We now state the analogue of Proposition 2.3 in this more general setting.

Proposition 4.11 (Rosenberg)**.** *Suppose that M admits a metric of positive scalar curvature. Then the index class* $\operatorname{Ind}\not{D}_{(M,r)}$ *vanishes for every Galois cover $\widetilde{M} \to M$ with classifying map $r : M \to B\Gamma$.*

Remark 4.12. Notice that the argument we used to prove Proposition 2.3 doesn't work in this situation. The problem is that the index class $\operatorname{Ind}\not{D}_{(M,r)}$ is defined after a perturbation of the operator $\not{D}_{(M,r)}$. Therefore, even if we prove that $\operatorname{Ker}\not{D}^{\pm}_{(M,r)} = \{0\}$, this fact doesn't imply that the class $\operatorname{Ind}\not{D}_{(M,r)}$ vanishes.

4.13. **Sobolev A-modules.** In order to prove Proposition 4.11, we need to construct some functional spaces and study the way the operator $\not{D}_{(M,r)}$ acts on such spaces.

Notice that the typical fiber of the bundle $\mathbb{S}_M \otimes r^*\mathcal{V}(\Gamma)$ is the $C^*_r(\Gamma)$-module

$$\mathbb{C}^m \otimes C^*_r(\Gamma) \;=\; \underbrace{C^*_r(\Gamma) \oplus \cdots \oplus C^*_r(\Gamma)}_{\text{m-times}} \;=\; C^*_r(\Gamma)^m, \tag{4.4}$$

where m is the dimension over $\mathbb{C}$ of the typical fiber of $\mathbb{S}_M$. In other words, the typical fiber of $\mathbb{S}_M \otimes r^*\mathcal{V}(\Gamma)$ is a free module over $C^*_r(\Gamma)$. Since the algebra $C^*_r(\Gamma)$ is closed under the $*$-operation (see Remark 4.2), on the free $C^*_r(\Gamma)$-module $C^*_r(\Gamma)^m$ there is a canonical $C^*_r(\Gamma)$-valued inner product defined by

$$\langle a_1 \oplus \cdots \oplus a_m | b_1 \oplus \cdots \oplus b_m\rangle \;:=\; a_1^*b_1 + \cdots a_m^*b_m, \tag{4.5}$$

where $(a_1 \oplus \cdots \oplus a_m)$, $(b_1 \oplus \cdots \oplus b_m)$ are arbitrary elements in $C^*_r(\Gamma)^m$.

The inner product (4.5) induces a $C^*_r(\Gamma)$-valued inner product on the fibers of $\mathbb{S}_M \otimes r^*\mathcal{V}(\Gamma)$. Fix a nonnegative integer l and define

$$\langle u, v\rangle_l \;:=\; \sum_{k=0}^{l} \int_M \langle (\not{D}^k_{(M,r)}u)(x), (\not{D}^k_{(M,r)}v)(x)\rangle_x \, dvol_g(x), \qquad u,\, v \in C^\infty_c(M, \mathbb{S}_M \otimes r^*\mathcal{V}(\Gamma)),$$

where $dvol_g$ denotes the volume form induced by the Riemannian metric g and $\langle\cdot,\cdot\rangle_x$ denotes the $C^*_r(\Gamma)$-valued inner product on the fiber of $\mathbb{S}_M \otimes r^*\mathcal{V}(\Gamma)$ at x. We use the product $\langle\cdot,\cdot\rangle_l$ to define the norm

$$\|u\|_l^2 \;:=\; \|\langle u, u\rangle_l\|, \qquad u \in C^\infty_c(M, \mathbb{S}_M \otimes r^*\mathcal{V}(\Gamma)), \tag{4.6}$$

where $\|\cdot\|$ denotes the operator norm on $C^*_r(\Gamma)$ coming from $\mathcal{B}(l^2(\Gamma))$. We denote by H^l the Banach space obtained as the completion of $C^\infty_c(M, \mathbb{S}_M \otimes r^*\mathcal{V}(\Gamma))$ with respect to the norm (4.6).

