Problem 1

a. Proof of Fermat's Theorem

Assuming that the local maximum is at $x = x_m$,

$$f(x) \leq f(x_m) \tag{1}$$

for all x close to x_m and for a small spacing h

$$(x_m + h) \leq f(x_m) \tag{2}$$

$$f(x_m + h) - f(x_m) \leq 0 \tag{3}$$

since any point around x_m, left or right, is smaller than $f(x_m)$.

Letting $h > 0$,

$$\frac{f(x_m + h) - f(x_m)}{h} \leq 0 \tag{4}$$

We then take the right-hand limit of the equation leading to

$$\lim_{h \to 0^+} \frac{f(x_m + h) - f(x_m)}{h} \leq 0 \tag{5}$$

$$f'(x_m) \leq 0 \tag{6}$$

From equation 3, we now let $h < 0$,

$$\frac{f(x_m + h) - f(x_m)}{h} \geq 0 \tag{7}$$

This time we take the left-hand limit of the equation to achieve

$$\lim_{h \to 0^-} \frac{f(x_m + h) - f(x_m)}{h} \geq 0 \tag{8}$$

$$f'(x_m) \geq 0 \tag{9}$$

To have both equations 6 and 9 satisfied,

$$f'(x_m) = 0 \tag{10}$$

Thus, at a local maximum,

$$f'(x) = 0 \tag{11}$$

b. Proof of Converse of the Second Derivative Test

For a local maximum at $x = x_m$, the Taylor expansion of $f(x)$ around x_m is

$$f(x) \approx f(x_m) + f'(x_m)(x - x_m) + \frac{f''(x_m)}{2!}(x - x_m)^2 + \cdots \tag{12}$$

Shifting $f(x_m)$ to the left hand side,

$$f(x) - f(x_m) \approx f'(x_m)(x - x_m) + \frac{f''(x_m)}{2!}(x - x_m)^2 + \cdots \tag{13}$$

Multiplying both sides by $\frac{2}{x - x_m}$ and rearranging,

$$f'' \approx [f(x) - f(x_m) - f'(x_m)(x - x_m]\frac{2}{(x - x_m)^2} \tag{14}$$

Since

$$f(x) - f(x_m) \leq 0 \tag{15}$$

$$f'(x_m) = 0 \tag{16}$$

$$(x - x_m)^2 \geq 0 \tag{17}$$

We can say that

$$f''(x_m) \leq 0 \tag{18}$$

c. Proof

Fermat's theorem states: if f has a local maximum or minimum at x_m, and if $f'(x_m)$ exists, $f'(x_m) = 0$. Hence for $u(x, y)$ which has a local maximum at (x_0, y_0),

$$\nabla u(x_0, y_0) = \langle \frac{du}{dx}, \frac{du}{dy} \rangle = \langle 0, 0 \rangle = \mathbf{0} \tag{19}$$

With this, the directional derivative of u in the direction of any unit vector v is

$$D_{\mathbf{v}} u(x_0, y_0) = \langle v_x \frac{\partial u}{\partial x}, v_y \frac{\partial u}{\partial y} \rangle = \langle v_x \cdot 0, v_y \cdot 0 \rangle = \langle 0, 0 \rangle = \mathbf{0} \tag{20}$$

d. Derivation

Let H be the Hessian matrix which is defined as

$$H = \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix} \tag{21}$$

and unit vector $\boldsymbol{v}$ is

$$\mathbf{v} = \langle p, q \rangle \tag{22}$$

With these two defined, we can start proving

$$\mathbf{v}^T H \mathbf{v} = D_{\mathbf{v}} \left(D_{\mathbf{v}} u \right) \tag{23}$$

