Signals and Systems HW#2

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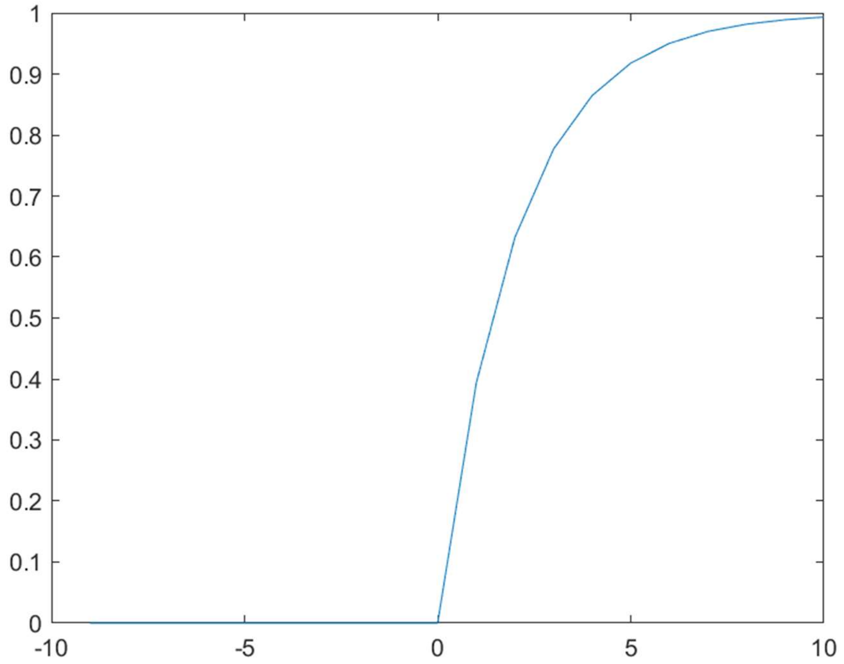
1.

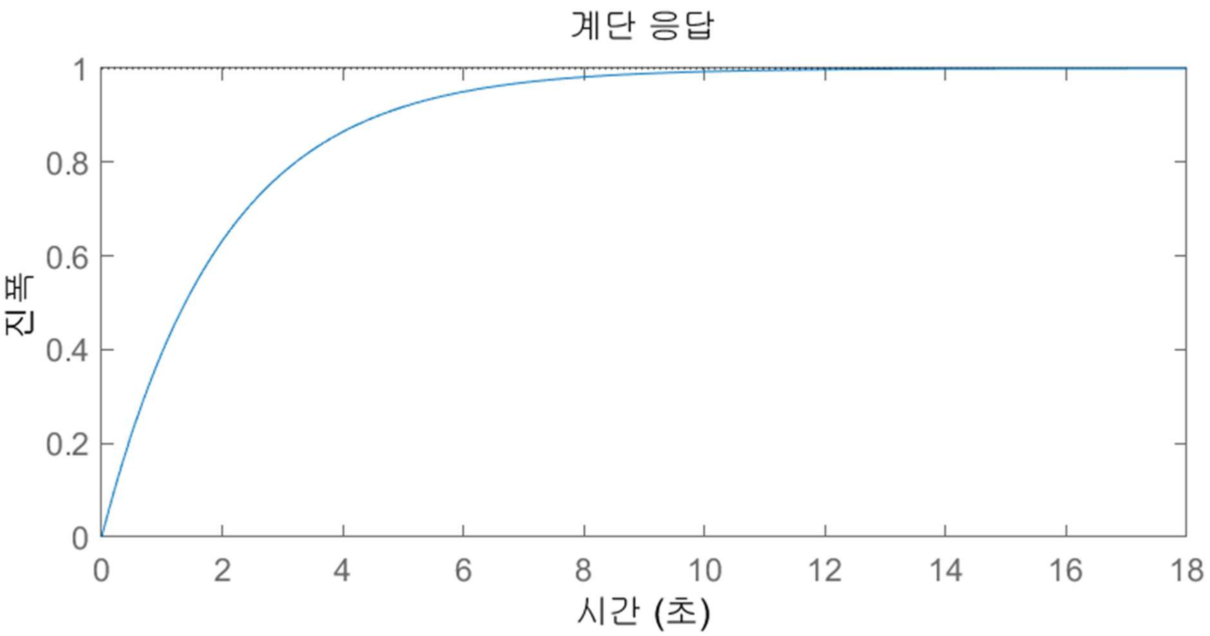
(a)

