Delft University of Technology

Advanced Numerical Methods - Take-home-exam 5

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Contents

1. Exact solution

In this section, the exact solutions for the Cauchy Problem of both the advection and acoustic problems are presented. The function $f(x)$ is to be used as part of both the problems in different ways thus its definition will first be presented for reference later in the section. $f(x)$ is defined as

$$f(x) = \begin{cases} \frac{1-\cos(\pi x)}{2} & : x \in [2,4], \\ 1 & : x \in [6,8], \\ 0 & : \text{otherwise.} \end{cases} \tag{1.1}$$

1 a. Advection problem

The given advection problem is

$$q_t + \bar{u} q_x = 0$$

where $\bar{u} = 1$ which makes the equation

$$q_t + q_x = 0. \tag{1.2}$$

The initial condition for this problem is

$$q(x,0) = \mathring{q}(x) = f(x) \tag{1.3}$$

where $f(x)$ has already been defined in (1.1).

As a first step, we let the solution be

$$\begin{aligned} q(x,t) &= q(x - \bar{u}t, 0) \\ &= \mathring{q}(x-t) \\ &= \mathring{q}(\xi) \end{aligned} \tag{1.4}$$

where $\xi(x,t) = x - t$ and check if it satisfies the initial condition given in (1.3). The solution for $t = 0$ is

$$\mathring{q}(\xi(x,0)) = \mathring{q}(x)$$

which satisfies the initial condition. We then check if it satisfies the problem itself by deriving q_t and q_x which are

$$\begin{aligned} q_t &= \frac{d\mathring{q}(\xi)}{d\xi} \cdot \frac{d\xi}{dt} \\ &= \frac{d\mathring{q}(\xi)}{d\xi} \cdot (-1) \\ &= -\frac{d\mathring{q}(\xi)}{d\xi} \qquad \text{and} \end{aligned} \tag{1.5}$$

$$\begin{aligned} q_x &= \frac{d\mathring{q}(\xi)}{d\xi} \cdot \frac{d\xi}{dx} \\ &= \frac{d\mathring{q}(\xi)}{d\xi} \cdot (1) \\ &= \frac{d\mathring{q}(\xi)}{d\xi} \end{aligned} \tag{1.6}$$

respectively. Substituting (1.5) and (1.6) into the partial differential equation (PDE), we get

$$-\frac{d\mathring{q}(\xi)}{d\xi} + \frac{d\mathring{q}(\xi)}{d\xi} = 0 \tag{1.7}$$

which satisfies the advection problem. Since the solution stated in (1.4) satisfies both the initial condition and the PDE, the solution of the advection problem can be said to be

$$\mathring{q}(x-t) = f(x-t)$$

where $f(x)$ is as shown in (1.1).

1 b. Acoustic problem

The acoustic problem is defined as

$$\begin{aligned} p_t - K_0 u_x &= 0 \\ \rho_0 u_t - p_x &= 0 \end{aligned}$$

where $K_0 = 4$ and $\rho_0 = 1$ which leads to the set of equations being

$$\begin{aligned} p_t - 4u_x &= 0 \\ u_t - p_x &= 0. \end{aligned}$$

The initial conditions for this problem are

$$\begin{aligned} u(x, 0) &= 1 \\ p(x, 0) &= f(x) \end{aligned} \tag{1.8}$$

where $f(x)$ has already been defined in (1.1). Letting $q = \begin{bmatrix} p \\ u \end{bmatrix}$, the problem can be expressed as

$$q_t + Aq_x = 0 \tag{1.9}$$

where the matrix A is

$$A = \begin{bmatrix} 0 & -4 \\ -1 & 0 \end{bmatrix}$$

The decoupled problem is then

$$\begin{aligned} w^1_t + cw^1_x &= 0 \\ w^2_t - cw^2_x &= 0 \end{aligned} \tag{1.10}$$

where c is a constant that can be deduced from the eigenvalues of matrix A.

First, the eigenvalues of matrix A are calculated as

$$\begin{aligned} |A - \lambda I| &= 0 \\ \lambda_{1,2} &= \pm 2 \end{aligned}$$

thus giving $c = 2$ for (1.10) and the corresponding eigenvectors are

$$r^1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$r^2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus, the leftward and rightward acoustic waves are

$$\begin{aligned} q(x, t) &= \mathring{w}^1(x + 2t)r^1 \\ q(x, t) &= \mathring{w}^2(x - 2t)r^2 \end{aligned}$$

respectively. The general solution is then the superposition of these waves and it is expressed as

$$q(x, t) = \mathring{w}^1(x + 2t)r^1 + \mathring{w}^2(x - 2t)r^2. \tag{1.11}$$

The initial condition of this solution can then be set as

$$
\begin{aligned}
q(x,0) &= \mathring{q}(x) \\
&= \begin{bmatrix} \mathring{p}(x) \\ \mathring{u}(x) \end{bmatrix} \\
&= \mathring{w}^1(x) r^1 + \mathring{w}^2(x) r^2.
\end{aligned}
$$

To express $\mathring{w}^1$ and $\mathring{w}^2$ in terms of p and u, matrix R needs to be setup whereby

$$
\begin{aligned}
R\mathring{\bar{w}}(x) &= \mathring{q}(x) \\
\mathring{\bar{w}}(x) &= R^{-1}\mathring{q}(x).
\end{aligned} \tag{1.12}
$$

The matrix R is defined as

$$
\begin{aligned}
R &= \begin{bmatrix} r^1 & r^2 \end{bmatrix} \\
&= \begin{bmatrix} -2 & 2 \\ 1 & 1 \end{bmatrix}.
\end{aligned}
$$

(1.12) can then be expanded into

$$
\mathring{\bar{w}}(x) = \frac{1}{4}\begin{bmatrix} -1 & 2 \\ 1 & 2 \end{bmatrix}\mathring{q}(x)
$$

$$
\mathring{w}^1(x) = -\frac{1}{4}\mathring{p}(x) + \frac{1}{2}\mathring{u}(x) \tag{1.13}
$$

$$
\mathring{w}^1(x) = \frac{1}{4}\mathring{p}(x) + \frac{1}{2}\mathring{u}(x) \tag{1.14}
$$

Substituting (1.13) and (1.14) into (1.11) and applying the initial conditions in (1.8), we have

$$
\begin{aligned}
p(x,t) &= \frac{1}{2}\mathring{p}(x+2t) - \mathring{u}(x+2t) + \frac{1}{2}\mathring{p}(x-2t) + \mathring{u}(x-2t) \\
&= \frac{1}{2}\left[f(x+2t) + f(x-2t)\right] \\
u(x,t) &= -\frac{1}{4}\mathring{p}(x+2t) + \frac{1}{2}\mathring{u}(x+2t) + \frac{1}{4}\mathring{p}(x-2t) + \frac{1}{2}\mathring{u}(x-2t) \\
&= -\frac{1}{4}\left[f(x+2t) - f(x-2t)\right] + 1
\end{aligned}
$$

where function $f(x)$ is defined in (1.1).

2. Stability analysis of method II for the advection problem

In this section, the stability of method II (Beam-Warming) in the 1-norm and 2-norm for the advection equation is investigated.

2 a. 2-norm stability

Just for 2-norm analysis, the capital letter 'I' will be used to indicate the spatial grid index. For the Beam-Warming method applied on the advection equation, we have

$$\begin{aligned} Q_I^{n+1} &= Q_I^n - \frac{\bar{u}\Delta t}{2\Delta x}\left(3Q_I^n - 4Q_{I-1}^n + Q_{I-2}^n\right) + \frac{1}{2}\left(\frac{\bar{u}\Delta t}{\Delta x}\right)^2\left(Q_I^n - 2Q_{I-1}^n + Q_{I-2}^n\right) \\ &= Q_I^n\left(1 - \frac{3}{2}\nu\right) + Q_{I-1}^n\left(2\nu - \nu^2\right) + Q_{I-2}^n\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right), \end{aligned} \tag{2.1}$$

where $\nu = \bar{u}\Delta t/\Delta x$, the Courant–Friedrichs–Lewy (CFL) number. Now, taking $Q_I^n = e^{i\xi I\Delta x}$ in which i is the imaginary number, ξ is the wave number, the above equation can be further expanded into

$$\begin{aligned} Q_I^{n+1} &= e^{i\xi I\Delta x}\left(1 - \frac{3}{2}\nu\right) + e^{i\xi(I-1)\Delta x}\left(2\nu - \nu^2\right) + e^{i\xi(I-2)\Delta x}\left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) \\ &= \left[1 - \frac{3}{2}\nu + \frac{\nu^2}{2} + (2\nu - \nu^2)e^{-i\xi\Delta x} + \left(\frac{\nu^2}{2} - \frac{\nu}{2}\right)e^{-2i\xi\Delta x}\right]e^{i\xi I\Delta x}. \end{aligned} \tag{2.2}$$

From the above derivation, the amplification g can be extracted and it is

$$g(\xi, \Delta x, \Delta t) = 1 - \frac{3}{2}\nu + \frac{\nu^2}{2} + (2\nu - \nu^2)e^{-i\xi\Delta x} + \left(\frac{\nu^2}{2} - \frac{\nu}{2}\right)e^{-2i\xi\Delta x}. \tag{2.3}$$

The condition for 2-norm stability is that

$$|g| \leq 1 \ \ \forall \ \ \xi.$$

Thus, for $\nu \in [0, 1]$ that suffices the CFL condition and $\xi \in [-100, 100]$, an arbitrary domain for testing, the values of $|g|$ have been plotted in Figure 1 and it can be clearly observed that for various combinations of ν and ξ, the magnitude $|g|$ remained equal to or less than 1 thus, satisfying the 2-norm stability condition. It is to be noted that many of the curves in Figure 1 overlap each other and thus only a few curves are visible even though a much greater number of ξ values have been computed for.

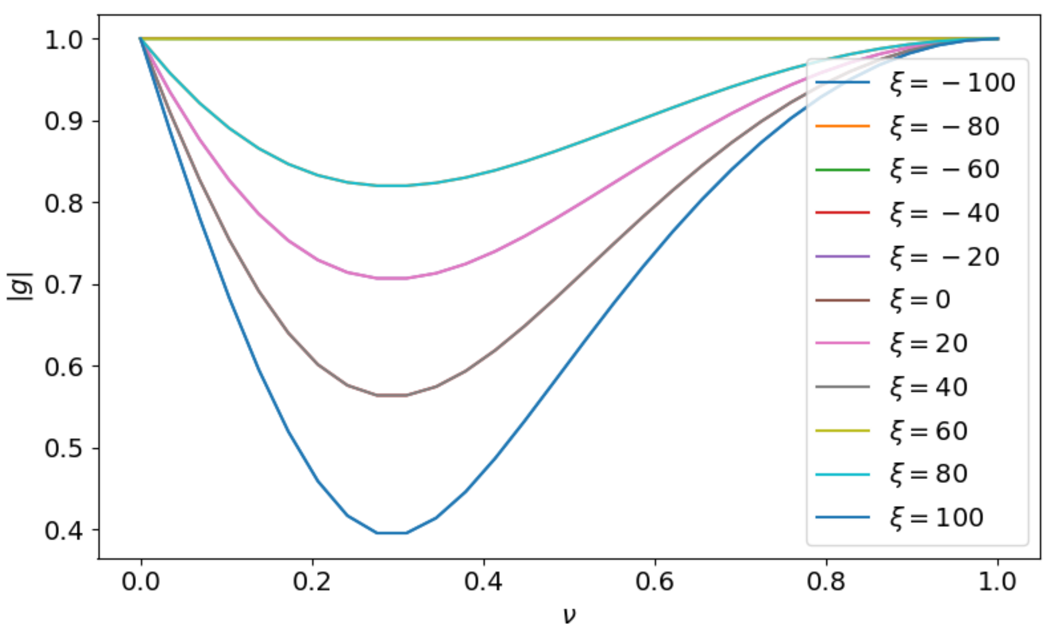


Fig. 1 $|g|$ **for various values of** $\nu \in [0, 1]$ **and** $\xi \in [-100, 100]$

2 b. 1-norm stability

Switching back to conventional small letter 'i' for spatial grid index, just like in (2.1), we again have Beam-Warming method applied on the advection equation as

$$Q_i^{n+1} = Q_i^n \left(1 - \frac{3}{2}\nu\right) + Q_{i-1}^n \left(2\nu - \nu^2\right) + Q_{i-2}^n \left(\frac{\nu^2}{2} - \frac{\nu}{2}\right). \tag{2.4}$$

We then represent

$$\left\|Q^{n+1}\right\|_1 = \Delta x \sum_i \left|Q_i^{n+1}\right|$$

in which we can substitute (2.4) to get

$$\begin{aligned}
\left\|Q^{n+1}\right\|_1 &= \Delta x \sum_i \left|Q_i^n \left(1 - \frac{3}{2}\nu\right) + Q_{i-1}^n \left(2\nu - \nu^2\right) + Q_{i-2}^n \left(\frac{\nu^2}{2} - \frac{\nu}{2}\right)\right| \\
&= \Delta x \sum_i \left[\left|Q_i^n\right| \left(1 - \frac{3}{2}\nu\right) + \left|Q_{i-1}^n\right| \left(2\nu - \nu^2\right) + \left|Q_{i-2}^n\right| \left(\frac{\nu^2}{2} - \frac{\nu}{2}\right)\right] \\
&= \|Q^n\|_1 \left(1 - \frac{3}{2}\nu\right) + \|Q^n\|_1 \left(2\nu - \nu^2\right) + \|Q^n\|_1 \left(\frac{\nu^2}{2} - \frac{\nu}{2}\right) \\
&= \left(1 - \frac{3}{2}\nu + \frac{\nu^2}{2} + 2\nu - \nu^2 + \frac{\nu^2}{2} - \frac{\nu}{2}\right) \|Q^n\|_1 \\
&= \|Q^n\|_1
\end{aligned}$$

which proves that the method is stable for $\nu \in [0, 1]$

3. Local truncation error investigation of method II for the advection problem

Again, the Beam-Warming method is applied onto the advection problem can be expressed in a different form as

$$\begin{aligned} q(x_i, t^{n+1}) = q(x_i, t^n) - \frac{\bar{u}\Delta t}{2\Delta x}\left[3q(x_i, t^n) - 4q(x_{i-1}, t^n) + q(x_{i-2}, t^n)\right] \\ + \frac{1}{2}\left(\frac{\bar{u}\Delta t}{2\Delta x}\right)^2\left[q(x_i, t^n) - 2q(x_{i-1}, t^n) + q(x_{i-2}, t^n)\right]. \end{aligned} \tag{3.1}$$

The local truncation error τ can then be defined as

$$\begin{aligned} \tau^n = \frac{1}{\Delta t}\Bigg\{ q(x_i, t^n) - \frac{\bar{u}\Delta t}{2\Delta x}\left[3q(x_i, t^n) - 4q(x_{i-1}, t^n) + q(x_{i-2}, t^n)\right] \\ + \frac{1}{2}\left(\frac{\bar{u}\Delta t}{2\Delta x}\right)^2\left[q(x_i, t^n) - 2q(x_{i-1}, t^n) + q(x_{i-2}, t^n)\right] - q(x_i, t^{n+1})\Bigg\}. \end{aligned} \tag{3.2}$$

Assuming q is smooth, we then apply Taylor expansions to the parameters in the above equation and they are

$$q(x_{i-1}, t^n) = q(x_i, t^n) - \Delta x q_x(x_i, t^n) + O(\Delta x^2), \tag{3.3}$$

$$q(x_{i-2}, t^n) = q(x_i, t^n) - 2\Delta x q_x(x_i, t^n) + O(\Delta x^2), \text{ and} \tag{3.4}$$

$$q(x_i, t^{n+1}) = q(x_i, t^n) + \Delta t q_x(x_i, t^n) + O(\Delta t^2). \tag{3.5}$$

Substituting (3.3), (3.4) and (3.5) into (3.2), we get

$$\tau^n = -(q_t + \bar{u}q_x) + O(\Delta x). \tag{3.6}$$

Since the advection equation states that

$$q_t + \bar{u}q_x = 0,$$

(3.6) can be further simplied into

$$\tau^n = O(\Delta x)$$

and this is the local truncation error of Beam-Warming method for the advection problem.