Starting from the left hand side,

$$\begin{aligned} \mathbf{v}^T H \mathbf{v} &= \begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial y \partial x} & \frac{\partial^2 u}{\partial y^2} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \\ &= \begin{bmatrix} p\frac{\partial^2 u}{\partial x^2} + q\frac{\partial^2 u}{\partial y \partial x} & p\frac{\partial^2 u}{\partial x \partial y} + q\frac{\partial^2 u}{\partial y^2} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \\ &= p^2 \frac{\partial^2 u}{\partial x^2} + pq \frac{\partial^2 u}{\partial y \partial x} + pq \frac{\partial^2 u}{\partial x \partial y} + q^2 \frac{\partial^2 u}{\partial y^2} \end{aligned} \tag{24}$$

Next we start off from the right hand side of the equation 23 and using the formula $D_u f = \nabla f \cdot u$,

$$\begin{aligned} D_{\mathbf{v}} \left(D_{\mathbf{v}} u \right) &= D_{\mathbf{v}} \, \langle p \frac{\partial u}{\partial x}, q \frac{\partial u}{\partial y} \rangle \\ &= \nabla \langle p \frac{\partial u}{\partial x}, q \frac{\partial u}{\partial y} \rangle \cdot \mathbf{v} \\ &= \langle p \frac{\partial^2 u}{\partial x^2} + q \frac{\partial^2 u}{\partial y \partial x}, p \frac{\partial^2 u}{\partial y^2} + q \frac{\partial^2 u}{\partial x \partial y} \rangle \cdot \langle p, q \rangle \\ &= p^2 \frac{\partial^2 u}{\partial x^2} + pq \frac{\partial^2 u}{\partial y \partial x} + pq \frac{\partial^2 u}{\partial x \partial y} + q^2 \frac{\partial^2 u}{\partial y^2} \end{aligned} \tag{25}$$

Hence it can be seen from above that the equations 24 and 25 end up to be the same, proving the equation 23 right.

e. Proof of Theorem 2.3.1

Given the fact that the intersection of any vertical plane going through the point of maximum (x_0, y_0) and the graph of $u(x, y)$ is a function of a single variable, the following equations can be deduced for this intersection.

$$\langle x, y \rangle = \langle x_0, y_0 \rangle + k^n \mathbf{v} \tag{26}$$

with k as the single variable of an unknown degree n and $\mathbf{v} = \langle p, q \rangle$. We then have

$$u(x, y) = u(x_0 + k^n p, y_0 + k^n q) = \tilde{u}(k) \tag{27}$$

with newly defined function $\tilde{u}$ for the intersection which has just 1 variable, k, and this function has its local maximum at the same location as u. This means

$$D_{\mathbf{v}} \left(D_{\mathbf{v}} u \right) = \frac{d^2 \tilde{u}}{dk^2} \leq 0 \tag{28}$$

according to equation 18 and with equation 23, it can be said that the Hessian matrix $H(x_0, y_0)$ of the function u is negative semi-definite.

Problem 2

a. Relation between h and N

On the interval $\Omega = [0, 1]$ and spacing of $h = 0.1$, the following uniform grid Ω_h can be constructed.

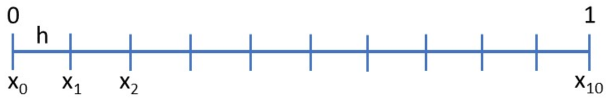

Figure 1: Uniform grid, $h = 0.1$

As it can be seen in the above figure, there are in total 11 points including the boundary points. Thus, with $x_{max} = 1$ and $x_{min} = 0$, the connection between h and the number of grid points (including boundary points), N, is as follows

$$N = \frac{x_{max} - x_{min}}{0.1} + 1 \tag{29}$$

b. Discretization of 1D Laplacian operator $\mathcal{L}$

The 1-dimensional Laplacian operator$\mathcal{L}$ is defined as

$$\mathcal{L} = -\frac{d^2}{dx^2} \tag{30}$$

First, we start off with Taylor expansion of f_{i+1} and f_{i-1}.