Notice that the spaces H^j are not Hilbert spaces. Nonetheless they are endowed with a $C^*_r(\Gamma)$-valued inner product that we use to define the notion of adjointable operator. We say that an $C_r^*(\Gamma)$-linear operator $T : H^k \to H^p$ is adjointable if there exists an A-linear operator $S : H^p \to H^k$ such that

$$\langle Tu|v\rangle_p = \langle u|Sv\rangle_k, \qquad u \in H^k,\, v \in H^p.$$

We denote by $\mathcal{B}(H^i, H^j)$ the space of bounded operators from H^i to H^j and by $\mathcal{L}_\Gamma(H^i, H^j)$ the space of bounded adjointable $C_r^*(\Gamma)$-linear operators from H^i to H^j. We also set $\mathcal{B}(H^i) := \mathcal{B}(H^i, H^i)$ and $\mathcal{L}_\Gamma(H^i) := \mathcal{L}_\Gamma(H^i, H^i)$. We finally notice (cf.[17, Lemma 15.1.3]) that the Cauchy-Schwartz inequality holds:

$$\|\langle u|v\rangle_0\| \leq \|u\|_0\|v\|_0, \qquad u,\, v \in H^0. \tag{4.7}$$

In the next lemma we collect some properties of the action of the operator $\not{D}_{(M,r)}$ on these spaces.

Lemma 4.14. *We have:*

(1) For every $f \in C^\infty(M)$ and every nonnegative integer k, l, the operator $\not{D}^k_{(M,r)}+f$ extends to a bounded adjointable operator

$$\not{D}^k_{(M,r)} + f\colon H^{k+l} \longrightarrow H^l. \tag{4.8}$$

(2) For every real number $\lambda > 0$ the operator $\not{D}^2_{(M,r)} + \lambda$ is invertible and $(\not{D}^2_{(M,r)} + \lambda)^{-1} \in \mathcal{L}_\Gamma(H^0, H^2) \cap \mathcal{L}_\Gamma(H^0)$.

(3) If the bounded adjointable operator

$$\not{D}^2_{(M,r)} : H^2 \longrightarrow H^0 \tag{4.9}$$

is invertible with inverse in $\mathcal{L}_\Gamma(H^0, H^2)$, then the index class $\operatorname{Ind}\not{D}_{(M,r)}$ *vanishes.*

The proof of this lemma is quite technical and requires the use of the theory of unbounded operators on the A-modules H^j. This is definitely beyond the scope of these notes. We refer the interested reader to [6, Section 9]. We are finally ready to prove Proposition 4.11.

4.15. **Proof of Proposition 4.11.** Pick $\lambda > 0$. By point (2) of Lemma 4.14, $\not{D}^2_{(M,r)} + \lambda$ is invertible and the inverse is in both $\mathcal{L}_\Gamma(H^0)$ and $\mathcal{L}_\Gamma(H^0, H^2)$. In particular, we can write

$$\begin{aligned} \not{D}^2_{(M,r)} &= \left(\not{D}^2_{(M,r)} + \lambda\right) - \lambda \\ &= \left\{I - \lambda\left(\not{D}^2_{(M,r)} + \lambda\right)^{-1}\right\}\left(\not{D}^2_{(M,r)} + \lambda\right). \end{aligned} \tag{4.10}$$

Since M is compact, there exists a constant $c > 0$ such that $\kappa(x) \geq 4c$, for every $x \in M$. From Proposition 4.9, we have

$$\langle(\not{D}^2_{(M,r)} + \lambda)u, u\rangle_0 = \langle\Delta_{\mathbb{S}\otimes\mathcal{V}}u|u\rangle + \langle\left(\frac{1}{4}\kappa + \lambda\right)u|u\rangle_0 \geq (c+\lambda)\,\langle u|u\rangle_0. \qquad u \in H^2$$

By using Cauchy-Schwartz inequality (4.7), we obtain

$$(c+\lambda)\|u\|_0^2 \leq \|\langle(\not{D}^2_{(M,r)} + \lambda)u, u\rangle_0\| \leq \|(\not{D}^2_{(M,r)} + \lambda)u\|_0\|u\|_0$$

from which it follows that

$$\left\|\lambda\left(\not{D}^2_{(M,r)} + \lambda\right)^{-1}\right\|_{H^0\to H^0} \leq \frac{\lambda}{c+\lambda} < 1.$$

Therefore, the operator $I - \lambda\big(\not{D}^2_{(M,r)} + \lambda\big)^{-1}$ is invertible with bounded inverse given by the Neumann series

$$\left\{I - \lambda\big(\not{D}^2_{(M,r)} + \lambda\big)^{-1}\right\}^{-1} = \sum_{k=0}^{\infty}(-1)^k\left\{\lambda\big(\not{D}^2_{(M,r)} + \lambda\big)^{-1}\right\}^k. \tag{4.11}$$

