Computer Vision

Photometric Stereo & Shape from Shading

Szeliski: 12.1.1

Trucco: 9.1 – 9.4 (Shape From Shading)

Horn: 215 - 232

Today: Photometric Stereo

Key Idea: use pixel brightness to understand shape

Photometric Stereo & Shape from Shading

- Technique for recovering 3-D shape information from image intensity (brightness)
- We will discuss:
  - Reflectance maps.
  - Photometric stereo.
  - Shape from shading.

Photometric Stereo

What results can you get?

Input (1 of 12) | Normals (RGB colormap) | Normals (vectors) | Shaded 3D rendering | Textured 3D rendering

Radiometry and Reflectance (REVIEW)

Image irradiance

Brightness falloff

$$E(\mathbf{p}) = L(\mathbf{P}) \frac{\pi}{4} \left(\frac{d}{f} \right)^2 \cos^4(\alpha)$$

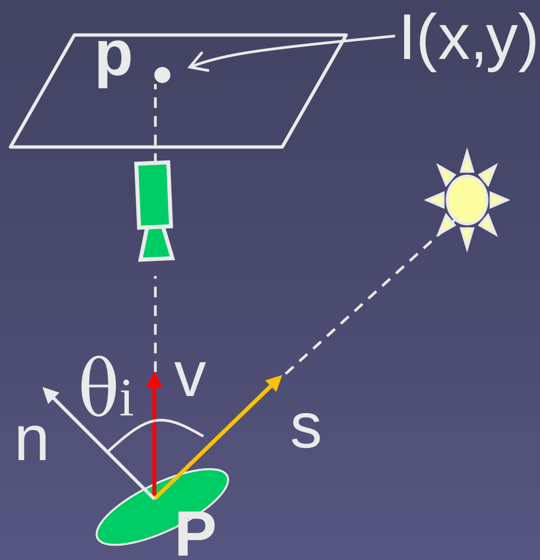
1 / effective F-number of lens

Scene radiance

We assume that:

$$E(\mathbf{p}) = L(\mathbf{P})$$

Lambertian Reflectance Model- Review



A Lambertian sphere

$$L(\mathbf{P}) = \frac{\rho'}{\pi} k \cos\theta_i = \frac{\rho'}{\pi} k (\mathbf{n} \cdot \mathbf{s})$$

$$L(\mathbf{P}) = \rho(\mathbf{n} \cdot \mathbf{s})$$

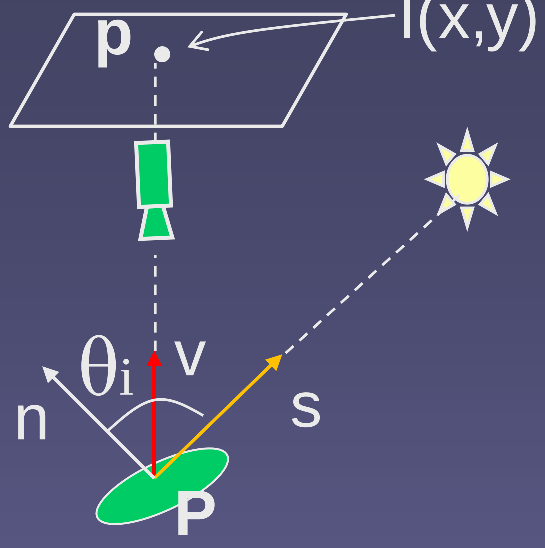
- k : Source brightness (intensity)
- ρ’ : Surface albedo (reflectance)
- ρ : Effective albedo (absorbs source brightness)

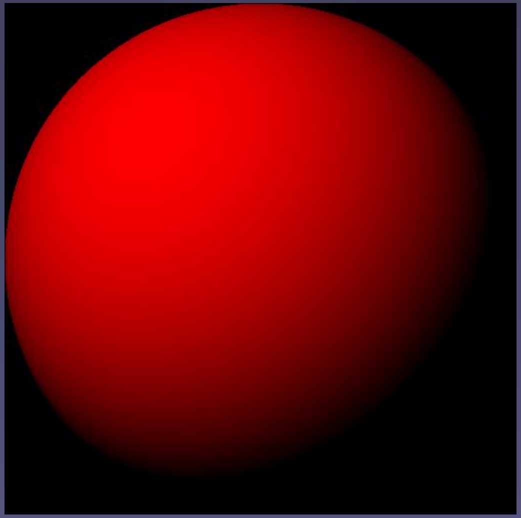
or:

$$E(\mathbf{p}) = \rho(\mathbf{n} \cdot \mathbf{s})$$

$$E(\mathbf{p}) = R_{\rho,\mathbf{s}}(\mathbf{n})$$: REFLECTANCE MAP

Lambertian Reflectance Model



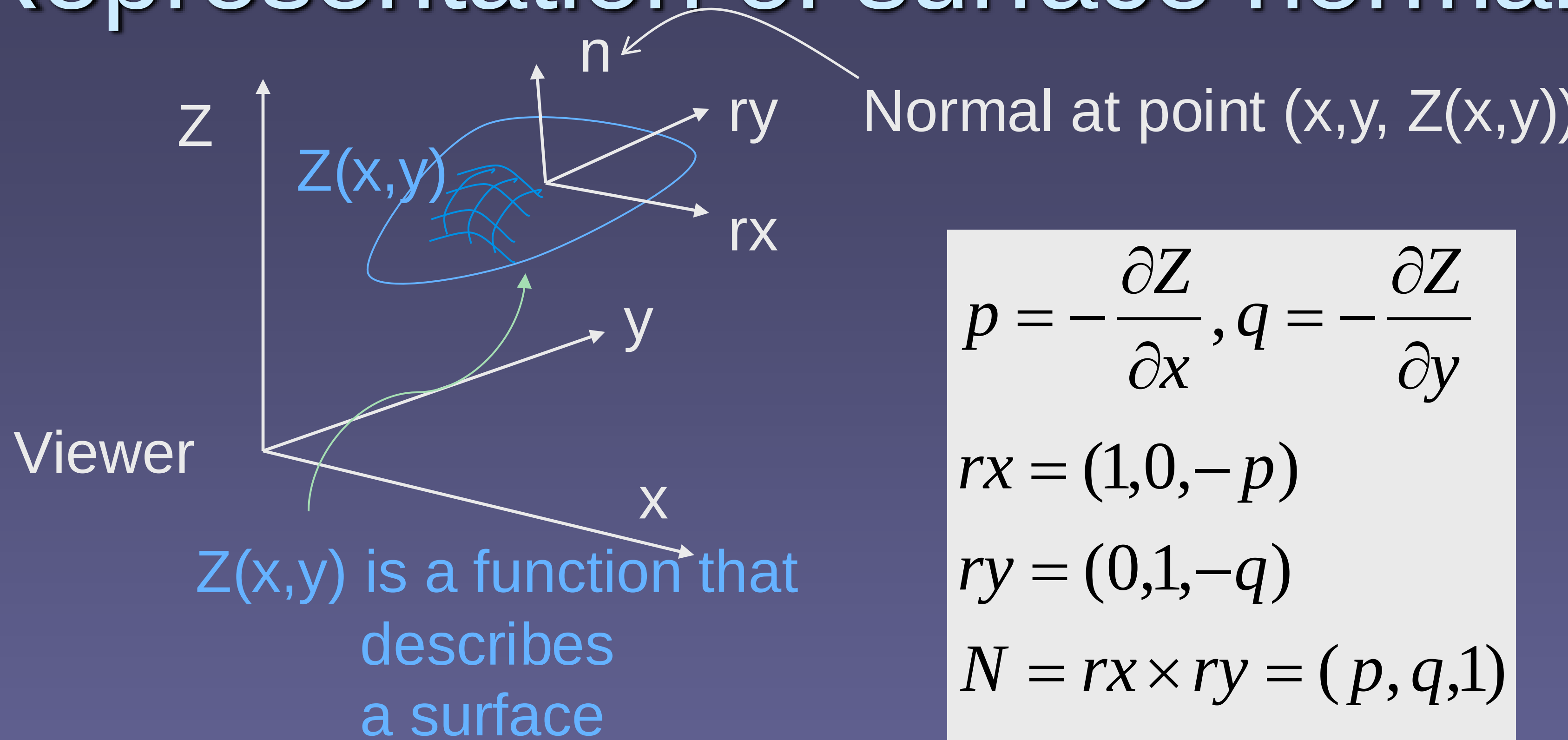


A Lambertian sphere

$$E(\mathbf{p}) = R_{\rho,\mathbf{s}}(\mathbf{n})$$: **REFLECTANCE MAP**

Relates Image Irradiance E to surface orientation for
given source direction
and surface reflectance

Representation of surface normal



$$p = -\frac{\partial Z}{\partial x}, q = -\frac{\partial Z}{\partial y}$$

$$rx = (1,0,-p)$$

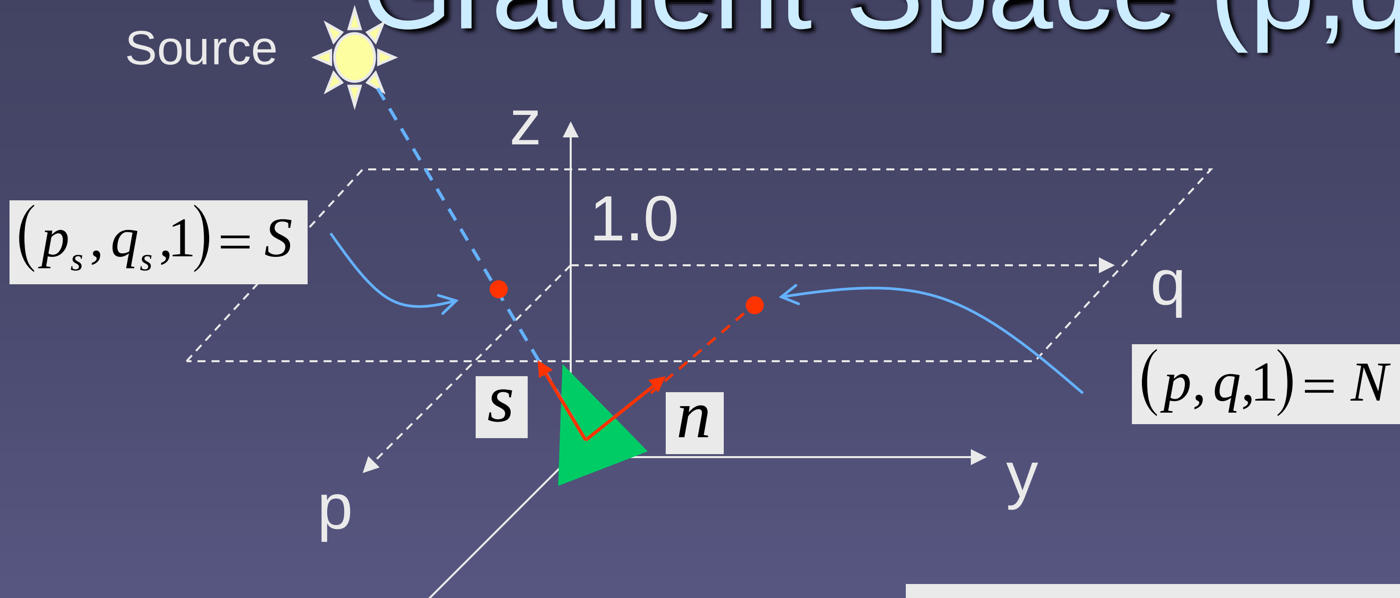
$$ry = (0,1,-q)$$

$$N = rx \times ry = (p, q, 1)$$

$$n = N / \| N \|$$

$$n = \frac{(p, q, 1)}{\sqrt{p^2 + q^2 + 1}}$$

Gradient Space (p,q)



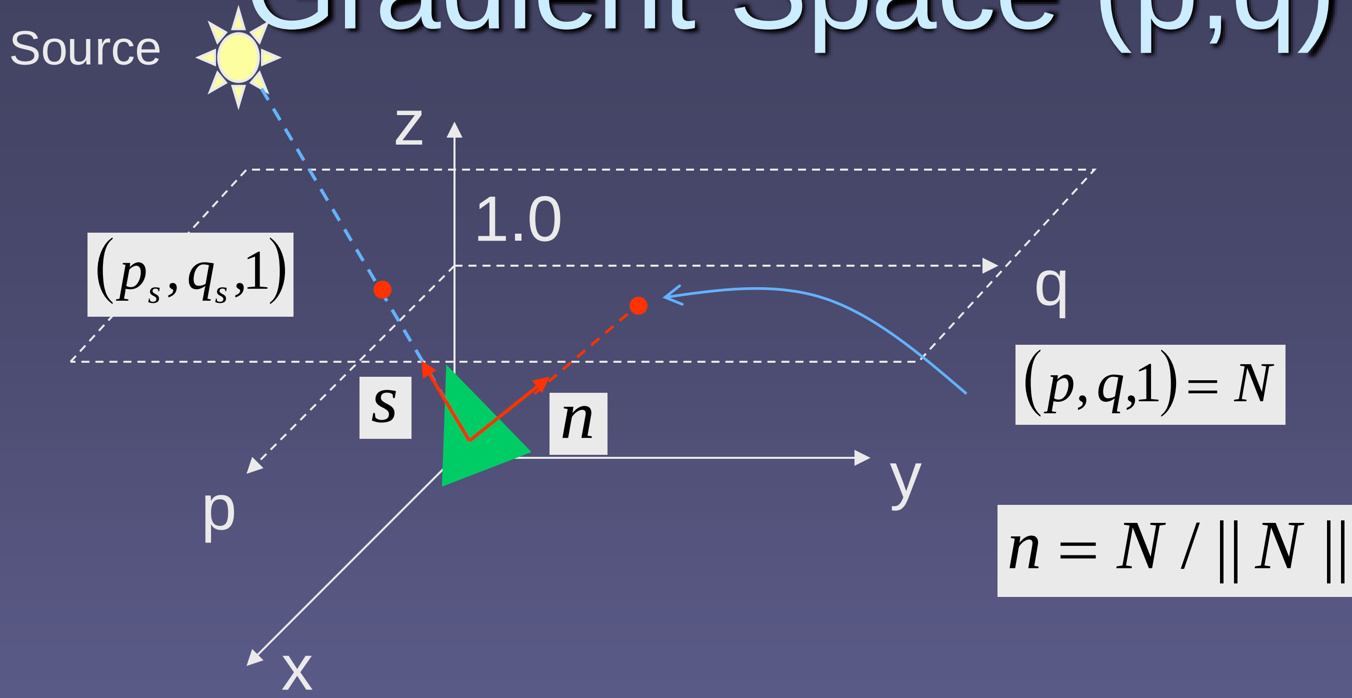
$$s = S / \| S \| = \frac{(p_s, q_s, 1)}{\sqrt{p_s^{\,2} + q_s^{\,2} + 1}}$$

$$n = N / \| N \| = \frac{(p, q, 1)}{\sqrt{p^2 + q^2 + 1}}$$

Surface normal can be represented by a point (p,q) on a plane! Source direction can be represented by a point (ps,qs), if we assume that source and normal are on the same side.

**We want to calculate (p,q) from pixel's intensity I

Gradient Space (p,q)



Surface normal $$n = \frac{(p, q, 1)}{\sqrt{p^2 + q^2 + 1}}$$

Source direction $$s = \frac{(p_s, q_s, 1)}{\sqrt{p_s^{\ 2} + q_s^{\ 2} + 1}}$$

$$\cos\vartheta_i = \mathbf{n} \cdot \mathbf{s} = \frac{1 + pp_s + qq_s}{\sqrt{1 + p^2 + q^2}\sqrt{1 + p_s^{\ 2} + q_s^{\ 2}}}$$

Assumption: SOURCE DIR. IS CONSTANT FOR ENTIRE SCENE.

Reflectance Map (Lambertian)

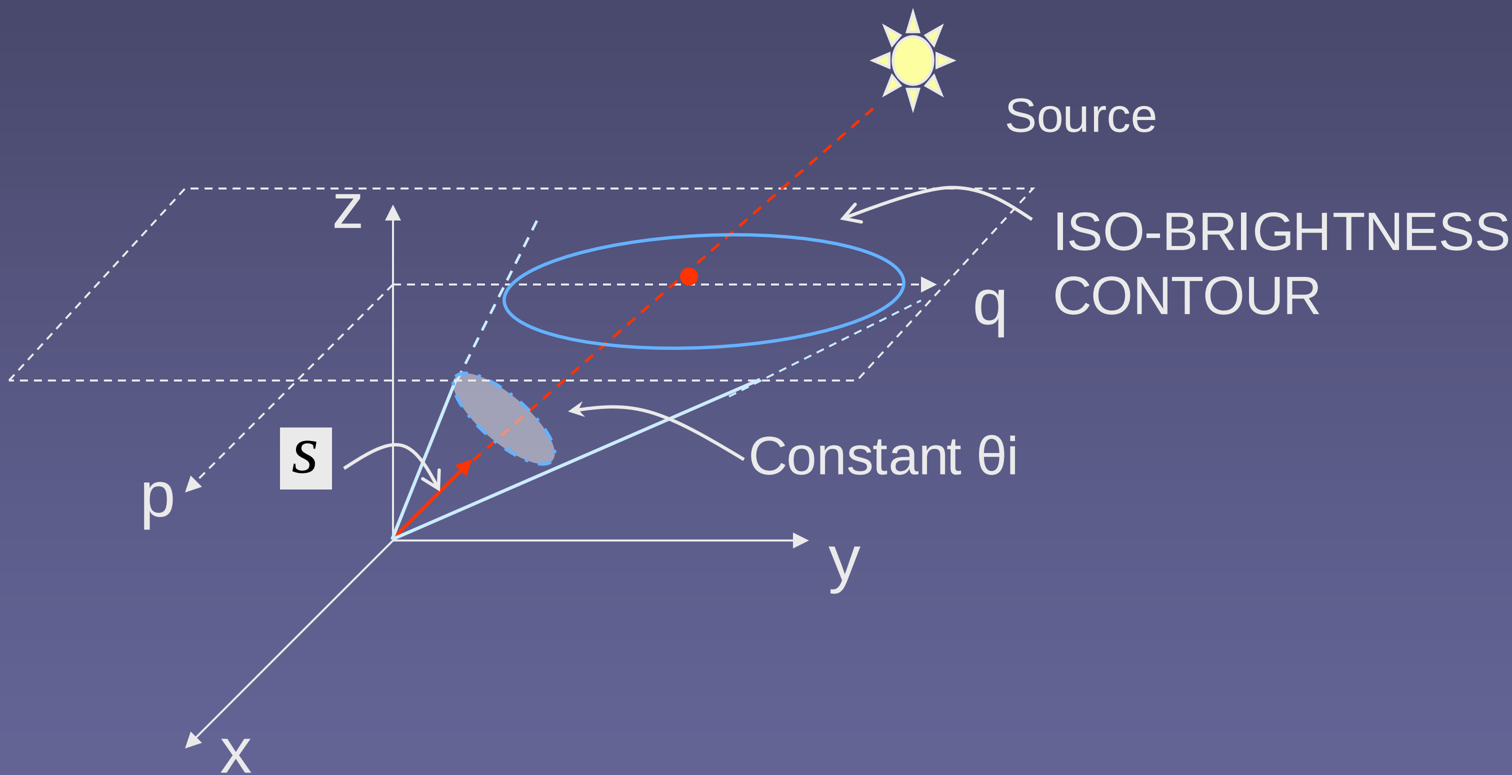
OR:

$$I(x,y) = E(\mathbf{p}) = \frac{1 + pp_s + qq_s}{\sqrt{1+p^2+q^2}\sqrt{1+p_s{}^2+q_s{}^2}} = R_{1,\mathbf{s}}(p,q)$$

Intensity at pixel (x,y) produced by 3D surface with normal (p,q)

Reflectance Map (Lambertian)

