1. *hp*-refinements

In Assignment 3, we have solved the 1d Poisson problem in just one element with different polynomial degree p. We have seen that, when we increase p (also denoted by N somewhere), the solution converges to the exact solution fast. This refinement is called p-refinement.

Another common refinement is the h-refinement. h usually refers to the element size. For example, if the domain is $[-1, 1]$ and it is uniformly divided into K elements, we have $h = 2/K$. If we increase K, h gets smaller, and the mesh is refined. Therefore, the solution will converge as well. This is the h-refinement.

An hp-refinement scheme is such a scheme that we can do both h- and p-refinement.

2. 1d Poisson problem with two different boundary conditions on the left and right boundaries

We still consider the exact solutions:

$$\phi = -\sin(\alpha\pi x),$$

$$u = \nabla\phi = -\alpha\pi\cos(\alpha\pi x),$$

$$f = -\nabla\cdot u = -\alpha^2\pi^2\sin(\alpha\pi x),$$

where $\alpha \in \{1, 2, 3, 4, ...\}$. The domain this time is set to $\Omega = [0, 1]$, not $[-1, 1]$ anymore, for simplicity.

The problem is then given as: In Ω, given f all over the domain and boundary conditions:

$$\phi(0) = 0,$$

and

$$u(1) = -\alpha\pi\cos(\alpha\pi),$$

solve the Poisson equation (See Assignment 3).

3. Transformations

Let the domain $\Omega = [0, 1]$ be divided into K uniform elements:

$$\Omega_k = [kh, (k+1)h], \quad k \in \{0, 1, ..., K-1\},$$

where $h = 1/K$ is the element size. We can easily derive the linear transformation, Φ_k, from the reference domain $\Omega_{\text{ref}} := \xi \in [-1, 1]$ to Ω_k:

$$\Phi_k := \Omega_{\text{ref}} \to \Omega_k.$$

With this transformation, we can switch between the element Ω_k and the reference domain Ω_{ref}.

3.1. From Ω_k to Ω_{ref}

Start from this moment, we use a bit differential forms to simplify the derivations. We first see how to transform from element Ω_k to the reference domain. In Ω_k, the ϕ is interpreted as a 1-form:

$$\phi^{(1)} = \phi(x)\ \mathrm{d}x,$$

and the u is interpreted as a 0-form:

$$u^{(0)} = u(x).$$

We can transform them back to the reference domain by:

$$\phi^{(1)}_{\text{ref}} = \left[\phi(\Phi_k(\xi))\ \frac{\mathrm{d}x}{\mathrm{d}\xi}\right] \mathrm{d}\xi,$$

$$u^{(0)}_{\text{ref}} = u(\Phi_k(\xi)).$$

3.2. From Ω_{ref} to Ω_k

Suppose we have $u^{(0)}_{\text{ref}}$ and $\phi^{(1)}_{\text{ref}}$ in Ω_{ref}:

$$u^{(0)}_{\text{ref}} = u_{\text{ref}}(\xi)$$

and

$$\phi^{(1)}_{\text{ref}} = \phi_{\text{ref}}(\xi)\ \mathrm{d}\xi.$$

Their transformation in Ω_k are

$$u^{(0)} = u_{\text{ref}}(\Phi_k^{-1}(x))$$

and

$$\phi^{(1)} = \left[\phi_{\text{ref}}(\Phi_k^{-1}(x))\frac{\mathrm{d}\xi}{\mathrm{d}x}\right] \mathrm{d}x.$$

4. Discrete 1d Poisson problem

Suppose in all elements, we use polynomial basis functions at $p = N$. In Ω_k, we can discretize f onto

$$f_k^h = \sum_{i=1}^{N} f_k^i e_i^k(x)\ \mathrm{d}x,$$

where

$$f_k^i = \int_{\Phi_k(\xi_{i-1})}^{\Phi_k(\xi_i)} f(x)\ \mathrm{d}x.$$

With the transformations, we can get the discrete 1d Poisson problem in Ω_k:

$$\begin{Bmatrix} \mathbb{M}_k^u & (\mathbb{M}_k^\phi \mathbb{E})^\mathsf{T} \\ \mathbb{M}_k^\phi \mathbb{E} & \end{Bmatrix} \begin{Bmatrix} \vec{u}_k \\ \vec{\phi}_k \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\mathbb{M}_k^\phi \vec{f}_k \end{Bmatrix}, \tag{1}$$

where $\vec{f}_k = \left\{ f_k^1, f_k^2, \cdots, f_k^N \right\}^\mathsf{T}$,

$$\mathbb{M}^u_{k;i,j} = \int_{-1}^{1} \frac{\mathrm{d}x}{\mathrm{d}\xi} h_i(\xi) h_j(\xi)\ \mathrm{d}\xi,$$

$$\mathbb{M}^\phi_{k;i,j} = \int_{-1}^{1} \frac{\mathrm{d}\xi}{\mathrm{d}x} e_i(\xi) e_j(\xi)\ \mathrm{d}\xi,$$

and $\mathbb{E}$ is the incidence matrix we have in assignment 3.

Above one is the one corresponding to continuous meshes (elements are not broken up). If we break the elements; make them discontinuous (also called hybrid), we will need introduce a new variable, the λ to re-enforce the continuity. As Marc has introduced during the meeting, in this hybrid setup, the hybrid discretization for element Ω_k will read as

$$\begin{Bmatrix} \mathbb{M}^u_k & (\mathbb{M}^\phi_k \mathbb{E})^\mathsf{T} & \mathbb{N}^\mathsf{T} \\ \mathbb{M}^\phi_k \mathbb{E} & & \\ \mathbb{N} & & \end{Bmatrix} \begin{Bmatrix} \vec{u}_k \\ \vec{\phi}_k \\ \vec{\lambda}_k \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\mathbb{M}^\phi_k \vec{f}_k \\ 0 \end{Bmatrix}. \tag{2}$$

The matrix $\mathbb{N}$, like the incidence matrix $\mathbb{E}$, is a topological matrix (is the same for all elements). And in 1d, it is extremely simple (two rows, each row has only one non-zero entry: -1 or 1).

Once we get the hybrid discretization for all elements, we can assemble them into a global system according to one numbering (gathering matrix). Boundary conditions then need to be applied to the global system before you send it to a solver.

Once it is solved, you need to use the numbering (gathering matrix) to distribute the solution to each element. With solutions in elements, you now can reconstruct the solution element-wise. Visualizing or measuring the error can follow.