4. Implementation of all 3 methods for both the advection and acoustic problem on a uniform grid

In this section, the solution for both the advection and acoustic problems that were computed on a uniform grid using three different methods: upwind (Godunov), Beam-Warming and van Leer are presented. The solution will be analysed for 1, 2 and 5 periods which corresponds to $T/5$, $2T/5$ and T respectively. For each of them, the CFL number will be varied to investigate the stability, dependence on the CFL number, wiggles and accuracy.

4 a. Advection problem

After 1 period Starting off with solution after 1 period, the plots are presented in Figure 2, Figure 3 and Figure 4.

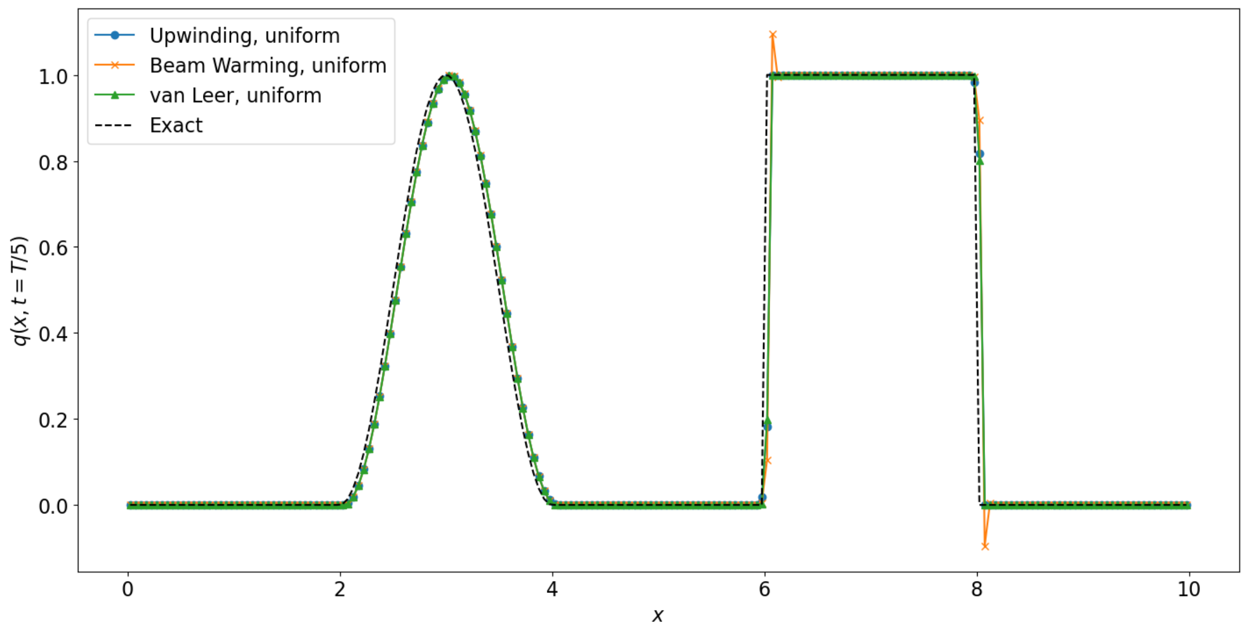


Fig. 2 Solution of advection problem after 1 period with CFL = 1.0

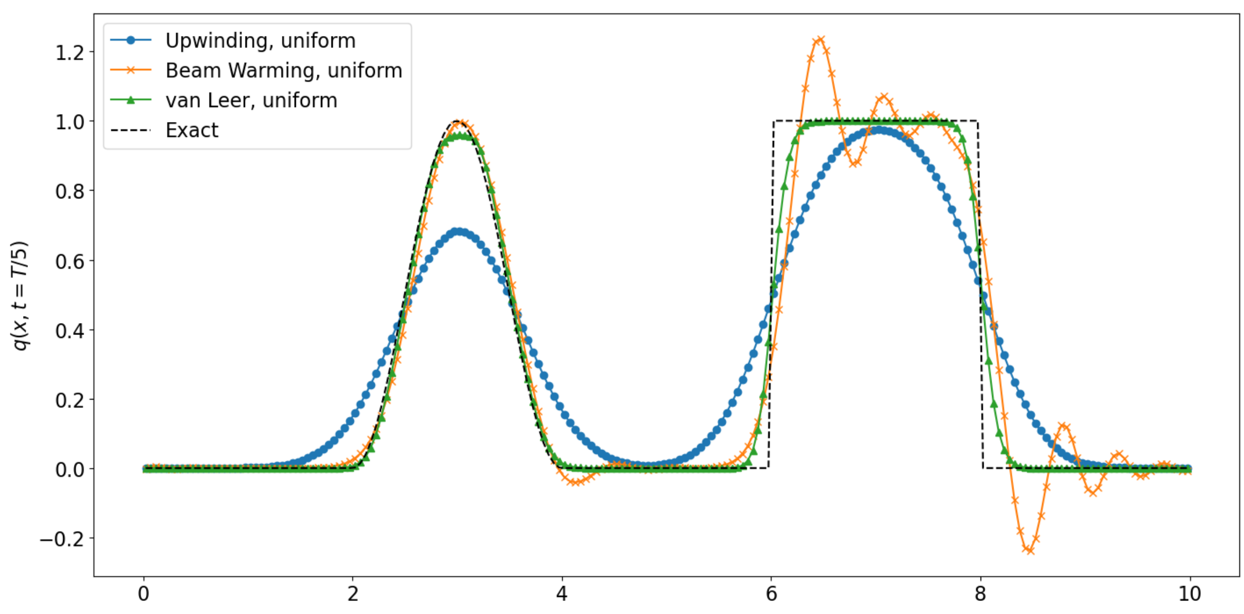


Fig. 3 Solution of advection problem after 1 period with CFL = 0.6

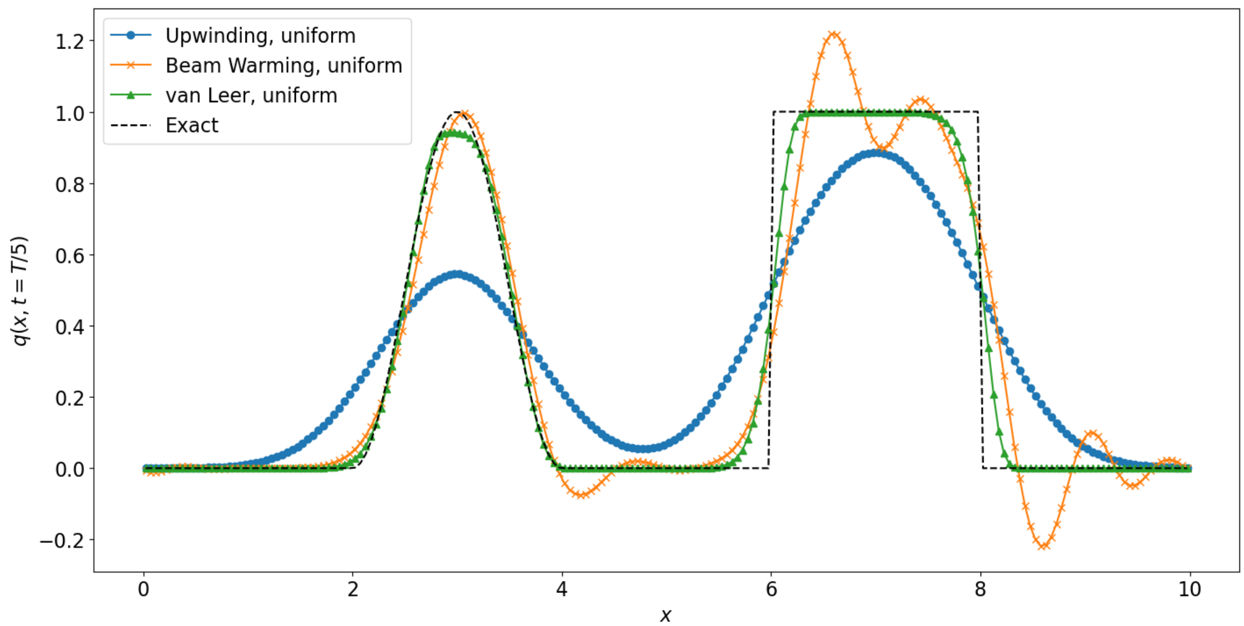


Fig. 4 Solution of advection problem after 1 period with CFL = 0.2

After 2 periods For solution after 2 periods, the plots are presented in Figure 5, Figure 6 and Figure 7.

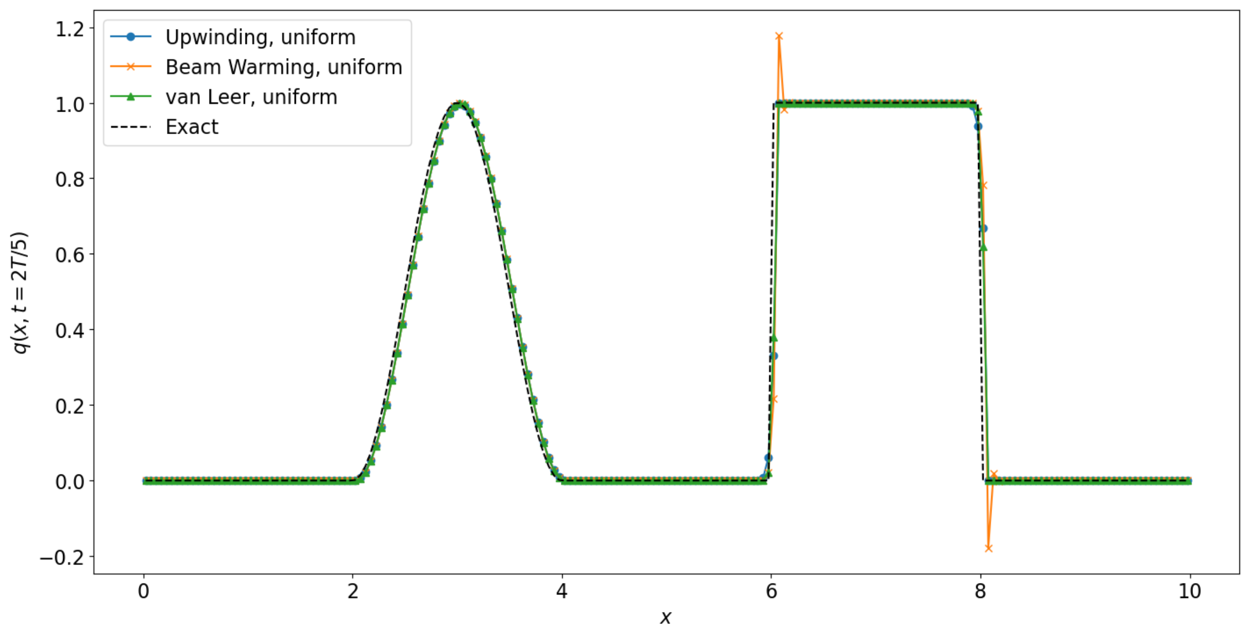


Fig. 5 Solution of advection problem after 2 periods with CFL = 1.0

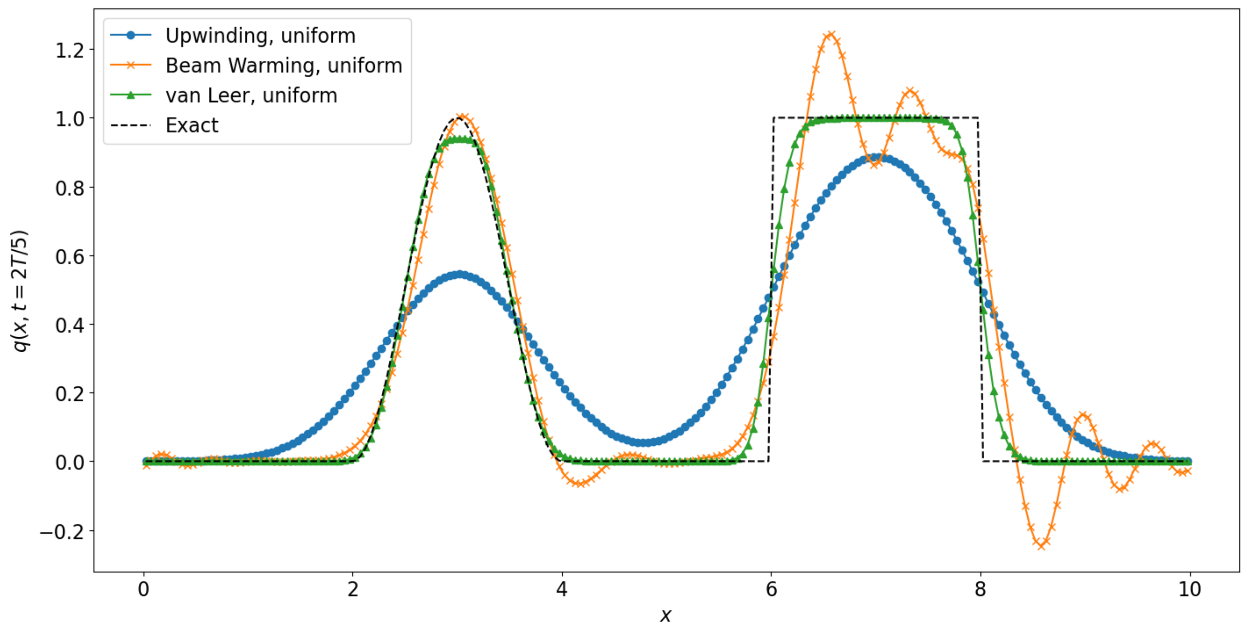


Fig. 6 Solution of advection problem after 2 periods with CFL = 0.6

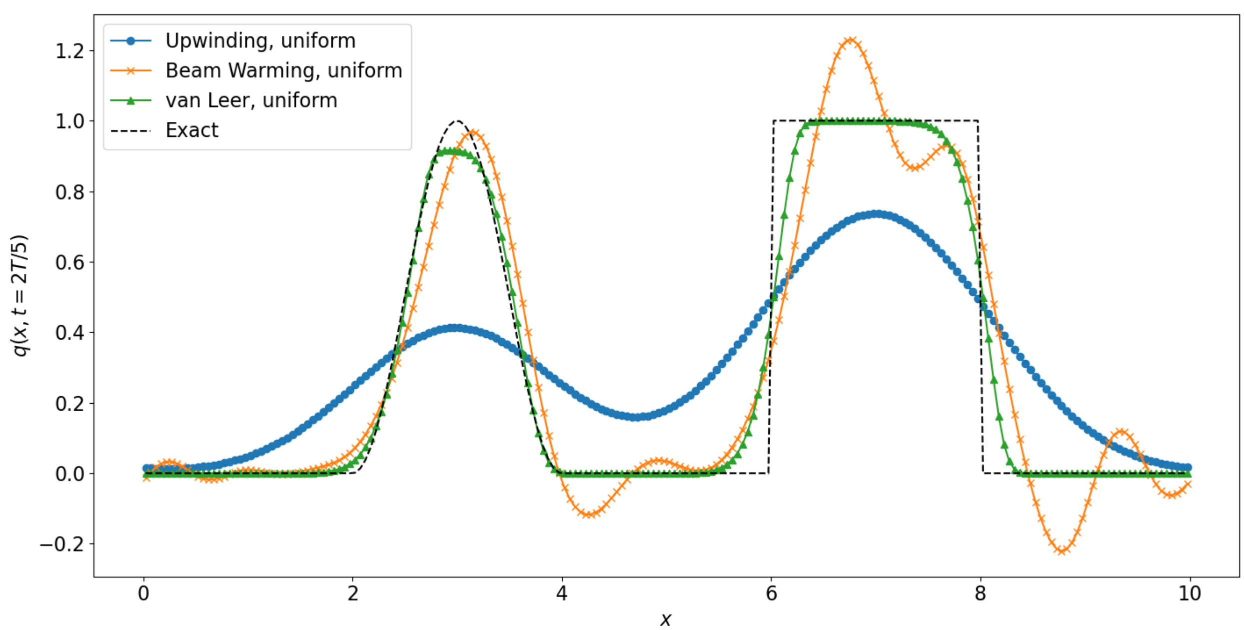


Fig. 7 Solution of advection problem after 2 periods with CFL = 0.2

After 5 periods Lastly for solution after 2 periods, the plots are presented in Figure 8, Figure 9 and Figure 10.