$$f_{i+1} \approx f_i + hf_i' + \frac{h^2}{2}f_i'' + \frac{h^3}{6}f_i''' \tag{31}$$

$$f_{i-1} \approx f_i - hf_i' + \frac{h^2}{2}f_i'' - \frac{h^3}{6}f_i''' \tag{32}$$

Adding the above two equations,

$$f_{i+1} + f_{i-1} = 2f_i + h^2 f_i'' \tag{33}$$

which can be rearranged into

$$f_i'' = \frac{f_{i+1} - 2f_i + f_{i-1}}{h^2} \tag{34}$$

Using this, the central difference 1-dimensional finite difference Laplacian matrix can be defined

$$\mathcal{L} = -f'' = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \tag{35}$$

c. Visualization of $\mathcal{L}$ matrix

The following figure describes the sparsity pattern of the matrix in the equation 48

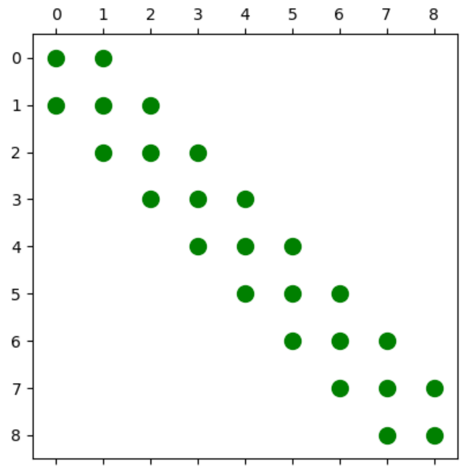

Figure 2: spy plot of $\mathcal{L}$

d. Comparison of the eigenvalues

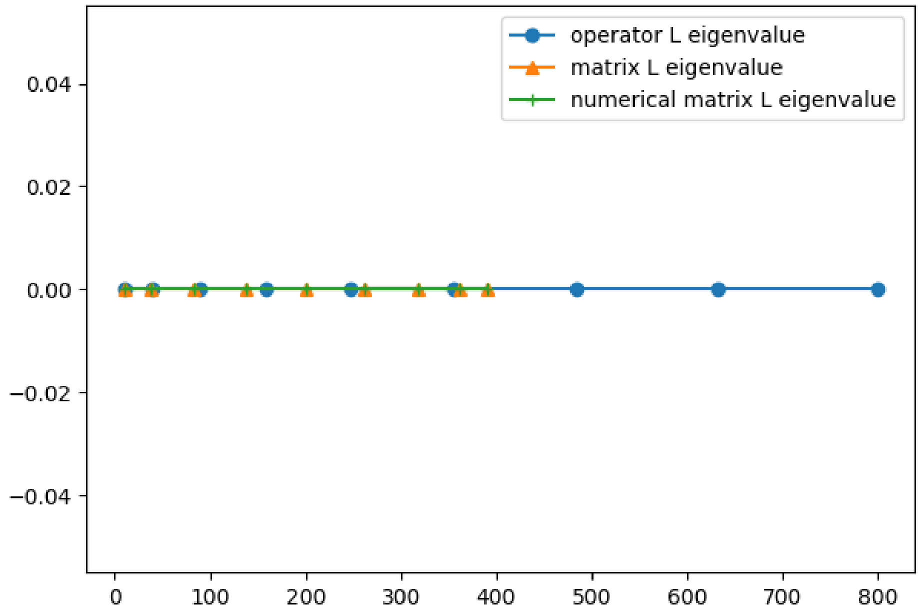

Figure 3: Comparing eigenvalues on Re-Im axes

As it can be seen from the above figure, all the eigenvalues are real numbers with none having an imaginary part.

