20.
The functions $\sin y$ and $x\cos y - \sin y$ have continuous first-order derivatives on $\mathbb{R}^2$ and

$\dfrac{\partial}{\partial y}(\sin y) = \cos y = \dfrac{\partial}{\partial x}(x\cos y - \sin y)$, so $\mathbf{F}(x, y) = \sin y\,\mathbf{i} + (x\cos y - \sin y)\,\mathbf{j}$ is a conservative vector field by Theorem 6 and hence the line integral is independent of path. Thus a potential function f exists, and $f_x(x, y) = \sin y$ implies $f(x, y) = x\sin y + g(y)$ and $f_y(x, y) = x\cos y + g'(y)$. But $f_y(x, y) = x\cos y - \sin y$ so $g'(y) = -\sin y \quad \Rightarrow \quad g(y) = \cos y + K$. We can take $K = 0$, so $f(x, y) = x\sin y + \cos y$. Then $\int_C \sin y\,dx + (x\cos y - \sin y)\,dy = f(1, \pi) - f(2, 0) = -1 - 1 = -2$.

29. Since $\mathbf{F}$ is conservative, there exists a function f such that $\mathbf{F} = \nabla f$, that is, $P = f_x$, $Q = f_y$, and $R = f_z$. Since P, Q, and R have continuous first order partial derivatives, Clairaut's Theorem says that $\partial P/\partial y = f_{xy} = f_{yx} = \partial Q/\partial x$, $\partial P/\partial z = f_{xz} = f_{zx} = \partial R/\partial x$, and $\partial Q/\partial z = f_{yz} = f_{zy} = \partial R/\partial y$.

30. Here $\mathbf{F}(x, y, z) = y\,\mathbf{i} + x\,\mathbf{j} + xyz\,\mathbf{k}$. Then using the notation of Exercise 29 , $\partial P/\partial z = 0$ while $\partial R/\partial x = yz$. Since these aren't equal, $\mathbf{F}$ is not conservative. Thus by Theorem 4, the line integral of $\mathbf{F}$ is not independent of path.

35. (a) $P = -\dfrac{y}{x^2 + y^2}$, $\dfrac{\partial P}{\partial y} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$ and $Q = \dfrac{x}{x^2 + y^2}$, $\dfrac{\partial Q}{\partial x} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$. Thus $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(b) C_1: $x = \cos t$, $y = \sin t$, $0 \le t \le \pi$, C_2: $x = \cos t$, $y = \sin t$, $t = 2\pi$ to $t = \pi$. Then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^\pi \frac{(-\sin t)(-\sin t) + (\cos t)(\cos t)}{\cos^2 t + \sin^2 t}\,dt = \int_0^\pi dt = \pi \text{ and } \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{2\pi}^\pi dt = -\pi$$

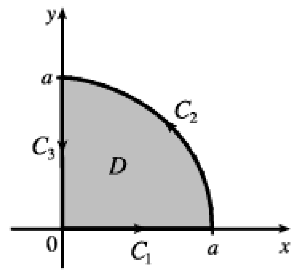
Since these aren't equal, the line integral of $\mathbf{F}$ isn't independent of path. (Or notice that $\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} dt = 2\pi$ where C_3 is the circle $x^2 + y^2 = 1$, and apply the contrapositive of Theorem 3.) This doesn't contradict Theorem 6, since the domain of $\mathbf{F}$, which is $\mathbb{R}^2$ except the origin, isn't simply-connected.

9. $\int_C y^3\,dx - x^3\,dy = \iint_D \left[\frac{\partial}{\partial x}(-x^3) - \frac{\partial}{\partial y}(y^3)\right] dA = \iint_D (-3x^2 - 3y^2)\,dA = \int_0^{2\pi} \int_0^2 (-3r^2)\,r\,dr\,d\theta$

$$= -3\int_0^{2\pi} d\theta \int_0^2 r^3\,dr = -3(2\pi)(4) = -24\pi$$

22. By Green's Theorem, $\frac{1}{2A}\oint_C x^2\,dy = \frac{1}{2A}\iint_D 2x\,dA = \frac{1}{A}\iint_D x\,dA = \overline{x}$ and $-\frac{1}{2A}\oint_C y^2\,dx = -\frac{1}{2A}\iint_D (-2y)\,dA = \frac{1}{A}\iint_D y\,dA = \overline{y}$.

23. We orient the quarter-circular region as shown in the figure.



$A = \frac{1}{4}\pi a^2$ so $\overline{x} = \dfrac{1}{\pi a^2/2}\oint_C x^2\,dy$ and $\overline{y} = -\dfrac{1}{\pi a^2/2}\oint_C y^2\,dx$.

Here $C = C_1 + C_2 + C_3$ where C_1: $x = t,\quad y = 0,\quad 0 \le t \le a$;

C_2: $x = a\cos t,\quad y = a\sin t,\quad 0 \le t \le \frac{\pi}{2}$; and

C_3: $x = 0$, $y = a - t$, $0 \le t \le a$. Then

$$\oint_C x^2\,dy = \int_{C_1} x^2\,dy + \int_{C_2} x^2\,dy + \int_{C_3} x^2\,dy = \int_0^a 0\,dt + \int_0^{\pi/2}(a\cos t)^2(a\cos t)\,dt + \int_0^a 0\,dt$$
$$= \int_0^{\pi/2} a^3\cos^3 t\,dt = a^3\int_0^{\pi/2}(1-\sin^2 t)\cos t\,dt = a^3\left[\sin t - \tfrac{1}{3}\sin^3 t\right]_0^{\pi/2} = \tfrac{2}{3}a^3$$

so $\overline{x} = \dfrac{1}{\pi a^2/2}\oint_C x^2\,dy = \dfrac{4a}{3\pi}$.

$$\oint_C y^2\,dx = \int_{C_1} y^2\,dx + \int_{C_2} y^2\,dx + \int_{C_3} y^2\,dx = \int_0^a 0\,dt + \int_0^{\pi/2}(a\sin t)^2(-a\sin t)\,dt + \int_0^a 0\,dt$$
$$= \int_0^{\pi/2}(-a^3\sin^3 t)\,dt = -a^3\int_0^{\pi/2}(1-\cos^2 t)\sin t\,dt = -a^3\left[\tfrac{1}{3}\cos^3 t - \cos t\right]_0^{\pi/2} = -\tfrac{2}{3}a^3,$$

so $\overline{y} = -\dfrac{1}{\pi a^2/2}\oint_C y^2\,dx = \dfrac{4a}{3\pi}$. Thus $(\overline{x},\overline{y}) = \left(\dfrac{4a}{3\pi}, \dfrac{4a}{3\pi}\right)$.