Gasdynamics

Task 1

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1.1 a)

b) Mass conservation

$$\cancel{\frac{\partial}{\partial t}\iiint_V \rho\, dV}^{\,0} + \iint_S \rho \bar{V}\cdot\bar{n}\, dS = 0$$

since the flow is steady

$$\Rightarrow \iint_S \rho \bar{V}\cdot\bar{n}\, dS = 0$$

$$\rho_2 V_2 A - \rho_1 V_1 A = 0$$

$$\rho_2 V_2 = \rho_1 V_1 \quad - \text{①}$$

Momentum conservation

$$\cancel{\frac{\partial}{\partial t}\iiint_V \rho \bar{V} dV}^{\,0} + \iint_S \rho \bar{V}\bar{V}\cdot\bar{n}\, dS + \iint_S p\cdot\bar{n}\, dS$$

due to steady flow

$$= \underbrace{\cancel{\iiint_V \rho f\, dV}}_{0} + \underbrace{\cancel{F_{viscous}}}_{0} + \underbrace{\cancel{F_{external}}}_{0}$$

due to absence of body forces   inviscid flow   no external forces.

$$\Rightarrow \iint_S \rho \bar{V}\bar{V}\cdot\bar{n}\, dS + \iint_S p\cdot\bar{n}\, dS = 0$$

$$\rho_2 V_2^2 \cancel{A} - \rho_1 V_1^2 \cancel{A} + p_2 \cancel{A} - p_1 \cancel{A} = 0$$

$$\left(p_2 + \rho_2 V_2^2\right) = \left(p_1 + \rho_1 V_1^2\right) \quad - \text{②}$$

substituting the variables,

$$①: (\rho + d\rho)(a - du) = \rho \cdot a$$

$$\cancel{\rho a} - \rho du + d\rho \cdot a - \underbrace{\cancel{d\rho du}}_{\approx 0} = \cancel{\rho a}$$

$$\rho du = d\rho \cdot a$$

$$a = \frac{\rho du}{d\rho} \quad — ③$$

$$②: (p + dp) + (\rho + d\rho)(a - du)^2 = p + \rho a^2$$

$$p + dp + (\rho + d\rho)(a^2 - 2a du + \underbrace{\cancel{(du)^2}}_{\approx 0}) = p + \rho a^2$$

$$\cancel{p} + dp + \cancel{\rho a^2} - 2\rho a du + a^2 d\rho - 2a \underbrace{\cancel{d\rho du}}_{\approx 0} = \cancel{p} + \cancel{\rho a^2}$$

$$dp - 2\rho a du + a^2 d\rho = 0$$

$$dp + a^2 d\rho = 2\rho a du$$

$$du = \frac{dp + a^2 d\rho}{2\rho a} \quad — ④$$

substituting ④ into ③:

$$a = \frac{\cancel{\rho}}{d\rho} \cdot \frac{dp + a^2 d\rho}{2\cancel{\rho} a}$$

$$2a^2 = \frac{dp}{d\rho} + a^2$$

$$a^2 = \frac{dp}{d\rho} \quad — ⑤$$

//

c) Energy conservation

$$\cancel{\frac{\partial}{\partial t}}\iiint_V \rho E \, dV + \iint_S \rho E \bar{V}\cdot\bar{n}\, dS + \iint_S p \bar{V}\cdot\bar{n}\, dS$$

0 steady flow

$$= \iiint \rho \cancel{\bar{f}}\cdot\bar{V}\, dV + \cancel{\dot{Q}} + \cancel{\dot{W}_{visc}} + \cancel{\dot{W}_{external}}$$

0 no body forces, 0 no heat added, 0 inviscid flow, 0 no external forces

$$\Rightarrow \iint_S \rho E \bar{V}\cdot\bar{n}\, dS + \iint_S p \bar{V}\cdot\bar{n}\, dS = 0$$

$$\cancel{\iint_S \rho \left(e + \frac{V^2}{2}\right)\bar{V}\cdot\bar{n}\, dS + \iint_S p \bar{V}\cdot\bar{n}\, dS = 0 \quad \text{since } E = e + \frac{V^2}{2}}$$

$$\cancel{\rho_2\left(e_2 + \frac{V_2^2}{2}\right)V_2 \cdot A - \rho_1\left(e_1 + \frac{V_1^2}{2}\right)V_1 \cdot A + p_2 V_2 A - p_1 V_1 A = 0}$$

$$\cancel{p_2 V_2 + \rho_2\left(e_2 + \frac{V_2^2}{2}\right)V_2 = p_1 V_1 + \rho_1\left(e_1 + \frac{V_1^2}{2}\right)V_1}$$

~~I realised it's better to keep the E, thus~~

$$\cancel{p_2 V_2 + \rho_2 E_2 V_2 = p_1 V_1 + \rho_1 E_1 V_1}$$

dividing throughout by $\rho \bar{V}$,

$$\iint_S E \cdot \bar{n}\, dS + \iint_S \frac{p}{\rho} \cdot \bar{n}\, dS = 0$$

$$E_2 \cancel{A} - E_1 \cancel{A} + \frac{p_2}{\rho_2}\cancel{A} - \frac{p_1}{\rho_1}\cancel{A} = 0$$

$$\left(E_2 + \frac{p_2}{\rho_2}\right) - \left(E_1 + \frac{p_1}{\rho_1}\right) = 0$$

$$H_2 - H_1 = 0$$

$$H_2 = H_1$$

continued...

Using $H = h + \frac{1}{2}V^2$,

$$h_2 + \frac{1}{2}V_2^2 = h_1 + \frac{1}{2}V_1^2$$

$$h_2 + \frac{1}{2}(a - du)^2 = h_1 + \frac{1}{2}a^2$$

$$h_2 + \frac{1}{2}(a^2 - 2a\,du + \underbrace{\cancel{(du)^2}}_{\approx 0}) = h_1 + \frac{1}{2}a^2$$

$$h_2 - a\,du = h_1$$

$$h_2 - h_1 = a\,du \approx 0$$

$$dh \approx 0$$

- Since it is a reversible process due to changes: dp, $d\rho$ and u being infintesimal, the 1st law of thermodynamics can be represented as:

$$\delta q - p\,dV = de, \quad \delta q = T \cdot ds \text{ for reversible process.}$$

$$\Rightarrow T \cdot ds = de + p\,dv$$

$$= dh - v\,dp.$$

$$ds = \frac{1}{T}(dh - v\,dp)$$

~~Since $dh \approx 0$ and $dp \approx 0$~~,

~~$ds \approx 0$~~

~~which proves that sound propagation is an isentropic process.~~

$$ds = \frac{1}{T}\left(a\,du - \frac{dp}{\rho}\right)$$

using (3): $$= \frac{1}{T}\left(\frac{a^2 d\rho}{\rho} - \frac{dp}{\rho}\right)$$

using (5): $$= \frac{1}{T}\left(\frac{dp}{\rho} - \frac{dp}{\rho}\right)$$

$$= 0 \Rightarrow \text{isentropic}$$

1.2 a) Aim: prove $[e] + \langle p \rangle [v] = 0$

$$\Rightarrow (e_2 - e_1) + \frac{1}{2}(p_1 + p_2)(v_2 - v_1) = 0$$

From steady jump eq$^{\underline{n}}$: $\frac{\partial F}{\partial x} = 0$, $F = \begin{pmatrix} \rho u \\ p + \rho u^2 \\ \rho u H \end{pmatrix}$

we have continuity eq$^{\underline{n}}$: $\rho_1 u_1 = \rho_2 u_2$ — ①

momentum eq$^{\underline{n}}$: $p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2$ — ②

energy eq$^{\underline{n}}$: $\rho_1 u_1 H_1 = \rho_2 u_2 H_2$ due to ①.

$H_1 = H_2$

$$E_1 + \frac{p_1}{\rho_1} = E_2 + \frac{p_2}{\rho_2} \qquad \frac{1}{\rho} = v$$

$$E_1 + p_1 v_1 = E_2 + p_2 v_2 \qquad E = e + \frac{1}{2}u^2$$

$$e_1 + \frac{1}{2}u_1^2 + p_1 v_1 = e_2 + \frac{1}{2}u_2^2 + p_2 v_2$$

$$(e_2 - e_1) + \frac{1}{2}\underbrace{(u_2^2 - u_1^2)}_{(u_2+u_1)(u_2-u_1)} + (p_2 v_2 - p_1 v_1) = 0 \quad — ③$$

substitute ① into ②: $p_1 + \rho_2 u_2 u_1 = p_2 + \rho_2 u_2^2$

$$p_1 - p_2 = \rho_2 u_2 (u_2 - u_1)$$

$$u_2 - u_1 = \frac{p_1 - p_2}{\rho_2 u_2} \quad — ④$$

substitute ④ into ③: $(e_2 - e_1) + \frac{1}{2}(u_2 + u_1)\frac{p_1 - p_2}{\rho_2 u_2} + (p_2 v_2 - p_1 v_1) = 0$

$$(e_2 - e_1) + \frac{1}{2}\left(\frac{u_2}{u_2} + \frac{u_1}{u_2}\right) v_2 \cdot (p_1 - p_2) + (p_2 v_2 - p_1 v_1) = 0$$

using ①

$$(e_2 - e_1) + \frac{1}{2}\left(1 + \frac{\rho_2}{\rho_1}\right) v_2 (p_1 - p_2) + (p_2 v_2 - p_1 v_1) = 0$$

$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2}$

$$(e_2 - e_1) + \frac{1}{2}(v_2 + v_1)(p_1 - p_2) + (p_2 v_2 - p_1 v_1) = 0$$

$$(e_2 - e_1) + \frac{1}{2}v_2 p_1 - \frac{1}{2}v_2 p_2 + \frac{1}{2}v_1 p_1 - \frac{1}{2}v_1 p_2 + p_2 v_2 - p_1 v_1 = 0$$

$$(e_2 - e_1) + \frac{1}{2}v_2 p_1 - \frac{1}{2}v_1 p_2 + \frac{1}{2}v_2 p_2 - \frac{1}{2}v_1 p_1 = 0$$

$$(e_2 - e_1) + \frac{1}{2}(p_1 + p_2)(v_2 - v_1) = 0$$

$$[e] + \langle p \rangle [v] = 0 \quad //$$

b) Aim: $\frac{[p]}{\langle p \rangle} + \gamma \frac{[v]}{\langle v \rangle} = 0$

$$\Rightarrow \frac{p_2 - p_1}{\frac{1}{2}(p_1 + p_2)} + \gamma \cdot \frac{v_2 - v_1}{\frac{1}{2}(v_1 + v_2)} = 0$$

