**Calc III: Workshop 1 Solutions, Fall 2018**

**Problem 1.** A vector $\mathbf{v}$ in $\mathbb{R}^2$ lies in the positive quandrant (i.e., where $x \geq 0$ and $y \geq 0$), makes an angle of $\pi/3$ with the positive $x$-axis, and satisfies $|\mathbf{v}| = 4$. Write $\mathbf{v}$ in component form.

*Solution.* We're given that $\cos\theta = \pi/3$, where $\theta$ is the angle off the $x$-axis. The $x$ component, $v_1$ of $\mathbf{v}$ is then given by $v_1 = \mathbf{v} \cdot \mathbf{i} = |v|\cos\theta = 4(\frac{1}{2}) = 2$. The $y$-component $v_2$ satisfies

$$\begin{aligned} 4 = |\mathbf{v}| &= \sqrt{v_1^2 + v_2^2} = \sqrt{4 + v_2^2} \\ \implies v_2 &= \sqrt{12} = 2\sqrt{3} \end{aligned}$$

Thus $\mathbf{v} = \left\langle 2, 2\sqrt{3} \right\rangle = 2\mathbf{i} + 2\sqrt{3}\mathbf{j}$. □

**Problem 2.** Find two unit vectors which are orthogonal to the vectors $\langle 3, 2, 1\rangle$ and $\langle -1, 1, 0\rangle$.

*Solution.* For this we use the cross product: calling the vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, we have

$$\mathbf{a} \times \mathbf{b} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \mathbf{i}\big(2(0) - 1(1)\big) - \mathbf{j}\big(3(0) - 1(-1)\big) + \mathbf{k}\big(3(1) - (-1)2\big) = \langle -1, 1, 5\rangle .$$

This vector is orthogonal to $\mathbf{a}$ and $\mathbf{b}$ by a property of the cross product, but is not a unit vector: $|\mathbf{a} \times \mathbf{b}| = \sqrt{1 + 1 + 25} = \sqrt{27}$. The two unit vectors are therefore

$$\pm\tfrac{1}{\sqrt{27}} \langle -1, 1, 5\rangle .$$

□

**Problem 3.** Let $A$, $B$ and $C$ be the vertices of a triangle in $\mathbb{R}^2$. Compute the vector $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

*Solution.* The sum is the zero vector. To see this, think in terms of displacement vectors. $\overrightarrow{AB}$ is the vector which, when added to the point $A$, gives the point $B$, etc. Thus the sum represents travelling from $A$, then to $B$, then to $C$, and back to $A$, which is a net displacement of 0. □

**Problem 4.** Find all vectors $\mathbf{v}$ such that $\langle 1, 2, 1\rangle \times \mathbf{v} = \langle 3, 1, -5\rangle$.

*Solution.* Let $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ have variable components. Writing the cross product, we have

$$\langle 1, 2, 1\rangle \times \mathbf{v} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 1 \\ v_1 & v_2 & v_3 \end{pmatrix} = \langle 2v_3 - v_2, v_1 - v_3, v_2 - 2v_1\rangle .$$

Setting this equal to $\langle 3, 1, -5\rangle$ gives the equations

$$\begin{aligned} 2v_3 - v_2 &= 3, \\ v_1 - v_3 &= 1, \\ v_2 - 2v_1 &= -5. \end{aligned}$$

The system has a free variable (say $v_3$, but this is not the only choice), in terms of which all solutions can be written as $\langle v_1, v_2, v_3\rangle = \langle -1 + v3, -3 + 2v_3, v_3\rangle$. $\square$

**Problem 5.** Prove the following for all vectors $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$:
(a) $|\mathbf{v}\times\mathbf{w}|^2 + |\mathbf{v}\cdot\mathbf{w}|^2 = |\mathbf{v}|^2\,|\mathbf{w}|^2$
(b) If $\mathbf{v}\cdot\mathbf{w} = 0$ and $\mathbf{v}\times\mathbf{w} = \mathbf{0}$, then either $\mathbf{v} = \mathbf{0}$ or $\mathbf{w} = \mathbf{0}$.

*Solution.*

(a) Using the fact that the magnitude of $\mathbf{v}\times\mathbf{w}$ is $|\mathbf{v}|\,|\mathbf{w}|\sin\theta$ while $\mathbf{v}\cdot\mathbf{w} = |\vec{v}|\,|\mathbf{w}|\cos\theta$, where $\theta$ is the angle between $\mathbf{v}$ and $\mathbf{w}$, we may square and add these to get

$$|\mathbf{v}\times\mathbf{w}|^2 + |\mathbf{v}\cdot\mathbf{w}|^2 = |\mathbf{v}|^2\,|\mathbf{w}|^2\left(\sin^2\theta + \cos^2\theta\right) == |\mathbf{v}|^2\,|\mathbf{w}|^2 .$$

(b) Using the previous result, if both $\mathbf{v}\times\mathbf{w} = \mathbf{0}$ and $\mathbf{v}\cdot\mathbf{w} = 0$, then

$$|\mathbf{v}|^2\,|\mathbf{w}|^2 = |\mathbf{v}\times\mathbf{w}|^2 + |\mathbf{v}\cdot\mathbf{w}|^2 = 0$$

which implies that either $|\mathbf{v}|$ or $|\mathbf{w}|$ vanishes; this in turn means that the corresponding vector is $\mathbf{0}$.

