Math 2321 Fall 2015: Quiz 8 Solutions

Problem 1. Compute $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y) = (x^2y + \sin x,\, xy + \cos y)$ and C_1 is the oriented curve consisting of straight line segments, from $(0, 0)$ to $(2, 0)$, and then to $(0, 3)$. (Hint: Note that C_1 is not closed. Use Green's Theorem to simplify the calculation by selecting a curve C_2 such that $C_1 + C_2$ is closed.)

Solution. If we let C_2 be the straight line segment from $(0, 0)$ to $(0, 3)$, then we have the closed curve $C_1 - C_2 = \partial R$, where R is the triangle with vertices $(0, 0)$, $(2, 0)$ and $(0, 3)$.

According to Green's Theorem,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial R} \mathbf{F} \cdot d\mathbf{r} = \iint_R \nabla \times \mathbf{F}\, dA,$$

where the curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \tfrac{\partial}{\partial x}(xy + \cos y) - \tfrac{\partial}{\partial y}(x^2y + \sin x) = y - x^2.$$

Rearranging, we solve for $\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$ to get

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \iint_R (y - x^2)\, dA.$$

The area integral is given by

$$\iint_R (y - x^2)\, dA = \int_0^2 \int_0^{3-(3/2)x} y - x^2\, dy\, dx = 1.$$

The line segment C_2 can be parameterized by $\mathbf{r}(t) = (0, t)$, $0 \le t \le 3$, and where we have $\mathbf{r}'(t) = (0, 1)$ and $\mathbf{F}(\mathbf{r}(t)) = (0, \cos t)$, so

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^3 \cos t\, dt = \sin t\Big|_0^3 = \sin 3.$$

Thus

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \sin 3 + 1$$

□

Problem 2. Compute the flux $\iint_M \mathbf{V} \cdot \mathbf{n}\, dS$, where $\mathbf{V}(x, y, z) = (-y, x, z)$, and M is the right circular cylinder (with no top or bottom), centered around the z-axis, of radius 5, between $z = -7$ and $z = 7$, oriented outward.

Solution. We can parameterize M via

$$\mathbf{r}(\theta, z) = (5\cos\theta, 5\sin\theta, z), \quad 0 \le \theta \le 2\pi,\ -7 \le z \le 7.$$

Then

$$\begin{aligned}\mathbf{n}\, dS &= \pm \mathbf{r}_\theta \times \mathbf{r}_z\, d\theta\, dz \\ &= \pm(-5\sin\theta, 5\cos\theta, 0) \times (0, 0, 1)\, d\theta\, dz \\ &= \pm(5\cos\theta, 5\sin\theta, 0)\, d\theta\, dz,\end{aligned}$$

and we clearly want to take the + sign for outward orientation. Then

$$\begin{aligned}\iint_M \mathbf{V} \cdot \mathbf{n}\, dS &= \int_0^{2\pi} \int_{-7}^{7} (-5\sin\theta, 5\cos\theta, z) \cdot (5\cos\theta, 5\sin\theta, 0)\, dz\, d\theta \\ &= \int_0^{2\pi} \int_{-7}^{7} -25\sin\theta\cos\theta + 25\sin\theta\cos\theta\, dz\, d\theta = 0.\end{aligned}$$

Alternatively, we could compute $\mathbf{n} = c(x, y, 0)$ for some constant c using geometric ideas, and then note that $\mathbf{V} \cdot \mathbf{n} = 0$. $\square$