~~Similarly, we have~~ $\cancel{\rho_1 u_1 = \rho_2 u_2}$

$$\cancel{p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2}$$

$$\cancel{\Rightarrow p_1 + \frac{u_1^2}{v_1} = p_2 + \frac{u_2^2}{v_2}}$$

$$\cancel{(e_2 - e_1) + \frac{1}{2}(u_2 + u_1)(u_2 - u_1) + (p_2 v_2 - p_1 v_1) = 0}$$

with $p = (\gamma - 1)\rho e. = \frac{(\gamma-1)}{v} e \Rightarrow e = \frac{pv}{\gamma - 1}$ — (5)

starting from part a)

$$e_2 - e_1 + \frac{1}{2}(p_1 + p_2)(v_2 - v_1) = 0$$

substituting (5): $\frac{p_2 v_2 - p_1 v_1}{\gamma - 1} + \frac{1}{2}(p_1 + p_2)(v_2 - v_1) = 0$

$$p_2 v_2 - p_1 v_1 + \frac{\gamma - 1}{2}(p_1 + p_2)(v_2 - v_1) = 0$$

$$p_2 v_2 - p_1 v_1 + \frac{\gamma}{2}(p_1 + p_2)(v_2 - v_1) - \frac{1}{2}(p_1 + p_2)(v_2 - v_1) = 0$$

$$p_2 v_2 - p_1 v_1 + \frac{\gamma}{2}(p_1 + p_2)(v_2 - v_1) - \frac{1}{2}p_1 v_2 + \frac{1}{2}p_1 v_1 - \frac{1}{2}p_2 v_2 + \frac{1}{2}p_2 v_1 = 0$$

$$\frac{1}{2}p_2 v_2 - \frac{1}{2}p_1 v_1 - \frac{1}{2}p_1 v_2 + \frac{1}{2}p_2 v_1 + \frac{\gamma}{2}(p_1 + p_2)(v_2 - v_1) = 0$$

$$\frac{1}{2}(p_2 - p_1)(v_1 + v_2) + \frac{\gamma}{2}(p_1 + p_2)(v_2 - v_1) = 0$$

$$\frac{p_2 - p_1}{\frac{1}{2}(p_1 + p_2)} + \gamma \frac{v_2 - v_1}{\frac{1}{2}(v_1 + v_2)} = 0$$

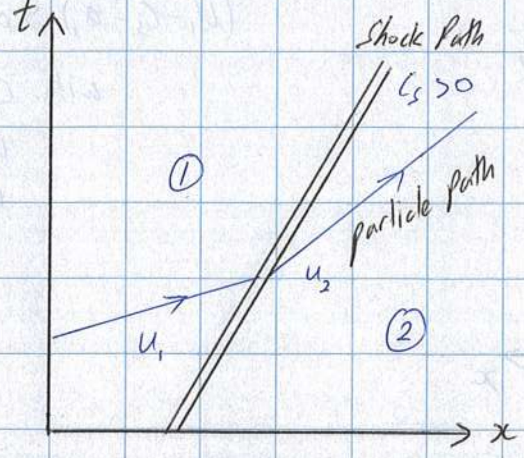
$$\frac{[p]}{\langle p \rangle} + \gamma \frac{[v]}{\langle v]} = 0 \quad //$$

c) equation in part a): $[e] + \langle p \rangle [v] = 0$ resembles the 1st law of thermodynamics differential relation of an adiabatic process ($dQ = 0$) which results in $de = -p dv \Rightarrow de + p dv$.

equation in part b), $\frac{[p]}{\langle p \rangle} + \gamma \frac{[v]}{\langle v \rangle} = 0$ resembles the 1st law of thermodynamics differential relation of isentropic process of a calorically perfect gas which from

$$ds = c_v \left(\frac{dp}{p} + \gamma \frac{dv}{v} \right) \text{ gives } \frac{dp}{p} + \gamma \frac{dv}{v} = 0.$$

1.3 a)

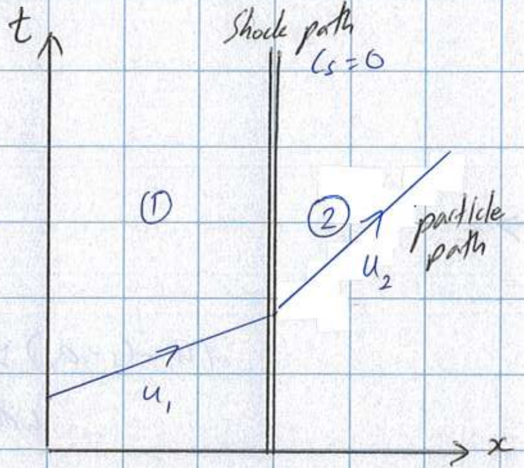


Condition

$$(u_1 - c_s - a_1) > 0 > (u_2 - c_s - a_2)$$

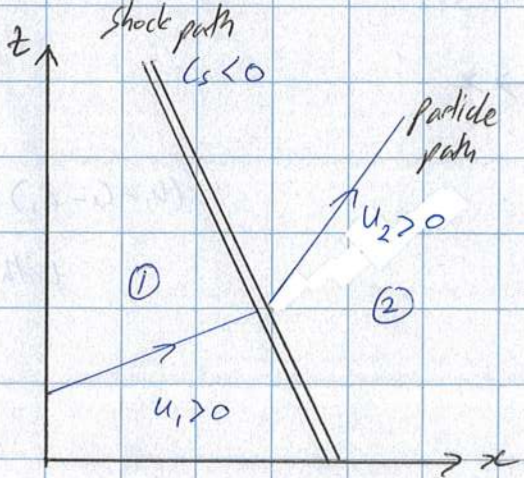
In the shock frame flow entering the shock has to be supersonic and flow leaving the shock has to be subsonic.

b)



$$u_1 - a_1 > 0 > u_2 - a_2$$

c)



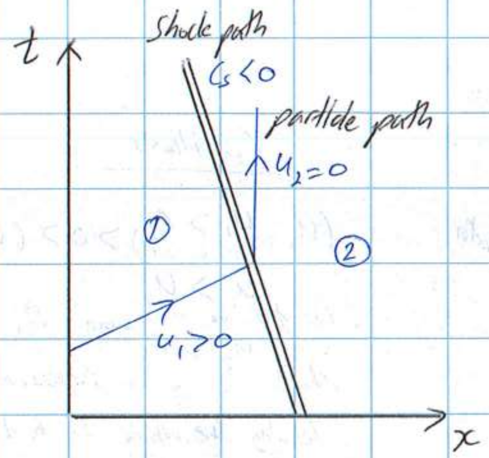
$$(u_1 - c_s - a_1) > 0 > (u_2 - c_s - a_2)$$

with $c_s < 0$

$u_1 > 0$

$u_2 > 0$

c2)



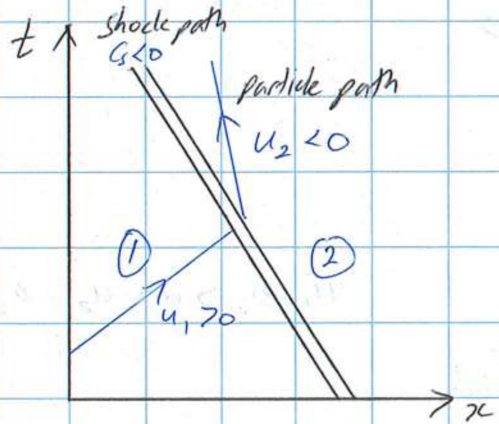
$(u_1-c_s-a_1)>0>(u_2-c_s-a_2)$

with $c_s<0$

$u_1>0$

$u_2=0$

c3)



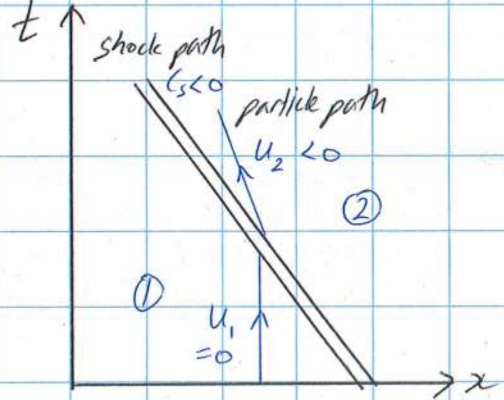
$(u_1-c_s-a_1)>0>(u_2-c_s-a_2)$

with $c_s<0$

$u_1>0$

$u_2<0$

c4)



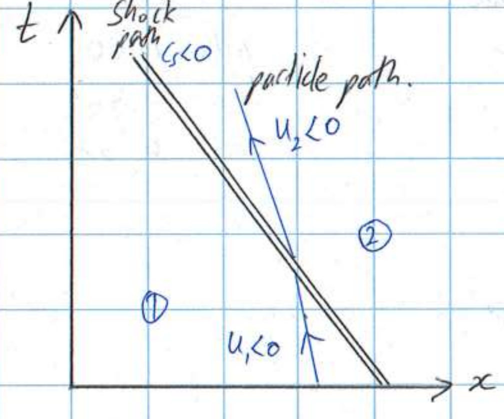
$(u_1-c_s-a_1)>0>(u_2-c_s-a_2)$

with $c_s<0$

$u_1=0$

$u_2<0$

c5)



$(u_1-c_s-a_1)>0>(u_2-c_s-a_2)$

with $c_s<0$

$u_1<0$

$u_2<0$

Typical velocities in limiting cases of strong shock in air with $\frac{\rho_2}{\rho_1} \to 6$.

sign convention $\longmapsto +$

a) $\overrightarrow{7} \; \| \xrightarrow{} 1 \; \overrightarrow{2}$

$u_1 = 1.4M$

$u_2 = 0.4M$

b) $\overrightarrow{6} \; \|^{0} \; \overrightarrow{1}$

$u_1 = 1.2M$

$u_2 = 0.2M$

c1) $\overrightarrow{11} \; \overleftarrow{1}\| \; \overrightarrow{1}$

(the leftward pointing arrow accounts for negativity)

$u_1 = 1.1M$

$u_2 = 0.1M$

c2) $\overrightarrow{5} \; \overleftarrow{1}\| \; 0$

$u_1 = 1.5M$

$u_2 = 0$

c3) $\overrightarrow{4} \; \overleftarrow{2}\| \; \overleftarrow{1}$

$u_1 = 1.2M$

$u_2 = -0.3M$

c4) $0 \; \overleftarrow{6}\| \; \overleftarrow{5}$

$u_1 = 0$

$u_2 = -0.9M$

c5) $\overleftarrow{1} \; \overleftarrow{7}\| \; \overleftarrow{6}$

$u_1 = -0.2M$

$u_2 = -1.2M$.

