2 WHEELED ROBOT DERIVATION

KIN CHANG

1. Symbols

L_x	length of the box
L_y	width of the box
W	distance between rotating center and sensors center
C_v	coefficient, distance traveled by the wheel per unit of delay time
C_r	coefficient, body orientation changed by the wheel per unit of delay time
l_x	range sensor measurement along x
l_y	range sensor measurement along y
α	MPU angle measurement
r_x	absolute x coordinate of the car
r_y	absolute y coordinate of the car
θ	absolute orientation of the car
τ_L	Servo input: delay time of the left wheel
τ_R	Servo input: delay time of the right wheel
θ_{tht}	top half threshold angle
θ_{thb}	bottom half threshold angle
θ_{thr}	bottom right threshold angle
θ_{thl}	bottom left threshold angle
ϵ_{str}	picking variable, 1 if going straight line, 0 otherwise
w_t	process noise
v_t	measurement noise

2. Assumption

For the sake of simplicity, we have the following assumption in the derivation below

(1) θ takes value only (-90, 90)
(2) τ_L and τ_R has the same magnitude, opposite sign if turning, otherwise same sign
(3) $\theta = \alpha$ assuming we have the the MPU calibrated at the 0 point

3. Introduction

State

$$x = \begin{bmatrix} r_x \\ r_y \\ \theta \end{bmatrix}$$

Sensor Measurements

$$y = \begin{bmatrix} l_x \\ l_y \\ \alpha \end{bmatrix}$$

Input

$$u = \begin{bmatrix} \tau_L \\ \tau_R \end{bmatrix}$$

4. Sensor Measurement

We want to find a function such that $y = h(x)$
We define the following threshold angles:

$$\theta_{tht} = \tan^{-1}(\frac{L_y - r_y}{L_x - r_x})$$
$$\theta_{thb} = \tan^{-1}(\frac{r_y}{L_x - r_x})$$
$$\theta_{thr} = \tan^{-1}(\frac{L_x - r_x}{r_y})$$
$$\theta_{thl} = \tan^{-1}(\frac{r_x}{r_y})$$

Under our assumption, our expression for l_x changes under these four cases:

$$l_x = \frac{L_x - r_x}{\cos|\theta|} \quad \theta > 0, |\theta| < \theta_{ttt}$$
$$l_x = \frac{L_y - r_y}{\sin|\theta|} \quad \theta > 0, |\theta| > \theta_{ttt}$$
$$l_x = \frac{L_x - r_x}{\cos|\theta|} \quad \theta < 0, |\theta| < \theta_{ttb}$$
$$l_x = \frac{r_y}{\sin|\theta|} \quad \theta < 0, |\theta| > \theta_{ttb}$$

Our expression for l_y changes under these four cases:

$$l_y = \frac{r_y}{\cos|\theta|} \quad \theta > 0, |\theta| < \theta_{ttr}$$
$$l_y = \frac{L_x - r_x}{\sin|\theta|} \quad \theta > 0, |\theta| > \theta_{ttr}$$
$$l_y = \frac{r_y}{\cos|\theta|} \quad \theta < 0, |\theta| < \theta_{ttl}$$
$$l_y = \frac{r_x}{\sin|\theta|} \quad \theta < 0, |\theta| > \theta_{ttl}$$

Our expression for $\alpha = \theta$ doesn't change under our assumption.

To summarize, we have total of 8 cases summarized below,

$\theta \geq 0, |\theta| < \theta_{ttt}, |\theta| < \theta_{ttr}$

$$y = \begin{bmatrix} \frac{L_x - r_x}{\cos|\theta|} \\ \frac{r_y}{\cos|\theta|} \\ \alpha = \theta \end{bmatrix}$$

$\theta > 0, |\theta| < \theta_{ttt}, |\theta| > \theta_{ttr}$

$$y = \begin{bmatrix} \frac{L_x - r_x}{\cos|\theta|} \\ \frac{L_x - r_x}{\sin|\theta|} \\ \alpha = \theta \end{bmatrix}$$

$\theta > 0, |\theta| > \theta_{ttt}, |\theta| < \theta_{ttr}$

$$y = \begin{bmatrix} \frac{L_y - r_y}{\sin|\theta|} \\ \frac{r_y}{\cos|\theta|} \\ \alpha = \theta \end{bmatrix}$$

$\theta > 0, |\theta| > \theta_{ttt}, |\theta| > \theta_{ttr}$

$$y = \begin{bmatrix} \frac{L_y - r_y}{\sin|\theta|} \\ \frac{L_x - r_x}{\sin|\theta|} \\ \alpha = \theta \end{bmatrix}$$

$\theta < 0, |\theta| < \theta_{ttb}, |\theta| < \theta_{ttl}$

$$y = \begin{bmatrix} \frac{L_x - r_x}{\cos|\theta|} \\ \frac{r_y}{\cos|\theta|} \\ \alpha = \theta \end{bmatrix}$$

