## Higher gerbes, loop spaces, and transgression

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Gerbes

Higher gerbes

Relation to loop spaces

## Line bundles : $H^2(X; \mathbb{Z})$

- ▶ A complex line bundle  $L \longrightarrow X$  has a Chern class  $c_1(L) \in H^2(X; \mathbb{Z})$ .
- Naturality:

$$c_1(\mathbb{C} \times X) = 0,$$
  $c_1(L \otimes L') = c_1(L) + c_1(L'),$   
 $c_1(L^{-1}) = -c_1(L),$   $c_1(f^*L) = f^*c_1(L)$ 

•  $c_1(L) = c_2(L')$  if and only if  $L \cong L'$ .

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- $c_1(L) = c_2(L')$  if and only if  $L \cong L'$ .
- ▶ Seen by Čech cohomology:  $[L] \in \check{C}^1(X; \mathbb{C}^*)$  satisfies d[L] = 1, unique up to dh,  $h \in \check{C}^0(X; \mathbb{C}^*)$ , so

$$[L] \in \check{H}^1(X; \mathbb{C}^*) \cong H^2(X; \mathbb{Z}).$$

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- ▶ Murray: a bundle gerbe (L, Y, X) is
  - a fiber bundle (more generally: locally split, i.e., surjective map admitting local sections)

$$p: Y \longrightarrow X$$
,

a line bundle

$$L \longrightarrow Y^{[2]} = Y \times_X Y = \{(y_1, y_2) : p(y_1) = p(y_2) \in X\}$$

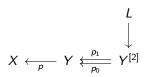
with a product

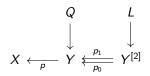
$$\phi: L_{(y_1,y_2)} \otimes L_{(y_2,y_3)} \xrightarrow{\cong} L_{(y_1,y_3)}, \quad (y_1,y_2,y_3) \in Y^{[3]}$$

satisfying associativity:

$$\phi \circ (1 \otimes \phi) = \phi \circ (\phi \otimes 1) : L_{(y_1, y_2)} \otimes L_{(y_2, y_3)} \otimes L_{(y_3, y_4)} \cong L_{(y_1, y_4)},$$
$$(y_1, y_2, y_3, y_4) \in Y^{[4]}$$

▶ (L, Y, X) has a Dixmier Douady class  $DD(L, Y, X) \in H^3(X; \mathbb{Z})$ .

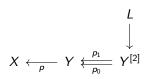




▶ Trivialization: an isomorphism  $L \cong \delta Q := p_0^* Q \otimes p_1^* Q^{-1}$  for some  $Q \longrightarrow Y$ .

$$X \xleftarrow{p} Y \xleftarrow{p_1} Y^{[2]}$$

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- ► Inverse:  $(L, Y, X)^{-1} = (L^{-1}, Y, X)$ .
- ▶ Product:  $(L, Y, X) \otimes (L', Y', X) = (\pi_1^* L \otimes \pi_2^* L', Y \times_X Y', X)$
- ▶ Pullback:  $f^*(L, Y, X) = (f^*L, f^*Y, X')$



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- ▶ Pullback:  $f^*(L, Y, X) = (f^*L, f^*Y, X')$
- Relation with DD class:
  - ▶ DD(L) = 0 if and only if L is trivial.
  - $DD(L^{-1}) = -DD(L)$
  - $DD(L \otimes L') = DD(L) + DD(L')$
  - $DD(f^*L) = f^*DD(L).$
  - ▶ DD(L) = DD(L') if and only if L and L' are stably isomorphic, i.e.,  $L \otimes Q \cong L' \otimes Q'$  for trivial gerbes Q and Q'.