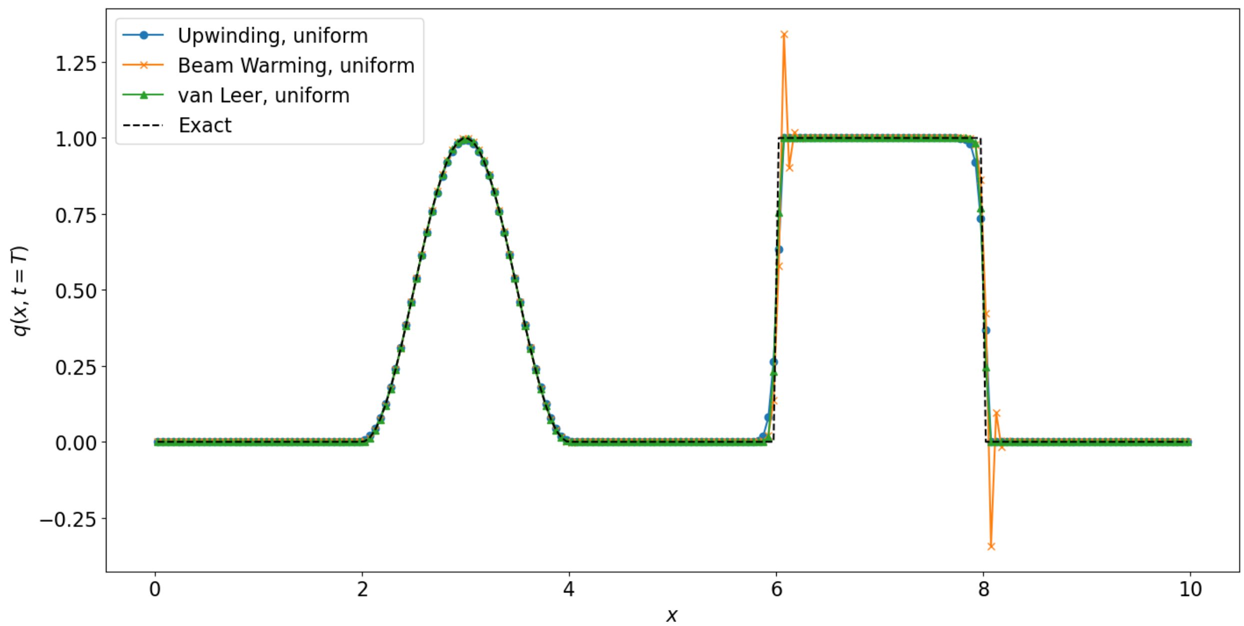


Fig. 8 Solution of advection problem after 5 periods with CFL = 1.0

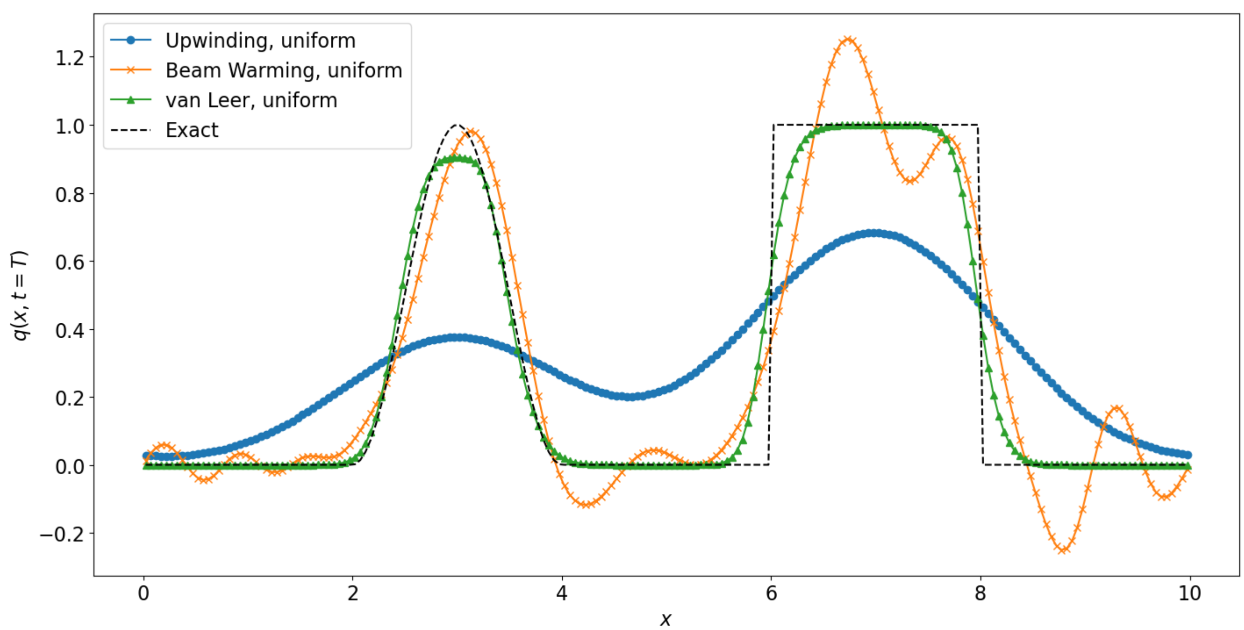


Fig. 9 Solution of advection problem after 5 periods with CFL = 0.6

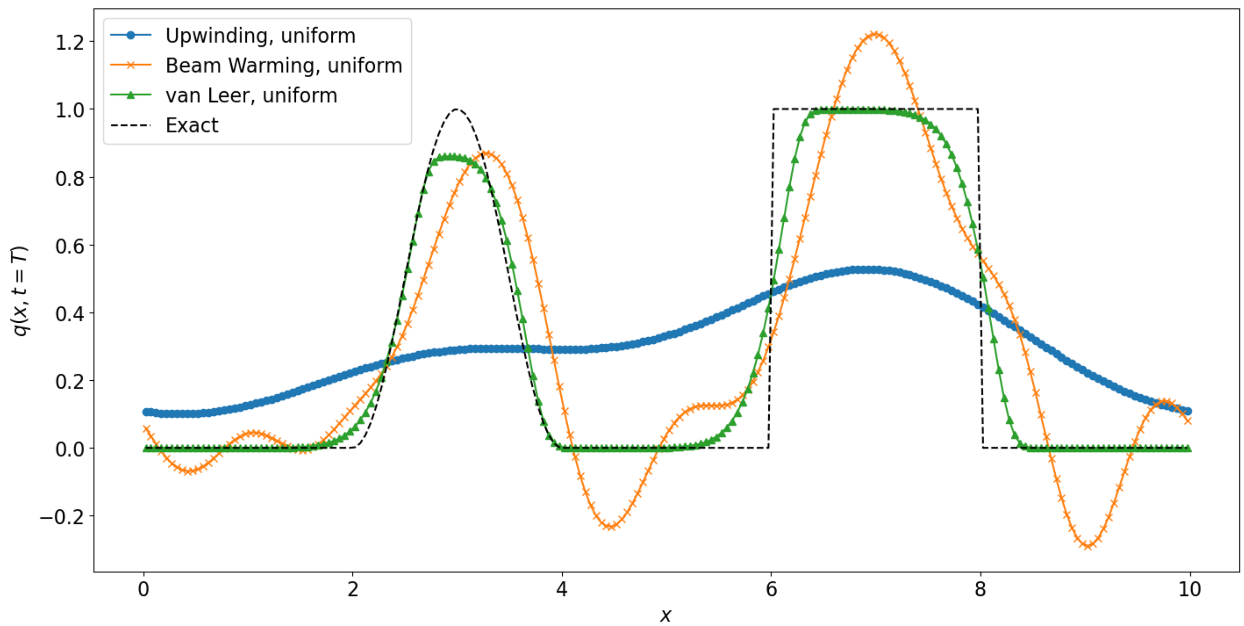


Fig. 10 Solution of advection problem after 5 periods with CFL = 0.2

It can be observed that all three methods are very accurate for CFL = 1.0 but for the Beam-Warming method, it is unstable near the sharp changes in gradients. As the CFL number is reduced, accuracy decreases for all methods. However, the van Leer method remained still highly accurate while the Beam-Warming method has introduced a lot of wiggles around the 0 gradients. although the upwinding method did not introduce any wiggles, its accuracy decreased to a large extent with the reduction in CFL number.

As the periods increased, the instability in the Beam-warming method worsened, with larger wiggles around the origin. furthermore, the solution of the Beam-warming method slowly shifts to the right with an increase in the number of periods, thus becoming less accurate. This phenomenon is not observed for upwinding and van Leer methods.

In conclusion, the van Leer method gives the most accurate solution out of the three methods and it also stays stable regardless of the number of periods and CFL number thus being the best method of the three. As for the Beam-Warming method, it generally gives an accurate solution that matches the shape of the exact solution well but it is the most unstable of the three methods and it also is expected to be increasing inaccurate with a greater number of periods. Lastly, for the upwind method, it manages to stay stable but that is the only positive aspect of it. With a reduction in CFL number, its accuracy drastically reduces thus not being a trustworthy method.

Acoustic problem For the acoustic problem, each solution is presented in two plots, $p(x,t)$ and $u(x,t)$. Since the acoustic problem consists of systems of equations, the upwind method is referred to as the Godunov method in this part.

After 1 period Starting off with solution after 1 period, the plots are presented in Figure 11, Figure 12 and Figure 13.

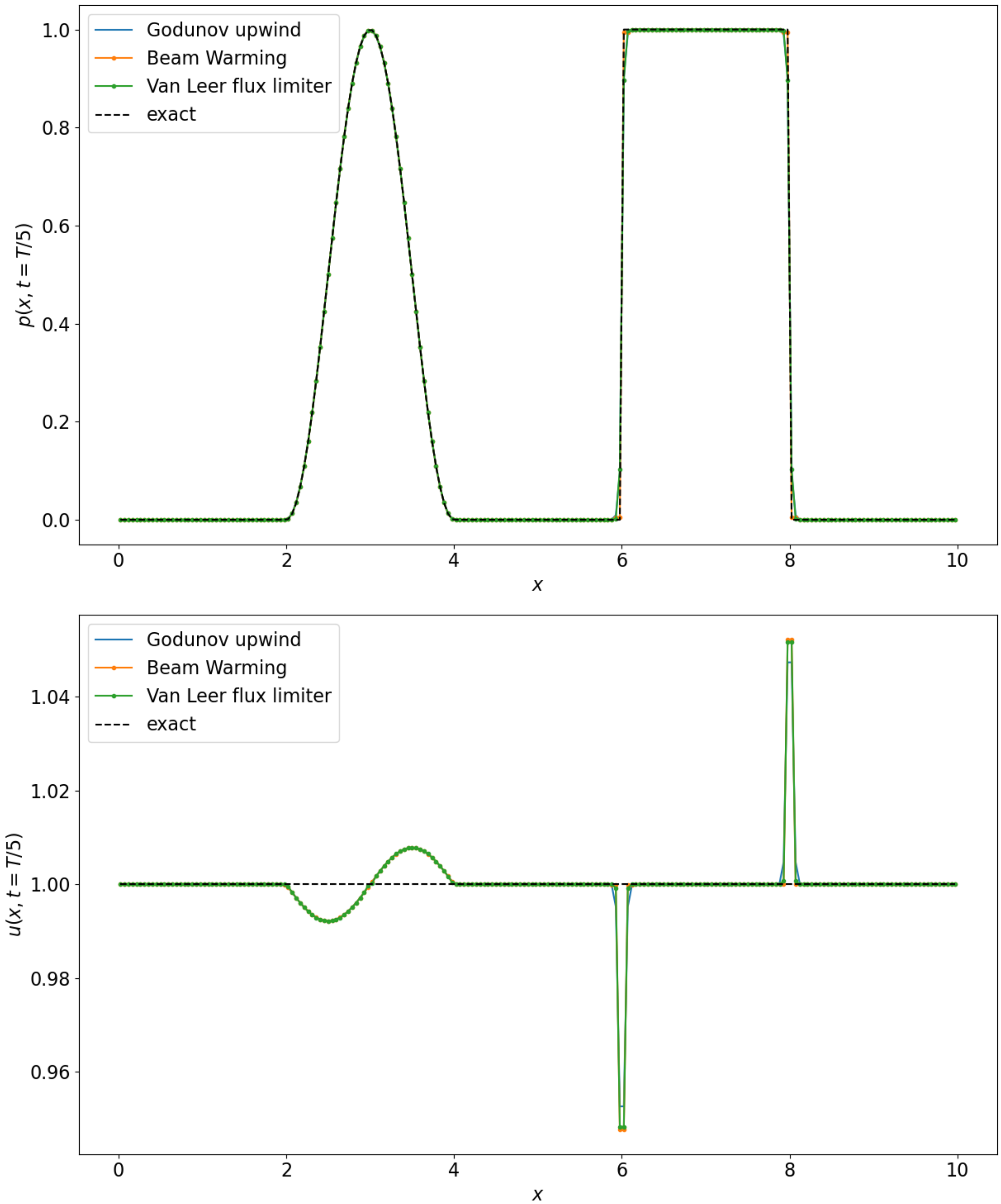
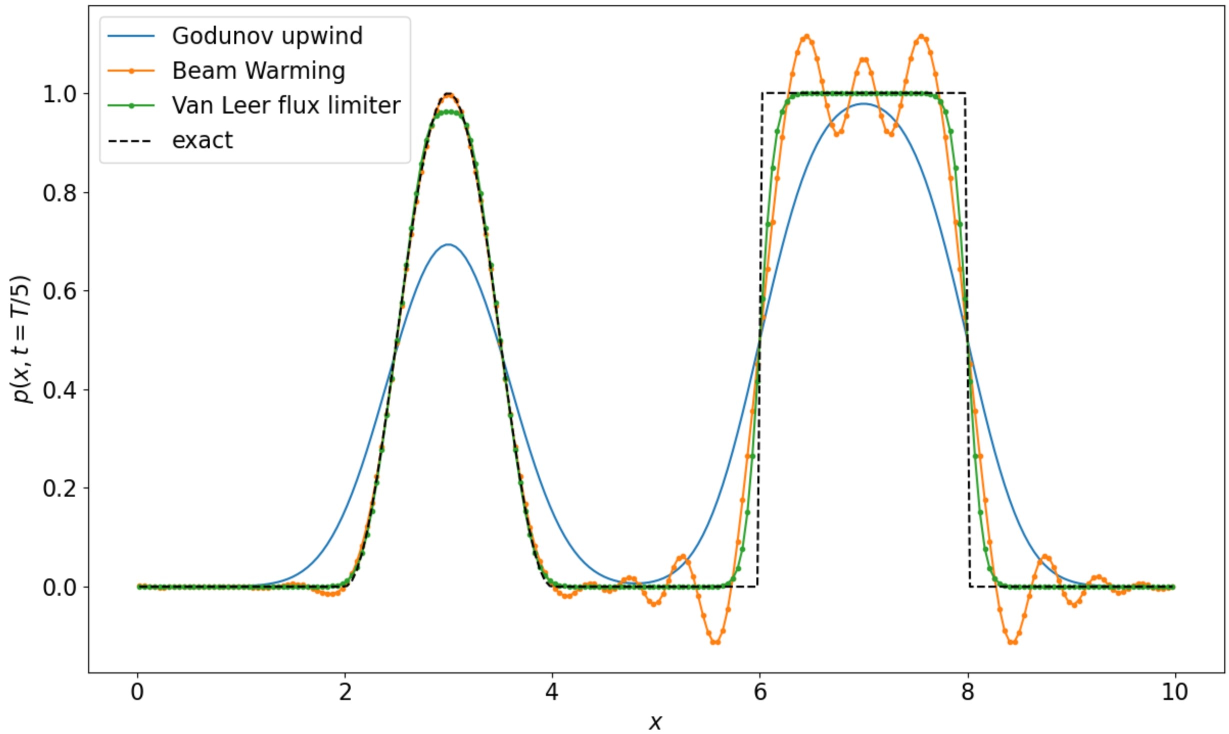