Notice that the series on the right hand side of (4.11) converges in norm and that each summand is in $\mathcal{L}_\Gamma(H^0)$. Since by [17, Proposition 15.2.4] the space $\mathcal{L}_\Gamma(H^0)$ is closed in the operator norm, the series (4.11) defines a bounded adjointable operator on H^0. Since $\left(\not{D}^2_{(M,r)} + \lambda\right)^{-1}$ is in $\mathcal{L}_\Gamma(H^0)$, from (4.10) we finally deduce that $\not{D}^2_{(M,r)}$ is invertible with inverse in $\mathcal{L}_\Gamma(H^0, H^2)$ given by the norm convergent series

$$\left(\not{D}^2_{(M,r)}\right)^{-1} = \left(\not{D}^2_{(M,r)} + \lambda\right)^{-1} \sum_{k=0}^{\infty} (-1)^k \left\{\lambda(\not{D}^2_{(M,r)} + \lambda)^{-1}\right\}^k.$$

Now the thesis follows from point (3) of Lemma 4.14. □

5. The assembly map and the strong Novikov conjecture

In this final section we connect the index of the operator $\not{D}_{(M,r)}$ studied in the previous section with the higher $\widehat{A}$-genera defined in Section 3.5. For this last part, we closely follow [14, Section 7], where the connection between the Novikov conjecture and the strong Novikov conjecture is discussed.

As in the previous section, we let M be a closed even dimensional spin manifold. Let us fix a Galois cover $\widetilde{M} \to M$ with classifying map $r : M \to B\Gamma$. The first problem is that on the analytic side we constructed a unique obstruction, namely $\operatorname{Ind} \not{D}_{(M,r)}$, whereas on the topological side we constructed the family of higher $\widehat{A}$-genera $\widehat{A}(M, [u])$, where $[u]$ varies between the cohomology classes in $H^*(B\Gamma, \mathbb{Q})$. The first step consists in selecting a unique element also on the topological side.

5.1. **K-homology.** Let $\widehat{\mathbf{A}}(X) \cap [X]$ be the Poincaré dual of $\widehat{\mathbf{A}}(X)$ [1]. Consider the class $r_*(\widehat{\mathbf{A}}(X) \cap [X]) \in H_*(B\Gamma, \mathbb{Q})$. We have the following

Lemma 5.2. *For every class $[u] \in H^*(B\Gamma, \mathbb{Q})$, we have*

$$\widehat{A}(X; [u]) = \langle [u], r_*(\widehat{\mathbf{A}}(X) \cap [X])\rangle \tag{5.1}$$

Remark 5.3. This lemma shows in particular that if any of the higher $\widehat{A}$-genera doesn't vanish, then the class $r_*(\widehat{\mathbf{A}}(X) \cap [X]$ doesn't vanish as well.

We recall that the topological K-theory is a generalized cohomology theory. This implies that it admits a dual theory, called K-homology. We denote with $K_*(X)$ the K-homology of a space X. Moreover, the usual Chern character has a homological version

$$Ch : K_*(X, \mathbb{Z}) \longrightarrow H_*(X, \mathbb{Z}),$$

that is an isomorphism modulo torsion. In particular, it defines an isomorphism

$$Ch^{-1} : H_*(B\Gamma, \mathbb{Q}) \longrightarrow K_*(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$$

5.4. **The assembly map.** It remains to connect the K-homology of $B\Gamma$ with the K-theory of $C^*_r\Gamma$ (see Section 4.4). To this end, we need to describe the elements of $K_*(B\Gamma)$. In [1], Atiyah characterizes cycles in K-homology as 'abstract elliptic operators'. This idea was further developed by Kasparov and other authors. We follow the approach of Baum and Douglas (cf.[4]). In this section we pretend that $B\Gamma$ is compact, even if by Example 3.4 we know that this is not always the case. All the constructions of this section can be extended to the case when $B\Gamma$ is not compact by making use of inductive limits. We will omit this very technical part.

Let X be a compact Hausdorff space. A K_0-cycle over X is a triple $(M, r : M \to X, E)$, where M is an even dimensional oriented manifold, $r : M \to X$ is a continous map and $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$-graded Clifford bundle over M. On cycles it is defined an equivalence relations $\sim$ given

[1]The choice of the symbol $\widehat{\mathbf{A}}(X) \cap [X]$ is motivated by the use of the cap product to construct this element.

by *bordism*, *direct sum - disjoint union* and *vector bundle modification* . For a description of these operations, we refer to the notes of Paul Baum [3]. It turns out that

$$\{K_0\text{-cycles } (M, r : M \to X, E) \text{ over } X\ \}/_{\sim} \;=\; K_0(X).$$

Given a K_0-cycle $(M, r : M \to X, E)$ over X, we denote by $[M, r : M \to X, E]$ the corresponding class in $K_0(X)$. For example, the triple $(M, \mathrm{id}, \mathbb{S}_M)$, where $\mathbb{S}_M = \mathbb{S}_M^+ \oplus \mathbb{S}_M^-$ is the spin bundle, defines a class $[M, \mathrm{id}, \mathbb{S}_M]$ in $K_0(M)$. In a similar way, if $r : M \to B\Gamma$ is a classifying map, then $[M, r : M \to B\Gamma, \mathbb{S}_M]$ defines an element in $K_0(B\Gamma)$.