1.4 a) Helium : $\gamma = \frac{5}{3}$, $\left(\frac{\rho_2}{\rho_1}\right)_{max} = 4. = \frac{\gamma+1}{\gamma-1}$

~~Since it is a~~ The very-strong-shock limit allows for usage of $\left(\frac{\rho_2}{\rho_1}\right)_{max}$ which is 4 for Helium.

Moving onto ~~lab~~ shock frame from lab frame, we have.

$\overset{+}{\mapsto}$

$\overrightarrow{u_1 - C_s} \quad \Vert \quad \overrightarrow{u_2 - C_s}$ whereby $C_s < 0$ and $u_2 = 0$

Applying conservation of mass gives:

$$\rho_1 (u_1 - C_s) = \rho_2 (u_2 - C_s)$$

$$u_1 - C_s = \frac{\rho_2}{\rho_1}(-C_s)$$

for this limit case: $u_1 - C_s = \left(\frac{\rho_2}{\rho_1}\right)_{max}(-C_s)$

$$u_1 - C_s = 4(-C_s) = -4C_s$$

$$u_1 = -3C_s$$

$$C_s = -\frac{1}{3}u_1 \quad // \text{shown.}$$

b) Conservation of momentum equation:

$$P_1 + \rho_1 \underline{u_1}^2 = P_2 + \rho_2 \underline{u_2}^2$$

$\underline{u_1} = u_1 - C_s = \frac{4}{3}u_1$

$\underline{u_2} = u_2 - C_s = -C_s = \frac{1}{3}u_1$

$$P = \rho R T = \frac{\rho}{\gamma} a^2$$

$$\Rightarrow \frac{\rho_1}{\gamma} a_1^2 + \rho_1 \underline{u_1}^2 = P_2 + \rho_2 \underline{u_2}^2$$

since $u_1 >> a_1$, and $\gamma > 1$ $\quad \rho_1 \underline{u_1}^2 \approx P_2 + \rho_2 \underline{u_2}^2$

tells us that $\frac{\rho_1}{\gamma} a_1^2$ will not blow up to a large number.

$$\rho_1 \frac{16}{9} u_1^2 \approx P_2 + \rho_2 \frac{1}{9} u_1^2$$

$$\frac{16}{9} u_1^2 \approx \frac{P_2}{\rho_1} + 4 \times \frac{1}{9} u_1^2$$

$$\frac{12}{9} u_1^2 \approx \frac{P_2}{\rho_1} \Rightarrow P_2 \approx \frac{4}{3}\rho_1 u_1^2 \quad // \text{shown.}$$

c) Conservation of energy: $\cancel{H_1 = H_2}$

$$\cancel{h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2}$$

Using eq1 for calorically perf. gas,

$$e_1 + \cancel{\frac{P_1}{\rho_1}} + \cancel{\frac{1}{2}u_1^2} = e_2 + \cancel{\frac{P_2}{\rho_2}} + \cancel{\frac{1}{2}u_2^2}$$

$$P_2 = (\gamma - 1)\rho_2 e_2$$

$$P_2 = \left(\frac{5}{3} - 1\right)\rho_2 e_2$$

$$P_2 = \frac{2}{3}\rho_2 e_2$$

substitute $P_2 \approx \frac{4}{3}\rho_1 u_1^2$ from part b)

$$\frac{4}{3}\rho_1 u_1^2 \approx \frac{2}{3}\rho_2 e_2$$

$$2u_1^2 \approx \frac{\rho_2}{\rho_1} e_2$$

$$2u_1^2 \approx 4 e_2$$

$$e_2 \approx \frac{1}{2}u_1^2$$

// shown.

d). Energy of the flow in 2 that has 0 velocity equals to the kinetic energy of flow in 1 since P_1, as proved in part b), is negligible.

2.1 a) Unperturbed: regions ①, ④, ⑥

- These regions are not enclosed by the two Γ^+ characteristic lines and two Γ^- characteristic lines.

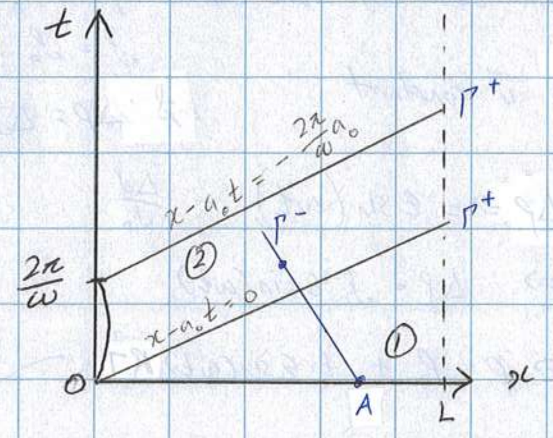
Simple waves: regions ②, ⑤

- The flow in these regions have only 1 Riemann invariant each. For ②, it has only the Γ^+ characteristic line's Riemann invariant while ⑤ has only the Γ^- characteristic line's Riemann invariant

Non-simple waves: region ③

- The flow in region ③ is influenced by 2 Riemann invariants, from both the Γ^+ and Γ^- characteristic lines.

b)



Point A is an initial condition and it has $M_A = 0$; $\tilde{S}_A = 0$

Thus, along Γ^-:

$M_A - \tilde{S}_A = M_2 - \tilde{S}_2$

$0 = M_2 - \tilde{S}_2$

$M_2 = \tilde{S}_2$ where 2 is a point lying on Γ^-, in region ②.

Since $u_p = \varepsilon \cdot a_0 \sin(\omega t)$ for $0 \le t \le \frac{2\pi}{\omega}$, $M_p = \varepsilon \cdot \sin(\omega t)$

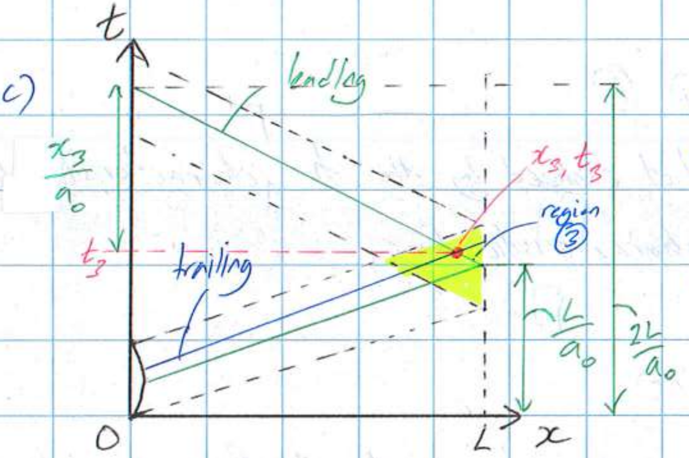
(a_0 ↳ speed of sound)

As shown in the graph, region ② is defined by: $-a_0 \frac{2\pi}{\omega} < (x - a_0 t) < 0$

Thus, $M_2 = \varepsilon \cdot \sin\left[\omega\left(t_2 - \frac{x_2}{a_0}\right)\right] = \tilde{S}_2$

where (x_2, t_2): coordinates of the point in region ②.

c)



We first consider the leading flow which has already been reflected off the end of the tube at $x = L$. This flow is represented in green. It can be observed from the plot on the left that this flow has $M = \varepsilon \cdot \sin\left\{\omega\left[t_3 - \left(\frac{2L - x_3}{a_0}\right)\right]\right\}$ in the negative direction due to reflection

Assumption: - Perfectly elastic collision at the static wall.

We then consider the trailing flow represented in blue in the plot. This is the same flow discussed in part b) with $M = \varepsilon \cdot \sin\left[\omega\left(t_3 - \frac{x_3}{a_0}\right)\right]$.

Thus, the M at ③ is, $M_3 = \varepsilon \sin\left[\omega\left(t_3 - \frac{x_3}{a_0}\right)\right] - \varepsilon \sin\left\{\omega\left[t_3 - \frac{2L}{a_0} + \frac{x_3}{a_0}\right]\right\}$

~~with~~ defining rightwards as positive.

d) let P_0, ρ_0 be the initial condition and that $P = P_0 + \Delta P$.

$\rho = \rho_0 + \Delta\rho$.

Also, assume temperature is constant. and $\Delta P = \Delta\rho\, a_0^2$.

at $x = 0$; $0 \leq t \leq \frac{2\pi}{\omega}$: $\tilde{S} = M = \varepsilon \sin(\omega t) = \frac{\Delta\rho}{\rho_0}$

$\Rightarrow \Delta\rho = \rho_0 \varepsilon \sin(\omega t)$

$\Rightarrow P = P_0 + \rho_0 \varepsilon \sin(\omega t)\, a_0^2$.

at $x = 0$; $t > \frac{2\pi}{\omega}$: $P = P_0$

at $x = L$; $0 \leq t \leq \frac{L}{a_0}$: $P = P_0$

at $x = L$; $\frac{L}{a_0} \leq t \leq \left(\frac{L}{a_0} + \frac{2\pi}{\omega}\right)$: since $M - \tilde{S}$ is constant along Γ^+,

and that $M = \tilde{S}$ in this region,

$P = P_0 + \rho_0 \varepsilon \sin\left(\omega \cdot \left(t - \frac{L}{a_0}\right)\right) \cdot a_0^2$

at $x = L$; $t > \left(\frac{L}{a_0} + \frac{2\pi}{\omega}\right)$: $P = P_0$.

e) the observer experiences a cycle of still flow - rightward flow - still flow - leftward flow with varying time intervals, depending on the value of x.

f) we analyse the change in system from $t=0$ to $t=\frac{2\pi}{\omega}$.

