1. Beta - Binomial conjugation:

proof: For convenience, we take coin tossing as an example.

Assume that we toss a coin N times, with m times resulting in heads.

Previously, there have been a heads a times and tails b times.

Denote the probability of tossing heads by θ.

Then $$P(\theta \mid \text{event}) = \frac{P(\text{event} \mid \theta) \cdot P(\theta)}{P(\text{event})}$$

$$(*) \quad = \frac{\cancel{\binom{N}{m}} \cdot \theta^{m} \cdot (1-\theta)^{N-m} \cdot \theta^{a-1} (1-\theta)^{b-1} \cdot \cancel{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}}}{\int_0^1 \cancel{\binom{N}{m}} \cdot \theta^{m} \cdot (1-\theta)^{N-m} \cdot \theta^{a-1} \cdot (1-\theta)^{b-1} \cdot \cancel{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}} \, d\theta}$$

$$\because \int_0^1 \beta(\theta \mid m+a, N-m+b) \, d\theta$$

$$= \int_0^1 \theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1} \cdot \frac{\Gamma(N+a+b)}{\Gamma(m+a)\Gamma(N-m+b)}$$

$$= 1$$

$$\therefore \int_0^1 \theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1} = \frac{\Gamma(m+a)\Gamma(N-m+b)}{\Gamma(N+a+b)}$$

Thus (*) becomes $\theta^{m+a-1} \cdot (1-\theta)^{N-m+b-1} \cdot \frac{\Gamma(N+a+b)}{\Gamma(m+a)\Gamma(N-m+b)}$

$$= \beta(\theta \mid m+a, N-m+b)$$ ※

2. Gamma - Poisson conjugation

Gamma distribution : $f(x|\alpha.\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \cdot x^{\alpha-1} \cdot e^{-\beta x}$

proof: By Bayes Rule, we have $P(\lambda|x) = \frac{P(x|\lambda) \cdot P(\lambda)}{P(x)}$

$$\propto P(x|\lambda) \cdot P(\lambda)$$

↳ x : event

Assume there are n data points.

$$P(x|\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{(x_i)!} = \frac{\lambda^{\sum_{i=1}^{n} x_i} \cdot e^{-n\lambda}}{\prod_{i=1}^{n} (x_i)!}$$

$$\propto \lambda^{\sum_{i=1}^{n} x_i} \cdot e^{-n\lambda}.$$

$$\underbrace{P(\lambda)}_{\text{prior}} \propto \lambda^{\alpha-1} e^{-\beta\lambda}$$

Thus, $P(\lambda|x) \propto \lambda^{\sum_{i=1}^{n} x_i} \cdot e^{-n\lambda} \cdot \lambda^{\alpha-1} \cdot e^{-\beta\lambda}$

$$= \lambda^{n\overset{\text{mean}}{\bar{x}}} \cdot \lambda^{\alpha-1} \cdot e^{-(\beta+n)\lambda}$$

$$= \lambda^{(n\bar{x}+\alpha-1)} \cdot e^{-(\beta+n)\lambda}$$

↳ $\sim \text{gamma}(n\bar{x}+\alpha, \beta+n)$

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