

### Calc III: Workshop 6 Solutions, Fall 2018

**Problem 1.** A lamina occupies the region in the positive quadrant (where  $x \geq 0$  and  $y \geq 0$ ) which lies inside the circle  $x^2 + y^2 = 2$  but outside the circle  $x^2 + y^2 = 1$ . Find the center of mass if the density at any point is inversely proportional to its distance from the origin.

*Solution.* First we compute the mass of the lamina, given by

$$\begin{aligned} M &= \int_0^{\pi/2} \int_1^{\sqrt{2}} \frac{k}{r} r \, dr \, d\theta \\ &= \frac{\pi}{2} \int_1^{\sqrt{2}} k \, dr \\ &= \frac{k(\sqrt{2} - 1)\pi}{2}. \end{aligned}$$

By symmetry, it follows that the center of mass  $(\bar{x}, \bar{y})$  lies on the line  $y = x$ , and so will satisfy  $\bar{x} = \bar{y}$ . Thus we compute

$$\begin{aligned} \bar{y} = \bar{x} &= \frac{1}{M} \int_0^{\pi/2} \int_1^{\sqrt{2}} r \cos \theta \frac{k}{r} r \, dr \, d\theta \\ &= \frac{1}{M} \int_1^{\sqrt{2}} kr \, dr \\ &= \frac{k(2 - 1)}{2M} \\ &= \frac{k}{2M} \\ &= \frac{1}{(\sqrt{2} - 1)\pi}. \end{aligned}$$

□

**Problem 2.** Find the volume of the solid enclosed by the cylinder  $x^2 + z^2 = 4$  and the planes  $y = -1$  and  $y + z = 4$ .

*Solution.* There are several ways to set up the integral, but it is convenient to integrate first in  $y$ :

$$\begin{aligned} \iiint_C dV &= \int_{-2}^2 \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{-1}^{4-z} dy \, dz \, dx \\ &= \int_{-2}^2 \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} (3 - z) \, dz \, dx \end{aligned}$$

At this point in the computation, it may be convenient to switch to polar coordinates in  $x$  and  $z$ :

$$\int_{-2}^2 \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} (3 - z) \, dz \, dx = \int_0^{2\pi} \int_0^2 (3 - r \cos \theta) r \, dr \, d\theta = 2\pi \frac{3}{2} (2)^2 = 12\pi.$$

□

**Problem 3.** Write five other iterated integrals that are equivalent to the iterated integral

$$\int_0^1 \int_y^1 \int_0^y f(x, y, z) dz dx dy.$$

*Solution.* Draw the 3D picture and all three 2D projections onto coordinate planes! Then it follows that

$$\begin{aligned} & \int_0^1 \int_y^1 \int_0^y f(x, y, z) dz dx dy \\ &= \int_0^1 \int_0^x \int_0^y f(x, y, z) dz dy dx \\ &= \int_0^1 \int_z^1 \int_z^x f(x, y, z) dy dx dz \\ &= \int_0^1 \int_0^x \int_z^x f(x, y, z) dy dz dx \\ &= \int_0^1 \int_z^1 \int_y^1 f(x, y, z) dx dy dz \\ &= \int_0^1 \int_0^y \int_y^1 f(x, y, z) dx dz dy \end{aligned}$$

□

**Problem 4.** Evaluate the triple integral  $\iiint_C (4 + 5x^2yz^2) dV$  (try using only geometric interpretation and symmetry), where  $C$  is the cylindrical region  $x^2 + y^2 \leq 4$ ,  $-2 \leq z \leq 2$ .

