Calculus III Concept Review

1 Scalar functions and vector fields

(1) **Partial derivatives**: of a scalar function $f(x, y, z)$: f_x, f_y, $f_{xy} = f_{yx}$, etc.

(2) **Gradient**: $\nabla f(\mathbf{p}) = \langle f_x(\mathbf{p}), f_y(\mathbf{p}), f_z(\mathbf{p})\rangle$.

(3) **Linear approximation**: of f near $\mathbf{p}_0 = (x_0, y_0, z_0)$:

$$f(\mathbf{p}) \approx f(\mathbf{p}_0) + \nabla f(\mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0), \qquad \mathbf{p} = (x, y, z), \quad \mathbf{p}_0 = (x_0, y_0, z_0).$$

(4) **Directional derivative**: $\nabla f(\mathbf{p}) \cdot \mathbf{v}$ of f at $\mathbf{p}$ in direction $\mathbf{v}$. (Remember: $\mathbf{v}$ must be a *unit* vector: $|\mathbf{v}| = 1$.)

   (a) Gradient points in the direction of maximum increase and is the directional derivative in this direction. In other words, $\nabla f(\mathbf{p}) \cdot \mathbf{v}$ is maximized for $=\nabla f(\mathbf{p}) / |\nabla f(\mathbf{p})|$ where it equals $|\nabla f(\mathbf{p})|$.

(5) **Critical points**: $\mathbf{p}$ such that $\nabla f(\mathbf{p}) = 0$. May be local maxima, minima, saddle points, or degenerate.

(6) **Second derivative test** in 2D: if $\mathbf{p} = (x, y)$ is a critical point of f, so $\nabla f(\mathbf{p}) = 0$, then with $D(\mathbf{p}) = \det \begin{vmatrix} f_{xx}(\mathbf{p}) & f_{xy}(\mathbf{p}) \\ f_{yx}(\mathbf{p}) & f_{yy}(\mathbf{p}) \end{vmatrix} = f_{xx}(\mathbf{p}) f_{yy}(\mathbf{p}) - \big(f_{xy}(\mathbf{p})\big)^2$:

   (i) If $D(\mathbf{p}) < 0$, then $\mathbf{p}$ is a *saddle point.*

   (ii) If $D(\mathbf{p}) > 0$ and $f_{xx}(\mathbf{p}) > 0$, then $\mathbf{p}$ is a *local minimum.*

   (iii) If $D(\mathbf{p}) > 0$ and $f_{xx}(\mathbf{p}) < 0$, then $\mathbf{p}$ is a *local maximum.*

   (iv) If $D(\mathbf{p}) = 0$, then $\mathbf{p}$ is a *degenerate critical point* and the test gives no information (the point may be a maximum, minimum, saddle point, or none of the above).

(7) **Level curves/surfaces**: curves $\big\{(x, y) \in \mathbb{R}^2 : f(x, y) = c\big\}$ in 2D and surfaces $\big\{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) = c\big\}$ in 3D,

   (a) Includes the case of *graphs* $y = g(x) \iff g(x) - y = 0$, or $z = g(x, y) \iff g(x, y) - z = 0$.

(8) **Tangent line/plane**: gradient is orthogonal to level lines/surfaces of f, so tangent line/plane to to level curve/surface $\{f = c\}$ at the point $\mathbf{p}_0$ is

$$\nabla f(\mathbf{p}_0) \cdot (\mathbf{p} - \mathbf{p}_0) = 0,$$

where $\mathbf{p} = (x, y)$, $\mathbf{p}_0 = (x_0, y_0)$ in 2D or $\mathbf{p} = (x, y, z)$, $\mathbf{p}_0 = (x_0, y_0, z_0)$ in 3D.

(9) **Vector fields**: $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k} = \langle P, Q, R\rangle$ in 3D or $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} = \langle P, Q\rangle$ in 2D.

(10) **Curl and divergence**: with $\nabla = \langle \partial/\partial x,\ \partial/\partial y,\ \partial/\partial z\rangle$:

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix}, \qquad \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = P_x + Q_y + R_z.$$

2 Parameterized curves and line integrals

(1) **Parameterized curves**: $\mathbf{r}(t) = \langle x(t),\, y(t)\rangle$ in 2D or $\mathbf{r}(t) = \langle x(t),\, y(t),\, z(t)\rangle$ in 3D, with derivative $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$.

(a) **Straight line segment**: between $\mathbf{p}_0$ and $\mathbf{p}_1$ parameterized by

$$\mathbf{r}(t) = \mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0), \quad 0 \le t \le 1.$$

(2) **Tangent line**: at $\mathbf{p}_0 = \mathbf{r}(t_0)$ is the parameterized line

$$\mathbf{l}(s) = \mathbf{r}(t_0) + s\mathbf{r}'(t_0), \qquad s \in \mathbb{R}.$$

(3) **Line integral of scalar function**:

$$\int_C f\, ds = \int_a^b f\big(\mathbf{r}(t)\big)\, \underbrace{\left|\mathbf{r}'(t)\right|\, dt}_{ds}$$

where C is parameterized by $\mathbf{r}(t)$, $t \in [a, b]$. Value is independent of orientation and parameterization. **Arc length** of C given by $\int_C ds$.

