2 WHEELED ROBOT DERIVATION

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1. Symbols

L_x	length of the box
L_y	width of the box
D_w	diameter of the wheel
D_b	distance between the two wheels
l_x	range sensor measurement along x
l_y	range sensor measurement along y
α	MPU angle measurement
r_x	absolute x coordinate of the car
r_y	absolute x coordinate of the car
θ	absolute orientation of the car
ϕ_L	Servo input: angle rotation of the left wheel
ϕ_R	Servo input: angle rotation of the right wheel
θ_{tht}	Top half threshold angle
θ_{thb}	Bottom half threshold angle
θ_{thr}	Bottom right threshold angle
θ_{thl}	Bottom left threshold angle

2. Assumption

For the sake of simplicity, we have the following assumption in the derivation below

(1) θ takes value only (-90, 90)
(2) ϕ_L and phi_R has the same magnitude, opposite sign if turning, otherwise same sign
(3) $\theta = \alpha$ assuming we have the the MPU calibrated at the 0 point

3. Introduction

State

$$x = \begin{bmatrix} r_x \\ r_y \\ \theta \end{bmatrix}$$

Sensor Measurements

$$y = \begin{bmatrix} l_x \\ l_y \\ \alpha \end{bmatrix}$$

Input

$$u = \begin{bmatrix} \phi_L \\ \phi_R \end{bmatrix}$$

4. Sensor Measurement

We want to find a function such that $y = h(x)$
We define the following threshold angles:

$$\theta_{tht} = \arctan\left(\frac{L_y - r_y}{L_x - r_x}\right)$$
$$\theta_{thb} = \arctan\left(\frac{r_y}{r_x}\right)$$
$$\theta_{thr} = \arctan\left(\frac{L_x - r_x}{r_y}\right)$$
$$\theta_{thl} = \arctan\left(\frac{r_x}{r_y}\right)$$

Under our assumption, our expression for l_x changes under these four cases:

$$l_x = \frac{L_x - r_x}{\cos\theta} \quad \theta > 0, \theta < \theta_{ttt}$$
$$l_x = \frac{L_y - r_y}{\cos(90 - \theta)} \quad \theta > 0, \theta > \theta_{ttt}$$
$$l_x = \frac{L_y - r_y}{\cos\theta} \quad \theta < 0, \theta < \theta_{ttb}$$
$$l_x = \frac{r_y}{\cos(90 - \theta)} \quad \theta < 0, \theta > \theta_{ttb}$$

Our expression for l_y changes under these four cases:

$$l_y = \frac{r_y}{\cos\theta} \quad \theta > 0, \theta < \theta_{ttr}$$
$$l_y = \frac{L_x - r_x}{\cos(90 - \theta)} \quad \theta > 0, \theta > \theta_{ttr}$$
$$l_y = \frac{r_y}{\cos\theta} \quad \theta < 0, \theta < \theta_{ttl}$$
$$l_y = \frac{r_x}{\cos(90 - \theta)} \quad \theta < 0, \theta > \theta_{ttl}$$

Our expression for $\alpha = \theta$ doesn't change under our assumption.

To summarize, we have total of 8 cases summarized below,

$\theta > 0, \theta < \theta_{ttt}, \theta < \theta_{ttr}$

$$y = \begin{bmatrix} \frac{L_x - r_x}{\cos\theta} \\ \frac{r_y}{\cos\theta} \\ \alpha = \theta \end{bmatrix}$$

$\theta > 0, \theta < \theta_{ttt}, \theta > \theta_{ttr}$

$$y = \begin{bmatrix} \frac{L_x - r_x}{\cos\theta} \\ \frac{L_x - r_x}{\cos(90 - \theta)} \\ \alpha = \theta \end{bmatrix}$$

$\theta > 0, \theta > \theta_{ttt}, \theta < \theta_{ttr}$

$$y = \begin{bmatrix} \frac{L_y - r_y}{\cos(90 - \theta)} \\ \frac{r_y}{\cos\theta} \\ \alpha = \theta \end{bmatrix}$$