*Solution.* The region is symmetric with respect to reflection about the coordinate planes, and in particular with respect to reflection about  $y = 0$ . In the integrand, the second term,  $5x^2yz^2$ , is odd with respect to reflection about  $y = 0$ , so this term will integrate to 0. This leaves the integral of the first term  $\iiint_C 4 dV = 4 \iiint_C dV$ , which computes 4 times the volume of  $C$ . Thus

$$\iiint_C (4 + 5x^2yz^2) dV = 4\text{Vol}(C) = 4(\pi(2)^2)(4) = 64\pi.$$

□

**Problem 5.** Find the center of mass of the solid  $E$  of constant density, which lies above the  $xy$ -plane and below the paraboloid  $z = 1 - x^2 - y^2$ . [Hint: it may be useful to switch to polar coordinates after integrating in  $z$ .]

*Solution.* We may as well assume the constant in the density is 1, since its precise value is canceled when computing the center of mass.

We can write  $E$  as the region  $0 \leq z \leq 1 - x^2 - y^2$ ,  $-\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}$ ,  $-1 \leq x \leq 1$ , so that the integral giving its mass is

$$M = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} 1 dz dy dx = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 - x^2 - y^2 dy dx$$

At this point, we note it is convenient to switch the remaining double integral over to polar coordinates:

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^1 (1 - r^2) r \, dr \, d\theta \\ &= 2\pi \left( \frac{1}{2} - \frac{1}{4} \right) \\ &= \frac{\pi}{2}. \end{aligned}$$

By symmetry, the  $x$  and  $y$  coordinates of the center of mass must vanish, so we need only compute

$$\begin{aligned} \bar{z} &= \frac{1}{M} \iiint_E z \, dV \\ &= \frac{1}{M} \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} z \, r \, dr \, d\theta \\ &= \frac{2\pi}{M} \int_0^1 \frac{1}{2} (1 - r^2)^2 r \, dr \\ &= -\frac{\pi}{2M} \frac{(1 - r^2)^2}{2} \Big|_{r=0}^1 \\ &= \frac{\pi}{4M} = \frac{1}{2}. \end{aligned}$$

□

**Problem 6** (Optional bonus). The *simplex of dimension  $n$*  is the set  $T_n$  of points  $(x_1, \dots, x_n)$  in  $\mathbb{R}^n$  bounded by the “hyperplanes”  $x_i = 0$  for  $i = 1, \dots, n$  and  $x_1 + x_2 + \dots + x_n = 1$ . For  $n = 2$  this is a triangle and for  $n = 3$  it is a tetrahedron. Compute the 4-dimensional volume of the 4-simplex:

$$\text{Vol}(T_4) = \int \int \int \int 1 \, dx_4 \, dx_3 \, dx_2 \, dx_1$$

Formulate a conjecture for the  $n$ -dimensional volume of the  $n$ -simplex, and try and prove it!

*Solution.* The 4-volume is

$$\begin{aligned}
 \text{Vol} &= \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} \int_0^{1-x_1-x_2-x_3} dx_4 dx_3 dx_2 dx_1 \\
 &= \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} (1-x_1-x_2-x_3) dx_3 dx_2 dx_1 \\
 &= -\frac{1}{2} \int_0^1 \int_0^{1-x_1} (1-x_1-x_2-x_3)^2 \Big|_{x_3=0}^{1-x_1-x_2} dx_2 dx_1 \\
 &= \frac{1}{2} \int_0^1 \int_0^{1-x_1} (1-x_1-x_2)^2 dx_2 dx_1 \\
 &= -\frac{1}{6} \int_0^1 (1-x_1-x_2)^3 \Big|_{x_2=0}^{1-x_1} dx_1 \\
 &= \frac{1}{6} \int_0^1 (1-x_1)^3 dx_1 \\
 &= -\frac{1}{24} (1-x_1)^4 \Big|_{x_1=0}^1 \\
 &= \frac{1}{24}.
 \end{aligned}$$

Note that it helps immensely to do each integral as a  $u$ -substitution where  $u = 1 - x_1 - x_2 - \dots - x_n$ .

The area of the 2-simplex is  $\frac{1}{2}$ , the volume of the 3-simplex is  $\frac{1}{6}$ , and in general the  $n$ -volume of the  $n$ -simplex is  $\frac{1}{n!}$ , which can be computed as above for general  $n$ .  $\square$