Calc III: Workshop 5 Solutions, Fall 2017

Problem 1. Find and classify the local maxima and minima of the function

$$f(x,y) = x^3 - 3xy + y^3 + 2.$$

Solution. Setting the gradient of f equal to zero yeilds

$$\nabla f(x,y) = (3x^2 - 3y, -3x + 3y^2) = (0,0), \quad \text{or} \quad \begin{cases} y = x^2 \\ x = y^2 \end{cases}.$$

Plugging in, we get $y = y^4$, with solutions $y = 0$ or $y = 1$. Plugging these into $x = y^2$ gives the two critical points $(x,y) = (0,0)$ and $(x,y) = (1,1)$.
We compute $f_{xx} = 6x$, $f_{yy} = 6y$, and $f_{xy} = -3$, so

$$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = 36xy - 9.$$

We have $D(0,0) = -9$, so $(0,0)$ is a saddle point, and $D(1,1) = 27$, $f_{xx}(1,1) = 6$, so $(1,1)$ is a local minimum. □

Problem 2. Find the max and min of $f(x,y) = x + 2y + 1$ on the ellipse $x^2 + 2y^2 = 1$.

Solution. Lagrange multipliers $(1,2) = \nabla f = \lambda \nabla g = \lambda(2x, 4y)$ gives the system of equations

$$\begin{aligned} 1 &= 2\lambda x \\ 2 &= 4\lambda y \\ x^2 + 2y^2 &= 1 \end{aligned}$$

Since $\lambda = 0$ leads to a contradiciton in the first two equations, we may freely divide by it, to get $x = \frac{1}{2\lambda}$ and $y = \frac{1}{2\lambda} = x$. Then setting $x = y$ in the last equation gives $3x^2 = 1$, or $x = \pm\frac{1}{\sqrt{3}}$. Thus the two constrained critical points are $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ and $(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$. Evaluating f at these points gives

$$\begin{aligned} f(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}) &= \tfrac{3}{\sqrt{3}} + 1 = 1 + \sqrt{3}, \\ f(-\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}}) &= -\tfrac{3}{\sqrt{3}} + 1 = 1 - \sqrt{3}. \end{aligned}$$

which are the maximum and minimum values, respectively. □

Problem 3. You want to construct an open top box (i.e., 4 sides and a bottom, but no top) having a volume of 4000 cubic centimeters, but using the least amount of materials (i.e., having minimal surface area). What are the dimensions of such a box?

Solution. Let x, y, and z denote the dimensions of the box. Then the surface area is $f(x,y,z) = xy + 2xz + 2yz$, which we want to minimize subject to the constraint $g(x,y,z) = xyz = 4000$. Via lagrange multipliers ($\nabla f = \lambda \nabla g$), this leads to the system of equations

$$\begin{aligned} y + 2z &= \lambda yz \\ x + 2z &= \lambda xz \\ 2x + 2y &= \lambda xy \\ xyz &= 4000 \end{aligned}$$

Solving for λ in the first three equations, we get

$$\frac{y+2z}{yz} = \frac{x+2z}{xz} = \frac{2x+2y}{xy}. \tag{1}$$

Cross multiplying in the first of these equations, we get

$$xz(y+2z) = yz(x+2z) \implies 2xz^2 = 2yz^2.$$

Note that z cannot be zero owing to the constraint equation $xyz = 4000$, so we can cancel it from both sides to get $x = y$. Similarly, from the second equation in (1), we get

$$2z = y.$$

Setting $x = y = 2z$ in the constraint equation gives $4z^3 = 4000$, or $z = 10$. Thus the optimal dimensions of the box are $(x, y, z) = (20, 20, 10)$. □

Problem 4. Find the global maximum and minimum of $f(x, y) = 4x^2 - 4x + y$ over the region where $0 \le y \le 4 - x^2$.

Solution. First we find any critical points in the interior. Trying to solve $\nabla f = (8x-4, 1) = (0, 0)$, we see that it has no solutions (since $1 \neq 0$), so there are no critical points of f in the interior of the region. The maximum and minimum must occur at the boundary, which consists of two two curves $y = 0$ and $y = 4 - x^2$, both where $-2 \le x \le 2$.
Proceeding with lagrange multipliers, we have in the first case $g(x, y) = y$, so $\nabla g = (0, 1)$ and lagrange multipliers gives

$$\begin{aligned} 8x - 4 &= 0 \\ 1 &= \lambda \\ y &= 0 \end{aligned}$$

with the unique solution $(x, y) = (1/2, 0)$.
With $g(x, y) = y - 4 + x^2$ lagrange multipliers gives

$$\begin{aligned} 8x - 4 &= 2\lambda x \\ 1 &= \lambda \\ y &= 4 - x^2 \end{aligned}$$

with the solution $(x, y) = (2/3, 32/9)$.
Alternatively, the constrained optimization problems could be solved by substituting $y = 0$ and $y = 4 - x^2$ respectively into $f(x, y)$ and optimizing the resulting functions of 1 variable. Due to the nonsmoothness of the boundary at the "corners" $(-2, 0)$ and $(2, 0)$, we need to add these to the list of points under consideration (these are "boundaries of the boundaries", if you like). Evaluating f at all the candidate points, we have

$$f(1/2, 0) = -1, \quad f(2/3, 32/9) = 8/3, \quad f(-2, 0) = 24, \quad f(2, 0) = 8.$$

So the maximum is 24, achieved at the point $(-2, 0)$, and the minimum is -1, achieved at the point $(1/2, 0)$. □