Revision Course in Asset Management: Statistics and Linear Algebra*

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Abstract

These notes provide an overview of the main concepts used in asset management. I list the most important estimators and decomposition rules in statistics in Section 1. Next, I introduce the concept of matrices and the most important computation rules in Section 2. In particular, I focus on matrix inversion which is essential in solving systems of linear equations, as I show in Section 3. Finally, I also apply the toolbox to a typical example in asset management, the computation of tangency portfolio weights, in Section 4. Most sections contain examples and exercises which facilitate the internalization of these concepts. I highly recommend spending one afternoon on going through the notes and solving some exercises.

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1 Basic Statistics

Consider two random variables X and Y with the means μ_x, μ_y and variances σ_x^2, σ_y^2. Assume that we can observe only n random draws of each random variable X and Y, i.e. we have a sample of realizations of the two variables:

$$\mathbf{X} = \{x_1, \ldots, x_n\} \text{ and } \mathbf{Y} = \{y_1, \ldots, y_n\}.$$

Sample statistics tell us something about the population moments. The main estimators of the population moments are:

1. Mean (Unbiased Estimator): $\mathbb{E}[\mathbf{X}] = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$
2. Variance (Biased Estimator): $\widetilde{var}(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mathbb{E}[\mathbf{X}])^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \tilde{\sigma}_x^2$
3. Variance (Unbiased Estimator): $\widehat{var}(\mathbf{X}) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \mathbb{E}[\mathbf{X}])^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \hat{\sigma}_x^2$
4. Standard Deviation (Biased Estimator): $\widetilde{sd}(\mathbf{X}) = \sqrt{\tilde{\sigma}_x^2} = \tilde{\sigma}_x$
5. Standard Deviation (Unbiased Estimator): $\widehat{sd}(\mathbf{X}) = \sqrt{\hat{\sigma}_x^2} = \hat{\sigma}_x$
6. Covariance: $\widehat{cov}(\mathbf{X}, \mathbf{Y}) = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \hat{\sigma}_{xy}$
7. Correlation: $\widehat{cor}(\mathbf{X}, \mathbf{Y}) = \hat{\sigma}_{xy}/\sqrt{\hat{\sigma}_x^2\hat{\sigma}_y^2} = \hat{\sigma}_{xy}/\hat{\sigma}_x\hat{\sigma}_y = \rho_{xy}$

Note: unbiased estimators are obtained by applying *Bessel's correction* ($\frac{n}{n-1}$) to biased estimators.

The following two decomposition can be very useful to work with random variables and their moments.:

- **Variance Decomposition:**

$$var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

Proof.

$$\begin{aligned} var(X) &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= \mathbb{E}\left[X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2\right] \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2 \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \end{aligned}$$

□

- **Covariance Decomposition**

$$cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

Proof.

$$\begin{aligned} cov(X, Y) &= \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] \\ &= \ldots \text{some algebra inside the square brackets} \ldots \\ &= \mathbb{E}\left[XY - X\mathbb{E}[Y] - \mathbb{E}[X]Y + \mathbb{E}[X]\mathbb{E}[Y]\right] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \cancel{- \mathbb{E}[X]\mathbb{E}[Y]} + \cancel{\mathbb{E}[X]\mathbb{E}[Y]} \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \end{aligned}$$

□

Note: These decompositions are equivalent to the *biased estimators* of the population statistics. The equivalent *unbiased estimators* are:

$$\begin{aligned} \widehat{var}(\mathbf{X}) &= \frac{n}{n-1}\left(\mathbb{E}[\mathbf{X}^2] - (\mathbb{E}[\mathbf{X}])^2\right), \\ \widehat{cov}(\mathbf{X}, \mathbf{Y}) &= \frac{n}{n-1}\left(\mathbb{E}[\mathbf{XY}] - \mathbb{E}[\mathbf{X}]\mathbb{E}[\mathbf{Y}]\right). \end{aligned}$$

Exercise 1. *Compute all estimators of population moments from above for the following two samples:*

$$\begin{aligned} \mathbf{X} &= \{5, 11, 9, 20, 27\} \\ \mathbf{Y} &= \{7, 12, 8, 18, 23\} \end{aligned}$$

Check the difference between biased and unbiased estimators. Check that the standard and the equivalent decompositions of variance and covariance formulas lead to the same results.

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2 Basic Linear Algebra

2.1 Matrices

Definition 1. *A* matrix *is a rectangular array of the elements arranged in rows and columns.*

Example:

$$\underset{(m\times n)}{\mathbf{A}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The dimensions of this matrix are $m \times n$ (read "m by n"), where m is the number of *rows* and n is the number of *columns*. Every element in the matrix has a unique position which is identified by the row and the column where it is located. For example, element a_{22} is located in the second row and in the second column.