Fix a K_0-cycle $(M, r : M \to B\Gamma, E^+ \oplus E^-)$ over $B\Gamma$. Let D^E be the Dirac operator associated to the bundle E and let $D^E_{(M,r)}$ be the operator D^E twisted by the flat bundle $r^*\mathcal{V}(\Gamma)$. Notice that the index class $\mathrm{Ind}\, D^E_{(M,r)} \in K_0(C^*_r\Gamma)$ depends only on the equivalence class $[M, r : M \to B\Gamma, E]$ in $K_0(B\Gamma)$ and not on the choice of a particular cycle. Therefore, we define a map

$$\mu : K_0(B\Gamma) \longrightarrow K_0(C^*_r\Gamma)$$

by setting

$$\mu([M, r : M \to B\Gamma, E]) \;:=\; \mathrm{Ind}\, D^E_{(M,r)} \in K_0(C^*_r\Gamma). \tag{5.2}$$

Our fundamental example is

$$\mu([M, r : M \to B\Gamma, \mathbb{S}_M]) \;=\; \mathrm{Ind}\ \not{D}_{(M,r)} \in K_0(C^*_r\Gamma).$$

A similar map can be defined from $K_1(B\Gamma)$ to $K_1(C^*_r\Gamma)$ by considering odd dimensional manifolds and ungraded Clifford modules. By the Bott periodicity theorem (see Section 4.4), this is enough to define a map $\mu : K_*(B\Gamma) \to K_*(C^*_r(\Gamma))$.

Definition 5.5. The *assembly map* is the map

$$\mu_{\mathbb{Q}} : K_0(B\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow K_0(C^*_r\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{5.3}$$

induced by μ.

5.6. **The strong Novikov conjecture.** In this last section we show the connection between assembly map and Conjecture 3.7.

Before stating the main theorem, we need to relate the assembly map $\mu_{\mathbb{Q}}$ with the map Ch^{-1}. By unwinding the definition, we obtain

$$Ch^{-1}(r_*(\widehat{\mathbf{A}}(M) \cap [M])) \;=\; [M, r : M \to B\Gamma, \mathbb{S}_M]. \tag{5.4}$$

Therefore,

$$\mu_{\mathbb{Q}}(Ch^{-1}(r_*(\widehat{\mathbf{A}}(M) \cap [M]))) \;=\; \mathrm{Ind}\ \not{D}_{(M,r)}. \tag{5.5}$$

Theorem 5.7. *Suppose that the map $\mu_{\mathbb{Q}}$ is injective. If M carries a positive scalar curvature metric, then the higher genera $\widehat{A}(M; [u])$ vanish for every class $[u] \in H^*(B\Gamma; \mathbb{Q})$.*

Proof. By Lemma 5.2, it suffices to prove that

$$r_*(\widehat{\mathbb{A}}(M) \cap [M]) \;=\; 0, \qquad\qquad \text{in } H_*(B\Gamma, \mathbb{Q}). \tag{5.6}$$

If M admits a Riemannian metric of positive scalar curvature, by Proposition 4.11 and (5.5), we get

$$\mu_{\mathbb{Q}}(Ch^{-1}(r_*(\widehat{\mathbf{A}}(M) \cap [M]))) \;=\; \mathrm{Ind}\ \not{D}_{(M,r)} \;=\; 0. \tag{5.7}$$

Since $\mu_{\mathbb{Q}}$ is injective and Ch^{-1} is bijective, (5.7) implies (5.6), from which the thesis follows. $\square$

The injectivity of the map $\mu_{\mathbb{Q}}$ is a famous open problem:

Conjecture 5.8 (Strong Novikov conjecture)**.** *The map $\mu_{\mathbb{Q}}$ is injective.*

This conjecture has been proved for many groups: see for instance [5, Theorem 24.2.3]. In all these cases, by Theorem 5.7 also Conjecture 3.7 holds. The strong Novikov conjecture was formulated in an attempt to solve a conjecture of Novikov about the homotopy invariance of the higher signatures (for an overview of this topic, we refer to [14]). In [15] it was observed that it also implies that the higher genera are obstructions to the existence of metrics of positive scalar curvature.

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