Calc III Fall 2017: Exam 2 Solutions

Problem 1. Find the center of mass $(\overline{x}, \overline{y})$ of the quarter disk

$$Q = \{(x, y) : x^2 + y^2 \leq 1,\ 0 \leq x,\ 0 \leq y\}$$

assuming Q has unit density ($\delta(x, y) = 1$).

Solution. By symmetry $\overline{x} = \overline{y}$, where $(\overline{x}, \overline{y})$ are the coordinates of the center of mass. Also, since the density is equal to 1, the total mass is just the area, which is $\frac{1}{4}\pi$, one quarter of the area of the unit disk. Thus

$$\begin{aligned}
\overline{x} &= \frac{1}{M} \iint_Q x\, dA \\
&= \frac{4}{\pi} \int_0^{\pi/2} \int_0^1 r \sin\theta\, r\, dr\, d\theta \\
&= \frac{4}{\pi} \int_0^{\pi/2} \sin\theta\, d\theta \int_0^1 r^2\, dr \\
&= \frac{4}{3\pi}
\end{aligned}$$

So the center of mass is $(4/3\pi, 4/3\pi)$. $\square$

Problem 2. The ant  from Exam 1 has found a DELICIOUS FOOD SOURCE: an ice cream cone occupying the region E above the cone $z = \sqrt{3(x^2 + y^2)}$ and below the sphere $x^2 + y^2 + z^2 = 4$. Find the total mass, M, of this frosty treat, assuming its mass density is given by $\delta(x, y, z) = 2z$.

Solution. The mass can be computed in any of the three coordinate systems, but cylindrical and spherical are the best choices. In cylindrical:

$$\begin{aligned}
M &= \iiint_E 2z\, dV \\
&= \int_0^{2\pi} \int_0^1 \int_{\sqrt{3}r}^{\sqrt{4-r^2}} 2zr\, dz\, dr\, d\theta \\
&= 2\pi \int_0^1 r\big((4 - r^2) - 3r^2\big)\, dr \\
&= 2\pi \int_0^1 4r - 4r^3\, dr \\
&= 2\pi(2 - 1) \\
&= 2\pi.
\end{aligned}$$

In spherical:

$$
\begin{aligned}
M &= \iiint_E 2z\,dV \\
&= \int_0^{2\pi}\int_0^{\pi/6}\int_0^2 (2\rho\cos\varphi)\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \\
&= 2\int_0^{2\pi} d\theta \int_0^{\pi/6} \sin\varphi\cos\varphi\,d\varphi \int_0^2 \rho^3\,d\rho \\
&= 2(2\pi)\left(\tfrac{1}{2}\sin^2(\pi/6)\right)\left(\tfrac{1}{4}2^4\right) \\
&= 2\pi.
\end{aligned}
$$

$\square$

Problem 3. Consider the double integral

$$\int_0^1\int_1^{e^x} f(x,y)\,dy\,dx$$

where $f(x,y)$ is some unspecified function.

(a) Draw the region of integration in the xy-plane.

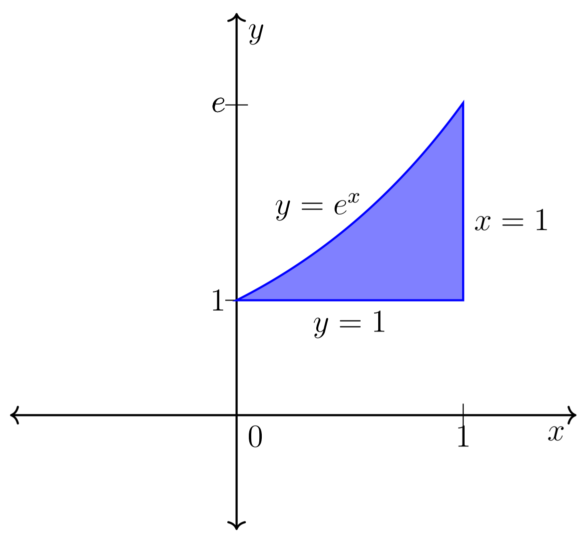
(b) Change the order of integration from $dy\,dx$ to $dx\,dy$, i.e., fill in the limits in the right hand side of the equation

$$\int_0^1\int_1^{e^x} f(x,y)\,dy\,dx = \int\quad\int\quad f(x,y)\,dx\,dy$$

(You are not meant to evaluate any integrals, just give the appropriate limits.)

