Computational Fluid Dynamics - Part I

Discretization Techniques - Elliptic Poblems

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September 25, 2007/ MSc Aerospace Engineering

Outline

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What we do and do not do

- General techniques for elliptic equations
- Finite Difference Method, Finite Volume Method and Finite Element Method
- Mixed problems
- Transformation to general domains
- We will not discuss Section 3.1.1 from the lecture notes.

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- Quantitative approximation (small errors)
- Qualitative approximation (preserve symmetries of differential equations)

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Elliptic problems

The Laplace equation

The Laplace equation

The Laplace equation for u is given by

$$\Delta u = \nabla \cdot \nabla u = 0 \quad \Longrightarrow \quad \begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & 2D \\[2ex] \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0 & 3D \end{cases}$$

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First order system

Introducing the vector $\vec{w}$ we can write this as a first order system (mixed problem):

$$\vec{\nabla} \cdot \vec{w} = 0 \; ,$$

$$\vec{w} = \vec{\nabla} u \; .$$

Finite Difference Method

Laplace Equation I

2D FD method

- Assume we have a 2D uniform grid
- Discretize solution at $u_{i,j} = u(i \star \Delta x, j \star \Delta y)$
- Relate the value in neighboring point by Taylor series expansion

$$u_{i-1} = u_i - \Delta x \left.\frac{\partial u}{\partial x}\right|_i + \frac{\Delta x^2}{2} \left.\frac{\partial^2 u}{\partial x^2}\right|_i - \frac{\Delta x^3}{6} \left.\frac{\partial^3 u}{\partial x^3}\right|_i + \frac{\Delta x^4}{24} \left.\frac{\partial^4 u}{\partial x^4}\right|_i + O\left(\Delta x^5\right)$$

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Graphical Representation

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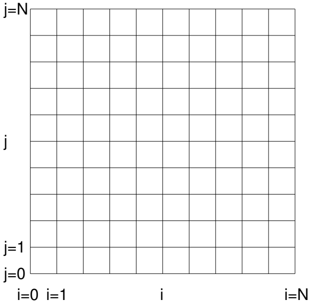

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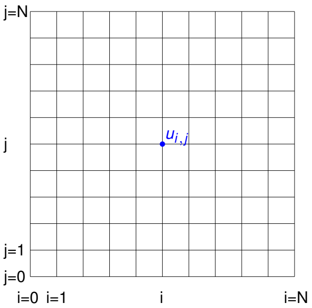

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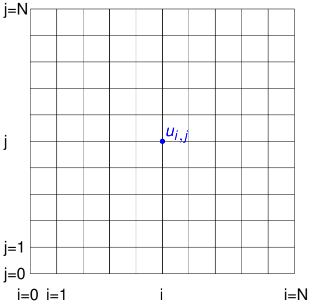

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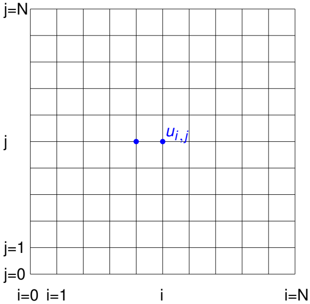

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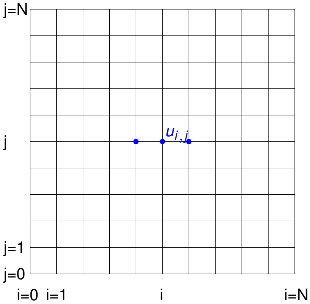

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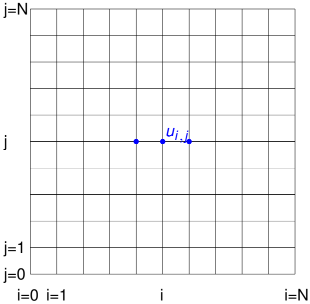

Exact expansion $\partial^2 u/\partial x^2$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} + O\left(\Delta x^4\right) .$$

Finite Difference Method

Laplace Equation II

Exact expansion $\partial^2 u/\partial x^2$

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Finite Difference Method

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Approximation $\partial^2 u/\partial x^2$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \approx \frac{\partial^2 u}{\partial x^2} .$$

Finite Difference Method

Laplace Equation II

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Truncation error $\partial^2 u/\partial x^2$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} - \frac{\partial^2 u}{\partial x^2} = \underbrace{\frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4}}_{\text{Truncation error}} + O\left(\Delta x^4\right) .$$

Finite Difference Method

Laplace Equation III

Laplace equation

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} \underbrace{- \frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} - \frac{\Delta y^2}{12}\frac{\partial^4 u}{\partial y^4} + O\left(\Delta x^4, \Delta y^4\right)}_{\text{Truncation Error}} .$$

Finite Difference Method

Laplace Equation III

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Consistency

A numerical scheme is called consistent if the truncation error goes to zero for the mesh to zero

$$\lim_{\Delta x, \Delta y \to 0} \left[\frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} - \frac{\Delta y^2}{12}\frac{\partial^4 u}{\partial y^4} + O\left(\Delta x^4, \Delta y^4\right) \right] = 0 \; .$$

Finite Difference Method

Laplace Equation IV

Modified/equivalent differential equation

By solving

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = 0 \ ,$$

we actually approximate

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} + \frac{\Delta y^2}{12}\frac{\partial^4 u}{\partial y^4} = 0 \ ,$$

up to $O\left(\Delta x^4, \Delta y^4\right)$.