Definition 2. *A matrix which has dimension* 1×1 *is called* scalar *(i.e. simply a number).*

Example:

$$\underset{(1\times 1)}{[a_{11}]}$$

Definition 3. *A matrix with only one column and several rows (or one row and several columns) is called* vector*. A row vector is a* $(1 \times n)$ *matrix and a column vector is a* $(n \times 1)$ *matrix.*

Example:

$$\underset{(1\times 3)}{a} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} \text{ or } \underset{(3\times 1)}{b} = \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}$$

Definition 4. *A matrix with the same number of rows and columns* $(m = n)$ *is called* square matrix*.*

Example:

$$\underset{(3\times 3)}{\mathbf{B}} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Definition 5. *A square matrix which has ones on its main diagonal*[1] *and zero elsewhere is called* an identity matrix *and is typically denoted as I.*

Example:

$$\underset{(3\times 3)}{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ or } \underset{(4\times 4)}{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Definition 6. *A* symmetric matrix *is a square matrix which entries are symmetric with respect to the main diagonal, i.e.* $a_{ij} = a_{ji}$*.*

Example:

$$\underset{(3\times 3)}{A} = \begin{pmatrix} 1 & 2 & -8 \\ 2 & 3 & -5 \\ -8 & -5 & 4 \end{pmatrix}$$

2.1.1 Elementary Row Operations

It is possible to make several operations within a given matrix, namely:

- Interchange rows of a matrix, e.g.:

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \\ 0 & 4 & 1 \end{pmatrix} \xrightarrow[\text{second to third, and third to first}]{\text{switch rows: first to second,}} \begin{pmatrix} 0 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & 0 & -1 \end{pmatrix}$$

[1] This is the diagonal that goes form *top-left*

- Change a row by adding to it a multiple of another row, e.g.:

$$\begin{pmatrix}1 & 2 & 3\\ 3 & 0 & -1\\ 0 & 4 & 1\end{pmatrix} \xrightarrow[\text{and add it to second row}]{\text{multiply first row by -3}} \begin{pmatrix}1 & 2 & 3\\ 3+(-3\cdot 1) & 0+(-3\cdot 2) & -1+(-3\cdot 3)\\ 0 & 4 & 1\end{pmatrix}$$

- Multiply each element in a row by the same nonzero number (k, m), e.g:

$$\begin{pmatrix}1 & 2 & 3\\ 0 & 0 & -1\\ 0 & 4 & 1\end{pmatrix} \xrightarrow[\text{multiplying the third row by m}]{\text{multiplying the first row by k}} \begin{pmatrix}k & k\cdot 2 & k\cdot 3\\ 0 & 0 & -1\\ 0 & m\cdot 4 & m\cdot 1\end{pmatrix}$$

These operations are useful to solve a system of linear equations.

2.1.2 Addition

Definition 7. *Let A and B be $(m \times n)$ matrices. Then their sum $C = A + B$ is an $(m \times n)$ matrix in which each element is the sum of the corresponding elements of the matrices A and B.*

Example:

$$\underset{(2\times 3)}{A} = \begin{pmatrix}1 & 2 & 3\\ 3 & 0 & -1\end{pmatrix} \text{ and } \underset{(2\times 3)}{B} = \begin{pmatrix}-2 & 0 & 3\\ 1 & -1 & 3\end{pmatrix}$$

Their sum is given by,

$$\underset{(2\times 3)}{C} = \begin{pmatrix}1 & 2 & 3\\ 3 & 0 & -1\end{pmatrix} + \begin{pmatrix}-2 & 0 & 3\\ 1 & -1 & 3\end{pmatrix} = \begin{pmatrix}-1 & 2 & 6\\ 4 & -1 & 2\end{pmatrix}$$

Note: Only matrices with the same dimensions can be added to each other!
The addition operation has several properties:

- **Commutativity**: $A + B = B + A$, conditional to be of the same dimension.
- **Associativity**: $(A+B)+C = A+(B+C)$, conditional to be of the same dimension.

2.1.3 Multiplication

It is always possible to multiply a matrix by a scalar. For example, we can write $cA = B$, where c is a scalar (any number) and A is a matrix. The result of this operation is the matrix B (that has the same size of the matrix A) whose elements are simply the product of the elements of A times c.