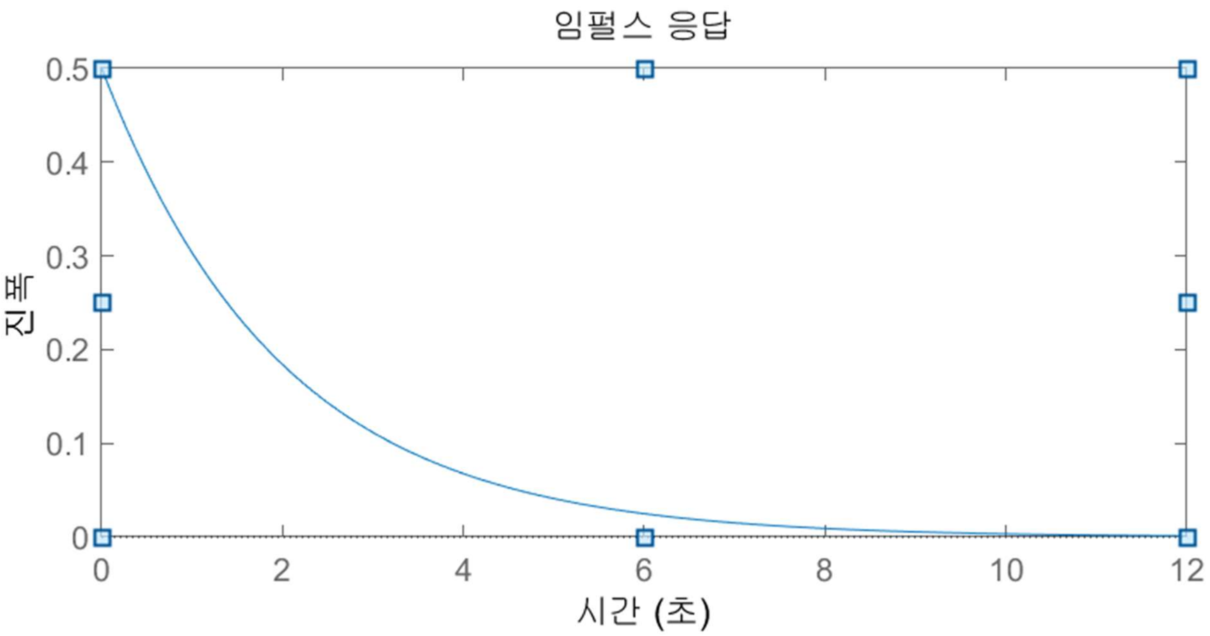
```
%% 1.(a)
clearvars();

a = [2, 1];
b = [0, 1];
t = [-9:10];
x = [zeros(1, length(t)/2-1) ones(1, length(t)/2+1)];
y = lsim(b, a, x, t);

plot(t, y)
```



(b)

```
%% 1.(b)
subplot(2, 1, 1);
step(tf(b, a));

subplot(2, 1, 2);
impulse(tf(b, a));
```





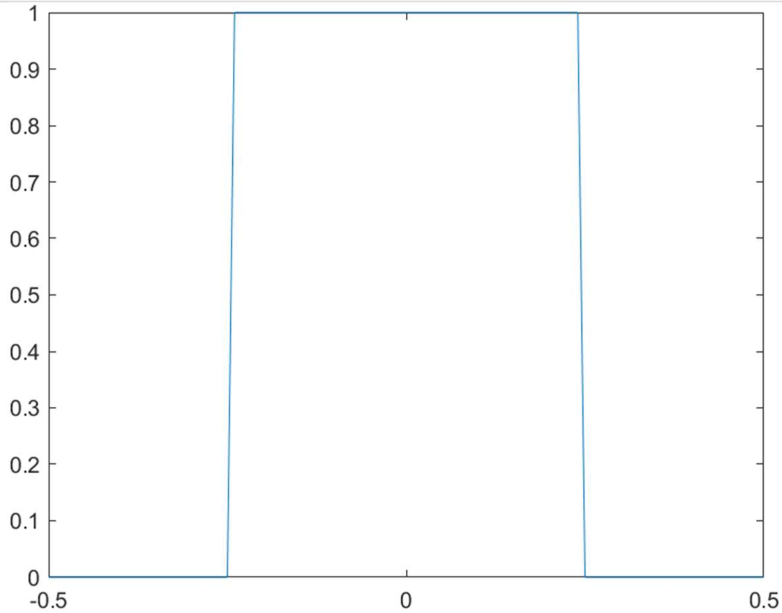
2.