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Dirichlet Boundary Conditions

If on the lefthand side of the domain $u_{0,j} = u_j^*$ is prescribed then

$$\frac{u_{2,j} - 2u_{1,j}}{\Delta x^2} + \frac{u_{1,j+1} - 2u_{1,j} + u_{1,j-1}}{\Delta y^2} = -\frac{u_j^*}{\Delta x^2} \ .$$

Analogously for other boundaries

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Finite Difference Method

Laplace Equation IV

Modified/equivalent differential equation

By solving

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{\Delta y^2} = 0 \; ,$$

we actually approximate

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} + \frac{\Delta y^2}{12}\frac{\partial^4 u}{\partial y^4} = 0 \; ,$$

up to $O\left(\Delta x^4, \Delta y^4\right)$.

Neumann Boundary Conditions

If on the lefthand side of the domain $\partial u/\partial x(0,j) = A_j$ is prescribed then

$$\frac{2u_{1,j} - 2u_{0,j}}{\Delta x^2} + \frac{u_{0,j+1} - 2u_{0,j} + u_{0,j-1}}{\Delta y^2} = \frac{2A_j}{\Delta x} \; .$$

Analogously for other boundaries

Finite Difference Method

Laplace Equation V

Example on a 3 $\times$ 3-grid, Dirichlet

Let $\Delta x = \Delta y = h$

$$\frac{1}{h^2}\begin{pmatrix} -4 & 1 & 0 & 1 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 1 & 0 & 1 & -4 \end{pmatrix}\begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{2,2} \\ u_{1,2} \end{pmatrix} = \frac{1}{h^2}\begin{pmatrix} -u^*_{0,1} - u^*_{1,0} \\ -u^*_{2,0} - u^*_{3,1} \\ -u^*_{3,2} - u^*_{2,3} \\ -u^*_{1,3} - u^*_{0,2} \end{pmatrix}$$

Finite Difference Method

Laplace Equation V

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Symmetry

Note that the matrix in this example is symmetric. This was to be expected, because the Laplace operator is a symmetric operator.

Finite Difference Method

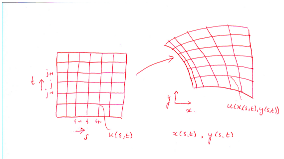
General domain

Transformation

- Suppose we can map square domain onto $x(s,t)$ and $y(s,t)$
- $$\begin{pmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$
- $$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \frac{\partial y}{\partial t} & -\frac{\partial y}{\partial s} \\ -\frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{pmatrix}$$

  $J = x_s y_t - x_t y_s$.

Mapping



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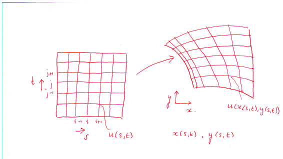

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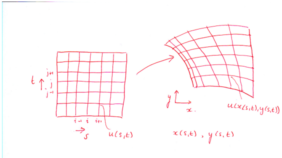

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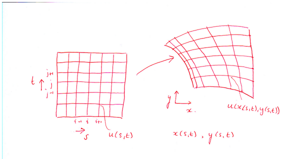

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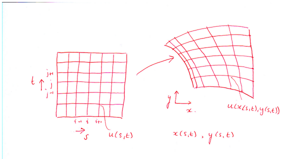
Transformation

- Suppose we can map square domain onto $x(s,t)$ and $y(s,t)$
- $$\begin{pmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$
- $$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \frac{1}{J} \begin{pmatrix} \frac{\partial y}{\partial t} & -\frac{\partial y}{\partial s} \\ -\frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial s} \\ \frac{\partial}{\partial t} \end{pmatrix}$$

  $J = x_s y_t - x_t y_s$.

Mapping



Laplace equation

$$\frac{1}{J}\frac{\partial y}{\partial t}\frac{\partial}{\partial s}\left[\frac{1}{J}\frac{\partial y}{\partial t}\frac{\partial u}{\partial s} - \frac{1}{J}\frac{\partial y}{\partial s}\frac{\partial u}{\partial t}\right] - \frac{1}{J}\frac{\partial y}{\partial s}\frac{\partial}{\partial t}\left[\frac{1}{J}\frac{\partial y}{\partial t}\frac{\partial u}{\partial s} - \frac{1}{J}\frac{\partial y}{\partial s}\frac{\partial u}{\partial t}\right] +$$
$$\frac{1}{J}\frac{\partial x}{\partial s}\frac{\partial}{\partial t}\left[\frac{1}{J}\frac{\partial x}{\partial s}\frac{\partial u}{\partial t} - \frac{1}{J}\frac{\partial x}{\partial t}\frac{\partial u}{\partial s}\right] - \frac{1}{J}\frac{\partial x}{\partial t}\frac{\partial}{\partial s}\left[\frac{1}{J}\frac{\partial x}{\partial s}\frac{\partial u}{\partial t} - \frac{1}{J}\frac{\partial x}{\partial t}\frac{\partial u}{\partial s}\right] = 0$$

Finite Volume Method

General Philosophy

- The finite volume method is a method for conservation laws, see Lecture 1.
- The idea is to divide the computational domain into volumes and impose the conservation laws on these volumes.
- Use the divergence theorem to convert volume integrals to boundary integrals
- Evaluate fluxes at boundaries.