Example: Take $c = 5$ and A the (2×3) matrix that we used before. The product between c and A is equal to:

$$\underset{(2\times3)}{B} = 5 \times \begin{pmatrix} 1 & 2 & 3 \\ 3 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 10 & 15 \\ 15 & 0 & -5 \end{pmatrix}.$$

The multiplication operation is more complex when two matrices are multiplied.

Definition 8. *Let A be an $(m \times p)$ matrix and B be a $(p \times n)$ matrix. Then their* product *is a $(m \times n)$ matrix C.*

Rule: For any multiplication, e.g. $A \cdot B$, we must have that the *number of columns of A is equal to the number of rows of B.*

Example: Consider two matrices:

$$\underset{(2\times3)}{A} = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 0 & 1 \end{pmatrix} \text{ and } \underset{(3\times3)}{B} = \begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix}.$$

We obtain $A \cdot B = C$, where C is a (2×3) matrix:

$$\underset{(2\times3)}{C} = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix},$$

and the elements of matrix C are:

- $c_{11} = 2 \cdot 3 + 1 \cdot 2 + -2 \cdot 1 = 6$
- $c_{12} = 2 \cdot 0 + 1 \cdot 1 + -2 \cdot 1 = -1$
- $c_{13} = 2 \cdot 1 + 1 \cdot 3 + -2 \cdot 0 = 5$
- $c_{21} = 3 \cdot 3 + 0 \cdot 2 + 1 \cdot 1 = 10$
- $c_{22} = 3 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1$

- $c_{23} = 3 \cdot 1 + 0 \cdot 3 + 1 \cdot 0 = 3$

By plugging every c_{ij} into the matrix C, we obtain:

$$\underset{(2\times 3)}{C} = \begin{pmatrix} 6 & -1 & 5 \\ 10 & 1 & 3 \end{pmatrix}.$$

Note: The multiplication is always *"rows times columns"*. Check the size of the matrix before starting your calculations. As you can see in the previous example, you can compute $A \cdot B$, but you cannot compute $B \cdot A$.

Properties of multiplication:

- **Commutativity**: we can compute $A \cdot B$ but we cannot compute $B \cdot A$. Even if we can compute both (think if they are both square matrices) the final result is not the same, unless one matrix is an **identity** matrix.
- **Associativity**: $A(BC) = (AB)C$. Note that the matrices do not switch their order, otherwise we get that $A(BC) \neq (BA)C$.

Note: The order in the multiplication is crucial!

2.1.4 Transpose of a Matrix

Definition 9. *Let A be an $(m \times n)$ matrix, we say that B $(n \times m)$ is the* transpose *of A if $b_{ij} = a_{ji}$. We write,*

$$B = A^\top.$$

Note: you may find other symbols indicating the transpose of a matrix: $A^\top \equiv A' \equiv A^{tr} \equiv A^t$.

Example:

$$\underset{(3\times 3)}{A} = \begin{pmatrix} 3 & 7 & 2 \\ 11 & 4 & 6 \\ 8 & 0 & 9 \end{pmatrix} \text{ and } \underset{(3\times 3)}{A^\top} = \begin{pmatrix} 3 & 11 & 8 \\ 7 & 4 & 0 \\ 2 & 6 & 9 \end{pmatrix}$$

The rows of the initial matrix A will be the columns of the transposed matrix $A^\top$.

Rule: You need to take each row and flip it 90° clockwise, then you put them into a new matrix with the a *1-to-1* relation between row and columns (the first row that you flip is the first column of the transposed matrix).

Properties: of transposition:

- $(A^\top)^\top = A$
- For any number α, $(\alpha A)^\top = \alpha(A)^\top$
- For any equal size matrices A and B, $(A+B)^\top = A^\top + B^\top$
- For any matrices A and B for which AB is defined, $(AB)^\top = B^\top A^\top$.

Exercise 2. *Consider the matrices A and B:*

$$\underset{(3\times 2)}{A} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \\ 1 & 0 \end{pmatrix} \quad \textit{and} \quad \underset{(2\times 4)}{B} = \begin{pmatrix} 5 & 1 & -2 & 3 \\ 0 & -1 & 3 & 1 \end{pmatrix}.$$

Compute AB and $B^\top A^\top$.

Exercise 3. *Consider the matrices A and B:*

$$\underset{(2\times 2)}{A} = \begin{pmatrix} 2 & 3 \\ 5 & 9 \end{pmatrix} \quad \textit{and} \quad \underset{(2\times 2)}{B} = \begin{pmatrix} 1 & 0 \\ 8 & 2 \end{pmatrix}.$$

Compute $(AB)^\top$ and show that $A^\top B^\top \neq (AB)^\top$, but $B^\top A^\top = (AB)^\top$.

