Calc III: Workshop 6 Solutions, Fall 2017

Problem 1. Compute the integral of $f(x, y) = x \cos y$ over the region in the xy-plane bounded by $y = 0$, $y = x^2$, and $x = 1$.

Solution. The easiest direction is to integrate in y first:

$$\begin{aligned} \int_0^1 \int_0^{x^2} x \sin y \, dy \, dx &= \int_0^1 x \sin y \Big|_{y=0}^{x^2} dx \\ &= \int_0^1 x \sin x^2 \, dx \\ &= \tfrac{1}{2} \int_0^1 \sin u \, du \\ &= \tfrac{1}{2}(1 - \cos 1). \end{aligned}$$

□

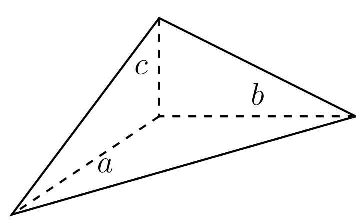
Problem 2. By setting up an appropriate double integral, find the area of the bounded region between the curves $x = y^2$ and $y = x^2$.

Solution. The area of a region R is given by the double integral of $f(x, y) = 1$ over R (think of the volume under the surface $z = 1$ over R), so

$$A = \iint_R 1 \, dA = \int_0^1 \int_{x^2}^{\sqrt{x}} dy \, dx = \int_0^1 \sqrt{x} - x^2 \, dx = \tfrac{2}{3} x^{3/2} - \tfrac{1}{3} x^3 \Big|_{x=0}^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

□

Problem 3. Prove that the volume of a tetrahedron with mututally perpindicular adjacent sides of length a, b, and c is $\frac{abc}{6}$.



Solution. First, we find the equation of the plane through the three points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, which has normal vector

$$(-a, b, 0) \times (-a, 0, c) = (bc, ac, ab),$$

so the equation of the plane is

$$bc(x - a) + ac(y - 0) + ab(z - 0) = 0.$$

Solving for z as a function of x and y, we get

$$z = \frac{abc - bcx - acy}{ab},$$

which we may integrate over the triangular region in the xy-plane bounded by $x = 0$, $y = 0$ and the line $y = b - (b/a)x$. This gives

$$\begin{aligned}
\int_0^a \int_0^{b-(b/a)x} \frac{abc - bcx - acy}{ab}\, dy\, dx &= \int_0^a \int_0^{b-(b/a)x} c - \tfrac{c}{a}x - \tfrac{c}{b}y\, dy\, dx \\
&= \int_0^a \int_0^{b-(b/a)x} c - \tfrac{c}{a}x - \tfrac{c}{b}y\, dy\, dx \\
&= \int_0^a c(b - \tfrac{b}{a}x) - \tfrac{c}{a}x(b - \tfrac{b}{a}x) - \tfrac{c}{b}\tfrac{1}{2}(b - \tfrac{b}{a}x)^2\, dx \\
&= \int_0^a \frac{bc}{2} - \frac{bc}{a}x + \frac{bc}{a^2}x^2\, dx \\
&= \frac{abc}{2} - \frac{abc}{2} + \frac{abc}{6} = \frac{abc}{6}.
\end{aligned}$$

□

Problem 4. Given an integral of the form

$$\int_0^2 \int_0^{-x^2+2x} f(x, y)\, dy\, dx,$$

change the order of integration from $dy\, dx$ to $dx\, dy$ and find the new limits.

Solution. It helps to draw the picture of the region, but then it is clear that the upper and lower limits for x are given by the two solutions produced by solving $y = -x^2 + 2x$ for x in terms of y (hint: complete the square), namely $x = 1 \pm \sqrt{1 - y}$. The limits in y are given by 0 and the maximum value of the parabola $y = -x^2 + 2x$, namely 1. Thus

$$\int_0^2 \int_0^{-x^2+2x} f(x, y)\, dy\, dx = \int_0^1 \int_{1-\sqrt{1-y}}^{1+\sqrt{1-y}} f(x, y)\, dx\, dy.$$

□

Problem 5. Compute the double integral

$$\int_0^4 \int_{x/2}^2 e^{y^2}, dy\, dx.$$

Solution. Since e^{y^2} does not have an elementary antiderivative, we want to switch the order of integration. The region of integration $R = \{(x, y) : x/2 \le y \le 2,\ 0 \le x \le 4\}$ is a triangle, which can also be written as $\{(x,y) : 0 \le x \le 2y,\ 0 \le y \le 2\}$. Thus

$$\begin{aligned}
\int_0^4 \int_{x/2}^2 e^{y^2}\,dy\,dx &= \int_0^2 \int_0^{2y} e^{y^2}\,dx\,dy \\
&= \int_0^2 xe^{y^2}\Big|_{x=0}^{2y}\,dy \\
&= \int_0^2 2ye^{y^2}\,dy \\
&= \int_0^4 e^u\,du \\
&= e^4 - 1.
\end{aligned}$$

$\square$