Domino Tiling an N-Queens Chessboard

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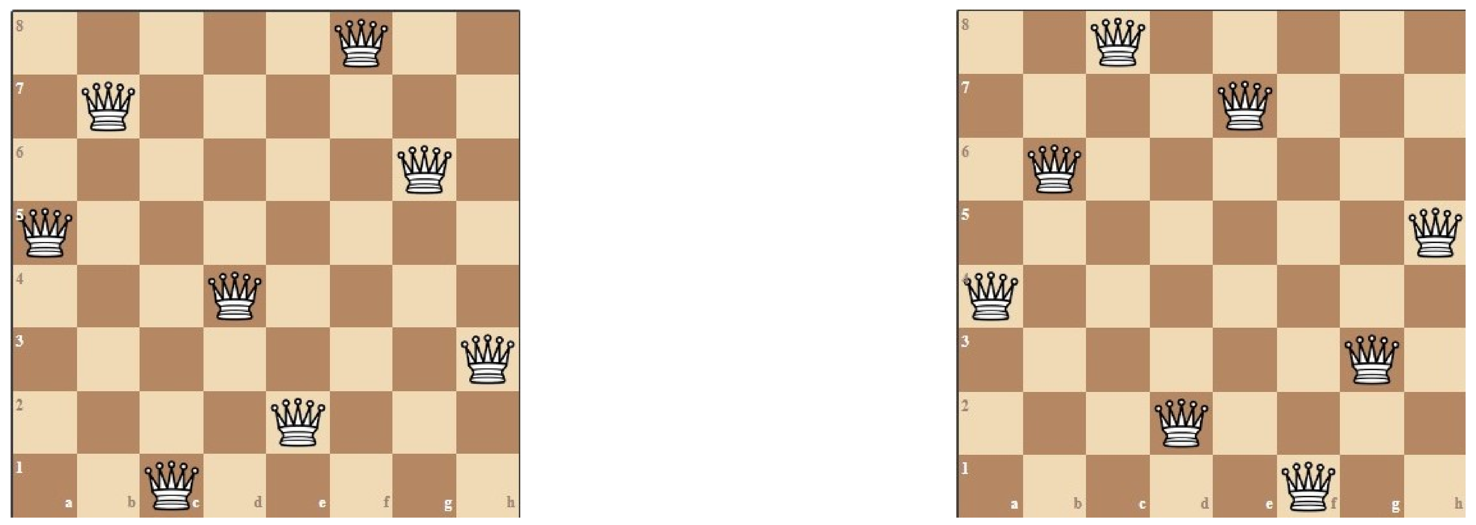
Introduction and Definition

N-Queens Problem
Place n queens on an n by n chessboard such that no two queens are attacking each other. Two queens are attacking each other if they are on the same row, column, or diagonal. The statement is equivalent to placing n queens on an n by n board such that they are all on different rows, different columns, and different diagonals. Although there are $\binom{64}{8} = 4,426,165,368$ ways to place 8 queens on an 8 by 8 chessboard without restrictions, only 92 of these are solutions to the problem. The number of fundamental solutions, or solutions that are not rotations or reflection of each other, has only been shown to be 12 for $n = 8$.



(a) No Three Queens in a Line

(b) Staircase Pattern

Figure 1: Two Solutions of 8-queens

Domino Tiling
The 1x2 domino tiling problem originates from Tatami, which are Japanese floor mats in the shape of a 1x2 rectangle. These mats are used to tile rooms, but with a bit more restriction on the tiling. On a n by n chessboard, 1x2 tiles are needed to be placed on the board such that they don't overlap and fill up the entire board. A form of this is mutilated tiling, in which some squares are removed from the board and the rest need to be tiled.

