# Calc III Fall 2017: Exam 1 Solutions

**Problem 1.** For the curve $\mathbf{f}(t) = \big(\cos^2(t), 3\sin(t), 2\sqrt{t}\big)$, compute the derivative $\mathbf{f}'(t)$ and find an equation for the tangent line to the curve at $t = \pi$.

*Solution.* The derivative is

$$\mathbf{f}'(t) = \big(-2\cos(t)\sin(t), 3\cos(t), t^{-1/2}\big).$$

The tangent line to the curve is given by

$$\mathbf{l}(s) = \mathbf{f}(\pi) + s\mathbf{f}'(\pi) = \big(1, -3s, 2\sqrt{\pi} + \tfrac{1}{\sqrt{\pi}}s\big).$$

$\square$

**Problem 2.** An ant  is located on the $xy$-plane at the point $(1, 0)$. The density of ant pheromone (which signals that a DELICIOUS FOOD SOURCE is nearby) is given by

$$f(x, y) = ye^{xy}.$$

(a) What is the rate of change (directional derivative) of pheromone percieved by the ant if it is heading in the $(-1, 1)$ direction?

(b) If the ant wants to head toward the DELICIOUS FOOD SOURCE as quickly as possible by maximizing the rate of change of pheromone, in which direction should it initially move?

*Solution.*

(a)

$$\nabla f(x, y) = \big(y^2 e^{xy},\ e^{xy} + xye^{xy}\big) \quad \nabla f(1, 0) = (0, 1)\,.$$

The directional derivative in the direction of $\mathbf{u} = (-1, 1)$, after normalizing to $\mathbf{v} = \mathbf{u}/\left\|\mathbf{u}\right\| = \frac{1}{\sqrt{2}}(-1, 1)$ is

$$\nabla f(1, 0) \cdot \mathbf{v} = (0, 1) \cdot \frac{1}{\sqrt{2}}(-1, 1) = \frac{1}{\sqrt{2}}.$$

(b) To get to the food source as quickly as possible, the ant should head in the direction of $\nabla f(1, 0)$, namely $(0, 1)$, for in this direction the rate of change is $\|\nabla f(1, 0)\| = 1$.

$\square$

**Problem 3.** Find an equation for the tangent plane to the hyperboloid $z^2 - x^2 - y^2 = 2$ at the point $(1, 1, 2)$.

*Solution.* The hyperboloid is a level surface of the function $g(x, y, z) = z^2 - x^2 - y^2$, the gradient of which is

$$\nabla g(x, y, z) = (-2x, -2y, 2z).$$

The gradient is normal to the surface at each point, so a normal vector to the tangent plane is given by $\mathbf{n} = \nabla g(1, 1, 2) = (-2, -2, 4)$. An equation for the plane is then

$$\mathbf{n} \cdot (x - 1, y - 1, z - 2) = -2(x - 1) - 2(y - 1) + 4(z - 2) = 0.$$

Alternatively, we could solve for $z$ (locally) as a function of $x$ and $y$ near the point $(1, 1, 2)$, namely $z = \sqrt{2 + x^2 + y^2} = f(x, y)$. Then the tangent plane is given by the formula

$$f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) - z + 2 = 0$$

where $f_x(x,y) = \frac{x}{\sqrt{2+x^2+y^2}}$ and $f_y(x,y) = \frac{y}{\sqrt{2+x^2+y^2}}$, so $f_x(1,1) = f_y(1,1) = \frac{1}{2}$, giving

$$\frac{1}{2}(x-1) + \frac{1}{2}(y-1) - z + 2 = 0.$$

□

**Problem 4.** Compute the arc length of the curve $\mathbf{f}(t) = (\frac{1}{3}t^3, \frac{1}{3}t^3+1, t^2+2)$, where $0 \leq t \leq 4$.