Solution.

(a) (Drawn with aspect ratio 2:1 for visual clarity:)



(b) Integrating in x first, the limits become $\ln y \le x \le 1$ and $1 \le y \le e$. Thus an equivalent integral is

$$\int_1^e\int_{\ln y}^1 f(x,y)\,dx\,dy.$$

$\square$

Problem 4. The vector field

$$\mathbf{F}(x,y) = (ye^{xy} + 2x)\mathbf{i} + (xe^{xy} + 2y)\mathbf{j}$$

is conservative.

(a) Find a potential function for $\mathbf{F}$, i.e., $f(x,y)$ such that $\nabla f(x,y) = \mathbf{F}(x,y)$.
(b) Use the Fundamental Theorem for Line Integrals to compute $\int_C \mathbf{F}(x,y)\,\mathbf{T}\,ds$ where C is the portion of the ellipse $x^2 + 4y^2 = 1$ from $(1,0)$ to $(0,\frac{1}{2})$.

Solution.

(a) We want to solve $\nabla f = \mathbf{F}$, which amounts to the system of equations

$$f_x(x,y) = ye^{xy} + 2x, \quad f_y(x,y) = xe^{xy} + 2y.$$

Integrating the first equation in x, we find $f(x,y) = e^{xy} + x^2 + c(y)$, where $c(y)$ is an unknown function of y. Plugging this into the second equation yields

$$f_y = xe^{xy} + c'(y) = xe^{xy} + 2y \quad \Longrightarrow \quad c'(y) = 2y \quad \Longrightarrow \quad c(y) = y^2$$

Thus $f(x,y) = e^{xy} + x^2 + y^2$ is a potential function.

(b) The FTCLI says

$$\int_C \nabla f \cdot \mathbf{T}\,ds = f(0,\tfrac{1}{2}) - f(1,0) = (1 + \tfrac{1}{4}) - (1 + 1) = -\tfrac{3}{4}.$$

$\square$

Problem 5. For the vector field

$$\mathbf{F}(x,y) = \tfrac{1}{2}(x^2 + y^2)\mathbf{i} + 2xy\mathbf{j}$$

(a) Compute the line integral $\int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds$, where C_1 is the straight line segment from $(-1,0)$ to $(1,0)$ along the x-axis.
(b) Use part (a) and Green's Theorem to evaluate the line integral $\int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds$, where C_2 is the path consisting of straight line segments from $(1,0)$ to $(1,1)$ to $(-1,1)$ to $(-1,0)$. (Note that C_2 is not closed.)

Solution. We can parameterize C_1 by $\mathbf{p}(t) = (x(t), y(t)) = (t, 0)$ where $-1 \le t \le 1$. Then $\mathbf{T}\,ds = \mathbf{p}'(t)\,dt = (1,0)\,dt$ and

$$\begin{aligned}
\int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds &= \int_{-1}^{1} (\tfrac{1}{2}(t^2 + 0), 0)\cdot(1,0)\,dt \\
&= \int_{-1}^{1} \tfrac{1}{2}t^2\,dt = \frac{1}{3}.
\end{aligned}$$

To use Green's Theorem, we set $P(x,y) = \frac{1}{2}(x^2 + y^2)$ and $Q(x,y) = 2xy$ and compute $Q_x - P_y = 2y - y = y$. The path C_2 is not closed, but $C_2 + C_1$ is the (correctly oriented) boundary of the rectangle $R = \{(x, y) : -1 \leq x \leq 1,\ 0 \leq y \leq 2\}$, so

$$\begin{aligned}
\int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds + \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds &= \iint_R Q_x - P_y\, dA \\
&= \int_{-1}^{1} \int_0^1 y\, dy\, dx \\
&= \int_{-1}^{1} \tfrac{1}{2}\, dx \\
&= 1.
\end{aligned}$$

Then

$$\int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds = 1 - \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds = 1 - \frac{1}{3} = \frac{2}{3}.$$

$\square$