Math 2321 Fall 2015: Exam 2 Solutions

Problem 1. Integrate the function $f(x, y) = (x^2 + y^2)^{3/2}$ over the annulus, $R \subset \mathbb{R}^2$, consisting of the points where $1 \le x^2 + y^2 \le 4$.

Solution. In polar coordinates, $f = (r^2)^{3/2} = r^3$ and the region of integration is given by $0 \le \theta \le 2\pi$ and $1 \le r \le 2$. Thus

$$\iint_A f(x, y)\, dA = \int_0^{2\pi} \int_1^2 r^3\, r\, dr\, d\theta = 2\pi\Big(\frac{2^5}{5} - \frac{1}{5}\Big) = \frac{62\pi}{5}. \qquad \square$$

Problem 2. Change the order of integration in the integral

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^4 dz\, dy\, dx$$

to $dx\, dy\, dz$. **You do not have to evaluate the integral.**

Solution. One really has to draw the region of integration to get this right. The region is bounded by the paraboloid $z = x^2 + y^2$, the plane $z = 4$, and the planes $x = 0$ and $y = 0$. Having drawn it, it is then clear that to integrate first in x, the limits should be $0 \le x \le \sqrt{z - y^2}$. Projecting onto the y-z plane, we get the 2D region bounded by $z = y^2$, $z = 4$ and $y = 0$, so to integrate next in y the limits are $0 \le y \le \sqrt{z}$. Finally, projecting onto the z axis, we have $0 \le z \le 4$. Thus the integral is equal to

$$\int_0^4 \int_0^{\sqrt{z}} \int_0^{\sqrt{z-y^2}} dx\, dy\, dz. \qquad \square$$

Problem 3. Find the surface area of the surface, S, given by $z = 1 - x^2 - y^2$, in the region where $z \ge 0$.

Solution. The surface is a downward opening paraboloid $z = f(x, y) = 1 - x^2 - y^2$ lying over the disk of radius 1 in the x-y plane. There are two reasonable parameterizations. The first is $\mathbf{r}(x, y) = (x, y, f(x, y)) = (x, y, 1 - x^2 - y^2)$, giving

$$dS = |\mathbf{r}_x \times \mathbf{r}_y|\; dx\, dy = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy = \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy$$

so the area is given by

$$\text{Area} = \iint_R \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy \tag{1}$$

where R is the disk of radius 1.

Alternatively, one could parameterize by polar coordinates, giving $\mathbf{r}(r, \theta) = (r\cos\theta, r\sin\theta, 1 - r^2)$ over $0 \le \theta \le 2\pi$ and $1 \le r \le 1$. The surface area element is

$$dS = |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^4(\cos^2\theta + \sin^2\theta)}\, dr\, d\theta = r\sqrt{4r^2 + 1}\, dr\, d\theta,$$

so the area is given by

$$\int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1}\, r\, dr\, d\theta = 2\pi\big(\tfrac{1}{8}\big)\big(\tfrac{2}{3}\big)(4r^2 + 1)^{3/2}\Big|_{r=0}^{1} = \frac{\pi}{6}(5^{3/2} - 1). \qquad \square$$

Problem 4. A solid region $S \subset \mathbb{R}^3$ occupies a quarter sphere, bounded by $x^2+y^2+z^2=4$ and the planes $z=0$ and $y=0$, and has mass density given by $\delta(x,y,z) = 1+x^2+y^2+z^2$. Compute the mass of S.

Solution. We parameterize S using spherical coordinates, with limits $0 \le \rho \le 2$, $0 \le \varphi \le \frac{\pi}{2}$, and $0 \le \theta \le \pi$. The density in spherical coordinates is given by $\delta = 1+\rho^2$. The mass is therefore given by

$$\text{Mass}(S) = \int_0^{\pi}\int_0^{\pi/2}\int_0^2 (1+\rho^2)\,\rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta = \pi\Big(\frac{2^3}{3}+\frac{2^5}{5}\Big). \qquad \square$$

Problem 5. Consider the vector field

$$\mathbf{F}(x,y,z) = (yz+1,\ xz+3y^2z,\ xy+y^3+2z).$$

(a) Verify that $\mathbf{F}$ is conservative by computing the curl $\nabla\times\mathbf{F}$.

Solution.

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz+1 & xz+3y^2z & xy+y^3+2z \end{vmatrix} = \big((x+3y^2)-(x+3y^2), y-y, z-z\big) = (0,0,0).$$

Since $\nabla\times\mathbf{F}$ is defined and vanishing for all (x,y,z), $\mathbf{F}(x,y,z)$ is conservative. $\square$

(b) Find a potential function $f(x,y,z)$, so that $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$.

Solution. We want to solve

$$\begin{aligned} f_x(x,y,z) &= yz+1 \\ f_y(x,y,z) &= xz+3y^2z \\ f_z(x,y,z) &= xy+y^3+2z. \end{aligned}$$

Integrating the first equation gives $f(x,y,z) = xyz+x+A(y,z)$ where $A(y,z)$ is unknown. Plugging this into the second equation gives $f_y = xz+A_y(y,z) = xz+3y^2z$, so $A_y(y,z) = 3y^2z$ and therefore $A(y,z) = y^3z+B(z)$, where $B(z)$ is unknown. Finally, plugging into the third equation gives $f_z = xy+y^3+B'(z) = xy+y^3+2z$, so $B'(z) = 2z$ and therefore $B(z) = z^2+c$ where c is a constant we may take to be 0. Thus a potential function is given by

$$f(x,y,z) = xyz+x+y^3z+z^2. \qquad \square$$

(c) Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the curve parameterized by $\mathbf{r}(t) = (\cos t, \sin t, t)$, $0 \le t \le 2\pi$.

Solution. Having found a potential function, we can use the fundamental theorem for line integrals to compute

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(2\pi)) - f(\mathbf{r}(0)) = f(1,0,2\pi) - f(1,0,0) = 1+4\pi^2-1 = 4\pi^2. \qquad \square$$