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Master Thesis

Multivariate modelling of the
dependency structure between
article sales of a sportswear
manufacturer

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1 Introduction

Write introduction here and "upper" subsections here (adidas, Motivation, etc...)

1.1 Data Sources

Throughout each season, transactional data are collected from online purchases of the sports brand's eCommerce website. Specifically, we are provided with weekly sales data for western European countries. A short description is depicted in Table 1.1.

Column	Description	Values
week_id	Calendar week of a specific year (YYYYWW)	Factors: 201648, ..., 201852
article_number	Unique article identification number (article ID)	Factors: 10669, 10, ...
min_date_of_week	Minimum date of the respective week; always a Monday (YYYY-MM-DD)	Dates: 2016-11-28, ..., 2018-12-24
art_min_price	Minimal recorded price of the article	Non-negative (integer) value
month_id	Calendar month of a specific year (YYYYMM)	Factors: 201612, ..., 201812
season	Season of year (format: SSYY) (Spring-Summer [SS]: December - May) Fall-Winter [FW]: June - November)	Factors: SS17, FW17, SS18, FW18, SS19
bf_w	Weekly "Black Friday" promotion intensity of the article	Between 0 and 1
ff_w	Weekly "Friends & Family" promotion intensity of the article	Between 0 and 1
ot_w	Weekly article promotion intensity of "Other" type	Between 0 and 1
gross_demand_quantity	Weekly amount of added articles to shopping cart	Non-negative (integer) value
base_price_locf	Retail price of the article without any discounts	Non-negative (integer) value
total_markdown_pct		
day_of_month	Day of the month	Integers: 1 - 31
month_of_year	Month of the year	Factors: January, ..., December
year	Year	Integers: 2016, 2017, 2018
week_of_year	Week of the year	Integers: 1 - 52

Table 1.1: Transactional raw data description from online purchases of western European countries

Due to legal regulations of the company, some columns had to undergo anonymization in order for the data to be released. To ensure data protection and confidentiality, numeric variables (with exception of time-indicating columns) were transformed. As a consequence for the analysis part, most integer values were converted to float numbers. This fact should be kept in mind by the reader, since the above table serves as a reminder and reference point for the data documentation.

Another peculiarity of this setup is to be considered, too. We will often refer to the variable *gross demand quantity* as *sales*, even though it is obviously not exactly the same. In the eCommerce environment, there are several stages before the purchase is complete, e.g. addition to cart, removal from cart, proceeding to checkout & even the return of bought articles. Targeting the articles added to cart, i.e. the (gross) demand quantity, provides the optimal data extraction for analytical purposes and is the closest to adequately model the dependency structure between net sales of articles.

Besides the transactional data, *article master data*, i.e. attributes of the articles, are provided and described in Table 1.2.

Column	Description	Values (all Factors)
article_number	Unique article identification number (article ID)	10669, 10, ...
gender	Gender type of the article (Men, Women, Unisex)	M, W, U
age_group	Age group of the article (Adult, Infant, Junior, Kids)	A, I, J, K
product_division_descr	Product division of the article	APPAREL, FOOTWEAR, HARDWARE
product_group_descr	Product group of the article	BAGS, BALLS, FOOTWEAR ACCESSORIES, SHOES, ...
color	Consolidated color group of the article	BEIGE, BLACK, BROWN, ORANGE, PINK, RED, ...
sports_category_descr	Sports category of the article	encoded: SC_1, ..., SC_22
sales_line_descr	Sales line of the article	encoded: SL_1, ..., SL_379
business_unit_descr	The article's Business Unit membership	encoded: BU_1, ..., BU_18
business_segment_descr	The article's Business Segment membership	encoded: BS_1, ..., BS_49
sub_brand_descr	Sub-brand of the article	encoded: sub-brand_1, ..., sub-brand_4
item_type	Item type of the article	encoded: IT_1, ..., IT_171
brand_element	Brand element of the article	encoded: BE_1, ..., BE_131
product_franchise_descr	Product franchise of the article	encoded: franchise_1, ..., franchise_72
product_line_descr	Product line of the article	encoded: PL_1, ..., PL_105
franchise_bin	Franchise indicator of the article	FRANCHISE, NON-FRANCHISE
category	Category of the article	encoded: category_1, category_2

Table 1.2: Article master data

Plenty of additional information is stored in the database, but we are neglecting columns omitted in these tables, as they are redundant, already summarized, transformed or simply do not provide any value.