$\square$

**Problem 6.** Find the point at which the line through the points $P_0(3,2,0)$ and $P_1(2,3,5)$ intersects the plane $x - y + 2z = 9$.

*Solution.* First we need either the vector or parametric equations for the line. We take the base point to be $P_0(3,2,0)$ and the direction vector to be

$$\overrightarrow{P_0P_1} = \langle 2-3, 3-2, 5-0\rangle = \langle -1, 1, 5\rangle\,,$$

which gives the vector equation

$$\langle x, y, z\rangle = \langle 3, 2, 0\rangle + t\,\langle -1, 1, 5\rangle = \langle 3-t, 2+t, 5t\rangle$$

or equivalently $x = 3 - t$, $y = 2 + t$, $z = 5t$.
We plug the parametric equations for the line into the equation for the plane to get

$$(3-t) - (2+t) + 2(5t) = 9.$$

Solving for $t$ gives $t = 1$, which we plug back into the equations for the line to get the point of intersection

$$(x, y, z) = (2, 3, 5).$$

$\square$

**Problem 7.** Determine whether the pairs of lines $L_1$ and $L_2$ are parallel, intersecting, or skew (neither parallel nor intersecting). If they intersect, find the point of intersection.
(a) $L_1$: $x = 3 + 2t$, $y = 4 - t$, $z = 1 + 3t$

$L_2$: $x = 1 + 4s$, $y = 3 - 2s$, $z = 4 + 5s$.
(b) $L_1$: $x = 5 - 12t$, $y = 3 + 9t$, $z = 1 - 3t$

$L_2$: $x = 3 + 8s$, $y = -6s$, $z = 7 + 2s$.

(c) $L_1$: $\frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3}$

$L_2$: $\frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7}$

*Solution.*

(a) To determine whether the lines are parallel, we read off direction vectors and see if these are parallel. A direction vector for $L_1$ is given by $\langle 2, -1, 3\rangle$ (the coefficients of $t$) and a direction vector for $L_2$ is given by $\langle 4, -2, 5\rangle$. These are not parallel since

$$\langle 2, -1, 3\rangle \times \langle 4, -2, 5\rangle = \langle 1, 2, 0\rangle \neq \mathbf{0}.$$

To find out if they intersect, we derive symmetric equations for one of them (say $L_2$) and plug in the parametric equations for $L_1$ to try and find a solution in $t$. Symmetric equations for $L_2$ are given by

$$L_2: \qquad \frac{x-1}{4} = \frac{y-3}{-2} = \frac{z-4}{5}.$$

Plugging in the parametric equations for $L_1$ gives equations

$$\frac{(3+2t)-1}{4} = \frac{(4-t)-3}{-2} = \frac{(1+3t)-4}{5}.$$

There are two equations and only one unknown, so there may be no solutions (in which case the lines are skew) or one solution (in which case they intersect). Simplifying the first equation leads to $-2 = 2$, so we see that there is no solution and the lines are skew.

(b) Direction vectors for $L_1$ and $L_2$ are given by $\langle -12, 9, -3\rangle$ and $\langle 8, -6, 2\rangle$. These are parallel, which we see either by computing

$$\langle -12, 9, -3\rangle \times \langle 8, -6, 2\rangle = \langle 0, 0, 0\rangle$$

or by noting that $\langle -12, 9, -3\rangle = -\frac{3}{2}\langle 8, -6, 2\rangle$. We conclude that the lines are parallel.

(c) We can read off direction vectors for the two lines from the denominators in the symemtric equations, so $\mathbf{v}_1 = \langle 1, -2, -3\rangle$ and $\mathbf{v}_2 = \langle 1, 3, -7\rangle$ are direction vectors for $L_1$ and $L_2$, respectively. These are not parallel since $\mathbf{v}_1 \times \mathbf{v}_2 \neq \mathbf{0}$.

To find out if they intersect, we need to transform one (say $L_1$) into parametric form. (To pass from symmetric form to parametric form, set $t$ equal to all of the expressions and solve for $x$, $y$, and $z$ in terms of $t$.) We have

$$L_1: \qquad x = 2+t, \quad y = 3-2t, \quad z = 1-3t.$$

Plugging this into the symmetric equations for $L_2$ gives

$$\frac{(2+t)-3}{1} = \frac{(3-2t)+4}{3} = \frac{(1-3t)-2}{-7}$$

which has the solution $t = 2$, which gives the intersection point $(4, -1, -5)$.

□