Fig. 11 Solution of acoustic problem after 1 period with CFL = 1.0



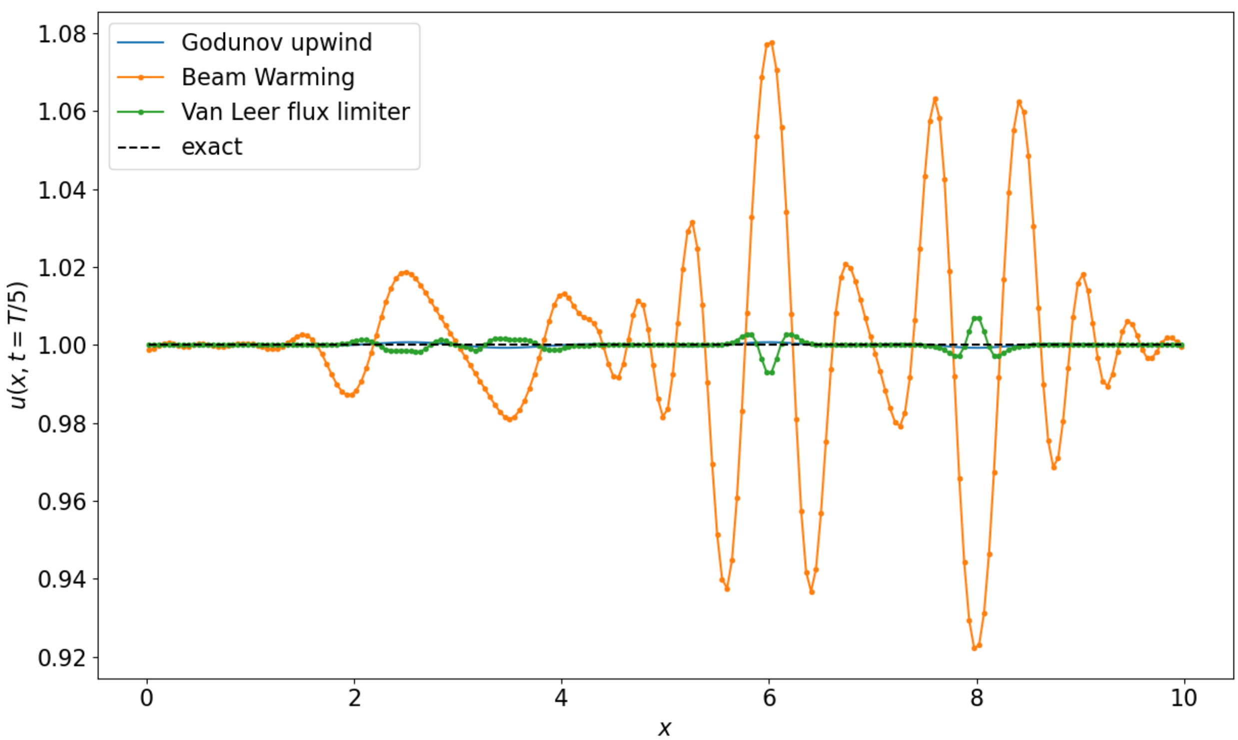


Fig. 12 Solution of acoustic problem after 1 period with CFL = 0.6

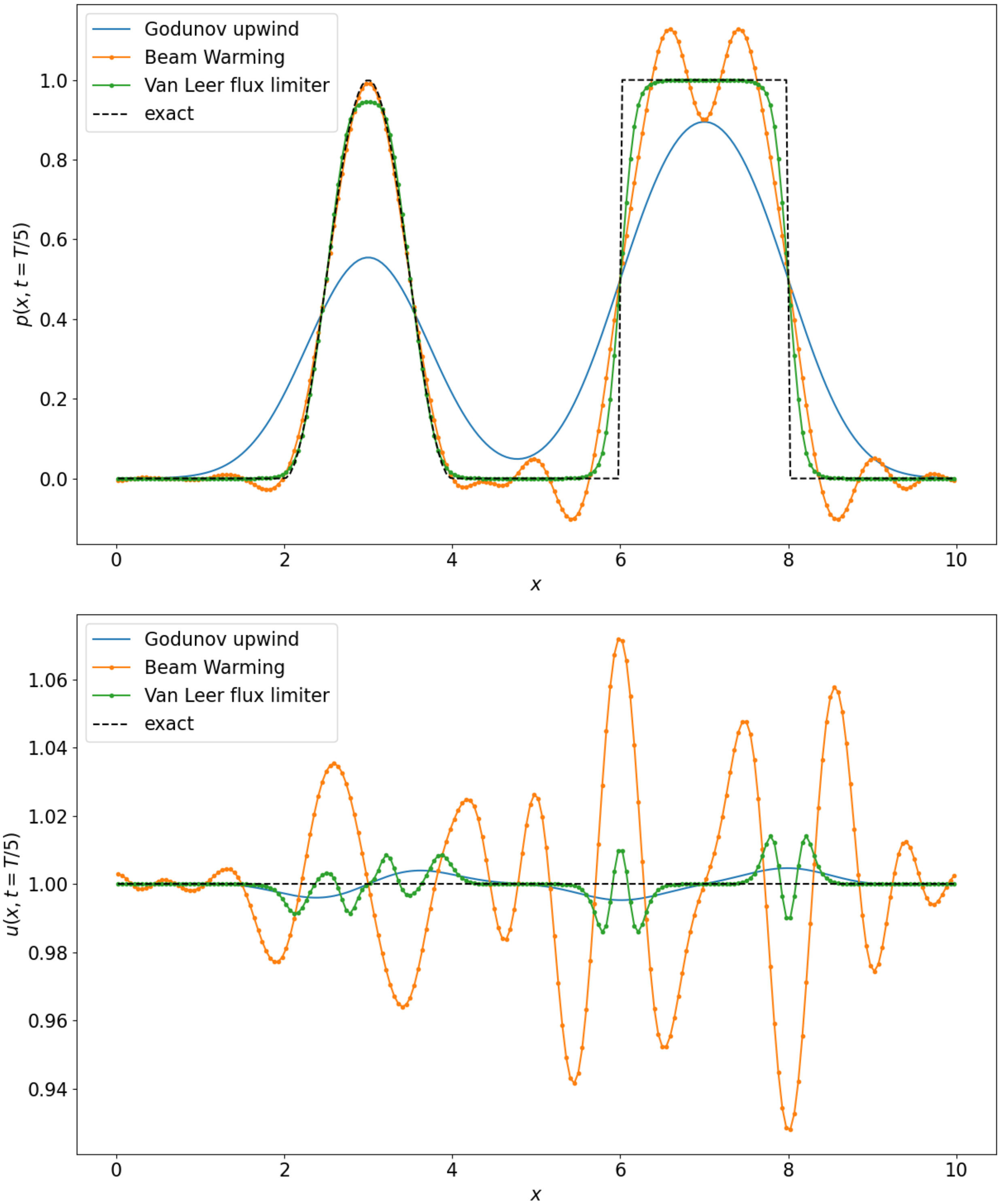


Fig. 13 Solution of acoustic problem after 1 period with CFL = 0.2

After 2 periods For solution after 2 periods, the plots are presented in Figure 14, Figure 15 and Figure 16.

