Take-home-exam 5, WI4212 Advanced Numerical Methods

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Consider the following problems:

Advection equation

$$q_t + \bar{u} q_x = 0, \text{ with } \bar{u} = 1,$$

and the

Acoustic equations:

$$p_t - K_0 u_x = 0,$$

$$\rho_0 u_t - p_x = 0,$$

where $K_0 = 4$ and $\rho_0 = 1$.

For both problems we take periodic boundary conditions in the space direction and $x \in [0, 10]$. We take $t \in [0, T]$, where T is chosen equal to 5 periods. As initial conditions, we choose for the first problem

$$q(x, 0) = f(x)$$

and for the second problem

$$u(x, 0) = 1, p(x, 0) = f(x),$$

where f is

$$f(x) = \begin{cases} \frac{1-\cos(\pi x)}{2} & : \quad x \in [2, 4], \\ 1 & : \quad x \in [6, 8], \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Furthermore, we consider the following numerical methods:

I First order upwind,

II Beam-Warming method,

III High resolution method with the Van Leer flux limiter.

Answer the following questions:

1. Give the exact solution for both problems for the Cauchy problem, so $x \in (-\infty, \infty)$.

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2. Investigate the stability of method II in the 1- and 2-norm for the advection equation.
3. Investigate the local truncation error of method II for the advection equation.
4. Implement the 3 methods for both problems and compare the solution of the numerical methods with the exact solution. Investigate the stability, dependence on the CFL number, wiggles and accuracy. Do this comparison after 1, 2 and 5 periods.
5. Implement method I and method II for the advection equation on a non-uniform grid. The grid size on [0,5] is 2 times as small as on [5,10]. Investigate the stability, dependence on the CFL number, wiggles and accuracy. Do this comparison after 1 and 5 periods.