Delft University of Technology

Viscous Flows take-home exam (44)

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Contents

1. F

a.

Starting with the basic momentum conservation equation,

$$\rho \frac{Du_i}{Dt} = (\rho g_i) - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j} \tag{1.1}$$

Substitute the following dimensionless parameters:

$$\rho = \rho^* \rho_0 \tag{1.2}$$
$$u_i = u_i^* U \tag{1.3}$$
$$p = p^* p_0 \tag{1.4}$$
$$\tau_{ij} = \tau_{ij}^* \frac{\mu_0 U}{L} \tag{1.5}$$
$$x_i = x_i^* L \tag{1.6}$$
$$x_j = x_j^* L \tag{1.7}$$
$$t = t^* \frac{L}{U} \tag{1.8}$$

where (1.4) is the new non-dimensional scaling of p instead of scaling by $\rho_0 U^2$ which then transforms the left hand side (LHS) of (1.1) into

$$\text{LHS} = \rho^* \rho_0 \frac{D(u_i^* U)}{D(t^* \frac{L}{U})} \tag{1.9}$$
$$= \rho_0 \frac{U^2}{L} \left(\rho^* \frac{Du_i^*}{Dt^*} \right) \tag{1.10}$$

The right hand side (RHS) will then be

$$\text{RHS} = \rho^* \rho_0 g - \frac{\partial (p^* p_0)}{\partial (x_i^* L)} + \frac{\partial \left(\tau_{ij}^* \frac{\mu_0 U}{L} \right)}{\partial (x_j^* L)} \tag{1.11}$$
$$= \rho_0 g \rho^* - \frac{p_0}{L} \frac{\partial p^*}{\partial x_i^*} + \frac{\mu_0 U}{L^2} \frac{\partial \tau_{ij}^*}{\partial x_j^*} \tag{1.12}$$

Substituting (1.10) and (1.12) back into (1.1),

$$\rho_0 \frac{U^2}{L} \left(\rho^* \frac{Du_i^*}{Dt^*} \right) = \rho_0 g \rho^* - \frac{p_0}{L} \frac{\partial p^*}{\partial x_i^*} + \frac{\mu_0 U}{L^2} \frac{\partial \tau_{ij}^*}{\partial x_j^*} \tag{1.13}$$
$$\rho^* \frac{Du_i^*}{Dt^*} = \left[\frac{gL}{U^2} \right] \rho^* - \left[\frac{p_0}{\rho_0 U^2} \right] \frac{\partial p^*}{\partial x_i^*} + \left[\frac{\mu_0}{\rho_0 U L} \right] \frac{\partial \tau_{ij}^*}{\partial x_j^*} \tag{1.14}$$

Comparing this to the non-dimensionalised form of momentum equation for which p is scaled by $\rho_0 U^2$ which gives

$$\rho^* \frac{Du_i^*}{Dt^*} = \left[\frac{gL}{U^2} \right] \rho^* - \frac{\partial p^*}{\partial x_i^*} + \left[\frac{\mu_0}{\rho_0 U L} \right] \frac{\partial \tau_{ij}^*}{\partial x_j^*} \tag{1.15}$$

we see that there is an additional dimensionless parameter for (1.14) compared to (1.15) and that is

$$\text{Additional dimensionless parameter} = \frac{p_0}{\rho_0 U^2} \tag{1.16}$$

To see how (1.16) relates to the Mach number for a perfect gas, we first start with the Mach number.

$$M = \frac{U}{a} \tag{1.17}$$

$$= \frac{U}{\sqrt{\gamma R T}} \tag{1.18}$$

For perfect gas, $p = \rho R T$ and thus

$$M = \frac{U}{\sqrt{\gamma \frac{p}{\rho}}} \tag{1.19}$$

$$M^2 = \frac{U^2 \rho}{\gamma p} \tag{1.20}$$

$$\frac{p}{\rho U^2} = \frac{1}{\gamma M^2} \tag{1.21}$$

This is then related to the additional dimensionless parameter of (1.16) in the following way.

$$\boxed{\text{Additional dimensionless parameter} = \frac{p_0}{\rho_0 U^2} = \frac{1}{\gamma M^2}} \tag{1.22}$$

b.

The Mach number is

$$M = \frac{U}{a} \tag{1.23}$$

where as the Reynolds number is

$$Re = \frac{\rho U L}{\mu} \tag{1.24}$$

The ratio M/Re is

$$\frac{M}{Re} = \frac{\frac{U}{a}}{\frac{\rho U L}{\mu}} \tag{1.25}$$

$$= \frac{\mu}{\rho L a} \tag{1.26}$$

where

$$\mu \approx 0.67 \rho \ell a \tag{1.27}$$

according to [1] in accordance with the predictions from kinetic gas theory where l: molecular mean free path which then leads to

$$\frac{M}{Re} \approx \frac{0.67 \rho \ell a}{\rho L a} \tag{1.28}$$

$$\boxed{\frac{M}{Re} \approx \frac{0.67 \ell}{L}} \tag{1.29}$$

Since L, the characteristic length scale $>> \ell$, the molecular mean free path, it can be said that

$$\frac{M}{Re} << 1 \tag{1.30}$$

$$M << Re \tag{1.31}$$

c.

The Grashoff number is said to be

$$Gr = \frac{\Delta\rho}{\rho}\frac{gL^3}{v^2} \tag{1.32}$$

We let the dimensionless density difference be labelled as

$$\tilde{S} = \frac{\Delta\rho}{\rho} \tag{1.33}$$

We also have the Reynolds number, Re, and Froude number, Fr, as the following

$$Re = \frac{\rho LU}{\mu} = \frac{LU}{\nu} \tag{1.34}$$

$$Fr = \frac{U^2}{gL} \tag{1.35}$$

Substituting (1.33), (1.34) and (1.35) into (1.32),

$$\begin{aligned} Gr &= \tilde{S}\,\frac{gL}{U^2}\,\frac{U^2L^2}{\nu^2} \\ &= \tilde{S}\,\frac{1}{Fr}\,\frac{L^2U^2}{\nu^2} \\ \end{aligned}$$

$$\boxed{Gr = \tilde{S}\,\frac{Re^2}{Fr}}$$

d.

Starting with the left hand side which is the Reynolds number,

$$\begin{aligned} Re &= \frac{\rho_0 V_0 L_0}{\mu_0} \\ &\quad \text{multiplying both numerator and denominator by } V_0, \\ &= \frac{\rho_0 V_0^2 L_0}{\mu_0 V_0} \\ &\quad \text{multiplying both numerator and denominator by } \frac{1}{L_0^2} \\ &= \frac{\frac{\rho_0 V_0^2}{L_0}}{\frac{\mu_0 V_0}{L_0^2}} \\ &= \frac{\frac{\text{Inertial force}}{\text{Volume}}}{\frac{\text{Viscous force}}{\text{Volume}}} \end{aligned}$$

$$\boxed{Re = \frac{\text{Inertial force}}{\text{Viscous force}}}$$

e.

Starting with the derivation dimensional analysis of gravity force / volume,

$$\begin{aligned}\frac{\text{Gravity force}}{\text{Volume}} &= \frac{\text{gravitational acceleration} \times \text{density} \times \text{Volume}}{\text{Volume}}\\ &= \frac{g\rho_0 L_0^3}{L_0^3}\\ &= g\rho_0\end{aligned}$$

As for buoyant force / volume, using the follow relation $\Delta\rho = \rho\beta\Delta T$ given by [1] where β: thermal expansion coefficient, we get

$$\begin{aligned}\frac{\text{Buoyant force (caused by temperature difference)}}{\text{Volume}} &= \frac{\beta\rho g\Delta T L_0^3}{L_0^3}\\ &= \frac{g\Delta\rho L_0^3}{L_0^3}\\ &= g\Delta rho\end{aligned}$$

First, for the Froude number,

$$\begin{aligned}Fr^2 &= \frac{V_0^2}{gL_0}\\ &\quad \text{multiplying both numerator and denominator by } \rho_0\\ &= \frac{\rho_0 V_0^2}{\rho_0 g L_0}\\ &\quad \text{multiplying both numerator and denominator by } \frac{1}{L_0}\\ &= \frac{\frac{\rho_0 V_0^2}{L_0}}{\rho_0 g}\\ &= \frac{\frac{\text{Inertial force}}{\text{Volume}}}{\frac{\text{Gravity force}}{\text{Volume}}}\end{aligned}$$

$$\boxed{Fr^2 = \frac{\text{Inertial force}}{\text{Gravity force}}}$$

Moving onto the Grashoff number, we use the fact that $\mu = \nu\rho$.

$$
\begin{aligned}
Gr &= \frac{\Delta\rho}{\rho}\frac{gL_0^3}{\nu^2} \\
&= \frac{\Delta\rho}{\rho}\frac{gL_0^3\rho^2}{\mu^2} \\
&= \frac{\Delta\rho g L_0^3\rho}{\mu^2} \\
&\quad \text{multiplying both numerator and denominator by } V_0 \\
&= \frac{\Delta\rho g}{\frac{\mu V_0}{L_0^2}}\frac{\rho L_0 V_0}{\mu} \\
&= \frac{\frac{\text{Buoyancy force}}{\text{Volume}}}{\frac{\text{Viscous force}}{\text{Volume}}} Re \\
&= \frac{\text{Buoyancy force}}{\text{Viscous force}}\frac{\text{Inertial force}}{\text{Viscous force}}
\end{aligned}
$$

$$
\boxed{Gr = \frac{\text{Buoyancy force} \times \text{Inertial force}}{(\text{Viscous force})^2}}
$$

2. D

Given are that

$$U(t) = \begin{cases} 0 & t \leq 0 \\ \frac{t}{\Delta t} U_0 & 0 \leq t \leq \Delta t \\ U_0 & t \geq \Delta t \end{cases} \tag{2.1}$$

Assuming similarity variable,

$$\eta = \frac{y}{L(t)} \tag{2.2}$$

$$L(t) = 2\sqrt{\mu t} \tag{2.3}$$

a.

Since linearity permits superposition of indicial solutions, the following equations hold.

$$u(y,t) = \int_0^t \frac{dU}{dt^*} \cdot f\left(\frac{y}{2\sqrt{v\,(t-t^*)}}\right) dt^* \tag{2.4}$$

$$\tau_w(t) = \mu \left(\frac{\partial u}{\partial y}\right)_w = -\frac{\mu}{\sqrt{\pi v}} \int_0^t \frac{dU}{dt^*} \frac{dt^*}{\sqrt{(t-t^*)}} \tag{2.5}$$

For $t \leq 0$,

$$\tau_w = 0 \tag{2.6}$$

For $0 \leq t \leq \Delta t$,

$$\frac{dU}{dt^*} = \frac{U_0}{\Delta t} \tag{2.7}$$

$$\tau_w = -\frac{\mu}{\sqrt{\pi v}} \int_0^t \frac{dU}{dt^*} \frac{dt^*}{\sqrt{(t-t^*)}} \tag{2.8}$$

$$= -\frac{\mu}{\sqrt{\pi \nu}} \int_0^t \frac{U_0}{\Delta t} \frac{dt^*}{\sqrt{t-t^*}} \tag{2.9}$$

$$= -\frac{\mu U_0}{\Delta t \sqrt{\pi \nu}} \int_0^t \frac{dt^*}{\sqrt{t-t^*}} \tag{2.10}$$

$$= -\frac{\mu U_0}{\Delta t \sqrt{\pi \nu}} \left[-2\sqrt{t-t^*}\right]_0^t \tag{2.11}$$

$$= \frac{\mu U_0}{\Delta t \sqrt{\pi \nu}} (-2\sqrt{t}) \tag{2.12}$$

$$\boxed{\tau_w = -\frac{2\mu U_0 \sqrt{t}}{\Delta t \sqrt{\pi \nu}}} \tag{2.13}$$

For $t \geq \Delta t$,

$$\frac{dU}{dt^*} = 0 \tag{2.14}$$

$$\tau_w = -\frac{\mu}{\sqrt{\pi v}} \int_0^t \frac{dU}{dt^*} \frac{dt^*}{\sqrt{(t-t^*)}} \tag{2.15}$$

$$= -\frac{\mu}{\sqrt{\pi \nu}} \left(\int_0^{\Delta t} \frac{dU}{dt^*} \frac{dt^*}{\sqrt{(t-t^*)}} + \int_{\Delta t}^{t^*} \frac{dU}{dt^*} \frac{dt^*}{\sqrt{(t-t^*)}}\right) \tag{2.16}$$

$$= -\frac{\mu U_0}{\sqrt{\pi \nu} \Delta t} \left[-2\sqrt{t-t^*}\right]_0^{\Delta t} \tag{2.17}$$

$$\boxed{\tau_w = \frac{2\mu U_0}{\sqrt{\pi \nu} \Delta t} \left(\sqrt{t-\Delta t} - \sqrt{t}\right)} \tag{2.18}$$

As for the impulsively started plate ($\Delta t \to 0$),

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_w \tag{2.19}$$

$$= \mu \frac{U_0}{2\sqrt{\nu t}} \left(\frac{df}{d\eta}\right)_0 \tag{2.20}$$

$$\boxed{\tau_w = -\frac{\mu U_0}{\sqrt{\pi \nu t}}} \tag{2.21}$$

The following Fig.1 shows the results of the cases. which was created using arbitrary but realistic values of $\mu = 1.81 \times 10^{-5}$, $\nu = 1.5 \times 10^{-5}$ and $U_0 = 5$.

