Math 2321 Fall 2015: Quiz 3 Solutions

Problem 1. Use the linear approximation of the function $g(s,t) = e^{s^2-t}$ at $(s,t) = (1,0)$ to estimate the value of $g(1.01, -0.04)$.

Solution. The linear approximation is

$$L_g(s,t) = g(1,0) + g_s(1,0)(s-1) + g_t(1,0)(t-0).$$

Using $g_s = 2se^{s^2-t}$, $g_t = -e^{s^2-t}$, and plugging in $g(1,0) = e$, we have

$$L_g(s,t) = e + 2e(s-1) - t$$
$$L_g(1.01, -0.04) = e + 0.02\,e + 0.04.$$

$\square$

Problem 2. Suppose the electric charge in the plane is given by

$$Q = xy + x^2 \sin(\pi y),$$

and a particle sits at the point $(1,1)$.

(a) In what direction relative to the particle does the charge increase most rapidly, and what is the rate of increase of the charge in this direction?

(b) What is the rate of increase of the charge if the particle moves from $(1,1)$ towards $(4,5)$?

Solution. We compute the gradient of Q:

$$\nabla Q = \big(y + 2x\sin(\pi y),\ x + \pi x^2 \cos(\pi y)\big).$$

Evaluated at $(1,1)$ we have $\nabla Q(1,1) = (1, 1-\pi)$. The particle moves in the direction

$$\mathbf{v} = \frac{\nabla Q(1,1)}{|\nabla Q(1,1)|} = \frac{(1, 1-\pi)}{\sqrt{1+(1-\pi)^2}}.$$

The rate of increase in this direction is $|\nabla Q(1,1)| = \sqrt{1+(1-\pi)^2}$.

If the particle moves from $(1,1)$ towards $(2,3)$ then it is moving in the direction

$$\mathbf{v} = \frac{(4,5)-(1,1)}{|(4,5)-(1,1)|} = (\tfrac{3}{5}, \tfrac{4}{5})$$

and the rate of increase is

$$\nabla Q(1,1) \cdot \mathbf{v} = (1, 1-\pi) \cdot (\tfrac{3}{5}, \tfrac{4}{5}) = \frac{7-4\pi}{5}.$$

$\square$

Problem 3. Find an equation for the tangent plane to the level surface of $F(x,y,z) = x^2 + y^2 + z$ at the point $(1,2,3)$.

Solution. The gradient of F is

$$\nabla F(x,y,z) = (2x, 2y, 1), \quad \nabla F(1,2,3) = (2,4,1),$$

The tangent plane is given by the equation

$$0 = \nabla F(1,2,3) \cdot (x-1, y-2, z-3) = 2(x-1) + 4(y-2) + (z-3).$$

$\square$