Overall, we will be dealing with data collected over two years, namely the years 2017 and 2018, while some transactions of late 2016 might be attached marginally. In summary, after joining the transactional observations to the article attributes by the article ID, this translates to a dataset of over 587,000 rows and over 30 variables.

2 Statistical Theory & Methods

This chapter introduces various statistical methods used during the conduction of this thesis. It is assumed that basic understanding and knowledge of the reader regarding mathematical foundations of statistics (like linear algebra, probability theory, etc) already exists.

2.1 Generalized Linear Models

Generalized Linear Models (GLMs) are an extension of the classical *Linear Regression Model (LM)*

$$y_i = \beta_0 + \beta_1 x_{i1} + \dots + \beta_k x_{ik} + \varepsilon_i, \quad i = 1, \dots, n$$

which in matrix notation can be written as

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where the response variable y_i can take values from several probability distributions (e.g. Poisson, Binomial, Gamma and others), which are members of the exponential family [Fahrmeir et al., 2003]. The linear predictor

$$\eta_i = \beta_0 + \beta_1 x_{i1} + \dots + \beta_k x_{ik} + \varepsilon_i = \mathbf{x}_i' \boldsymbol{\beta} \quad (2.1)$$

is passed through a *response function* h (a one-to-one, twice differentiable transformation), such that

$$E(y_i) = h(\eta_i)$$

i.e. the expected value of the response variable belongs to the value range of h . The inverse of the response function,

$$g = h^{-1}$$

is called the *link function*.

2.2 Generalized Additive Models

2.3 Mixed Effects Models

Linear Mixed Models (LMMs) are powerful tools when dealing with clustered data or data with a longitudinal structure (repeated measurements of individuals). As

in the classical LM, there are population-specific effects, namely the parameter vector of *fixed effects* β , as well as the cluster- or individual-specific effects of such models called *random effects* [Fahrmeir et al., 2003]. In the following, we will refer to our clusters or individuals as "groups" for briefness. Mathematically speaking, the linear predictor $\eta_{ij} = \mathbf{x}'_{ij}\beta$ is extended to

$$\eta_{ij} = \mathbf{x}'_{ij}\beta + \mathbf{u}'_{ij}\gamma_i, \quad j = 1, \dots, m, \quad i = 1, \dots, n_i, \quad (2.2)$$

where

- i is the number of groups
- j is the number of observations per group
- β is the vector of fixed effects
- γ_i is the vector of random effects
- \mathbf{x}'_{ij} is the vector of covariates and
- \mathbf{u}'_{ij} is a subvector of \mathbf{x}'_{ij} .

$\mathbf{x}'_{ij} = (1, x_{ij1}, \dots, x_{ijk})$ and $\mathbf{u}'_{ij} = (1, u_{ij1}, \dots, u_{ijk})$ are therefore the design vectors and ε_{ij} are the error terms of the *measurement model*

$$y_{ij} = \mathbf{x}'_{ij}\beta + \mathbf{u}'_{ij}\gamma_i + \varepsilon_{ij}, \quad \varepsilon_{ij} \stackrel{i.i.d.}{\sim} N(0, \sigma^2) \quad (2.3)$$

or in matrix notation

$$\mathbf{y}_i = \mathbf{X}_i\beta + \mathbf{U}_i\gamma_i + \boldsymbol{\varepsilon}_i \quad (2.4)$$

for group $i = 1, \dots, m$ with $E(\boldsymbol{\varepsilon}_i) = \mathbf{0}$.

Similar to GLMs, *Generalized Linear Mixed Models (GLMMs)* relate the linear mixed predictor 2.2 to the conditional mean $\mu_{ij} = E(y_{ij}|\gamma_i)$ via a suitable response function h , such that $\mu_{ij} = h(\eta_{ij})$ and thus the conditional density of y_{ij} belongs to the exponential family.

3 Copulas & Dependence Modelling

Multivariate distributions consist of the marginal distributions and the dependence structure between those marginals. These components can be specified separately in a single framework with the help of copula functions. This chapter introduces the concept of modelling such dependency structures with copulas, which is the main focus of this thesis.

