31. The solid lies below the surface $z = 2 + x^2 + (y-2)^2$ and above the plane $z = 1$ for $-1 \le x \le 1$, $0 \le y \le 4$. The volume of the solid is the difference in volumes between the solid that lies under $z = 2 + x^2 + (y-2)^2$ over the rectangle $R = [-1, 1] \times [0, 4]$ and the solid that lies under $z = 1$ over R.

$$\begin{aligned} V &= \int_0^4 \int_{-1}^1 [2 + x^2 + (y-2)^2]\,dx\,dy - \int_0^4 \int_{-1}^1 (1)\,dx\,dy = \int_0^4 \left[2x + \tfrac{1}{3}x^3 + x(y-2)^2\right]_{x=-1}^{x=1} dy - \int_{-1}^1 dx \int_0^4 dy \\ &= \int_0^4 \left[\left(2 + \tfrac{1}{3} + (y-2)^2\right) - \left(-2 - \tfrac{1}{3} - (y-2)^2\right)\right] dy - [x]_{-1}^1\,[y]_0^4 \\ &= \int_0^4 \left[\tfrac{14}{3} + 2(y-2)^2\right] dy - [1-(-1)][4-0] = \left[\tfrac{14}{3}y + \tfrac{2}{3}(y-2)^3\right]_0^4 - (2)(4) \\ &= \left[\left(\tfrac{56}{3} + \tfrac{16}{3}\right) - \left(0 - \tfrac{16}{3}\right)\right] - 8 = \tfrac{88}{3} - 8 = \tfrac{64}{3} \end{aligned}$$

35. R is the rectangle $[-1, 1] \times [0, 5]$. Thus, $A(R) = 2 \cdot 5 = 10$ and

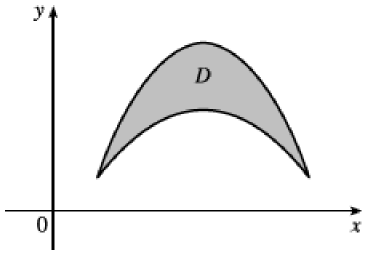
$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x,y)\,dA = \tfrac{1}{10} \int_0^5 \int_{-1}^1 x^2 y\,dx\,dy = \tfrac{1}{10} \int_0^5 \left[\tfrac{1}{3}x^3 y\right]_{x=-1}^{x=1} dy = \tfrac{1}{10} \int_0^5 \tfrac{2}{3}y\,dy = \tfrac{1}{10}\left[\tfrac{1}{3}y^2\right]_0^5 = \tfrac{5}{6}.$$

37. $\displaystyle \iint_R \frac{xy}{1+x^4}\,dA = \int_{-1}^1 \int_0^1 \frac{xy}{1+x^4}\,dy\,dx = \int_{-1}^1 \frac{x}{1+x^4}\,dx \int_0^1 y\,dy$ [by Equation 5] but $f(x) = \dfrac{x}{1+x^4}$ is an odd function so $\displaystyle\int_{-1}^1 f(x)\,dx = 0$ by (6) in Section 4.5 [ET (7) in Section 5.5]. Thus $\displaystyle\iint_R \frac{xy}{1+x^4}\,dA = 0 \cdot \int_0^1 y\,dy = 0.$

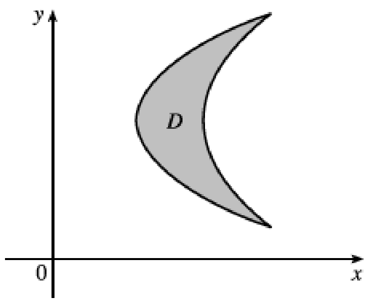
38.
$$\begin{aligned} \iint_R (1 + x^2 \sin y + y^2 \sin x)\,dA &= \iint_R 1\,dA + \iint_R x^2 \sin y\,dA + \iint_R y^2 \sin x\,dA \\ &= A(R) + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^2 \sin y\,dy\,dx + \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} y^2 \sin x\,dy\,dx \\ &= (2\pi)(2\pi) + \int_{-\pi}^{\pi} x^2 dx \int_{-\pi}^{\pi} \sin y\,dy + \int_{-\pi}^{\pi} \sin x\,dx \int_{-\pi}^{\pi} y^2\,dy \end{aligned}$$

But $\sin x$ is an odd function, so $\int_{-\pi}^{\pi} \sin x\,dx = \int_{-\pi}^{\pi} \sin y\,dy = 0$ by (6) in Section 4.5 [ET (7) in Section 5.5] and $\iint_R (1 + x^2 \sin y + y^2 \sin x)\,dA = 4\pi^2 + 0 + 0 = 4\pi^2$.

