Signals and Systems
Homework Two

Due Wednesday, April 26

1. `lsim` with differential equations
   The function `lsim` can be used to simulate the output of continuous-time, causal LTI systems described by linear constant-coefficient differential equations of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}.$$

   To use `lsim`, the coefficients a_k and b_m must be stored in MATLAB vectors `a` and `b`, respectively, in descending order of the indices k and m. Rewriting the above equation in terms of the vectors `a` and `b` gives

$$\sum_{k=0}^{N} \mathtt{a}(N+1-k) \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} \mathtt{b}(M+1-m) \frac{d^m x(t)}{dt^m}.$$

   Note that `a` must contain $N+1$ elements, which might require appending zeros to `a` to account for coefficients a_k that equal zero. Similarly, the vector `b` must contain $M+1$ elements.
   Executing `y = lsim(b,a,x,t);` simulates the response of the LTI system to the input signal specified by the vectors `x` and `t`. The vector `t` contains the time samples for the input and output, `x` contains the values of the input $x(t)$ at each time in `t`, and `y` contains the simulated values of the output $y(t)$ at each time in `t`. Basically, `lsim` interpolates the pair `t`, `x` to represent the true function $x(t)$.

   (a) Consider the causal LTI system described by the first-order differential equation

$$\frac{y(t)}{dt} = -\frac{1}{2} y(t) + x(t).$$

   Compute and plot the step response of the system. Use `lsim` and `plot`. The input step function can be defined as

```
>> t=[0:10];
>> x=ones(1, length(t));
```

(b) The functions `impulse` and `step` can be used to compute the impulse and step responses of the systems. Use `step` and `impulse` to compute and plot the step and impulse responses of the system described above.

2. Fun with Gibbs Phenomenon!
   In this problem we will investigate the Gibbs Phenomenon: the peculiar manner Fourier series representations behave at certain discontinuities.

   (a) Briefly explain the Gibbs Phenomenon. What is it? When does it happen?

   (b) Consider the square function $f(t)$ with period T defined as follows for $t \in [-T/2, T/2]$,

   $$f(t) = \begin{cases} 1, & \text{if} |t| < T_1 \\ 0, & \text{otherwise}, \end{cases}$$

   where $T_1 < T/2$. From here on we will set $T_1 = 0.25, T = 1$. Using MATLAB, plot $f(t)$ for $|t| < 0.5$.

   (c) What are the Fourier coefficients of $f(t)$?

   (d) We have learned from class that Fourier series representation of a continuous time signal $x(t)$ is given as follows,

   $$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T}kt}. \tag{1}$$

   Consider $x_N(t)$ which is the $2N+1$ partial sum of the Fourier series,

   $$x_N(t) = \sum_{k=-N}^{N} a_k e^{j\frac{2\pi}{T}kt}. \tag{2}$$

   Plot $x_5(t)$.

   (e) Plot $x_{25}(t), x_{125}(t)$. Can you observe the Gibbs Phenomenon taking place?

   (f) Sigma approximation, also known as Lanczos σ approximation, alleviates the spurious peaks induced by Gibbs Phenomenon. After reading the explanation from Wikipedia, briefly explain what sigma approximation is.

   (g) Let $s_N(t)$ be the $2N+1$ partial sum of Fourier series with sigma approximation,

   $$s_N(t) = \sum_{k=-N}^{N} a_k s_k e^{j\frac{2\pi}{T}kt}, \tag{3}$$

   where s_k is the Lanczos σ factor. Implement sigma approximation with MATLAB, and plot $s_5(t), s_{25}(t), s_{125}(t)$. Does your implementation attenuate the Gibbs Phenomenon?

3. Heat Equation and Fourier Series

In this problem we will investigate the relationship between the heat equation and Fourier Series. Heat equation is a partial differential equation given as follows,

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial^2 x}, \tag{4}$$

where $u(t)$ satisfies the following initial condition for a continuous function $f(x)$,

$$u(x,0) = f(x), \tag{5}$$

$$u(0,t) = u(l,t) = 0. \tag{6}$$

(a) Using separation of variables, we have

$$u(x,t) = X(x)T(t). \tag{7}$$

Express $\partial u/\partial t$, $\partial^2 u/\partial^2 x$ using $X(\cdot), T(\cdot)$.

(b) Using the results from problem (a), derive two ordinary differential equations with respect to $X(x), T(t)$. Also, solve the ordinary differential equations. The two differential equations will yield solutions indexed by a whole number n, namely $X_n(x), T_n(t)$ for $n = 1, 2, 3, \ldots$ Express your answers in the form $X_n(x), T_n(t)$.

(c) Now we can express $u(x,t)$ as follows,

$$u(x,t) = \sum_n a_n X_n(x) T_n(t) \tag{8}$$

Show that for any configuration for a_n, the above equation solves the heat equation *if* the initial conditions are ignored.

(d) What is the appropriate value of a_n that makes $u(x,t)$ to satisfy the initial condition? Can you find any links with Fourier series?