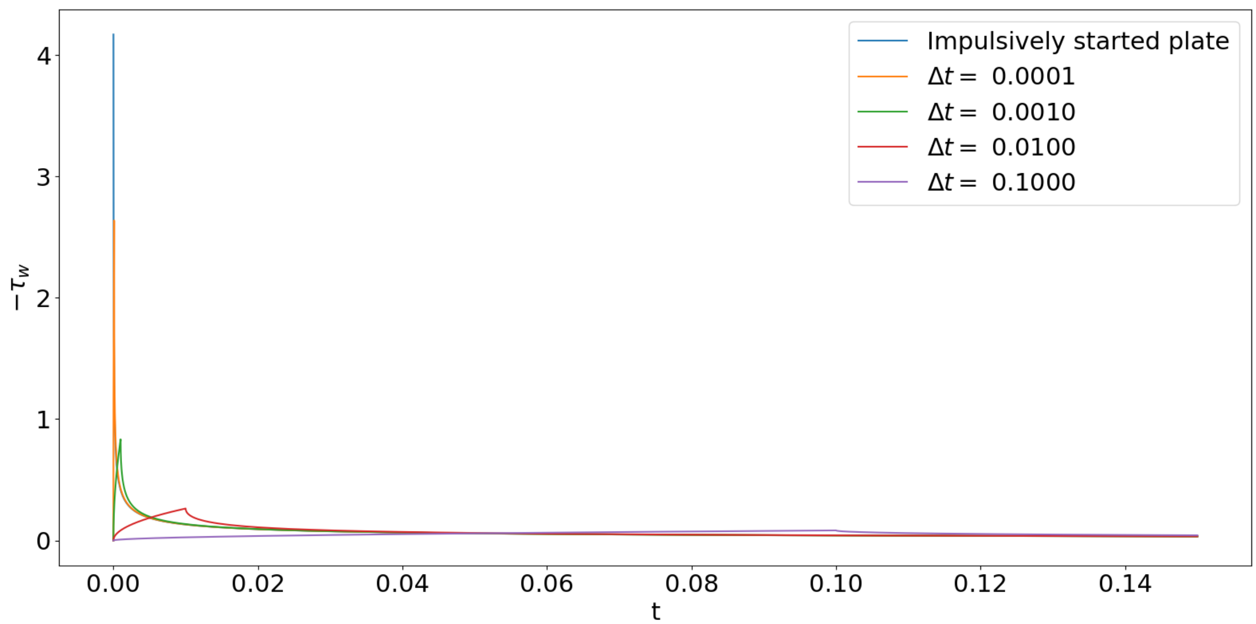

Fig. 1 Skin friction, τ_w, at various Δt and for impulsively started plate against time, t

It can be seen that as Δt approaches 0, the plot becomes increasingly similar to the plot of impulsively started plate.

b.

The given equation for the mechanical work required to move the plate against the friction forces is as follows.

$$W(t) = \int_0^t -\tau_w(t) U(t) dt \tag{2.22}$$

For $t \leq 0$,

$$\boxed{W(t) = 0} \tag{2.23}$$

For $0 \le t \le \Delta t$,

$$W(t) = \int_0^t -\left(-\frac{2\mu U_0\sqrt{t}}{\Delta t\sqrt{\pi\nu}}\right)\frac{\tilde{t}}{\Delta t}U_0 d\,\tilde{t} \tag{2.24}$$

$$= \int_0^t \frac{2\mu U_0^2\tilde{t}^{3/2}}{\Delta t^2\sqrt{\pi\nu}}d\tilde{t} \tag{2.25}$$

$$= \frac{2\mu U_0^2}{\Delta t^2\sqrt{\pi\nu}}\frac{2}{5}\left[\tilde{t}^{5/2}\right]_0^t \tag{2.26}$$

$$\boxed{W(t) = \frac{4}{5}\frac{\mu U_0^2}{\Delta t^2\sqrt{\pi\nu}}t^{5/2}} \tag{2.27}$$

For $t \ge \Delta t$,

$$W(t) = \int_0^{\Delta t} -\tau_2(t)U(t)dt + \int_{\Delta t}^t -\tau_w(t)U(t)dt \tag{2.28}$$

$$= \frac{4}{5}\frac{\mu U_0^2}{\Delta t^2\sqrt{\pi\nu}}\Delta t^{5/2} - \int_{\Delta t}^t \frac{2\mu U_0}{\sqrt{\pi\nu}\Delta t}\left(\sqrt{\tilde{t}-\Delta t} - \sqrt{\tilde{t}}\right)U_0\,d\tilde{t} \tag{2.29}$$

$$= \frac{4}{5}\frac{\mu U_0^2}{\Delta t^2\sqrt{\pi\nu}}\Delta t^{5/2} - \frac{2\mu U_0^2}{\sqrt{\pi\nu}\Delta t}\left[\frac{2}{3}(\tilde{t}-\Delta t)^{3/2} - \frac{2}{3}t^{\tilde{3}/2}\right]_{\Delta t}^t \tag{2.30}$$

$$\boxed{W(t) = \frac{4}{5}\frac{\mu U_0^2}{\Delta t^2\sqrt{\pi\nu}}\Delta t^{5/2} - \frac{4}{3}\frac{\mu U_0^2}{\sqrt{\pi\nu}\Delta t}\left[(t-\Delta t)^{3/2} - t^{3/2} + \Delta t^{3/2}\right]} \tag{2.31}$$

As for the impulsively started plate ($\Delta t \to 0$),

$$U(t) = U_0 \tag{2.32}$$

$$W(t) = \int_0^t -\left(-\frac{\mu U_0}{\sqrt{\pi\nu\tilde{t}}}\right)U_0\,d\tilde{t} \tag{2.33}$$

$$= \frac{\mu U_0^2}{\sqrt{\pi\nu}}\int_0^t \frac{1}{\sqrt{\tilde{t}}}d\tilde{t} \tag{2.34}$$

$$= \frac{\mu U_0^2}{\sqrt{\pi\nu}}2\left[\sqrt{t}\right]_0^t \tag{2.35}$$

$$\boxed{W(t) = \frac{2\mu U_0^2\sqrt{t}}{\sqrt{\pi\nu}}} \tag{2.36}$$

The following Fig.2 shows the results of the cases.

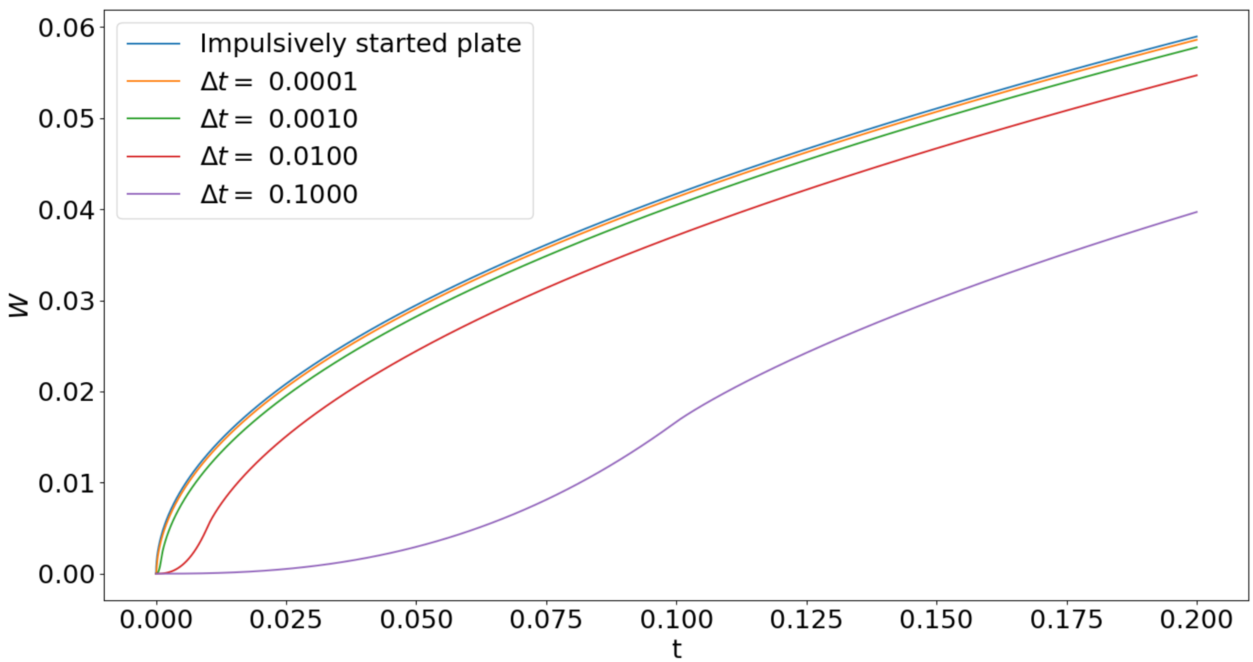

Fig. 2 Work required, W, at various Δt and for impulsively started plate against time, t

Again, it can be seen that as Δt approaches 0, the plot becomes increasingly similar to the plot of impulsively started plate.

3. D

a.

For $\beta = -1$, the interpretation is that the retarded flows pass around an expansion corner with turning angle $\beta\pi/2$ which equals to $-\pi/2$. This is visualised in the Fig.3.

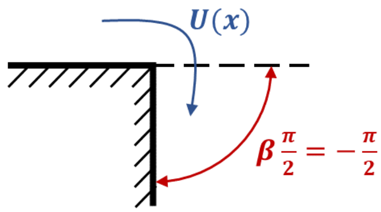

Fig. 3 Expansion geometry of $\beta = -1$

b.

The Falkner-Skan equation is as follows:

$$f''' + ff'' + \beta\left(1 - f'^2\right) = 0 \tag{3.1}$$

With $\beta = -1$, we have

$$f''' + ff'' + f'^2 - 1 = 0 \tag{3.2}$$

Integrating (3.2) with respect to η and using integration by parts,

$$\int \left(f''' + ff'' + f'^2 - 1\right) d\eta = 0 \tag{3.3}$$

$$f'' + ff' - \int f'^2 d\eta + \int f'^2 d\eta - \eta + C_1 = 0 \tag{3.4}$$

$$f'' + ff' - \eta + C_1 = 0 \tag{3.5}$$

where C_1 is an integration constant.

Integration (3.5) once more with respect to η,

$$\int (f'' + ff' - \eta + C_1)\, d\eta = 0 \tag{3.6}$$

$$f' + \frac{1}{2}f^2 - \frac{1}{2}\eta^2 + C_1\eta + C_2 = 0 \tag{3.7}$$

where C_2 is another integration constant.

At the wall, $\eta = 0$ and thus we have

$$f(0) = f_w \tag{3.8}$$

$$f'(0) = 0 \tag{3.9}$$

$$f''(0) = f''_w \tag{3.10}$$

Substituting $\eta = 0$ into (3.5),

$$f''(0) + f(0)f'(0) - 0 + C_1 = 0 \tag{3.11}$$

$$f''_w + 0 + C_1 = 0 \tag{3.12}$$

$$C_1 = -f''_w \tag{3.13}$$

Substituting $\eta = 0$ into (3.7),

$$f'(0) + \frac{1}{2}f^2(0) - 0 + 0 + C_2 = 0 \tag{3.14}$$

$$0 + \frac{1}{2}f_w^2 + C_2 = 0 \tag{3.15}$$

$$C_2 = -\frac{1}{2}f_w^2 \tag{3.16}$$

Substituting (3.13) and (3.16) into (3.5) and (3.7), we ultimately have

$$f'' + ff' - \eta - f_w'' = 0 \tag{3.17}$$

$$f' + \frac{1}{2}f^2 - \frac{1}{2}\eta^2 - f_w''\eta - \frac{1}{2}f_w^2 = 0 \tag{3.18}$$

$$\tag{3.19}$$

c.

The boundary layer character is described by η approaching a large value, say ∞. As η approaches ∞ ($\eta \longrightarrow \infty$), we have

$$u(x, y) \longrightarrow u_e \tag{3.20}$$

$$f' = \frac{u(x, y)}{u_e} \longrightarrow 1 \tag{3.21}$$

$$f'' \longrightarrow 0 \tag{3.22}$$

Starting from (3.17), as $\eta \longrightarrow \infty$, we have

$$0 + f \cdot 1 - \eta - f_w'' = 0 \tag{3.23}$$

$$f = \eta + f_w'' \tag{3.24}$$

Now for (3.18 using (3.24), as $\eta \longrightarrow \infty$, we have

$$1 + \frac{1}{2}\left(\eta + f_w''\right)^2 - \frac{1}{2}\eta^2 - f_w''\eta = \frac{1}{2}f_w^2 = 0 \tag{3.25}$$

$$1 + \frac{1}{2}f_w''^2 - \frac{1}{2}f_w^2 = 0 \tag{3.26}$$

$$2 + f_w''^2 - f_w^2 = 0 \tag{3.27}$$

$$f_w''^2 = f_w^2 - 2 \tag{3.28}$$

Positive values of f_w allow for boundary layer solutions, with reference to the plot of shear-stress profiles ($f'' v.s.\eta$).

The displacement thickness, δ^*, is defined as

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{u_e}\right) dy \tag{3.29}$$

where δ: boundary layer thickness which is characterised by y approaching a large enough value. We can then formulate (3.29 as follows.

$$\delta^* \lim_{\tilde{\eta} \to 0} \int_0^{\tilde{\eta}} (1 - f')\, dy \tag{3.30}$$

where $\tilde{\eta}$ is an arbitrary variable describing y approaching a large value. Now, using integration by substitution where

$$\frac{dy}{d\eta} = x\sqrt{\frac{2\nu}{u_e x(m + 1)}} \tag{3.31}$$

(3.30) can then be rewritten as

$$\delta^* = x\sqrt{\frac{2\nu}{u_e x(m+1)}} \lim_{\tilde{\eta}\to 0} \int_0^{\tilde{\eta}} (1 - f')\, d\eta \tag{3.32}$$

$$= x\sqrt{\frac{2\nu}{u_e x(m+1)}} \lim_{\tilde{\eta}\to 0} [\eta - f(\eta)]_0^{\tilde{\eta}} \tag{3.33}$$

$$= x\sqrt{\frac{2\nu}{u_e x(m+1)}} \lim_{\tilde{\eta}\to 0} (\tilde{\eta} - f(\tilde{\eta}) - 0 + f(0)) \tag{3.34}$$

We now replace $\tilde{\eta}$ by η as we also take $\eta \longrightarrow \infty$,

$$\delta^* = x\sqrt{\frac{2\nu}{u_e x(m+1)}} \lim_{\eta\to 0} (\eta - f(\eta) + f(0)) \tag{3.35}$$

$$= x\sqrt{\frac{2\nu}{u_e x(m+1)}} \lim_{\eta\to 0} (\eta - f(\eta) + f_w) \tag{3.36}$$

From (3.24), we know the behaviour of f as η approaches a large value. Thus, we have that

$$\delta^* = x\sqrt{\frac{2\nu}{u_e x(m+1)}} \left(\eta - \eta - f_w'' + f_w\right) \tag{3.37}$$

$$\boxed{\delta^* = x\sqrt{\frac{2\nu}{u_e x(m+1)}} \left(f_w - f_w''\right)} \tag{3.38}$$

d.