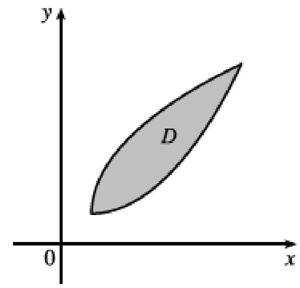
11. (a) At the right we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of x (a type I region) but not as lying between graphs of two continuous functions of y (a type II region). The regions shown in Figures 6 and 8 in the text are additional examples.



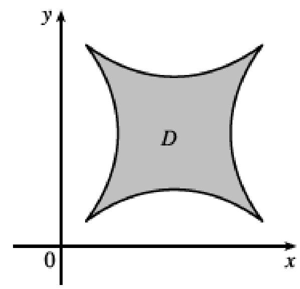
(b) Now we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of y but not as lying between graphs of two continuous functions of x. The first region shown in Figure 7 is another example.



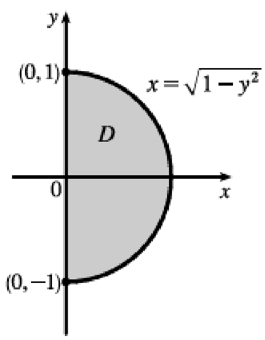
12. (a) At the right we sketch an example of a region D that can be described as lying between the graphs of two continuous functions of x (a type I region) and also as lying between graphs of two continuous functions of y (a type II region). For additional examples see Figures 9, 10, 12, and 14–16 in the text.



(b) Now we sketch an example of a region D that can't be described as lying between the graphs of two continuous functions of x or between graphs of two continuous functions of y. The region shown in Figure 18 is another example.



20.



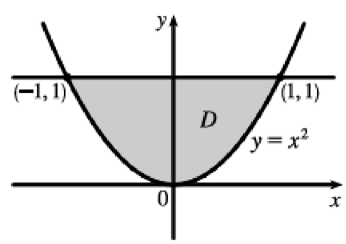
$$\iint_D xy^2\,dA = \int_{-1}^{1}\int_0^{\sqrt{1-y^2}} xy^2\,dx\,dy$$

$$= \int_{-1}^{1} y^2\left[\tfrac{1}{2}x^2\right]_{x=0}^{x=\sqrt{1-y^2}} dy = \tfrac{1}{2}\int_{-1}^{1} y^2(1-y^2)\,dy$$

$$= \tfrac{1}{2}\int_{-1}^{1}(y^2 - y^4)\,dy = \tfrac{1}{2}\left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5\right]_{-1}^{1}$$

$$= \tfrac{1}{2}\left(\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{2}{15}$$

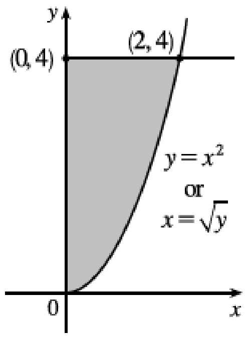
36. The two planes intersect in the line $y = 1$, $z = 3$, so the region of integration is the plane region enclosed by the parabola $y = x^2$ and the line $y = 1$. We have $2 + y \geq 3y$ for $0 \leq y \leq 1$, so the solid region is bounded above by $z = 2 + y$ and bounded below by $z = 3y$.



$$V = \int_{-1}^{1}\int_{x^2}^{1}(2+y)\,dy\,dx - \int_{-1}^{1}\int_{x^2}^{1}(3y)\,dy\,dx = \int_{-1}^{1}\int_{x^2}^{1}(2+y-3y)\,dy\,dx = \int_{-1}^{1}\int_{x^2}^{1}(2-2y)\,dy\,dx$$

$$= \int_{-1}^{1}\left[2y - y^2\right]_{y=x^2}^{y=1} dx = \int_{-1}^{1}(1 - 2x^2 + x^4)\,dx = x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\Big]_{-1}^{1} = \tfrac{16}{15}$$

44.

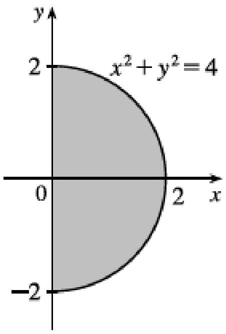


Because the region of integration is

$$D = \{(x,y) \mid x^2 \le y \le 4, 0 \le x \le 2\}$$
$$= \{(x,y) \mid 0 \le x \le \sqrt{y}, 0 \le y \le 4\}$$

we have $\int_0^2 \int_{x^2}^4 f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA = \int_0^4 \int_0^{\sqrt{y}} f(x,y)\,dx\,dy$.

46.



Because the region of integration is

$$D = \left\{(x,y) \mid 0 \le x \le \sqrt{4-y^2}, -2 \le y \le 2\right\}$$
$$= \left\{(x,y) \mid -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, 0 \le x \le 2\right\}$$

we have

$$\int_{-2}^2 \int_0^{\sqrt{4-y^2}} f(x,y)\,dx\,dy = \iint_D f(x,y)\,dA = \int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy\,dx.$$