As for kinetic energy, KE, $KE=\frac{1}{2}mu^2$, m: mass, u: velocity

$$\Delta KE = KE_{t=\frac{2\pi}{\omega}} - KE_{t=0} = KE_{t=\frac{2\pi}{\omega}} - 0$$

from $x-a_0t=0$, we know that gas has moved a distance of $\frac{2\pi a_0}{\omega}$ for $t=\frac{2\pi}{\omega}$.

$$\Delta KE = KE_{t=\frac{2\pi}{\omega}} = \int_0^{\frac{2\pi a_0}{\omega}} \frac{1}{2} A_t \rho_{t=\frac{2\pi}{\omega}} u^2_{t=\frac{2\pi}{\omega}}\, dx$$

where $\rho_{t=\frac{2\pi}{\omega}} = \rho_0 + \rho_0 \varepsilon \cdot \sin\left[\omega\left(\frac{2\pi}{\omega}-\frac{x}{a_0}\right)\right]$

and $u_{t=\frac{2\pi}{\omega}} = a_0 \varepsilon \sin\left[\omega\left(\frac{2\pi}{\omega}-\frac{x}{a_0}\right)\right]$

$$= \frac{1}{2} A_t \rho_0 a_0^2 \varepsilon^2 \int_0^{\frac{2\pi a_0}{\omega}} \left[1+\varepsilon \sin\left(2\pi - \frac{\omega}{a_0}x\right)\right]\left[\sin\left(2\pi-\frac{\omega}{a_0}x\right)\right]^2 dx$$

$$= \frac{1}{2} A_t \rho_0 a_0^2 \varepsilon^2 \int_0^{\frac{2\pi a_0}{\omega}} \Big[\underbrace{\sin^2\left(2\pi - \frac{\omega}{a_0}x\right)}_{①} + \underbrace{\varepsilon \sin^3\left(2\pi-\frac{\omega}{a_0}x\right)}_{②}\Big] dx$$

$$① = \int_0^{\frac{2\pi a_0}{\omega}} \frac{1}{2}\left(1-\cos\left[2\left(2\pi-\frac{\omega}{a_0}x\right)\right]\right) dx$$

$$= \frac{\pi a_0}{\omega}$$

$$② = \varepsilon \int_0^{2\pi \frac{a_0}{\omega}} \sin\left(2\pi - \frac{\omega}{a_0}x\right)\left(1 - \cos^2\left(2\pi - \frac{\omega}{a_0}x\right)\right) dx$$

$$= \left[\frac{a_0}{\omega} \cos\left(2\pi - \frac{\omega}{a_0}x\right) + \frac{a_0}{3\omega}\cos^3\left(2\pi - \frac{\omega}{a_0}x\right)\right]_0^{2\pi \frac{a_0}{\omega}}$$

$$= 0.$$

$$\therefore \Delta KE = \frac{1}{2} A_t \rho_0 a_0^2 \varepsilon^2 \times \frac{\pi a_0}{\omega}$$

$$= \frac{1}{2\omega} \pi a_0^3 \varepsilon^2 A_t \rho_0 .$$

For work done. $W = \int p A_t \, dx. \quad -\circledast$

where p is the pressure by gas on piston

$$u_p = \varepsilon a_0 \sin(\omega t) = \frac{dx_p}{dt}$$

$$\rho_{t=\frac{2\pi}{\omega}} = \rho_0 + d\rho \quad , \quad dp = a_0^2 d\rho$$

$$d\rho = \rho_0 \cdot \underbrace{\varepsilon \sin(\omega t)}_{\tilde{s}_p = u_p}$$

$$\Rightarrow dp = a_0^2 \rho_0 \varepsilon \sin(\omega t)$$

$$\circledast : \quad W = \int_0^{2\pi/\omega} \left(p_0 + a_0^2 \rho_0 \varepsilon \sin(\omega t)\right) A_t \cdot \varepsilon a_0 \sin(\omega t) \, dt$$

$$= \int_0^{2\pi/\omega} \left[\underbrace{p_0 A_t \varepsilon a_0 \cdot \sin(\omega t)}_{③} + \underbrace{a_0^3 \rho_0 \varepsilon^2 A_t \sin^2(\omega t)}_{④} \right] dt$$

$$③ = p_0 A_t \varepsilon a_0 \int_0^{2\pi/\omega} \sin(\omega t)\, dt$$

$$= 0$$

$$④ = a_0^3 \rho_0 \varepsilon^2 A_t \int_0^{2\pi/\omega} \sin^2(\omega t)\, dt$$

$$= a_0^3 \rho_0 \varepsilon^2 A_t \int_0^{\frac{2\pi}{\omega}} \frac{1}{2}(1 - \cos(2\omega t))\, dt$$

$$= \frac{a_0^3 \varepsilon^2 A_t \pi \rho_0}{\omega}$$

$$\therefore KE = \frac{\pi a_0^3 \varepsilon^2 A_t \rho_0}{2\omega} \neq \frac{\pi a_0^3 \varepsilon^2 A_t \rho_0}{\omega} = \text{Work done.}$$

$$\Rightarrow KE = \frac{1}{2} WD$$

This is due to omitting effect of change in other parameters. In reality, some heat is lost to surroundings as the piston compresses the gas. Furthermore, the energy stored in the gas of higher pressure and density is not accounted for.

