Calc III: Workshop 9.5 (was 10, got renumbered), Fall 2018

Problem 1.

(a) Verify that the vector field $\mathbf{F}(x, y, z) = (2xy + ye^x)\mathbf{i} + (x^2 + e^x)\mathbf{j}$ is conservative, and find a potential function $f(x, y)$.

(b) Compute the line integral $\int_C \mathbf{F}(x, y) \cdot d\mathbf{r}$, where C is any curve from $(0, 1)$ to $(1, 2)$.

Problem 2.

(a) The vector field $\mathbf{F}(x, y, z) = \sin y\,\mathbf{i} + (x\cos y + \cos z)\,\mathbf{j} - y\sin z\,\mathbf{k}$ is conservative. Find a potential function $f(x, y, z)$.

(b) Compute the line integral $\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r}$, where C is the parameterized curve $\mathbf{r}(t) = \sin t\,\mathbf{i} + t\,\mathbf{j} + 2t\,\mathbf{k}$, $0 \le t \le \pi/2$.

Problem 3. Show that if the vector field $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is conservative and P, Q, and R have continuous first order partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}.$$

Problem 4. Use the previous exercise to show that the line integral $\int_C (y\,\mathbf{i} + x\,\mathbf{j} + xyz\,\mathbf{k}) \cdot d\mathbf{r}$ is not independent of path.

Problem 5. Let $\mathbf{F}(x, y) = \frac{-y\,\mathbf{i} + x\,\mathbf{j}}{x^2 + y^2}$.

(a) Show that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

(b) Show that $\int_C \mathbf{F} \cdot \mathbf{T}\,ds$ is not independent of path. [Hint: compute $\int_{C_1} \mathbf{F} \cdot \mathbf{T}\,ds$ and $\int_{C_2} \mathbf{F} \cdot \mathbf{T}\,ds$ where C_1 and C_2 are the upper and lower halves of the unit circle from $(1, 0)$ to $(-1, 0)$.] Does this contradict the theorem that says if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on a simply connected region, then $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is path independent? Why not?

Problem 6. Let $\mathbf{F} = \nabla f$, where $f(x, y) = \sin(x - 2y)$. Find curves C_1 and C_2 that are not closed and satisfy the equation

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0, \qquad \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 1.$$