OR: $$I(x,y) = E(\mathbf{p}) = \frac{1 + pp_s + qq_s}{\sqrt{1+p^2+q^2}\sqrt{1+{p_s}^2+{q_s}^2}} = R_{1,\mathbf{s}}(p,q)$$



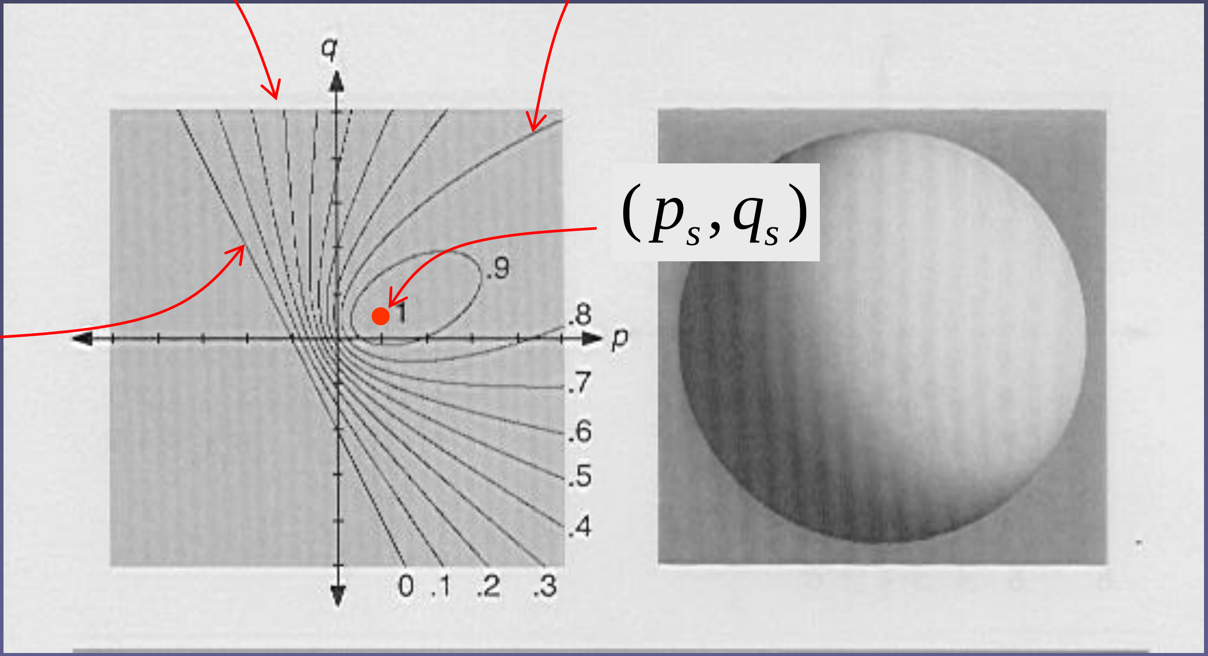
Reflectance Map (Lambertian)

ISO-BRIGHTNESS
CONTOURS

$R_{1,\mathbf{s}}(p,q)=0.8$

(p_s, q_s)

$\theta_i = 90^o$

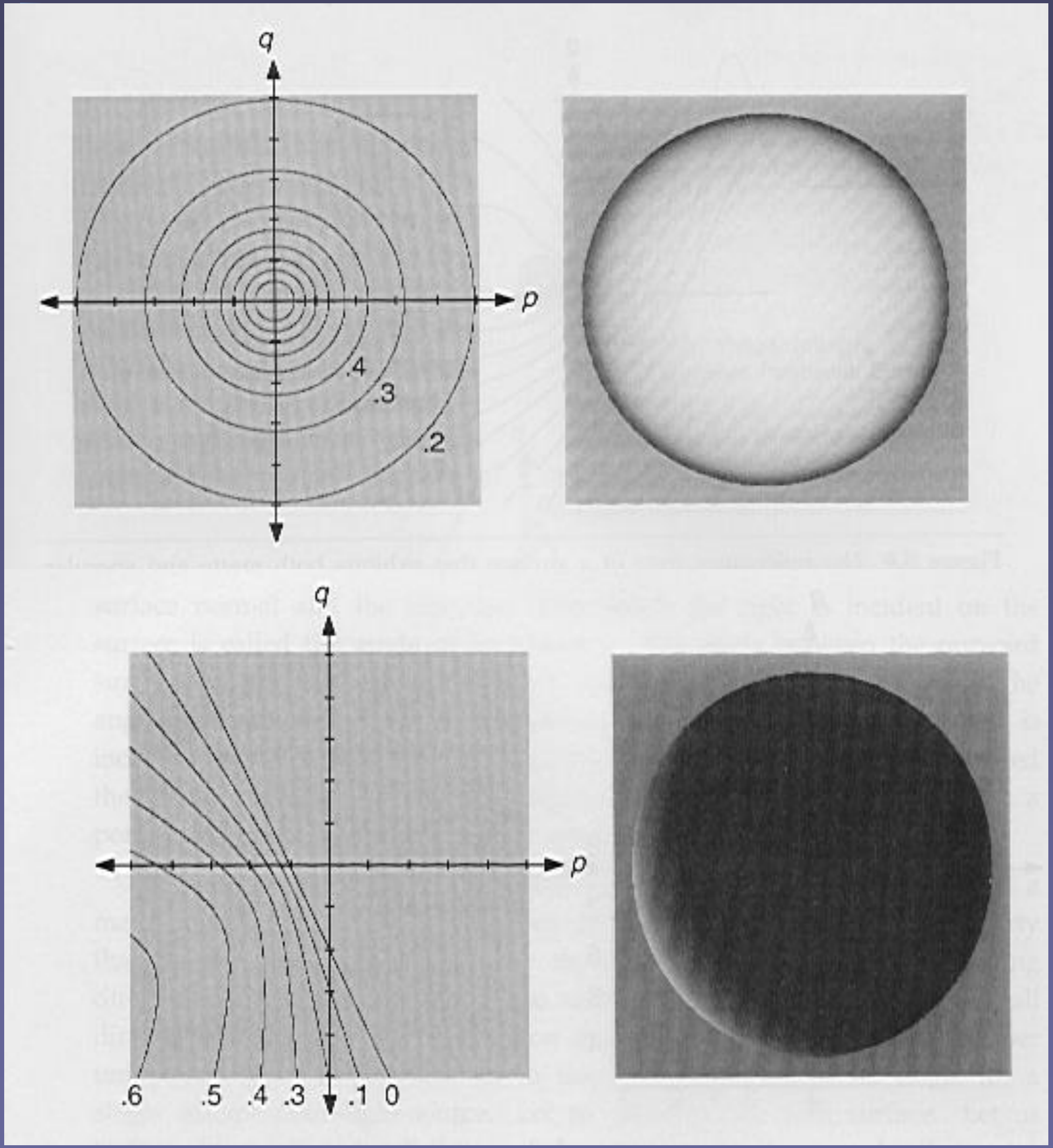


NOTE: R(p,q) is maximum when (p,q)=(ps,qs)

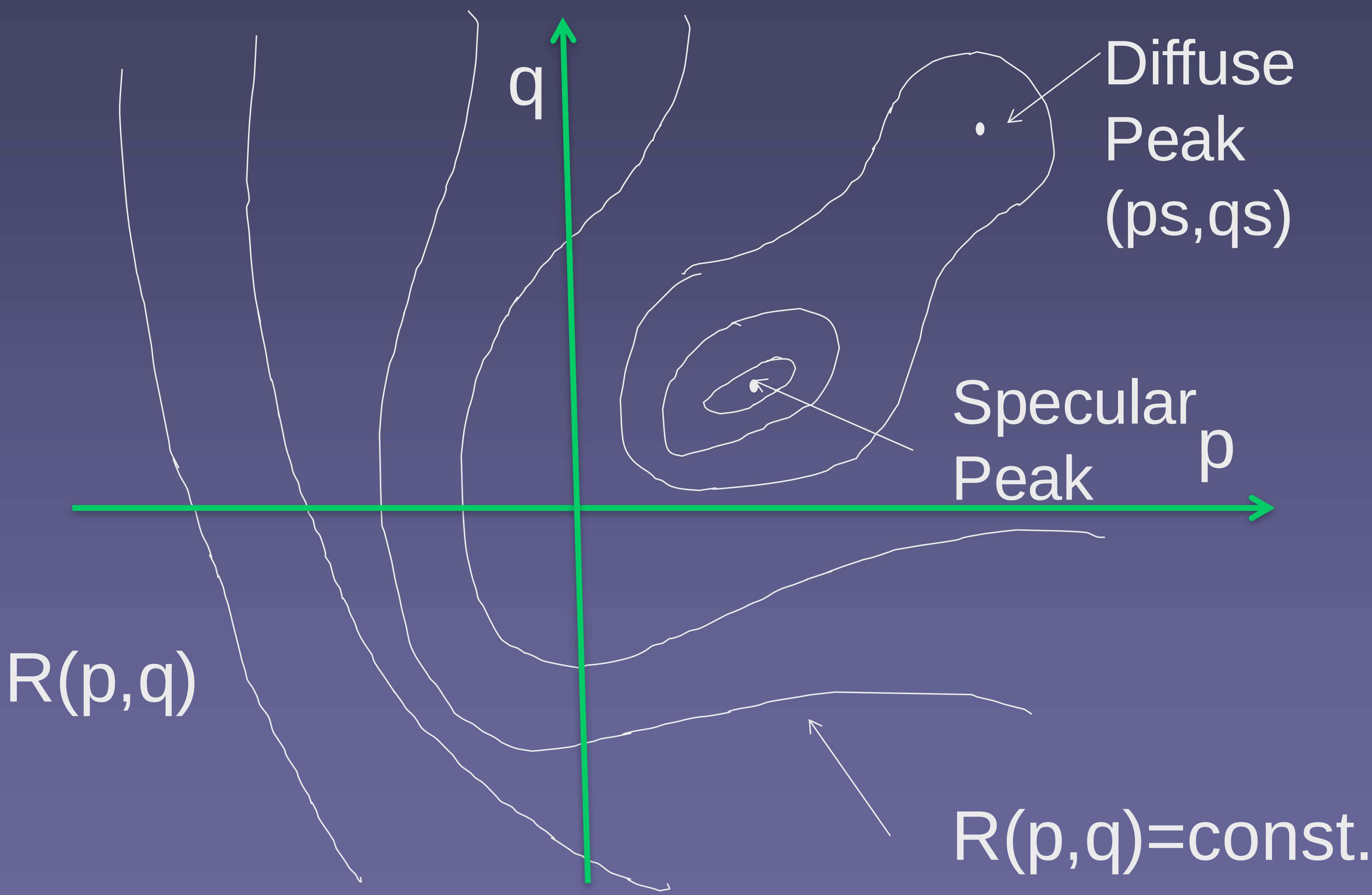
Reflectance Map (Lambertian)

Examples.

Where is the source with respect to the sphere?



Reflectance Map (Glossy Surfaces)



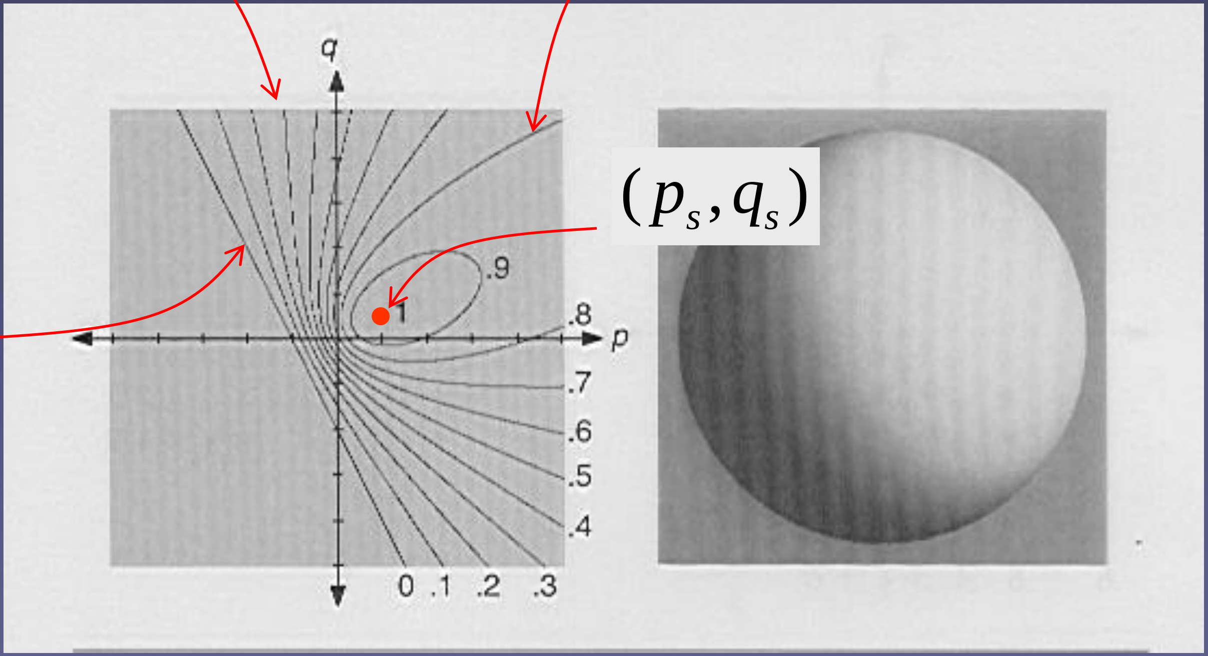
Shape from Shading

ISO-BRIGHTNESS CONTOURS

$R_{1,\mathbf{s}}(p,q) = 0.8$

$\theta_i = 90^o$

(p_s, q_s)



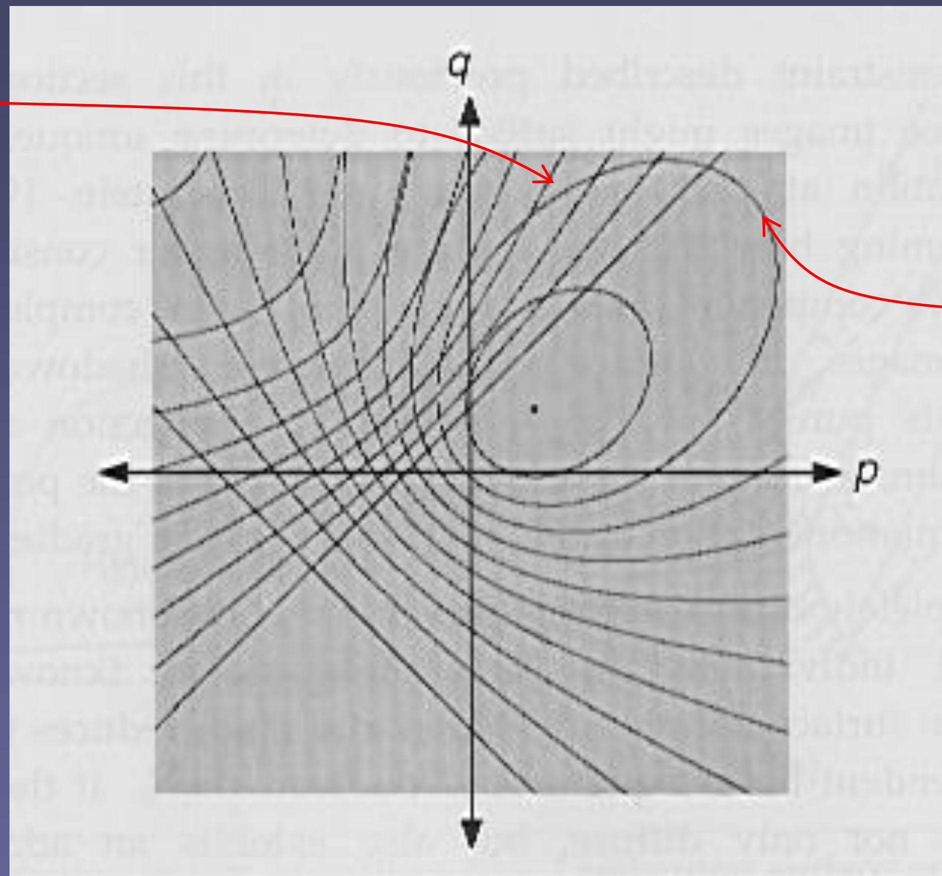
PROBLEM: Given 1) source direction (p_s, q_s)
2) surface albedo (ρ)
3) one intensity image I(x,y)

$\Rightarrow R_{\rho,\mathbf{s}}(p,q)$

Can we find unique surface orientation (p,q) per pixel?

Two reflectance maps?

$$R2(p,q) = r$$



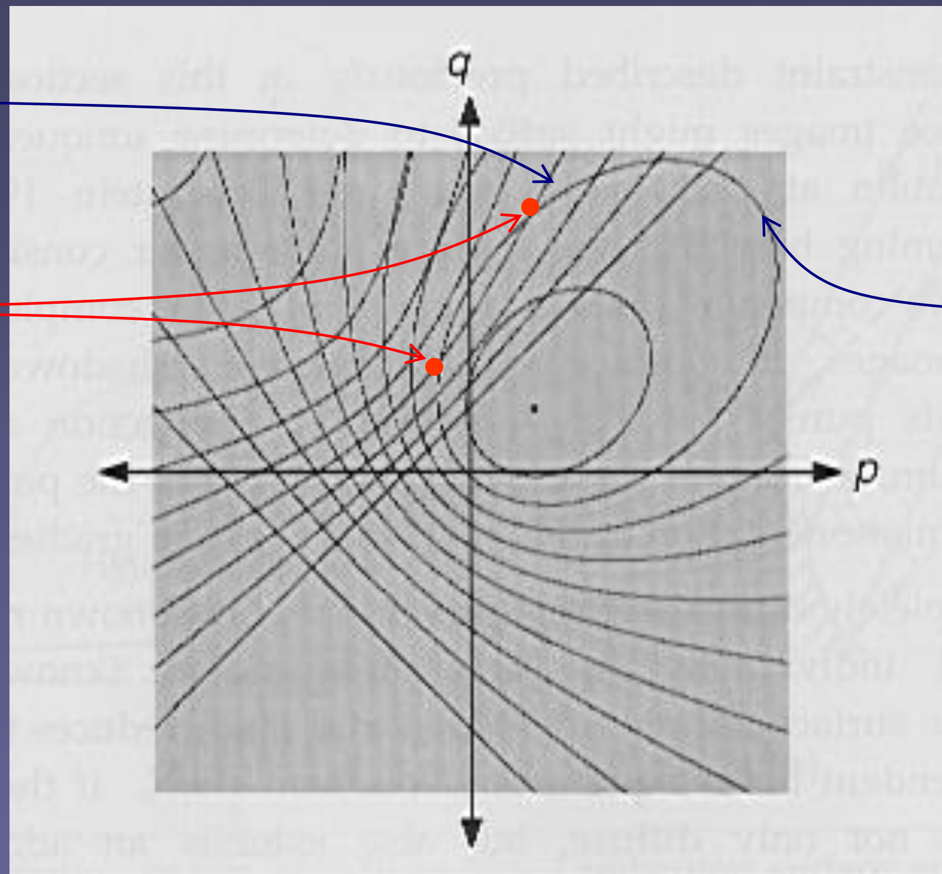
$$R1(p,q) = r'$$

For example:

r = 0.8 &
r’ = 0.4

Two reflectance maps?