2.2 a) Diagram 1

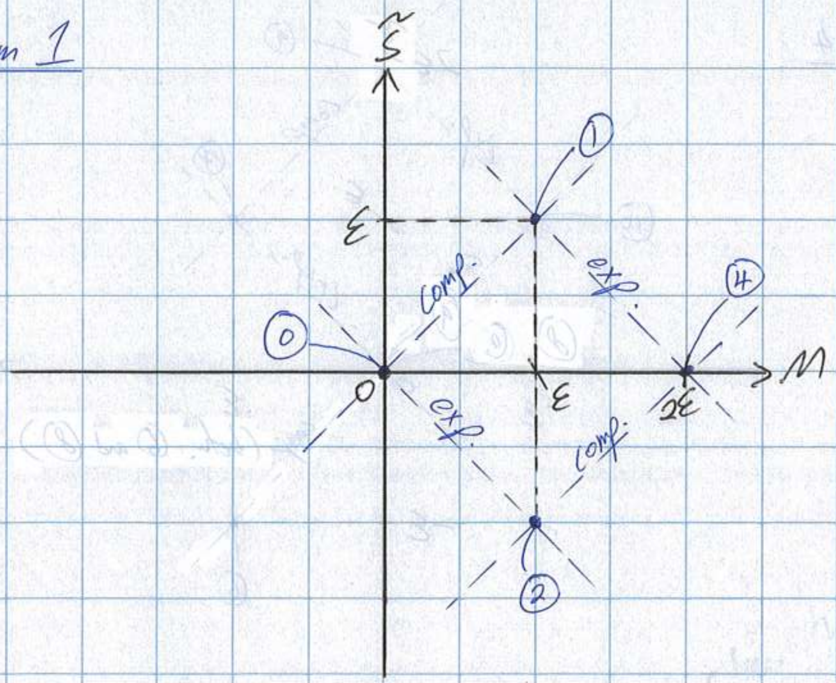


Diagram 2

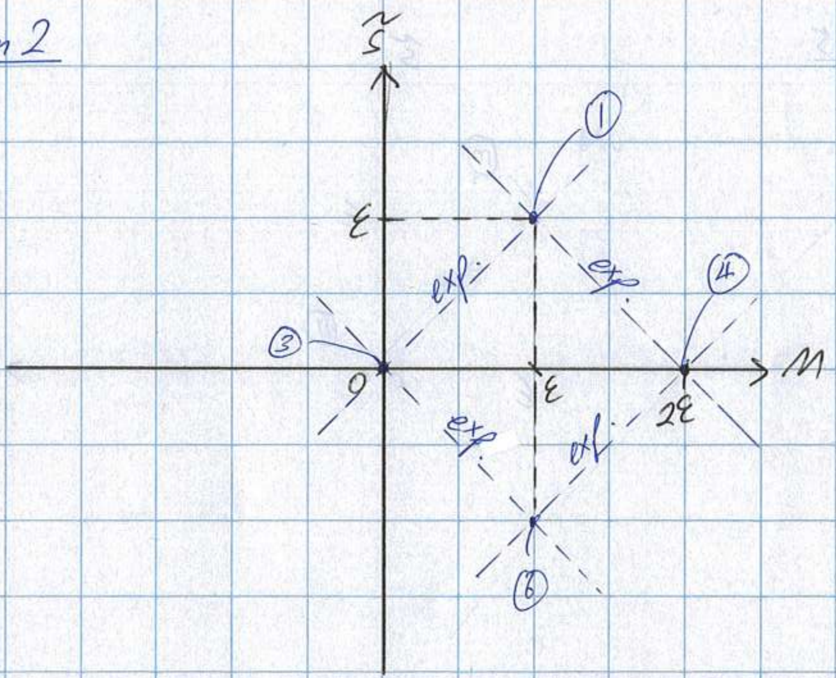


Diagram 3

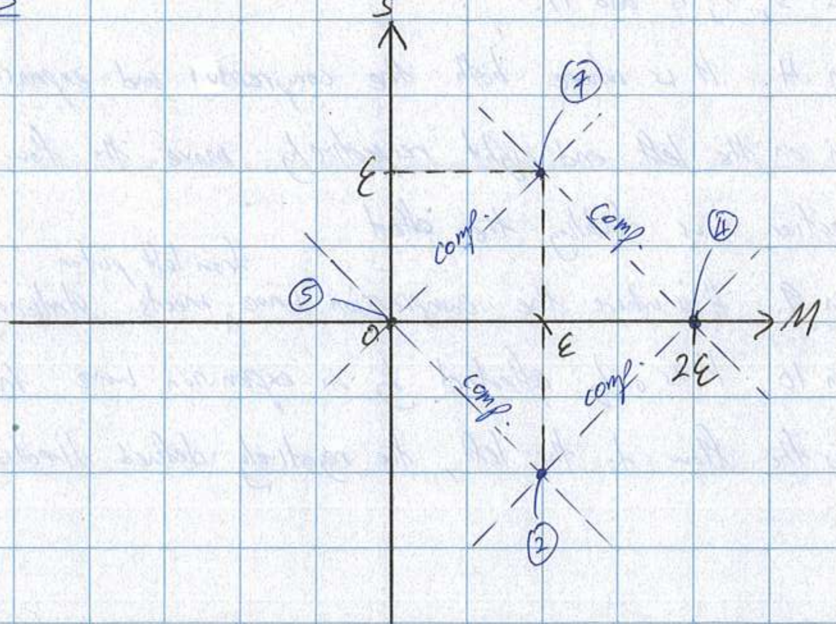


Diagram 4

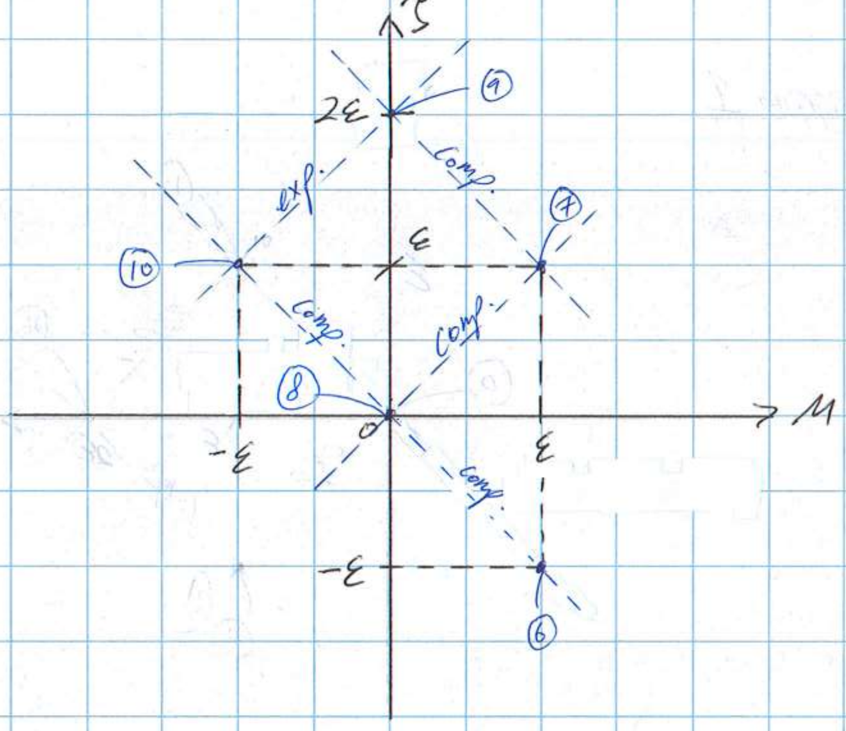
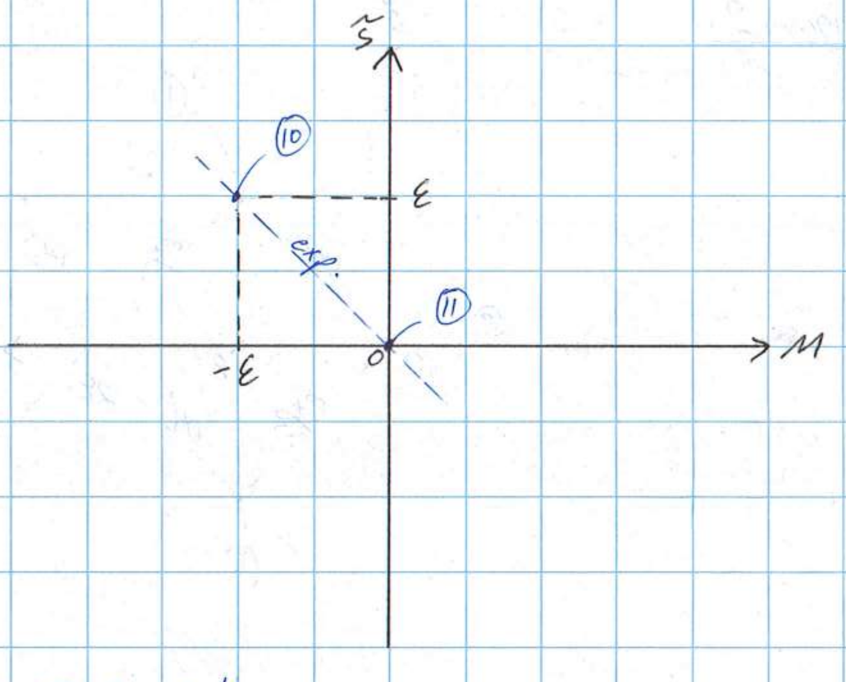


Diagram 5



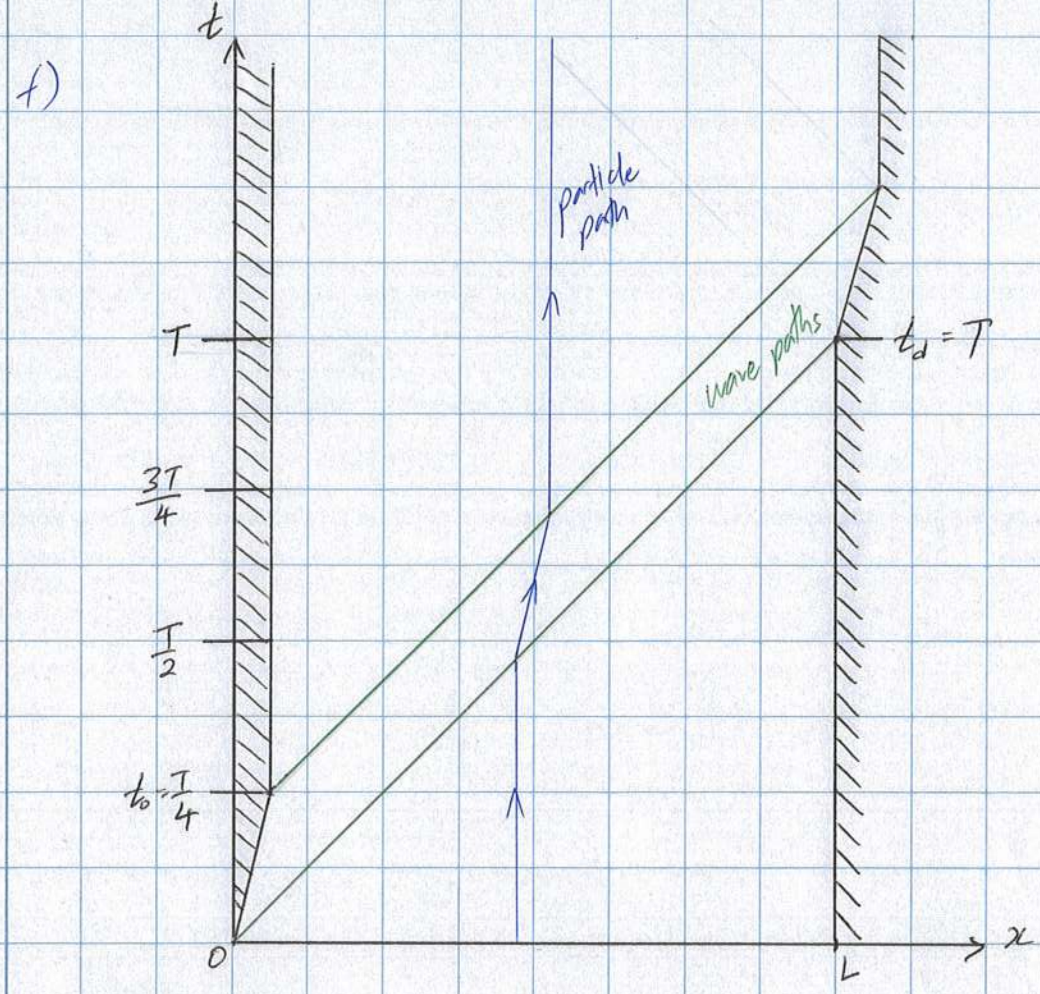
b) Regions 3, 5, 8 and 11.

c) Region 4. It is where both the compression and expansion waves from piston on the left and right respectively move the flow in the same direction thus doubling the effect.

d) Region 9. It is where the compression wave from left piston meets stationary wall.

e) Region 10. It is only affected by an expansion wave from left piston which moves the flow to the left, the negatively defined direction.

f)



The system can be modified in a way that an expansion wave is induced from piston on the right at the time when the compression wave from piston on the left reaches the right piston. ~~This~~ Furthermore these 2 pistons should move with the same velocity and have an equal duration of movement.

2.3 a) $a^2 = \gamma R T$, $p = \rho R T$

$a^2 = \gamma \cdot \frac{p}{\rho}$

$a = \sqrt{\gamma \cdot \frac{p}{\rho}}$

before reflection

slope of characteristic to the left of c.d. $= a_0 = \sqrt{\gamma \cdot \frac{p_0}{\rho_0}}$

slope of characteristic to the left of c.d. after reflection $= -a_0 = -\sqrt{\gamma \cdot \frac{p_0}{\rho_0}}$

slope of characteristic to the right of c.d (transmitted) $= a_4 = \sqrt{\gamma \cdot \frac{p_4}{\rho_4}}$

- If $\rho_4 > \rho_0$, $a_4 < a_0$. This means that this slope (to the right of c.d.) is steeper in the $t-x$ plot
  (vert - horiz)

- If $\rho_4 < \rho_0$, $a_4 > a_0$ and the outcome is opposite. This slope (to the right of c.d.) is less steep in the $t-x$ plot.

b) It is given that the same gas in regions 0 and 4 is at rest which means, $V_{n,0} = V_{t,0} = V_{n,4} = V_{t,4} = 0$. ($V_n$: normal vel., V_t: transverse vel.)
For contact discontinuity, $V_{n,0} = V_{n,4} = C_s$ (C_s: vel. of contact discontinuity)
Hence, $C_s = 0$.