$\theta > 0, \theta > \theta_{ttt}, \theta > \theta_{ttr}$

$$y = \begin{bmatrix} \frac{L_y - r_y}{\cos(90 - \theta)} \\ \frac{L_x - r_x}{\cos(90 - \theta)} \\ \alpha = \theta \end{bmatrix}$$

$\theta < 0, \theta < \theta_{ttb}, \theta < \theta_{ttl}$

$$y = \begin{bmatrix} \frac{L_y - r_y}{\cos\theta} \\ \frac{r_y}{\cos\theta} \\ \alpha = \theta \end{bmatrix}$$

$\theta < 0, \theta < \theta_{ttb}, \theta > \theta_{ttl}$

$$y = \begin{bmatrix} \frac{L_x - r_x}{\cos\theta} \\ \frac{r_x}{\cos(90 - \theta)} \\ \alpha = \theta \end{bmatrix}$$

$\theta < 0, \theta > \theta_{ttb}, \theta < \theta_{ttl}$

$$y = \begin{bmatrix} \frac{r_y}{\cos(90 - \theta)} \\ \frac{r_y}{\cos\theta} \\ \alpha = \theta \end{bmatrix}$$

$\theta < 0, \theta > \theta_{ttb}, \theta > \theta_{ttl}$

$$y = \begin{bmatrix} \frac{r_y}{\cos(90 - \theta)} \\ \frac{r_x}{\cos(90 - \theta)} \\ \alpha = \theta \end{bmatrix}$$

We need to linearize all these cases such that, $y = Hu$, where H is a matrix. Here we will demonstrate how to d linearization for one case, $\theta > 0, \theta < \theta_{ttt}, \theta < \theta_{ttr}$,

$$l_x = \frac{L_x - r_x}{\cos\theta}$$
$$l_y = \frac{r_y}{\cos\theta}$$

Linearization around $l_{x0}, l_{y0}, \alpha_0 = h(r_{x0}, r_{y0}, \theta_0)$

$$l_x = l_{x0} + \frac{-1}{\cos\theta_0}(r_x - r_{x0}) + \frac{(L_x - r_{x0})\sin(\theta_0)}{(\cos\theta_0)^2}(\theta - \theta_0)$$
$$l_y = l_{y0} + \frac{1}{\cos\theta}(r_y - r_{y0}) + \frac{r_y\sin(\theta_0)}{(\cos\theta_0)^2}(\theta - \theta_0)$$
$$l_x = \frac{-1}{\cos\theta_0}r_x + \frac{(L_x - r_{x0})\sin(\theta_0)}{(\cos\theta_0)^2}\theta + [\frac{r_{x0}}{\cos\theta_0} - \frac{(L_x - r_{x0})\sin(\theta_0)}{(\cos\theta_0)^2}\theta_0 + l_{x0}]$$
$$l_y = \frac{1}{\cos\theta_0}r_y + \frac{r_y\sin(\theta_0)}{(\cos\theta_0)^2}\theta + [\frac{-r_{y0}}{\cos\theta_0} - \frac{r_y\sin(\theta_0)}{(\cos\theta_0)^2}\theta_0 + l_{y0}]$$

In matrix form, the complete form for this case is

$$y = \begin{bmatrix} \frac{-1}{\cos\theta_0} & 0 & \frac{(L_x - r_{x0})\sin(\theta_0)}{(\cos\theta_0)^2} \\ 0 & \frac{1}{\cos\theta_0} & \frac{r_y\sin(\theta_0)}{(\cos\theta_0)^2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ \theta \end{bmatrix} + \begin{bmatrix} \frac{r_{x0}}{\cos\theta_0} - \frac{(L_x - r_{x0})\sin(\theta_0)}{(\cos\theta_0)^2}\theta_0 + l_{x0} \\ \frac{-r_{y0}}{\cos\theta_0} - \frac{r_y\sin(\theta_0)}{(\cos\theta_0)^2}\theta_0 + l_{y0} \\ 0 \end{bmatrix}$$