(4) **Line (work) integral of a vector field**:

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \underbrace{\mathbf{r}'(t)\, dt}_{\mathbf{T}\, ds}$$

where C is parameterized by $\mathbf{r}(t)$, $t \in [a, b]$. Value depends on orientation, but not on parameterization.

3 Parameterized surfaces

(1) **Parameterized surfaces**: $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, with $(u, v) \in D \subset \mathbb{R}^2$. Two partial derivatives $\mathbf{r}_u = \langle x_u, y_u, z_u\rangle$ and $\mathbf{r}_v = \langle x_v, y_v, z_v\rangle$ give two tangent vectors. (Parameterization is *regular* if these exist, are nonzero, and not parallel. Otherwise bad things may happen.)

(2) **Tangent plane** to parameterized surface at $\mathbf{p}_0 = \mathbf{r}(u_0, v_0)$ spanned by $\mathbf{r}_u(u_0, v_0)$, $\mathbf{r}_v(u_0, v_0)$, thus parameterized by

$$\mathbf{l}(s, t) = \mathbf{r}(u_0, v_0) + s\mathbf{r}_u(u_0, v_0) + t\mathbf{r}_v(u_0, v_0)$$

or equivalently given by the equation $\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$, where $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v$.

(3) **Surface integral of a scalar function**:

$$\iint_S f\, dS = \iint_D f\big(\mathbf{r}(u, v)\big)\, \underbrace{\left|\mathbf{r}_u \times \mathbf{r}_v\right|\, du\, dv}_{dS}$$

where S is parameterized by $\mathbf{r}(u, v)$, $(u, v) \in D \subset \mathbb{R}^2$. Value is independent of orientation and parameterization. **Surface area** of S given by $\iint_S dS$.

(4) **Surface (flux) integral of a vector field**:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_D \mathbf{F}\big(\mathbf{r}(u,v)\big)\cdot \underbrace{\pm\mathbf{r}_u\times\mathbf{r}_v\,du\,dv}_{dS}$$

where S is parameterized by $\mathbf{r}(u,v)$, $(u,v)\in D\subset\mathbb{R}^2$ and sign $\pm\mathbf{r}_u\times\mathbf{r}_v$ given by orientation (choice of $\mathbf{n}$). Value depends on orientation but not on parameterization.

(5) For a graph $z=z(x,y)$ with graph parameterization $\mathbf{r}(x,y)=\langle x,y,z(x,y)\rangle$,

$$\mathbf{n}\,dS = \pm\langle -z_x,-z_y,1\rangle\;dx\,dy,\quad dS=\sqrt{z_x^2+z_y^2+1}\,dx\,dy.$$

4 Multiple integration

(1) **Iterated integrals**:

$$\int_a^b\int_{c(x)}^{d(x)} f(x,y)\,dy\,dx = \int_a^b g\big(x,d(x)\big)-g\big(x,c(x)\big)\,dx,$$

where $g_y(x,y)=f(x,y)$, etc.

(a) In case all limits are constant and $f(x,y)=g(x)h(y)$:

$$\int_a^b\int_c^d g(x)h(y)\,dy\,dx = \int_a^b g(x)\,dx\int_c^d h(y)\,dy.$$

(2) **Area integrals in cartesian coordinates**:

$$\iint_D f(x,y)\,dA = \int_a^b\int_{c(x)}^{d(x)} f(x,y)\,dy\,dx,\qquad D=\{(x,y): a\le x\le b,\ c(x)\le y\le d(x)\},$$

or

$$\iint_D f(x,y)\,dA = \int_c^d\int_{a(y)}^{b(y)} f(x,y)\,dx\,dy,\qquad D=\{(x,y): a(y)\le x\le b(y),\ c\le y\le d\}.$$

(3) **Polar coordinates**: $(x,y)=(r\cos\theta,r\sin\theta)$, $dA=r\,dr\,d\theta$.

$$\iint_D f(x,y)\,dA = \int_{\theta_0}^{\theta_1}\int_{r_0}^{r_1} f(r\cos\theta,r\sin\theta)\,r\,dr\,d\theta$$

where $D=\{\theta_0\le\theta\le\theta_1,\ r_0\le r\le r_1\}$.

(4) **Volume integrals in cartesian coordinates**:

$$\iiint_E f(x,y,z)\,dV = \int_a^b\int_{c(x)}^{d(x)}\int_{e(x,y)}^{g(x,y)} f(x,y,z)\,dz\,dy\,dx,$$
$$E=\{(x,y,z): a\le x\le b, c(x)\le y\le d(x), e(x,y)\le z\le g(x,y)\},$$

and similarly with other orders.