The c.d. starts moving when it intersects with the characteristic lines and this is at $t = \frac{L}{a_0}$

$M_0 = \tilde{S}_0 = 0 \Rightarrow M_1 = \tilde{S}_1 = \varepsilon = \frac{U_p}{a_0}$

$M_1 = \frac{U_1}{a_0} = \varepsilon \Rightarrow U_1 = \varepsilon a_0$

$\tilde{S}_1 = \frac{\Delta \rho_1}{\rho_0} = \varepsilon \Rightarrow \Delta \rho_1 = \varepsilon \rho_0$ $\quad p_1 = p_0 + \varepsilon \rho_0 a_0^2$ since $a_0^2 = \frac{\Delta p}{\Delta \rho}$

c) Consider characteristic Γ^+ going through region 1 and 2

$\Gamma^+: \mu_1 + \tilde{S}_1 = \mu_2 + \tilde{S}_2$

$$\frac{U_p}{a_0} + \frac{U_p}{a_0} = \frac{u_2}{a_0} + \frac{\Delta\rho_2}{\rho_0}$$

using result from part b)

$$\frac{2U_p}{a_0} = \frac{u_2}{a_0} + \frac{\Delta\rho_2}{\rho_0} \quad ; \quad P_2 = P_0 + \Delta P_2$$

$$= P_0 + \Delta\rho_2 a_0^2$$

$$\Delta\rho_2 = \frac{P_2 - P_0}{a_0^2}$$

$$\frac{2U_p}{a_0} - \frac{u_2}{a_0} = \frac{P_2 - P_0}{\rho_0 a_0^2}$$

$$P_2 = \rho_0 a_0 (2U_p - u_2) + P_0 \quad — \quad ①$$

Consider characteristic Γ^- going through region 3 and 4

$\Gamma^-: \mu_3 - \tilde{S}_3 = \cancel{\mu_4}_0 - \cancel{\tilde{S}_4}_0 = 0$

$$\mu_3 = \tilde{S}_3$$

$$\frac{u_3}{a_4} = \frac{\Delta\rho_3}{\rho_4} \quad ; \quad P_3 = P_4 + \Delta\rho_3 a_4^2$$

$$\Delta\rho_3 = \frac{P_3 - P_4}{a_4^2}$$

$$\frac{u_3}{a_4} = \frac{P_3 - P_4}{\rho_4 a_4^2}$$

$$P_3 = u_3 a_4 \rho_4 + P_4 \quad — \quad ②$$

given $P_2 = P_3$: ① = ②

since $P_0 = P_4$

$$\rho_0 a_0 (2U_p - u_2) + \cancel{P_0} = u_3 a_4 \rho_4 + \cancel{P_4}$$

$$\rho_0 a_0 (2U_p - u_{cd}) = u_{cd} a_4 \rho_4 \quad \text{since } u_2 = u_3 = u_{cd}$$

using characteristic of a c.d.

$$2\rho_0 a_0 U_p - \rho_0 a_0 u_{cd} = u_{cd} a_4 \rho_4$$

$$u_{cd} (a_0 \rho_0 + a_4 \rho_4) = 2\rho_0 a_0 U_p$$

$$u_{cd} = \frac{2\rho_0 a_0 U_p}{a_0 \rho_0 + a_4 \rho_4} = \frac{\rho_0 a_0}{\frac{1}{2}(\rho_0 a_0 + \rho_4 a_4)} U_p$$

// shown.

c (continued)

② $M_2 = \frac{u_2}{a_0} = \frac{u_{c.d}}{a_0}$ where $u_{c.d} = \frac{\rho_0 a_0}{\frac{1}{2}(\rho_0 a_0 + \rho_4 a_4)} U_p$

$$\tilde{S}_2 = M_1 + \tilde{S}_1 - M_2$$

$$= \frac{2U_p}{a_0} - \frac{u_{c.d.}}{a_0}$$

$$= \frac{1}{a_0}(2U_p - u_{c.d})$$

③ $M_3 = \frac{u_3}{a_4} = \frac{u_{c.d}}{a_4}$ where $a_4 = \sqrt{\gamma \cdot \frac{P_4}{\rho_4}}$

$$\tilde{S}_3 = M_3 = \frac{u_{c.d}}{a_4}$$

⑥ $\Gamma^-_{(2)-(6)}$: $M_6 - \tilde{S}_6 = M_2 - \tilde{S}_2$

$$= \frac{u_{c.d}}{a_0} - \frac{2U_p - u_{c.d}}{a_0}$$

$$= \frac{2(U_p + u_{c.d})}{a_0}$$

$\Gamma^+_{(5)-(6)}$: $M_6 + \tilde{S}_6 = M_5 + \tilde{S}_5$

$$= 0$$

$$M_6 = -\tilde{S}_6$$

$$\therefore \quad 2M_6 = \frac{2(U_p + u_{c.d})}{a_0}$$

$$M_6 = \frac{U_p + u_{c.d}}{a_0}$$

$$\tilde{S}_6 = -\frac{U_p + u_{c.d}}{a_0}$$

d) for $\rho_4 = \rho_0$ that has no c.d., there will not be any reflected wave and thus region 6 will be unperturbed. The transmitted wave will have the same slope as incoming wave (from left). Thus, region 3 will have the same property as region 1.

for $\rho_4 >> \rho_0$, point B will lie directly above A, so at $x = L$ resulting in a smaller region 6. ~~According to equations for M_6 and $\tilde{\rho}_6$, region 6 will have a smaller velocity and less negative change in density~~ Flow in region 6 will have velocity of same magnitude as that of region 1 but in opposite direction. Density change will be the same as that of region 1. As for transmitted wave, there will not be any transmitted wave and thus region 3 will be unperturbed.

2.4 $\quad u_0 = \tilde{s}_0 = 0 \quad , \quad u_\ell = \varepsilon a_0$

a) $\Gamma^-: u_1 - \tilde{s}_1 = u_0 - \tilde{s}_0$

$u_1 - \tilde{s}_1 = 0$

$u_1 = \tilde{s}_1 = \varepsilon$

$\Gamma^+: u_2 + \tilde{s}_2 = u_1 + \tilde{s}_1$

$u_2 + \tilde{s}_2 = 2\varepsilon \quad - ①$

Region 2 is affected by expansion wave induced by $u_r(t)$

$\therefore u_2 = \dfrac{u_r(t)}{a_0}$

substituting into ①: $\tilde{s}_2 = 2\varepsilon - \dfrac{u_r(t)}{a_0}$

Region 3 is affected by compressive wave on the other side of right piston.

$\therefore u_3 = \dfrac{u_r(t)}{a_0}$

$\Gamma^-: u_3 - \tilde{s}_3 = u_0 - \tilde{s}_0$

$= 0$

$u_3 = \tilde{s}_3$

$\tilde{s}_3 = \dfrac{u_r(t)}{a_0}$

b) Force $= ma$

pressure diff. betw. ② and ③ $\times A = m \cdot \dfrac{d u_r(t)}{dt}$

$$\frac{d u_r(t)}{dt} = \frac{A}{m} \cdot \left(\cancel{P_0} + \Delta P_2 - (\cancel{P_0} + \Delta P_3)\right)$$

$$= \frac{A}{m} \cdot \left(\Delta \rho_2 a_0^2 - \Delta \rho_3 a_0^2\right)$$

$$= \frac{A}{m} a_0^2 \left(\tilde{s}_2 \times \rho_0 - \tilde{s}_3 \times \rho_0\right)$$

$$= \frac{A}{m} a_0^2 \rho_0 \left(2\varepsilon - \frac{u_r(t)}{a_0} - \frac{u_r(t)}{a_0}\right)$$

$$= \frac{2A}{m} a_0^2 \rho_0 \left(\varepsilon - \frac{u_r(t)}{a_0}\right)$$

continued.

$$= \frac{2A}{m} a_0 \rho_0 (a_0 \varepsilon - u_r(t))$$

$$= \lambda (a_0 \varepsilon - u_r(t)) \quad \text{for } \lambda = \frac{2A a_0 \rho_0}{m}.$$

$$\Rightarrow \frac{du_r}{dt} + \lambda u_r = \lambda \varepsilon a_0 \quad \text{// shown.}$$

c) initial condition: @ $t = \frac{L}{a_0}$, $u_r = 0$ since $u_r = 0$ for $t \leq \frac{L}{a_0}$.

$$u_r = C e^{-\lambda t} + a_0 \varepsilon$$

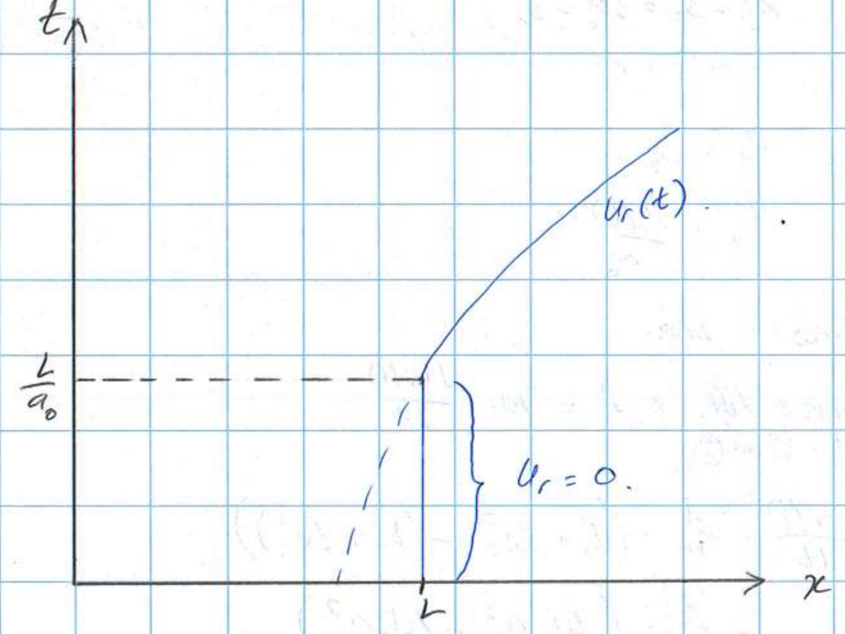
$$0 = C e^{-\lambda \frac{L}{a_0}} + a_0 \varepsilon$$

$$-a_0 \varepsilon = C e^{-\lambda \frac{L}{a_0}}$$

$$C = -a_0 \varepsilon \cdot e^{\lambda \frac{L}{a_0}}$$

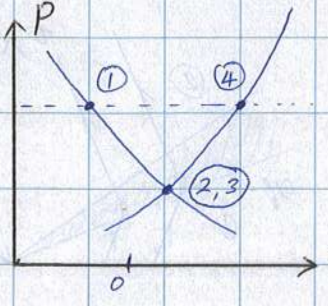
d) $$u_r(t) = -a_0 \varepsilon e^{\lambda(\frac{L}{a_0} - t)} + a_0 \varepsilon$$