2.1.5 Multiplicative Inverse

Note: This is a very important operation! Later we will use the **inverse** of a matrix to solve a system of linear equations.

Definition 10. *Matrix B is called the* inverse *of the square matrix A if $BA = AB = I$.*

Definition 11. *The $(n \times n)$ square matrix A is called* invertible *if there exists a $(n \times n)$ square matrix B such that*

$$\underset{(n\times n)}{A} \times \underset{(n\times n)}{B} = \underset{(n\times n)}{B} \times \underset{(n\times n)}{A} = \underset{(n\times n)}{I}.$$

Example:

$$\underset{(2\times 2)}{A} = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \text{ and } \underset{(2\times 2)}{A^{-1}} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

Exercise 4. *Compute AA^{-1} where A is given in the example above to check if it is actually equal to an identity matrix.*

It is important to develop some practical skill for computing the *inverse* of a matrix. It gets really messy as the size of the matrix increases. Already by going from a (3×3) to a (4×4) entails a huge increase in complexity.

How do we know whether the matrix is invertible? Let's start by considering a generic (2×2) matrix A:

$$\underset{(2\times 2)}{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We can compute the *determinant* of matrix A as

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc,$$

which can be interpreted as the scaling factor of the transformation described by matrix A. The following notations are equivalent $|A| = \det(A) = \det A$. The (2×2) matrix A is invertible if and only if $ad - bc \neq 0$, i.e. the *determinant* of the matrix is different from zero. In other words, the determinant *determines* whether or not the square matrix is invertible. The inverse of matrix A is then (2×2) matrix B:

$$\underset{(2\times 2)}{B} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Similarly, we can compute the determinant of a (3×3) matrix

$$\underset{(2\times 2)}{C} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

as

$$|C| = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh.$$

This method that yields the determinat of a (3×3) matrix is called *Sarrus' rule.*

Exercise 5. *Compute the following determinants*

$$|A| = \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}, \quad |B| = \begin{vmatrix} 1 & 2 \\ 5 & 10 \end{vmatrix}, \quad |C| = \begin{vmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}.$$

Now we have only have to come up with a procedure that allows us to find the inverse of an invertible matrix. Consider the following matrix,

$$\underset{(3\times 3)}{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}$$

Is matrix A invertible? Let's calculate the determinant of matrix A to check if it is not equal to 0. To do this we use *Sarrus Rule*, according to which the determinant of the matrix A is given by:

$$\begin{aligned} det(A) =& [(1 \cdot 2 \cdot 4) + (1 \cdot 3 \cdot 1) + (1 \cdot 1 \cdot 2)] \\ & - [(1 \cdot 2 \cdot 1) + (2 \cdot 3 \cdot 1) + (4 \cdot 1 \cdot 1)] \\ =& (8 + 3 + 2) - (2 + 6 + 4) \\ =& 1 \neq 0 \end{aligned}$$

As determinant of the matrix A is not equal to 0, this matrix is invertible.

We want to compute the inverse of this matrix, that is A^{-1}. For this we will use the **Gauss-Jordan elimination method**:

1) Gauss Elimination method allows us to reduce a system of linear equations (matrix) to its **Row Echelon** form (REF).

2) Gauss-Jordan Elimination method allows us to further reduce such a system to its **Reduced Row Echelon** form (RREF) $\rightarrow$ we need to get this form to find the inverse of a matrix.

Let's see some examples to understand when a system is in its **Row Echelon** form and what is the difference between **Row Echelon** and **Reduced Row Echelon** form. For example, the (3×3) matrix is:

$$\underset{(3\times 3)}{No - REF} = \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{3} & 1 \\ 0 & \boxed{1} & 2 \end{pmatrix}, \underset{(3\times 3)}{REF} = \begin{pmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{3} \end{pmatrix}, \text{ and } \underset{(3\times 3)}{RREF} = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix}.$$

Note: To check if a matrix is in its REF, go from the left-top to the right-bottom and check the position of the **leading coefficient** (i.e. the first non-zero number from the left) in each row. If the leading coefficient of every row is strictly to the right of the leading coefficient of the row above, the matrix is in its REF; otherwise, it is not in its REF. Moreover, if the matrix is in its REF and every leading coefficient is equal to 1 and all the other entries are equal to 0, the matrix is in its RREF. In both REF and RREF, if there are zero-rows (i.e. all its elements are equal to 0), they are placed at the bottom of the matrix.