### Example: lifting bundle gerbes

 $ightharpoonup E \longrightarrow X$  principal G bundle, where G has a central extension

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \widehat{\mathsf{G}} \longrightarrow \mathsf{G} \longrightarrow 1$$

- ▶  $\widehat{G} \longrightarrow G$  defines an associated line bundle  $L = \widehat{G} \times_{\mathbb{C}^*} \mathbb{C} \longrightarrow G$
- ▶ Difference map  $u: E^{[2]} \longrightarrow G$ , where " $u(y_0, y_1) = y_0^{-1} y_1$ " i.e.,  $u(y_0, y_1) = g$  such that  $y_1 = y_0 g$ .
- $(u^*L, E, X)$  is the *lifting bundle gerbe* for E.

$$\begin{array}{ccc}
u^*L & L \\
\downarrow & \downarrow \\
E^{[2]} & \xrightarrow{u} & G \\
\downarrow & \chi
\end{array}$$

▶  $DD(u^*L, E, X) \in H^3(X; \mathbb{Z})$  is the obstruction to lifting E to a  $\widehat{G}$  bundle  $\widehat{E} \longrightarrow X$ .

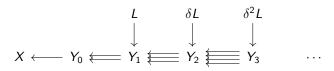
$$X \longleftarrow Y \longleftarrow Y^{[2]} \longleftarrow Y^{[3]} \longleftarrow Y^{[4]} \cdots$$

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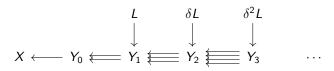
▶ A simplicial space (over X) is a sequence  $\{Y_n : n \in \mathbb{N}_0\}$  of spaces with face maps  $p_j : Y_n \longrightarrow Y_{n-1}, j = 0, \ldots, n$  and degeneracy maps  $s_j : Y_{n-1} \longrightarrow Y_n, j = 0, \ldots, n-1$  (commuting with maps to X), satisfying relations derived from those of standard simplices.

$$X \longleftarrow Y_0 \Longleftarrow Y_1 \oiint Y_2 \oiint Y_3 \cdots$$

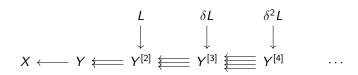
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- ▶ [Brylinski-McLaughlin]: A simplicial line bundle is a line bundle  $L \longrightarrow Y_1$  with a trivialization of  $\delta L = p_0^*L \otimes p_1^*L^{-1} \otimes p_2^*L$  pulling back to the canonical trivialization of  $\delta \delta L$ .



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$$\begin{array}{cccc}
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\check{C}^{0}(Y^{[2]}; \mathbb{C}^{*}) \stackrel{d}{\to} \check{C}^{1}(Y^{[2]}; \mathbb{C}^{*}) \stackrel{d}{\to} \check{C}^{2}(Y^{[2]}; \mathbb{C}^{*}) \stackrel{d}{\to} \cdots \\
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\uparrow\\
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Then  $DD(L) = [\alpha] \in \check{H}^2(X; \mathbb{C}^*) \cong H^3(X; \mathbb{Z}).$ 

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Then  $DD(L) = [\alpha] \in \check{H}^2(X; \mathbb{C}^*) \cong H^3(X; \mathbb{Z})$ . Also Y supports a bundle gerbe with class  $[\alpha] \in H^3(X; \mathbb{Z})$  iff  $\delta[\alpha] = 0 \in H^3(Y; \mathbb{Z})$ .

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- A gerbe  $\mathbb{L} = (L, Z, Y^{[2]}),$
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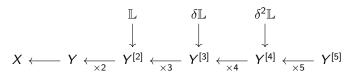
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- ▶ A 2-morphism (did I mention gerbes have 2-morphisms?) relating the induced trivialization of  $\delta^2\mathbb{L}$  to the canonical one,
- A coherency condition on pulled back 2-morphisms over Y<sup>[5]</sup>.

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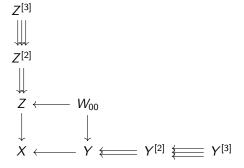
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- ▶ (L, Z, Y, X) has a well-defined class  $DD(L, Z, Y, X) \in H^4(X; \mathbb{Z})$ .
- ▶ For higher gerbes  $(H^{\geq 5}(X; \mathbb{Z}))$ , higher and more complicated coherency conditions will appear.
- Y and Z are not on equal footing.

$$Z \longleftarrow W_{00}$$
 $\downarrow$ 
 $X \longleftarrow Y$ 

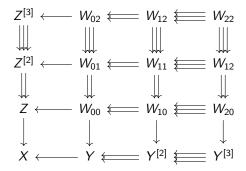
- ▶ Start with  $Y \longrightarrow X$  and  $Z \longrightarrow X$  fiber bundles (or locally split).
- ▶ Take  $W \longrightarrow Y$ ,  $W \longrightarrow Z$  fiber bundles forming a commutative square. Minimal choice:  $W = Y \times_X Z$ , but typically W will be larger. Set  $W_{00} = W$ .