Starting from (3.28), we have that when $f_w = 1.5$,

$$f_w''^2 = f_w^2 - 2 \tag{3.39}$$

$$f_w''^2 = 1.5^2 - 2 = 0.25 \tag{3.40}$$

$$\boxed{f_w'' = \pm 0.5} \tag{3.41}$$

Substituting (3.41) into (3.18), we have

$$f' + \frac{1}{2}f^2 - \frac{1}{2}\eta^2 - f_w''\eta - \frac{1}{2}f_w^2 = 0 \tag{3.42}$$

$$f' = -\frac{1}{2}f^2 + \frac{1}{2}\eta^2 + f_w''\eta + \frac{1}{2}f_w^2 \tag{3.43}$$

Additionally, we have the following initial conditions

$$f(0) = f_w = 1.5 \quad \text{(given)} \tag{3.44}$$

$$f'(0) = 0 \tag{3.45}$$

$$f''(0) = f_w'' = \pm 0.5 \tag{3.46}$$

Ultimately, we solve (3.43) numerically using Heun's method and the following velocity profiles are achieved as shown in Fig.4.

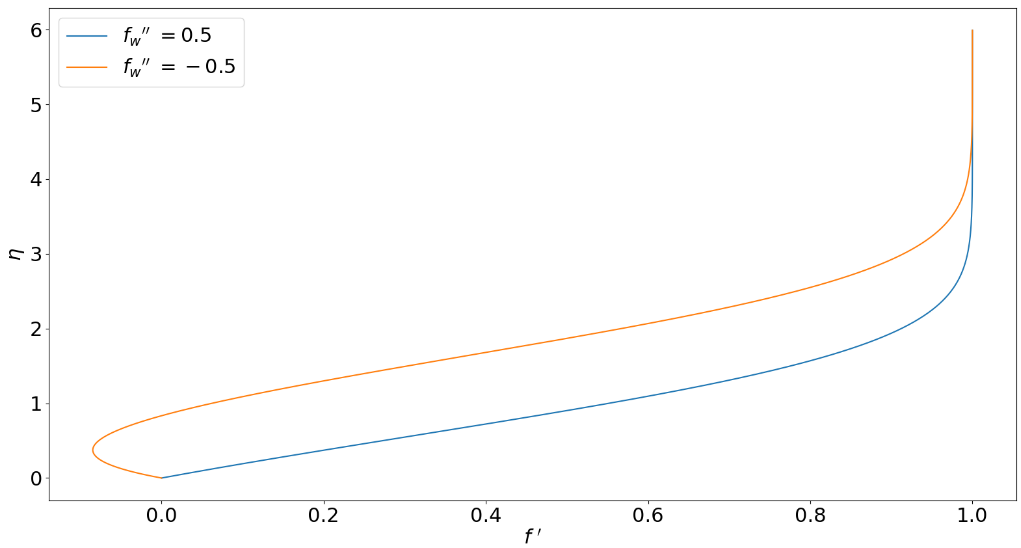

Fig. 4 Velocity profiles of Falkner-Skan equation for $\beta = -1$

4. B

a.

Determining $\theta(x)$ and showing λ remains constant

From the closed-form solution for the Thwaites method, we have the following equation for the square of the momentum thickness, θ.

$$\theta^2 = \frac{a\nu}{u_e^b}\int_0^x u_e^{b-1}dx \tag{4.1}$$

where

$$u_e(x) = U_0\left(\frac{x}{L}\right)^2 \tag{4.2}$$

as given in the problem. So we have

$$\theta^2 = \frac{a\nu}{U_0^b\left(\frac{x}{L}\right)^{mb}}\int_0^x -U_0^{b-1}\left(\frac{x}{L}\right)^{m(b-1)}dx \tag{4.3}$$

$$= \frac{a\nu}{U_0^b\left(\frac{x}{L}\right)^{mb}}\frac{U_0^{b-1}}{L^{m(b-1)}}\frac{1}{m(b-1)+1}\left[x^{m(b-1)+1}\right]_0^x \tag{4.4}$$

$$= \frac{a\nu L^m}{U_0 x^{mb}}\frac{1}{m(b-1)+1}x^{m(b-1)+1} \tag{4.5}$$

$$\boxed{\theta^2 = \frac{1}{m(b-1)+1}\frac{a\nu L^m x^{1-m}}{U_0}} \tag{4.6}$$

λ is defined as follows.

$$\lambda = \frac{\theta^2}{\nu}\frac{du_e}{dx} \tag{4.7}$$

Differentiating (4.2),

$$\frac{du_e}{dx} = \frac{U_0}{L^m}mx^{m-1} \tag{4.8}$$

Thus for λ we have,

$$\lambda = \frac{1}{m(b-1)+1}\frac{a\nu L^m x^{1-m}}{U_0}\frac{1}{\nu}\frac{U_0}{L^m}mx^{m-1} \tag{4.9}$$

$$\boxed{\lambda = \frac{ma}{m(b-1)+1}} \tag{4.10}$$

and this is independent of x and thus remains constant.

Expressing m in terms of λ, a and b

Rearranging the above equation,

$$[m(b-1)+1]\lambda = ma \tag{4.11}$$

$$m(b-1)\lambda + \lambda = ma \tag{4.12}$$

$$m\left((b-1)\lambda - a\right) = -\lambda \tag{4.13}$$

$$m = \frac{\lambda}{a-(b-1)\lambda} \tag{4.14}$$

Properties of the 2 self-similar boundary layer solutions for $m = 0$ and $m = -1$

For $m = 0$ which corresponds to flat plate boundary layer:

$$\boxed{\lambda = 0} \tag{4.15}$$

$$\boxed{\theta^2 = \frac{a\nu x}{U_0}} \tag{4.16}$$

For $m = -1$ which corresponds to planar stagnation flow:

$$\boxed{\lambda = \frac{a}{b-2}} \tag{4.17}$$

$$\boxed{\theta^2 = \frac{a\nu x^2}{(2-b)LU_0}} \tag{4.18}$$

b.

As for the non-similar boundary layer, the potential flow velocity is

$$u_e = \frac{U_0}{1 - \frac{x}{L}} \tag{4.19}$$

Its derivative is:

$$\frac{du_e}{dx} = \frac{U_0}{L\left(1 - \frac{x}{L}\right)^2} \tag{4.20}$$

Substituting (4.19) into (4.1),

$$\theta^2 = \frac{a\nu}{U_0^b}\left(1 - \frac{x}{L}\right)^b \int_0^x U_0^{b-1}\left(1 - \frac{x}{L}\right)^{1-b} dx \tag{4.21}$$

$$= \frac{a\nu}{U_0^b}\left(1 - \frac{x}{L}\right)^b \frac{-L}{2-b}\left[\left(1 - \frac{x}{L}\right)^{-b+2}\right]_0^x \tag{4.22}$$

$$= \frac{a\nu L}{U_0(b-2)}\left(1 - \frac{x}{L}\right)^b \left(\left(1 - \frac{x}{L}\right)^{2-b} - 1\right) \tag{4.23}$$

As for λ, substitute (4.20) into (4.7) giving

$$\lambda = \frac{\theta^2}{U_0}\nu L\left(1 - \frac{x}{L}\right)^2 \tag{4.24}$$

Using the above relations, the plots for $\theta/L\sqrt{U_0 L/v}$, $\theta/x\sqrt{u_e x/v\,[L/(L-x)]}$ and $lambda$ are produced as shown below using values $a = 0.45$ and $b = 6$ and are compared with the exact data.

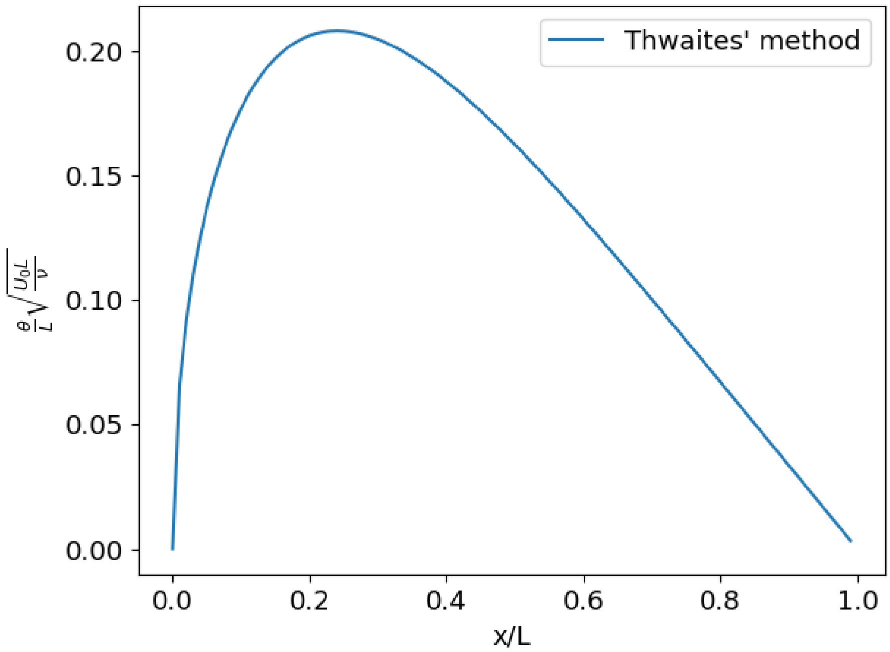

Fig. 5 a = 0.45 **and** b = 6

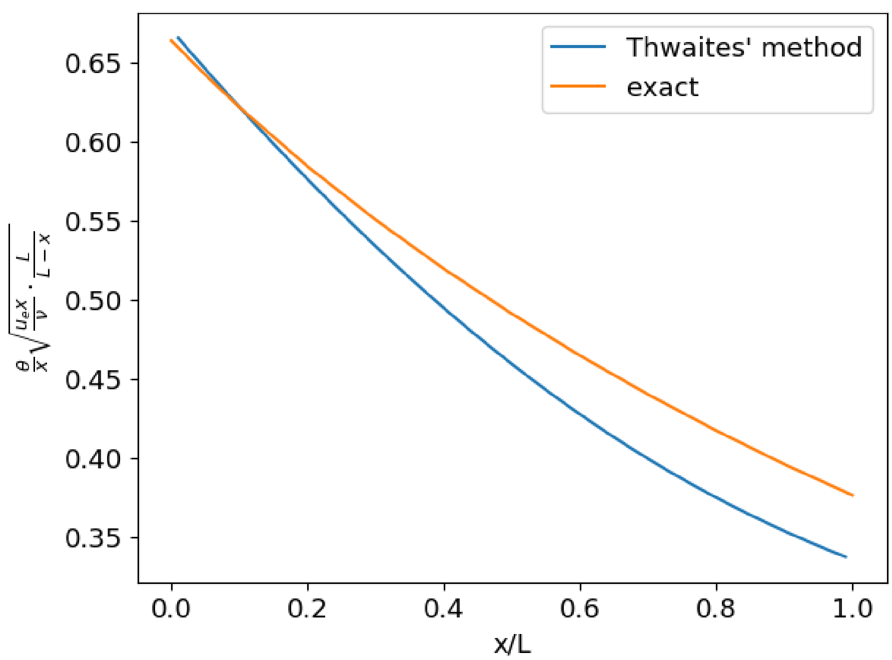

Fig. 6 a = 0.45 **and** b = 6

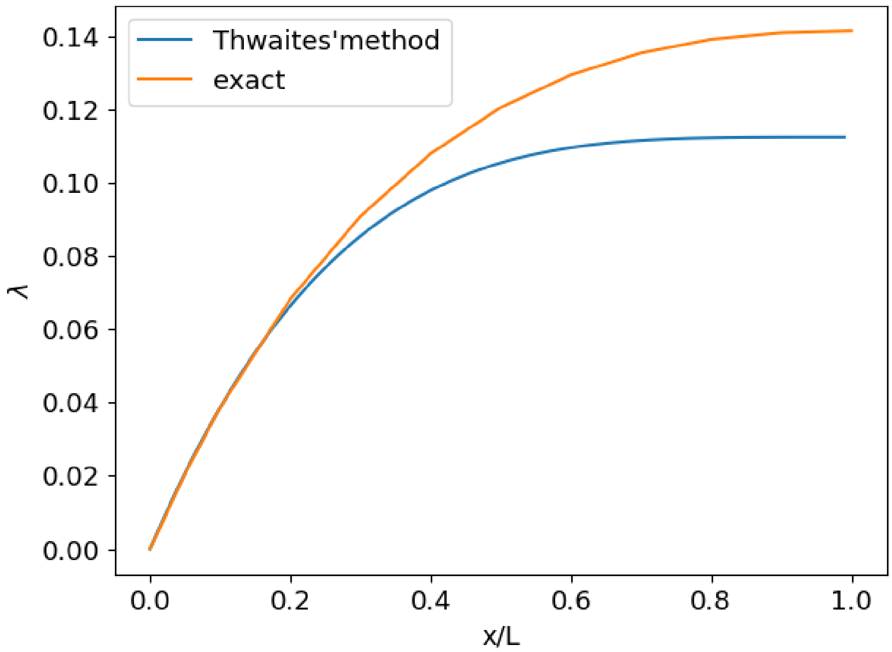

Fig. 7 $a = 0.45$ **and** $b = 6$

c.