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Finite Volume Method

Laplace Equation I

2D FV method

- Use again a grid with the unknowns defined at the vertices
- Surround the point (i, j) with a volume $\Omega_{i,j}$
- Integrate the Laplace equation over $\Omega_{i,j}$

$$0 = \int_{\Omega_{i,j}} \nabla \cdot \nabla u \, d\Omega = \int_{\partial\Omega_{i,j}} \nabla u \cdot n \, dS$$

- Split this boundary integral over the sides of the volume
- This is still exact. The numerical approximation enters into the scheme if approximate the integrals and the derivatives $\partial u / \partial x$, $\partial u / \partial y$ numerically.

Graphical Representation

Finite Volume Method

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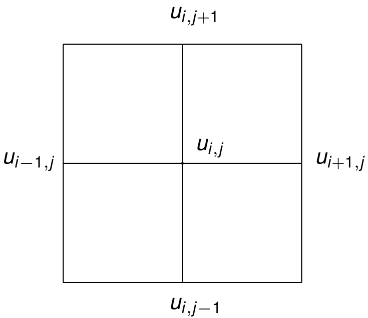

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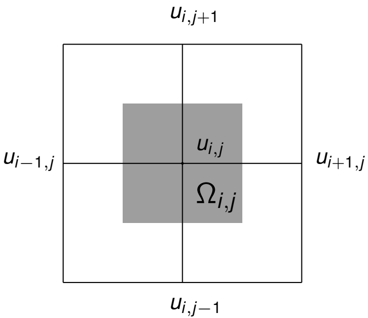

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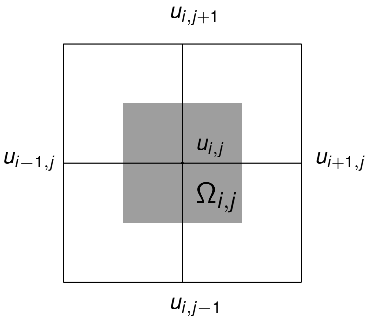

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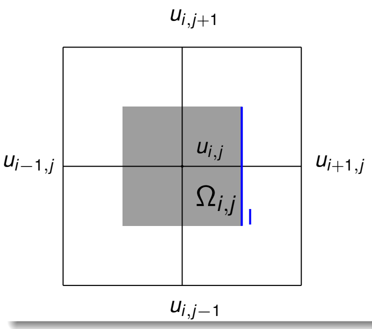

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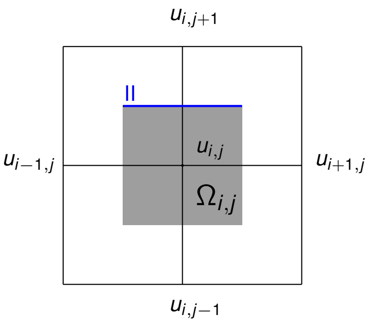

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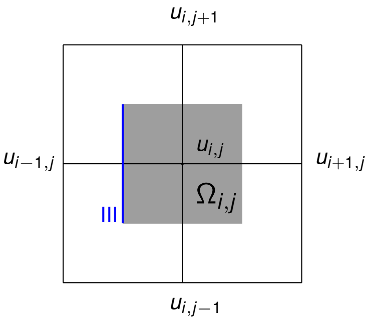

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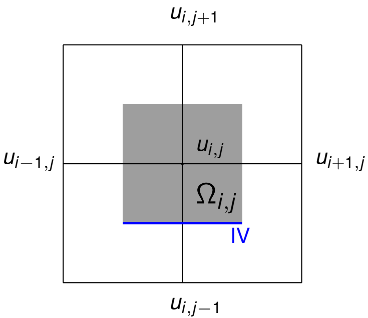

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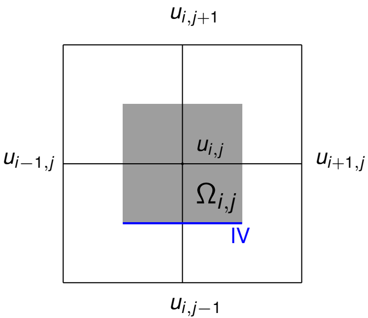

Finite Volume Method

Laplace Equation II

Discretization

$$0 = \int_I \frac{\partial u}{\partial x} dy + \int_{II} \frac{\partial u}{\partial y} dx - \int_{III} \frac{\partial u}{\partial x} dy - \int_{IV} \frac{\partial u}{\partial y} dx$$

- Approximate integrals by midpoint rule: Let $y = f(x)$, $x \in [a, b]$ then

$$\int_a^b f(x)\, dx = f\left(\frac{a+b}{2}\right) h + \frac{h^2}{24} f''(c) \approx f\left(\frac{a+b}{2}\right) h\,, \quad h = b - a\,, \quad c \in (a, b)$$

- The derivatives at the midpoints are approximated by finite differences:
- So the finite volume discretization becomes:

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta x} \Delta y + \frac{u_{i,j+1} - u_{i,j}}{\Delta y} \Delta x - \frac{u_{i,j} - u_{i-1,j}}{\Delta x} \Delta y - \frac{u_{i,j} - u_{i,j-1}}{\Delta y} \Delta x = 0\,.$$

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Finite Volume Method

Laplace Equation II

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Finite Volume Method

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- The derivatives at the midpoints are approximated by finite differences:

$$\left.\frac{\partial u}{\partial x}\right|_{\text{midpoint I}} \approx \frac{u_{i+1,j} - u_{i,j}}{\Delta x}$$