3.1 Introduction to Copulas

A d -dimensional function $C : [0, 1]^d \rightarrow [0, 1]$ is called a *copula*, if it is a Cumulative Distribution Function (CDF) with uniform margins, i.e.

$$P(U_1 \leq u_1, \dots, U_d \leq u_d) = C(u_1, \dots, u_d)$$

where $U_i, i = 1, \dots, d$ are uniformly distributed Random Variables (RVs) in $[0, 1]$.

Sklar's Theorem [Sklar, 1959]

Let F be a d -dimensional CDF with marginal distributions $F_i, i = 1, \dots, d$. Then there exists a copula C such that

$$F(x_1, \dots, x_d) = C(F_1(x_1), \dots, F_d(x_d)) \quad (3.1)$$

for all $x_i \in \mathbb{R}, i = 1, \dots, d$.

The copula C is unique, if $\forall i = 1, \dots, d, F_i$ is continuous. Otherwise C is uniquely determined only on $Ran(F_1) \times \dots \times Ran(F_d)$, where $Ran(F_i)$ is the range of F_i .

Conversely, if C is a d -dimensional copula and F_1, \dots, F_d are univariate CDF's, then F as defined in Equation 3.1 is a d -dimensional CDF.

□

If the copula has a Probability Density Function (PDF), then the *copula density* is defined as

$$c(\mathbf{u}) = \frac{\partial^d C(u_1, \dots, u_d)}{\partial u_1 \dots \partial u_d} \quad (3.2)$$

for a differentiable copula function C and the realization of a random vector $\mathbf{u} = (u_1, \dots, u_d)$.

By virtue of Equation 3.1 in Sklar's theorem and given that

$$C(\mathbf{u}) = F\left(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)\right), \quad (3.3)$$

i.e. invertible CDFs F_i , $i = 1, \dots, d$, we can rewrite the copula density to

$$c(u_1, \dots, u_d) = \frac{f\left(F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)\right)}{\prod_{i=1}^d f_i\left(F_i^{-1}(u_i)\right)} \quad (3.4)$$

for densities f of F and f_1, \dots, f_d of the corresponding marginals.

Invariance Principal

Suppose the RVs X_1, \dots, X_d have continuous marginals and copula C . For strictly increasing functions $T_i : \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, \dots, d$, the RVs $T_1(X_1), \dots, T_d(X_d)$ also have copula C .

□

Fréchet-Hoeffding Bounds

Let $C(\mathbf{u}) = C(u_1, \dots, u_d)$ be any d -dimensional copula.

Then, for

$$W(\mathbf{u}) = \max \left\{ \sum_{i=1}^d u_i - d + 1, 0 \right\} \quad \text{as well as} \quad M(\mathbf{u}) = \min_{1 \leq i \leq d} \{u_i\},$$

it holds that

$$W(\mathbf{u}) \leq C(\mathbf{u}) \leq M(\mathbf{u}), \quad \mathbf{u} \in [0, 1]^d. \quad (3.5)$$

We call W the *lower Fréchet-Hoeffding bound* and M the *upper Fréchet-Hoeffding bound*.

Note that W is a copula if and only if $d = 2$, whereas M is a copula for all $d \geq 2$ (more on this later in Section 3.2.1).

□

MORE ON COPULA THEORY (NOTES)

3.2 Copula Classes

In this section we will take a look at three very popular *copula classes*, namely *fundamental*, *elliptical* and *archimedean copulas*. For each class, we will present a few (parametric) *copula families* which are widely used.

3.2.1 Fundamental Copulas

Fundamental copulas are a basic class of copulas, which emerge directly from the copula framework and do not depend on any parametric components.

Independence Copula

It is well known that the joint probability of a finite set of events $E_i, i = 1, \dots, k$, is equal to the product of the marginals if and only if the Events E_i are independent.

$$P\left(\bigcap_{i=1}^k E_i\right) = \prod_{i=1}^k P(E_i).$$

Analogously, the same concept applies when we talk about the *independence copula*

$$\Pi(\mathbf{u}) = \prod_{i=1}^d u_i. \quad (3.6)$$

As a result of Sklar's theorem the RVs u_i are independent if and only if their copula is the independence copula, i.e.

$$C(\mathbf{u}) = \Pi(\mathbf{u})$$

and thus the copula density would be

$$c(\mathbf{u}) = 1, \quad \mathbf{u} \in [0, 1]^d.$$

□

From Equation 3.5, it is obvious that the Fréchet-Hoeffding bounds correspond to the extreme cases of perfect dependence between the RVs $X_i, i = 1, \dots, d$.