$\theta < 0, |\theta| < \theta_{ttb}, |\theta| > \theta_{ttl}$

$$y = \begin{bmatrix} \frac{L_x - r_x}{\cos|\theta|} \\ \frac{r_x}{\sin|\theta|} \\ \alpha = \theta \end{bmatrix}$$

$\theta < 0, |\theta| > \theta_{ttb}, |\theta| < \theta_{ttl}$

$$y = \begin{bmatrix} \frac{r_y}{\sin|\theta|} \\ \frac{r_y}{\cos|\theta|} \\ \alpha = \theta \end{bmatrix}$$

$\theta < 0, |\theta| > \theta_{ttb}, |\theta| > \theta_{ttl}$

$$y = \begin{bmatrix} \frac{r_y}{\sin|\theta|} \\ \frac{r_x}{\sin|\theta|} \\ \alpha = \theta \end{bmatrix}$$

We need to linearize all these cases such that, $y = Hx$, where H is a matrix. Here we will demonstrate how to do linearization for one case, $\theta > 0, |\theta| < \theta_{ttt}, |\theta| < \theta_{ttr}$,

$$l_x = \frac{L_x - r_x}{\cos\theta}$$
$$l_y = \frac{r_y}{\cos\theta}$$

Linearization around $l_{x0}, l_{y0}, \alpha_0 = h(r_{x0}, r_{y0}, \theta_0)$

$$l_x = l_{x0} + \frac{-1}{\cos\theta_0}(r_x - r_{x0}) + \frac{(L_x - r_{x0})\sin(\theta_0)}{(\cos\theta_0)^2}(\theta - \theta_0)$$
$$l_y = l_{y0} + \frac{1}{\cos\theta_0}(r_y - r_{y0}) + \frac{r_{y0}\sin(\theta_0)}{(\cos\theta_0)^2}(\theta - \theta_0)$$
$$l_x = \frac{-1}{\cos\theta_0}r_x + \frac{(L_x - r_{x0})\sin(\theta_0)}{(\cos\theta_0)^2}\theta + [\frac{r_{x0}}{\cos\theta_0} - \frac{(L_x - r_{x0})\sin(\theta_0)}{(\cos\theta_0)^2}\theta_0 + l_{x0}]$$
$$l_y = \frac{1}{\cos\theta_0}r_y + \frac{r_{y0}\sin(\theta_0)}{(\cos\theta_0)^2}\theta + [\frac{-r_{y0}}{\cos\theta_0} - \frac{r_{y0}\sin(\theta_0)}{(\cos\theta_0)^2}\theta_0 + l_{y0}]$$

In matrix form, the complete form for this case is

$$y = \begin{bmatrix} \frac{-1}{\cos\theta_0} & 0 & \frac{(L_x - r_{x0})\sin(\theta_0)}{(\cos\theta_0)^2} \\ 0 & \frac{1}{\cos\theta_0} & \frac{r_{y0}\sin(\theta_0)}{(\cos\theta_0)^2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ \theta \end{bmatrix} + \begin{bmatrix} \frac{r_{x0}}{\cos\theta_0} - \frac{(L_x - r_{x0})\sin(\theta_0)}{(\cos\theta_0)^2}\theta_0 + l_{x0} \\ \frac{-r_{y0}}{\cos\theta_0} - \frac{r_{y0}\sin(\theta_0)}{(\cos\theta_0)^2}\theta_0 + l_{y0} \\ 0 \end{bmatrix}$$

4.1. **Complete Case Analysis.**

$\theta \geq 0, |\theta| < \theta_{ttt}, |\theta| < \theta_{ttr}$

$$H = \begin{bmatrix} \frac{-1}{\cos\theta_0} & 0 & \frac{(L_x - r_{x0})\sin(\theta_0)}{(\cos\theta_0)^2} \\ 0 & \frac{1}{\cos\theta_0} & \frac{r_{y0}\sin(\theta_0)}{(\cos\theta_0)^2} \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} \frac{r_{x0}}{\cos\theta_0} - \frac{(L_x - r_{x0})\sin(\theta_0)}{(\cos\theta_0)^2}\theta_0 + l_{x0} \\ \frac{-r_{y0}}{\cos\theta_0} - \frac{r_{y0}\sin(\theta_0)}{(\cos\theta_0)^2}\theta_0 + l_{y0} \\ 0 \end{bmatrix}$$