Using the limit cases, we find new values for a and b. As for $m = 0$ which corresponds to the flat-plate boundary layer, for $\lambda = 0$,

$$\sqrt{a} = 0.664 \quad \text{(Blasius)} \tag{4.25}$$

$$\boxed{a = 0.4409} \tag{4.26}$$

For $m = -1$ which corresponds to the planar stagnation flow, using (4.17 and the exact data of $\lambda = 0.1415$, we have

$$\frac{a}{b-2} = 0.1415 \tag{4.27}$$

$$\boxed{b = 5.116} \tag{4.28}$$

We then achieve the new set of following plots

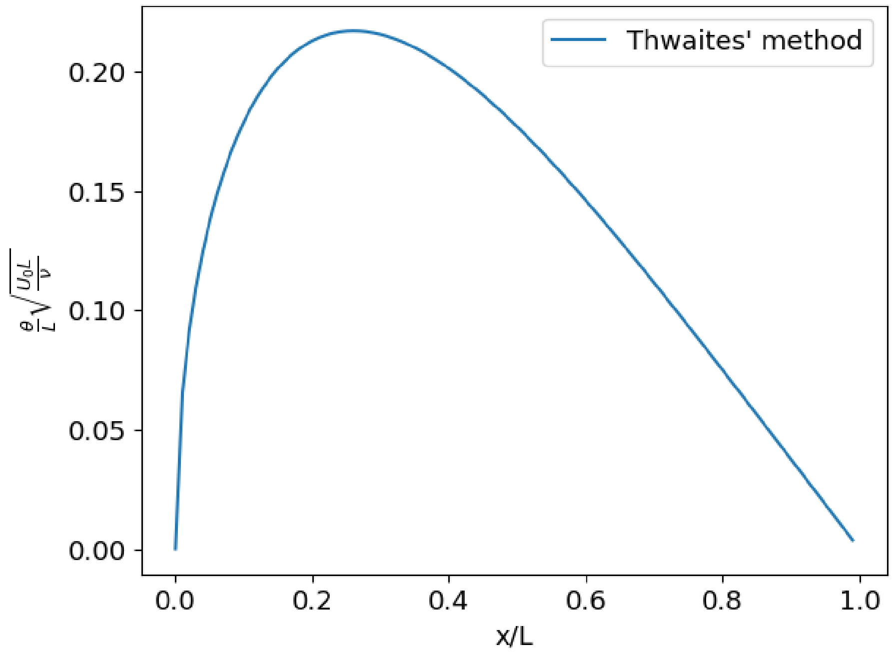

Fig. 8 a = 0.4409 **and** b = 5.116

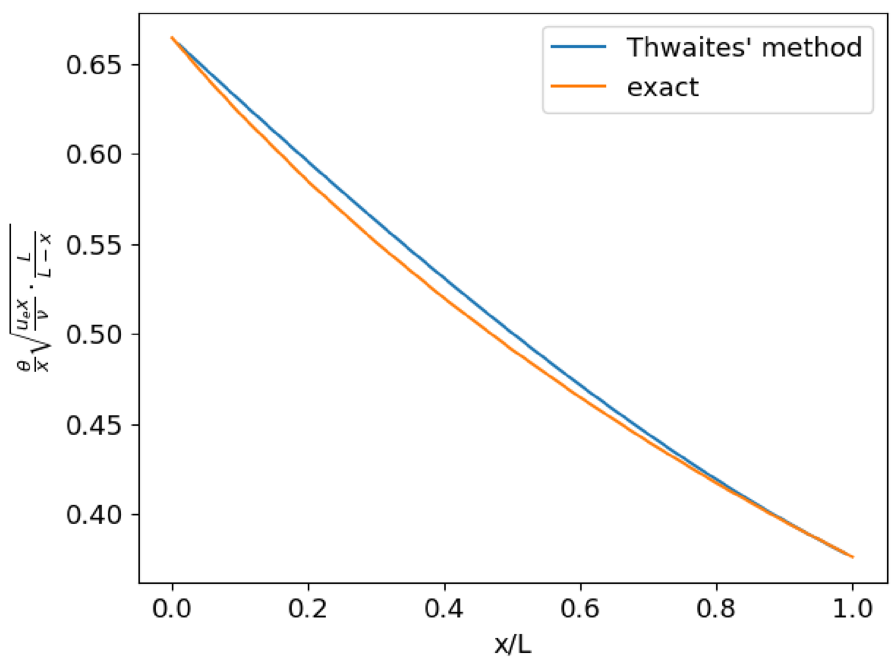

Fig. 9 a = 0.4409 **and** b = 5.116

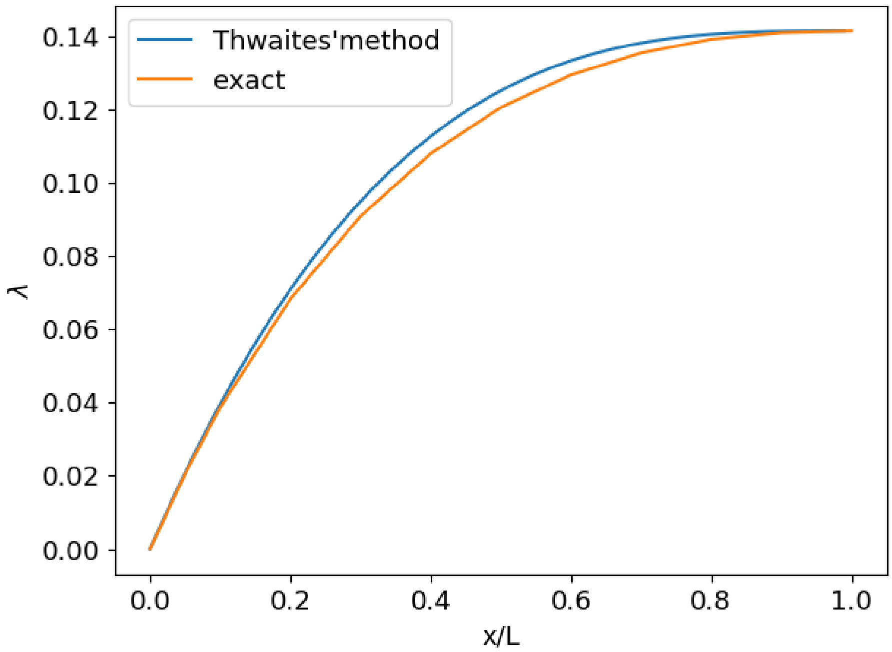

Fig. 10 a = 0.4409 **and** b = 5.116

It can be clearly observed that for Fig.9 and Fig.10, the limits coincides with the exact data and can be said to be improved from the previous set of plots.

5. E

*The exponentials are indicated by $\exp(x)$ instead of e^x to clearly visualise the superscripts.

The given governing momentum and energy equations of the adiabatic suction boundary layer are:

$$\rho v_w \frac{\partial u}{\partial y} = \mu \frac{\partial^2 u}{\partial y^2} \tag{5.1}$$

$$\rho c_p v_w \frac{\partial T}{\partial y} = \mu \left(\frac{\partial u}{\partial y}\right)^2 + k \frac{\partial^2 T}{\partial y^2} \tag{5.2}$$

where v_w is the constant transverse velocity and it is negative as the equations deal with suction.

The velocity profile is

$$\frac{u}{u_e} = 1 - \exp(-\eta) \tag{5.3}$$

where

$$\eta = -\frac{y v_w}{v} \tag{5.4}$$

Additionally the Prandtl number is defined as

$$\mathrm{Pr} = \frac{\mu c_p}{k} \tag{5.5}$$

$$= \frac{\rho \nu c_p}{k} \tag{5.6}$$

a.

From (5.2), we solve the following homogeneous equation:

$$k \frac{\partial^2 T}{\partial y^2} - \rho c_p v_w \frac{\partial T}{\partial y} = 0 \tag{5.7}$$

$$k r^2 - \rho c_p v_w r = 0 \tag{5.8}$$

$$r(kr - \rho c_p v_w) = 0 \tag{5.9}$$

$$r_1 = 0, \quad r_2 = \frac{\rho c_p v_w}{k} = \frac{\mathrm{Pr}\, v_w}{\nu} \tag{5.10}$$

The homogeneous solution, T_h, is then

$$\mathrm{T}_h = c_1 + c_2 \cdot \exp(r_2 y) \tag{5.11}$$

$$= c_1 + c_2 \cdot \exp\left(\frac{\mathrm{Pr}\, v_w}{\nu} y\right) \tag{5.12}$$

where c_1 and c_2 are constants

As for the particular solution, T_p, we first differentiate the velocity profile

$$\frac{du}{dy} = u_e \eta \cdot \exp(-\eta) \tag{5.13}$$

$$= -\frac{u_e v_w}{\nu} \cdot \exp\left(\frac{v_w}{\nu} y\right) \tag{5.14}$$

From (5.2), we have

$$k \frac{\partial^2 T}{\partial y^2} - \rho c_p v_w \frac{\partial T}{\partial y} = -\mu \left(\frac{u_e v_w}{\nu}\right)^2 \exp\left(\frac{2 v_w}{\nu} y\right) \tag{5.15}$$

The guess for T_p is then

$$T_p = c_3 \cdot \exp\left(\frac{2v_w}{\nu}y\right) \tag{5.16}$$

$$\frac{\partial T_p}{\partial y} = c_3\frac{2v_w}{\nu} \cdot \exp\left(\frac{2v_w}{\nu}y\right) \tag{5.17}$$

$$\frac{\partial^2 T_p}{\partial y^2} = c_3\left(\frac{2v_w}{\nu}\right)^2 \exp\left(\frac{2v_w}{\nu}y\right) \tag{5.18}$$

$$= 4c_3\left(\frac{v_w}{\nu}\right)^2 \exp\left(\frac{2v_w}{\nu}y\right) \tag{5.19}$$

where c_3 is a constant as well. Substituting these into (5.15), the equation is reduced to

$$c_3\left[4k\left(\frac{v_w}{\nu}\right)^2 - 2\rho c_p\frac{v_w^2}{\nu}\right] = -\mu\left(\frac{u_e^2 v_w}{\nu}\right) \tag{5.20}$$

$$c_3\left(4k - 2\rho c_p\nu\right) = -\mu u_e^2 \tag{5.21}$$

$$c_3 = \frac{-\mu u_e^2}{4k - 2\rho c_p \nu} \tag{5.22}$$

$$= \frac{-\rho\nu u_e^2}{2k(2-\mathrm{Pr})} \tag{5.23}$$

$$= \frac{-\mathrm{Pr}\, u_e^2}{2c_p(2-\mathrm{Pr})} \tag{5.24}$$

The solution for T is then

$$T = T_h + T_p \tag{5.25}$$

$$= c_1 + c_2 \cdot \exp\left(\frac{\mathrm{Pr}\, v_w}{\nu}y\right) - \frac{\mathrm{Pr}\, u_e^2}{2c_p(2-\mathrm{Pr})} \cdot \exp\left(\frac{2v_w}{\nu}y\right) \tag{5.26}$$

In order to solve for the unknown constants c_3 and c_4, we use the following boundary conditions:

$$\text{Adiabatic wall: } \left.\frac{\partial T}{\partial y}\right|_{y=0} = 0 \tag{5.27}$$

$$\text{External flow: } T(y \to \infty) = T_e \tag{5.28}$$

To evaluate the first boundary condition, we first differentiate (5.25).

$$\frac{\partial T}{\partial y} = c_2\mathrm{Pr}\,\frac{v_w}{\nu} \cdot \exp\left(\mathrm{Pr}\,\frac{v_w}{\nu}y\right) - \frac{\mathrm{Pr}\, u_e^2}{2c_p(2-\mathrm{Pr})}\left(\frac{2v_w}{\nu}\right) \cdot \exp\left(\frac{2v_w}{\nu}y\right) \tag{5.29}$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = c_2\mathrm{Pr}\,\frac{v_w}{\nu} - \frac{u_e^2 v_w \mathrm{Pr}}{\nu c_p(2-\mathrm{Pr})} = 0 \tag{5.30}$$

$$c_2 = \frac{u_e^2}{c_p(2-\mathrm{Pr})} \tag{5.31}$$

which gives

$$T = c_1 + \frac{u_e^2}{c_p(2-\mathrm{Pr})} \cdot \exp\left(\frac{\mathrm{Pr}\, v_w}{\nu}y\right) - \frac{\mathrm{Pr}\, u_e^2}{2c_p(2-\mathrm{Pr})} \cdot \exp\left(\frac{2v_w}{\nu}y\right) \tag{5.32}$$

Using the second boundary condition and the fact that $v_w < 0$,

$$\text{As} \qquad y \to \infty, \tag{5.33}$$

$$\exp\left(\frac{\mathrm{Pr}\, v_w}{\nu} y\right) \to 0 \tag{5.34}$$

$$\exp\left(\frac{2v_w}{\nu} y\right) \to 0 \tag{5.35}$$

giving

$$\mathrm{T}_e = c_1 \tag{5.36}$$

Thus (5.25) with the found constants is now

$$\mathrm{T} = \mathrm{T}_e + \frac{u_e^2}{c_p(2-\mathrm{Pr})} \cdot \exp\left(\frac{\mathrm{Pr}\, v_w}{\nu} y\right) - \frac{\mathrm{Pr}\, u_e^2}{2c_p(2-\mathrm{Pr})} \cdot \exp\left(\frac{2v_w}{\nu} y\right) \tag{5.37}$$

$$= \mathrm{T}_e + \frac{u_e^2}{c_p(2-\mathrm{Pr})} \left[\exp\left(\frac{\mathrm{Pr}\, v_w}{\nu} y\right) - \frac{\mathrm{Pr}}{2} \cdot \exp\left(\frac{2v_w}{\nu} y\right)\right] \tag{5.38}$$

Defining T as a function of η instead,

$$\boxed{\mathrm{T}(\eta) = \mathrm{T}_e + \frac{u_e^2}{c_p(2-\mathrm{Pr})} \left[\exp\left(-\eta \mathrm{Pr}\right) - \frac{\mathrm{Pr}}{2} \cdot \exp\left(-2\eta\right)\right]} \tag{5.39}$$

b.