Definition 12. *The system is said to be in REF if,*

- *all non-zero rows are above zero-rows, and*
- *the leading coefficient of a non-zero row is always strictly to the right of the leading coefficient of the row above it.*

Definition 13. *The system is said to be in RREF if,*

- *it is in REF,*
- *every leading coefficient of the non-zero row is equal to 1, and*
- *all the other entries are equal to 0.*

Now we are ready to compute the inverse of the (3×3) matrix A:

$$\underset{(3\times3)}{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}$$

The computation requires two steps:

1. Construct the augmented matrix $(A|I)$:

$$\underset{(3\times3)}{(A|I)} = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array} \right)$$

2. Perform elementary row operations on the matrix $(A|I)$ to get the RREF for matrix A. The result is that the left part of matrix $(A|I)$ will be an identity matrix and what now is an identity matrix will be our inverse matrix A^{-1}.

We begin by subtracting the first row from the second and the third, we get,

$$\underset{(3\times3)}{(A|I)} = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 3 & -1 & 0 & 1 \end{array} \right)$$

This is still not a REF of the matrix A. Then we subtract the second row from the third row

$$\underset{(3\times3)}{(A|I)} = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right),$$

and we get the REF of the matrix A. However, to get the inverse matrix of A we need to transform matrix A to its RREF.
Note that there is no need to change the third row, it is already good for the RREF. So we perform backward elimination (starting from the last row). Let's subtract the third

row multiplied by two from the second row,

$$\underset{(3\times3)}{(A|I)} = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right),$$

Now subtract the third row from the first row,

$$\underset{(3\times3)}{(A|I)} = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$$

To get the RREF for matrix A, we need to perform an operation that makes the element in the first row and second column equal to zero. This is obtained by subtracting the second row from the first row,

$$\underset{(3\times3)}{(A|I)} = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right).$$

Matrix A is now an identity matrix and the right part of $(A|I)$ is our inverse matrix. That is,

$$\underset{(3\times3)}{A^{-1}} = \begin{pmatrix} 2 & -2 & 1 \\ -1 & 3 & -2 \\ 0 & -1 & 1 \end{pmatrix}.$$

Exercise 6. *Show that we actually found the inverse of matrix A in the previous example.*

Exercise 7. *Compute the inverse of the following matrices*

$$W = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & -1 \\ -1 & 3 & 1 \end{pmatrix}.$$

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3 Linear Equations

3.1 Definitions and Basic Properties

Definition 14. *A* linear equation *is an equation of the form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

There are three elements: a_i (coefficients), b (constant) and x_i (variables). In many cases, however, we have to solve a couple of linear equations simultaneously.

Definition 15. *A* system of linear equations *is a group of linear equations considered together*

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Finding a solution to such a system of liner equations means to find coefficients for each x_i that satisfy all equations simultaneously. If a system of linear equation admits more than one solution, then we have infinitely many solutions.

We find the solutions to a system of linear equations by using matrix operations developed above. First, let us rewrite the system of linear equations in matrix notation as

$$Ax = b.$$

We know A and b and want to find x that satisfies this equation.

Just like with matrices, there are several operations that can be done by obtaining an equivalent system.

- **Interchanging equations**.

$$\begin{cases} x_1 + 2x_2 = 4 \\ 2x_1 - x_2 = 3 \end{cases}$$

and

$$\begin{cases} 2x_1 - x_2 = 3 \\ x_1 + 2x_2 = 4 \end{cases}$$

- **Multiplication**. You can multiply by any number not equal to zero.

$$\begin{cases} x_1 + 2x_2 = 4 \\ 2x_1 - x_2 = 3 \end{cases}$$

and

$$\begin{cases} 3x_1 + 6x_2 = 12 \\ 6x_1 - 3x_2 = 9 \end{cases}$$

- **Addition**. You can add any equation multiplied by some number to any other equation.

$$\begin{cases} x_1 + 2x_2 = 4 \\ 2x_1 - x_2 = 3. \end{cases}$$

Take the first equation and multiply it by 3. You get $3x_1 + 6x_2 = 12$. Now add this equation to the second equation in the following way,

$$\begin{cases} 3x_1 + 6x_2 = 12 \\ \underline{2x_1 - x_2 = 3} \\ 5x_1 + 5x_2 = 15. \end{cases}$$

The new (equivalent) system of linear equations is,

$$\begin{cases} x_1 + 2x_2 = 4 \\ 5x_1 + 5x_2 = 15. \end{cases}$$

3.2 Solving a System of Linear Equations

In this section we apply the tools that we have developed in matrix algebra to solve systems of linear equations.