e. Eigenfunctions and eigenvectors

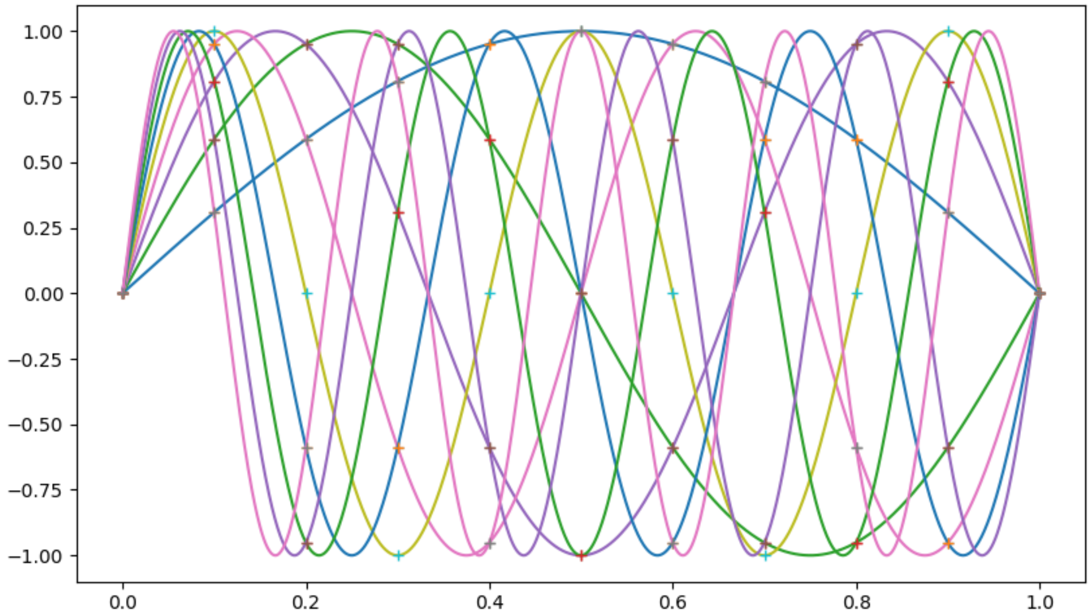

Figure 4: Eigenfunctions and Eigenvectors(+)

It is observed that the eigenvectors are positioned on their respective eigenfunctions, showing that they are the sampled eigenfunctions of $\mathcal{L}$

f. 10^{th} pair

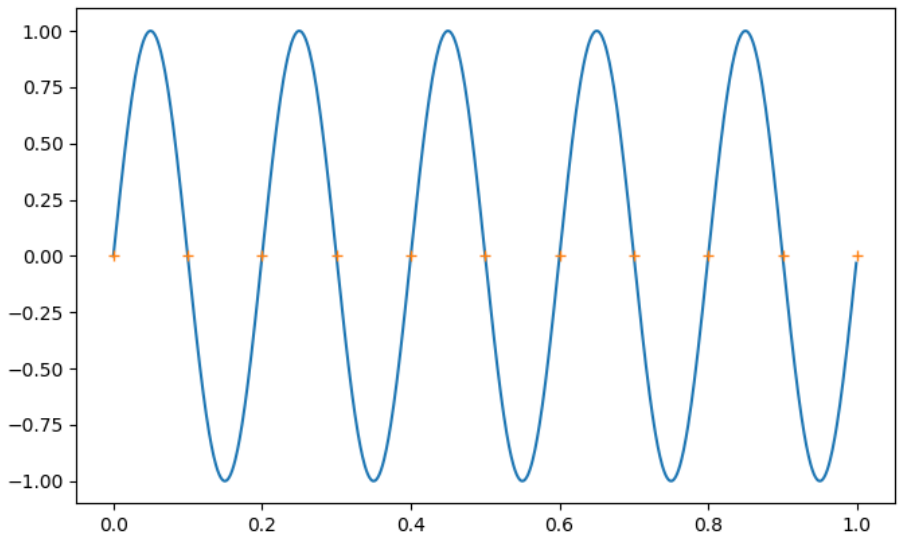

Figure 5: 10^{th} pair of eigenfunction and eigenvectors(+)

For the eigenfunction-eigenvectors pair plotted in the above figure, it can be observed that the eigenfunction is only sampled at its roots. For the grid formed with the given value of $h = 0.1$, this 10^{th} pair is the first pair unable to be resolved.