Comonotonicity Copula

Consider the RVs X_1, \dots, X_d and strictly increasing transformations T_1, \dots, T_d and

$X_i = T(X_i)$ for $i = 2, \dots, d$. Making use of the *invariance principal*, it can be shown that these RVs have as copula the upper Fréchet-Hoeffding bound

$$M(\mathbf{u}) = \min\{u_1, \dots, u_d\}.$$

Since there is perfect positive dependence between those RVs, we call M the *comonotonicity copula*. The number of dimensions d can be any finite number greater than or equal to 2 for M to be a copula, as the minimum remains well defined.

□

Countermonotonicity Copula

Similar to the comonotonic case, it can be shown that if two RVs X_1 and X_2 are perfectly negatively dependent, their copula is the lower Fréchet-Hoeffding bound

$$W(\mathbf{u}) = \max \left\{ \sum_{i=1}^d u_i - d + 1, 0 \right\}.$$

Therefore, W is known as the *countermonotonicity copula*. Because of the fact that countermonotonicity is not valid for a dimension greater than 2, we end up with the restriction $d = 2$ for W to be indeed a copula.

□

3.2.2 Elliptical Copulas

Copulas which can be derived from known multivariate distributions like for example the *Multivariate Normal (or Gaussian) Distribution* or the *Multivariate Student's t-Distribution* are called *implicit copulas*. *Elliptical copulas* are implicit copulas which arise via Sklar's theorem from elliptical distributions like the mentioned examples.

Gaussian Copula

Without loss of generality (w.l.o.g.), for a random vector $\mathbf{X} \sim \mathcal{N}_d(\mathbf{0}, \mathbf{P})$ and correlation matrix \mathbf{P} , the *Gaussian copula (family)* is given by

$$C_{\mathbf{P}}^{Ga}(\mathbf{u}) = \Phi_{\mathbf{P}} \left(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d) \right), \quad (3.7)$$

where Φ is the CDF of $\mathcal{N}(0, \sigma^2)$ and $\Phi_{\mathbf{P}}$ is the CDF of $\mathcal{N}_d(\mathbf{0}, \mathbf{P})$.

There are special cases to this copula family, namely for $d = 2$ and correlation ρ , the *bivariate Gaussian copula* C_{ρ}^{Ga} is equivalent to

- the independence copula Π if $\rho = 0$,
- the comonotonicity copula M if $\rho = 1$ and
- the countermonotonicity copula W if $\rho = -1$

The density of the Gaussian copula is given by

$$c_{\mathbf{P}}^{Ga}(\mathbf{u}) = \frac{1}{\sqrt{\det \mathbf{P}}} \exp \left(-\frac{1}{2} \mathbf{x}' (\mathbf{P}^{-1} - \mathbf{I}_d) \mathbf{x} \right), \quad (3.8)$$

where $\mathbf{x} = (\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_d))$.

□

t-Copula

Consider w.l.o.g. $\mathbf{X} \sim t_d(\nu, \mathbf{0}, \mathbf{P})$ (multivariate Student's t-distribution) with ν Degrees of Freedom (d.o.f.) and \mathbf{P} a correlation matrix, then the *t-copula (family)* is given by

$$C_{\nu, \mathbf{P}}^t(\mathbf{u}) = t_{\nu, \mathbf{P}}(t_{\nu}^{-1}(u_1), \dots, t_{\nu}^{-1}(u_d)), \quad (3.9)$$

where t_{ν} is the CDF of the univariate Student's t-distribution and $t_{\nu, \mathbf{P}}$ is the CDF of the multivariate Student's t-distribution (both with ν d.o.f.).

For the *bivariate t-copula* ($d = 2$), the special cases are the same as for the Gaussian copula except that $d = 0$ does not yield the independence copula (unless $\nu \rightarrow \infty$ in which case $C_{\nu, \rho}^t = C_{\rho}^{Ga}$).