The static temperature, T, and the total temperature, T_t, are related by the following equation

$$c_p \mathrm{T} + \frac{1}{2} u^2 = c_p \mathrm{T}_t \tag{5.40}$$

$$\mathrm{T}_t = \mathrm{T} + \frac{u^2}{2c_p} \tag{5.41}$$

To calculate u^2 we square (5.3)

$$u^2 = u_e^2 \left(\frac{u}{u_e}\right)^2 \tag{5.42}$$

$$= u_e^2 \left(1 - \exp\left(-\eta\right)\right)^2 \tag{5.43}$$

$$= u_e^2 \left(1 - 2 \cdot \exp\left(-\eta\right) + \exp\left(-2\eta\right)\right) \tag{5.44}$$

Substituting u^2 and T into (5.41),

$$\boxed{\mathrm{T}_t(\eta) = \mathrm{T}_e + \frac{u_e^2}{c_p(2-\mathrm{Pr})} \left[\exp\left(-\eta \mathrm{Pr}\right) - \frac{\mathrm{Pr}}{2} \cdot \exp\left(-2\eta\right)\right] + \frac{u_e^2}{2c_p} \left[1 - 2 \cdot \exp\left(-\eta\right) + \exp\left(-2\eta\right)\right]} \tag{5.45}$$

c.

Firstly for Pr = 1 we have

$$\boxed{\mathrm{T} = \mathrm{T}_e + \frac{u_e^2}{c_p} \left[\exp\left(-\eta\right) - \frac{1}{2} \cdot \exp\left(-2\eta\right)\right]} \tag{5.46}$$

$$\mathrm{T}_t = \mathrm{T}_e + \frac{u_e^2}{c_p} \left[\exp\left(-\eta\right) - \frac{1}{2} \cdot \exp\left(-2\eta\right)\right] + \frac{u_e^2}{2c_p} \left[1 - 2 \cdot \exp\left(-\eta\right) + \exp\left(-2\eta\right)\right] \tag{5.47}$$

$$\boxed{\mathrm{T}_t = \mathrm{T}_e + \frac{u_e^2}{c_p}} \tag{5.48}$$

which shows that the total temperature does not vary with the distance from the wall.

As for Pr = 2, since the denominator (2 − Pr) goes to 0, Pr is left as a variable in the equation and the limit of the equation as Pr goes to 2 is evaluated using Wolfram Alpha.

It was found that

$$\lim_{\mathrm{Pr}\to 2}\left[\frac{1}{(2-\mathrm{Pr})}\left(\exp\left(-\eta\mathrm{Pr}\right)-\frac{\mathrm{Pr}}{2}\cdot\exp\left(-2\eta\right)\right)\right]=\frac{1}{2}\cdot\exp\left(-2\eta\right)\left(1+2\eta\right) \tag{5.49}$$

which then gives the following values for static and total temperature as Pr → 2.

$$\boxed{\mathrm{T}=\mathrm{T}_e+\frac{1}{2}\cdot\exp\left(-2\eta\right)\left(1+2\eta\right)} \tag{5.50}$$

$$\boxed{\mathrm{T}_t=\mathrm{T}_e+\frac{1}{2}\cdot\exp\left(-2\eta\right)\left(1+2\eta\right)+\frac{u_e^2}{2c_p}\left[1-2\cdot\exp\left(-\eta\right)+\exp\left(-2\eta\right)\right]} \tag{5.51}$$

Here we can see that the total temperature does vary with η.

d.

As for the adiabatic wall, $T_0 = T_{aw}$ and hence the recovery factor, r, is defined as

$$r=\frac{\mathrm{T}_0-\mathrm{T}_e}{\frac{u_e^2}{2c_p}} \tag{5.52}$$

where

$$\mathrm{T}_0=\mathrm{T}(\eta=0)=\mathrm{T}_e+\frac{u_e^2}{c_p(2-\mathrm{Pr})}\left(1-\frac{\mathrm{Pr}}{2}\right) \tag{5.53}$$

Thus, the recovery factor, r, is then

$$r=\frac{2}{2-\mathrm{Pr}}\left(1-\frac{\mathrm{Pr}}{2}\right) \tag{5.54}$$

$$=\frac{2-\mathrm{Pr}}{2-\mathrm{Pr}} \tag{5.55}$$

$$\boxed{r=1} \tag{5.56}$$

It is observed that the recovery factor, $r = 1$, does not change for Pr = 2 compared to Pr = 1.

e.

The temperature profiles for static and total temperature are

$$\Theta(\eta)=\frac{T-T_e}{\frac{u_e^2}{2c_p}} \tag{5.57}$$

$$=\frac{2}{2-\mathrm{Pr}}\left(\exp\left(-\eta\mathrm{Pr}\right)-\frac{\mathrm{Pr}}{2}\cdot\exp\left(-2\eta\right)\right) \tag{5.58}$$

$$\Theta_t(\eta)=\frac{T_t-T_e}{\frac{u_e^2}{2c_p}} \tag{5.59}$$

$$=\frac{2}{2-\mathrm{Pr}}\left(\exp\left(-\eta\mathrm{Pr}\right)-\frac{\mathrm{Pr}}{2}\cdot\exp\left(-2\eta\right)\right)+1-2\cdot\exp\left(-\eta\right)+\exp\left(-2\eta\right) \tag{5.60}$$

where for Pr $\rightarrow$ 2,

$$\frac{2}{2-\mathrm{Pr}}\left(\exp\left(-\eta\mathrm{Pr}\right)-\frac{\mathrm{Pr}}{2}\cdot\exp\left(-2\eta\right)\right)=\exp\left(-2\eta\right)\left(1+2\eta\right) \tag{5.61}$$

with the limit found in (5.49). The following plots are then achieved.

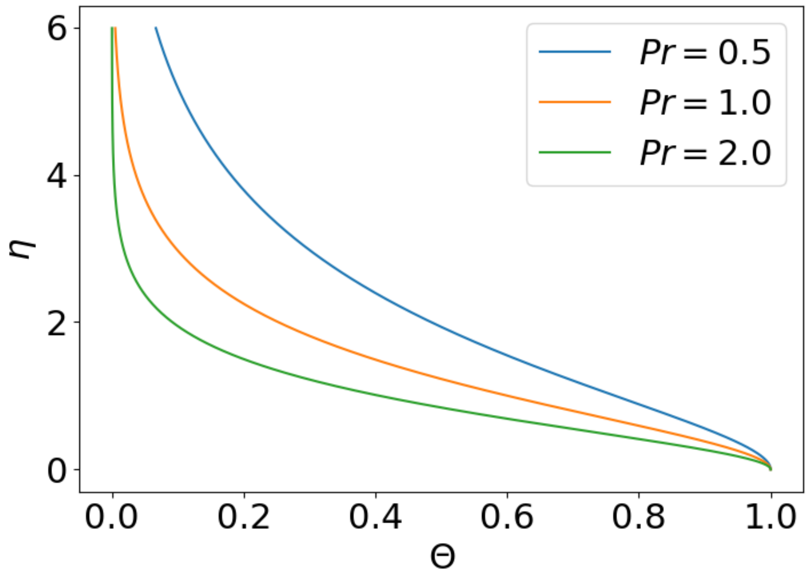

Fig. 11 Static temperature profile

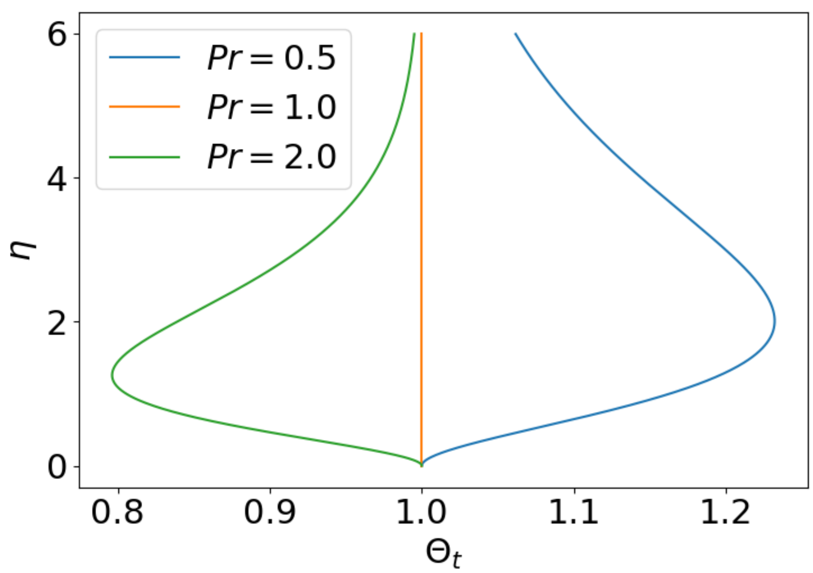

Fig. 12 Total temperature profile

6. A

Given are

$$\text{Point of instability: } Re_{\theta,\text{ crit}} = \exp(26.3 - 8H) \quad \text{(Wieghardt)} \tag{6.1}$$

$$\text{Point of transition: } Re_{\theta,\text{ trans}} = 2.9\left(Re_{x,\text{ trans}}\right)^{0.4} \quad \text{(Michel)} \tag{6.2}$$

for the approximate estimate of the locations of instability and transition points of a boundary layer. The external flow velocity is given as:

$$u_e(x) = U_0(1 + cx) \tag{6.3}$$

where $U_0 = 50\,m/s$, $v = 15{\times}10^{-}6\,m^2/s$ and $c = 0\,m^{-1}$ when pressure gradient is zero, $c = -0.05\,m^{-1}$ for adverse-pressure gradient and $c = 0.05\,m^{-1}$ for favourable-pressure gradient.

a.

The following are the equations for the momentum thickness, θ and the shape factor H, as defined by Thwaites [1].

$$\theta^2(x) = \frac{0.45v}{u_e^6}\int_0^x u_e^5 dx \tag{6.4}$$

$$H(z) = 2 + 4.14z - 83.5z^2 + 854z^3 - 3337z^4 + 4576z^5 \tag{6.5}$$

$$z(\lambda) = 0.25 - \lambda \tag{6.6}$$

$$\lambda(x) = \frac{\theta^2}{v}\frac{du_e}{dx} \tag{6.7}$$

The following plots of θ and H were produced as shown in Fig.13 and 14.

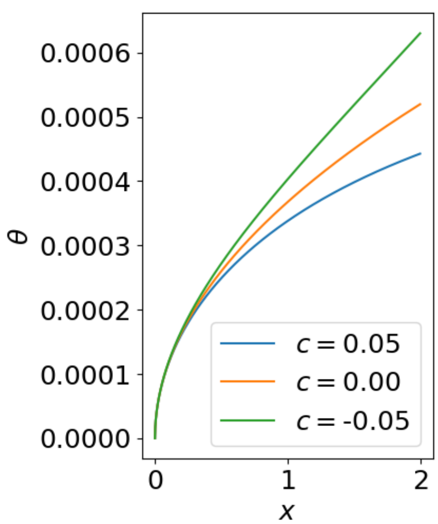

Fig. 13 Momentum Thickness, θ, for 3 cases of pressure-gradient

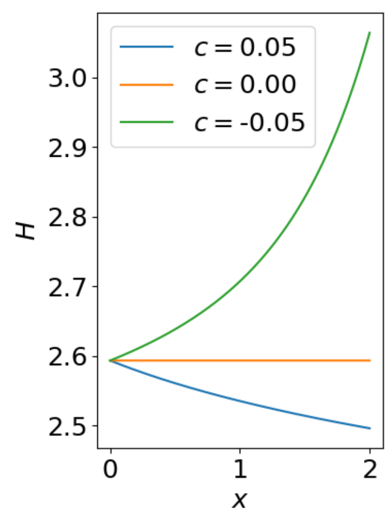

Fig. 14 Shape factor, H, for 3 cases of pressure-gradient

b.

Using the given empirical relations for point of instability and transition in (6.1) and (6.2), the following plots were obtained.

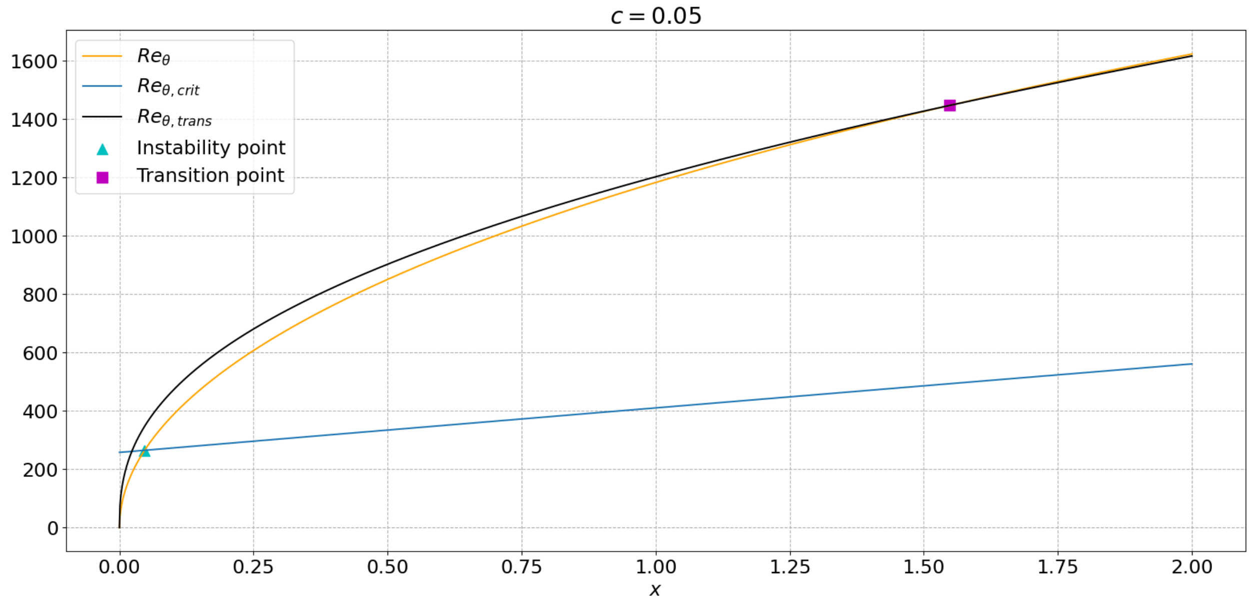

Fig. 15 Development of Re_θ, $Re_{\theta,crit}$ and $Re_{\theta,trans}$ wih respect to x for $c = +0.05$