$R2(p,q) = r$

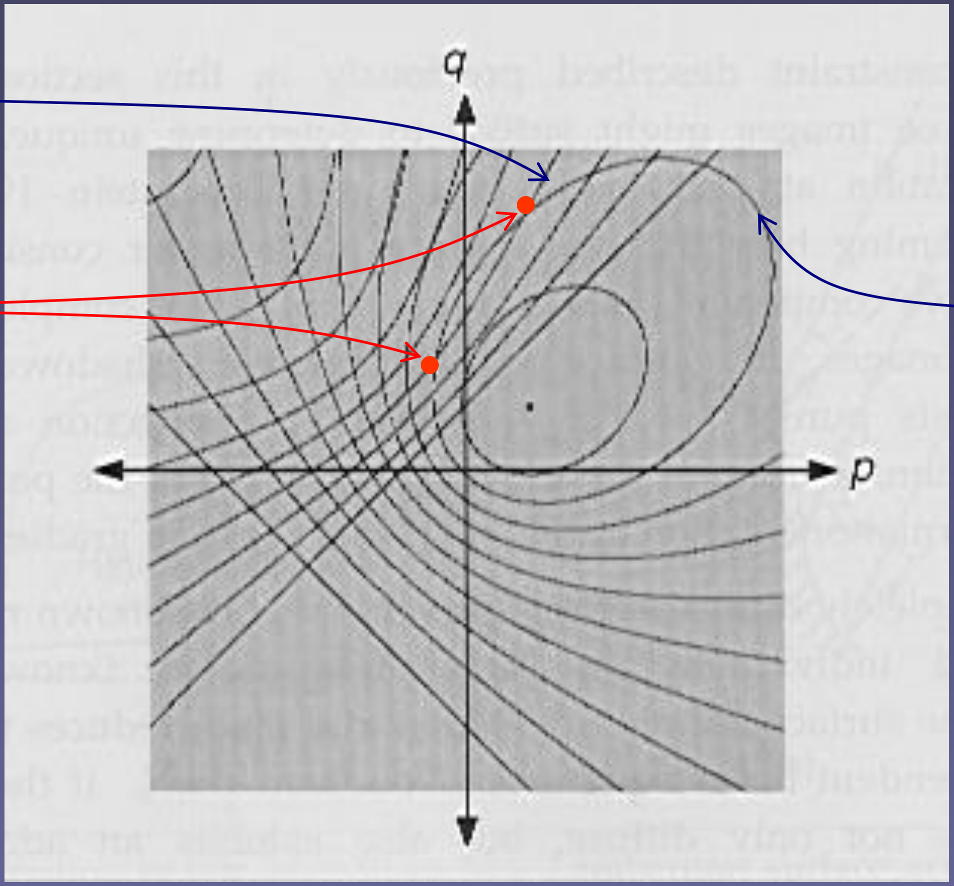


$R1(p,q) = r'$

Intersections:
2 solutions for
p and q.

Two reflectance maps?

$$R2(p,q) = r$$

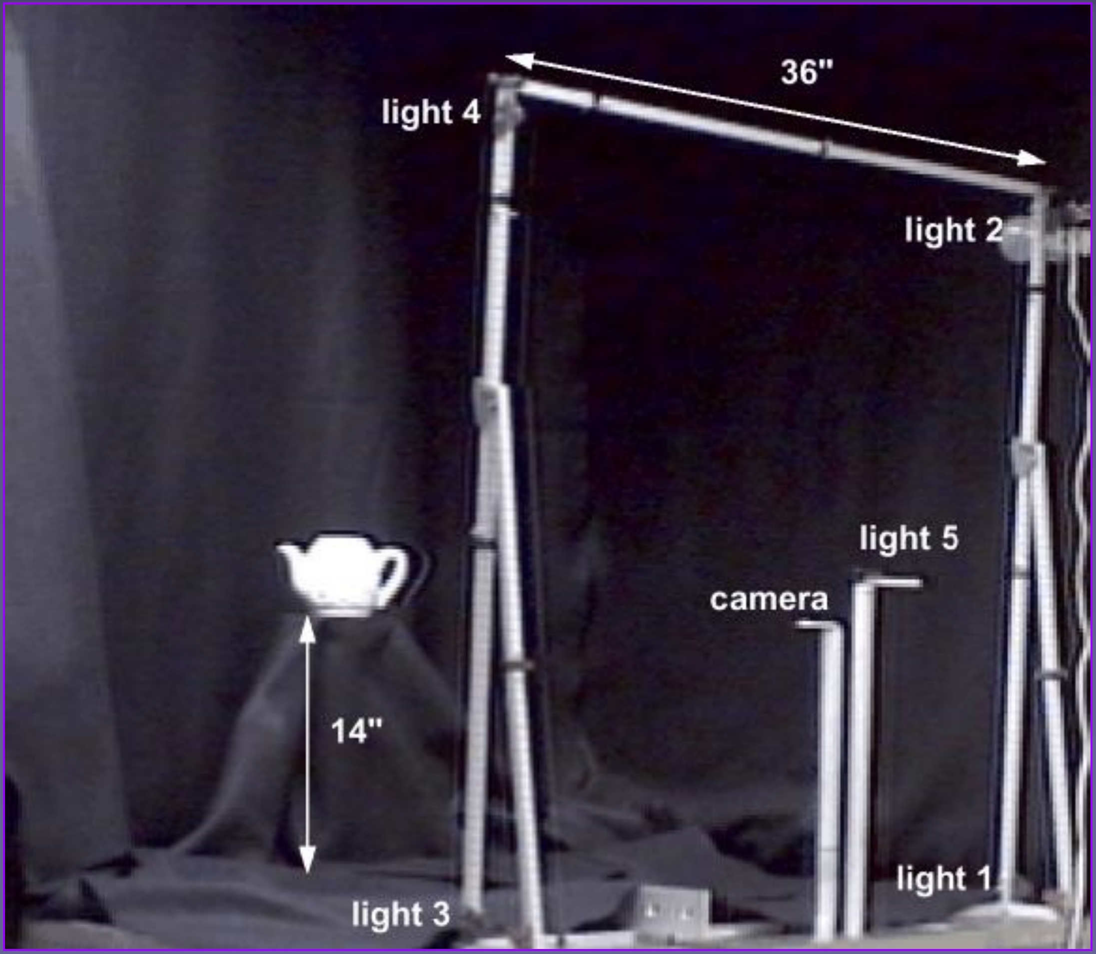


$$R1(p,q) = r'$$

Intersections:
2 solutions for
p and q.

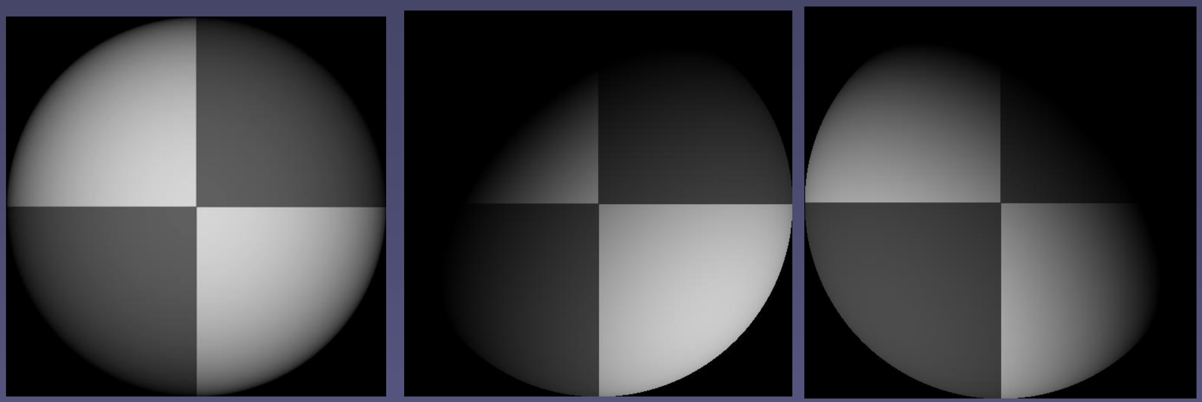
What if we don't know the albedo?

Photometric Stereo Setup



[Rushmeier et al., 1997]

Photometric Stereo

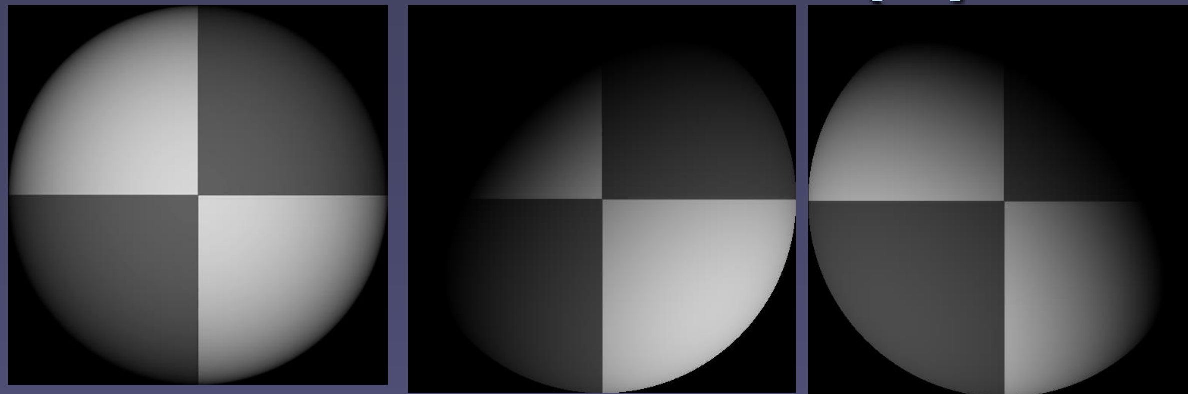


Use multiple light sources to resolve ambiguity
In surface orientation.

Note: Scene does not move – Correspondence
between points in different images is easy!

Notation: Direction of source i: $\mathbf{s}_i$ or (p_{s_i}, q_{s_i})
Image intensity produced by source i: $I_i(x, y)$

Lambertian Surfaces (special case)



$$\mathbf{n} = (n_x, n_y, n_z)$$

$$\mathbf{s_i} = (s_{x_i}, s_{y_i}, s_{z_i})$$

Use THREE sources in directions $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$

Image Intensities measured at pixel (x,y):

$$I_1 = \rho(\mathbf{s}_1 \cdot \mathbf{n})$$
$$I_2 = \rho(\mathbf{s}_2 \cdot \mathbf{n})$$
$$I_3 = \rho(\mathbf{s}_3 \cdot \mathbf{n})$$

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \rho \underbrace{\begin{bmatrix} \mathbf{s}_1^T \\ \mathbf{s}_2^T \\ \mathbf{s}_3^T \end{bmatrix}}_{\mathbf{S}} \mathbf{n}$$

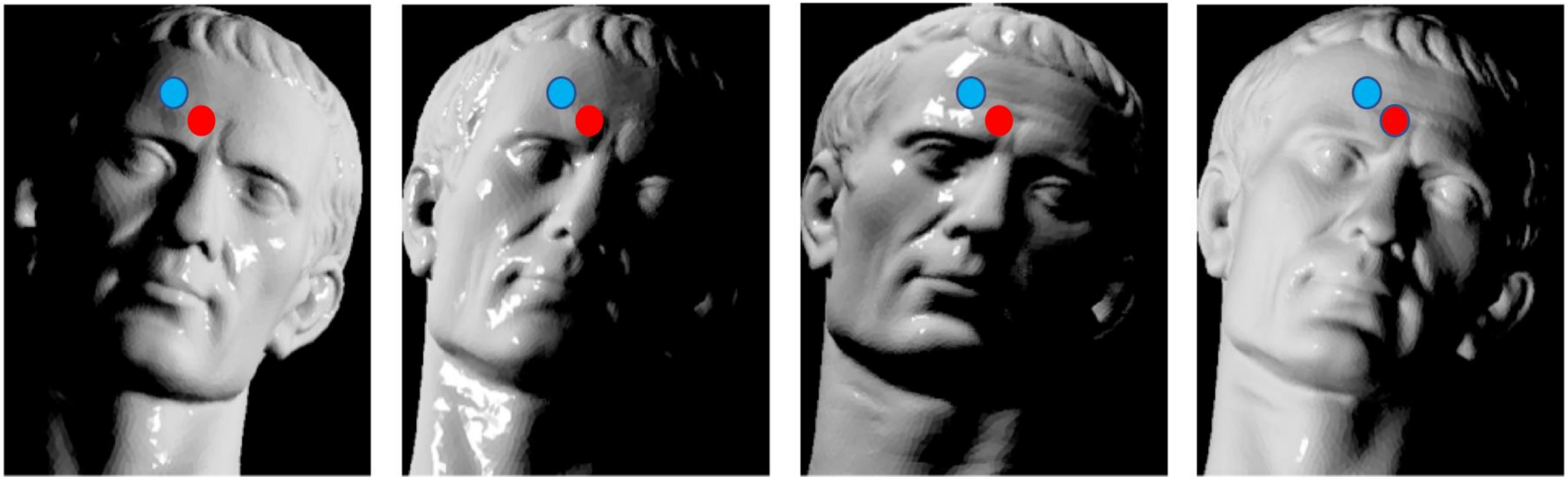
$$\rho\mathbf{n} = \mathbf{S}^{-1} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \mathbf{N}$$

$$\mathbf{n} = \frac{\mathbf{N}}{|\mathbf{N}|}$$ orientation

$$\rho = |\mathbf{N}|$$ albedo

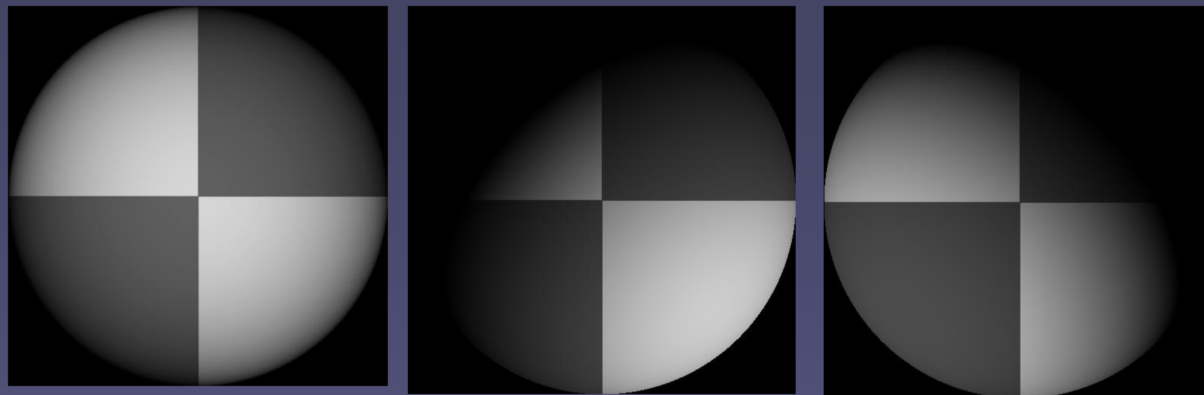
Multiple pixels

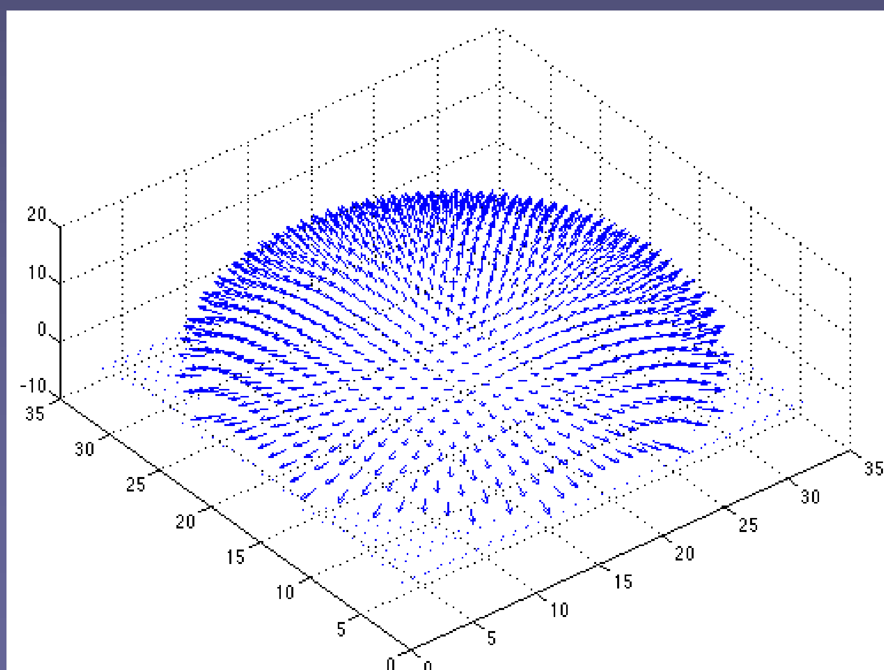
- We've looked at a single pixel till now
- How do we handle multiple pixels?
- Essentially independent equations!



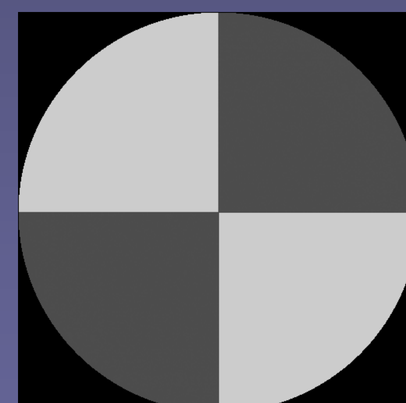
Photometric Stereo: RESULT

INPUT





orientation



albedo

Photometric Stereo

What results can you get?

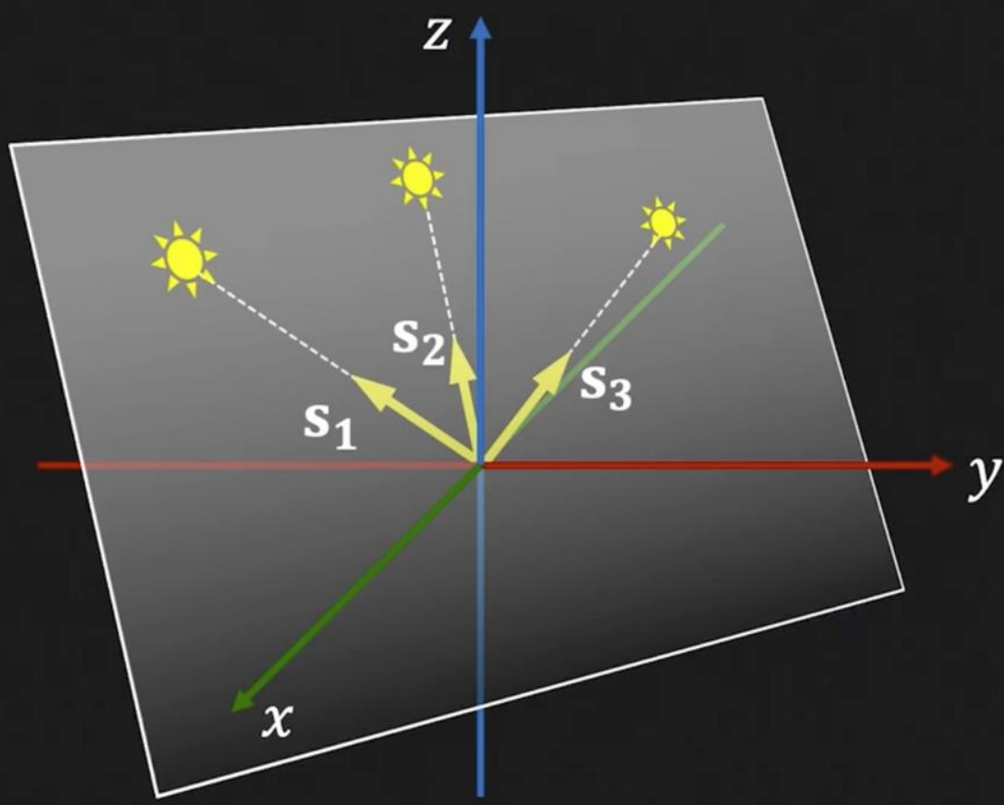
Input (1 of 12) | Normals (RGB colormap) | Normals (vectors) | Shaded 3D rendering | Textured 3D rendering

Bad light configurations

When $S_{3\times 3}$ is not invertible.