- So the finite volume discretization becomes:

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta x} \Delta y + \frac{u_{i,j+1} - u_{i,j}}{\Delta y} \Delta x - \frac{u_{i,j} - u_{i-1,j}}{\Delta x} \Delta y - \frac{u_{i,j} - u_{i,j-1}}{\Delta y} \Delta x = 0\,.$$

Finite Volume Method

Laplace Equation II

Discretization

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- Approximate integrals by midpoint rule: Let $y = f(x)$, $x \in [a, b]$ then

$$\int_a^b f(x)\, dx = f\left(\frac{a+b}{2}\right) h + \frac{h^2}{24} f''(c) \approx f\left(\frac{a+b}{2}\right) h\,, \quad h = b - a\,, \quad c \in (a, b)$$

- The derivatives at the midpoints are approximated by finite differences:

$$\left.\frac{\partial u}{\partial y}\right|_{\text{midpoint II}} \approx \frac{u_{i,j+1} - u_{i,j}}{\Delta y}$$

- So the finite volume discretization becomes:

$$\frac{u_{i+1,j} - u_{i,j}}{\Delta x} \Delta y + \frac{u_{i,j+1} - u_{i,j}}{\Delta y} \Delta x - \frac{u_{i,j} - u_{i-1,j}}{\Delta x} \Delta y - \frac{u_{i,j} - u_{i,j-1}}{\Delta y} \Delta x = 0\,.$$

Finite Volume Method

Laplace Equation II

Discretization

$$0 = \int_I \frac{\partial u}{\partial x} dy + \int_{II} \frac{\partial u}{\partial y} dx - \int_{III} \frac{\partial u}{\partial x} dy - \int_{IV} \frac{\partial u}{\partial y} dx$$

- Approximate integrals by midpoint rule: Let $y = f(x)$, $x \in [a, b]$ then

$$\int_a^b f(x)\,dx = f\left(\frac{a+b}{2}\right) h + \frac{h^2}{24} f''(c) \approx f\left(\frac{a+b}{2}\right) h\,, \quad h = b - a\,, \quad c \in (a, b)$$

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$$\left.\frac{\partial u}{\partial x}\right|_{\text{midpoint III}} \approx \frac{u_{i,j} - u_{i-1,j}}{\Delta x}$$

- So the finite volume discretization becomes:

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Finite Volume Method

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$$\left.\frac{\partial u}{\partial y}\right|_{\text{midpoint IV}} \approx \frac{u_{i,j} - u_{i,j-1}}{\Delta y}$$

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Finite Volume Method

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Finite Volume Method

Laplace Equation III

Comparison with FDM

$$\frac{u_{i+1,j}-u_{i,j}}{\Delta x}\Delta y+\frac{u_{i,j+1}-u_{i,j}}{\Delta y}\Delta x-\frac{u_{i,j}-u_{i-1,j}}{\Delta x}\Delta y-\frac{u_{i,j}-u_{i,j-1}}{\Delta y}\Delta x=0\ .$$

- If we divide the Finite Volume scheme by the area of $\Omega_{i,j}=\Delta x\Delta y$ we obtain

$$\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{\Delta x^2}+\frac{u_{i,j+1}-u_{i,j}+u_{i,j-1}}{\Delta y^2}=0\ .$$

  This is, in this particular case, the same as the Finite Difference scheme obtained earlier.
- Note that we started with a second order PDE, but due to the conversion to boundary integrals we only need to evaluate first order derivatives.
- Note that the midpoint rule is second order accurate and the approximation of the derivatives is second order accurate, so the overall scheme is second order.

Finite Volume Method

Laplace Equation III

Comparison with FDM

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Finite Volume Method

Laplace Equation IV

Mixed Formulation I

- Recall the mixed formulation of the Laplace equation

$$\vec{\nabla} \cdot \vec{w} = 0 \; , \quad \text{with} \quad \vec{w} = (w_1 \, , \, w_2)^T \; ,$$

$$\vec{w} = \vec{\nabla} u \; .$$

- Use the same volume $\Omega_{i,j}$
- Integrate the $\vec{\nabla} \cdot \vec{w}$ equation over $\Omega_{i,j}$

$$0 = \int_{\Omega_{i,j}} \vec{\nabla} \cdot \vec{w} \, d\Omega = \int_{\partial\Omega_{i,j}} \vec{w} \cdot \vec{n} \, dS$$

- Split this boundary integral over the sides of the volume
- Note that both components of the vector $\vec{w}$ are located at different places in the gird. u at the vertices, w_1 at the vertical edges, w_2 at the horizontal edges. Such a grid is called a staggered grid.

Graphical Representation

Finite Volume Method

Laplace Equation IV

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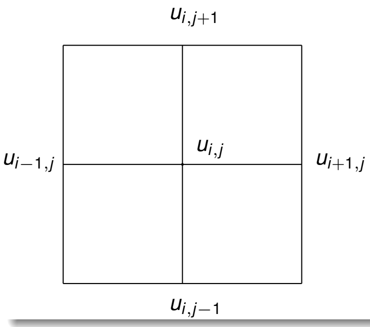

Finite Volume Method

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$$\vec{\nabla} \cdot \vec{w} = 0 \ , \quad \text{with} \quad \vec{w} = (w_1 \, , w_2)^T \ ,$$

$$\vec{w} = \vec{\nabla} u \ .$$

- Use the same volume $\Omega_{i,j}$
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$$0 = \int_{\Omega_{i,j}} \vec{\nabla} \cdot \vec{w} \, d\Omega = \int_{\partial\Omega_{i,j}} \vec{w} \cdot \vec{n} \, dS$$

- Split this boundary integral over the sides of the volume
- Note that both components of the vector $\vec{w}$ are located at different places in the gird. u at the vertices, w_1 at the vertical edges, w_2 at the horizontal edges. Such a grid is called a staggered grid.

