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Master Thesis

Multivariate modelling of the dependency structure between article sales of a sportswear manufacturer

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1 Introduction

Write introduction here and "upper" subsections here (adidas, Motivation, etc...)

1.1 Data Sources

Throughout each season, transactional data are collected from online purchases of the sports brand's eCommerce website. Specifically, we are provided with weekly sales data for western European countries. A short description is depicted in Table 1.1.

Column	Description	Values
week_id	Calender week of a specific year (YYYYWW)	Factors: *201648, ..., 201852*
article_number	Unique article identification number (article ID)	Factors: *10669, 10, ...*
min_date_of_week	Minimum date of the respective week; always a Monday (YYYY-MM-DD)	Dates: *2016-11-28, ..., 2018-12-24*
art_min_price	Minimal recorded price of the article	Non-negative (integer) value
month_id	Calender month of a specific year (YYYYMM)	Factors: *201612, ..., 201812*
season	Season of year (format: SSYY) (Spring-Summer [SS]: December - May) Fall-Winter [FW]: June - November)	Factors: *SS17, FW17,* *SS18, FW18, SS19*
bf_w	Weekly "Black Friday" promotion intensity of the article	Between 0 and 1
ff_w	Weekly "Friends & Family" promotion intensity of the article	Between 0 and 1
ot_w	Weekly article promotion intensity of "Other" type	Between 0 and 1
gross_demand_quantity	Weekly amount of added articles to shopping cart	Non-negative (integer) value
base_price_locf	Retail price of the article without any discounts	Non-negative (integer) value
total_markdown_pct		
day_of_month	Day of the month	Integers: *1 - 31*
month_of_year	Month of the year	Factors: *January, ..., December*
year	Year	Integers: *2016, 2017, 2018*
week_of_year	Week of the year	Integers: *1 - 52*

Table 1.1: Transactional raw data description from online purchases of western European countries

Due to legal regulations of the company, some columns had to undergo anonymization in order for the data to be released. To ensure data protection and confidentiality, numeric variables (with exception of time-indicating columns) were transformed. As a consequence for the analysis part, most integer values were converted to float numbers. This fact should be kept in mind by the reader, since the above table serves as a reminder and reference point for the data documentation.

Another peculiarity of this setup is to be considered, too. We will often refer to the variable *gross demand quantity* as *sales*, even though it is obviously not exactly the same. In the eCommerce environment, there are several stages before the purchase is complete, e.g. addition to cart, removal from cart, proceeding to checkout & even the return of bought articles. Targeting the articles added to cart, i.e. the (gross) demand quantity, provides the optimal data extraction for analytical purposes and is the closest to adequately model the dependency structure between net sales of articles.

Besides the transactional data, *article master data*, i.e. attributes of the articles, are provided and described in Table 1.2.

Column	**Description**	**Values (all Factors)**
article_number	Unique article identification number (article ID)	*10669, 10, ...*
gender	Gender type of the article (Men, Women, Unisex)	*M, W, U*
age_group	Age group of the article (Adult, Infant, Junior, Kids)	*A, I, J, K*
product_division_descr	Product division of the article	*APPAREL, FOOTWEAR, HARDWARE*
product_group_descr	Product group of the article	*BAGS, BALLS, FOOTWEAR ACCESSORIES, SHOES, ...*
color	Consolidated color group of the article	*BEIGE, BLACK, BROWN, ORANGE, PINK, RED, ...*
sports_category_descr	Sports category of the article	encoded: *SC_1, ..., SC_22*
sales_line_descr	Sales line of the article	encoded: *SL_1, ..., SL_379*
business_unit_descr	The article's Business Unit membership	encoded: *BU_1, ..., BU_18*
business_segment_descr	The article's Business Segment membership	encoded: *BS_1, ..., BS_49*
sub_brand_descr	Sub-brand of the article	encoded: *sub-brand_1, ..., sub-brand_4*
item_type	Item type of the article	encoded: *IT_1, ..., IT_171*
brand_element	Brand element of the article	encoded: *BE_1, ..., BE_131*
product_franchise_descr	Product franchise of the article	encoded: *franchise_1, ..., franchise_72*
product_line_descr	Product line of the article	encoded: *PL_1, ..., PL_105*
franchise_bin	Franchise indicator of the article	*FRANCHISE, NON-FRANCHISE*
category	Category of the article	encoded: *category_1, category_2*

Table 1.2: Article master data

Plenty of additional information is stored in the database, but we are neglecting columns omitted in these tables, as they are redundant, already summarized, transformed or simply do not provide any value.
Overall, we will be dealing with data collected over two years, namely the years 2017 and 2018, while some transactions of late 2016 might be attached marginally. In summary, after joining the transactional observations to the article attributes by the article ID, this translates to a dataset of over 587,000 rows and over 30 variables.

2 Statistical Theory & Methods

This chapter introduces various statistical methods used during the conduction of this thesis. It is assumed that basic understanding and knowledge of the reader regarding mathematical foundations of statistics (like linear algebra, probability theory, etc) already exists.