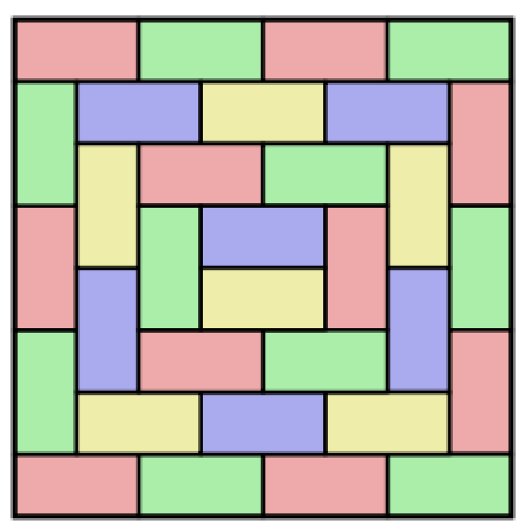


Figure 2: Tatami Tiling of an 8x8 square

Mutilated Chessboards

Mutilated chessboards are chessboard in which some squares are missing/mutilated. Since a chessboard is made up of alternating black and white squares, each 1x2 tile will cover one black square and one white square. The necessary condition for a mutilated chessboard to be tileable is that the number of black and white squares are equal. However, the mutilations can split the board into groups which are disjoint and not tileable. In the solutions to the n-queens puzzle, the queens (mutilated squares) are so far apart that the entire board is connected.

Algorithm

```
placeTiles(current x, current y)
{
    if all tiles have been placed
    {
        add one to counter of solutions
        break
    }
    if the current square has been covered
    {
        placeTiles(next x, next y)
    }
    set the tile to be covered in the boolean array
    if it is possible to place a left-right tile
    {
        place the left-right tile
        placeTiles(next x, next y)
        remove the tile
    }
    if it is possible to place an up-down tile
    {
        place the up-down tile
        placeTiles(next x, next y)
        remove the tile
    }
    set the current square to not be covered
}
placeQueens(current column)
{
    if current column is n
    {
        if number of black squares = number of white squares
        {
            placeTiles(0, 0)
        }
        break
    }
    for all squares in the current column
    {
        check if the square is attacked by any queens in the same row
        check if the square is attacked by any queens on both diagonals
        if the square is not attacked
        {
            place the queen
            mark the square as covered
            placeQueens(next column)
            remove the queen
            mark the square as not covered
        }
    }
}
```

Necessary Condition

n	All Squares	White Squares	Black Squares	White Queens	Black Queens
$4k$	$16k^2$	$8k^2$	$8k^2$	$2k$	$2k$
$4k+1$	$16k^2+8k+1$	$8k^2+4k+1$	$8k^2+4k$	$2k+1$	$2k$
$4k+2$	$16k^2+16k+4$	$8k^2+8k+2$	$8k^2+8k+2$	$2k+1$	$2k+1$
$4k+3$	$16k^2+24k+9$	$8k^2+12k+5$	$8k^2+12k+4$	$2k+2$	$2k+1$

Table 1: Exact Number of White/Black Queens and Squares When Placing n Queens on n by n board, to be 1x2 Tileable

Results

n	All solutions	Tileable Solutions
1	1	0
2	0	0
3	0	0
4	2	2
5	10	8
6	4	0
7	40	0
8	92	92
9	352	304
10	724	0
11	2,680	0
12	14,200	12,996
13	73,712	58,392
14	365,596	0
15	2,279,184	0

Table 2: Exact Number of Solutions for Placing n Queens on n by n Board

Tiling Arrangements	Frequency
382	8
653	8
3,806	8
5,271	8
6,378	8
6,705	8
7,408	8
8,416	8
9,424	8
9,981	8
15,012	8
21,940	4

Table 3: Tiling Arrangements and Frequency on an 8 by 8 chessboard

White Queens	Black Queens	White Squares	Black Squares	Arrangements
5	10	108	102	33,912
7	8	106	104	1,739,936
9	6	104	106	504,776
11	4	102	108	560

Table 4: Possible Numbers of White and Black Queens Along with the Possible Numbers of Remaining White and Black Squares and Arrangements on an 15 by 15 chessboard.

Conclusion

It's hypothesized that as long as the necessary condition of having equal white and black squares left in a given n-queen solution, the remaining chessboard will 1x2 tileable. This is not always true for other mutilated chessboards, as the mutilations could split the board into untileable groups. In the example of n-queens, the queens, or mutilations, are spread out so far apart that the entire board is connected. The number of black squares and number of white squares being equal seems to sometimes occur when $n \equiv 0, 1 \mod 4$. On the other hand, for $n \equiv 2, 3 \mod 4$, none of the placements of queens satisfy this property of equivalence.