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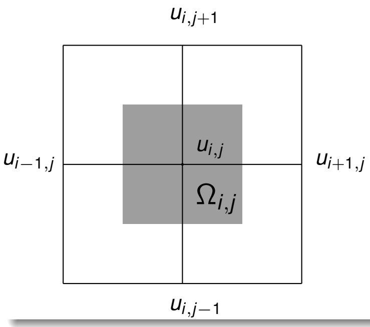

Finite Volume Method

Laplace Equation IV

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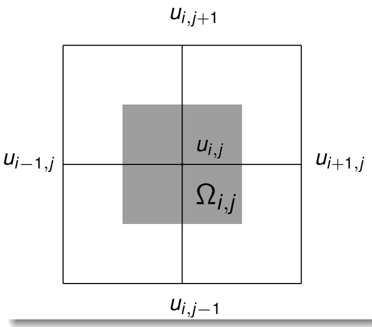

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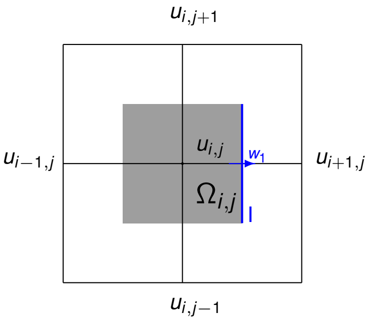

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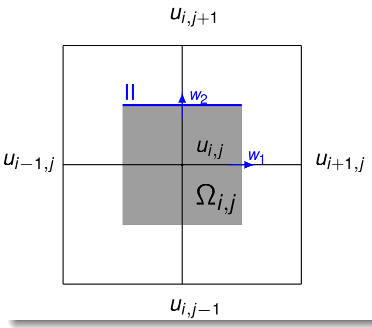

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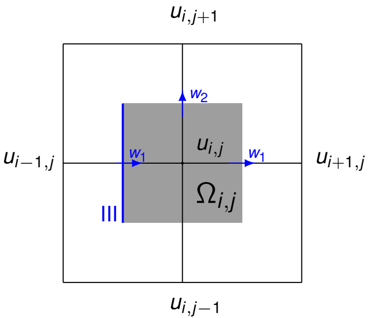

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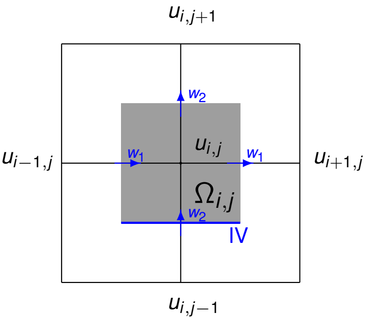

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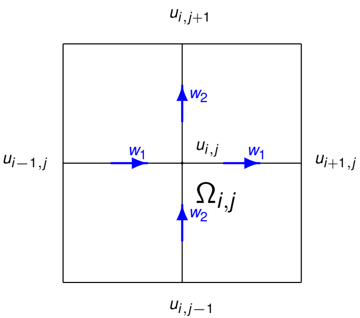

Finite Volume Method

Laplace Equation V

Mixed Formulation II

- The staggered grid arises naturally from the finite volume approach. Not so easy to deduce in finite difference method.
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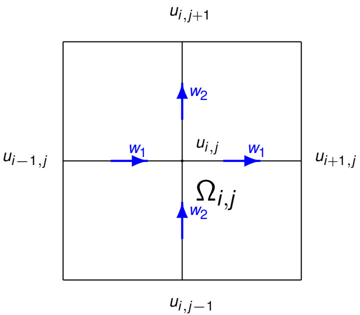

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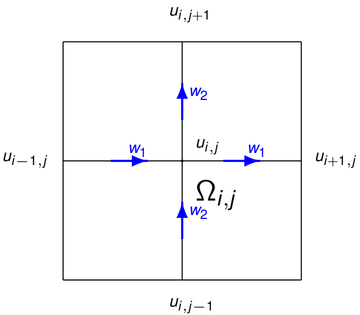

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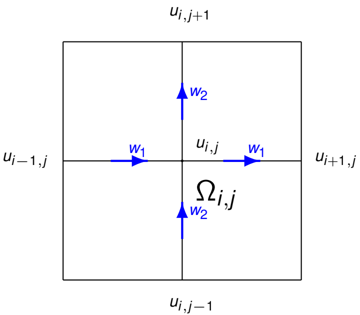

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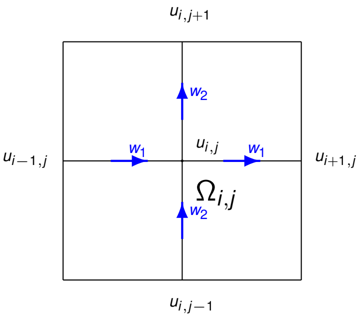