3.2.1 System of Linear Equations With a Unique Solution.

Let's start with an easy system. Whenever the number of variables is equal to the number of equations (and the matrix is full rank[2]), then this system admits a **unique solution**. Consider the system given by,

$$\underset{(n\times n)}{A} \cdot \underset{(n\times 1)}{x} = \underset{(n\times 1)}{b}, \tag{1}$$

where A is the matrix of the coefficients of x (which is the vector of variables) and bB is the vector of constants terms. The aim is to find the value of each element of X such that the equality is satisfied. In the previous section we have seen that whenever a square matrix is of full rank, then this is invertible. Let pre-multiply both sides of equation (1) by the inverse of A, that is

$$\underbrace{A^{-1} \cdot A}_{=I} \cdot x = A^{-1} \cdot b,$$

so we get that $x = A^{-1}b$.

Example:

$$\begin{cases} x_1 - x_2 = 2 \\ 4x_1 - 6x_2 = 4. \end{cases}$$

Let's rewrite this system of equations in matrix notation,

$$A = \begin{pmatrix} 1 & -1 \\ 4 & -6 \end{pmatrix}; \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}; \quad b = \begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

We know that $x = A^{-1}b$, we need to compute the inverse of A which is,

$$A^{-1} = \begin{pmatrix} 3 & -1/2 \\ 2 & -1/2 \end{pmatrix}.$$

[2]All of the vectors in a matrix are linearly independent.

We only need to compute the solution,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 & -1/2 \\ 2 & -1/2 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \rightarrow \quad x_1 = 4; \;\; x_2 = 2.$$

3.2.2 System of Linear Equations Without Unique Solution

Next, we deal with systems in which either the number of equations is smaller than the number of variables or the matrix of the coefficients is not invertible. We need to learn the difference between **leading** and **free** variables. We already introduced the concept of **leading** coefficient as the first nonzero number from the left. The **free** variables are the ones which are not leading. Let us see an example, consider the system

$$\begin{cases} \boxed{x_1} & +2x_2 & +3x_3 & +x_4 & = 4 \\ & & \boxed{2x_3} & -x_4 & = 3. \end{cases} \tag{2}$$

The system in equation (2) has

- Leading variables: x_1 and x_3
- Free variables: x_2 and x_4

We are going to use the **Gauss elimination method** to get the Row Echelon form and then the **Gauss-Jordan elimination method** to solve the system.

Remember: a system is said to be in row echelon form if the subscripts of the leading variables in its equations form a strictly increasing sequence. Let's see two examples.

Example:
System in Row Echelon form (REF),

$$\begin{cases} x_1 & +2x_2 & +3x_3 & +x_4 & = 4 \\ & 5x_2 & +2x_3 & -x_4 & = 3 \\ & & & 2x_4 & = 10 \end{cases}$$

the subscripts of the leading variables form a strictly increasing sequence: x_1, x_2, x_4.
Example:

This system is not in Row Echelon form (No-REF),

$$\begin{cases} x_1 & +2x_2 & +3x_3 & +x_4 & = 4 \\ & 5x_2 & +2x_3 & -x_4 & = 3 \\ & 3x_2 & + & 2x_4 & = 10 \end{cases}$$

the subscripts of the leading variables are not strictly increasing: x_1, x_2, x_2.

The algorithm for solving linear system consists of **two steps**:

1. Reduce the system to the equivalent system in row echelon form by using elementary row operations.
2. Solve the system in (reduced) row echelon form.

3.2.3 System of Linear Equations With Many Solutions

Consider the following system,

$$\begin{cases} x_1 & +4x_2 & -2x_3 & +7x_4 & = 10 \\ 2x_1 & +5x_2 & -x_3 & +8x_4 & = 14 \\ 3x_1 & +6x_2 & & +9x_4 & = 18 \end{cases}$$

First step is to write the augmented matrix of coefficients:

$$\underset{(3\times 4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 4 & -2 & 7 & 10 \\ 2 & 5 & -1 & 8 & 14 \\ 3 & 6 & 0 & 9 & 18 \end{array}\right)$$

Now multiply the first row by 2 and subtract it from the second row. We get,

$$\underset{(3\times 4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 4 & -2 & 7 & 10 \\ 0 & -3 & 3 & -6 & -6 \\ 3 & 6 & 0 & 9 & 18 \end{array}\right)$$

multiply the first row by 3 and subtract it from the third row, we obtain

$$\underset{(3\times4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 4 & -2 & 7 & 10 \\ 0 & -3 & 3 & -6 & -6 \\ 0 & -6 & 6 & -12 & -12 \end{array}\right).$$