Problem 3

a. Exact solutions

Given

$$-\frac{d^2u}{dx^2} = f_i, \quad x \in (0,1) \tag{36}$$

$$u(0) = 1, \quad u(1) = 2 \tag{37}$$

$$f_1(x) = 1, \quad f_2(x) = e^x, \quad x \in [0,1] \tag{38}$$

Starting off by dealing with f_1,

$$-\frac{d^2u_1}{dx^2} = 1 \tag{39}$$

$$\frac{d^2u_1}{dx^2} = -1$$

$$\frac{du_1}{dx} = -x + const_1$$

$$u_1 = -\frac{1}{2}x^2 + const_1 x + const_2$$

Substituting in the boundary conditions,

$$u_1(0) = 0 + 0 + const_2 = 1$$

$$const_2 = 1$$

$$u_1(1) = -\frac{1}{2} + const_1 + 1 = 2$$

$$const_1 = \frac{3}{2}$$

Giving

$$u_1(x) = -\frac{1}{2}x^2 + \frac{3}{2}x + 1 \tag{40}$$

Moving on to f_2,

$$-\frac{d^2u_2}{dx^2} = e^x \tag{41}$$

$$\frac{d^2u_2}{dx^2} = -e^x$$

$$\frac{du_2}{dx} = -e^x + const_3$$

$$u_2 = -e^x + const_3 x + const_4$$

Substituting in the boundary conditions,

$$u_2(0) = -1 + 0 + const_4 = 1$$

$$const_4 = 2$$

$$u_2(1) = -e + const_3 + 2 = 2$$

$$const_3 = e$$

Giving

$$u_2(x) = -e^x + ex + 2 \tag{42}$$

b. Derivation of linear algebraic problem

With spacing $h = 0.2$, the following grid is formed with total of 6 grid points including the boundary points

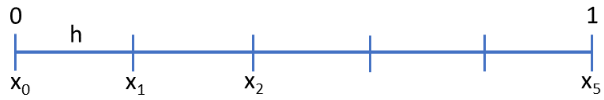

Figure 6: Uniform grid, $h = 0.2$

Starting off by Taylor expansion like equation 12, we have

$$u_{i+1} = u_i + hu_i' + \frac{h^2}{2}u_i'' + \frac{h^3}{6}u_i''' + \mathcal{O}\left(h^4\right) \tag{43}$$

$$u_{i-1} = u_i - hu_i' + \frac{h^2}{2}u_i'' - \frac{h^3}{6}u_i''' + \mathcal{O}\left(h^4\right) \tag{44}$$

Using these two expansions, we can arrive at

$$\mathcal{D}_x^+ u_i = u'\left(x_i\right) + \frac{h}{2}u''\left(x_i\right) + \frac{h^2}{6}u'''\left(x_i\right) + \mathcal{O}\left(h^3\right) \tag{45}$$

$$\mathcal{D}_x^- u_i = u'\left(x_i\right) - \frac{h}{2}u''\left(x_i\right) + \frac{h^2}{6}u'''\left(x_i\right) + \mathcal{O}\left(h^3\right) \tag{46}$$

We then have

$$\mathcal{D}_{xx}^{(2)} u_i = \frac{\mathcal{D}_x^+ u_i - \mathcal{D}_x^- u_i}{h} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{47}$$

Hence the Laplacian matrix for this can be defined similarly to the one in equation 48 for $h = 0.2$

$$\mathcal{L} = \begin{bmatrix} 50 & -25 & 0 & 0 \\ -25 & 50 & -25 & 0 \\ 0 & -25 & 50 & -25 \\ 0 & 0 & -25 & 50 \end{bmatrix} \tag{48}$$