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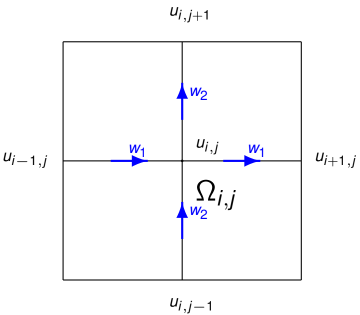

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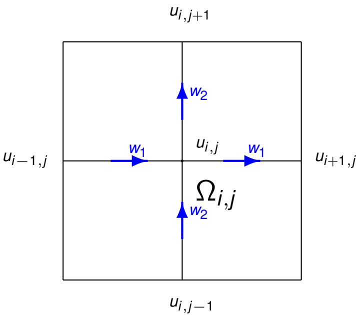

Finite Element Method I

What is different

- The Finite Difference Method is a collocation method. This is a purely nodal method in which all unknowns are defined in discrete points
- The Finite Volume Method is based on an integral formulation.
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- The idea in Finite Element Methods is to expand the unknown solution in terms of known basis functions and to determine the coefficients of this expansion such that the residual is 'small'.

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Finite Element Method II

One Element Case

Basic idea

- Suppose we have N basis functions $\phi_1(x,y), \ldots \phi_N(x,y)$ and we expand the solution as
$$u^N(x,y) = \sum_{i=1}^{N} \alpha_i \phi_i(x,y) \,.$$
- The residual is then defined as
$$R(x,y) = \Delta u^N(x,y) = \frac{\partial^2 u^N}{\partial x^2} + \frac{\partial^2 u^N}{\partial y^2}$$
- Galerkin: Set the projection of R onto the basis functions to zero:
$$0 = (R, \phi_i) = \int_\Omega \Delta u^N \phi_i \, d\Omega = -\int_\Omega \vec{\nabla} u^N, \vec{\nabla}\phi_i \, d\Omega + \int_{\partial\Omega} \frac{\partial u^N}{\partial n} \phi_i \, dS \quad i = 1, \ldots, N \,.$$
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Finite Element Method III

One Element Case - continued

Basic idea

- If Dirichlet boundary conditions are prescribed, ϕ_i needs to be zero at the boundary. Therefore

$$0 = (R, \phi_i) = \int_\Omega \Delta u^N \phi_i \, d\Omega = -\int_\Omega \vec{\nabla} u^N : \vec{\nabla} \phi_i \, d\Omega \quad i = 1, \ldots, N \, .$$

- If we insert the expansion for u^N, we get

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- The matrix K_{ij} is called the element matrix.
- The element matrix is again symmetric, i.e. $K_{ij} = K_{ji}$.

TUDelft

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Finite Element Method IV

One Element Case - continued

Linear basis functions

- Let us take the domain [0, 1] × [0, 1] and the 4 basis functions

$$\phi_1(x,y) = (1-x)(1-y)$$
$$\phi_2(x,y) = x(1-y)$$
$$\phi_3(x,y) = xy \text{ and}$$
$$\phi_4(x,y) = (1-x)y.$$

- The element K_{ij} is then given by

$$K_{ij} = \int_0^1 \int_0^1 \left[\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right] dxdy$$

For instance,

$$K_{11} = \int_0^1 \int_0^1 \left[(1-y)^2 + (1-x)^2 \right] dxdy = \frac{2}{3} \ .$$

- $$K = \frac{1}{6} \begin{pmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix} .$$

Finite Element Method IV

One Element Case - continued

Linear basis functions

- Let us take the domain $[0,1] \times [0,1]$ and the 4 basis functions

$$
\begin{aligned}
\phi_1(x,y) &= (1-x)(1-y) \\
\phi_2(x,y) &= x(1-y) \\
\phi_3(x,y) &= xy \text{ and} \\
\phi_4(x,y) &= (1-x)y.
\end{aligned}
$$

- The element K_{ij} is then given by

$$K_{ij} = \int_0^1 \int_0^1 \left[\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right] dxdy$$

For instance,

$$K_{11} = \int_0^1 \int_0^1 \left[(1-y)^2 + (1-x)^2 \right] dxdy = \frac{2}{3} .$$

-

$$K = \frac{1}{6} \begin{pmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix} .$$

Finite Element Method IV

One Element Case - continued

Linear basis functions

- Let us take the domain $[0, 1] \times [0, 1]$ and the 4 basis functions

$$\begin{aligned}\phi_1(x, y) &= (1 - x)(1 - y)\\ \phi_2(x, y) &= x(1 - y)\\ \phi_3(x, y) &= xy \text{ and}\\ \phi_4(x, y) &= (1 - x)y.\end{aligned}$$

- The element K_{ij} is then given by

$$K_{ij} = \int_0^1 \int_0^1 \left[\frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right] dxdy$$

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-

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Finite Element Method IV

One Element Case - continued

Linear basis functions

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$$\begin{aligned}\phi_1(x,y) &= (1-x)(1-y)\\ \phi_2(x,y) &= x(1-y)\\ \phi_3(x,y) &= xy \text{ and}\\ \phi_4(x,y) &= (1-x)y.\end{aligned}$$

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- $$K = \frac{1}{6} \begin{pmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix} \; .$$

Finite Element Method V

Multiple elements

- Show that if we evaluate the basis functions on a domain $[0, h] \times [0, h]$ we get the same element matrix.
- Consider the 3×3 mesh on the right.
- Choose an element numbering, see Fig blue numbers.
- Choose a global node numbering, see Fig red numbers
- Set up a table/matrix which relates local node numbers to global node numbers $GM(e, LN)$