We can multiply the second row by 2 and subtract it from the third row,

$$\underset{(3\times4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 4 & -2 & 7 & 10 \\ 0 & -3 & 3 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

<u>**Warning:**</u> the third row being equal to 0 is fine. We can erase it and not consider for the solving process. **However, sometimes it can happen that all the coefficient are zero but the constant term is different from zero. This implies that the system does not have a solution!**

Let's continue by considering only the first and the second row:

$$\underset{(3\times4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 4 & -2 & 7 & 10 \\ 0 & -3 & 3 & -6 & -6 \end{array}\right).$$

The system is now in its Row Echelon form (REF). We can solve it already. The leading variables are x_1 and x_2, while the free variables are x_3 and x_4. We can divide the second equation by -3 and then we can rewrite the system as follows:

$$\begin{cases} x_1 & = 10 - 4x_2 + 2x_3 - 7x_4 \\ x_2 & = 2 + x_3 - 2x_4 \end{cases}$$

The solution to this system of linear equation then is

$$\begin{cases} x_1 & = 2 - 2x_3 + x_4 \\ x_2 & = 2 + x_3 - 2x_4 \\ x_3 & \text{is free} \\ x_4 & \text{is free} \end{cases}$$

This means that you can pick any value you want for x_3 and x_4, according to that you

will get the corresponding values for x_1 and x_2 which solve the system.
We can go on and reach the Reduced Row Echelon form (RREF) for the matrix in equation (**??**). Let's divide the third row by -3, we get

$$\underset{(3\times 4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 4 & -2 & 7 & 10 \\ 0 & 1 & -1 & 2 & 2 \end{array}\right)$$

for reaching the RREF we need to make the 4 in the first row equal to 0. We multiply the second row by 4 and subtract from the first row,

$$\underset{(3\times 4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 2 \\ 0 & 1 & -1 & 2 & 2 \end{array}\right)$$

the exercise is solved. The set of solutions is:

$$\begin{cases} x_1 & = 2 - 2x_3 + x_4 \\ x_2 & = 2 + x_3 - 2x_4 \\ x_3 & \text{is free} \\ x_4 & \text{is free} \end{cases}$$

Next, consider the following system,

$$\begin{cases} x_1 + 2x_2 + x_3 & = 2 \\ x_1 + 3x_2 + 2x_3 - x_4 & = 4 \\ 2x_1 + x_2 - x_3 + 3x_4 & = -2 \\ 2x_1 - 2x_3 + 3x_4 & = 1 \end{cases}$$

First of all let's write the augmented matrix of the coefficients:

$$\underset{(3\times 4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 2 \\ 1 & 3 & 2 & -1 & 4 \\ 2 & 1 & -1 & 3 & -2 \\ 2 & 0 & -2 & 3 & 1 \end{array}\right)$$

Apply the following steps to the matrix:

- Subtract the first row from the second

- Multiply the first row by 2 and subtract it from the third row
- Subtract the third row from the forth

then the result is given by,

$$\underset{(3\times4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & -3 & -3 & 3 & -6 \\ 0 & -1 & -1 & 0 & 3 \end{array}\right)$$

- Multiply second row by 3 and add to the third row
- Add the second row to the fourth row

$$\underset{(3\times4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 5 \end{array}\right)$$

Placing all-zero row at the bottom of the matrix:

$$\underset{(3\times4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

the matrix is now in its REF. The third row can be not considered. Notice that we already find the value of one variable. If you look at the last row, the you realize that $x_4 = -5$. This is also one of those case in which solving the system from the REF is faster than continuing and getting the RREF. Here I continue till the RREF, you may try to solve from the previous matrix and you will realize that is much easier.

$$\underset{(3\times4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 2 \\ 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right)$$

... after simple row operations, we obtain

$$\underset{(3\times 4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 8 \\ 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & -5 \end{array}\right)$$

there are three leading variables (x_1,x_2, and x_4) and only one free variable, x_3.

$$\begin{cases} x_1 & = 8 + x_3 \\ x_2 & = -3 - x_3 \\ x_3 & \text{is free} \\ x_4 & = -5 \end{cases}$$

3.2.4 System of Linear Equations Without a Solution

Consider the following system,

$$\begin{cases} x_1 + 2x_2 - x_4 & = 1 \\ -2x_1 - 3x_2 + 4x_3 + 5x_4 & = 2 \\ 2x_1 + 4x_2 - 2x_4 & = 3 \end{cases}$$

First of all, let's write the augmented matrix of the coefficients

$$\underset{(3\times 4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ -2 & -3 & 4 & 5 & 2 \\ 2 & 4 & 0 & -2 & 3 \end{array}\right)$$

Apply the following steps to the matrix:

- Multiply the first row by 2 and subtract it from the second row
- Multiply the first row by 2 and subtract it from the third row

Now the matrix is in the following form:

$$\underset{(3\times 4)}{(A|B)} = \left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 1 \\ 0 & -1 & 4 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

In the third row all the coefficients are zero but the constant term is different from zero – the system does not have a solution.