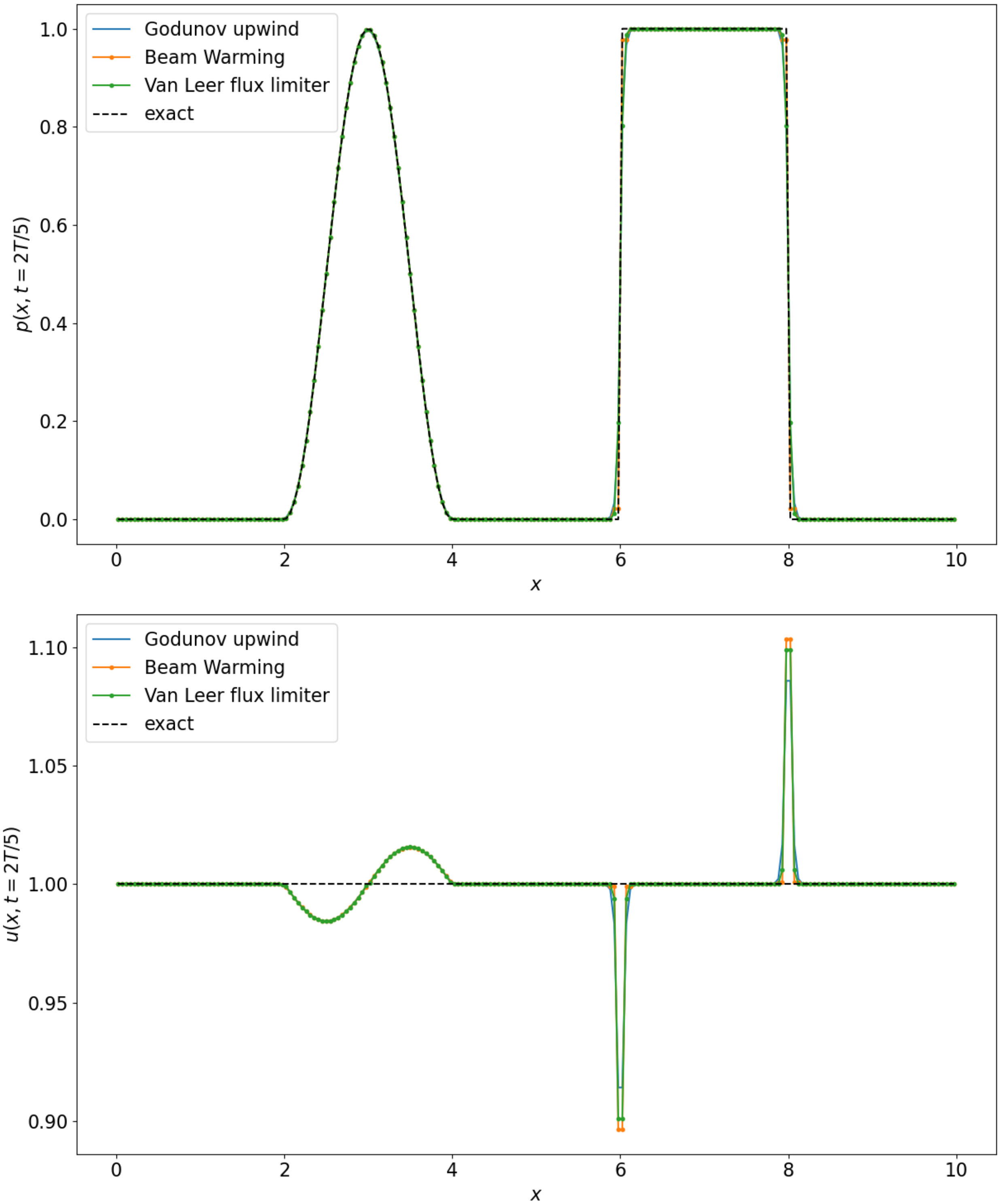


Fig. 14 Solution of acoustic problem after 2 periods with CFL = 1.0

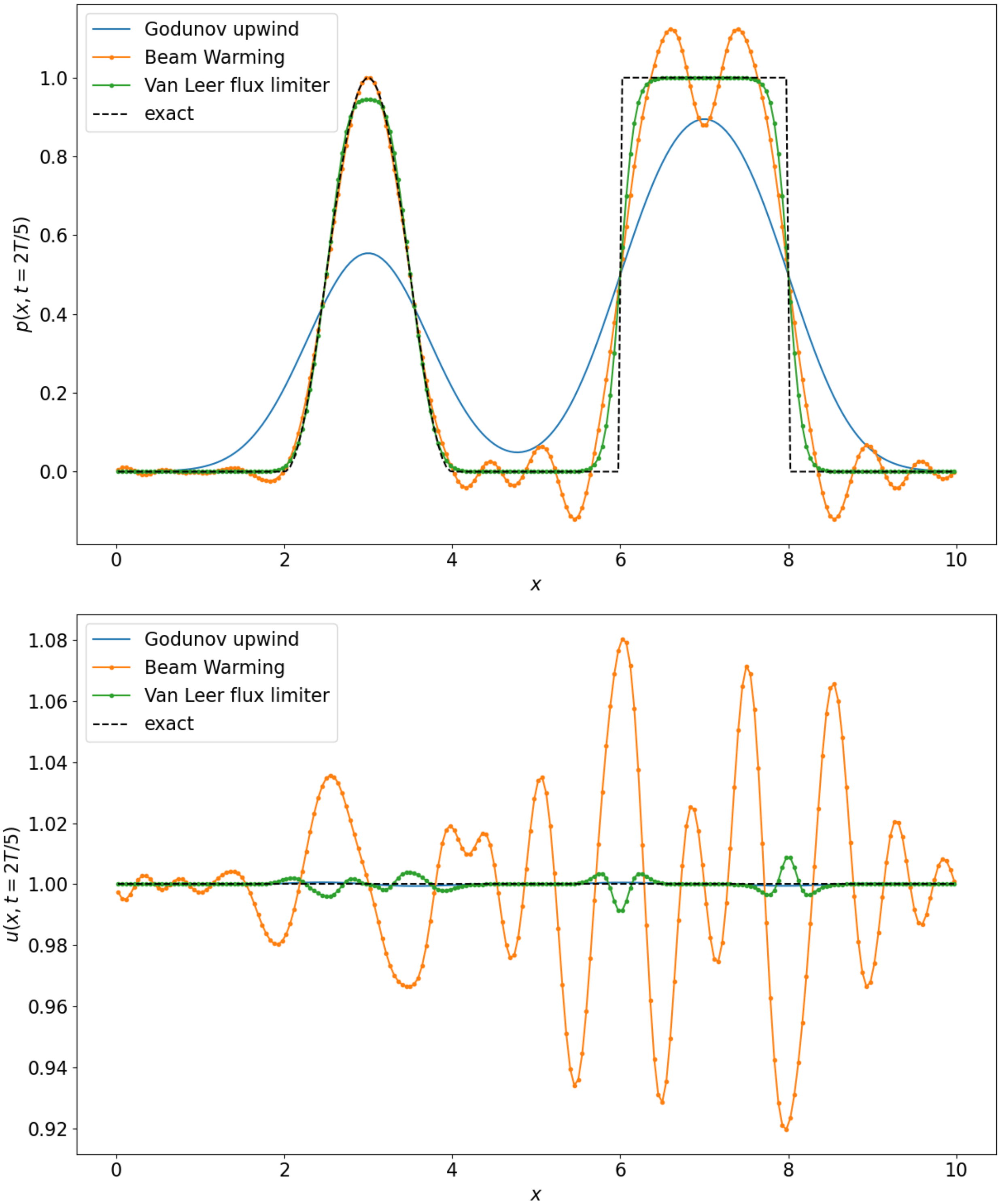


Fig. 15 Solution of acoustic problem after 2 s with CFL = 0.6

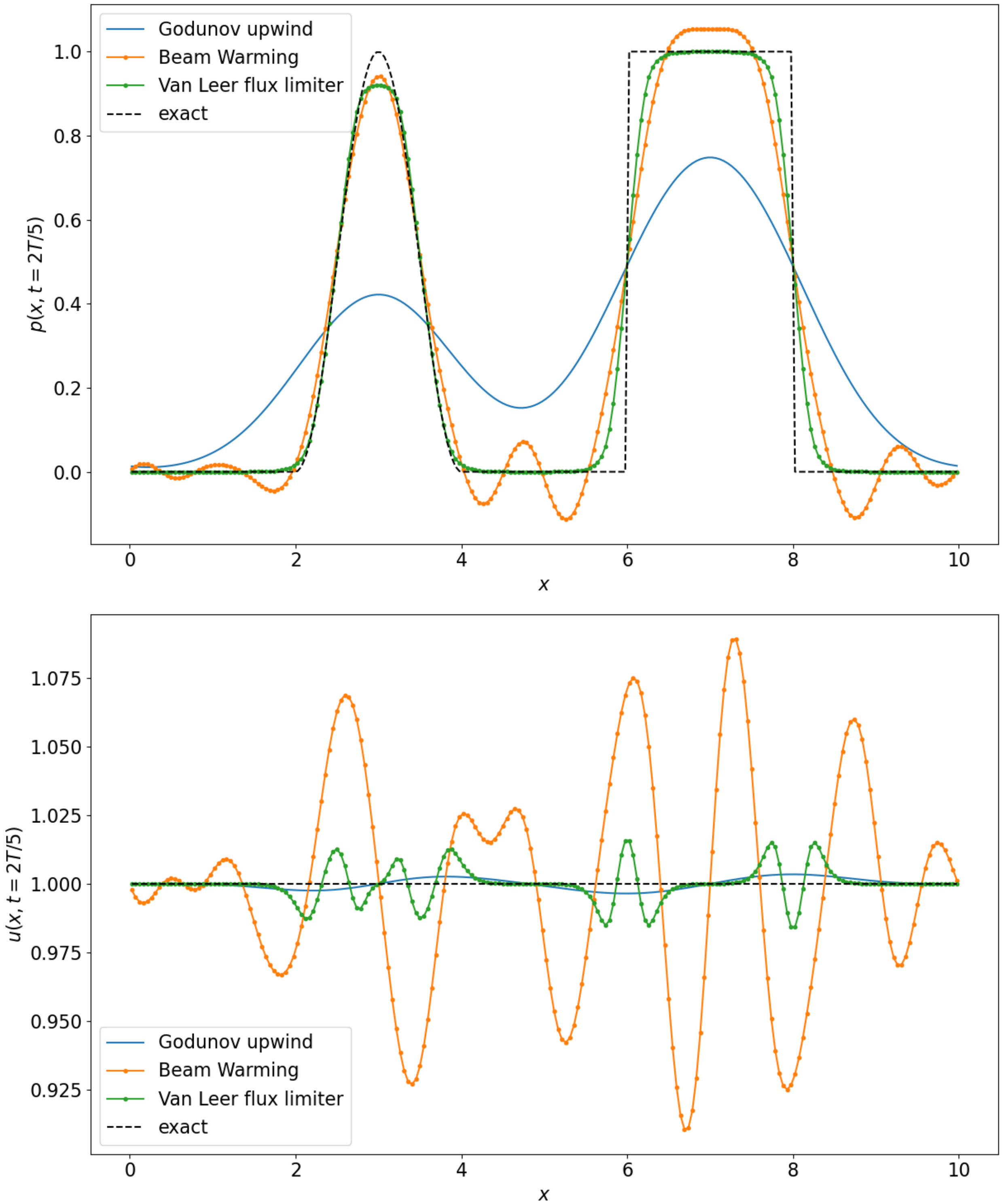


Fig. 16 Solution of acoustic problem after 2 periods with CFL = 0.2

After 5 periods Lastly for solution after 2 periods, the plots are presented in Figure 17, Figure 18 and Figure 19.

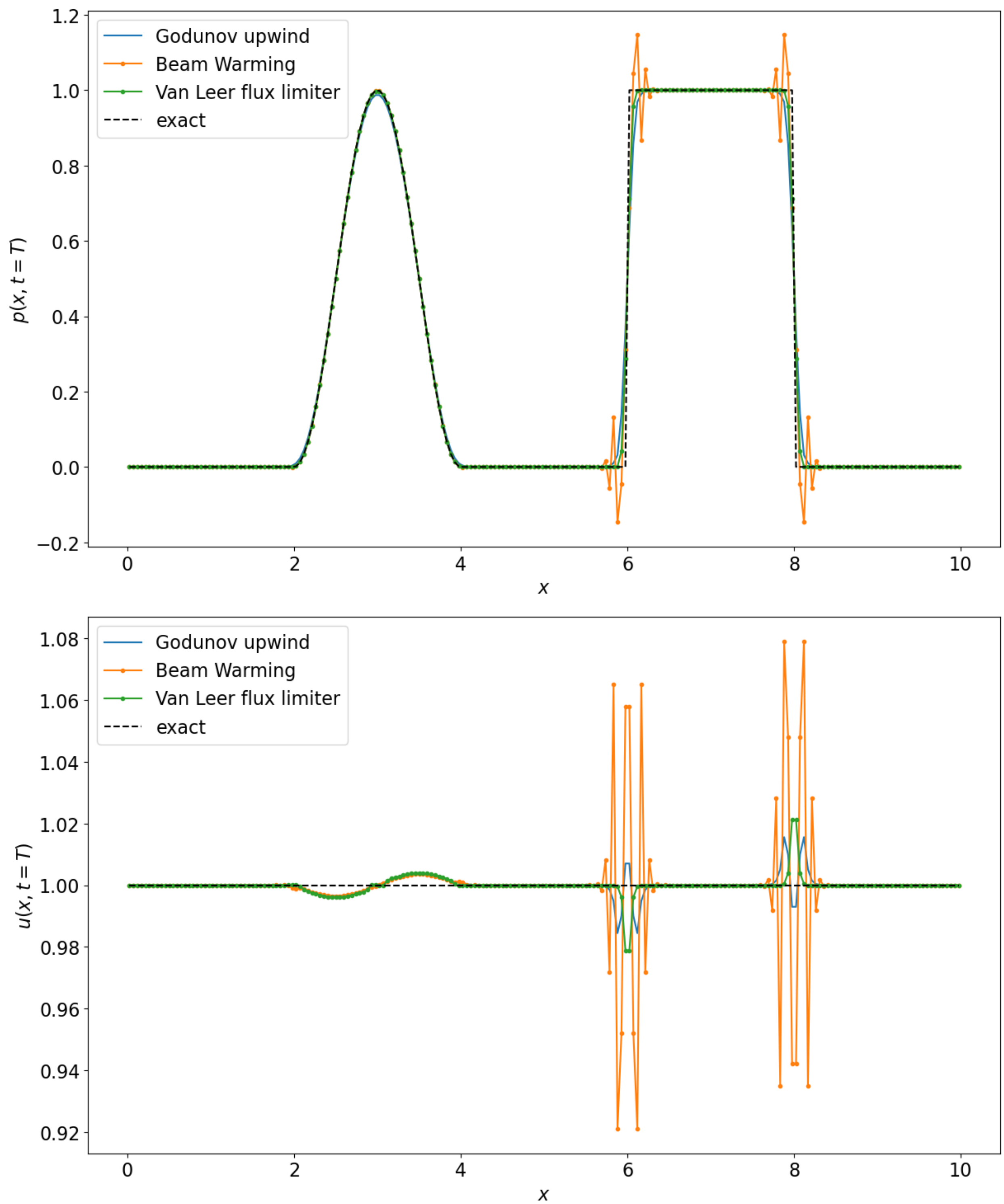


Fig. 17 Solution of acoustic problem after 5 periods with CFL = 1.0

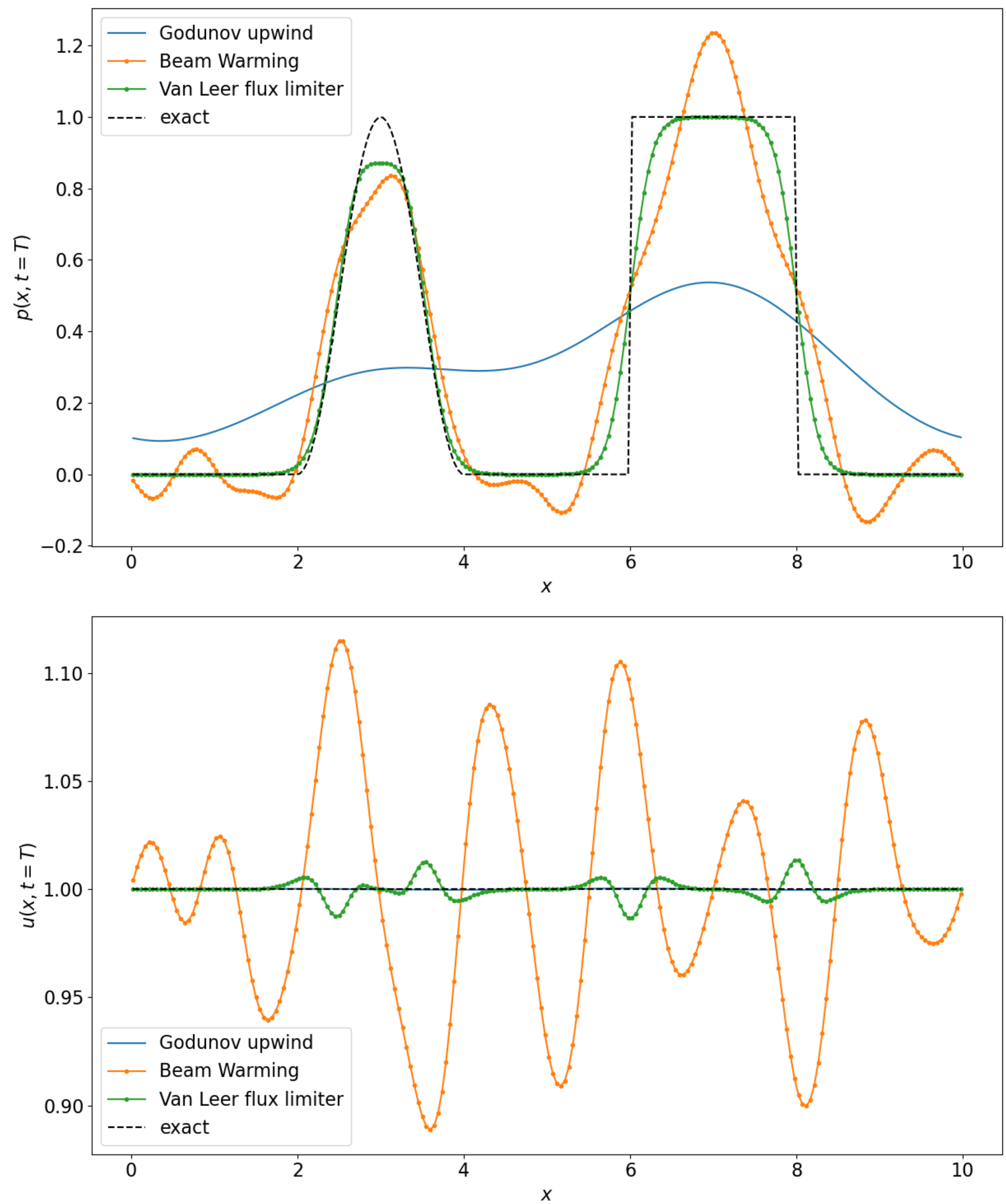


Fig. 18 Solution of acoustic problem after 5 periods with CFL = 0.6

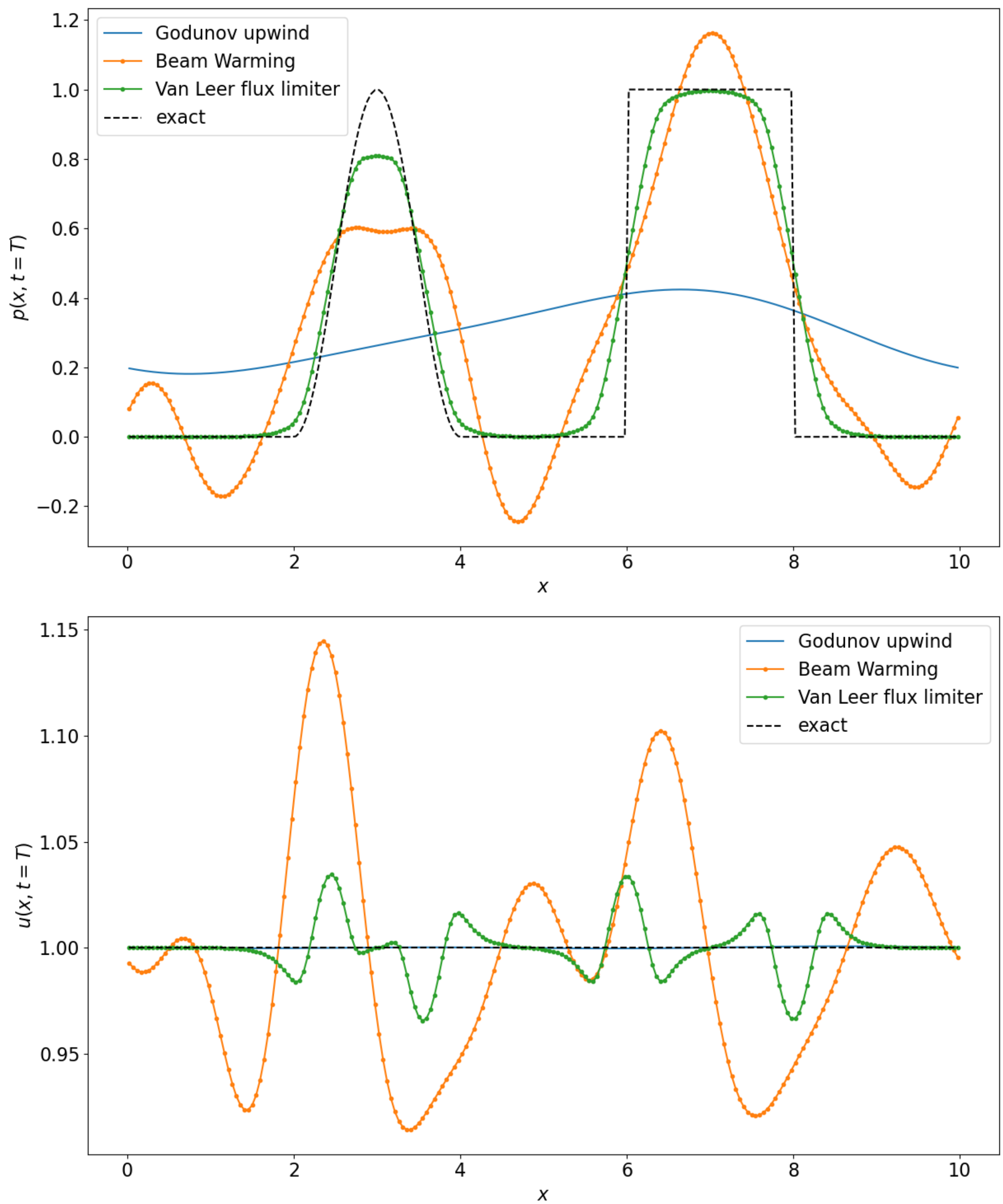


Fig. 19 Solution of acoustic problem after 5 periods with CFL = 0.2

A similar conclusion can be made for the observed solution of the acoustic problem as it was made for the advection problem. All three methods show high accuracy at CFL = 1.0. However, among the three methods, the Beam-Warming method introduces wiggles that causes instability. One difference from the conclusion of the advection problem is that for solution $u(x, t)$, the Godunov method also introduces a small amount of wiggle. However, it does not relate to instability like the Beam-Warming method which imposed negative pressure $p(x,t)$ values.