The density of $C_{\nu, \mathbf{P}}^t$ is given by

$$c_{\nu, \mathbf{P}}^t(\mathbf{u}) = \frac{\Gamma((\nu + d)/2)}{\Gamma(\nu/2)\sqrt{\det \mathbf{P}}} \left(\frac{\Gamma(\nu/2)}{\Gamma((\nu + 1)/2)} \right)^d \frac{(1 + \mathbf{x}' \mathbf{P}^{-1} \mathbf{x}/\nu)^{-(\nu+d)/2}}{\prod_{j=1}^d (1 + x_j^2/\nu)^{-(\nu+1)/2}}, \quad (3.10)$$

where $\mathbf{x} = (t_{\nu}^{-1}(u_1), \dots, t_{\nu}^{-1}(u_d))$.

□

3.2.3 Archimedean Copulas

Unlike implicit copulas, *explicit copulas* can be specified directly by taking into account certain constructional principles. The most important aspects of a such

explicit copulas, in particular *archimedean copulas*, are showcased in this subsection. Archimedean copulas are of the general form

$$C(\mathbf{u}) = \phi^{-1}(\phi(u_1) + \cdots + \phi(u_d)) \quad (3.11)$$

or equivalently

where the function $\phi : [0, 1] \rightarrow [0, \infty)$ is the (*archimedean*) *generator* and satisfies the following properties:

- ϕ is strictly decreasing in the entire domain $[0, 1]$
- We set $\phi(1) = 0$
- If $\phi(0) = \lim_{u \rightarrow 0^-} \phi(u) = \infty$, then ϕ is called *strict*.

Based on Equation 3.11 and according to the form of the generator, we can construct several copula families. Three of the most popular ones are the *Gumbel*, the *Clayton* and the *Frank copula*, which will be discussed.¹ The advantage of such copulas lies in the fact that they interpolate between certain fundamental dependency structures.

Clayton Copula

If the generator takes on the form

$$\phi_{Cl}(u) = \frac{1}{\theta} (u^{-\theta} - 1) \quad (3.12)$$

then we obtain the *Clayton copula* given by

$$C_{\theta}^{Cl}(u_1, u_2) = (\max\{u_1^{-\theta} + u_2^{-\theta} - 1, 0\})^{-\frac{1}{\theta}}, \quad (3.13)$$

where $\theta \in [-1, \infty) \setminus \{0\}$. For $\theta > 0$ the generator of the Clayton copula is strict and we arrive at

$$C_{\theta}^{Cl}(u_1, u_2) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-\frac{1}{\theta}}. \quad (3.14)$$

Note that for $\theta = -1$, we obtain the lower Fréchet-Hoeffding bound, whereas for the limits $\theta \rightarrow 0$ and $\theta \rightarrow \infty$ we arrive at the independence copula and the comonotonicity copula respectively.

□

¹We will look into these copulas for the bivariate case ($d = 2$) only.

Gumbel Copula

If the generator takes on the form

$$\phi_{Gu}(u) = (-\ln u)^\theta, \quad \theta \in [1, \infty), \quad (3.15)$$

then we arrive at the *Gumbel copula* given by

$$C_\theta^{Gu}(u_1, u_2) = \exp \left[- \left((-\ln u_1)^\theta + (-\ln u_2)^\theta \right)^{\frac{1}{\theta}} \right]. \quad (3.16)$$

Note that for $\theta = 1$, we obtain the independence copula, while for $\theta \rightarrow \infty$ the Gumbel copula converges to the comonotonicity copula.

□

Frank Copula

If the generator takes on the form

$$\ln(e^{-\theta} - 1) - \ln(e^{-\theta u} - 1), \quad \theta \in \mathbb{R} \setminus \{0\}, \quad (3.17)$$

we obtain the *Frank copula* given by

$$C_\theta^{Fr}(u_1, u_2) = -\frac{1}{\theta} \ln \left(1 + \frac{(e^{-\theta u_1} - 1) \cdot (e^{-\theta u_2} - 1)}{e^{-\theta} - 1} \right) \quad (3.18)$$

□

MAYBE MORE ON THESE WITH SOME PRETTY PLOTS

3.3 Dependence Measures

Appendix

Include appendix here...

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western European countries

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1.2 Article master data

2

List of Abbreviations

BIC Bayesian Information Criterion

GLM Generalized Linear Model

LM Linear Regression Model

LMM Linear Mixed Model

GLMM Generalized Linear Mixed Model

CDF Cumulative Distribution Function

PDF Probability Density Function

RV Random Variable

w.l.o.g. Without loss of generality

d.o.f. Degrees of Freedom

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