Complex Analysis, Final Oral Exam Questions Spring 2017

Problem 1. Define a function $\eta(s)$ by the series

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}.$$

(a) Show that $\eta(s)$ converges locally uniformly for $\operatorname{Re} s > 0$ and defines a holomorphic function.
(b) Show that in the region $\operatorname{Re} s > 1$, $\eta(s)$ and $\zeta(s)$ are related by

$$\eta(s) = (1 - 2^{1-s})\zeta(s).$$

(c) Show that $\eta(s)$ has no zeros for $s \in \mathbb{R}$, $0 < s < 1$ (Hint: alternating series have many lower bounds). Extend this to $s = 0$ by the functional equation for ζ.
(d) Show that $\zeta(s) \in \mathbb{R}$ for all $s \in \mathbb{R} \setminus \{1\}$.

Problem 2. Prove the fudamental theorem of algebra using Rouché's theorem. More precisely, if $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ with $a_n \neq 0$, prove that p has n roots (counted with multiplicity) in $\mathbb{C}$.

Problem 3. Compute

$$\frac{1}{2\pi i} \oint_{|z|=3/2} \frac{1}{(z-2)(z^5-1)}\, dz.$$

(Hint: Show that $\oint_{|z|=R} \frac{1}{(z-2)(z^5-1)}\, dz = 0$ for large R. Why?)

Problem 4. Let $L = \mathbb{Z}\langle \omega_1, \omega_2 \rangle \subset \mathbb{C}$ be a nondegenerate lattice. Construct a meromorphic function having poles of order 1 precisely at each lattice point, with all residues equal to 1. Is this function an elliptic function? What is its derivative?

Problem 5. Suppose $f(z)$ is an entire function which is periodic along the real and imaginary axes: $f(x+1) = f(x)$ for all $x \in \mathbb{R}$, and $f(iy + i) = f(iy)$ for all $y \in \mathbb{R}$. Prove that f is constant. (Hint: show that $f(z+1) = f(z+i) = f(z)$ for all z.)

Problem 6. For a fixed nondegenerate lattice $L \subset \mathbb{C}$, and $k \geq 3$ let

$$G_k(L) = \sum_{\omega \in L \setminus 0} \omega^{-k}$$

denote the Eisenstein series.
(a) Prove that $G_k(L) = 0$ for odd k.
(b) Prove the recursion formula

$$(2m+1)(m-3)(2m-1)G_{2m} = 3\sum_{j=2}^{m-2}(2j-1)(2m-2j-1)G_{2j}G_{2m-2j}$$

by expressing $\wp''(z)$ as a polynomial in $\wp'(z)$ and $\wp(z)$ and equating Laurent coefficients. (If the full recursion formula proves to be too much, just find expressions for G_8 and G_10 in terms of G_4 and G_6.)

Problem 7. Let D be the open unit disk in $\mathbb{C}$. Prove that there exists an analytic function $f : D \longrightarrow \mathbb{C}$ which admits *no* analytic extension to any strictly larger connected open domain $D \cup \Omega$. (Hint: Show that there exists f analytic which vanishes on a countable discrete set $S \subset D$ for which every point in ∂D is an accumulation point.)