That is, when one source direction can be represented as a linear combination of the other two.

$$\mathbf{s}_3 = \alpha \mathbf{s}_1 + \beta \mathbf{s}_2$$



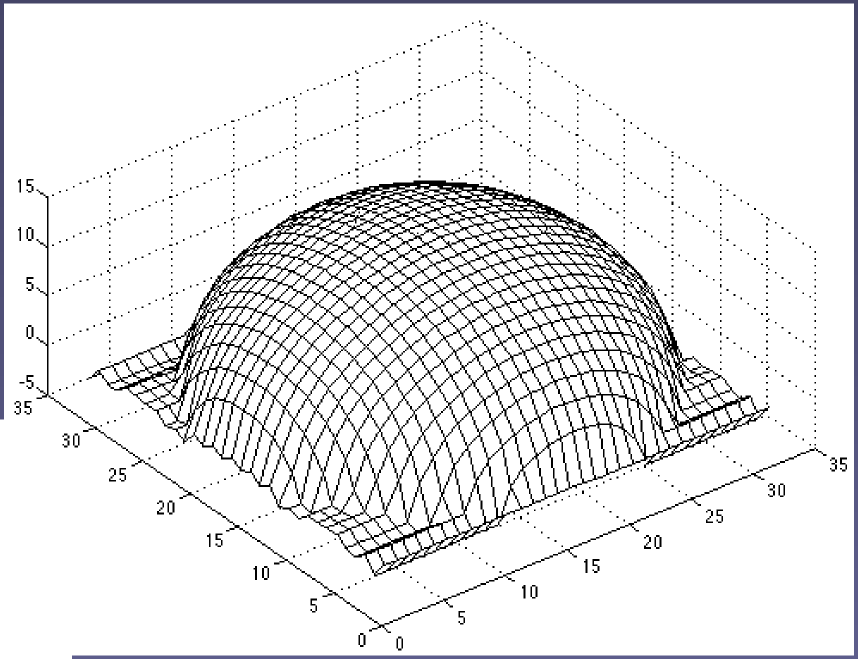
Shree Nayar

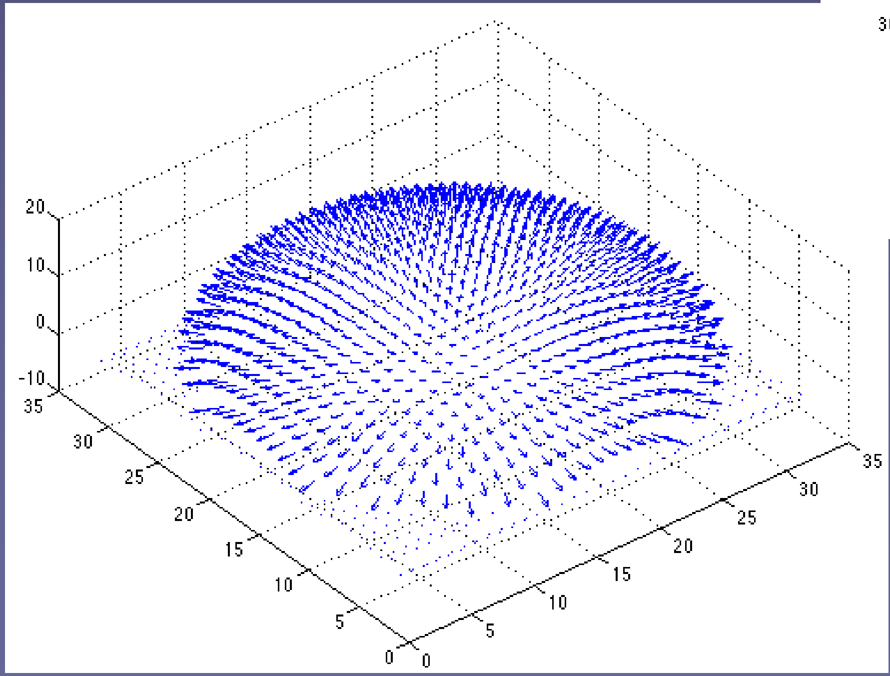
From Surface Orientations to Shape

$$(p, q) = \left(-\frac{\partial Z}{\partial x}, -\frac{\partial Z}{\partial y}\right)$$

$$Z(x, y) = Z(x_0, y_0) + \int_{(x_0,\ y_0)}^{(x,y)} -(p dx + q dy)$$

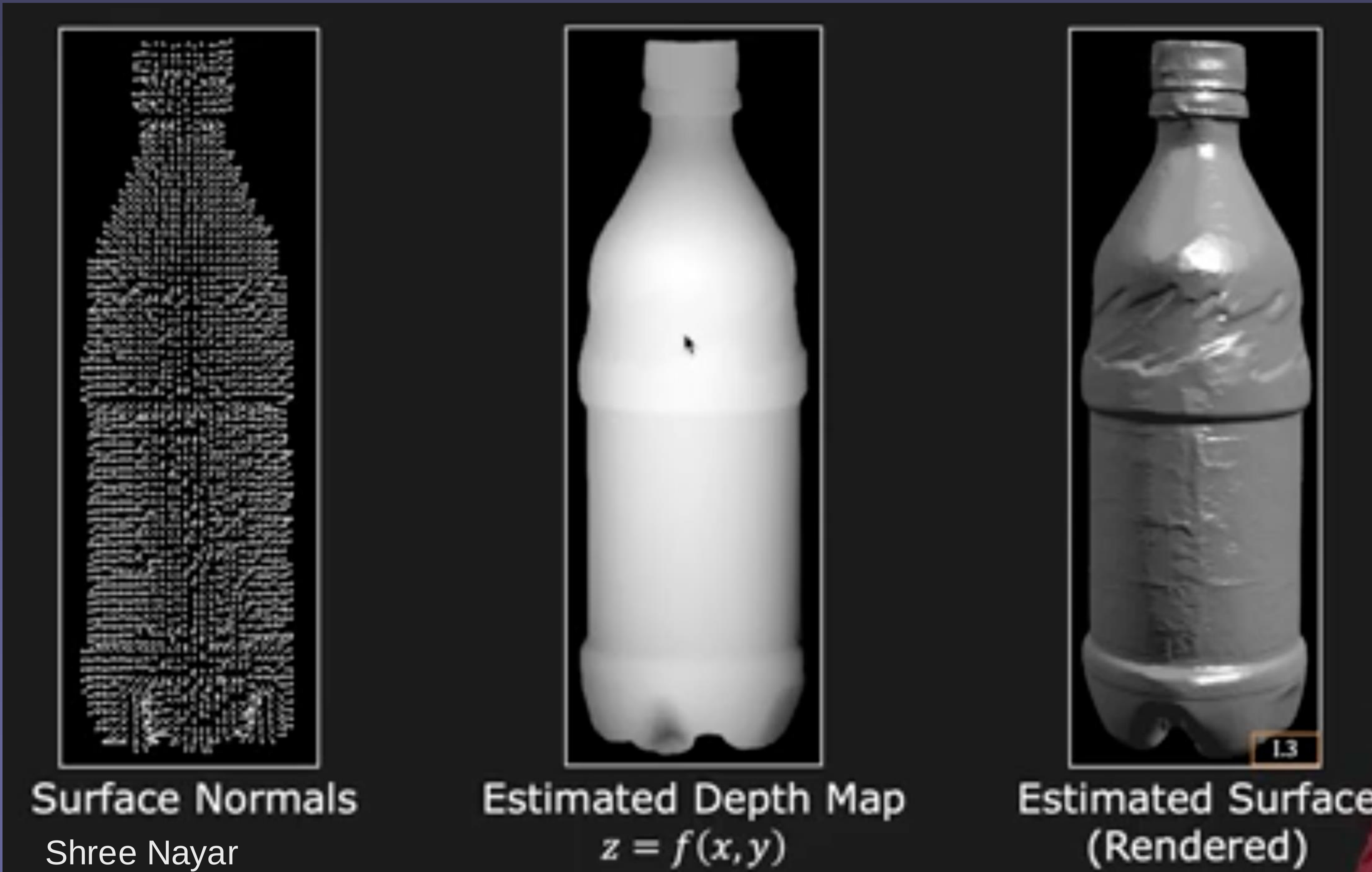
Where Z(x0, y0) is reference depth (0).





Integrate needle map

Result



Shree Nayar

Photometric Stereo Results

Input images

Recovered normals (re-lit)

Recovered color

[Rushmeier et al., 1997]

Photometric Stereo
Capturing details

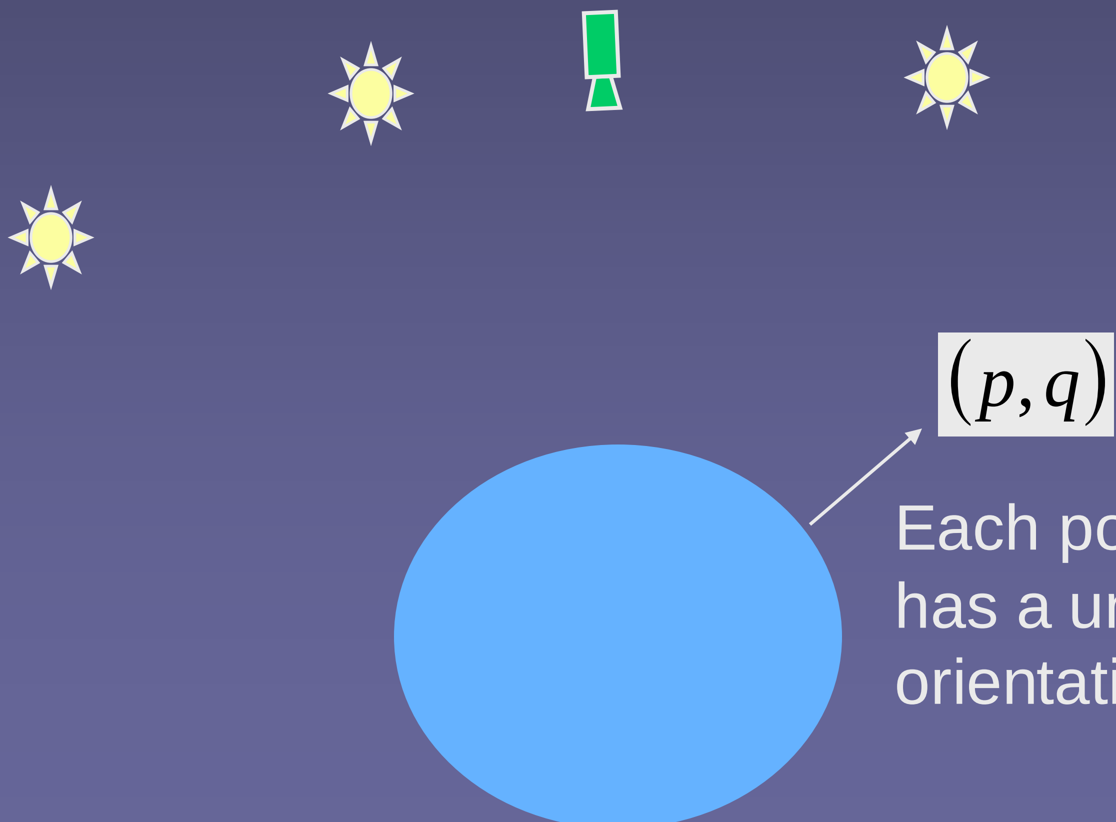
Bernardini, etal. Range sensing, combined w/ photometric Stereo.

(a) Color images taken with four of the five light sources. (b) Synthetic picture computed using the surface normals obtained with the photometric system.

Calibration Objects & Look-up Tables

Calibration: Useful when Reflectance Map Equations are uknown.

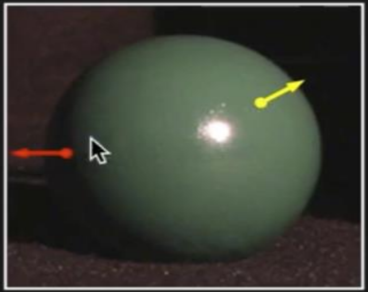
Use calibration sphere of known size and same reflectance as scene objects.



Each point on the sphere has a unique known orientation.

Calibration Objects

Use a calibration object of known size, shape (e.g. sphere) and same reflectance as the scene objects.



Calibration Sphere

Scene

Calibration Objects

Image 1 Image 2 ... Image K

Step 1: Capture $K \geq 3$ images under K different light sources.

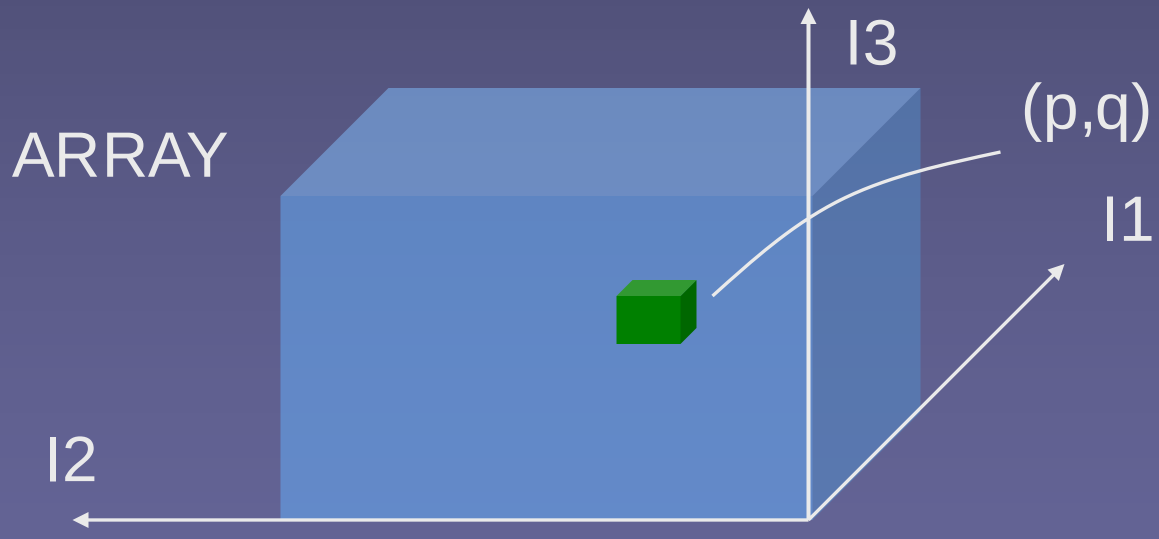
Each point on the sphere produces K image intensities $(I_1, I_2, \ldots, I_K)$ corresponding to the K light sources.

Calibration Objects & Look-Up Tables (LUT - hashing)

Illuminate the sphere with one source at a time and obtain an image.

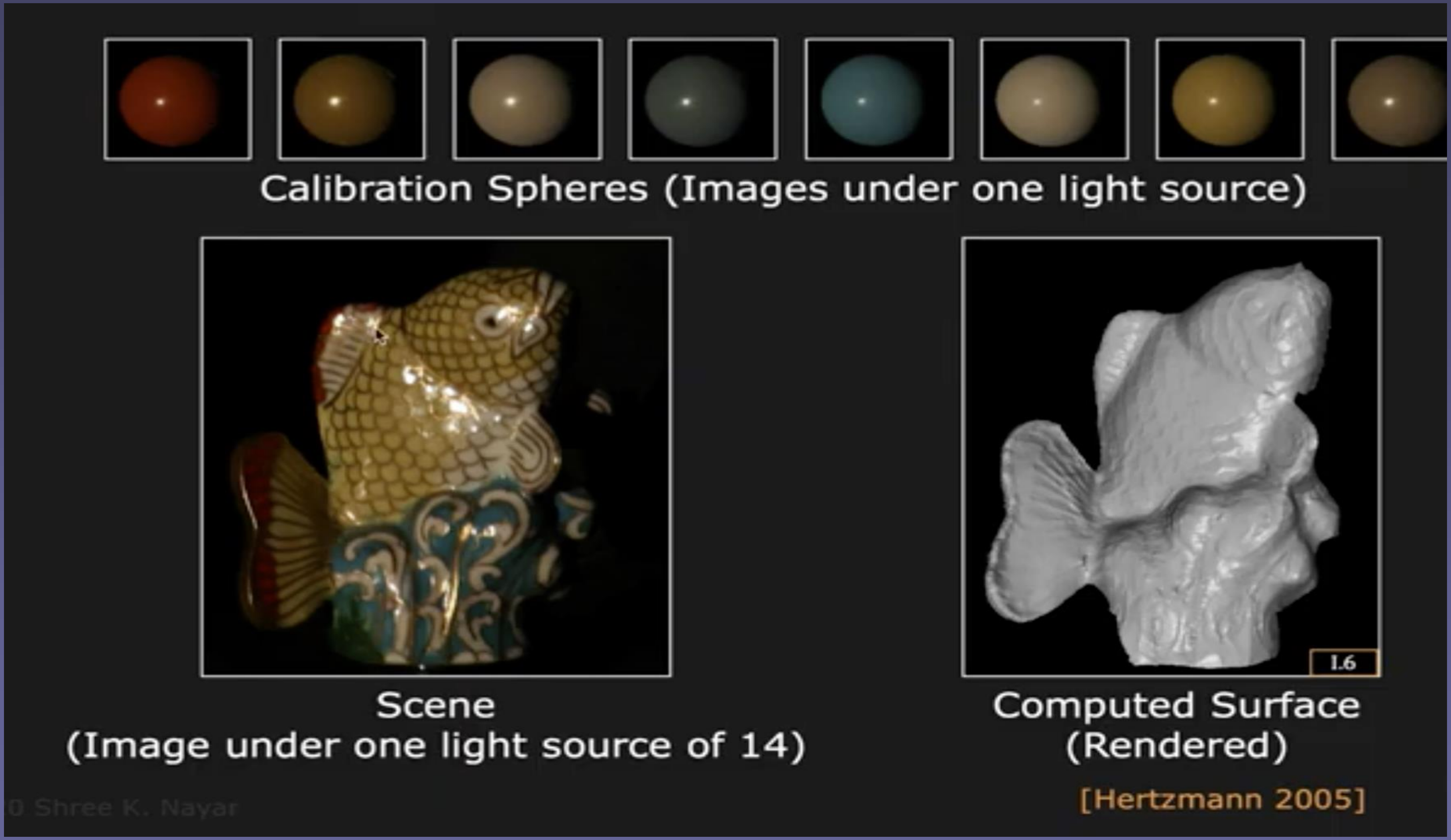
Each surface point with orientation (p,q) produces three images (I1, I2, I3)

Generate a LOOK-UP TABLE (I1, I2, I3) -> (p,q)



For an object of uknown shape but same reflectance, obtain 3 images using same sources. For each image point use LUT to map (I1, I2, I3) -> (p,q)

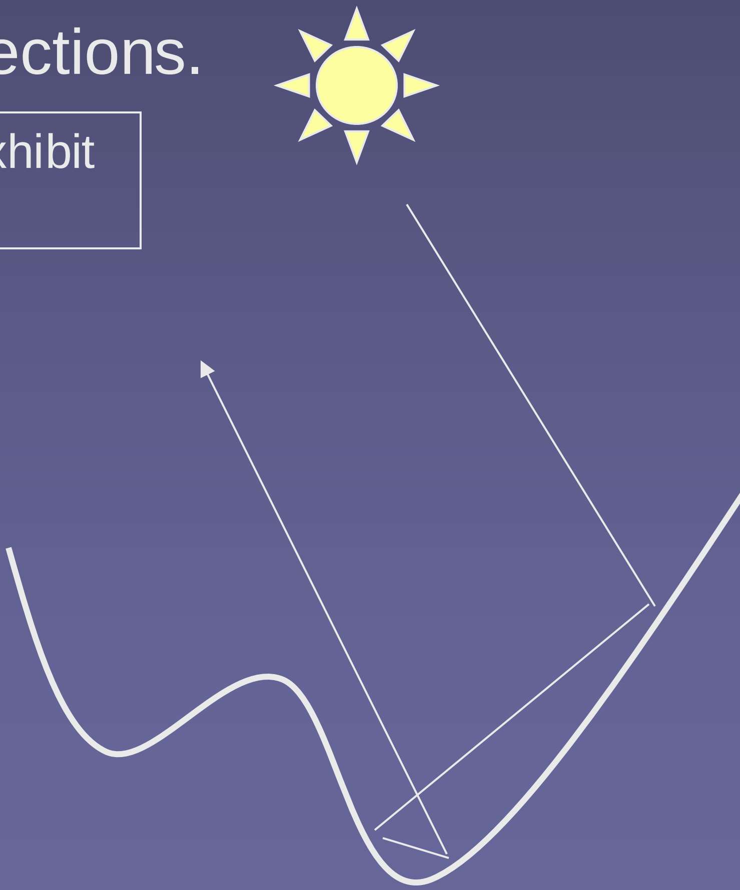
Calibration objects



Photometric Stereo: Remarks

(1) Reflectance & illumination must be known a-priori.
(2) Local Method.
(3) Major Assumption: No interreflections.

