23. In spherical coordinates, E is represented by $\{(\rho, \theta, \phi) \mid 2 \le \rho \le 3, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$ and

$x^2 + y^2 = \rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta = \rho^2 \sin^2\phi \left(\cos^2\theta + \sin^2\theta\right) = \rho^2 \sin^2\phi$. Thus

$$\iiint_E (x^2+y^2)\,dV = \int_0^\pi \int_0^{2\pi} \int_2^3 (\rho^2 \sin^2\phi)\,\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi = \int_0^\pi \sin^3\phi\,d\phi\,\int_0^{2\pi} d\theta\,\int_2^3 \rho^4\,d\rho$$

$$= \int_0^\pi (1-\cos^2\phi)\,\sin\phi\,d\phi\,\left[\,\theta\,\right]_0^{2\pi}\,\left[\tfrac{1}{5}\rho^5\right]_2^3 = \left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^\pi\,(2\pi)\cdot\tfrac{1}{5}(243-32)$$

$$= \left(1 - \tfrac{1}{3} + 1 - \tfrac{1}{3}\right)(2\pi)\left(\tfrac{211}{5}\right) = \tfrac{1688\pi}{15}$$

30. In spherical coordinates, the sphere $x^2+y^2+z^2=4$ is equivalent to $\rho = 2$ and the cone $z = \sqrt{x^2+y^2}$ is represented by $\phi = \frac{\pi}{4}$. Thus, the solid is given by $\left\{(\rho,\theta,\phi) \,\middle|\, 0 \le \rho \le 2, 0 \le \theta \le 2\pi, \frac{\pi}{4} \le \phi \le \frac{\pi}{2}\right\}$ and

$$V = \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^2 \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/4}^{\pi/2} \sin\phi\,d\phi\,\int_0^{2\pi} d\theta\,\int_0^2 \rho^2\,d\rho$$

$$= \left[-\cos\phi\right]_{\pi/4}^{\pi/2}\left[\,\theta\,\right]_0^{2\pi}\left[\tfrac{1}{3}\rho^3\right]_0^2 = \left(\tfrac{\sqrt{2}}{2}\right)(2\pi)\left(\tfrac{8}{3}\right) = \tfrac{8\sqrt{2}\,\pi}{3}$$

36.

Place the center of the sphere at $(0,0,0)$, let the diameter of intersection be along the z-axis, one of the planes be the xz-plane and the other be the plane whose angle with the xz-plane is $\theta = \frac{\pi}{6}$. Then in spherical coordinates the volume is given by

$V = \int_0^{\pi/6} \int_0^\pi \int_0^a \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/6} d\theta\,\int_0^\pi \sin\phi\,d\phi\,\int_0^a \rho^2\,d\rho = \frac{\pi}{6}(2)\left(\frac{1}{3}a^3\right) = \frac{1}{9}\pi a^3.$

39.

The region E of integration is the region above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 2$ in the first octant. Because E is in the first octant we have $0 \le \theta \le \frac{\pi}{2}$. The cone has equation $\phi = \frac{\pi}{4}$ (as in Example 4), so $0 \le \phi \le \frac{\pi}{4}$, and $0 \le \rho \le \sqrt{2}$. So the integral becomes

$$\int_0^{\pi/4} \int_0^{\pi/2} \int_0^{\sqrt{2}} (\rho\sin\phi\cos\theta)\,(\rho\sin\phi\sin\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \int_0^{\pi/4} \sin^3\phi\,d\phi\,\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta\,\int_0^{\sqrt{2}} \rho^4\,d\rho = \left(\int_0^{\pi/4} \left(1-\cos^2\phi\right)\sin\phi\,d\phi\right)\,\left[\tfrac{1}{2}\sin^2\theta\right]_0^{\pi/2}\,\left[\tfrac{1}{5}\rho^5\right]_0^{\sqrt{2}}$$

$$= \left[\tfrac{1}{3}\cos^3\phi - \cos\phi\right]_0^{\pi/4} \cdot \tfrac{1}{2} \cdot \tfrac{1}{5}\left(\sqrt{2}\right)^5 = \left[\tfrac{\sqrt{2}}{12} - \tfrac{\sqrt{2}}{2} - \left(\tfrac{1}{3} - 1\right)\right] \cdot \tfrac{2\sqrt{2}}{5} = \tfrac{4\sqrt{2}-5}{15}$$

40. The region of integration is the solid sphere $x^2+y^2+z^2 \le a^2$, so $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, and $0 \le \rho \le a$. Also $x^2z + y^2z + z^3 = (x^2+y^2+z^2)z = \rho^2 z = \rho^3\cos\phi$, so the integral becomes

$$\int_0^\pi \int_0^{2\pi} \int_0^a \left(\rho^3\cos\phi\right)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^\pi \sin\phi\cos\phi\,d\phi\,\int_0^{2\pi} d\theta\,\int_0^a \rho^5\,d\rho = \left[\tfrac{1}{2}\sin^2\phi\right]_0^\pi\,\left[\,\theta\,\right]_0^{2\pi}\,\left[\tfrac{1}{6}\rho^6\right]_0^a = 0$$

29.

$f(x,y) = x^2 + y^2 \quad \Rightarrow \quad \nabla f(x,y) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. Thus, each vector $\nabla f(x,y)$ has the same direction and twice the length of the position vector of the point (x,y), so the vectors all point directly away from the origin and their lengths increase as we move away from the origin. Hence, ∇f is graph III.

30.

$f(x,y)=x(x+y)=x^2+xy \quad \Rightarrow \quad \nabla f(x,y)=(2x+y)\,\mathbf{i}+x\,\mathbf{j}$. The y-component of each vector is x, so the vectors point upward in quadrants I and IV and downward in quadrants II and III. Also, the x-component of each vector is 0 along the line $y=-2x$ so the vectors are vertical there. Thus, ∇f is graph IV.

31.

$f(x,y)=(x+y)^2 \quad \Rightarrow \quad \nabla f(x,y)=2(x+y)\,\mathbf{i}+2(x+y)\,\mathbf{j}$. The x- and y-components of each vector are equal, so all vectors are parallel to the line $y=x$. The vectors are $\mathbf{0}$ along the line $y=-x$ and their length increases as the distance from this line increases. Thus, ∇f is graph II.

32.

$f(x,y)=\sin\sqrt{x^2+y^2} \quad \Rightarrow$

$$\begin{aligned}\nabla f(x,y)&=\left[\cos\sqrt{x^2+y^2}\cdot\tfrac{1}{2}(x^2+y^2)^{-1/2}(2x)\right]\mathbf{i}+\left[\cos\sqrt{x^2+y^2}\cdot\tfrac{1}{2}(x^2+y^2)^{-1/2}(2y)\right]\mathbf{j}\\&=\frac{\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,x\,\mathbf{i}+\frac{\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,y\,\mathbf{j}\ \text{ or }\ \frac{\cos\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\,(x\,\mathbf{i}+y\,\mathbf{j})\end{aligned}$$

Thus each vector is a scalar multiple of its position vector, so the vectors point toward or away from the origin with length that changes in a periodic fashion as we move away from the origin. ∇f is graph I.