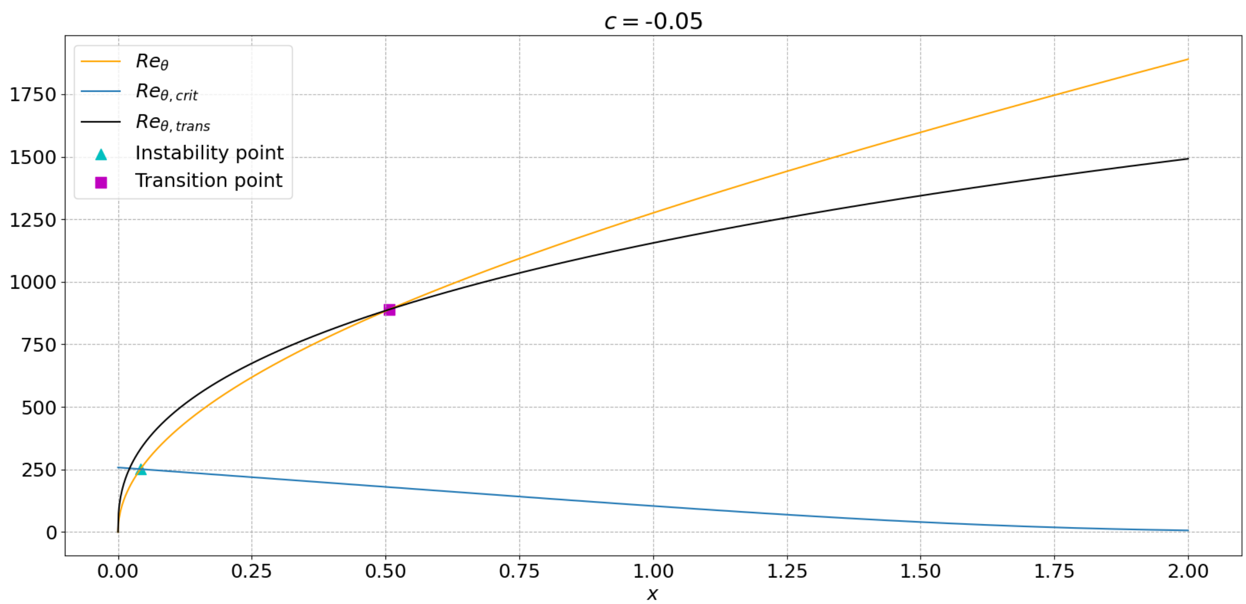

Fig. 16 Development of Re_θ, $Re_{\theta,crit}$ and $Re_{\theta,trans}$ wih respect to x for $c = -0.05$

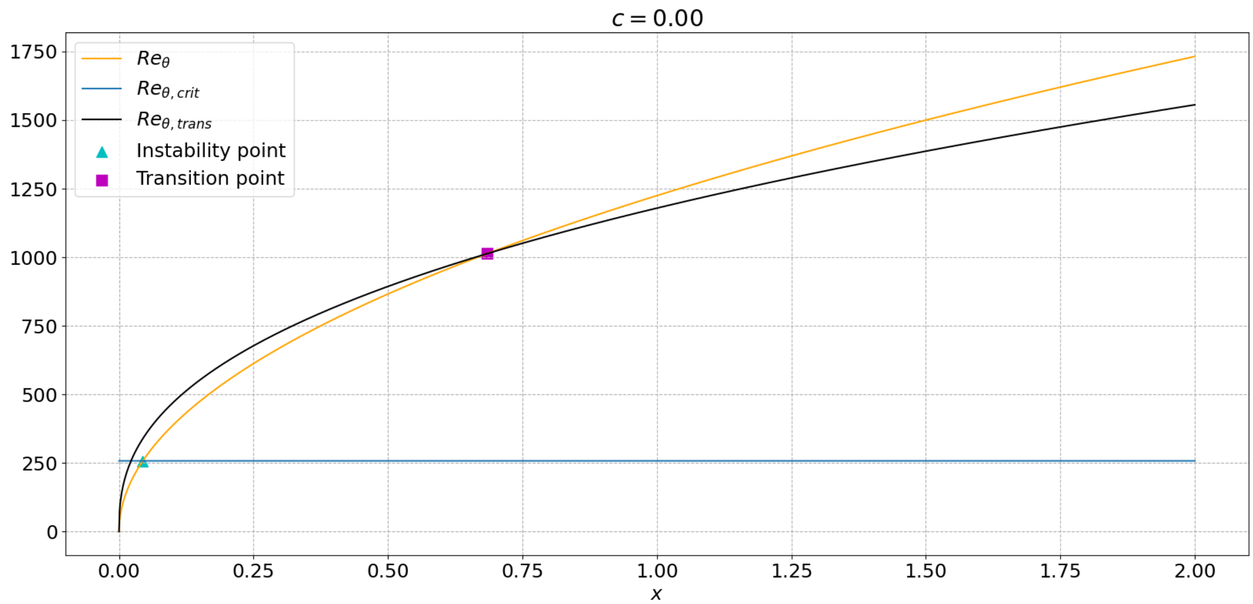

Fig. 17 Development of Re_θ, $Re_{\theta,crit}$ and $Re_{\theta,trans}$ wih respect to x for $c = 0.0$

The following table records the values of the instability and transition points for the 3 different pressure gradients.

Table 1 Instability and transition points for $c = 0.05, -0.05$ annd 0

	Instability point		Transition point	
	x	Re	x	Re
$c = 0.05$	0.047	265.05	1.548	1446.79
$c = -0.05$	0.042	251.39	0.507	889.45
$c = 0.0$	0.044	256.90	0.684	1012.92

c.

From the Table 1, it can realised that

- Favorable pressure gradient ($c = 0.05$) has both the instability and transition points furthest away from the leading edge.
- Adverse pressure gradient ($c = -0.05$) has both the instability and transition points closes to the leading edge.
- For flat-plate boundary layer ($c = 0$), the instability and transition points are in between the values for favorable and adverse pressure gradients. However, they are not exactly in the middle. They are closer to the values of adverse pressure gradient.

7. C

a.

The given data points plotted in a semi-logarithmic plot is shown in Fig.18.

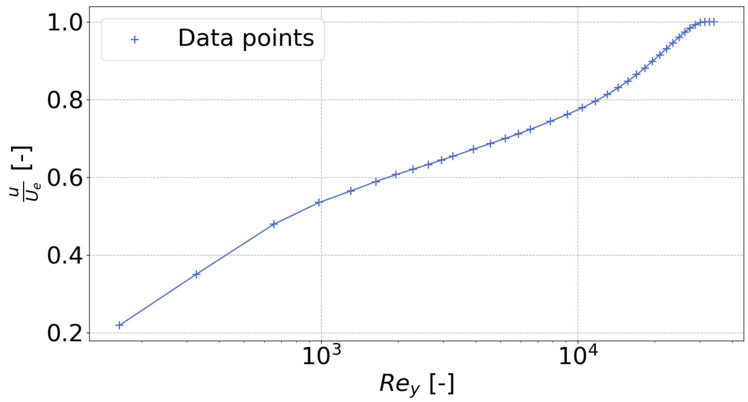

Fig. 18 Semi-logarithmic velocity profile plot of given data points, $\frac{u}{U_e}$ v.s. Re_y

u is the local mean velocity in the boundary layer, U_e is the velocity in the external flow and Re_y is the Reynolds number which is defined as

$$Re_y = \frac{yU_e}{\nu} \tag{7.1}$$

where y is the distance from the wall and ν is the kinematic viscosity.

b.

The velocity profile for the logarithmic overlap layer is defined as

$$u^+ = \frac{1}{\kappa} ln(y^+) + B \tag{7.2}$$

where κ is the von Karman constant that has a value of 0.41 and B is an integration constant that has a value of 5.0 while u^+ and y^+ are defined as the following.

$$u^+ = \frac{u}{v^*} \tag{7.3}$$

$$y^+ = \frac{yv^*}{\nu} \tag{7.4}$$

where v^* is the wall friction velocity. Substituting these into (7.2), we have

$$\frac{u}{v^*} = \frac{1}{\kappa} ln\left(\frac{yv^*}{\nu}\right) + B \tag{7.5}$$

The given relation for wall-friction velocity is

$$\frac{v^*}{U_e} = \sqrt{\frac{C_f}{2}} \tag{7.6}$$

where C_f is the skin friction coefficient. Substituting this back into (7.5) and using (7.1), we have

$$\frac{u}{u_e} = \sqrt{\frac{C_f}{2}} \left[\frac{1}{\kappa} ln\left(Re_y \sqrt{\frac{C_f}{2}}\right) + B\right] \tag{7.7}$$

This function is then used to fit to the data point. In order to account for only the overlap layer, only 6^{th} till the 16^{th} data points were used in fitting the curve. The skin friction coefficient was then found to be

$$\boxed{C_f = 0.0030339 = 3.303 \times 10^{-3}} \tag{7.8}$$

with a standard deviation of 4.65379×10^{-6}. The extrapolated fitted curve is plotted together with the data points as shown in Fig.19 below.

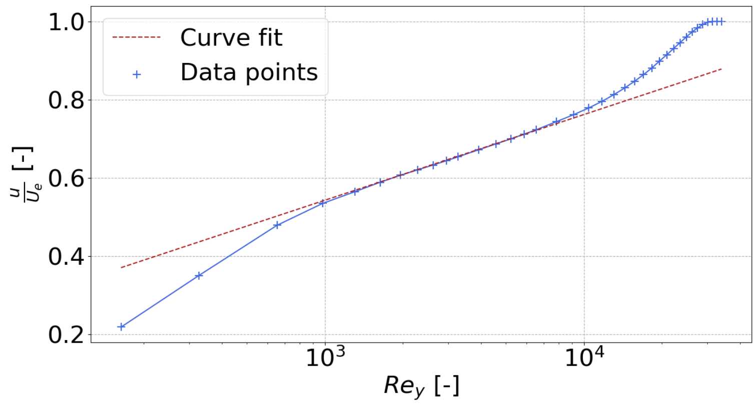

Fig. 19 Semi-logarithmic velocity profile plot of given data points and the fitted curve

c.

It is given that the strength of the wake component of the velocity profile is measured as the maximum difference between u and the law of the wall expression, so the fitted curve. The maximum difference of the two curves are pointed out in Fig.20 below.

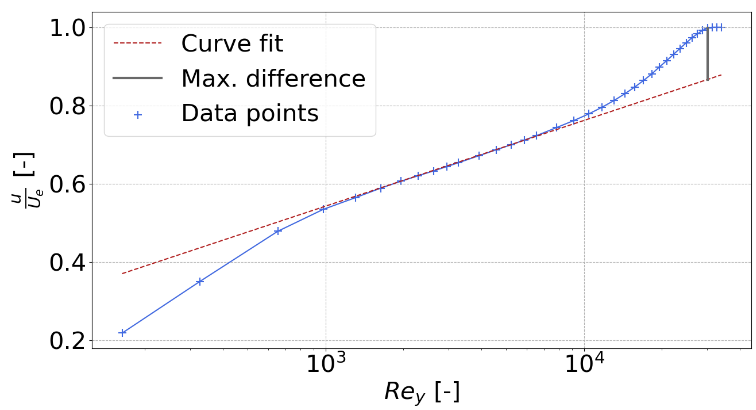

Fig. 20 Semi-logarithmic velocity profile plot of given data points and the fitted curve with max. difference

The maximum difference is found to be 0.13135. Since the $y - axis$ of the plot shown in Fig.20 is u/U_e, this value of maximum difference was multiplied with U_e to result in the following value.

$$\boxed{\text{Strength of wake component of velocity profile} = 1.2878} \tag{7.9}$$

8. C

a.

The given Spalding's implicit expression of the velocity profile in the entire wall layer is as follows:

$$y^+ = u^+ + e^{-\kappa B}\left(e^{\kappa u^+} - 1 - \kappa u^+ - \frac{(\kappa u^+)^2}{2} - \frac{(\kappa u^+)^3}{6}\right) \tag{8.1}$$

The Taylor expansion of du^+/dy^+ for small y^+ is:

$$\frac{du^+}{dy^+} \approx \left.\frac{du^+}{dy^+}\right|_{y^+=0} + \left.\frac{d^2u^+}{dy^{+2}}\right|_{y^+=0} y^+ + \frac{1}{2}\left.\frac{d^3u^+}{dy^{+3}}\right|_{y^+=0} y^{+2} + \frac{1}{6}\left.\frac{d^4u^+}{dy^{+4}}\right|_{y^+=0} y^{+3} + \cdots \tag{8.2}$$

The four derivative terms on the right hand side (RHS) of (8.2) are found using implicit differentiation of (8.1) and substitution of $u^+ = 0$ which is implied by the small y^+.