(a)

Gibbs Phenomenon (깁스 현상) 이란, 원래 함수의 푸리에 급수 중 부분합을 구하였을 때 원래 함수의 불연속한 값 근처에서 나타나는 불일치 현상이다. 부분합에 사용되는 항의 수를 늘려도 불일치의 폭은 줄어들지 않고 다만 불연속한 값 근처로 좁혀든다.

(b)

```
%% 2.(b)
clearvars();

T1 = 0.25;
T = 1;

t = -0.5:0.01:0.5;
f = zeros(1, length(t));
for i = 1:length(t)
    if abs(t(i)) < T1
        f(i) = 1;
    end
end

subplot(1, 1, 1);
plot(t, f);
```



(c)

$$a_k = \int_{-1/4}^{1/4} e^{-j2\pi kt}\, dt = \left[-\frac{1}{j2\pi k} e^{-j2\pi kt}\right]_{-1/4}^{1/4}$$

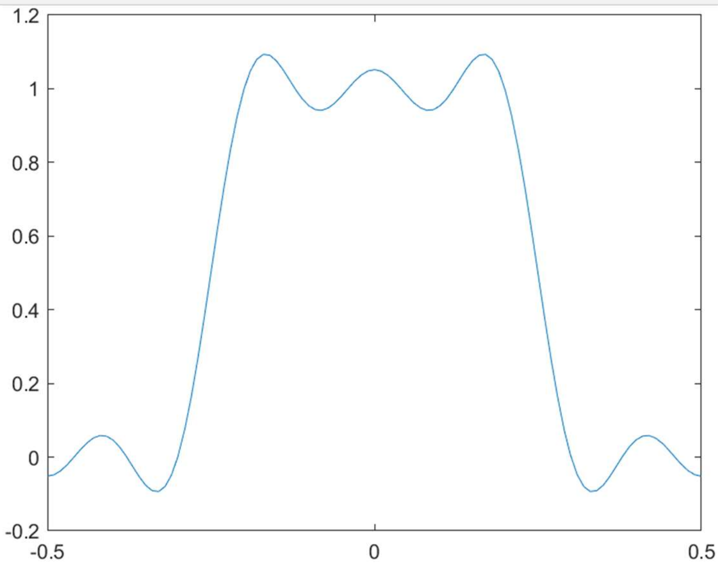
$$= -\frac{1}{j2\pi k}\cdot\left(e^{-j\pi k/2} - e^{j2\pi k/2}\right)$$

$$= \frac{1}{\pi k}\cdot \sin(\pi k/2)$$

$$= \frac{1}{2}\,\mathrm{sinc}(k/2)$$

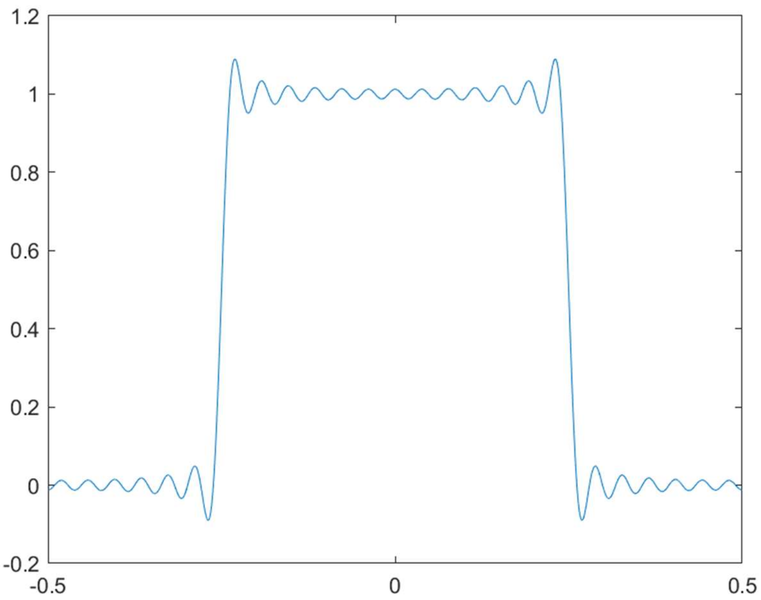
(d)

```
%% 2.(d)
clearvars();

T1 = 0.25;
T = 1;
f0 = 1/T;

t = -0.5:0.01:0.5;
f = zeros(1, length(t));

N = 5;
a = zeros(1, 2*N+1);
for i = 1:length(a)
    a(i) = sinc((i-N-1)/2)/2;
end

for k = -N:N
    f = f + a(k+N+1)*exp(j*2*pi*f0*k*t);
end

plot(t, f)
```