Concave surfaces exhibit interreflections.



Outdoor scenes

- Photometric Stereo for outdoor webcams:

http://www.gris.informatik.tu-darmstadt.de/~flanggut/downloads/Ackermann-2012-PSO.pdf

Figure 9. Some input images of the *church* dataset, and one input image for the *castle* dataset (scanline region marked in red).

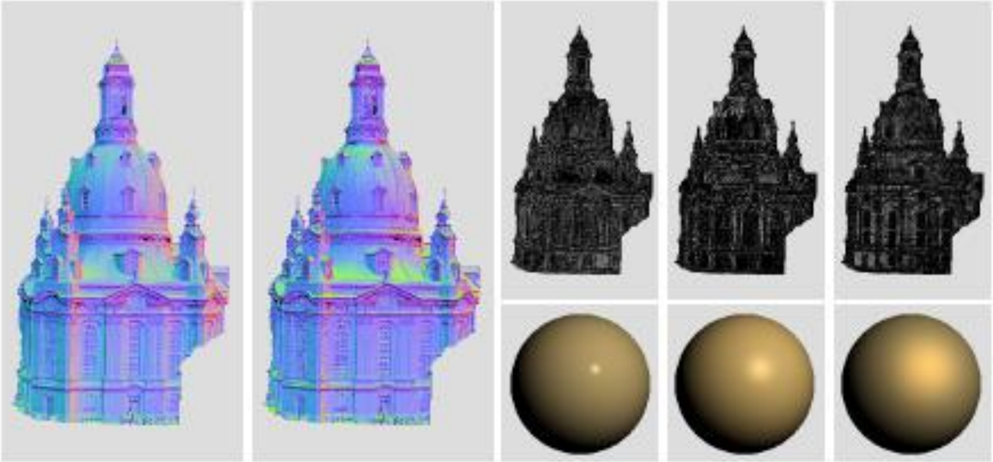
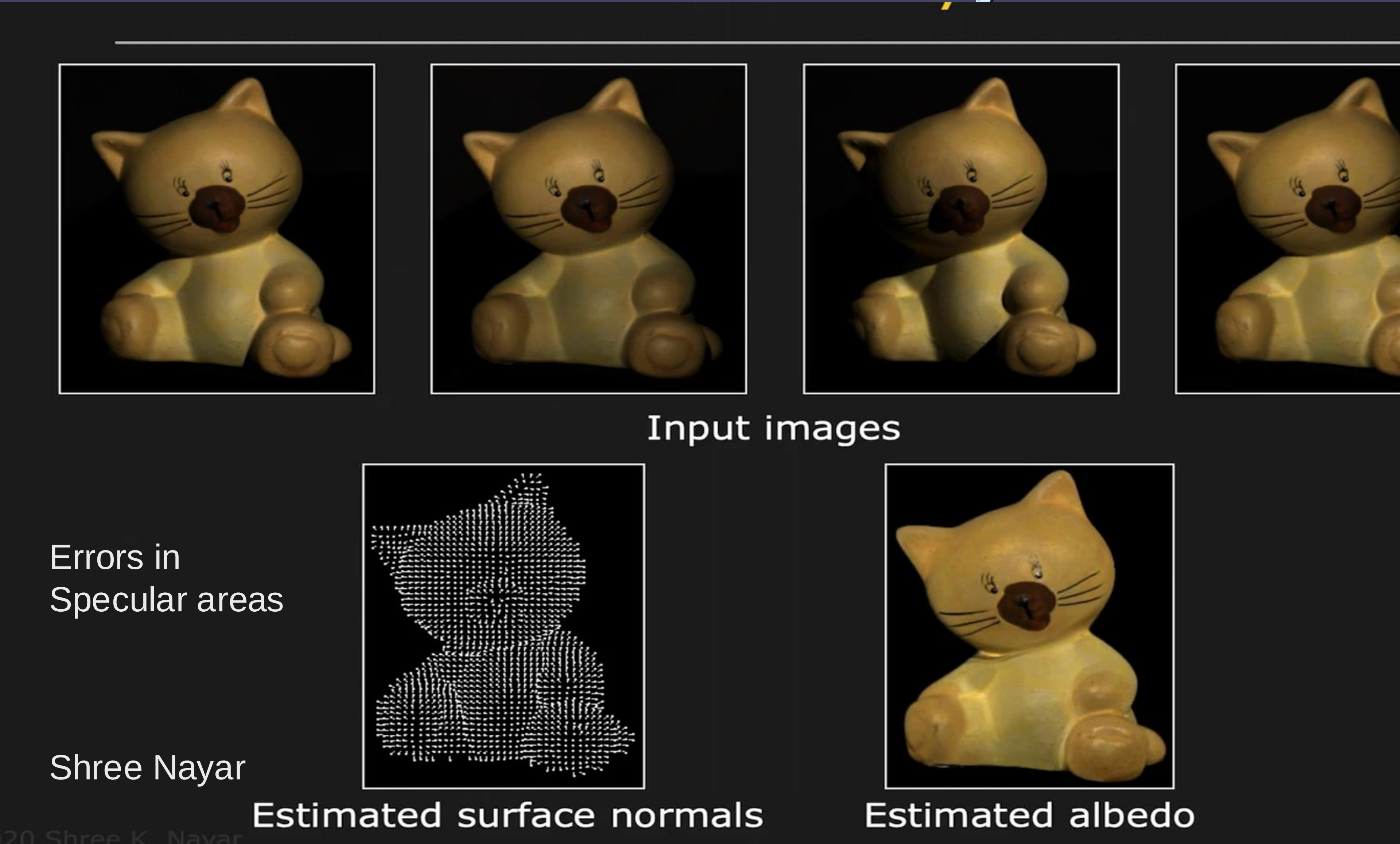


Figure 10. The initial normal map, the final normal map, and the three recovered BRDFs with corresponding material map.

Non-Lambertian objects



Errors in
Specular areas

Shree Nayar

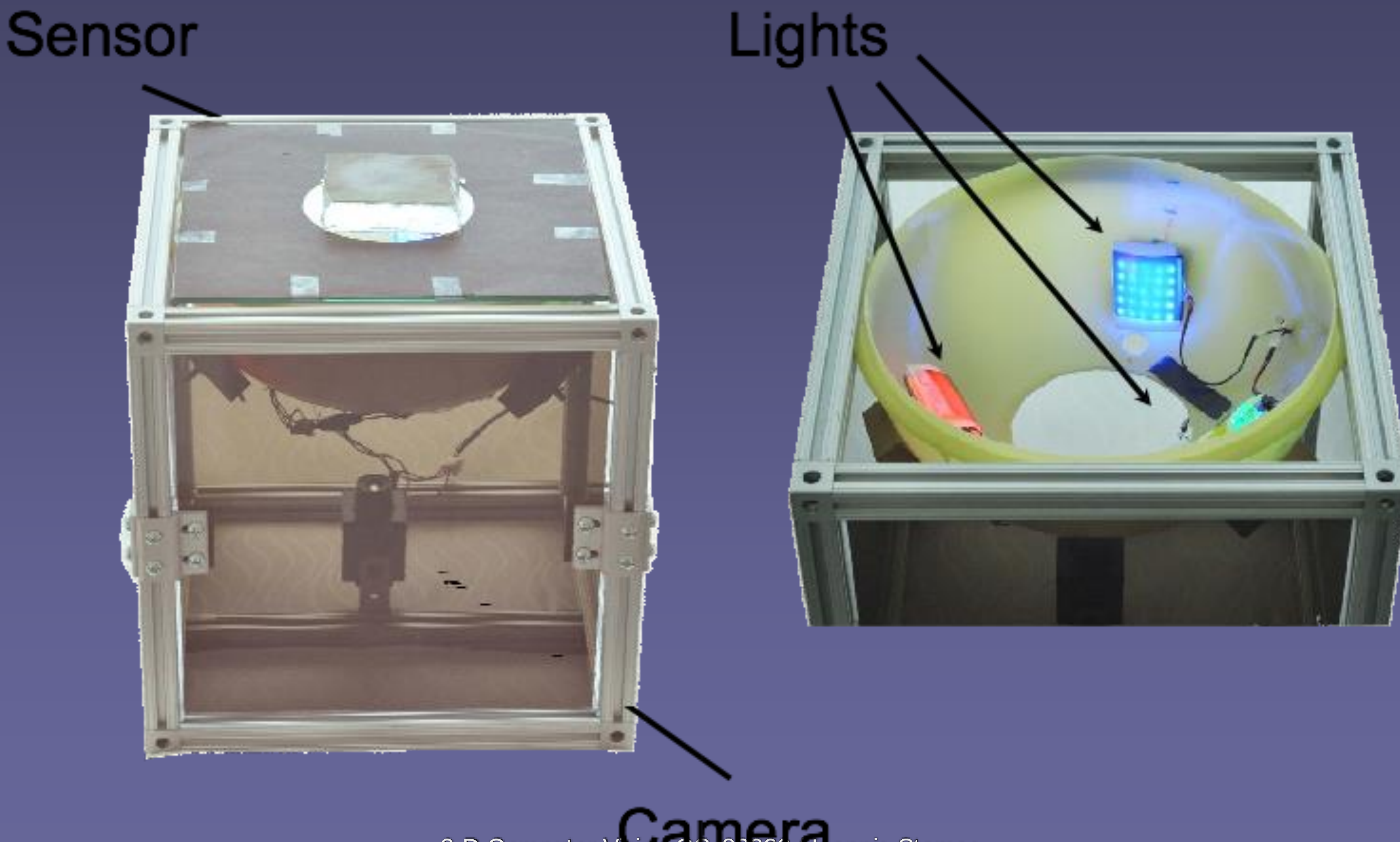
Photometric Stereo – Dyson Robotics Lab Imperial College

https://www.youtube.com/watch?v=PdJTIx7d0HA&ab_channel=DysonRoboticsLaboratoryatImperialCollege

Johnson and Adelson, 2009

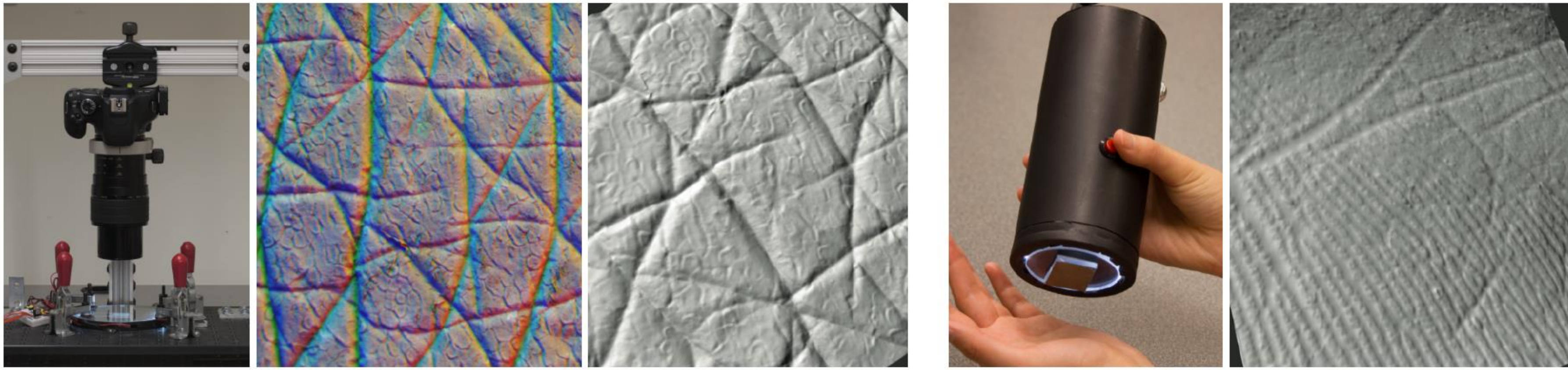

OREO

Lights, camera, action



(a) (b)

Figure 2. (a) This decorative pin consists of a glass bas-relief portrait mounted in a shiny gold setting. (b) The RGB image provided by the retrographic sensor. The pin is pressed into the elastomer skin, and colored lights illuminate it from three directions.



(a) bench configuration (b) captured (c) reconstruction (d) portable configuration (e) reconstruction

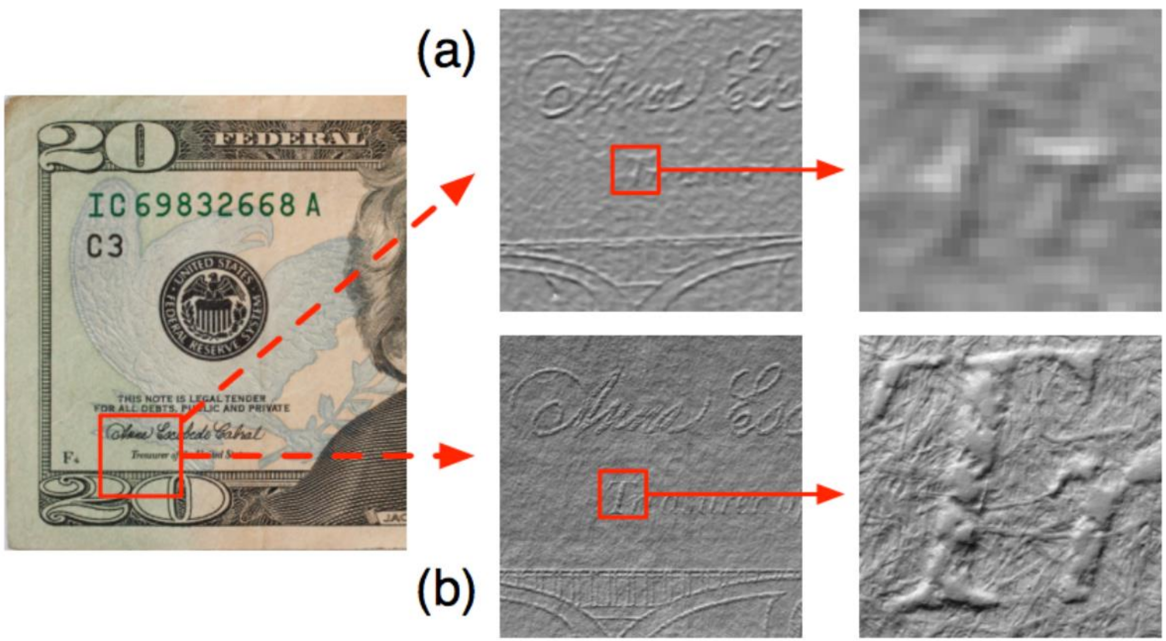


Figure 7: *Comparison with the high-resolution result from the original retrographic sensor. (a) Rendering of the high-resolution \$20 bill example from the original retrographic sensor with a close-up view. (b) Rendering of the captured geometry using our method.*

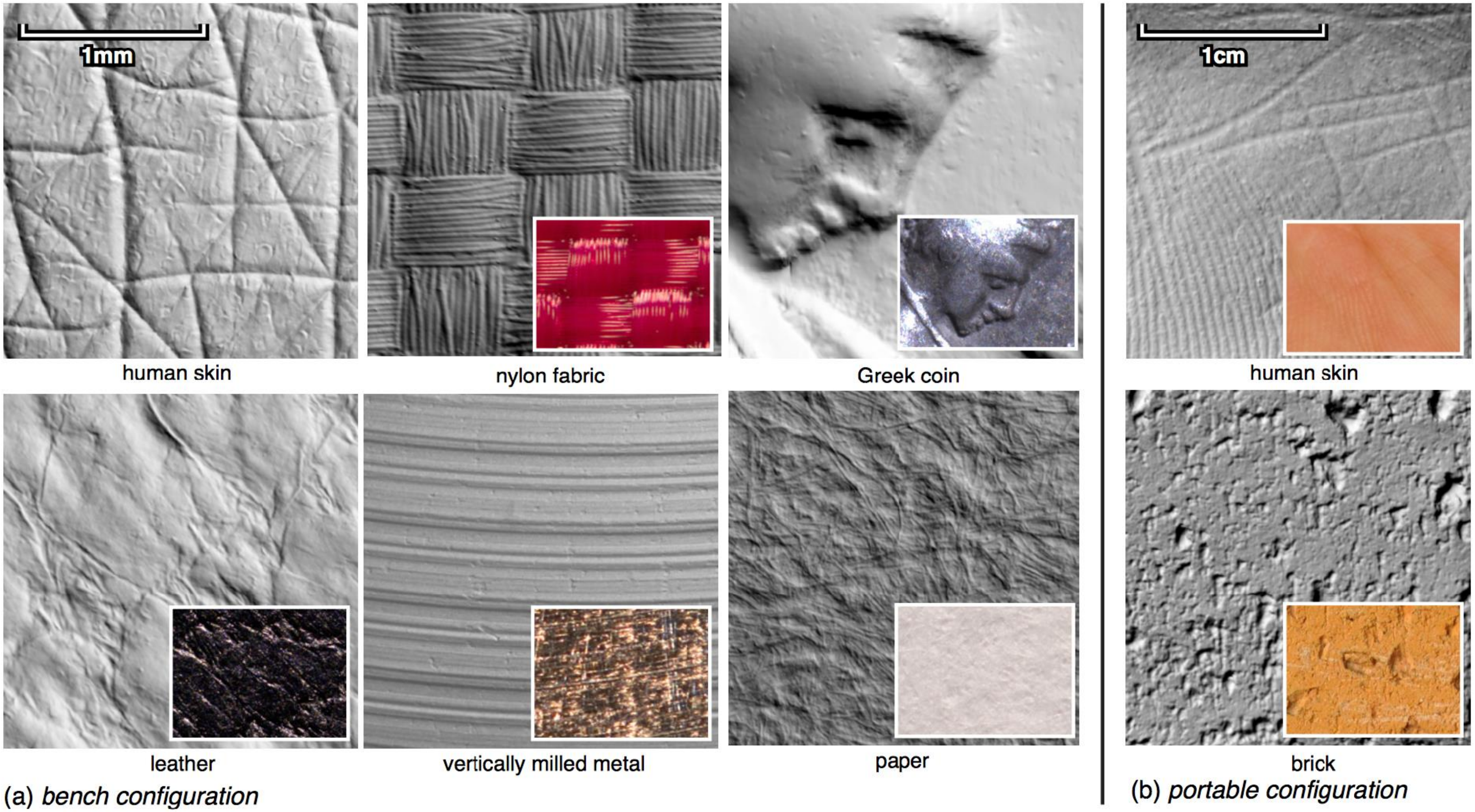


Figure 9: *Example geometry measured with the bench and portable configurations. Outer image: rendering under direct lighting. Inset: macro photograph of original sample. Scale shown in upper left. Color images are shown for context and are to similar, but not exact scale.*

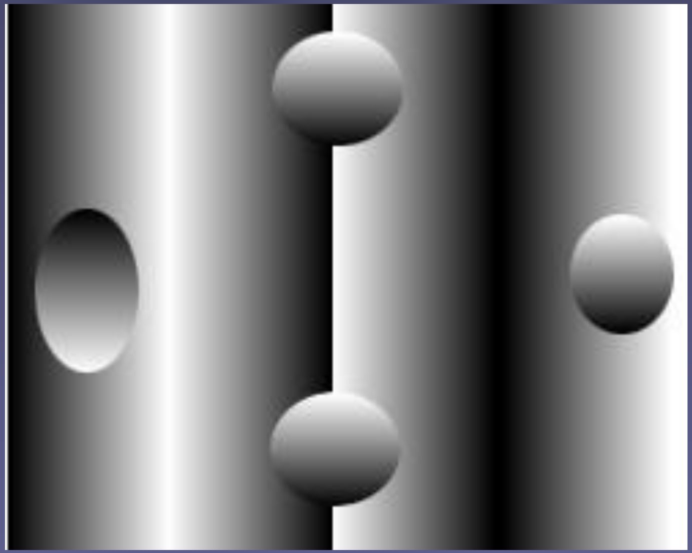
Photometric Stereo & Relighting (Optional)

- Example: https://www.youtube.com/watch?v=soZ8DLyRqo8
- More information:
  - https://www.youtube.com/watch?v=m-GG92moxnM
  - https://www.youtube.com/watch?v=qHUi_q0wkq4
  - Recent work on relighting (SIGGRAPH 2021): https://augmentedperception.github.io/total_relighting/

Shape from Shading

- Can we recover shape from a Single Image?