The entries for $\mathbf{f_1}$ and $\mathbf{f_2}$ for $h = 0.2$ are shown below

$$\mathbf{f_1} = \begin{bmatrix} 26 \\ 1 \\ 1 \\ 51 \end{bmatrix}, \mathbf{f_2} = \begin{bmatrix} 26.2214 \\ 1.49182 \\ 1.82212 \\ 52.2255 \end{bmatrix} \tag{49}$$

c. Comparison for $\mathbf{u}_1$ and $\mathbf{u}_2$

First we compare the exact and numerical values of $\mathbf{u}_1$ and $\mathbf{u}_2$ graphically

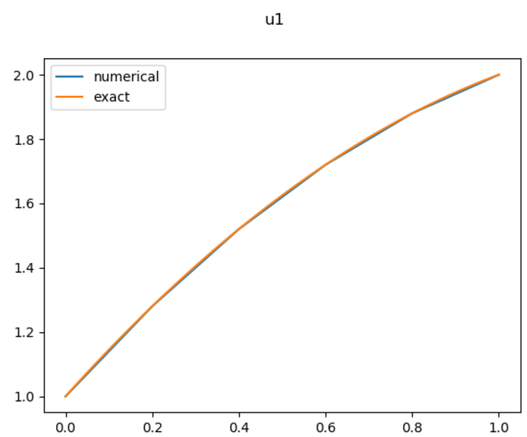

Figure 7: $\mathbf{u}_1$

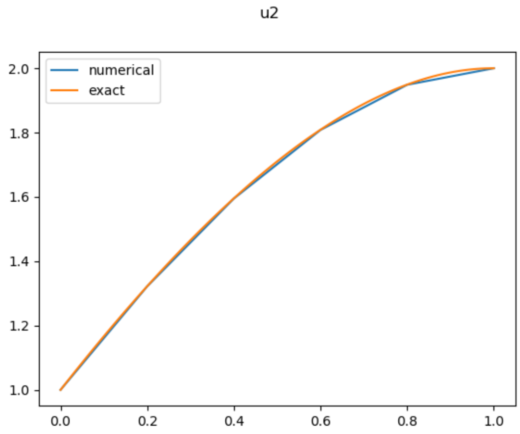

Figure 8: $\mathbf{u}_2$

Now we compare the exact and numerical values of each function at grid nodes using a table

	$\mathbf{u}_1$ exact	$\mathbf{u}_1$ numerical	$\mathbf{u}_2$ exact	$\mathbf{u}_2$ numerical
0	1	1	1	1
0.2	1.28	1.28	1.32225	1.32185
0.4	1.52	1.52	1.59549	1.59484
0.6	1.72	1.72	1.80885	1.80816
0.8	1.88	1.88	1.94908	1.94859
1.0	2	2	2	2
global error	1.4043333874306804e-16		0.0005132280521659377	

d. Varying grid-step

$$h_k = \frac{0.2}{2^k} \tag{50}$$

By varying the k of the above equation, the grid-step h is varied to give the following table of global errors

k	$\mathbf{u}_1$ global error	$\mathbf{u}_2$ global error
0	1.4043333874306804e-16	0.0005132280521659377
1	3.1401849173675503e-16	0.0001286144770556241
2	9.524652439874976e-16	3.2167484237603126e-05
3	1.271193242799326e-15	8.042653267534536e-06
4	4.136083689532441e-15	2.0107108881189185e-06

It can be observed that the global error of $\mathbf{u}_1 \approx= 0$ regardless of the step size as the numerical solution on the grid nodes lie exactly on the exact values and global error of $\mathbf{u}_2$ decreases with increasing k since the grid-step decreases allowing for better approximation of the curve.

e. Non-uniform grid

In my experiment, I have removed the first non-boundary point to create the non-uniform grid For non-uniform grid, global error of $\mathbf{u}_1$ showed little difference with being approximately 0 and that of $\mathbf{u}_2$ was about only six-fold greater than uniform grid. This small difference could be due to the nature of the graph. On the left side where I have removed a point, the graph is close to linear which makes the numerical approximation still very possible and accurate.