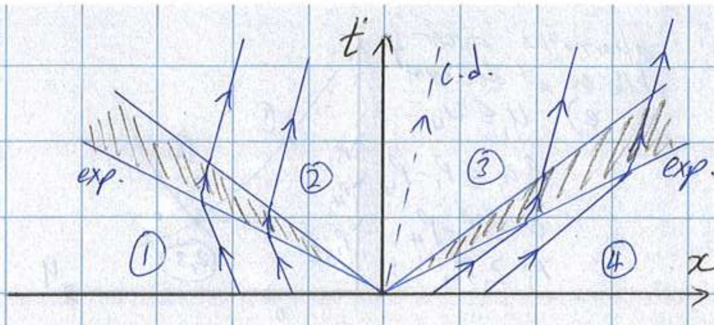
$$= a_0 \varepsilon \left(1 - e^{\lambda(\frac{L}{a_0} - t)}\right) \quad \text{for } t \geq \frac{L}{a_0}$$



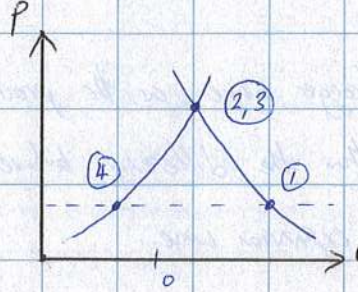
as $t \to \infty$, $u_r \to \varepsilon a_0 = u_l$

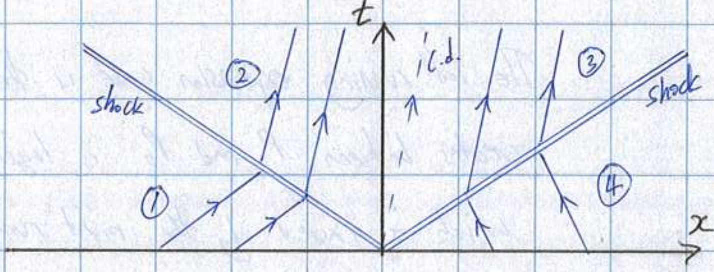
3.1 a) $u_1 < u_4$
$p_1 = p_4$





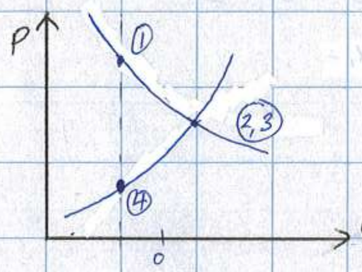
b) $u_1 > u_4$
$p_1 = p_4$

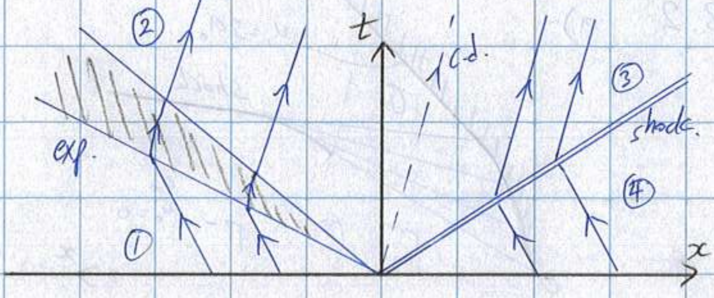




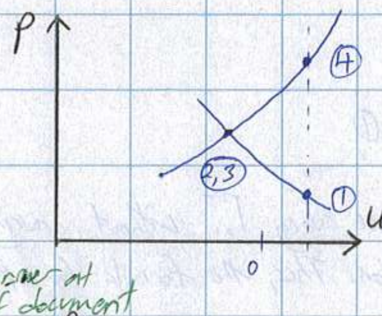
c) $u_1 > u_4$
$p_1 \neq p_4$

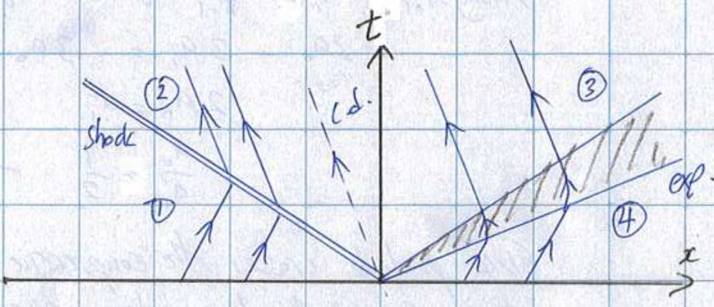
Case 1: $p_1 > p_4$





Case 2: $p_1 < p_4$





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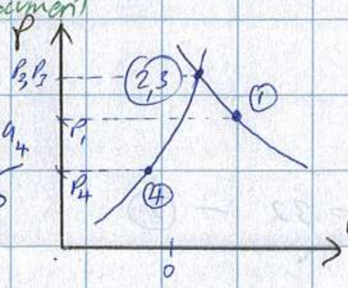
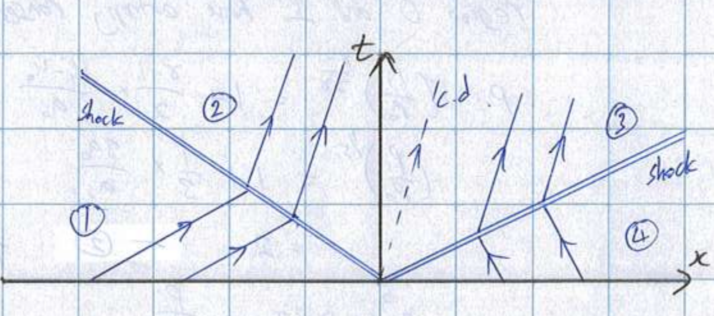
d) $u_1 > u_4$

$p_1 > p_4 + \underbrace{\rho_4 u_4}_{>0}$

$\Rightarrow p_1 > p_4$





The right running shock wave is the stronger wave as the pressure difference it creates between p_3 and p_4 is larger than the difference between p_1 and p_2 which is created by the left running shock wave

continued...

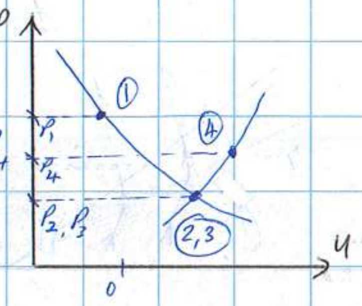
alternative answer at the end of document

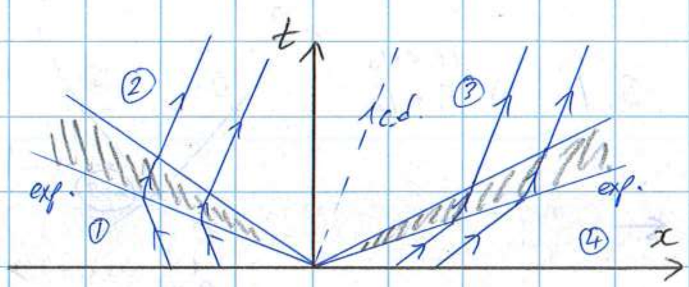
e) $u_1 \le u_4$

$\rho_1 a_1 \le P_1 - P_4$

$0 \le P_1 - P_4$

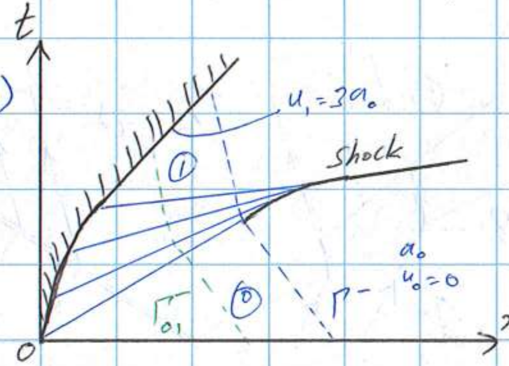
$P_1 \ge P_4$





The left running expansion wave is the stronger wave as the pressure difference it creates between P_1 and P_2 is larger than the difference between P_3 and P_4 which is caused by the right running expansion wave.