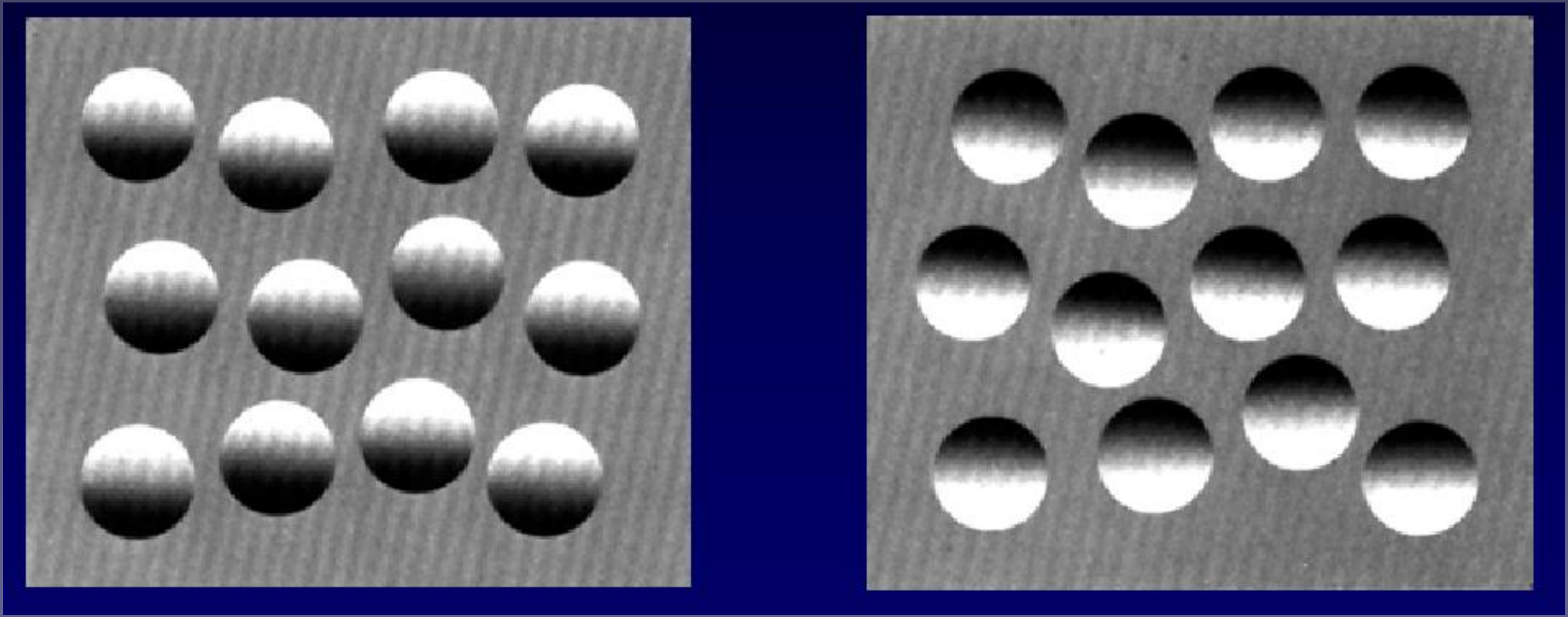
Human Perception of Shape from Shading



We assume light source is above us.

Ramachandran 88

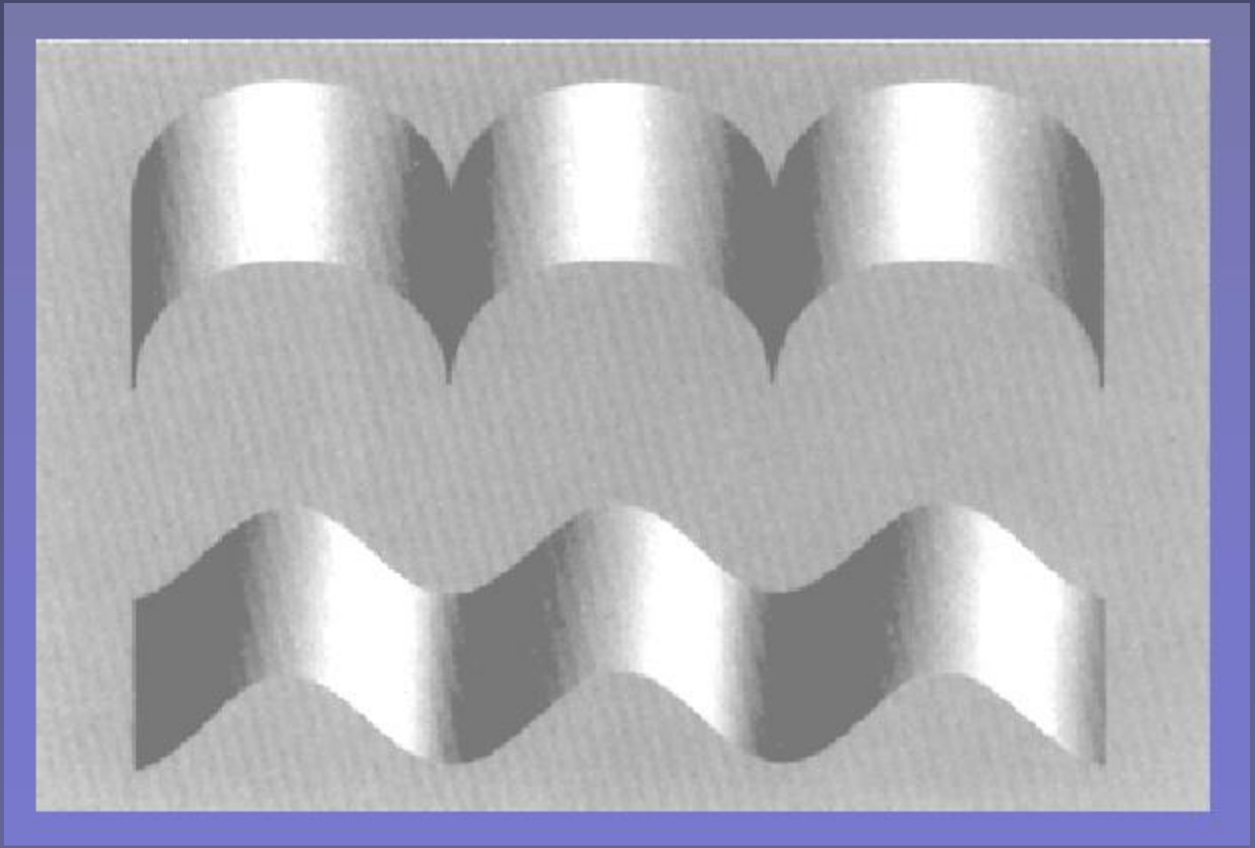
Human Perception of Shape from Shading



We assume light source is above us.

Ramachandran 88

Human Perception of Shape from Shading



Surface boundaries have strong influence on perceived shape

Variational Shape from Shading

- Approach: energy minimization
- Given observed I(x,y), find shape normals (p,q) per pixel that minimize the following:

$$\mathcal{E} = \sum_x \sum_y (I(x,y) - R(p,q))^2 + \lambda(p_x^2 + p_y^2 + q_x^2 + q_y^2)$$

- Minimize combination of disparity wrt data and smoothness of normal
  - Note that using only data term is not enough

Variational Shape from Shading

Energy without the smoothness term:

$$\mathcal{E} = \sum_x \sum_y \left(I(x,y) - R(p,q)\right)^2$$

Under-constrained problem

Number of unknowns: (p, q) per pixel

Some assumptions about normal needed!

Variational Shape from Shading

Energy without the smoothness term:

$$\mathcal{E} = \sum_x \sum_y (I(x, y) - R(p, q))^2$$

Under-constrained problem

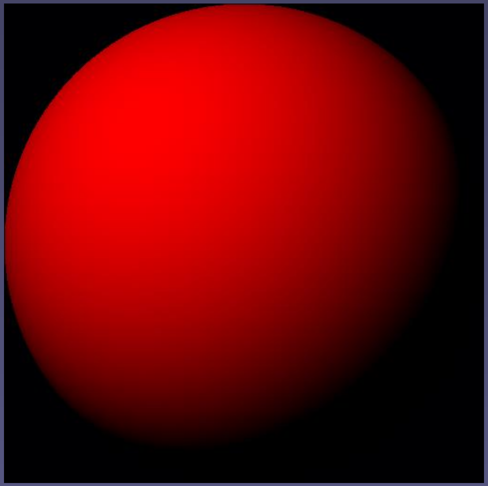
Number of unknowns: (p, q) per pixel

Some assumptions about normal needed!

=➔ Added smoothness constraint

Shape from Shading

$$R(p,q)=\frac{1+pp_s+qq_s}{\sqrt{1+p^2+q^2}\sqrt{1+p_s^{\ 2}+q_s^{\ 2}}}$$



Data term

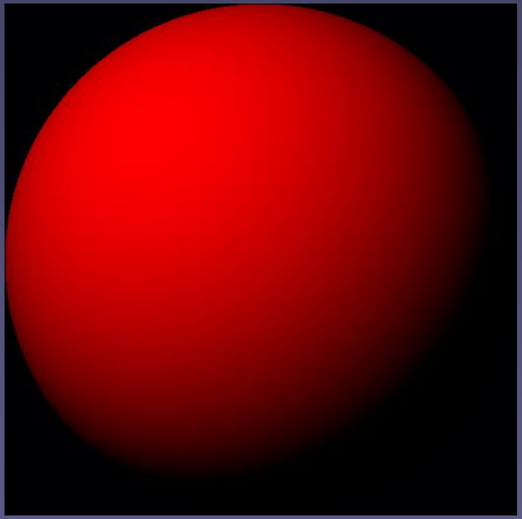
$$\sum\sum\lambda\underbrace{(p_x^{\ 2}+p_y^{\ 2}+q^2{}_x+q^2{}_y)}_{\text{Smoothness constraint}}+(I(x,y)-R(p,q))^2$$

$$p_x=\frac{\partial p}{\partial x},\ p_y=\frac{\partial p}{\partial y},\ q_x=\frac{\partial p}{\partial x},\ q_{xy}=\frac{\partial q}{\partial y}$$

Scalar parameter (>0) to “weigh” smoothness constraint and data term

Shape from Shading

$$e = \sum\sum \lambda(p_x{}^2 + p_y{}^2 + q^2{}_x + q^2{}_y) + (I(x,y) - R(p,q))^2$$



Calculus of Variations -> Discrete Case:

$$\frac{\partial e}{\partial p_{kl}} = 2\lambda\left(4p_{kl} - \bar{p}_{kl}\right) - 2\left(I(k,l) - R(p_{kl}, q_{kl})\right)\frac{\partial R}{\partial p} = 0$$

$$\frac{\partial e}{\partial q_{kl}} = 2\lambda\left(4q_{kl} - \bar{q}_{kl}\right) - 2\left(I(k,l) - R(p_{kl}, q_{kl})\right)\frac{\partial R}{\partial q} = 0$$

$\bar{p}_{kl}$ Is the average of the 4 neighboring values

CHECK NEXT SLIDES FOR DETAILS

$$p_{kl}^{n+1} = \bar{p}_{kl}^{n} + \frac{1}{4\lambda}\left(I(k,l) - R(p_{kl}^{n}, q_{kl}^{n})\right)\frac{\partial R}{\partial p}$$

$$q_{kl}^{n+1} = \bar{q}_{kl}^{n} + \frac{1}{4\lambda}\left(I(k,l) - R(p_{kl}^{n}, q_{kl}^{n})\right)\frac{\partial R}{\partial q}$$

Iterative solution

Shape from Shading

Smoothness constraint at pixel (k, l):

$$\varepsilon_{s_{k,l}} = \frac{1}{4}\left((p_{k+1,l} - p_{k,l})^2 + (p_{k,l+1} - p_{k,l})^2 + (q_{k+1,l} - q_{k,l})^2 + (q_{k,l+1} - q_{k,l})^2\right)$$

Note that derivates of p and q at pixel (k, l) are approximated by differences!

Shape from Shading

Smoothness constraint at pixel (k, l):

$$\varepsilon_{s_{k,l}} = \frac{1}{4}\left((p_{k+1,l} - p_{k,l})^2 + (p_{k,l+1} - p_{k,l})^2 + (q_{k+1,l} - q_{k,l})^2 + (q_{k,l+1} - q_{k,l})^2\right)$$

Data (image irradiance) term at pixel (k, l):

$$\varepsilon_{r_{k,l}} = (I_{k,l} - R(p_{k,l}, q_{k,l}))^2$$

Shape from Shading

Smoothness constraint at pixel (k, l):

$$\varepsilon_{s_{k,l}} = \frac{1}{4}\left((p_{k+1,l} - p_{k,l})^2 + (p_{k,l+1} - p_{k,l})^2 + (q_{k+1,l} - q_{k,l})^2 + (q_{k,l+1} - q_{k,l})^2\right)$$

Data (image irradiance) term at pixel (k, l):

$$\varepsilon_{r_{k,l}} = (I_{k,l} - R(p_{k,l}, q_{k,l}))^2$$

Total error to be minimized:

$$\varepsilon = \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} (\lambda\, \varepsilon_{s_{k,l}} + \varepsilon_{r_{k,l}})$$

Number of uknowns?

Shape from Shading

Smoothness error at (k, l):

$$\varepsilon_{s_{k,l}} = \frac{1}{4}\left((p_{k+1,l} - p_{k,l})^2 + (p_{k,l+1} - p_{k,l})^2 + (q_{k+1,l} - q_{k,l})^2 + (q_{k,l+1} - q_{k,l})^2\right)$$

Data (image irradiance) error at (k, l):

$$\varepsilon_{r_{k,l}} = (I_{k,l} - R(p_{k,l}, q_{k,l}))^2$$

Total error to be minimized:

$$\varepsilon = \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} (\lambda\, \varepsilon_{s_{k,l}} + \varepsilon_{r_{k,l}})$$

	(k, l-1)	
(k-1, l)	(k,l) x x	(k+1, l) x
	(k, l+1) x	

Number of uknowns? 2*M*N, for MxN image
(For a 400x600 image: 480,000 unknowns)

But note: Each $(p_{k,l}, q_{k,l})$ appears in just 4 terms

Shape from Shading

Minimizing error => Setting 2*M*N derivatives to 0:

$$\frac{\partial \varepsilon}{\partial p_{k,l}} = 0 \text{ and } \frac{\partial \varepsilon}{\partial q_{k,l}} = 0$$

Shape from Shading

Minimizing error => Setting 2*M*N derivatives to 0:

$$\frac{\partial \varepsilon}{\partial p_{k,l}} = 0 \text{ and } \frac{\partial \varepsilon}{\partial q_{k,l}} = 0 \Rightarrow$$

$$\frac{\partial e}{\partial p_{kl}} = 2\lambda\left(4p_{kl} - \bar{p}_{kl}\right) - 2\left(I(k,l) - R(p_{kl}, q_{kl})\right)\frac{\partial R}{\partial p} = 0$$

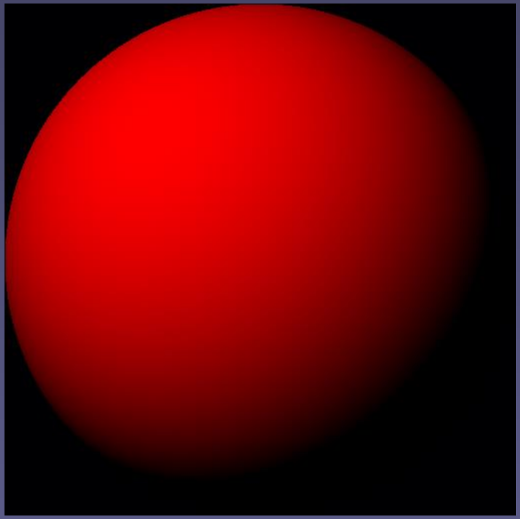
$$\frac{\partial e}{\partial q_{kl}} = 2\lambda\left(4q_{kl} - \bar{q}_{kl}\right) - 2\left(I(k,l) - R(p_{kl}, q_{kl})\right)\frac{\partial R}{\partial q} = 0$$

$\bar{p}_{kl}$ Is the average of the 4 neighboring values

2*M*N equations, 2*M*N unknowns. How to solve?

Shape from Shading – Iterative Solution

$$e = \sum\sum \lambda(p_x{}^2 + p_y{}^2 + q^2{}_x + q^2{}_y) + (I(x,y) - R(p,q))^2$$



Calculus of Variations -> Discrete Case:

$$\frac{\partial e}{\partial p_{kl}} = 2\lambda\left(4p_{kl} - \overline{p}_{kl}\right) - 2\left(I(k,l) - R(p_{kl}, q_{kl})\right)\frac{\partial R}{\partial p} = 0$$

$$\frac{\partial e}{\partial q_{kl}} = 2\lambda\left(4q_{kl} - \overline{q}_{kl}\right) - 2\left(I(k,l) - R(p_{kl}, q_{kl})\right)\frac{\partial R}{\partial q} = 0$$

$\overline{p}_{kl}$ Is the average of the 4 neighboring values

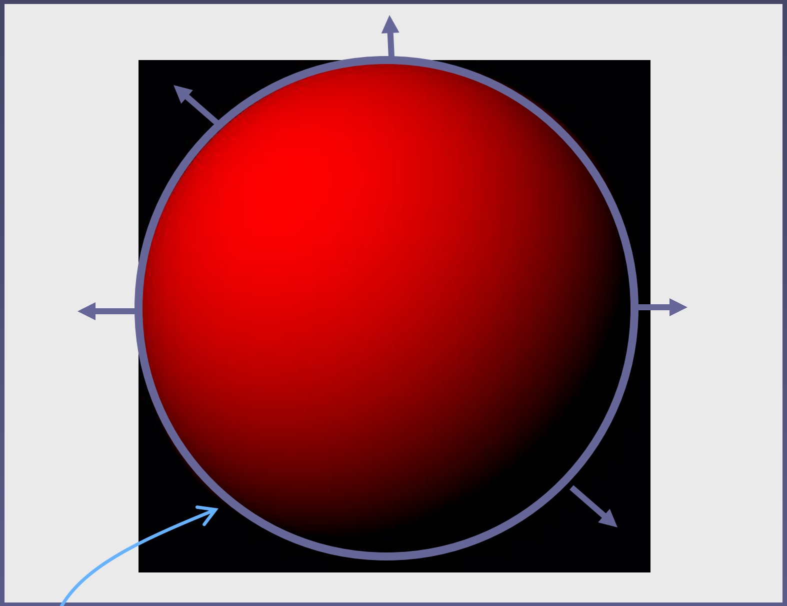
$$p_{kl}^{n+1} = \overline{p}_{kl}^{n} + \frac{1}{4\lambda}\left(I(k,l) - R(p_{kl}^{n}, q_{kl}^{n})\right)\frac{\partial R}{\partial p}$$

$$q_{kl}^{n+1} = \overline{q}_{kl}^{n} + \frac{1}{4\lambda}\left(I(k,l) - R(p_{kl}^{n}, q_{kl}^{n})\right)\frac{\partial R}{\partial q}$$

Iterative solution

Shape from Shading

We know the normal at the contour.
This provides boundary conditions.