2.1 Generalized Linear Models

Generalized Linear Models (GLMs) are an extension of the *Classical Linear Regression Model (LM)*

$$y_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_k x_{ik} + \varepsilon_i, \quad i = 1, \ldots, n$$

which in matrix notation can be written as

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

where the response variable y_i can take values from several probability distributions (e.g. Poisson, Binomial, Gamma and others), which are members of the exponential family [Fahrmeir et al., 2003]. The linear predictor

$$\eta_i = \beta_0 + \beta_1 x_{i1} + \ldots + \beta_k x_{ik} + \varepsilon_i = \boldsymbol{x}_i' \boldsymbol{\beta} \tag{2.1}$$

is passed through a *response function h* (a one-to-one, twice differentiable transformation), such that

$$E(y_i) = h(\eta_i)$$

i.e. the expected value of the response variable belongs to the value range of *h*. The inverse of the response function,

$$g = h^{-1}$$

is called the *link function*.

2.2 Generalized Additive Models

2.3 Mixed Effects Models

Linear Mixed Models (LMMs) are powerful tools when dealing with clustered data or data with a longitudinal structure (repeated measurements of individuals). As in the classical LM, there are population-specific effects, namely the parameter vector of *fixed effects* $\boldsymbol{\beta}$, as well as the cluster- or individual-specific effects of such models called *random effects* [Fahrmeir et al., 2003]. In the following, we will refer to our clusters or individuals as "groups" for briefness. Mathematically speaking, the linear predictor $\eta_{ij} = \boldsymbol{x}'_{ij}\boldsymbol{\beta}$ is extended to

$$\eta_{ij} = \boldsymbol{x'}_{ij}\boldsymbol{\beta} + \boldsymbol{u}'_{ij}\gamma_i, \quad j = 1, \ldots, m, \quad i = 1, \ldots, n_i, \tag{2.2}$$

where

- i is the number of groups
- j is the number of observations per group
- $\boldsymbol{\beta}$ is the vector of fixed effects
- γ_i is the vector of random effects
- $\boldsymbol{x}'_{ij}$ is the vector of covariates and
- $\boldsymbol{u}'_{ij}$ is a subvector of $\boldsymbol{x}'_{ij}$.

$\boldsymbol{x}'_{ij} = (1, x_{ij1}, \ldots, x_{ijk})$ and $\boldsymbol{u}'_{ij} = (1, u_{ij1}, \ldots, u_{ijk})$ are therefore the design vectors and ε_{ij} are the error terms of the *measurement model*

$$y_{ij} = \boldsymbol{x}'_{ij}\boldsymbol{\beta} + \boldsymbol{u}'_{ij}\boldsymbol{\gamma}_i + \epsilon_{ij}, \quad \varepsilon_{ij} \overset{i.i.d.}{\sim} N\left(0, \sigma^2\right) \tag{2.3}$$

or in matrix notation

$$\boldsymbol{y}_i = \boldsymbol{X}_i\boldsymbol{\beta} + \boldsymbol{U}_i\boldsymbol{\gamma}_i + \boldsymbol{\varepsilon}_i \tag{2.4}$$

for group $i = 1, \ldots, m$ with $E(\boldsymbol{\varepsilon}_i) = \boldsymbol{0}$.

Similar to GLMs, *Generalized Linear Mixed Models (GLMMs*) relate the linear mixed predictor 2.2 to the conditional mean $\mu_{ij} = E(y_{ij}|\boldsymbol{\gamma}_i)$ via a suitable response function h, such that $\mu_{ij} = h(\eta_{ij})$ and thus the conditional density of y_{ij} belongs to the exponential family.

3 Copulas & Dependence Modelling

Multivariate distributions consist of the marginal distributions and the dependence structure between those marginals. These components can be specified separately in a single framework with the help of copula functions. This chapter introduces the concept of modelling such dependency structures with copulas, which is the main focus of this thesis.

3.1 Copulas

A d-dimensional function $C : [0,1]^d \to [0,1]$ is called a *copula*, if it is a Cumulative Distribution Function (CDF) with uniform margins, i.e.

$$P\left(U_1 \leq u_1, \ldots, U_d \leq u_d\right) = C\left(u_1, \ldots, u_d\right)$$

where $U_i, i = 1, \ldots, d$ are uniformly distributed in $[0,1]$.

Sklar's Theorem

Let F be a d-dimensional CDF with marginal distributions $F_i, i = 1, \ldots, d$. Then there exists a copula C such that

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)) \tag{3.1}$$

for all $x_i \in \mathbb{R}, i = 1, \ldots, d$. The copula C is unique, if $\forall i = 1, \ldots, d$, F_i is continuous. Otherwise C is uniquely determined only on $Ran(F_1) \times \ldots \times Ran(F_d)$, where $Ran(F_i)$ is the range of F_i.

Conversely, if C is a d-dimensional copula and $F_1, \ldots, F_d$ are univariate CDF's, then F as defined in Equation 3.1 is a d-dimensional CDF.

Appendix

Include appendix here...

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List of Abbreviations

BIC Bayesian Information Criterion

GLM Generalized Linear Model

LM Linear Model

LMM Linear Mixed Model

GLMM Generalized Linear Mixed Model

CDF Cumulative Distribution Function

References

[Fahrmeir et al., 2003] Fahrmeir, L., Kneib, T., Lang, S., and Marx, B. (2003). Regression; Models, Methods and Applications. 2013.

[Lütkepohl, 2005] Lütkepohl, H. (2005). *New introduction to multiple time series analysis*. Springer Science & Business Media.