Calc III: Quiz 6 Solutions, Fall 2017

Problem 1. Find the volume of the solid region lying inside the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$, using spherical coordinates.

Solution. The sphere is given by $\rho = 1$ in spherical coordinates, while the cone is given by $\varphi = \pi/4$. The region of integration is

$$E = \{(\rho, \varphi, \theta) : 0 \le \rho \le 1,\ 0 \le \varphi \le \pi/4,\ 0 \le \theta \le 2\pi\},$$

and the volume is given by (remembering that $dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$)

$$\begin{aligned} \mathrm{Vol}(E) &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi \, d\varphi \int_0^1 \rho^2 \, d\rho \\ &= 2\pi\big(\cos(0) - \cos(\pi/4)\big)(1/3) \\ &= \frac{\pi}{3}(2 - \sqrt{2}) \end{aligned}$$

□

Problem 2. Compute the center of mass of the region $R = \{(x, y) : x^2 + y^2 \le a^2,\ y \ge 0\}$, assuming the density $\delta(x, y) = 1$.

Solution. Since the density of the region is 1, the mass is equal to the area, which is $M = \frac{1}{2}\pi a^2$. By symmetry, the x coordinate of the center of mass vanishes, and the y coordinate is given by

$$\begin{aligned} \overline{y} &= \frac{1}{M} \iint_R y \, dA \\ &= \frac{1}{M} \int_0^{\pi} \int_0^a (r \sin\theta)\, r \, dr \, d\theta \\ &= \frac{1}{M} \int_0^{\pi} \sin\theta \, d\theta \int_0^a r^2 \, dr \\ &= \frac{1}{M} \frac{2}{3} a^3 \\ &= \frac{4a}{3\pi}. \end{aligned}$$

□

Problem 3. Compute the line integral $\int_C xy \, ds$ where C is the curve given by $x = \cos t$, $y = \sin t$, $0 \le t \le \pi/2$.

Solution. The integrand becomes $xy = \cos t \sin t$, while

$$\begin{aligned} ds &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{x'(t)^2 + y'(t)^2} \, dt \\ &= \sqrt{(-\sin t)^2 + (\cos t)^2} \, dt = dt. \end{aligned}$$

Thus

$$\int_C xy\,ds = \int_0^{\pi/2} \cos t\ \sin t\,dt = \frac{1}{2}\sin^2 t\Big|_{t=0}^{\pi/2} = \frac{1}{2}.$$

$\square$