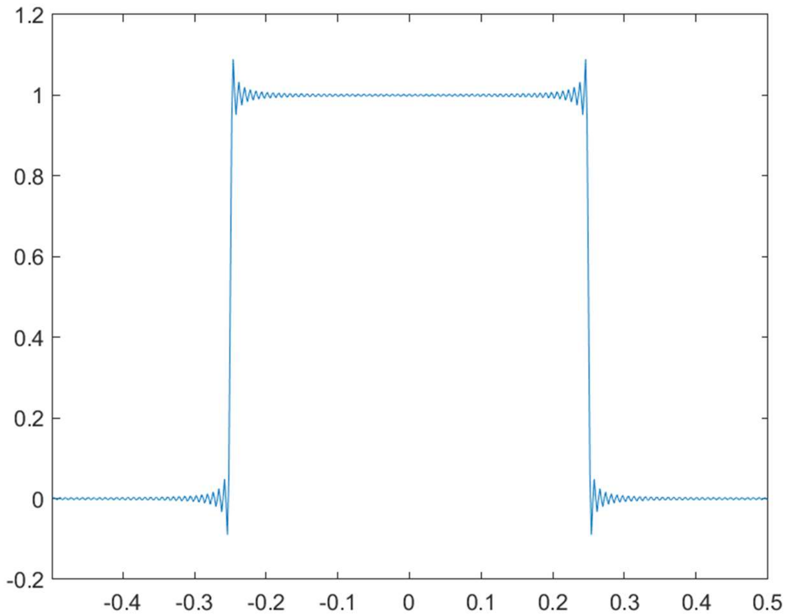


(e)

```
%% 2.(e) - x_25(t)
clearvars();

T1 = 0.25;
T = 1;
f0 = 1/T;

dt = 0.001
t = -0.5:dt:0.5;
f = zeros(1, length(t));

N = 25;
a = zeros(1, 2*N+1);
for i = 1:length(a)
    a(i) = sinc((i-N-1)/2)/2;
end

for k = -N:N
    f = f + a(k+N+1)*exp(j*2*pi*f0*k*t);
end

plot(t, f);
```



```
%% 2.(e) - x_125(t)
dt = 0.002
t = -0.5:dt:0.5;
f = zeros(1, length(t));

N = 125;
a = zeros(1, 2*N+1);
for i = 1:length(a)
    a(i) = sinc((i-N-1)/2)/2;
end

for k = -N:N
    f = f + a(k+N+1)*exp(j*2*pi*f0*k*t);
end

plot(t, f);
```



(f)

Lanczos σ approximation 은 푸리에 급수의 각 항에 sinc$\frac{k}{m}$로 표현되는 Lanczos σ factor 를 곱하는 방법을 말한다. 이 방법은 푸리에 급수의 부분합에서 불연속 값 근처에 발생하는 깁스 현상을 크게 감소시킨다.

3.

(a).

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(X(x)T(t)) = X(x)\frac{\partial}{\partial t}T(t) = X(x)T'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2}(X(x)T(t)) = T(t)\frac{\partial^2}{\partial x^2}X(x) = X''(x)T(t)$$

(b)

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$$ (α is thermal diffusivity, so $\alpha > 0$)

$$X(x)T'(t) = \alpha X''(x)T(t)$$

$$\frac{X''(x)}{X(x)} = \frac{1}{\alpha}\cdot\frac{T'(t)}{T(t)} = K.$$ → 역수로 바꾸고

아래 쓴거도 고쳐라~!!!

$$\begin{cases} X''(x) - KX(x) = 0 & \text{— (1)} \\ T'(t) - \alpha K T(t) = 0 & \text{— (2)} \end{cases}$$

If $K \geq 0$, $X(t) = 0$ is the only solution.

So let $K = -p^2 < 0$, $p > 0$. Then (1) is like

$$X''(x) + p^2 X(x) = 0.$$

Assume $X(x) = A\cos(px) + B\sin(px)$. It satisfies equation above. To satisfy initial condition,

$$u(0,t) = X(0)T(t) = AT(t) = 0. \rightarrow A = 0.$$
$$u(l,t) = X(l)T(t) = (0\cdot\cos(pl) + B\sin(pl))T(t) = 0.$$
$$\rightarrow \sin(pl) = 0.$$

So $pl = \pi n$, $n \in \mathbb{Z}$. Lets express variables as follow:

$$X_n(x) = B_n \sin(p_n x) = B_n \sin\left(\frac{\pi n}{l} x\right).$$

As we exchanged $K = -p^2$, $p = \frac{\pi n}{l}$, equation (2) is like:

$$T(t)' + \alpha p^2 T(t) = 0.$$

$$T(t)' + \frac{\alpha \pi^2 n^2}{l^2} T(t) = 0.$$

Let $T(t) = C \exp\left(-\frac{\alpha \pi^2 n^2}{l^2} t\right)$. It satisfies equation above.