3.2 a)



He: $\gamma = \frac{5}{3}$

Along Γ^-_{01}: $u_1 - \frac{2}{\gamma-1} a_1 = -\frac{2}{\gamma-1} a_0$

$$3a_0 - 3a_1 = -3a_0$$

$$2a_0 = a_1$$

$$\frac{a_0}{a_1} = \frac{1}{2} \quad — ①$$

Since particle crosses the compressive wave along Γ^-_{01} without crossing the shock wave, region 0 and 1 have entropy conservation. Thus, the formula for Poisson curve is used.

$$P: \left(\frac{P_1}{P_0}\right)^{\frac{\gamma-1}{2\gamma}} = 1 + \frac{\gamma-1}{2} \times \frac{u_1 - u_0}{a_0}$$

$$\left(\frac{P_1}{P_0}\right)^{1/5} = 1 + \frac{1}{3} \times \frac{3a_0}{a_0}$$

$$= 2 \quad \Rightarrow \quad \frac{P_1}{P_0} = 32 \quad — ②$$

$$a^2 = \gamma R T = \gamma \frac{P}{\rho}$$

$$\rho = \frac{\gamma P}{a^2}$$

$$\frac{\rho_1}{\rho_0} = \frac{P_1}{P_0} \times \frac{a_0^2}{a_1^2}$$

substituting ① and ②: $\frac{\rho_1}{\rho_0} = 32 \times \left(\frac{1}{2}\right)^2$

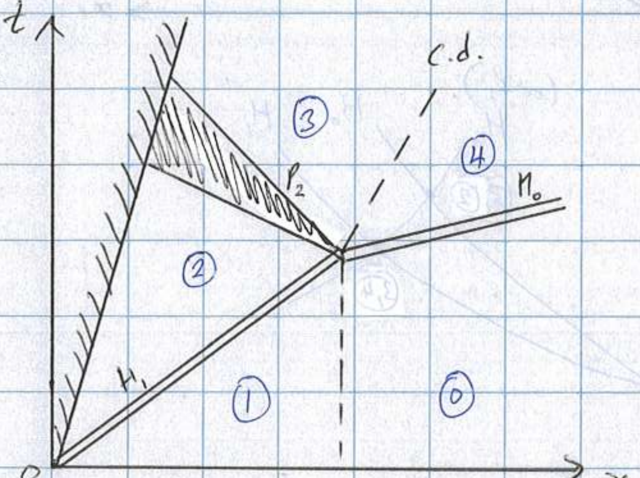
$$= 8.$$

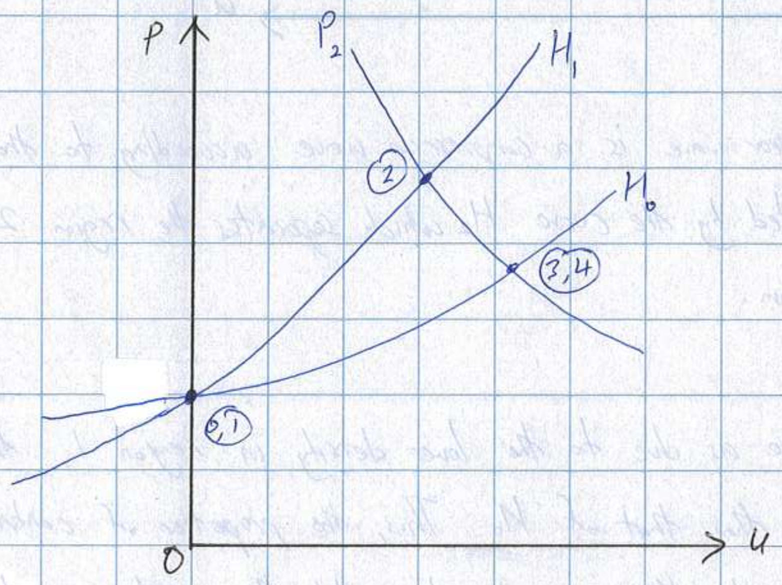
b). In this case, the shock is said to be a strong shock which limits the density ratio as follows:

$$\left(\frac{\rho_1}{\rho_0}\right)_{max} = \frac{\gamma+1}{\gamma-1} = 4 \quad \text{for He: } \gamma = \frac{5}{3}.$$

This, this shock will give density ratio of 4 or less which is smaller than that of (a).

3.3 for $\rho_1 > \rho_0$:

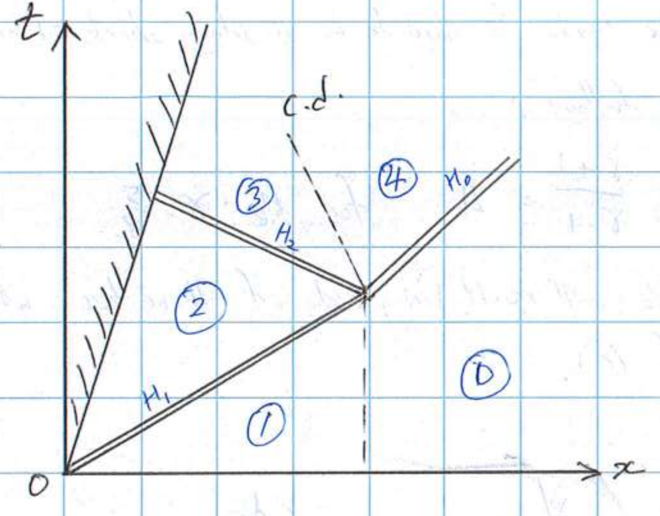


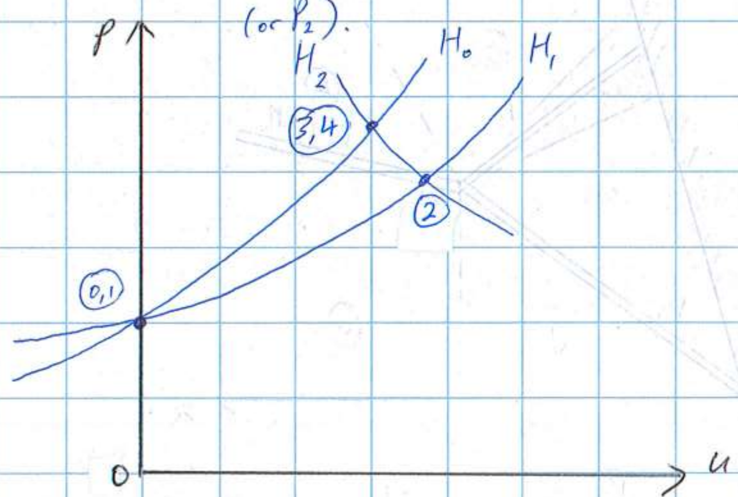


The second non-linear wave is an expansion wave according to the P-u diagram, represented by curve P_2. which separates regions 2 and 3 of the t-x diagram.

This makes sense as due to the higher density at region 1, the Hugoniot curve H_1 is of higher gradient than H_0. Thus, in order for properties of a contact discontinuity to be conserved, an expansion wave is required to compensate for the difference in P and u between regions 2 and 4.

for $\rho_1 < \rho_0$





The second non-linear wave is a compression wave according to the P-u diagram, represented by the curve H_2 which separates the region 2 and 3 of the t-x diagram.

This makes sense as due to the lower density in region 1, the slope of H_1 is smaller than that of H_0. Thus, the properties of contact discontinuity are not conserved. Hence, in order to match the properties, a compression wave is created, conserving the properties of contact discontinuity.

3.4 a)

Increasing entropy ↑

I: B, D_2, E, G_3, H ; II: G_1

III: Domains: A, C, F

These domains are not enclosed by the two Γ^0 dotted lines (particle paths), which means they are not subjected to varying shock thus having consistent entropies.

As for regions I and II, due to the shock, it is obvious that they have higher entropy than region III. However the difference between I and II require more information.

b) Domains B and F have velocity of u_p and uniform conditions

c) C: J^+ is constant in all regions except E and H

D_2: J^+ is constant in all regions except E and H

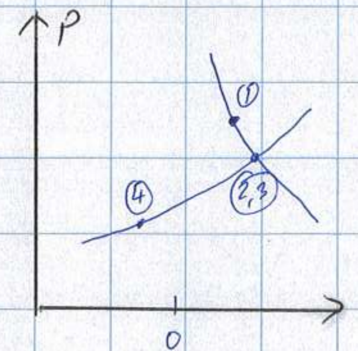
This is because at E and H, varying J^- is reflected off the left boundary

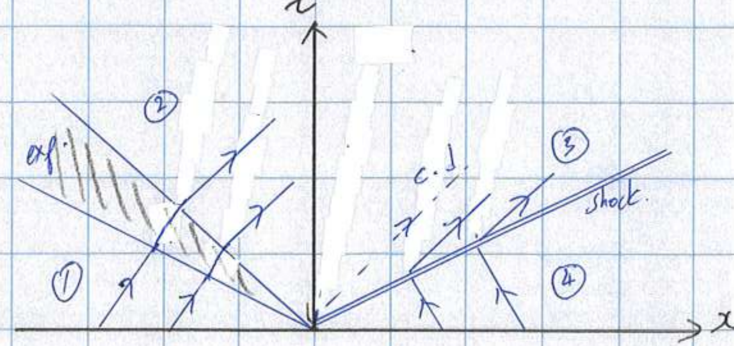
H: J^- is constant in all regions except C, D_1, D_2 and E.

d) Domains D_1 and G_2

d) Alternative answer for 3.1d).

$u_1 > u_4$
$p_1 > p_4$

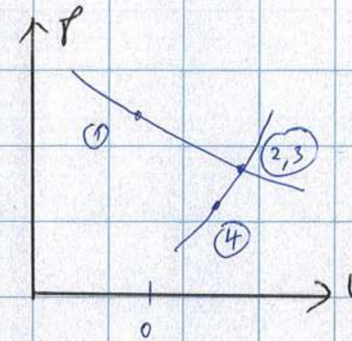


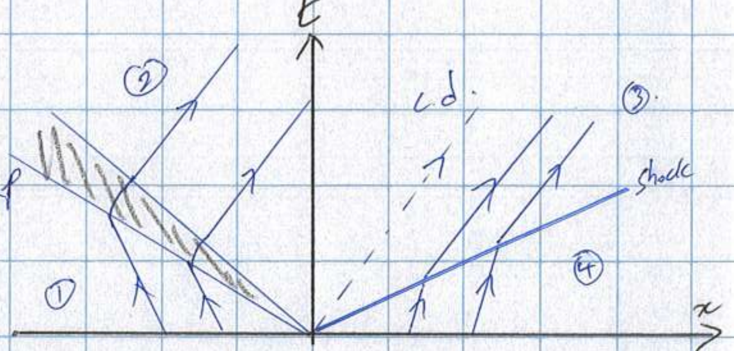


There is not enough information to support any claim on which wave is stronger.

e) Alternative answer for 3.1e)

$u_1 \leq u_4$
$p_1 \geq p_4$





Again, there's not enough information to select a stronger wave.