Differentiating (8.1) with respect to y^+ for the first RHS term of (8.2):

$$\begin{aligned} 1 &= \frac{du^+}{dy^+} + e^{-\kappa B}\left(\kappa e^{\kappa u^+}\frac{du^+}{dy^+} - \kappa\frac{du^+}{dy^+} - \kappa^2 u^+\frac{du^+}{dy^+} - \frac{\kappa^3 u^{+2}}{2}\frac{du^+}{dy^+}\right) \\ &= \frac{du^+}{dy^+}\left[1 + \kappa e^{-\kappa B}\left(e^{\kappa u^+} - 1 - \kappa u^+ - \frac{\kappa^2 u^{+2}}{2}\right)\right] \end{aligned} \tag{8.3}$$

Substituting $u^+ = 0$ since y^+ is small which implies that u^+ is small,

$$\begin{aligned} 1 &= \left.\frac{du^+}{dy^+}\right|_{y^+=0}\left[1 + \kappa e^{-\kappa B}(1 - 1 - 0 - 0)\right] \\ 1 &= \left.\frac{du^+}{dy^+}\right|_{y^+=0}[1 + 0] \\ \left.\frac{du^+}{dy^+}\right|_{y^+=0} &= 1 \end{aligned} \tag{8.4}$$

We then differentiate (8.3) with respect to y^+ for the second RHS term of (8.2):

$$\begin{aligned} 0 &= \frac{d^2u^+}{dy^{+2}}\left[1 + \kappa e^{-\kappa B}\left(e^{\kappa u^+} - 1 - \kappa u^+ - \frac{\kappa^2 u^{+2}}{2}\right)\right] + \frac{du^+}{dy^+}\left[\kappa e^{-\kappa B}\left(\kappa e^{\kappa u^+}\frac{du^+}{dy^+} - \kappa\frac{du^+}{dy^+} - \kappa^2 u^+\frac{du^+}{dy^+}\right)\right] \\ &= \frac{d^2u^+}{dy^{+2}}\left[1 + \kappa e^{-\kappa B}\left(e^{\kappa u^+} - 1 - \kappa u^+ - \frac{\kappa^2 u^{+2}}{2}\right)\right] + \frac{du^+}{dy^+}\left[\kappa^2 e^{-\kappa B}\left(e^{\kappa u^+}\frac{du^+}{dy^+} - \frac{du^+}{dy^+} - \kappa u^+\frac{du^+}{dy^+}\right)\right] \end{aligned} \tag{8.5}$$

Substituting $u^+ = 0$ once again,

$$\begin{aligned} 0 &= \left.\frac{d^2u^+}{dy^{+2}}\right|_{y^+=0}\left[1 + \kappa e^{-\kappa B}(1 - 1 - 0 - 0)\right] + 1\left[\kappa^2 e^{-\kappa B}(1 - 1 - 0)\right] \\ 0 &= \left.\frac{d^2u^+}{dy^{+2}}\right|_{y^+=0}[1 + 0] + 0 \\ \left.\frac{d^2u^+}{dy^{+2}}\right|_{y^+=0} &= 0 \end{aligned} \tag{8.6}$$

Since the second derivative of u^+ is found to equal to 0 in (8.6), the terms that are to be multiplied with it are merely expressed as '$\cdots$' for conciseness in further derivations.

To find the third RHS term of (8.2), we differentiate (8.5) with respect to y^+:

$$
\begin{aligned}
0 = {} & \frac{d^3u^+}{dy^{+3}}\left[1+\kappa e^{-\kappa B}\left(e^{\kappa u^+}-1-\kappa u^+-\frac{\kappa^2u^{+2}}{2}\right)\right]+\frac{d^2u^+}{dy^{+2}}\left(\cdots\right)+\frac{d^2u^+}{dy^{+2}}\left(\cdots\right) \\
& +\frac{du^+}{dy^+}\left\{\kappa^2e^{-\kappa B}\left[\kappa e^{\kappa u^+}\left(\frac{du^+}{dy^+}\right)^2+e^{\kappa u^+}\frac{d^2u^+}{dy^{+2}}-\frac{d^2u^+}{dy^{+2}}-\kappa\left(\frac{du^+}{dy^+}\right)^2-\kappa u^+\frac{d^2u^+}{dy^{+2}}\right]\right\}
\end{aligned}
\tag{8.7}
$$

Substituting $u^+ = 0$,

$$
\begin{aligned}
0 &= \left.\frac{d^3u^+}{dy^{+3}}\right|_{y^+=0}\left[1+\kappa e^{-\kappa B}\left(1-1-0-0\right)\right]+0+0+1\left\{\kappa^2e^{-\kappa B}\left[\kappa+0-0-\kappa-0\right]\right\} \\
\left.\frac{d^3u^+}{dy^{+3}}\right|_{y^+=0} &= 0
\end{aligned}
\tag{8.8}
$$

Similarly, since the third derivative of u^+ is found to equal to 0 in (8.8), the terms that are to be multiplied with it are merely expressed as '$\cdots$' for conciseness in further derivations.

Lastly, we find the fourth RHS term of (8.2) by differentiating (8.7) with respect to y^+:

$$
\begin{aligned}
0 = {} & \frac{d^4u^+}{dy^{+4}}\left[1+\kappa e^{-\kappa B}\left(e^{\kappa u^+}-1-\kappa u^+-\frac{\kappa^2u^{+2}}{2}\right)\right]+\frac{d^3u^+}{dy^{+3}}\left(\cdots\right)+\frac{d^3u^+}{dy^{+3}}\left(\cdots\right)+\frac{d^2u^+}{dy^{+2}}\left(\cdots\right)+\frac{d^3u^+}{dy^{+3}}\left(\cdots\right) \\
& +\frac{d^2u^+}{dy^{+2}}\left(\cdots\right)+\frac{d^2u^+}{dy^{+2}}\left(\cdots\right)+\frac{du^+}{dy^+}\left\{\kappa^2e^{-\kappa B}\left[\kappa^2e^{\kappa u^+}\left(\frac{du^+}{dy^+}\right)^3+2\kappa e^{\kappa u^+}\frac{du^+}{dy^+}\cdot\frac{d^2u^+}{dy^{+2}}+\kappa e^{\kappa u^+}\frac{du^+}{dy^+}\cdot\frac{d^2u^+}{dy^{+2}}\right.\right. \\
& \left.\left.+e^{\kappa u^+}\frac{d^3u^+}{dy^{+3}}-\frac{d^3u^+}{dy^{+3}}-2\kappa\frac{du^+}{dy^+}\cdot\frac{d^2u^+}{dy^{+2}}-\kappa\frac{du^+}{dy^+}\cdot\frac{d^2u^+}{dy^{+2}}-\kappa u^+\ \frac{d^3u^+}{dy^{+3}}\right]\right\}
\end{aligned}
\tag{8.9}
$$

Substituting $u^+ = 0$,

$$
\begin{aligned}
0 = \left.\frac{d^4u^+}{dy^{+4}}\right|_{y^+=0}&\left[1+\kappa e^{-\kappa B}\left(1-1-0-0\right)\right]+0+0+0+0+0+0 \quad +1\left\{\kappa^2e^{-\kappa B}\left[\kappa^2+0+0+0-0-0-0-0\right]\right\} \\
0 &= \left.\frac{d^4u^+}{dy^{+4}}\right|_{y^+=0}+\kappa^4e^{-\kappa B} \\
\left.\frac{d^4u^+}{dy^{+4}}\right|_{y^+=0} &= -\kappa^4e^{-\kappa B}
\end{aligned}
\tag{8.10}
$$

Substituting the values of the derivatives found at $y^+ = 0$ and thus at $u^+ = 0$ in (8.4), (8.6), (8.8) and (8.10) into the Taylor expansion equation in (8.2), we get

$$
\begin{aligned}
\frac{du^+}{dy^+} &\approx 1+0\cdot y^+ +0\cdot y^{+2}+\frac{1}{6}\left(-\kappa^4e^{-\kappa B}\right)y^{+3}+\cdots \\
&= 1-\frac{1}{6}\kappa^4e^{-\kappa B}y^{+3}+\cdots
\end{aligned}
\tag{8.11}
$$

while the given expression with constant A is

$$
\frac{du^+}{dy^+} = 1-Ay^{+3}+\ldots \tag{8.12}
$$

Comparing (8.11) and (8.12), we realise that A is

$$
\boxed{A = \frac{1}{6}\kappa^4e^{-\kappa B}} \tag{8.13}
$$

Additionally, with the given standard values of $\kappa = 0.41$ and $B = 5.0$, we get that the value of A is

$$
\boxed{A = 6.0629\times10^{-4}} \tag{8.14}
$$

b.

The following are the given scaling of the fluctuating velocity components:

$$\frac{u'(t)}{v^*} \approx A_1 y^+ \phi_1(t) \tag{8.15}$$
$$\frac{v'(t)}{v^*} \approx A_2 y^{+2} \phi_2(t) \tag{8.16}$$
$$\frac{w'(t)}{v^*} \approx A_3 y^+ \phi_3(t) \tag{8.17}$$

Additionally it is given that

$$\overline{\phi_1(t)^2} = \overline{\phi_2(t)^2} = \overline{\phi_3(t)^2} = 1 \tag{8.18}$$

Starting with the derivation for $\overline{u'^2}/v^{*2}$ using (8.15) and (8.18),

$$\begin{aligned}
\frac{u'(t)}{v^*} &= A_1 y^+ \phi_1(t) \\
\frac{u'(t)^2}{v^{*2}} &= A_1^2 y^{+2} \phi_1(t)^2 \\
\frac{\overline{u'(t)^2}}{v^{*2}} &= \overline{A_1^2 y^{+2} \phi_1(t)^2} \\
\frac{\overline{u'(t)^2}}{v^{*2}} &= A_1^2 y^{+2} \overline{\phi_1(t)^2} \\
&\boxed{\frac{\overline{u'(t)^2}}{v^{*2}} = A_1^2 y^{+2}}
\end{aligned} \tag{8.19}$$

Similarly for $\overline{v'^2}/v^{*2}$, the derivation is as follows:

$$\begin{aligned}
\frac{v'(t)}{v^*} &= A_2 y^{+2} \phi_2(t) \\
\frac{v'(t)^2}{v^{*2}} &= A_2^2 y^{+4} \phi_2(t)^2 \\
\frac{\overline{v'(t)^2}}{v^{*2}} &= \overline{A_2^2 y^{+4} \phi_2(t)^2} \\
\frac{\overline{v'(t)^2}}{v^{*2}} &= A_2^2 y^{+4} \overline{\phi_2(t)^2} \\
&\boxed{\frac{\overline{v'(t)^2}}{v^{*2}} = A_2^2 y^{+4}}
\end{aligned} \tag{8.20}$$

Lastly, for the derivation of $\overline{w'^2}/v^{*2}$,

$$\begin{aligned}
\frac{w'(t)}{v^*} &= A_3 y^+ \phi_3(t) \\
\frac{w'(t)^2}{v^{*2}} &= A_3^2 y^{+2} \phi_3(t)^2 \\
\frac{\overline{w'(t)^2}}{v^{*2}} &= \overline{A_3^2 y^{+2} \phi_3(t)^2} \\
\frac{\overline{w'(t)^2}}{v^{*2}} &= A_3^2 y^{+2} \overline{\phi_3(t)^2} \\
&\boxed{\frac{\overline{w'(t)^2}}{v^{*2}} = A_3^2 y^{+2}}
\end{aligned} \tag{8.21}$$

The kinetic energy, K, consists of mean flow component and also the turbulent component. It is defined as follows:

$$\begin{aligned} K &= K_{\text{mean}} + K_{\text{turb}} \\ K &= \frac{1}{2}\left(\overline{u}^2 + \overline{v}^2 + \overline{w}^2\right) + \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) \end{aligned} \tag{8.22}$$

Thus, using the derived expressions for $\overline{u'^2}/v^{*2}$, $\overline{v'^2}/v^{*2}$ and $\overline{w'^2}/v^{*2}$ as shown in (8.19), (8.20) and (8.21) and the definition of kinetic energy in (8.22), the leading-order approximation for the near-wall kinetic energy, K/v^{*2}, is determined as follows:

$$\boxed{\frac{K}{v^{*2}} = \frac{1}{2v^{*2}}\left(\overline{u}^2 + \overline{v}^2 + \overline{w}^2\right) + \frac{y^{+2}}{2}\left(A_1^2 + A_2^2 y^{+2} + A_3^2\right)} \tag{8.23}$$

c.

The total stress which is assumed to be constant, expressed as τ_w close to the wall, is a sum of viscous stress and turbulent stress as shown below.

$$\begin{aligned} \text{total stress} &= \text{viscous stress} + \text{turbulent streses} \\ \tau_w &= \tau_{\text{visc}} + \tau_{\text{turb}} \end{aligned} \tag{8.24}$$

Since

$$\frac{\tau_{\text{visc}}}{\tau_w} = \frac{\partial u^+}{\partial y^+} \tag{8.25}$$

$$\tau\text{turb} = -\rho\overline{u'v'} \tag{8.26}$$

(8.24) can be expressed as

$$1 = \frac{\partial u^+}{\partial y^+} - \frac{\rho}{\tau_w}\overline{u'v'} \tag{8.27}$$

Furthermore, using the given relation of $v^* = \sqrt{\tau_w/\rho}$, we have

$$\begin{aligned} 1 &= \frac{\partial u^+}{\partial y^+} - \frac{\rho}{v^{*2}\rho}\overline{u'v'} \\ 1 &= \frac{\partial u^+}{\partial y^+} - \frac{\overline{u'v'}}{v^{*2}} \\ &\boxed{-\frac{\overline{u'v'}}{v^{*2}} = 1 - \frac{\partial u^+}{\partial y^+}} \end{aligned} \tag{8.28}$$

which shows that the expression for the turbulent shear stress can be related directly to the velocity gradient.

Substituting (8.12) into (8.28),

$$\begin{aligned} -\frac{\overline{u'v'}}{v^{*2}} &= 1 - \left(1 - Ay^{+3}\right) \\ &\boxed{-\frac{\overline{u'v'}}{v^{*2}} = Ay^{+3}} \end{aligned} \tag{8.29}$$

which is in agreement with the following given turbulent shear stress near the wall

$$\frac{-\overline{u'v'}}{v^{*2}} \approx A_{12}y^{+3} \tag{8.30}$$

which implies that

$$\begin{gathered} A_{12} y^{+3} = A y^{+3} \\ \boxed{A_{12} = A} \end{gathered} \tag{8.31}$$

It was found from (8.14) that $A = 6.0629 \times 10^{-4}$. Additionally, it is given that $A_{12} = 6 \times 10^{-4}$. Thus, it can be said that the results are in agreement.

