Calc III Fall 2018: Exam 1 Solutions

Problem 1. For the curve $\mathbf{r}(t) = \langle te^{t-\pi},\, \cos t,\, 3t^2 \rangle$, compute the derivative $\mathbf{r}'(t)$ and find an equation for the tangent line to the curve at the point $(\pi, -1, 3\pi^2)$.

Solution. The derivative is

$$\mathbf{r}'(t) = \left\langle te^{(t-\pi)} + e^{t-\pi}, -\sin t, 6t \right\rangle .$$

The tangent line to the curve at $(\pi, -1, 3\pi^2) = \mathbf{r}(\pi)$ is given by

$$\mathbf{l}(s) = \mathbf{r}(\pi) + s\mathbf{r}'(\pi) = \left\langle \pi, -1, 3\pi^2 \right\rangle + s \left\langle \pi + 1, 0, 6\pi \right\rangle = \left\langle \pi + s(1+\pi), -1, 3\pi^2 + s(6\pi) \right\rangle .$$

$\square$

Problem 2. Find an equation for the tangent plane to the surface

$$z - \ln(x + y^2) = 0$$

at the point $(0, 1, 0)$.

Solution. There are two ways to proceed. We could view the surface as the level set $g(x, y, z) = 0$ of the function $g(x, y, z) = z - \ln(x + y^2)$, the gradient of which is

$$\nabla g(x, y, z) = \Big\langle -\frac{1}{x + y^2}, -\frac{2y}{x + y^2}, 1 \Big\rangle$$

and which gives the normal vector

$$\mathbf{n} = \nabla g(0, 1, 0) = \langle -1, -2, 1 \rangle$$

from which we can write the equation

$$0 = \mathbf{n} \cdot \langle x - 0, y - 1, z - 0 \rangle = -x - 2(y - 1) + z.$$

Alternatively, we can view the surface as the graph $z = f(x, y)$ where $f(x, y) = \ln(x + y^2)$. The linear approximation to f is given by

$$L(x, y) = f(0, 1) + f_x(0, 1)(x - 0) + f_y(0, 1)(y - 1) = 0 + \tfrac{1}{0+1^2}(x - 0) + \tfrac{2(1)}{0+1^2}(y - 1) = x + 2(y - 1).$$

Then the equation for the plane is the graph

$$z = L(x, y) = x + 2(y - 1).$$

$\square$

Problem 3. For the function

$$f(x, y) = xy^2 + \cos(xy) + 3x + 2y$$

(a) Compute the linear approximation $L(x, y)$ to f at the point $(0, 0)$.
(b) Use your answer from part (a) to compute the approximation $L(0.1, -0.2)$ to the value $f(0.1, -0.2)$. (*Do not* compute the exact value of f; please use the linear approximation).

Solution.

(a) We compute the partial derivatives

$$f_x(x,y) = y^2 + y\cos(xy) + 3 \quad f_y(x,y) = 2xy + x\cos(xy) + 2,$$

which, evaluated at $(0,0)$ give

$$f_x(0,0) = 3, \quad f_y(0,0) = 2.$$

The linear approximation is

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 1 + 3x + 2y.$$

(b) The linear approximation gives

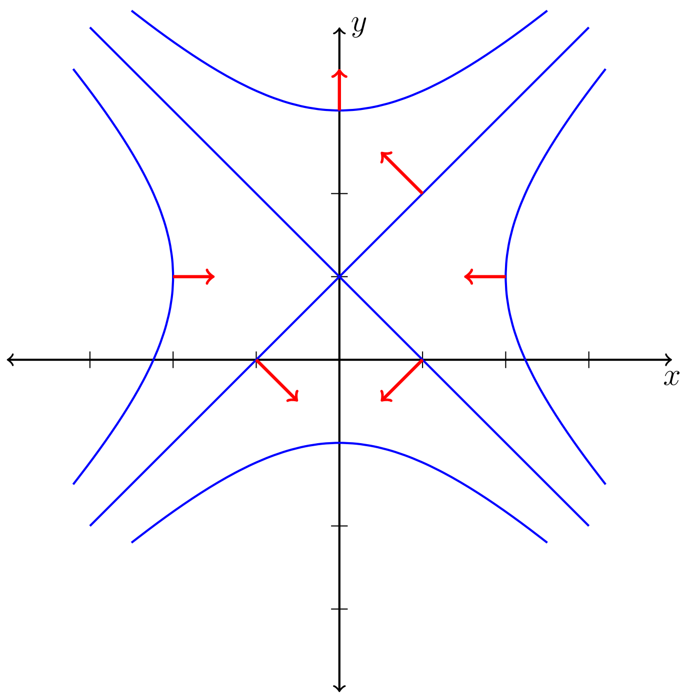
$$f(0.1,-0.2) \approx L(0.1,-0.2) = 1 + 3(0.1) + 2(-0.2) = 0.9.$$

□

Problem 4. For the function $f(x,y) = (y-1)^2 - x^2 + 1$, plot the following:

(a) the level curves where $f = -3, 1$, and 5. (Hint: think about what the function would be if $(y-1)$ were replaced by y and the $+1$ at the end was removed.)