Node numbering

Finite Element Method V

Multiple elements

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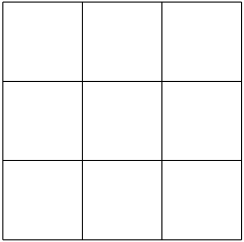

Finite Element Method V

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- Set up a table/matrix which relates local node numbers to global node numbers $GM(e, LN)$

Node numbering

1	2	3
1	2	3
1	2	3

TUDelft

Finite Element Method V

Multiple elements

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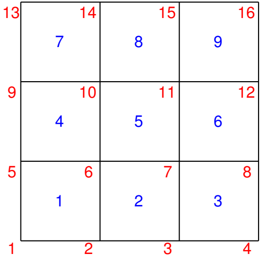
Node numbering



TUDelft

Finite Element Method V

Multiple elements

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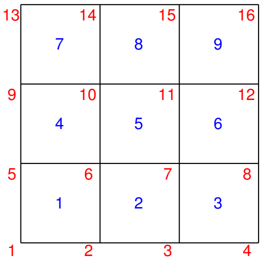
Node numbering



Gathering matrix

GM	1	2	3	4	5	6	7	8	9
1	1	2	3	5	6	7	9	10	11
2	2	3	4	6	7	8	10	11	12
3	6	7	8	10	11	12	14	15	16
4	5	6	7	9	10	11	13	14	15

Finite Element Method VI

Multiple elements - continued

Gathering matrix

GM	1	2	3	4	5	6	7	8	9
1	1	2	3	5	6	7	9	10	11
2	2	3	4	6	7	8	10	11	12
3	6	7	8	10	11	12	14	15	16
4	5	6	7	9	10	11	13	14	15

- The global matrix is now constructed from the element matrices and gathering matrix by
- $A = 0$
  for $e = 1$ to NoE
  for $i = 1$ to 4
  for $j = 1$ to 4
  $A(GM(e,i), GM(e,j)) = A(GM(e,i), GM(e,j)) + K^e(i,j)$
  end for
  end for
  end for

Global matrix structure

Finite Element Method VI

Multiple elements - continued

Gathering matrix

GM	1	2	3	4	5	6	7	8	9
1	1	2	3	5	6	7	9	10	11
2	2	3	4	6	7	8	10	11	12
3	6	7	8	10	11	12	14	15	16
4	5	6	7	9	10	11	13	14	15

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  end for
  end for

Global matrix structure

Finite Element Method VI

Multiple elements - continued

Gathering matrix

GM	1	2	3	4	5	6	7	8	9
1	1	2	3	5	6	7	9	10	11
2	2	3	4	6	7	8	10	11	12
3	6	7	8	10	11	12	14	15	16
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  end for
  end for
  end for

Global matrix structure

Finite Element Method VI

Multiple elements - continued

Gathering matrix

GM	1	2	3	4	5	6	7	8	9
1	1	2	3	5	6	7	9	10	11
2	2	3	4	6	7	8	10	11	12
3	6	7	8	10	11	12	14	15	16
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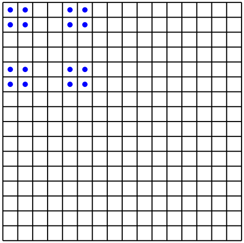
  end for
  end for
  end for

Global matrix structure

Contribution of element 1 to matrix



Finite Element Method VI

Multiple elements - continued

Gathering matrix

GM	1	2	3	4	5	6	7	8	9
1	1	2	3	5	6	7	9	10	11
2	2	3	4	6	7	8	10	11	12
3	6	7	8	10	11	12	14	15	16
4	5	6	7	9	10	11	13	14	15

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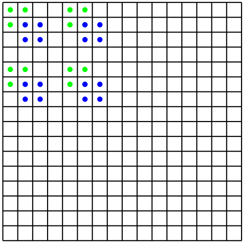
  for $j = 1$ to 4
  $$A(GM(e,i), GM(e,j)) = A(GM(e,i), GM(e,j)) + K^e(i,j)$$
  end for
  end for
  end for

Global matrix structure

Contribution of element 2 to matrix



Finite Element Method VI

Multiple elements - continued

Gathering matrix

GM	1	2	3	4	5	6	7	8	9
1	1	2	3	5	6	7	9	10	11
2	2	3	4	6	7	8	10	11	12
3	6	7	8	10	11	12	14	15	16
4	5	6	7	9	10	11	13	14	15

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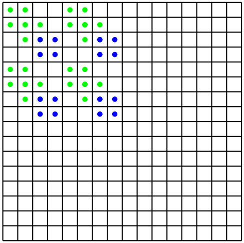
  for $i = 1$ to 4
  for $j = 1$ to 4
  $$A(GM(e,i), GM(e,j)) = A(GM(e,i), GM(e,j)) + K^e(i,j)$$
  end for
  end for
  end for