Exercise 8. *Solve the following systems of equations*

$$(a) \begin{cases} x_1 + x_2 + 2x_3 & = 9 \\ 2x_1 + 4x_2 - 3x_3 & = 1 \\ 3x_1 + 6x_3 - 5x_3 & = 0 \end{cases}, \quad (b) \begin{cases} 2x_1 + x_2 - 2x_3 & = 10 \\ 3x_1 + 2x_2 + 2x_3 & = 1 \\ 5x_1 + 4x_3 - 3x_3 & = 4 \end{cases}$$

Exercise 9. *Find the solutions to the system of equations $Ax = b$ where*

$$A = \begin{pmatrix} 1 & -2 & 0 & -1 & 0 \\ 0 & -1 & 0 & -4 & 1 \\ 0 & -3 & 1 & -2 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}.$$

Exercise 10. *Solve the following equation with the Gauss-Jordan method:*

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

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4 Applications to Asset Management

4.1 Compute the Tangency Portfolio

Now we apply the methods developed above to solve a typical problem in asset management. I assume that you discussed the theoretical background in class before. Suppose we have three assets with the variance-covariance matrix

$$\Sigma = \begin{pmatrix} 0.01 & 0.02 & 0.03 \\ 0.02 & 0.03 & 0.01 \\ 0.03 & 0.01 & 0.02 \end{pmatrix},$$

the risk-free rate $r_f = 0.01$ and the expected returns $\mathbb{E}[r] = (0.03, 0.04, 0.05)'$. From the asset management lecture, we know that we want to find the tangency portfolio w such that

$$w = \Sigma^{-1}\left(\mathbb{E}[r] - r_f\right).$$

As a first step, we invert the variance-covariance matrix. Since working with decimals can get nasty very quickly, we invert the matrix

$$Z = 100 \cdot \Sigma$$

using the Gauss-Jordan algorithm introduced above. We get the inverse

$$Z^{-1} = \begin{pmatrix} -\frac{5}{18} & \frac{1}{18} & \frac{7}{18} \\ \frac{1}{18} & \frac{7}{18} & -\frac{5}{18} \\ \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{pmatrix} = \frac{1}{100}\Sigma^{-1}.$$

Note that the inverse of a symmetric matrix is also symmetric, so you can already check whether you made a mistake at this point. Next we plug in the solution to get calculate the portfolio weights

$$\begin{aligned} w &= 100 \begin{pmatrix} -\frac{5}{18} & \frac{1}{18} & \frac{7}{18} \\ \frac{1}{18} & \frac{7}{18} & -\frac{5}{18} \\ \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{pmatrix} \left[\begin{pmatrix} 0.03 \\ 0.04 \\ 0.05 \end{pmatrix} - 0.01 \right] \\ &= 100 \begin{pmatrix} -\frac{5}{18} & \frac{1}{18} & \frac{7}{18} \\ \frac{1}{18} & \frac{7}{18} & -\frac{5}{18} \\ \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{pmatrix} \begin{pmatrix} 0.02 \\ 0.03 \\ 0.04 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{5}{18} & \frac{1}{18} & \frac{7}{18} \\ \frac{1}{18} & \frac{7}{18} & -\frac{5}{18} \\ \frac{7}{18} & -\frac{5}{18} & \frac{1}{18} \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} \frac{21}{18} \\ \frac{3}{18} \\ \frac{3}{18} \end{pmatrix}. \end{aligned}$$

As a last step, we have to apply the condition that weights have to sum up to one (i.e. we normalize the weights by the sum over the weights we calculated before) and we get the weight of the tangency portfolio

$$w^* = \begin{pmatrix} \frac{21}{18} \\ \frac{3}{18} \\ \frac{3}{18} \end{pmatrix} \frac{1}{\frac{21}{18} + \frac{3}{18} + \frac{3}{18}} = \begin{pmatrix} \frac{21}{18} \\ \frac{3}{18} \\ \frac{3}{18} \end{pmatrix} \frac{18}{27} = \begin{pmatrix} \frac{21}{27} \\ \frac{3}{27} \\ \frac{3}{27} \end{pmatrix}.$$