*Solution.* The arc length is given by the integral of the magnitude of the derivative:

$$\begin{aligned} s &= \int_0^4 \|\mathbf{f}'(t)\| \, dt \\ &= \int_0^4 \left\|(t^2, t^2, 2t)\right\| \, dt \\ &= \int_0^4 \sqrt{t^4 + t^4 + 4t^2} \, dt \\ &= \int_0^4 t\sqrt{2t^2+4} \, dt \\ &= \frac{1}{4}\int_4^{36} \sqrt{u} \, du \\ &= \frac{1}{6}(36^{3/2} - 4^{3/2}) \\ &= \frac{104}{3}. \end{aligned}$$

□

**Problem 5.** Find the critical points of the function

$$f(x,y) = x^2y - 2xy^2 + 6xy + 4$$

and classify them as local minima, local maxima, saddle points, or undetermined.

*Solution.* We set the gradient of $f$ equal to zero:

$$\nabla f(x,y) = (2xy - 2y^2 + 6y, x^2 - 4xy + 6x) = (0,0)$$

which yields the pair of equations

$$2y(x-y+3) = 0 \quad x(x-4y+6) = 0.$$

There are four possibilities:
(1) $y = 0$, $x = 0$, giving the critical point $(0,0)$,
(2) $y = 0$, $(x+6) = 0$, giving the critical point $(-6,0)$,
(3) $x = 0$, $(-y+3) = 0$, giving the critical point $(0,3)$, and
(4) $(x-y+3) = 0$, $(x-4y+6) = 0$, giving the critical point $(-2,1)$.

We then apply the second derivative test, with $f_{xx} = 2y$, $f_{yy} = -4x$, $f_{xy} = 2x - 4y + 6$, so

$$D(x,y) = f_{xx}f_{yy} - (f_{xy})^2 = (2y)(-4x) - (2x-4y+6)^2.$$

(1) $D(0,0) = -(6)^2 < 0$, so $(0,0)$ is a *saddle point*.

(2) $D(-6,0) = -(-12+6)^2 < 0$, so $(-6,0)$ is a *saddle point*.
(3) $D(0,3) = -(-12+6)^2 < 0$, so $(0,3)$ is a *saddle point*.
(4) $D(-2,1) = (2)(8) - (-4-4+6)^2 = 16 - (-2)^2 = 12 > 0$, while $f_{xx}(-2,1) = 2 > 0$, so $(-2,1)$ is a *local minimum*.

□

**Problem 6.** Find the maximum and minimum values of the function $f(x,y) = xy$ on the ellipse $8x^2 + 2y^2 = 1$, and the points at which these values occur.

*Solution.* We use Lagrange multipliers, setting $\nabla f = \lambda \nabla g$, where $g(x,y) = 8x^2 + 2y^2$, to get the system of equations

$$\begin{aligned} y &= 16\lambda x \\ x &= 4\lambda y \\ 8x^2 + 2y^2 &= 1 \end{aligned}$$

Solving for $\lambda$ in the first two equations and setting them equal leads to $\frac{y}{16x} = \frac{x}{4y}$, or $y^2 = 4x^2$. This has two solutions $y = 2x$ and $y = -2x$. Plugging either of these into the third equation gives $16x^2 = 1$, or $x = \pm\frac{1}{4}$. Plugging back into $y$, we have the four solutions $(x,y) = (\frac{1}{4}, \frac{1}{2})$, $(\frac{1}{4}, -\frac{1}{2})$, $(-\frac{1}{4}, \frac{1}{2})$, and $(-\frac{1}{4}, -\frac{1}{2})$.

Evaluating $f$ at these four points gives

$$\begin{aligned} f\big(\tfrac{1}{4}, \tfrac{1}{2}\big) &= \frac{1}{8} & f\big(\tfrac{1}{4}, -\tfrac{1}{2}\big) &= -\frac{1}{8} \\ f\big(-\tfrac{1}{4}, \tfrac{1}{2}\big) &= -\frac{1}{8} & f\big(-\tfrac{1}{4}, -\tfrac{1}{2}\big) &= \frac{1}{8} \end{aligned}$$

So the minumum of $f$ is $-\frac{1}{8}$, achieved at the two points $(\frac{1}{4}, -\frac{1}{2})$ and $(-\frac{1}{4}, \frac{1}{2})$, while the maximum is $\frac{1}{8}$, achieved at the two points $(\frac{1}{4}, \frac{1}{2})$ and $(-\frac{1}{4}, -\frac{1}{2})$. □