Expressing variables with n:

$$T_n(t) = C_n \exp\left(-\frac{\alpha \pi^2 n^2}{l^2} t\right)$$

(c).

$u_n(x,t) = X_n(x)\,T_n(t)$, $\quad u_m(x,t) = X_m(x)\,T_m(t)$

are the solution for given equation. So the followings hold:

$$X_n(x)\,T'_n(t) = \alpha X_n''(x)\,T_n(t), \qquad \text{—— ①}$$

$$X_m(x)\,T'_m(t) = \alpha X_m''(x)\,T_m(t). \qquad \text{—— ②}$$

Let's check if $a u_n(x,t) + b u_m(x,t)$ is also solution for the equation.

$$\frac{\partial}{\partial t}\left(a u_n(x,t) + b u_m(x,t)\right)$$

$$= \frac{\partial}{\partial t}\left(a X_n(x) T_n(t) + b X_m(x) T_m(t)\right)$$

$$= a X_n(x) T_n'(t) + b X_m(x) T'_m(t). \qquad \text{—— } (*)$$

$$\alpha \frac{\partial^2}{\partial t^2}\left(a u_n(x,t) + b u_m(x,t)\right)$$

$$= \alpha \frac{\partial^2}{\partial t^2}\left(a X_n(x) T_n(t) + b X_m(x) T_m(t)\right)$$

$$= \alpha\left(a X_n(x) T''_n(t) + b X_m(x) T''_m(t)\right)$$

$$= a\cdot \alpha X_n(x) T''_n(t) + b\cdot \alpha X_m(x) T''_m(t). \qquad \text{—— } (**)$$

At $(*)$, by substituting ① & ②,

$$(*) = a\cdot \alpha X_n(x) T_n''(t) + b\cdot \alpha X_m(x) T_m''(t) = (**).$$

As a result, $a u_n(x,t) + b u_m(x,t)$ is a solution for given equation. So this equation satisfies superposition property for its solution.

$u(x,t) = \sum_n a_n X_n(x) T_n(t)$ is a linear combination of solutions for the equation, so $u(x,t)$ solves the equation with arbitrary a_n.

(d).

Let $u(x,t) = \sum_n B_n \sin(\frac{\pi n}{l}x) \cdot C_n \exp(-\frac{\alpha \pi^2 n^2}{l^2}t)$

$$= \sum_{n=1}^{\infty} a_n \sin(\frac{\pi n}{l}x) \exp(-\alpha \frac{\pi^2 n^2}{l^2} t).$$

Initial condition says,

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin(\frac{\pi n}{l}x) = f(x).$$

$$\sum_{n=1}^{\infty} a_n \sin(\frac{\pi n}{l}x) \cdot \sin(\frac{\pi m}{l}x) = f(x) \sin(\frac{\pi m}{l}x)$$

$$\int_0^l \sum_{n=1}^{\infty} a_n \sin(\frac{\pi n}{l}x) \cdot \sin(\frac{\pi m}{l}x)\,dx = \int_0^l f(x) \sin(\frac{\pi m}{l}x)\,dx$$

$$\sum_{n=1}^{\infty} a_n \int_0^l \sin(\frac{\pi n}{l}x) \sin(\frac{\pi m}{l}x)\,dx = \int_0^l f(x) \sin(\frac{\pi m}{l}x)\,dx$$

$$\int_0^l \sin(\frac{\pi n}{l}x) \sin(\frac{\pi m}{l}x)\,dx = \begin{cases} \frac{1}{2}l & \text{if } n=m \\ 0 & \text{else} \end{cases}$$

$$a_m \cdot \frac{l}{2} = \int_0^l f(x) \sin(\frac{\pi m}{l}x)\,dx$$

$$\therefore a_m = \frac{2}{l} \int_0^l f(x) \sin(\frac{\pi m}{l}x)\,dx.$$

구한 a_m의 식은 함수 $f(x)$에 대한 Fourier 사인 급수의 계수이다.