Calc III Fall 2018: Exam 2 Solutions

Problem 1. Evaluate the integral $\iint_R (x^2+y^2)^{3/2}\,dA$, where R is the region between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution. In polar coordinates,

$$\begin{aligned}\iint_R (x^2+y^2)^{3/2}\,dA &= \int_0^{2\pi}\int_1^2 (r^3)r\,dr\,d\theta \\ &= 2\pi\left(\frac{2^5}{5}-\frac{1}{5}\right) = \frac{62\pi}{5}\end{aligned}$$

□

Problem 2. Find the volume of the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $x+2y+z=2$.

Solution.

$$\begin{aligned}\text{Vol} &= \int_0^1\int_0^{2-2y}\int_0^{2-2y-x} dz\,dx\,dy \\ &= \int_0^1\int_0^{2-2y} 2-2y-x\,dx\,dy \\ &= \int_0^1 (2-2y)(2-2y)-\tfrac{1}{2}(2-2y)\,dy \\ &= \frac{1}{2}\int_0^1 (2-2y)^2\,dy \\ &= -\frac{1}{12}(2-2y)^3\Big|_{y=0}^1 \\ &= \frac{2^3}{12} = \frac{2}{3}.\end{aligned}$$

□

Problem 3. Consider the double integral

$$\int_0^{\sqrt{2}}\int_2^{4-x^2} f(x,y)\,dy\,dx$$

where $f(x,y)$ is some unspecified function.

(a) Draw the region of integration in the xy-plane.

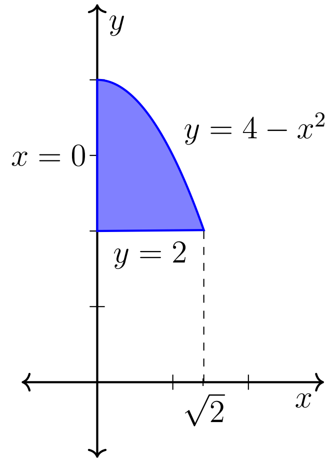
(b) Change the order of integration from $dy\,dx$ to $dx\,dy$, i.e., fill in the limits in the right hand side of the equation

$$\int_0^{\sqrt{2}}\int_2^{4-x^2} f(x,y)\,dy\,dx = \int\quad\int\quad f(x,y)\,dx\,dy$$

(You are not meant to evaluate any integrals, just give the appropriate limits.)

Solution.

(a)



(b) Integrating in x first, the limits become $0 \le x \le \sqrt{4-y}$ and $2 \le y \le 4$. Thus an equivalent integral is

$$\int_2^4 \int_0^{\sqrt{4-y}} f(x,y)\,dx\,dy.$$

$\square$

Problem 4. The ant from Exam 1 has found a DELICIOUS FOOD SOURCE: an ice cream cone occupying the region above the cone $z = \sqrt{3(x^2+y^2)}$ and below the sphere $x^2+y^2+z^2 = 4$. Find the total mass of this frosty treat, assuming its mass density is given by $\delta(x,y,z) = 2z$.

Solution. The mass can be computed in any of the three coordinate systems, but cylindrical and spherical are the best choices. In cylindrical:

$$\begin{aligned}
M &= \iiint_E 2z\,dV \\
&= \int_0^{2\pi} \int_0^1 \int_{\sqrt{3}r}^{\sqrt{4-r^2}} 2zr\,dz\,dr\,d\theta \\
&= 2\pi \int_0^1 r\big((4-r^2) - 3r^2\big)\,dr \\
&= 2\pi \int_0^1 4r - 4r^3\,dr \\
&= 2\pi(2-1) \\
&= 2\pi.
\end{aligned}$$

In spherical:

$$
\begin{aligned}
M &= \iiint_E 2z\, dV \\
&= \int_0^{2\pi} \int_0^{\pi/6} \int_0^2 (2\rho\cos\varphi)\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \\
&= 2\int_0^{2\pi} d\theta \int_0^{\pi/6} \sin\varphi\cos\varphi\, d\varphi \int_0^2 \rho^3\, d\rho \\
&= 2(2\pi)\left(\tfrac{1}{2}\sin^2(\pi/6)\right)\left(\tfrac{1}{4}2^4\right) \\
&= 2\pi.
\end{aligned}
$$

$\square$

Problem 5. Evaluate the line integral $\int_C \mathbf{F}(x,y)\cdot d\mathbf{r}$, where

$$\mathbf{F}(x,y) = xy^2\mathbf{i} - x^2\mathbf{j}$$

is the vector field and and C is the curve given by $\mathbf{r}(t) = \langle t^3, t^2\rangle$, $0 \le t \le 1$.

Solution. Evaluating $\mathbf{F}(\mathbf{r}(t))$ gives

$$\mathbf{F}(\mathbf{r}(t)) = \left\langle x(t)y^2(t),\, -x^2(t)\right\rangle = \left\langle t^7, -t^6\right\rangle.$$

Then

$$d\mathbf{r} = \mathbf{T}\, ds = \mathbf{r}'(t)\, dt = \left\langle 3t^2,\, 2t\right\rangle\, dt,$$

so

$$
\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \left\langle t^7,\, -t^6\right\rangle \cdot \left\langle 3t^2,\, 2t\right\rangle\, dt \\
&= \int_0^1 3t^9 - 2t^7\, dt \\
&= \frac{3}{10} - \frac{2}{8} = \frac{1}{20}.
\end{aligned}
$$

$\square$