Just like for the advection problem, the reduction in CFL number decreases the accuracy of all three methods as well. Moreover, with a larger number of periods, the Beam-Warming method introduces more and larger wiggles, and thus more instability. Not only does it introduces more instability but also more inaccuracy. One noticeable difference from the solution of the advection problem is that the accuracy of the Godunov method decreases much more quickly with the increase in the number of periods. After 5 periods and for CFL < 1.0, the solution almost loses its true shape and does not resemble the exact solution to a large extent.

Thus, the van Leer method is still the best method among the three ultimately as it stays stable regardless of CFL number and the number of periods. Although its accuracy reduces with a reduction in CFL number, it still resembles the exact solution to a large extent. As for the Beam-Warming method, it also does resemble the shape of the exact solution to a small extent but is extremely unstable, even for CFL = 1.0. As for the Godunov method, it has good accuracy and stability at CFL = 1.0 but its dependence on the CFL number is the highest among the three methods. With a reduction in CFL number, its accuracy decreases drastically and for a larger number of periods, it loses the shape of the exact solution almost completely.

5. Implementation of method I and II for advection problem on a non-uniform grid

Lastly, method I (First order upwind) and method II (Beam-Warming) are implemented for the advection problem on a non-uniform grid. This non-uniform grid is defined with the spacing between the nodes for $x \in [0,5]$ half the size of that for $x \in [5,10]$. The solution will be analysed for 1 and 5 periods which corresponds to $T/5$ and T respectively. Similar to the previous section, the CFL number will be varied for the analyses of stability, dependence on CFL number, wiggles and accuracy.

Since the different methods have already been compared against each other, the two methods will be analysed separately. For each method, the solution for a non-uniform grid will be compared against two different solutions of the uniform grid, one that is computed using the smaller spacing and the other using the larger spacing of the two different magnitudes of spacings but with the same CFL number.

5 a. Method I (First order upwind)

After 1 period The solution for the first order upwind method after 1 period are presented in Figure 20, Figure 21 and Figure 22. The dependence on CFL number has been visualised in Figure 23.

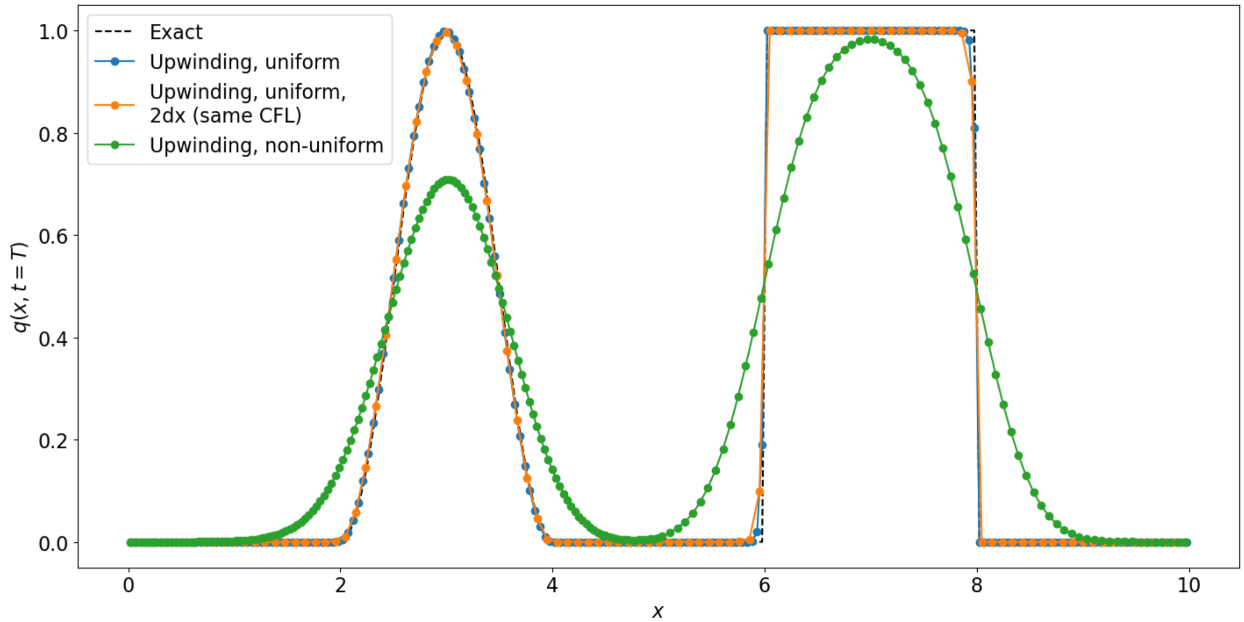


Fig. 20 Solution of advection problem on non-uniform grid using first order upwind method after 1 period with CFL = 1.0

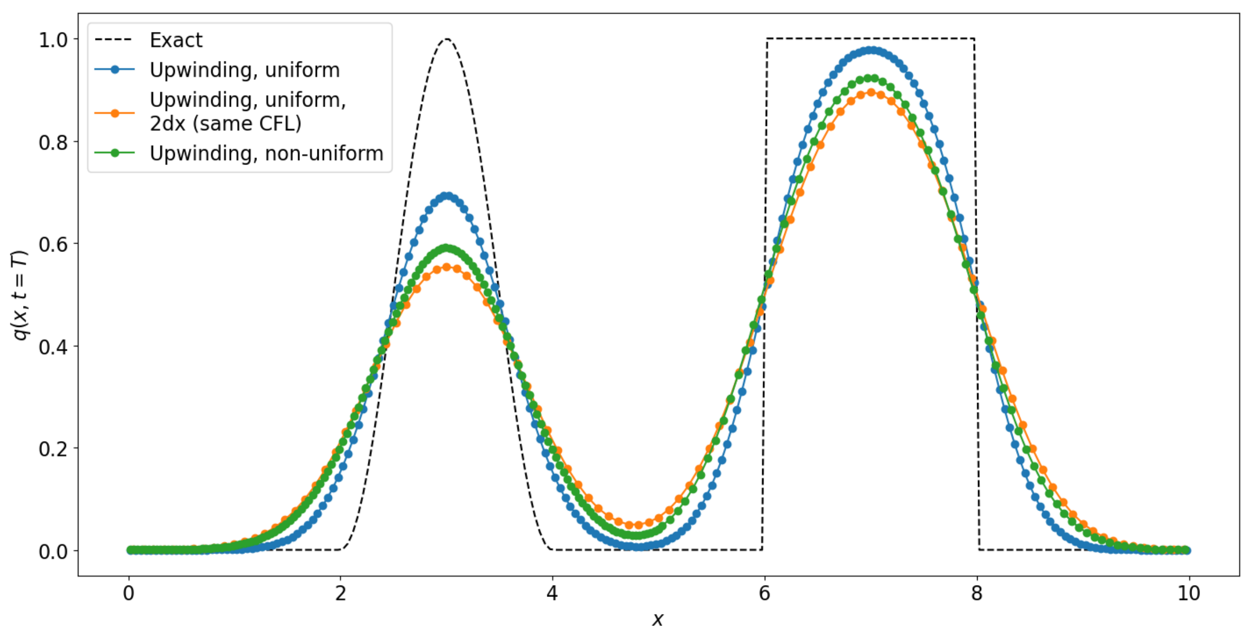


Fig. 21 Solution of advection problem on non-uniform grid using first order upwind method after 1 period with CFL = 0.6

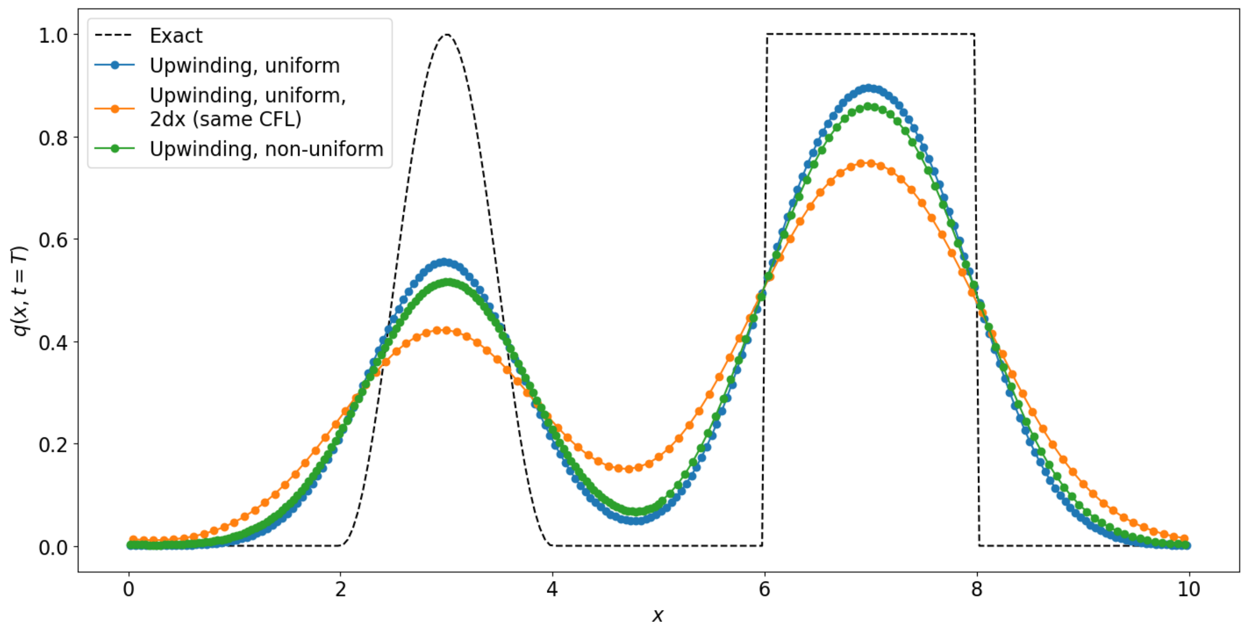


Fig. 22 Solution of advection problem on non-uniform grid using first order upwind method after 1 period with CFL = 0.2

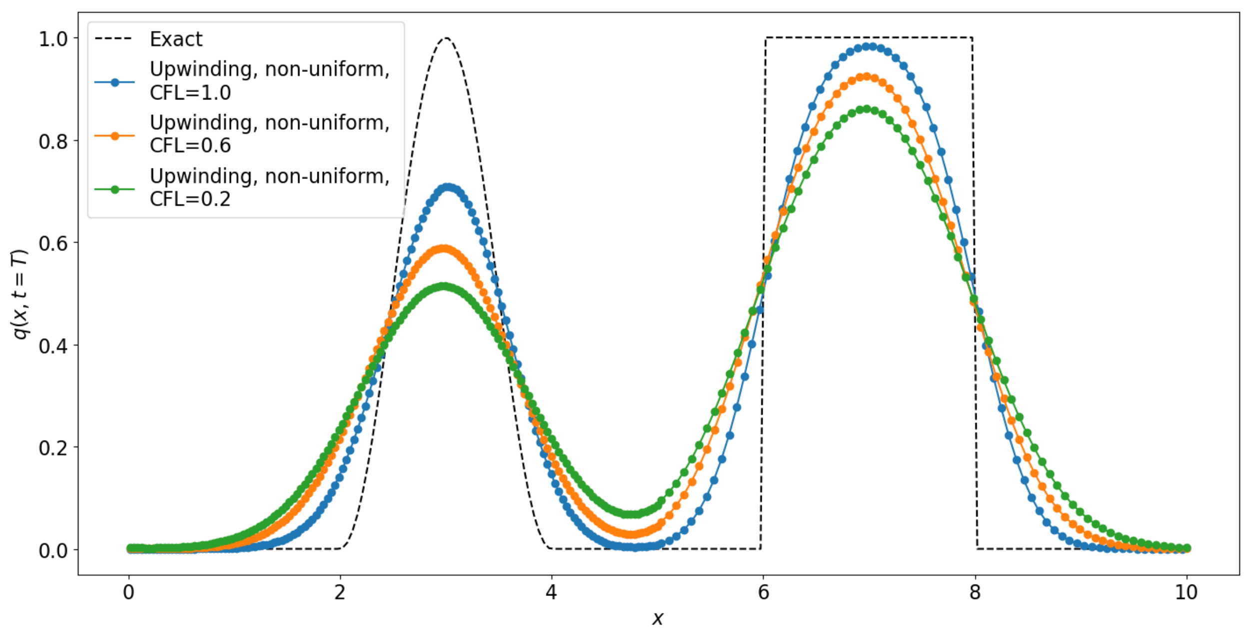


Fig. 23 Solution of advection problem on non-uniform grid using first order upwind method after 1 period for various CFL values

After 5 periods The solution for the first order upwind method after 5 periods are presented in Figure 24, Figure 25 and Figure 26. The dependence on CFL number has been visualised in Figure 27.

