n append operations, followed by n pop operations on an initially empty array

appending

For n append operations, if the underlying array initially has size 1, and doubles in size each time when capacity is reached, then we will have $\lceil \log_2 n \rceil$ resizing operations, the underlying array sizes will be:

$$1, 2, 4, 8, 16 \dots 2^{\lceil \log_2 n \rceil}$$

$$= 2^0, 2^1, 2^2, 2^3 \dots 2^{\lceil \log_2 n \rceil}$$

and for each capacity, except the last one, we will need to copy over all the elements in the array, giving:

$$2^0, 2^1, 2^2 \dots 2^{\log_2 n} \quad \text{copy operations}$$

because we want first n+1 terms, since from 0 to n is n+1

$$\sum_{0}^{\log_2 n} 2^i = \frac{1-2^{\log_2 n+1}}{1-2} = \frac{1-2\cdot 2^{\log_2 n}}{1-2} = \frac{1-2n}{-1} = 2n-1$$

and then we also have all the other appending operations, which do not require resizing, of which there are $n - (\log_2 n + 1)$. So in total we have $\approx 3n$ operations for doing n appends, which is $O(n)$

~~Popping~~

For popping elements from the list, we will get the following sequence:

no. of elements $\quad 40, 39, 38 \dots \underline{16}, 15, 14 \dots \underline{8}, 7, 6 \dots \underline{4}, 3, \underline{2}, \underline{1}$ $\quad 16 + 8 + 4 + 2 + 1$ copy operations $= 31$

size of array $\quad 64 \quad 32 \quad 16 \quad 8 \quad 4 \quad 2$

Generally:

no. of elements $\quad n \quad 2^{\lceil \log_2 n \rceil - 2} \dots\dots 1$

size of array $\quad 2^{\lceil \log_2 n \rceil} \quad 2^{\lceil \log_2 n \rceil - 1} \dots\dots 2^1$

$$\sum_{i=0}^{\lceil \log_2 n \rceil - 2} = \frac{1 - 2^{\lceil \log_2 n \rceil - 2 + 1}}{1 - 2} = \frac{1}{2} \cdot \frac{1 - 2^{\lceil \log_2 n \rceil}}{-1} = \frac{n - 1}{2}$$

- the number of copy operations for n appends can vary between n to $2n$, consider the cases when $n = 32$ and $n = 33$. For $n = 32$, we would have $\sum_{i=0}^{4} 2^i = 31 \simeq n$ whereas for $n = 33$ we would have $\sum_{i=0}^{5} 2^i = 63 \simeq 2n$

- same for n pops, consider $n = 32$, we would have $\sum_{i=0}^{3} 2^i = 15 \simeq \frac{n}{2}$, whereas for $n = 33$, we would have $\sum_{i=0}^{4} 2^i = 31 \simeq n$

- So for n appends, followed by n pops, we would have:

$$\text{appends}: \underbrace{(n < k < 2n)}_{\text{copying}} + \underbrace{n}_{\text{appends w/o copying}} = 2n < k < 3n$$

$$\text{pops}: \underbrace{\left(\frac{n}{2} < k < n\right)}_{\text{copying}} + \underbrace{n}_{\text{pops without copying}} = \frac{3n}{2} < k < 2n$$

nevertheless, we would have $O(n)$ running time