Given equation for the degree of correlation between u-fluctuation and v-fluctations is

$$R_{uv} = \frac{\overline{u'v'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}}} \tag{8.32}$$

For the numerator, it can be derived from (8.30) as follows

$$\overline{u'v'} = -v^{*2} A_{12} y^{+3} \tag{8.33}$$

As for the first term of the denominator, it can be derived from (8.19) as follows

$$\begin{aligned} \frac{\overline{u'^2}}{v^{*2}} &= A_1^2 y^{+2} \\ \overline{u'^2} &= v^{*2} A_1^2 y^{+2} \\ \sqrt{\overline{u'^2}} &= v^{*} A_1 y^{+} \end{aligned} \tag{8.34}$$

Similarly, the second term of the denominator can be derived from (8.20):

$$\begin{aligned} \frac{\overline{v'^2}}{v^{*2}} &= A_2^2 y^{+4} \\ \overline{v'^2} &= v^{*2} A_2^2 y^{+4} \\ \sqrt{\overline{v'^2}} &= v^{*} A_2 y^{+2} \end{aligned} \tag{8.35}$$

Substituting (8.33), (8.34) and (8.35) into (8.32), we have

$$\begin{gathered} R_{uv} = \frac{-v^{*2} A_{12} y^{+3}}{v^{*} A_1 y^{+} \cdot v^{*} A_2 y^{+2}} \\ \boxed{R_{uv} = \frac{-A_{12}}{A_1 \cdot A_2}} \end{gathered} \tag{8.36}$$

in which y^+ is cancelled out and is no longer a variable for R_{uv}. Thus, this correlation does not vanish in the limit when $y \to 0$. The given values are $A_1 \approx 0.35$, $A_2 \approx 0.01$, $A_3 \approx 0.20$ and $A_{12} \approx 6 \times 10^{-4}$. Substituting these values into (8.36), we have

$$\begin{gathered} R_{uv} = \frac{-6 \times 10^{-4}}{0.35 \times 0.01} \\ \boxed{R_{uv} = -0.17143} \end{gathered} \tag{8.37}$$

d.

Given is the definition of the turbulent (viscous) dissipation:

$$\varepsilon = \nu \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}} \tag{8.38}$$

where in this case the u_i' represents u', v' and w' while x_j represents just y since the derivatives are only a function of y and not x nor z. For the relationship between y^+ and y, we have

$$y^+ = \frac{v^*}{v} y \tag{8.39}$$

Starting from u' that is defined in (8.15) and (8.39),

$$\begin{aligned}
u' &= A_1 v^* y^+ \phi_1 \\
&= A_1 \frac{v^{*2}}{v} y \phi_1 \\
\frac{\partial u'}{\partial y} &= A_1 \frac{v^{*2}}{v} \phi_1 \\
\overline{\frac{\partial u'}{\partial y} \frac{\partial u'}{\partial y}} &= A_1^2 \frac{v^{*4}}{v^2} \overline{\phi_1^2} \\
&= A_1^2 \frac{v^{*4}}{v^2}
\end{aligned} \tag{8.40}$$

Similarly for v' defined in (8.16), we have

$$\begin{aligned}
v' &= A_2 v^* y^{+2} \phi_2 \\
&= A_2 \frac{v^{*3}}{v^2} y^2 \phi_2 \\
\frac{\partial v'}{\partial y} &= 2A_2 \frac{v^{*3}}{v^2} y \phi_2 \\
&= 2A_2 \frac{v^{*2}}{v} y^+ \phi_2 \\
\overline{\frac{\partial v'}{\partial y} \frac{\partial v'}{\partial y}} &= 4A_2^2 \frac{v^{*4}}{v^2} y^{+2} \overline{\phi_2^2} \\
&= 4A_2^2 \frac{v^{*4}}{v^2} y^{+2}
\end{aligned} \tag{8.41}$$

Lastly for w' defined in (8.17),

$$\begin{aligned}
w' &= A_3 v^* y^+ \phi_3 \\
&= A_3 \frac{v^{*2}}{v} y \phi_3 \\
\frac{\partial w'}{\partial y} &= A_3 \frac{v^{*2}}{v} \phi_3 \\
\overline{\frac{\partial w'}{\partial y} \frac{\partial w'}{\partial y}} &= A_3^2 \frac{v^{*4}}{v^2} \overline{\phi_3^2} \\
&= A_3^2 \frac{v^{*4}}{v^2}
\end{aligned} \tag{8.42}$$

Substituting (8.40), (8.41) and (8.42) into (8.38),

$$\begin{aligned}
\varepsilon &= v \overline{\frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j}} \\
&= v \left(A_1^2 \frac{v^{*4}}{v^2} + 4A_2^2 \frac{v^{*4}}{v^2} y^{+2} + A_3^2 \frac{v^{*4}}{v^2} \right) \\
&\boxed{\varepsilon = \frac{v^{*4}}{v} \left(A_1^2 + 4A_2^2 y^{+2} + A_3^2 \right)}
\end{aligned} \tag{8.43}$$

where it can be verified that the dissipation scales with v^{*4}/v.

Introducing the scaled production $\varepsilon^{+} = \varepsilon v/v^{*4}$, we can rewrite (8.43) as

$$\varepsilon^{+} = A_1^2 + 4A_2^2 y^{+2} + A_3^2 \tag{8.44}$$

At the wall,

$$\begin{aligned}
\varepsilon_{wall} &= \lim_{y^{+}\to 0}\left[\frac{v^{*4}}{v}\left(A_1^2 + 4A_2^2 y^{+2} + A_3^2\right)\right] \\
&= \frac{v^{*4}}{v}\left(A_1^2 + A_3^2\right) \\
\varepsilon_{wall}^{+} &= \lim_{y^{+}\to 0}\left(A_1^2 + 4A_2^2 y^{+2} + A_3^2\right) \\
&= A_1^2 + A_3^2 \\
&\boxed{\varepsilon_{wall}^{+} = 0.1625 \neq 0}
\end{aligned} \tag{8.45}$$

where it is shown that the dissipation does not go to zero at the wall itself.

Given is the viscous diffusion term

$$T_{K,\mathrm{visc}} = v\frac{\partial^2 K}{\partial x_j \partial x_j} \tag{8.46}$$

where K is the turbulent kinetic energy, K_{turb}. Likewise, the x_j represents y only in this case. From (8.23) we have that the K_{turb} is

$$\begin{aligned}
K_{\mathrm{turb}} &= \frac{v^{*2}y^{+2}}{2}\left(A_1^2 + A_2^2 y^{+2} + A_3^2\right) \\
&= \frac{v^{*2}}{2}\left(A_1^2 y^{+2} + A_2^2 y^{+4} + A_3^2 y^{+2}\right) \\
&= \frac{v^{*2}}{2}\left[A_1^2\left(\frac{v^*}{v}\right)^2 y^2 + A_2^2\left(\frac{v^*}{v}\right)^4 y^4 + A_3^2\left(\frac{v^*}{v}\right)^2 y^2\right] \\
&= \frac{v^{*4}}{2v^2}\left[A_1^2 y^2 + A_2^2\left(\frac{v^*}{v}\right)^2 y^4 + A_3^2 y^2\right]
\end{aligned} \tag{8.47}$$

Differentiating once with respect to y, we have

$$\begin{aligned}
\frac{\partial K_{\mathrm{turb}}}{\partial y} &= \frac{v^{*4}}{2v^2}\left[2A_1^2 y + 4A_2^2\left(\frac{v^*}{v}\right)^2 y^3 + 2A_3^2 y\right] \\
&= \frac{v^{*4}}{v^2}\left[A_1^2 y + 2A_2^2\left(\frac{v^*}{v}\right)^2 y^3 + A_3^2 y\right]
\end{aligned}$$

Differentiating once more with respect to y,

$$\begin{aligned}
\frac{\partial^2 K_{\mathrm{turb}}}{\partial y \partial y} &= \frac{v^{*4}}{v^2}\left[A_1^2 + 6A_2^2\left(\frac{v^*}{v}\right)^2 y^2 + A_3^2\right] \\
\frac{\partial^2 K_{\mathrm{turb}}}{\partial y \partial y} &= \frac{v^{*4}}{v^2}\left(A_1^2 + 6A_2^2 y^{+2} + A_3^2\right)
\end{aligned} \tag{8.48}$$

Substituting (8.48) into (8.46),

$$\boxed{T_{K,\mathrm{visc}} = \frac{v^{*4}}{v}\left(A_1^2 + 6A_2^2 y^{+2} + A_3^2\right)} \tag{8.49}$$

For the behaviour of (8.49) at the wall, we have

$$\left(T_{K,\text{visc}}\right)_{wall} = \lim_{y^+\to 0}\left[\frac{v^{*4}}{v}\left(A_1^2 + 6A_2^2 y^{+2} + A_3^2\right)\right]$$

$$\boxed{\left(T_{K,\text{visc}}\right)_{wall} = \frac{v^{*4}}{v}\left(A_1^2 + A_3^2\right)} \tag{8.50}$$

where it can be observed that (8.50) is equivalent to (8.45).

It is given that an extra term, D, is used to describe the behaviour of dissipation in the viscous sublayer near the wall which is subtracted from the initial dissipation, ε, resulting in $\tilde{\varepsilon}$. It is also said that $\tilde{\varepsilon}$ goes to zero at the wall.

$$\tilde{\varepsilon} = \varepsilon - D \tag{8.51}$$

$$\tilde{\varepsilon}_{wall} = \varepsilon_{wall} - D_{wall} \tag{8.52}$$

The two popular choices of the additional term D are

$$D_1 = 2v\frac{K}{y^2} \tag{8.53}$$

$$D_2 = 2v\left(\frac{\partial\sqrt{K}}{\partial y}\right)^2 \tag{8.54}$$

Starting with the evaluation of the first choice, D_1, using (8.47) and also its behaviour at the wall we have

$$\begin{aligned}
D_1 &= 2v\frac{K_{\text{turb}}}{y^2} \\
&= \frac{2v}{y^2}\cdot\frac{v^{*4}}{2v^2}\left[A_1^2 y^2 + A_2^2\left(\frac{v^*}{v}\right)^2 y^4 + A_3^2 y^2\right] \\
&= \frac{v^{*4}}{v}\left[A_1^2 + A_2^2\left(\frac{v^*}{v}\right)^2 y^2 + A_3^2\right] \\
D_{1,wall} &= \lim_{y^+\to 0}\left\{\frac{v^{*4}}{v}\left[A_1^2 + A_2^2\left(\frac{v^*}{v}\right)^2 y^2 + A_3^2\right]\right\}
\end{aligned}$$

$$\boxed{D_{1,wall} = \frac{v^{*4}}{v}\left(A_1^2 + A_3^2\right)} \tag{8.55}$$

Substituting (8.45) and (8.55) into 8.52,

$$\tilde{\varepsilon}_{wall} = \frac{v^{*4}}{v}\left(A_1^2 + A_3^2\right) - \frac{v^{*4}}{v}\left(A_1^2 + A_3^2\right)$$

$$\boxed{\tilde{\varepsilon}_{wall} = 0}$$

where it is proved that for $D = D_1$, $\tilde{\varepsilon}$ goes to zero at the wall.

Similarly for the evaluation of D_2, we have

$$
\begin{aligned}
K_{\text{turb}} &= \frac{v^{*4}}{2v^2}\left[A_1^2 y^2 + A_2^2\left(\frac{v^*}{v}\right)^2 y^4 + A_3^2 y^2\right] \\
\sqrt{K_{\text{turb}}} &= \frac{v^{*2}}{\sqrt{2}v}\left[A_1^2 y^2 + A_2^2\left(\frac{v^*}{v}\right)^2 y^4 + A_3^2 y^2\right]^{\frac{1}{2}} \\
\frac{\partial\sqrt{K_{\text{turb}}}}{\partial y} &= \frac{1}{2}\frac{v^{*2}}{\sqrt{2}v}\left[A_1^2 y^2 + A_2^2\left(\frac{v^*}{v}\right)^2 y^4 + A_3^2 y^2\right]^{-\frac{1}{2}}\left(2A_1^2 y + 4A_2^2\left(\frac{v^*}{v}\right)^2 y^3 + 2A_3^2 y\right) \\
\left(\frac{\partial\sqrt{K_{\text{turb}}}}{\partial y}\right)^2 &= \frac{v^{*4}}{8v^2}\frac{\left(2A_1^2 y + 4A_2^2\left(\frac{v^*}{v}\right)^2 y^3 + 2A_3^2 y\right)^2}{A_1^2 y^2 + A_2^2\left(\frac{v^*}{v}\right)^2 y^4 + A_3^2 y^2} \\
&= \frac{v^{*4}}{2v^2}\frac{\left(A_1^2 + 2A_2^2\left(\frac{v^*}{v}\right)^2 y^2 + A_3^2\right)^2}{A_1^2 + A_2^2\left(\frac{v^*}{v}\right)^2 y^2 + A_3^2} \\
2v\left(\frac{\partial\sqrt{K_{\text{turb}}}}{\partial y}\right)^2 &= \frac{v^{*4}}{v}\frac{\left(A_1^2 + 2A_2^2\left(\frac{v^*}{v}\right)^2 y^2 + A_3^2\right)^2}{A_1^2 + A_2^2\left(\frac{v^*}{v}\right)^2 y^2 + A_3^2} \\
&\boxed{D_2 = \frac{v^{*4}}{v}\frac{\left(A_1^2 + 2A_2^2 y^{+2} + A_3^2\right)^2}{A_1^2 + A_2^2 y^{+2} + A_3^2}}
\end{aligned}
$$

At the wall, D_2 has the following behaviour

$$
\begin{aligned}
D_{2,wall} &= \lim_{y^+\to 0}\left[\frac{v^{*4}}{v}\frac{\left(A_1^2 + 2A_2^2 y^{+2} + A_3^2\right)^2}{A_1^2 + A_2^2 y^{+2} + A_3^2}\right] \\
&= \frac{v^{*4}}{v}\frac{\left(A_1^2 + A_3^2\right)^2}{A_1^2 + A_3^2} \\
&= \frac{v^{*4}}{v}\left(A_1^2 + A_3^2\right) \qquad (8.56)
\end{aligned}
$$

Substituting (8.45) and (8.56) into (8.52), we again have

$$
\begin{aligned}
\tilde{\varepsilon}_{wall} &= \frac{v^{*4}}{v}\left(A_1^2 + A_3^2\right) - \frac{v^{*4}}{v}\left(A_1^2 + A_3^2\right) \\
&\boxed{\tilde{\varepsilon}_{wall} = 0}
\end{aligned}
$$

where it is proved that for $D = D_2$, $\tilde{\varepsilon}$ goes to zero at the wall.

References

[1] White, F. M., *Viscous Fluid Flow*, 3rd ed., McGraw Hill, 2006.