(b) the direction of the gradient of f at the points (1,0), (2,1), (1,2), (0,3), (-2,1) and (-1,0).



Problem 5. An ant is located on the xy-plane at the point $(1,0)$. The density of ant pheromone (which signals that a DELICIOUS FOOD SOURCE is nearby) is given by

$$f(x,y) = ye^{xy}.$$

(a) What is the rate of change (directional derivative) of pheromone percieved by the ant if it is heading in the $\langle -1, 1\rangle$ direction?

(b) If the ant wants to head toward the DELICIOUS FOOD SOURCE as quickly as possible by maximizing the rate of change of pheromone, in which direction should it initially move?

Solution.

(a)

$$\nabla f(x,y) = \left\langle y^2 e^{xy},\, e^{xy} + xye^{xy} \right\rangle \quad \nabla f(1,0) = \langle 0, 1 \rangle\,.$$

The directional derivative in the direction of $\mathbf{u} = \langle -1, 1 \rangle$, after normalizing to $\mathbf{v} = \mathbf{u}/\|\mathbf{u}\| = \frac{1}{\sqrt{2}}\langle -1, 1 \rangle$ is

$$\nabla f(1,0) \cdot \mathbf{v} = \langle 0, 1 \rangle \cdot \frac{1}{\sqrt{2}} \langle -1, 1 \rangle = \frac{1}{\sqrt{2}}.$$

(b) To get to the food source as quickly as possible, the ant should head in the direction of $\nabla f(1,0)$, namely $\langle 0, 1 \rangle$, for in this direction the rate of change is $\|\nabla f(1,0)\| = 1$.

□

Problem 6. Find the critical points of the function $f(x,y) = 2x^2y - 2x^2 - y^2$ and classify them as local minima, maxima, saddle points, etc.

Solution. Solving $\nabla f(x,y) = \mathbf{0}$ gives the system of equations

$$(4xy - 4x, 2x^2 - 2y) = (0,0) \iff \begin{cases} x(y-1) = 0 \\ x^2 = y \end{cases}.$$

Plugging the second equation into the first, we have $x(x^2 - 1) = 0$ so $x = 0, \pm 1$. Plugging these values into the second equation gives the three critical points

$$(0,0), \quad (-1,1), \quad (1,1).$$

Next we compute the discriminant

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 4(y-1)(-2) - (4x)^2 = -8(y-1) - 16x^2$$

Evaluated at the critical points, we have

$$\begin{aligned} D(0,0) &= 8, \quad f_{xx}(0,0) = -4 \implies (0,0) \text{ is a maximum,} \\ D(-1,1) &= -16, \implies (-1,1) \text{ is a saddle,} \\ D(1,1) &= -16, \implies (1,1) \text{ is a saddle.} \end{aligned}$$

□

Problem 7. Find the maximum and minimum values of the function $f(x,y) = x^2 - y^2$ on the unit disk $x^2 + y^2 \leq 1$, and the points at which these values occur.

Solution. The function has a single critical point inside the disk:

$$\nabla f(x,y) = \langle 2x, -2y \rangle = \langle 0, 0 \rangle \iff (x,y) = (0,0).$$

On the boundary, we may either use $x^2 + y^2 = 1$ to eliminate a variable, parameterize the circle using $\langle x(t), y(t) \rangle = \langle \cos t, \sin t \rangle$, or use Lagrange multipliers. In the latter, we would set $g(x,y) = x^2 + y^2$ and solve $\nabla f(x,y) = \lambda \nabla g(x,y)$ to get the system

$$\begin{aligned} x &= \lambda x \\ -y &= \lambda y \\ x^2 + y^2 &= 1. \end{aligned}$$

From the first equation, either $x = 0$ or $\lambda = 1$. In the first case we get $y^2 = 1$ or $y = \pm 1$ from the third equation, and in the second case we must have $y = 0$ in the second equation, and then $x^2 = 1$ or $x = \pm 1$ from the third. This gives the additional candidate points

$$(1, 0), \quad (-1, 0), \quad (0, 1), \quad (0, -1).$$

Evaluating f at all points shows that the maximum and minimum values are 1 and -1, respectively, at the points

$$f(\pm 1, 0) = 1, \quad f(0, \pm 1) = -1.$$

$\square$