Problem 4

a. Derivation of 2-dimensional finite difference discretization

Given,

$$-\Delta u(x,y) = f(x,y), \quad (x,y) \in \Omega \tag{51}$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{52}$$

$$f(x,y) = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} \tag{53}$$

Using

$$-\frac{\partial^2 u}{\partial x^2} = \frac{-u_{i-1,j} + 2u_{i,j} - u_{i+1,j}}{h_x^2}, -\frac{\partial^2 u}{\partial y^2} = \frac{-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}}{h_y^2} \tag{54}$$

which are derived similarly to equation 47, we have

$$f_{i,j} = \frac{-u_{i-1,j} + 2u_{i,j} - u_{i+1,j}}{h_\chi^2} + \frac{-u_{i,j-1} + 2u_{i,j} - u_{i,j+1}}{h_y^2} = \mathcal{L}u \tag{55}$$

b. Lexicographic grid

Firstly, the 2-dimensional Laplacian matrix is defined as follows

$$L = I_y \otimes L_{xx} + L_{yy} \otimes I_x \tag{56}$$

where $\otimes$ is the Kronecker product. L_{xx}, which is of size $(N_x - 1, N_x - 1)$, where $N_x = \frac{x_{max} - x_{min}}{h}$, is defined as follows

$$L_{xx} = \frac{1}{h^2} \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \tag{57}$$

L_{yy} is defined similarly with size $(N_y - 1, N_y - 1)$. I_x and I_y are identity matrices of sizes $(N_x - 1, N_x - 1)$ and $(N_y - 1, N_y - 1)$ respectively. Substituting these back into the equation 56, we get

$$L = \frac{1}{h^2} \begin{bmatrix} 4 & -1 & 0 & \cdots & -1 & \cdots & 0 & 0 & 0 \\ -1 & 4 & -1 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 4 & \cdots & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & \cdots & -1 & 0 & 0 \\ -1 & 0 & 0 & \cdots & 0 & \cdots & 0 & -1 & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -1 & \cdots & 0 & \cdots & 4 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & -1 & 4 & -1 \\ 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & -1 & 4 \end{bmatrix} \tag{58}$$

which is a square matrix of size $[(N_x - 1)(N_y - 1), (N_x - 1)(N_y - 1)]$.
Also, given D_x is a matrix representing one-sided finite difference approximations of the 1st derivatives, we have

$$D_x = \frac{1}{h_x} \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 \end{bmatrix} \tag{59}$$

$$
L_{xx} = D_x^T D_x = \frac{1}{h_x}\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 \end{bmatrix}^T \frac{1}{h_x}\begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 \end{bmatrix} \tag{60}
$$