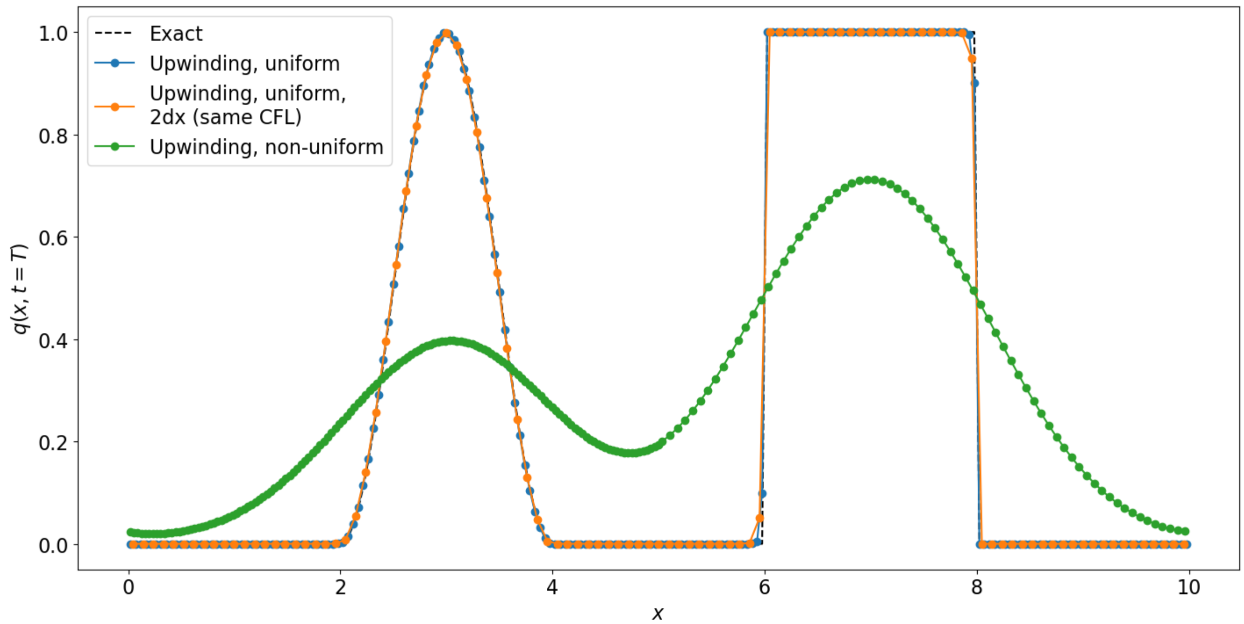


Fig. 24 Solution of advection problem on non-uniform grid using first order upwind method after 5 periods with CFL = 1.0

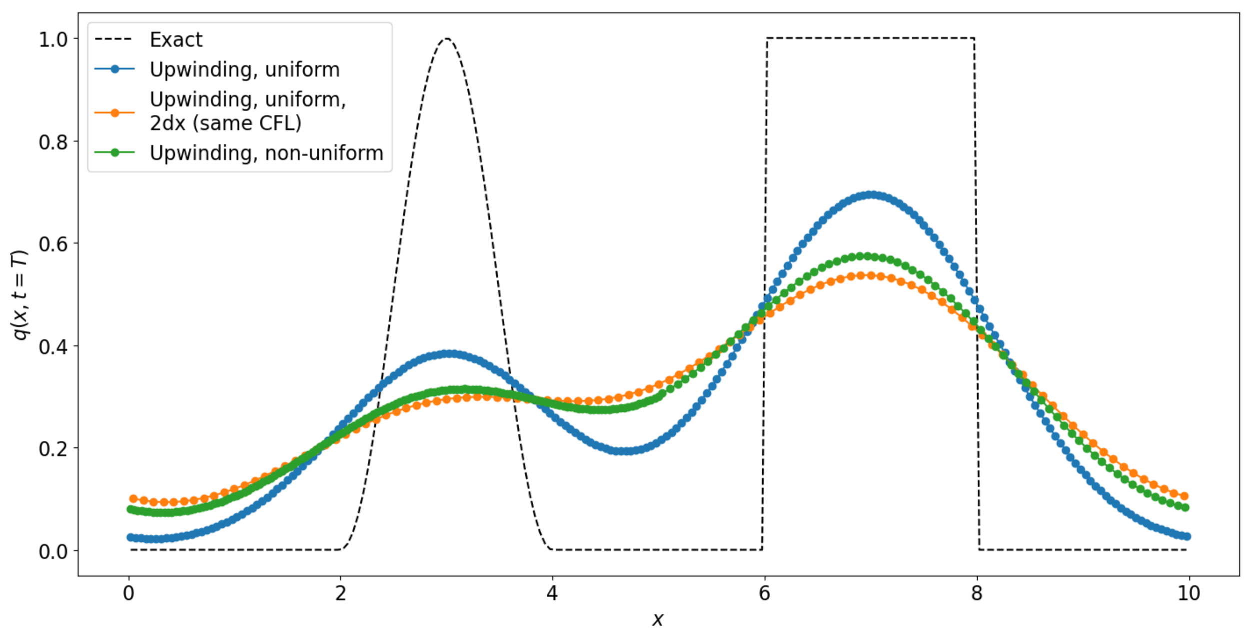


Fig. 25 Solution of advection problem on non-uniform grid using first order upwind method after 5 periods with CFL = 0.6

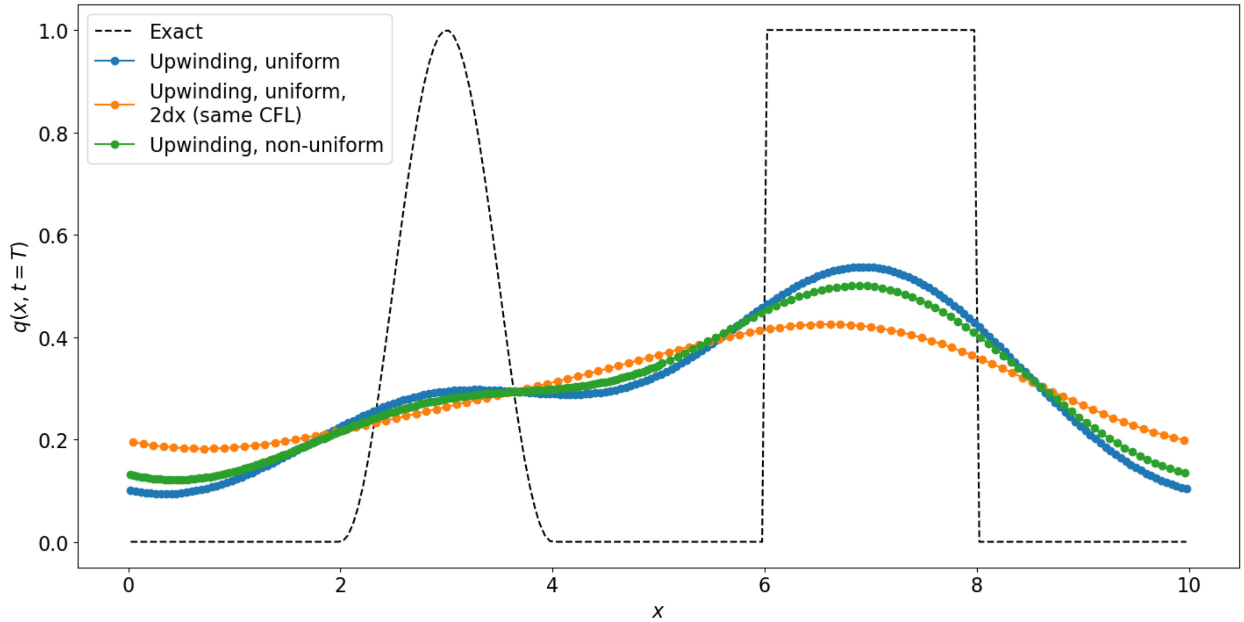


Fig. 26 Solution of advection problem on non-uniform grid using first order upwind method after 5 periods with CFL = 0.2

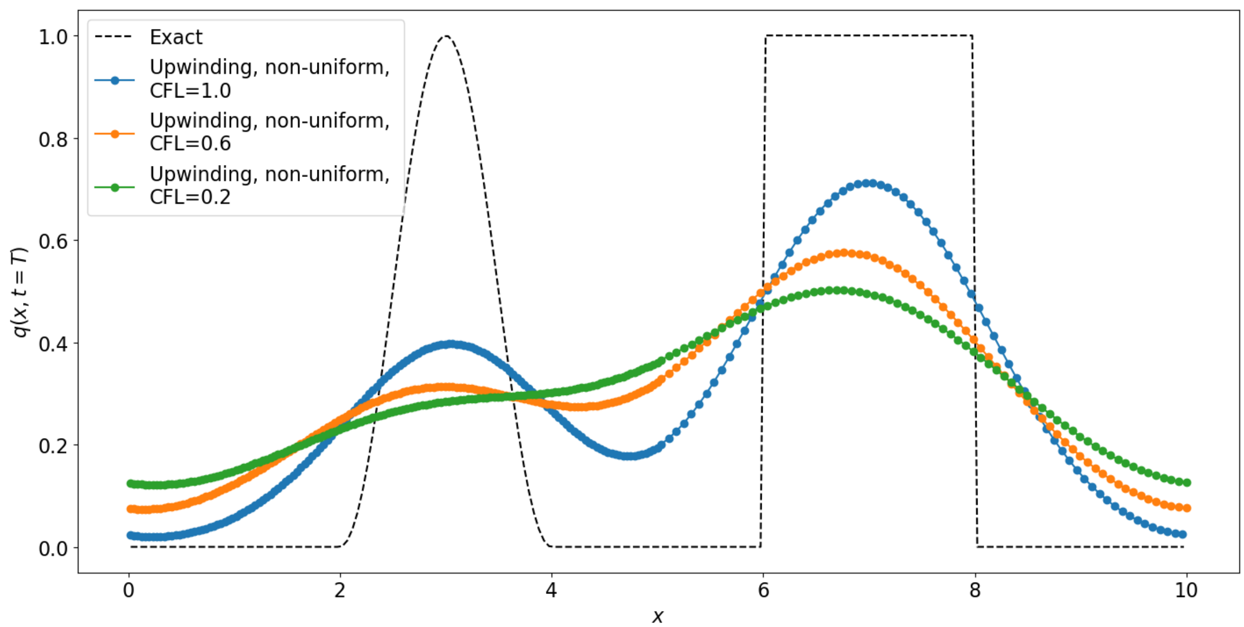


Fig. 27 Solution of advection problem on non-uniform grid using first order upwind method after 5 periods for various CFL values

5 b. Method II (Beam Warming)

After 1 period The solution for the Beam Warming method after 1 period are presented in Figure 28, Figure 29 and Figure 30. The dependence on CFL number has been visualised in Figure 31.

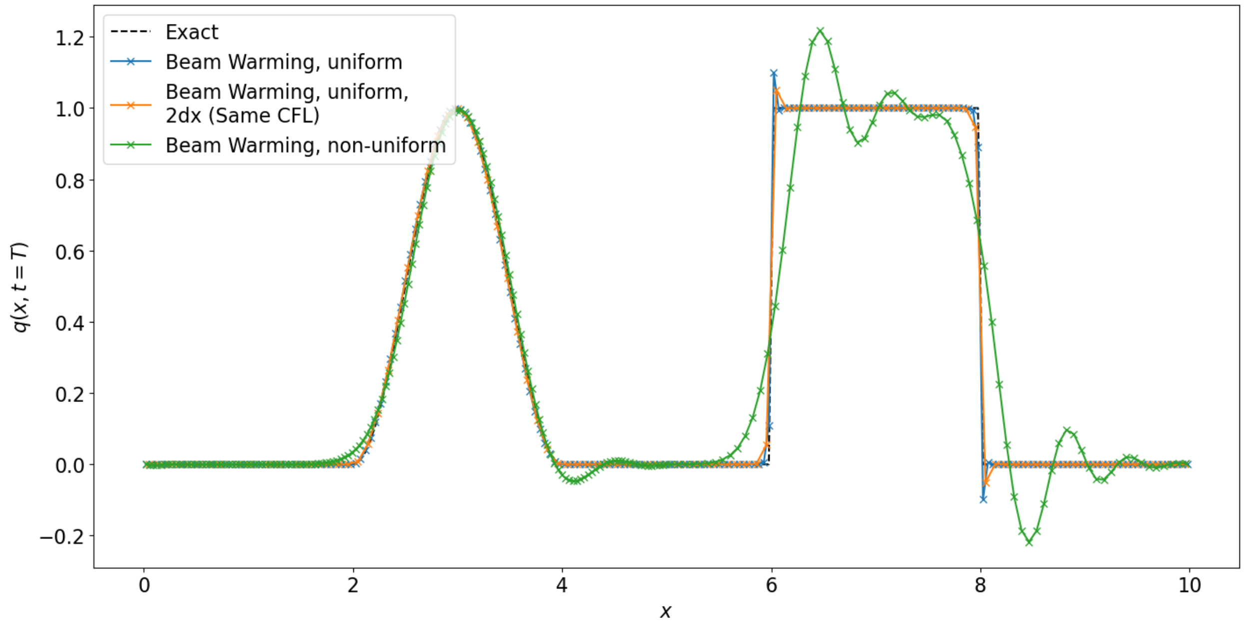


Fig. 28 Solution of advection problem on non-uniform grid using Beam Warming method after 1 period with CFL = 1.0

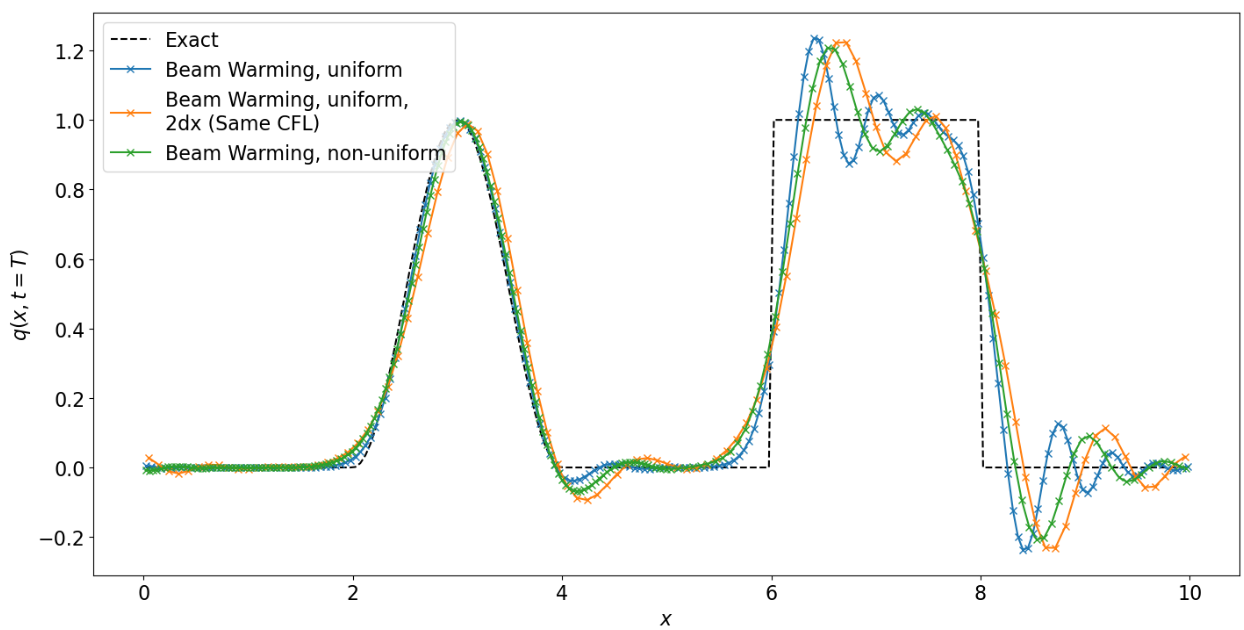


Fig. 29 Solution of advection problem on non-uniform grid using Beam Warming method after 1 period with CFL = 0.6

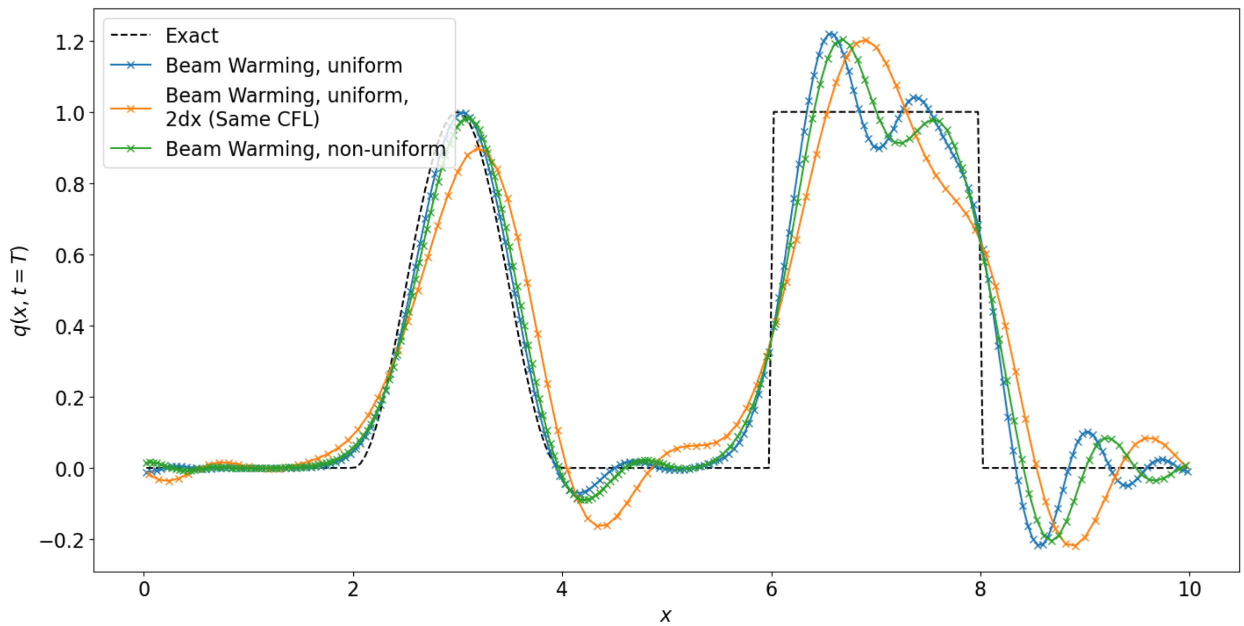


Fig. 30 Solution of advection problem on non-uniform grid using Beam Warming method after 1 period with CFL = 0.2