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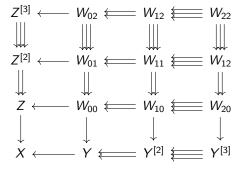
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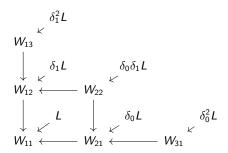
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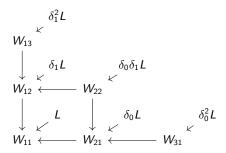


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- ▶ Fill out the diagram by fiber products.
- $\blacktriangleright$   $W_{\bullet\bullet}$  forms a bisimplicial space over X.



#### Definition

A bundle bigerbe is a "bisimplicial line bundle" over  $W_{\bullet \bullet}$ , i.e., a line bundle L over  $W_{11}$ , with trivializations of  $\delta_0 L$  and  $\delta_1 L$ , such that the induced trivializations of  $\delta_0 \delta_1 L$  agree and which induce the canonical trivializations of  $\delta_1^2 L$  and  $\delta_0^2 L$ .



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- Products, inverses, pull backs straightforward to define.
- ▶ A *trivialization* is an isomorphism  $L \cong \delta_0 \delta_1 Q$  for a line bundle Q over  $W_{00}$ .

#### **Theorem**

A bundle bigerbe (L,W,X) has a well-defined Dixmier-Douady class  $DD(L) \in H^4(X;\mathbb{Z})$ , with

$$DD(L^{-1}) = -DD(L),$$
  

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 Definition generalizes in a straightforward manner to higher degree (Exercise).

#### Theorem

A bundle multigerbe L of degree n has a well-defined Dixmier-Douady class  $DD(L) \in H^{2+n}(X;\mathbb{Z})$ , with

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 Definition generalizes in a straightforward manner to higher degree (Exercise).

- ▶ Suppose *X* is connected, and take  $Y = \mathcal{P}_*X$ , the based path space.
- ▶ Then  $Y^{[2]} = \mathcal{P}_*^{[2]} X \cong \Omega X$ , the based loop space.
- ▶ Every class in  $H^3(X; \mathbb{Z})$  is represented by a bundle gerbe  $(L, \mathcal{P}_*X, X)$ , i.e., a simplicial line bundle L on  $\Omega X$ .

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- ▶ In this case, the gerbe product is known as the "fusion product" (Stolz-Teichner, Waldorf), and *L* is a "fusion line bundle".

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- Likewise, if X is simply connected, with  $Y = Z = \mathcal{P}_*X$  and  $W_{00} = \mathcal{P}_*\mathcal{P}_*X$ , Then  $W_{11} = \Omega^2X$ , the double based loop space of X.
- ▶ Every class in  $H^4(X; \mathbb{Z})$  is represented by a bisimplicial (or doubly fusion) line bundle  $L \longrightarrow \Omega^2 X$ .

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- ▶ Every class in  $H^4(X; \mathbb{Z})$  is represented by a bisimplicial (or doubly fusion) line bundle  $L \longrightarrow \Omega^2 X$ .

#### Proposition

If X is k-connected, then every class in  $H^{3+k}(X; \mathbb{Z})$  is represented by a multisimplicial line bundle on  $\Omega^{2+k}X$ .

## Existence: free loop spaces

- ▶ Alternatively, take  $Y = \mathcal{P}X$ , the free path space, fibering over  $X^2$ .
- ▶ Then  $Y^{[2]} = \mathcal{P}^{[2]}X \cong \mathcal{L}X$ , the free loop space.