Global matrix structure

Contribution of element 3 to matrix



Finite Element Method VI

Multiple elements - continued

Gathering matrix

GM	1	2	3	4	5	6	7	8	9
1	1	2	3	5	6	7	9	10	11
2	2	3	4	6	7	8	10	11	12
3	6	7	8	10	11	12	14	15	16
4	5	6	7	9	10	11	13	14	15

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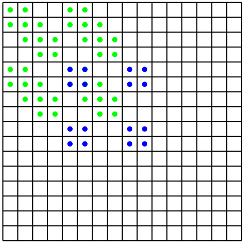
  for $j = 1$ to 4
  $A(GM(e,i), GM(e,j)) = A(GM(e,i), GM(e,j)) + K^e(i,j)$
  end for
  end for
  end for

Global matrix structure

Contribution of element 4 to matrix



Finite Element Method VI

Multiple elements - continued

Gathering matrix

GM	1	2	3	4	5	6	7	8	9
1	1	2	3	5	6	7	9	10	11
2	2	3	4	6	7	8	10	11	12
3	6	7	8	10	11	12	14	15	16
4	5	6	7	9	10	11	13	14	15

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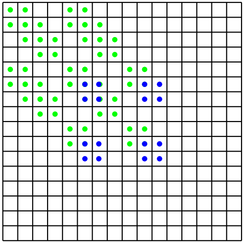
      for $j = 1$ to 4
        $A(GM(e,i), GM(e,j)) = A(GM(e,i), GM(e,j)) + K^e(i,j)$
      end for
    end for
  end for

Global matrix structure

Contribution of element 5 to matrix



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Finite Element Method VI

Multiple elements - continued

Gathering matrix

GM	1	2	3	4	5	6	7	8	9
1	1	2	3	5	6	7	9	10	11
2	2	3	4	6	7	8	10	11	12
3	6	7	8	10	11	12	14	15	16
4	5	6	7	9	10	11	13	14	15

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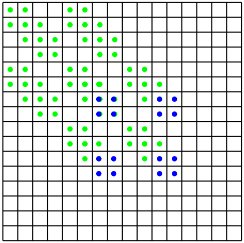
  end for
  end for
  end for

Global matrix structure

Contribution of element 6 to matrix



Finite Element Method VI

Multiple elements - continued

Gathering matrix

GM	1	2	3	4	5	6	7	8	9
1	1	2	3	5	6	7	9	10	11
2	2	3	4	6	7	8	10	11	12
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- The global matrix is now constructed from the element matrices and gathering matrix by
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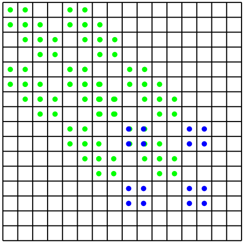
  $$A(GM(e,i), GM(e,j)) = A(GM(e,i), GM(e,j)) + K^e(i,j)$$
  end for
  end for
  end for

Global matrix structure

Contribution of element 7 to matrix



Finite Element Method VI

Multiple elements - continued

Gathering matrix

GM	1	2	3	4	5	6	7	8	9
1	1	2	3	5	6	7	9	10	11
2	2	3	4	6	7	8	10	11	12
3	6	7	8	10	11	12	14	15	16
4	5	6	7	9	10	11	13	14	15

- The global matrix is now constructed from the element matrices and gathering matrix by
- $A = 0$

  for $e = 1$ to NoE

  for $i = 1$ to 4

  for $j = 1$ to 4

  $$A(GM(e,i), GM(e,j)) = A(GM(e,i), GM(e,j)) + K^e(i,j)$$

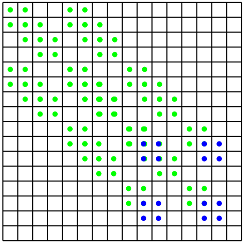
  end for

  end for

  end for

Global matrix structure

Contribution of element 8 to matrix



Finite Element Method VI

Multiple elements - continued

Gathering matrix

GM	1	2	3	4	5	6	7	8	9
1	1	2	3	5	6	7	9	10	11
2	2	3	4	6	7	8	10	11	12
3	6	7	8	10	11	12	14	15	16
4	5	6	7	9	10	11	13	14	15

- The global matrix is now constructed from the element matrices and gathering matrix by
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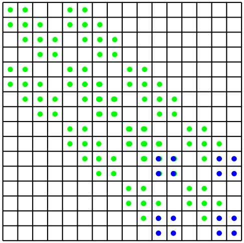
      end for
    end for
  end for

Global matrix structure

Contribution of element 9 to matrix



Finite Element Method VI

Multiple elements - continued

Gathering matrix

GM	1	2	3	4	5	6	7	8	9
1	1	2	3	5	6	7	9	10	11
2	2	3	4	6	7	8	10	11	12
3	6	7	8	10	11	12	14	15	16
4	5	6	7	9	10	11	13	14	15

- The global matrix is now constructed from the element matrices and gathering matrix by
- $A = 0$

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  for $j = 1$ to 4

  $$A(GM(e,i), GM(e,j)) = A(GM(e,i), GM(e,j)) + K^e(i,j)$$

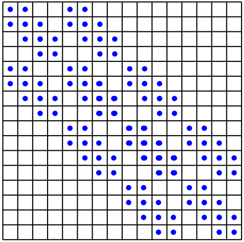
  end for

  end for

  end for

Global matrix structure

Sparsity pattern FEM Matrix



Summary

What you should have learned

Basic idea

- The three basic methods – FDM, FVM and FEM – to discretize an elliptic problem on a square domain
- How a mapping allows you to convert problems on a non-square domain to a square domain.
- That a mixed formulation should be discretized on a staggered grid
- That all methods lead to a symmetric linear, algebraic system.

Summary

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