OCCLUDING BOUNDARY

Difficulties with Shape from Shading

- Shadows
- Non-Lambertian surfaces
- More than 1 light, or “diffuse illumination”
- Interreflections

Shape from Shading Results

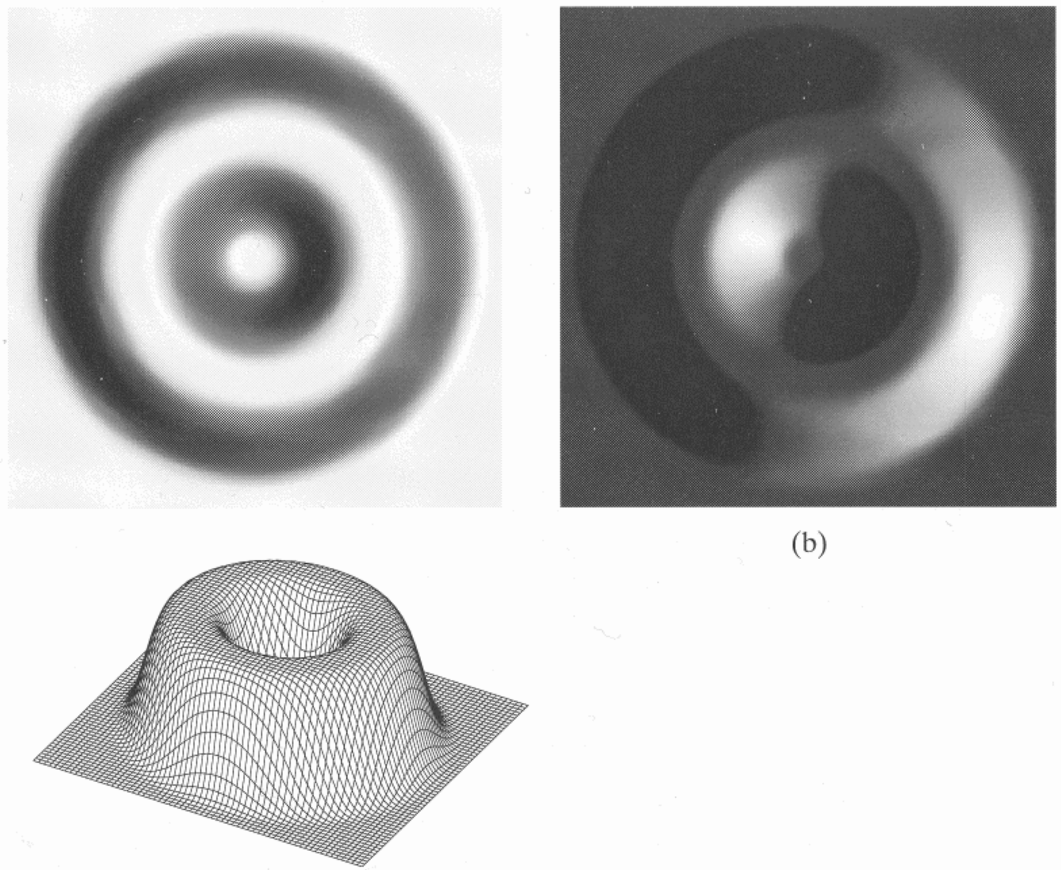


Figure 9.2 Two images of the same Lambertian surface seen from above but illuminated from different directions and 3-D rendering of the surface. Practically all the points in the top left image receive direct illumination ($\mathbf{i} = [0.20, 0, 0.98]^\top$); some regions of the top right image are in the dark due to self-shadowing effects ($\mathbf{i} = [0.94, 0.31, 0.16]^\top$).

[Trucco & Verri]

Shape from Shading Results

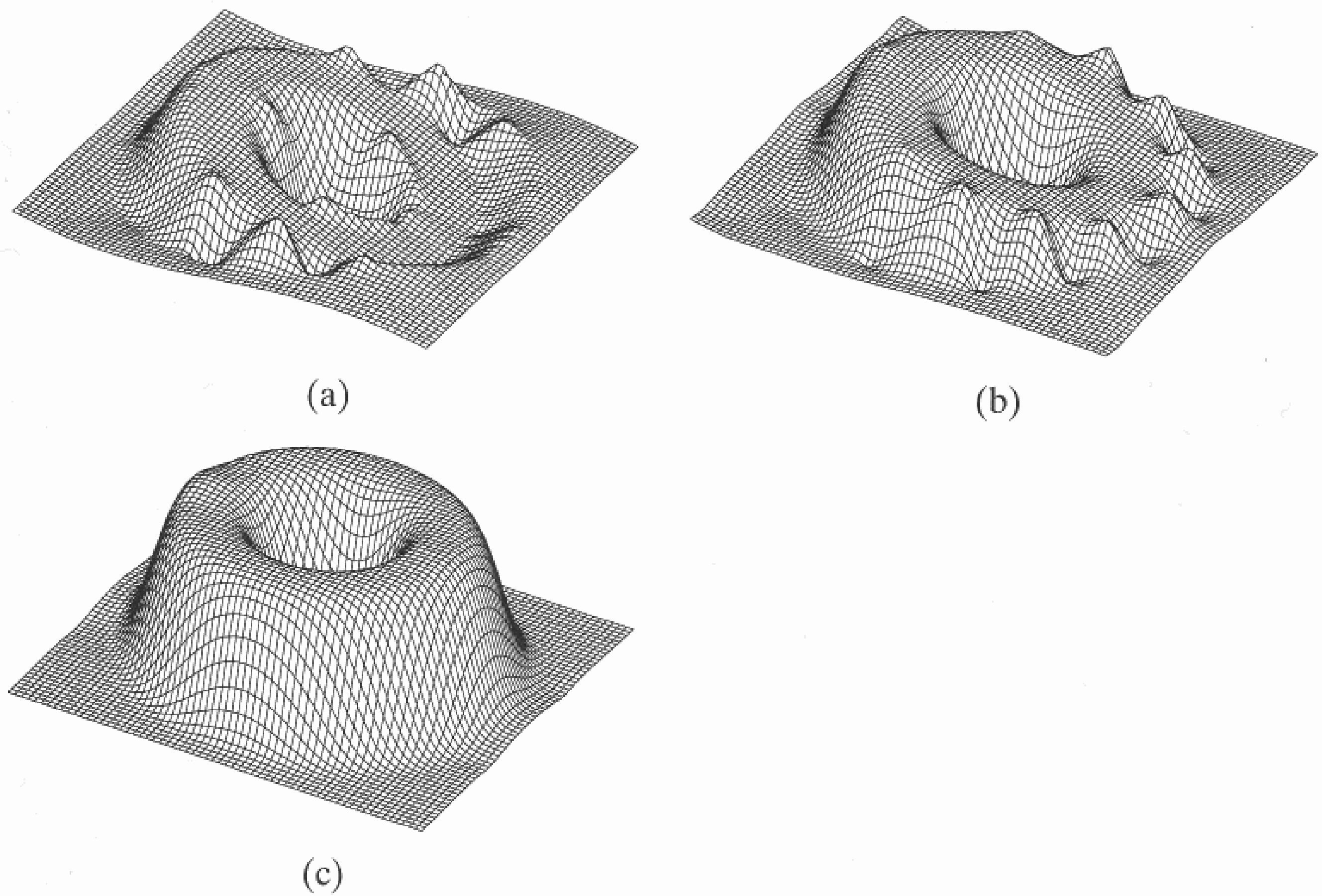


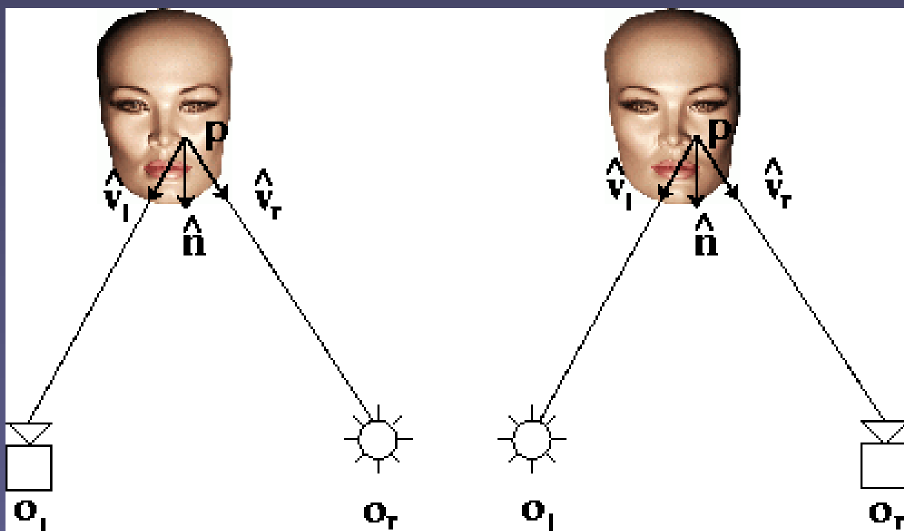
Figure 9.4 Reconstructions of the surface in Figure 9.2 after 100 (a), 1000 (b) and 2000 (c) iterations. The initial surface was a plane of constant height. The asymmetry of the first two reconstruction is due to the illuminant direction.

Appendix 0 (optional): Helmholtz Stereopsis

Helmholtz Stereopsis

- Based on Helmholtz reciprocity: any BRDF is the same under interchange of light, viewer
- So, take pairs of observations w. viewer, light interchanged
- Ratio of the observations in a pair is independent of BRDF

Helmholtz reciprocity

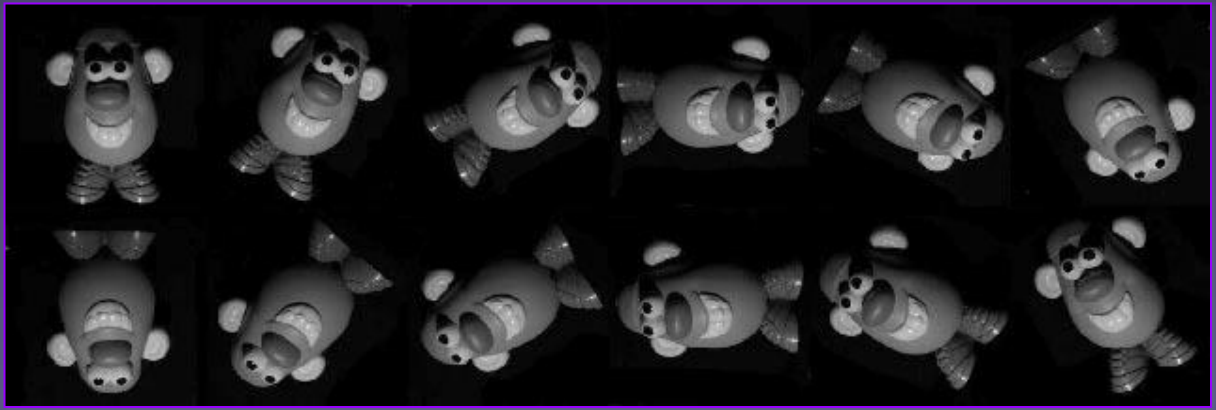


Consider taking two images as shown below. First we capture an image of the object illuminated by a point light source, and then we capture another with the light source and camera positions swapped exactly

For corresponding pixels, the ratio of incident irradiance (onto the object) to emmitted radiance (from the object) is the same. Put more simply, we can derive a relationship between the intensities of corresponding pixels that *does not depend on the BRDF of the surface*.

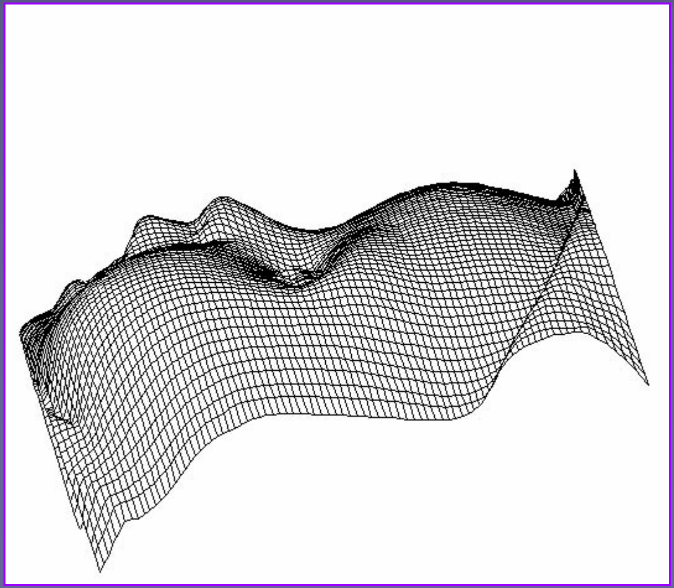
Todd Zickler, Harvard

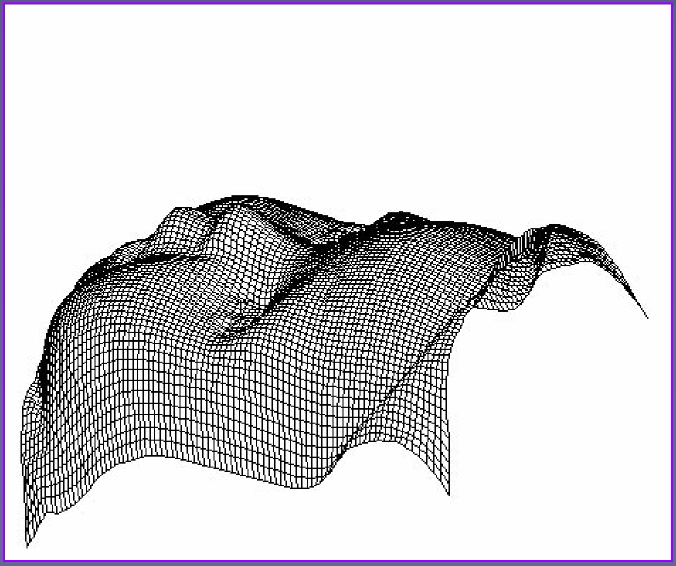
Helmholtz Stereopsis



[Zickler, Belhumeur, & Kriegman]

Helmholtz Stereopsis





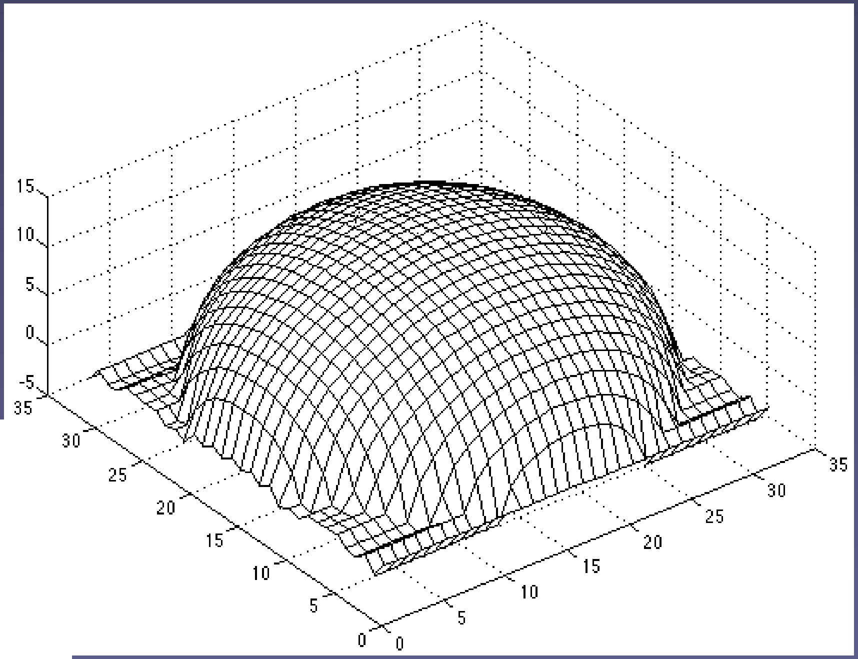
Appendix 1 (optional): From normal to depth

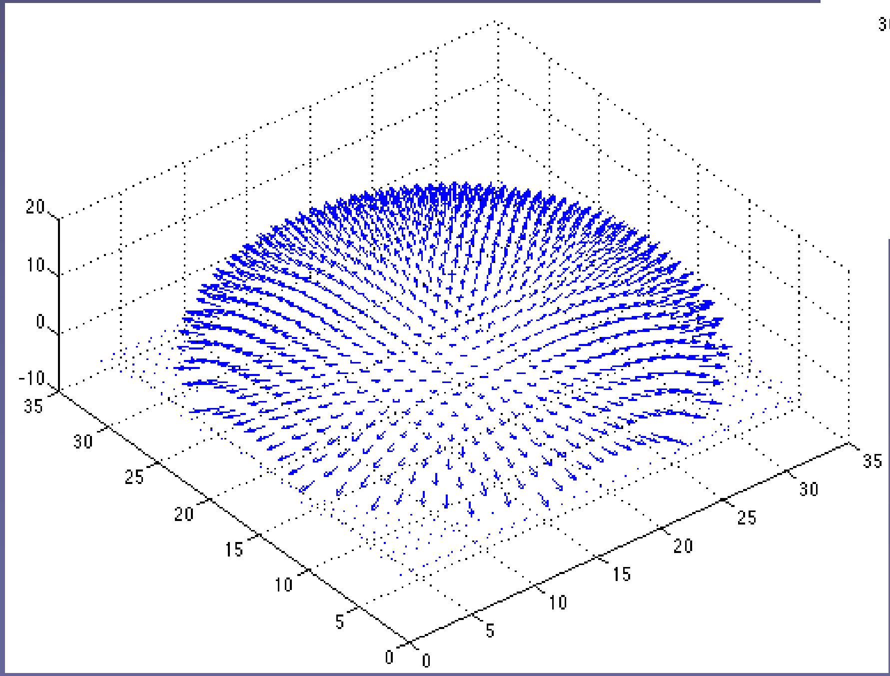
From Surface Orientations to Shape

$$(p,q) = \left(-\frac{\partial Z}{\partial x}, -\frac{\partial Z}{\partial y}\right)$$

$$Z(x,y) = Z(x_0, y_0) + \int_{(x_0,\, y_0)}^{(x,y)} -(p\,dx + q\,dy)$$

Where Z(x0, y0) is reference depth (0).