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- ▶ Every class in  $H^3(X; \mathbb{Z})$  is represented by a simplicial line bundle  $L = (L, \mathcal{P}X, X^2)$  on  $\mathcal{L}X$ , with the additional condition of a trivialization of the alternating product of pullbacks to the "figure-of-eight" loop space [K-Melrose, 2013].
- ► Figure-of-eight is yet another simplicial condition "over" the simplicial space

$$X \longleftarrow X^2 \longleftarrow X^3 \cdots$$

guaranteeing that the class in  $H^3(X^2; \mathbb{Z})$  comes from  $H^3(X; \mathbb{Z})$ .

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- ▶ Every class in  $H^3(X; \mathbb{Z})$  is represented by a simplicial line bundle  $L = (L, \mathcal{P}X, X^2)$  on  $\mathcal{L}X$ , with the additional condition of a trivialization of the alternating product of pullbacks to the "figure-of-eight" loop space [K-Melrose, 2013].
- Figure-of-eight is yet another simplicial condition "over" the simplicial space

$$X \longleftarrow X^2 \longleftarrow X^3 \cdots$$

guaranteeing that the class in  $H^3(X^2; \mathbb{Z})$  comes from  $H^3(X; \mathbb{Z})$ .

#### Proposition

Every class in  $H^{3+k}(X;\mathbb{Z})$  is represented by a multisimplicial (and multi figure-of-eight) line bundle on  $\mathcal{L}^{2+k}X$ .

- ▶ Take  $\alpha \in H^3(X; \mathbb{Z})$  and  $L \longrightarrow \mathcal{L}X$  with  $DD(L, \mathcal{P}X, X^2) = \alpha$ .
- ▶  $c_1(L) \in H^2(\mathcal{L}X; \mathbb{Z})$  is the *transgression* of  $\alpha$ :

$$H^k(X;\mathbb{Z}) \xrightarrow{\operatorname{ev}^*} H^k(\mathbb{S}^1 \times \mathcal{L}X;\mathbb{Z})$$

$$\downarrow^{\int_{\mathbb{S}^1}} H^{k-1}(\mathcal{L}X;\mathbb{Z})$$

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#### **Theorem**

On  $\mathcal{L}^{\ell}X$  there is a well-defined loop-fusion cohomology  $\check{H}^{\bullet}_{lf}(\mathcal{L}^{\ell}X;\mathbb{Z})$  through which iterated transgression factors as an isomorphism:

$$H_{\mathrm{lf}}^{k}(\mathcal{L}^{\ell}X;\mathbb{Z}) \xrightarrow{\cong} H_{\mathrm{lf}}^{k-n}(\mathcal{L}^{\ell+n}X;\mathbb{Z}).$$

### Spin structures on loop space

- ▶ Let X be a spin manifold and  $E \longrightarrow X$  the principal  $G = \operatorname{Spin}_n$  bundle.
- ▶ Then  $LE \longrightarrow LX$  is a LG bundle, and LG has a central extension

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \widehat{\mathcal{L}G} \longrightarrow \mathcal{L}G \longrightarrow 1$$

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#### Proposition

The lifting bundle gerbe  $(u^*\widehat{\mathcal{L}G},\mathcal{L}E,\mathcal{L}X)$  is a bundle bigerbe associated to the bisimplicial space generated by

$$\begin{array}{ccc}
E^2 & \longleftarrow & \mathcal{P}E \\
\downarrow & & \downarrow \\
X^2 & \longleftarrow & \mathcal{P}X
\end{array}$$

with Dixmier-Douady class  $\frac{1}{2}p_1(X) \in H^4(X; \mathbb{Z})$ .

c.f. McLaughlin, Redden, Waldorf, K.-Melrose.

#### Questions and future directions

- ▶ Connection structures, representations of differential cohomology.
- ▶ On  $\mathcal{L}X$  (and generally  $\mathcal{L}^kX$ ), equivariance of L with respect to action of  $\operatorname{Diffeo}^+(\mathbb{S}^1)$  (and its central extension) [c.f. Brylinski]. This may have an important role to play in elliptic cohomology theories.
- ▶ Loop-fusion K-theory of  $\mathcal{L}X$  and  $\mathcal{L}^{\ell}X$ .