$$
= \frac{1}{h_x^2}\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}
$$

L_{yy} can be similarly derived with D_y

c. Sparsity of Laplacian matrix

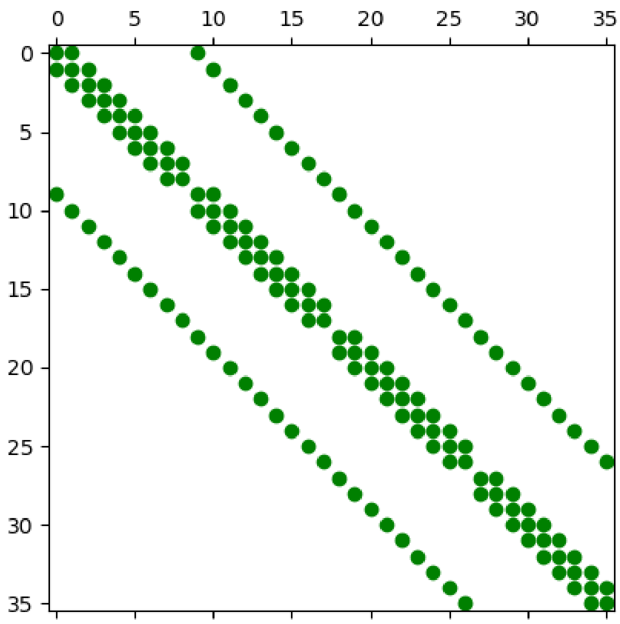

Figure 9: spy plot of $\mathcal{L}$, $h = 0.2$

d. Source function

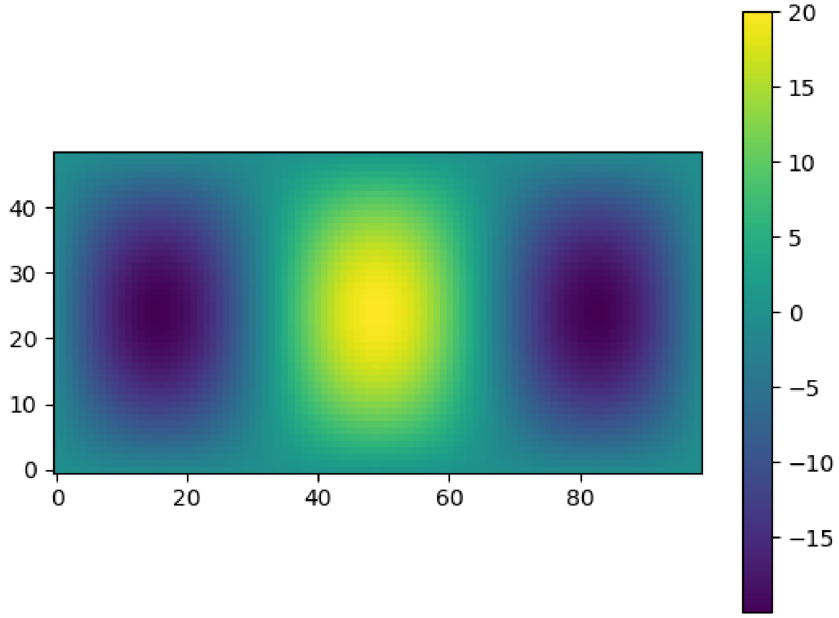

Figure 10: Visualization of the source function, $h = 0.02$

e. Solution

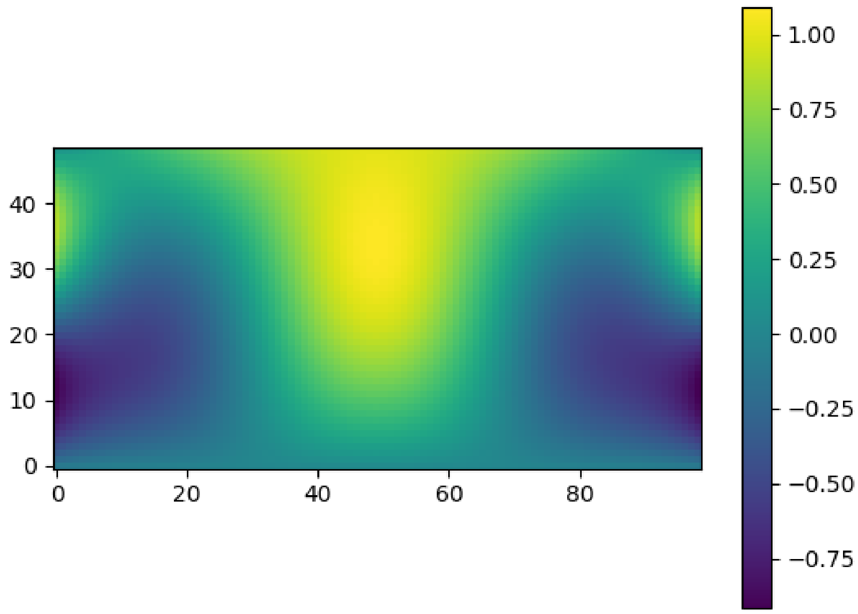

Figure 11: Visualization of the solution, $h = 0.02$