Calc III: Quiz 7 Solutions, Fall 2017

Problem 1. Use a surface integral in spherical coordinates to show that the surface area of a sphere of radius R is $4\pi R^2$.

Solution. The sphere is parameterized by

$$\mathbf{r}(\varphi,\theta) = (R\sin\varphi\cos\theta, R\sin\varphi\sin\theta, R\cos\varphi)$$

with partial derivatives

$$\mathbf{r}_\varphi(\varphi,\theta) = (R\cos\varphi\cos\theta, R\cos\varphi\sin\theta, -R\sin\varphi), \qquad \mathbf{r}_\theta(\varphi,\theta) = (-R\sin\varphi\sin\theta, R\sin\varphi\cos\theta, 0),$$

$$\mathbf{r}_\varphi \times \mathbf{r}_\theta = (R^2\sin^2\varphi\cos\theta, R^2\sin^2\varphi\sin\theta, R^2\sin\varphi\cos\varphi)$$

and surface area element

$$\begin{aligned} dS &= \|\mathbf{r}_\varphi \times \mathbf{r}_\theta\|\ d\varphi\, d\theta \\ &= \sqrt{R^4\sin^4\varphi\cos^2\theta + R^4\sin^4\varphi\sin^2\theta + R^4\sin^2\varphi\cos^2\varphi}\, d\varphi\, d\theta \\ &= R^2\sin\varphi\, d\varphi\, d\theta. \end{aligned}$$

The surface area is given by

$$\begin{aligned} \iint_S 1\, dS &= \int_0^{2\pi}\int_0^{\pi} R^2\sin\varphi\, d\varphi\, d\theta \\ &= R^2\int_0^{2\pi} d\theta \int_0^{\pi}\sin\varphi\, d\varphi \\ &= R^2(2\pi)(2) \\ &= 4\pi R^2. \end{aligned}$$

□

Problem 2. Use the Divergence Theorem to evaluate the integral $\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\, dS$ where $\mathbf{F}(x,y,z) = x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$ and Σ is the closed surface $x^2+y^2+z^2=9$, oriented with $\mathbf{n}$ pointing out.

Solution. We note that Σ is the correctly oriented boundary of the solid ball

$$B = \{(x,y,z) : x^2+y^2+z^2 \le 9\}$$

of radius 3, so by the Divergence Theorem,

$$\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_B \nabla\cdot\mathbf{F}\, dV.$$

We compute

$$\nabla\cdot\mathbf{F} = (\tfrac{\partial}{\partial x}\mathbf{i} + \tfrac{\partial}{\partial y}\mathbf{j} + \tfrac{\partial}{\partial z}\mathbf{k})\cdot(x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}) = 1+2+3 = 6.$$

Thus

$$\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_B 6\, dV = 6\mathrm{Vol}(B) = 8\pi(3)^3 = 216\pi.$$

□