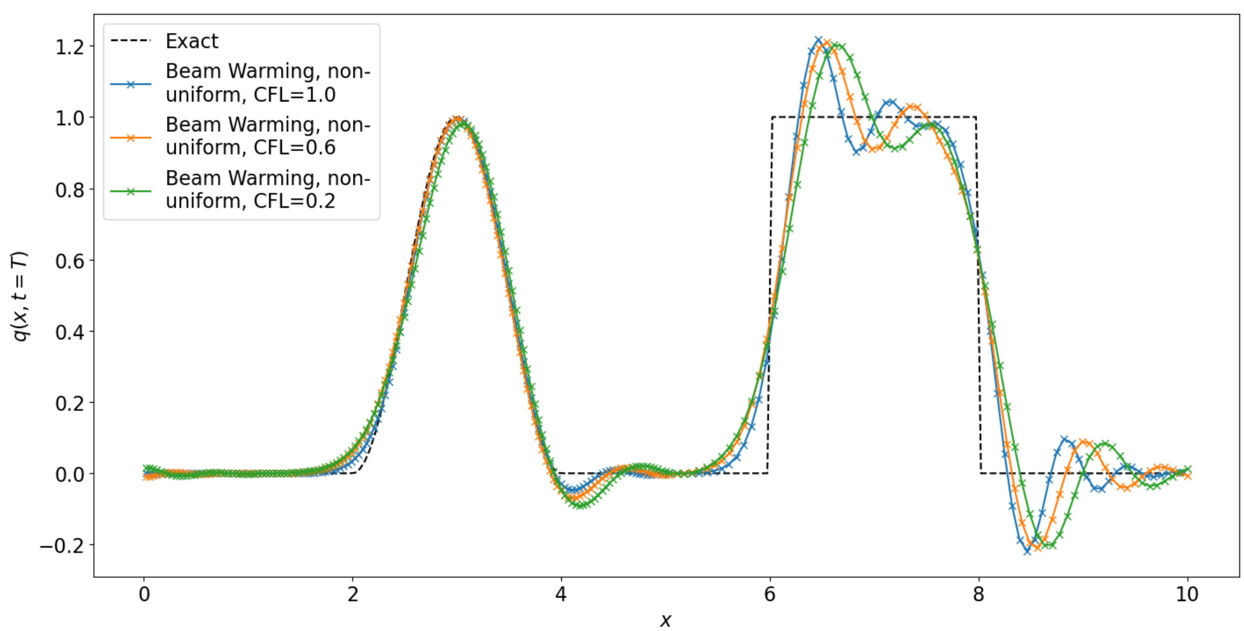


Fig. 31 Solution of advection problem on non-uniform grid using Beam Warming method after 1 period for various CFL values

After 5 periods The solution for the Beam Warming method after 5 periods are presented in Figure 32, Figure 33 and Figure 34. The dependence on CFL number has been visualised in Figure 35.

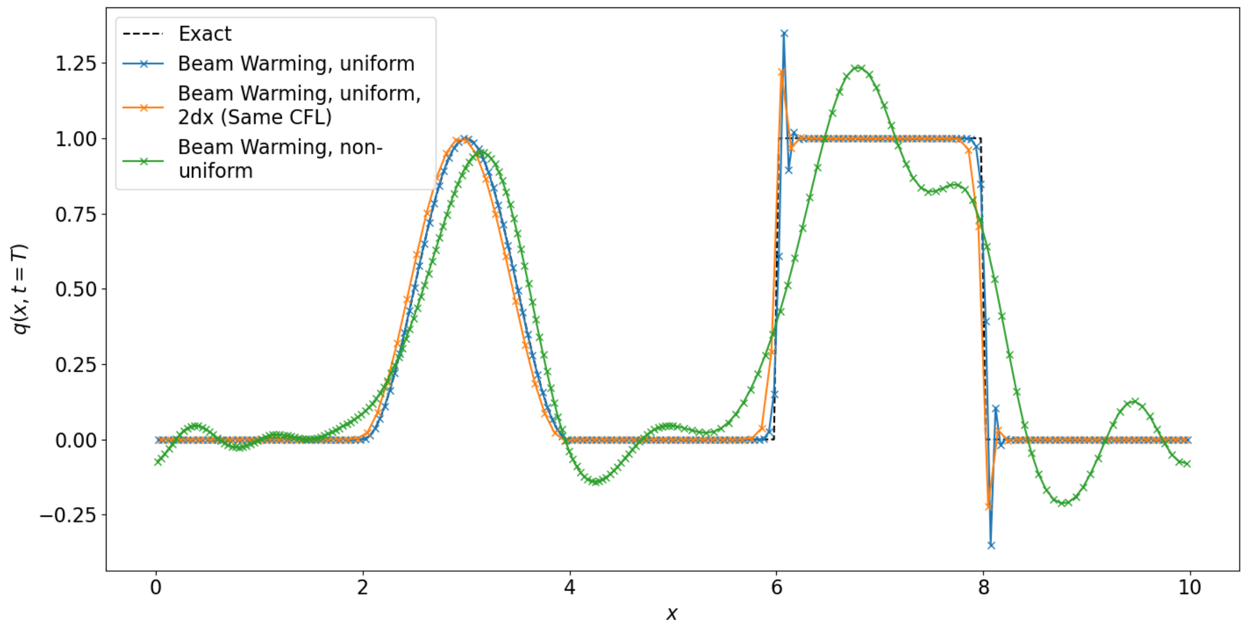


Fig. 32 Solution of advection problem on non-uniform grid using Beam Warming method after 5 periods with CFL = 1.0

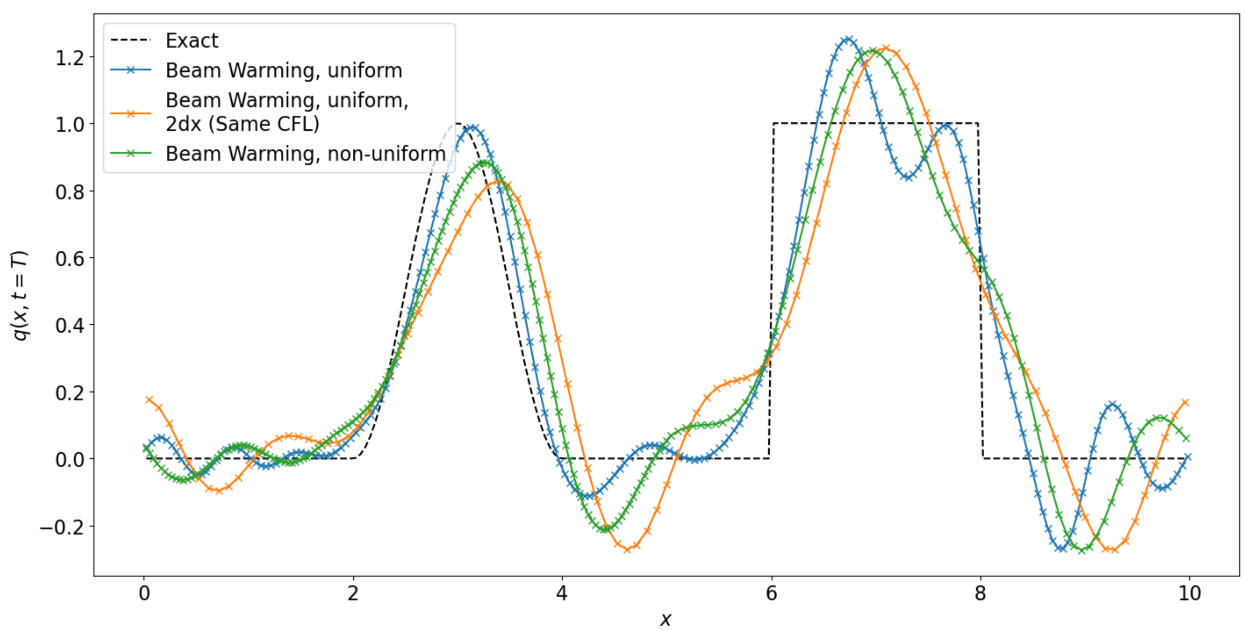


Fig. 33 Solution of advection problem on non-uniform grid using Beam Warming method after 5 periods with CFL = 0.6

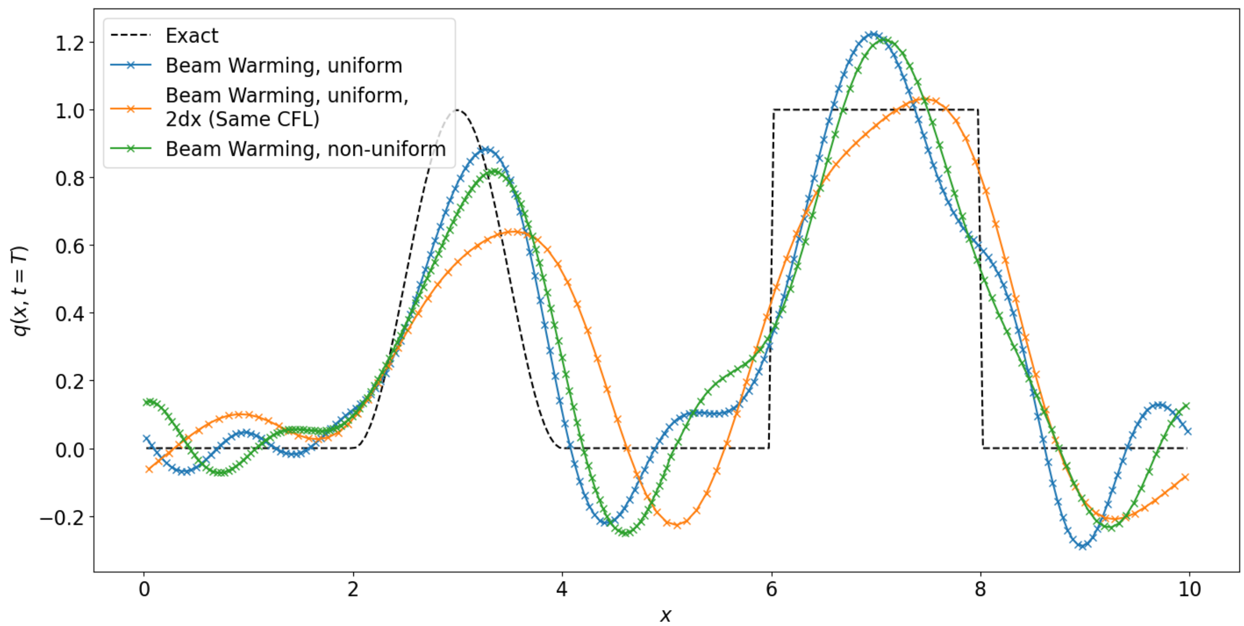


Fig. 34 Solution of advection problem on non-uniform grid using Beam Warming method after 5 periods with CFL = 0.2

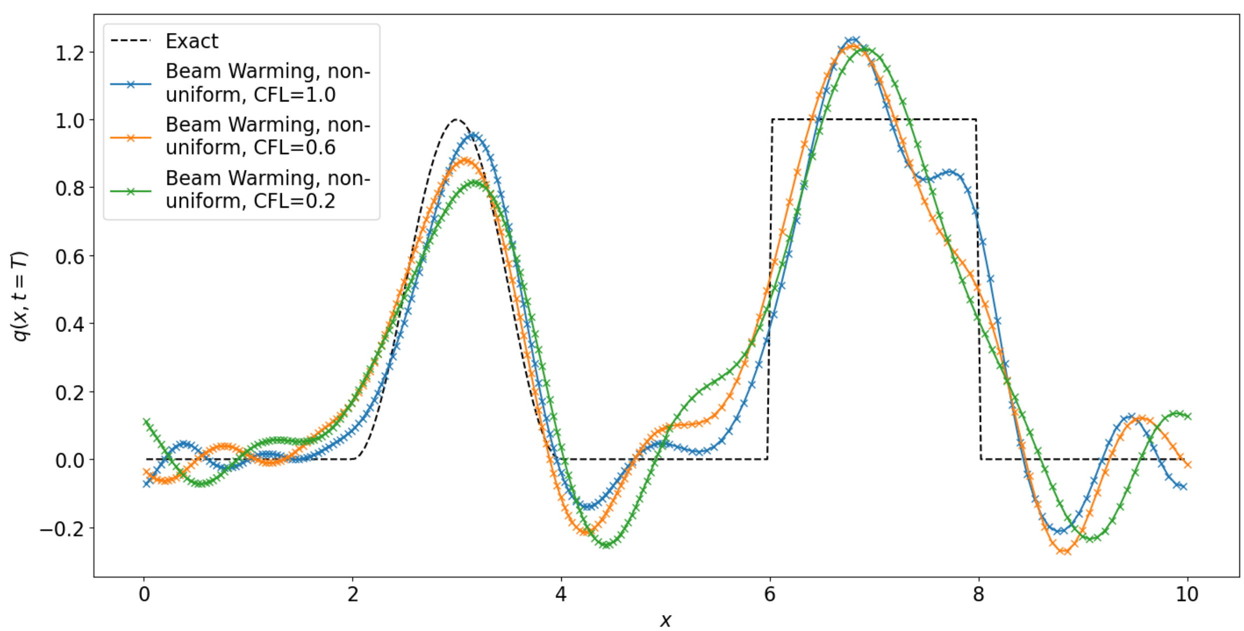


Fig. 35 Solution of advection problem on non-uniform grid using Beam Warming method after 5 periods for various CFL values

5 c. Analysis of the non-uniform grid solution

The first very obvious observation that could be made was that even at CFL = 1.0, the solutions of a non-uniform grid do not come very close to the exact solution, unlike the uniform grid's solutions. The second obvious observation was that for a lower value of CFL < 1.0, the solution of non-uniform grid seems to be in between the solution of the uniform grid with spacings dx and $2dx$. Thus, in the perspective of accuracy, the solution of the non-uniform grid seems to match up to the accuracy of the uniform-grid solution that uses the same method.

As for the wiggles, the upwind method implemented on a non-uniform grid seems to have the same amount of wiggles as the uniform grid counterpart and thus the difference in the grid itself does not induce any different wiggles. These wiggles in the upwind method, however, are mainly due to the solution not having the same shape as the exact solution. Thus, the wiggles have large wavelengths and are not rapidly oscillation around the origin.

The same can also be said for the Beam Warming method whereby the amount of wiggles is in between the ones present in the uniform grid. Additionally, the solution of the Beam-Warming method for non-uniform grid also has instability that was present in the uniform grid. This instability is due to the wiggles that push the solution into negative values. The instability slightly worsens with the reduction of the CFL number.

The dependence on CFL number for the solutions on the non-uniform grid are also just like what the solutions on the uniform grid were as shown in the previous section. Thus, a non-uniform grid does not introduce any new behaviour with respect to the CFL number.

Once again, just like it was concluded for the solution of the uniform grid, the increase in the number of periods (from 1 period to 5 periods) decreases the solution accuracy. The solution of the upwind method drastically moves away from the exact solution while the solution of the Beam-Warming method also moves away from the exact solution but to a smaller extent. However, for the Beam warming method, the solution seems to have shifted a lot more to the positive x values with an increase in the number of periods.