Integrate needle map

Algorithm

- Initilialize Z(0, 0) = 0
- Compute depth for 1st column:

For y = 1 to H -1:

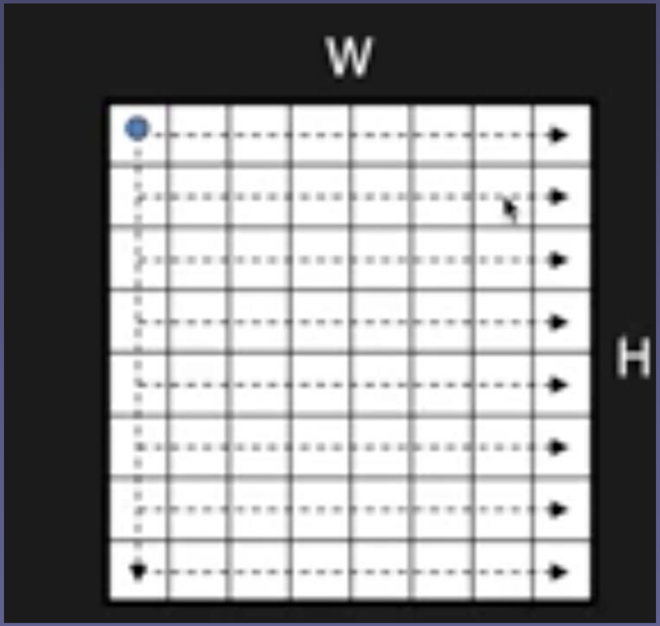
$Z(0, y) = Z(0, y - 1) - q(0, y)$

- Compute depth for each row:

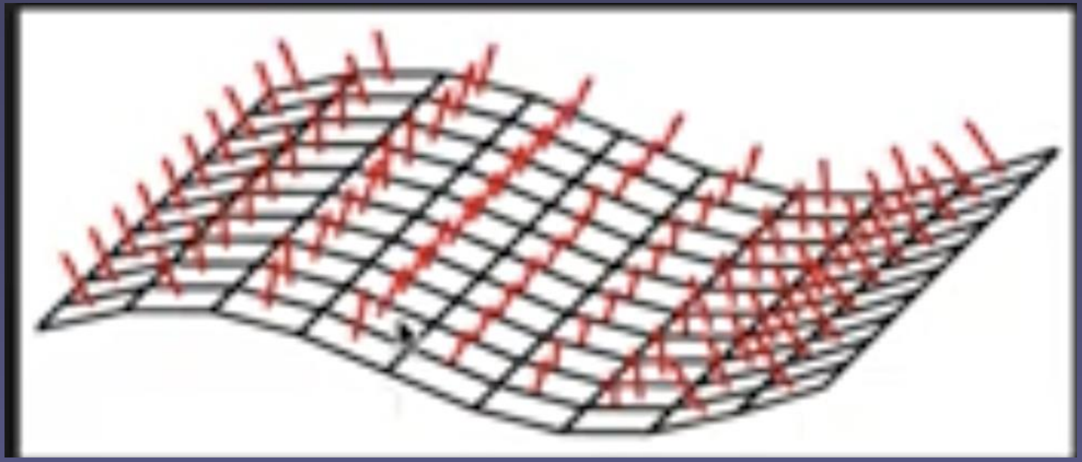
For y = 1 to H - 1:

for x = 1 to W – 1:

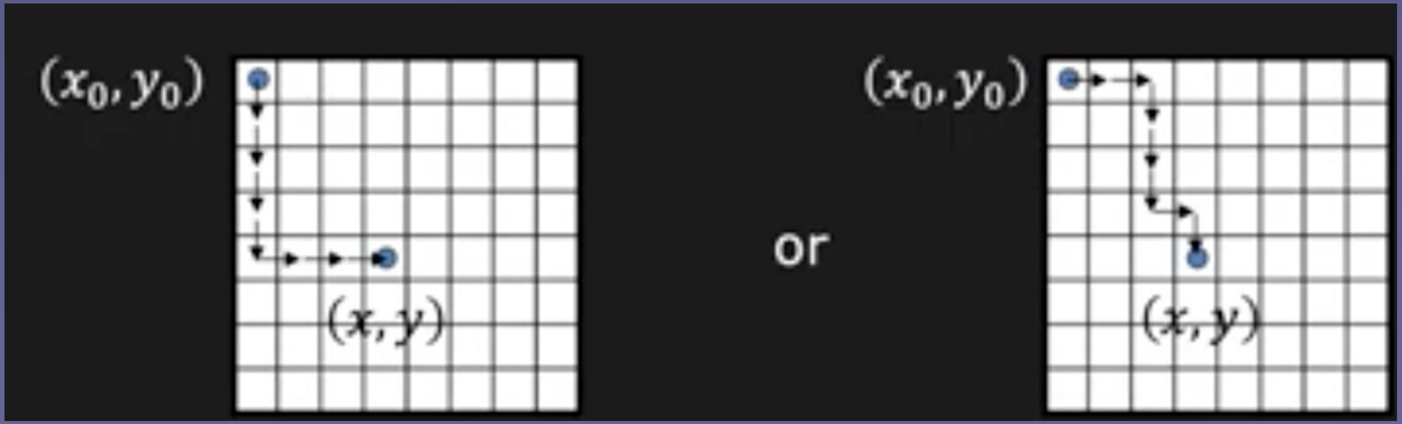
$Z(x,y) = Z(x-1, y) - p(x,y)$



Problem: Noisy normals

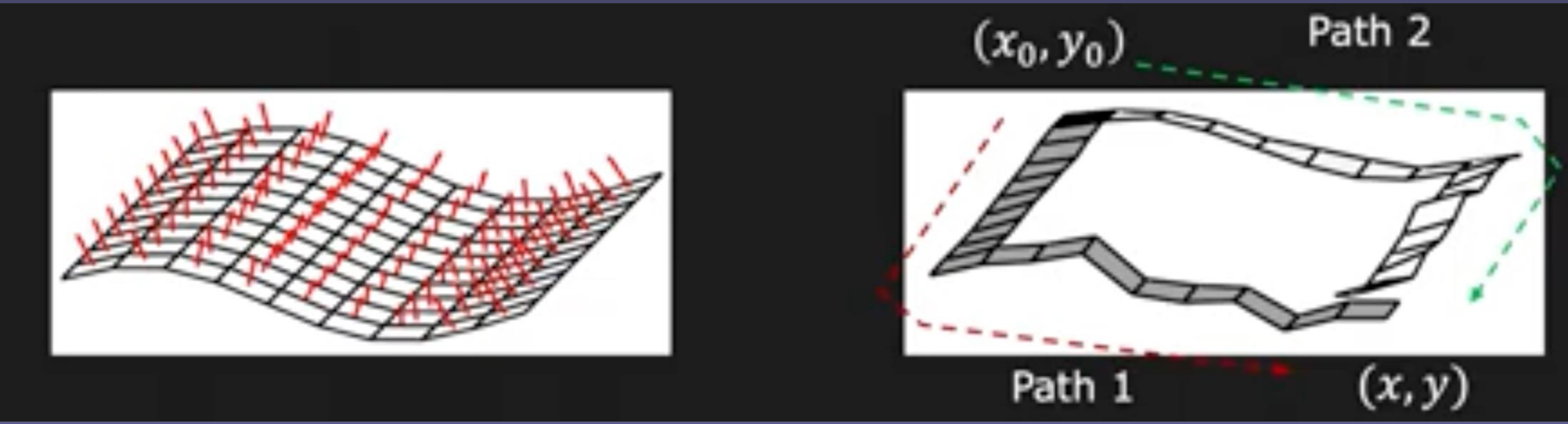


Multiple possible paths of integration to compute Z(x,y)



Shree Nayar

Problem: Noisy Normals



Different integration paths lead to different Z(x,y) results

Shree Nayar

Optimization

Minimize the errors between measured surface gradients (p, q) and surface gradients of estimated surface $z(x, y)$.

Error Measure:

$$D = \iint_{Image} \left(\frac{\partial z}{\partial x} + p\right)^2 + \left(\frac{\partial z}{\partial y} + q\right)^2 dxdy$$

where $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are gradients of the estimated surface.

Evaluated at each (x,y), i.e. p(x,y), q(x, y) – Same for the partial derivatives

Shree Nayar

Optimization in Fourier Domain

Minimize objective function D in Fourier Domain.

Let $Z(u,v)$, $P(u,v)$ and $Q(u,v)$ be the Fourier Transforms of $z(x,y)$, $p(x,y)$ and $q(x,y)$, respectively. Then:

$$z(x,y) = \iint_{-\infty}^{\infty} Z(u,v)e^{i2\pi(ux+vy)}dudv$$

$$p(x,y) = \iint_{-\infty}^{\infty} P(u,v)e^{i2\pi(ux+vy)}dudv$$

$$q(x,y) = \iint_{-\infty}^{\infty} Q(u,v)e^{i2\pi(ux+vy)}dudv$$

Optimization in Fourier Domain

Find $Z(u,v)$ that minimizes D using $\frac{\partial D}{\partial z} = 0$.

Solution: $\tilde{Z}(u,v) = \dfrac{iuP(u,v) + ivQ(u,v)}{u^2 + v^2}$

This is the Fourier Transform of the best fit surface.

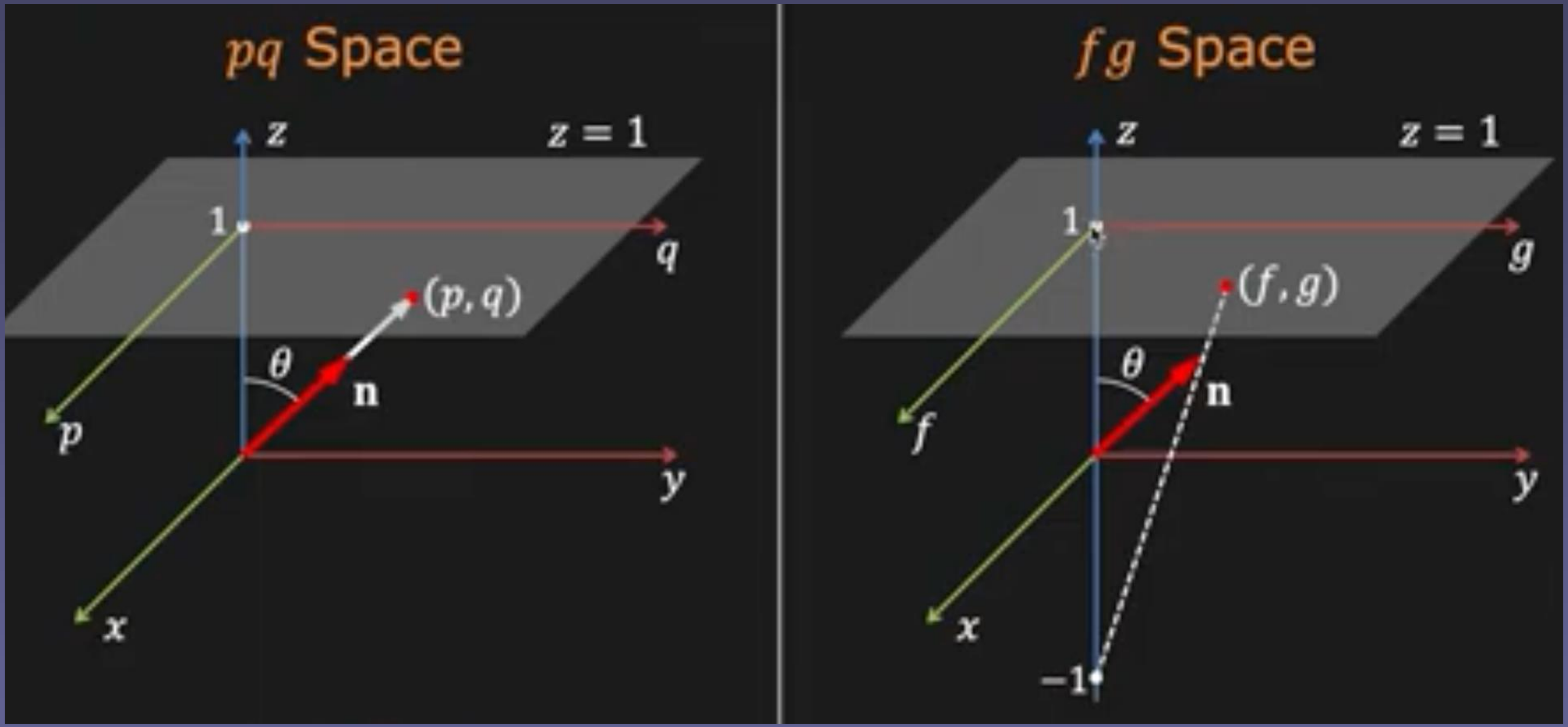
Compute Inverse Fourier Transform to obtain $\tilde{z}(x,y)$.

$$\tilde{Z}(u,v) \longrightarrow \boxed{\text{IFT}} \longrightarrow \tilde{z}(x,y)$$

[Frankot 1988]

Appendix 2 (Optional): Stereographic projection

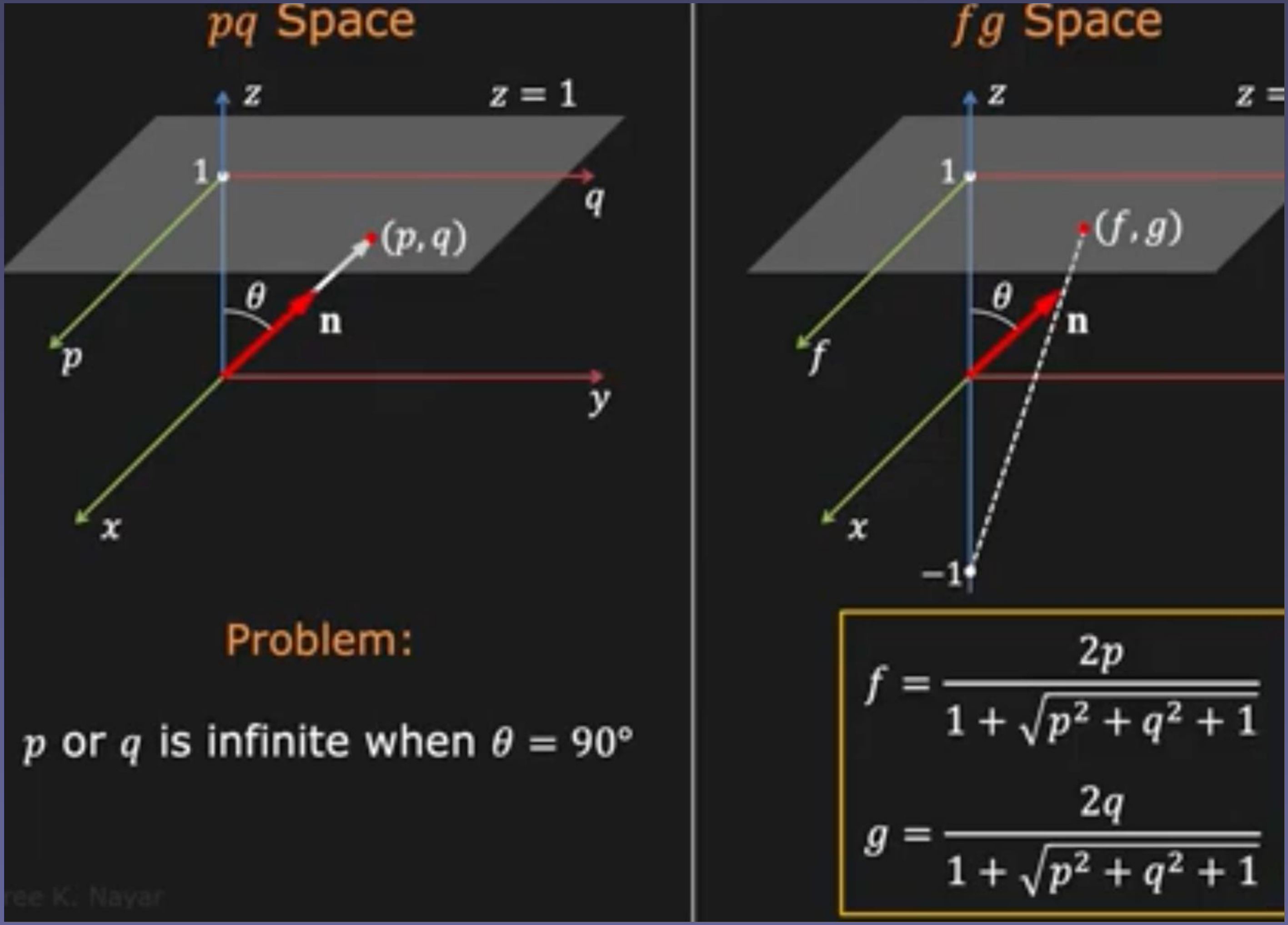
Stereographic projection



Problem: pq space can't represent 90 degree angles: (p, q) goes to infinity

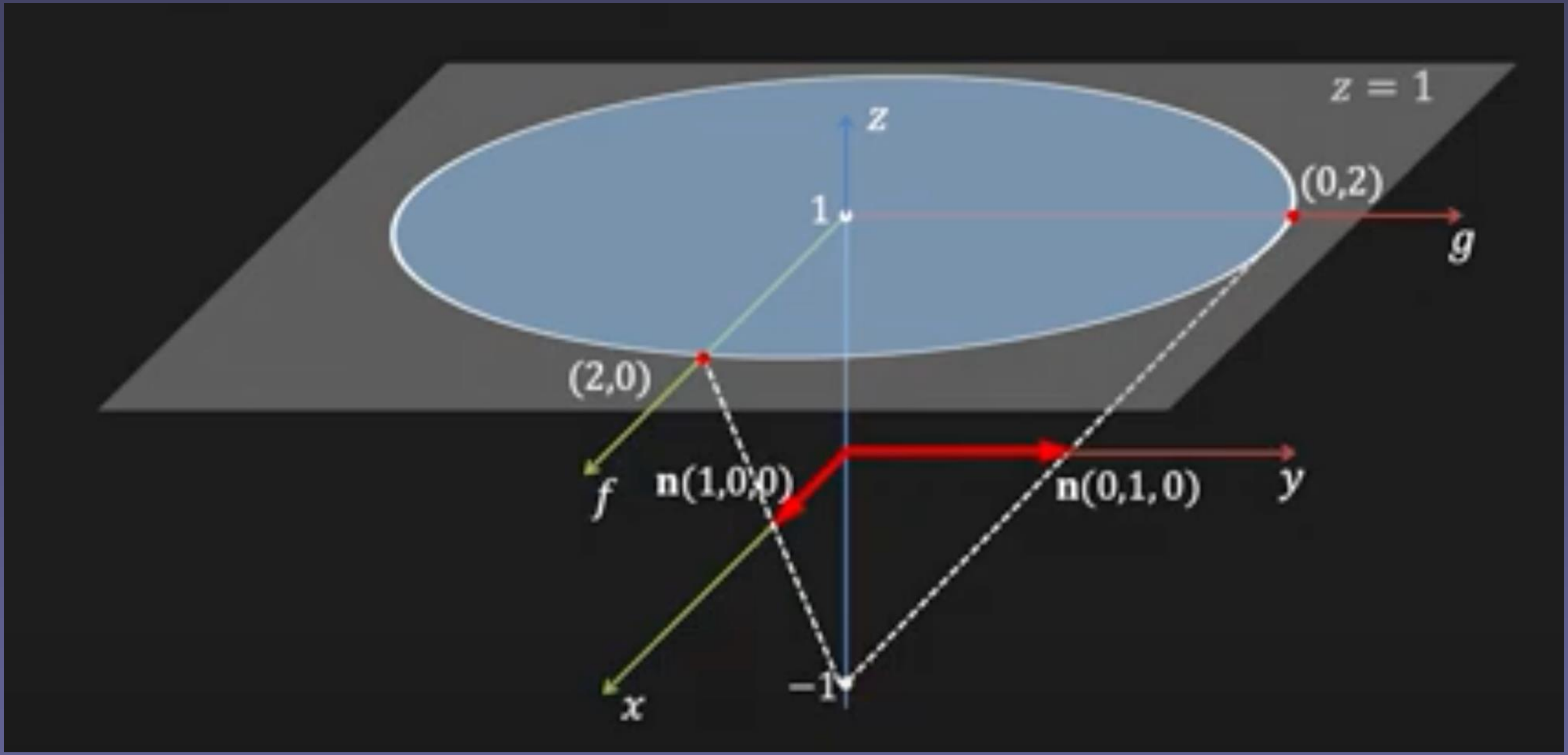
Solution: fg space

Stereographic projection



Mapping from (p, q) to (f, g)

Stereographic projection



Extent of (f,g) space