(5) **Cylindrical coordinates**:

$$(x, y, z) = (r\cos\theta, r\sin\theta, z), \qquad dV = r\,dz\,dr\,d\theta.$$

(6) **Spherical coordinates**:

$$(x, y, z) = (\rho\,\sin\phi\cos\theta,\ \rho\,\sin\phi\,\sin\theta,\ \rho\cos\phi), \qquad dV = \rho\sin^2\phi\,d\rho\,d\phi\,d\theta.$$

(7) **Mass**: If $E \subset \mathbb{R}^3$ has density $\delta(x, y, z)$, then

$$\text{Mass}(E) = \iiint_E \delta\,dV.$$

5 Vector Calculus

(1) **Fundamental Theorem for Line integrals**:

$$\underbrace{f(\mathbf{p}_1) - f(\mathbf{p}_0)}_{=\sum_{\mathbf{p}\in\partial C} f(\mathbf{p})} = \int_C \nabla f \cdot \mathbf{T}\,ds$$

where C starts at $\mathbf{p}_0$ and ends at $\mathbf{p}_1$

(a) A vector field $\mathbf{F}$ is **conservative** ($\mathbf{F} = \nabla f$ for some potential function f) if and only if $\nabla\times\mathbf{F} = \mathbf{0}$ everywhere.

(2) **Stokes' (Green's) Theorem**:

$$\oint_{C=\partial S} \mathbf{F}\cdot\mathbf{T}\,ds = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$$

with $C = \partial S$ oriented so that S is on the left if $\mathbf{n}$ is up.

(a) **Green's Theorem** is the special case that S is in the xy-plane, with $\mathbf{n} = \mathbf{k}$, and $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j} + 0\mathbf{k}$, in which case $\nabla\times F = (Q_x - P_y)\mathbf{k}$.

(3) **Divergence Theorem**:

$$\oiint_{S=\partial E} \mathbf{G}\cdot\mathbf{n}\,dS = \iiint_E \nabla\cdot\mathbf{G}\,dV$$

with $S = \partial E$ oriented with $\mathbf{n}$ pointing out of E.

6 Formula sheet to appear on final exam

Line/plane through $\mathbf{p}_0$ normal to $\mathbf{n}$:

$$\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0, \quad \mathbf{p} = (x, y) \text{ or } (x, y, z)$$

Line though $\mathbf{p}_0$ tangent to $\mathbf{v}$:

$$\mathbf{r}(t) = \mathbf{p}_0 + t\mathbf{v}, \quad t \in \mathbb{R}$$

Straight line segment from $\mathbf{p}_0$ to $\mathbf{p}_1$:

$$\mathbf{r}(t) = \mathbf{p}_0 + t(\mathbf{p}_1 - \mathbf{p}_0), \quad 0 \le t \le 1.$$

Curl and divergence:

$$\begin{gathered} \operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}, \quad \operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} \\ \nabla = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle \end{gathered}$$

Polar coordinates:

$$\begin{aligned} (x, y) &= (r\cos\theta, r\sin\theta), \\ dA &= r\,dr\,d\theta \end{aligned}$$

Cylindrical coordinates:

$$\begin{aligned} (x, y, z) &= (r\cos\theta, r\sin\theta, z), \\ dV &= r\,dz\,dr\,d\theta \end{aligned}$$

Spherical coordinates:

$$\begin{aligned} (x, y, z) &= (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi), \\ dV &= \rho\sin^2\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

Arc length elements:

$$\begin{aligned} ds &= \left|\mathbf{r}'(t)\right|\,dt, \\ \mathbf{T}\,ds &= \mathbf{r}'(t)\,dt \end{aligned}$$

Surface area elements:

$$\begin{aligned} dS &= \left|\mathbf{r}_u \times \mathbf{r}_v\right|\,du\,dv, \\ \mathbf{n}\,dS &= \pm\mathbf{r}_u \times \mathbf{r}_v\,du\,dv \end{aligned}$$

Fundamental Theorem for Line Integrals:

$$f(\mathbf{p}_1) - f(\mathbf{p}_0) = \int_C \nabla f \cdot \mathbf{T}\,ds$$

Stokes'/Green's Theorem:

$$\oint_{\partial S} \mathbf{F} \cdot \mathbf{T}\,ds = \iint_S \operatorname{curl} \mathbf{F} \cdot \mathbf{n}\,dS$$

Divergence Theorem:

$$\oiint_{\partial E} \mathbf{G} \cdot \mathbf{n}\,dS = \iiint_E \operatorname{div} \mathbf{G}\,dV$$

Trig identities:

$$\begin{aligned} \cos^2\theta &= \frac{1 + \cos 2\theta}{2} \\ \sin^2\theta &= \frac{1 - \cos 2\theta}{2} \end{aligned}$$

(More if deemed necessary.)