$\theta \geq 0, |\theta| < \theta_{ttt}, |\theta| > \theta_{ttr}$

$$H = \begin{bmatrix} \frac{-1}{\cos\theta_0} & 0 & \frac{(L_x - r_{x0})\sin(\theta_0)}{(\cos\theta_0)^2} \\ \frac{-1}{\sin\theta_0} & 0 & -\frac{L_x - r_{x0}\cos(\theta_0)}{(\sin\theta_0)^2} \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} \frac{r_{x0}}{\cos\theta_0} - \frac{(L_x - r_{x0})\sin(\theta_0)}{(\cos\theta_0)^2}\theta_0 + l_{x0} \\ \frac{r_{x0}}{\sin\theta_0} + \frac{(L_x - r_{x0})\cos(\theta_0)}{(\sin\theta_0)^2}\theta_0 + l_{y0} \\ 0 \end{bmatrix}$$

$\theta > 0, |\theta| > \theta_{ttt}, |\theta| < \theta_{ttr}$

$$H = \begin{bmatrix} 0 & \frac{-1}{\sin\theta_0} & -\frac{L_y - r_{y0}\cos(\theta_0)}{(\sin\theta_0)^2} \\ 0 & \frac{1}{\cos\theta_0} & \frac{r_{y0}\sin(\theta_0)}{(\cos\theta_0)^2} \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} \frac{r_{y0}}{\sin\theta_0} + \frac{(L_y - r_{y0})\cos(\theta_0)}{(\sin\theta_0)^2}\theta_0 + l_{x0} \\ \frac{-r_{y0}}{\cos\theta_0} - \frac{r_{y0}\sin(\theta_0)}{(\cos\theta_0)^2}\theta_0 + l_{y0} \\ 0 \end{bmatrix}$$

$\theta > 0, |\theta| > \theta_{ttt}, |\theta| > \theta_{ttr}$

$$H = \begin{bmatrix} 0 & \frac{-1}{\sin\theta_0} & -\frac{L_y - r_{y0}\cos(\theta_0)}{(\sin\theta_0)^2} \\ \frac{-1}{\sin\theta_0} & 0 & -\frac{L_x - r_{x0}\cos(\theta_0)}{(\sin\theta_0)^2} \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} \frac{r_{y0}}{\sin\theta_0} + \frac{(L_y - r_{y0})\cos(\theta_0)}{(\sin\theta_0)^2}\theta_0 + l_{x0} \\ \frac{r_{x0}}{\sin\theta_0} + \frac{(L_x - r_{x0})\cos(\theta_0)}{(\sin\theta_0)^2}\theta_0 + l_{y0} \\ 0 \end{bmatrix}$$

$\theta < 0, |\theta| < \theta_{ttb}, |\theta| < \theta_{ttl}$

$\theta < 0, |\theta| < \theta_{ttb}, |\theta| > \theta_{ttl}$

$\theta < 0, |\theta| > \theta_{ttb}, |\theta| < \theta_{ttl}$

$\theta < 0, |\theta| > \theta_{ttb}, |\theta| > \theta_{ttl}$

5. State Evolution

We aim to find a function such that $x_{t+1} = f(x_t, u_t)$.
First of all, 2 cases

Case 1: the car is going in a straight line (either backward or forward): $(\tau_{L,t} * \tau_{R,t}) > 0$

$$\begin{aligned} r_{x,t+1} &= r_{x,t} + C_v * \tau_{R,t}\cos\theta \\ r_{y,t+1} &= r_{y,t} + C_v * \tau_{R,t}\sin\theta \\ \theta_{t+1} &= \theta_t \end{aligned}$$

Case 2: the car is turning (either left or right): $(\tau_{L,t} * \tau_{R,t}) > 0$

$$\begin{aligned} r_{x,t+1} &= r_{x,t} - W[1 - \cos(C_r\tau_{R,t})] \\ r_{y,t+1} &= r_{y,t} + W[\sin(C_r\tau_{R,t})] \\ \theta_{t+1} &= \theta_t + C_r\tau_{R,t} \end{aligned}$$

noticed that all the signs worked out in the above expressions either going forward or backward, or turning left or right

To combine these two cases into one expression, we define a picking variable $\epsilon_{str} = \tau_{L,t} * \tau_{R,t} > 0)$ taking value 0 or 1. Then,

$$\begin{aligned} r_{x,t+1} &= r_{x,t} + \epsilon_{str} * C_v \tau_{R,t} \cos\theta - (1-\epsilon_{str}) * W[1 - \cos(C_r \tau_{R,t})] \\ r_{y,t+1} &= r_{y,t} + \epsilon_{str} * C_v \tau_{R,t} \sin\theta + (1-\epsilon_{str}) * W[\sin(C_r \tau_{R,t})] \\ \theta_{t+1} &= \theta_t + (1-\epsilon_{str}) * C_r \tau_{R,t} \end{aligned}$$

In matrix form,

$$\begin{aligned} x_{t+1} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{x,t} \\ r_{y,t} \\ \theta_t \end{bmatrix} + \begin{bmatrix} 0 & \epsilon_{str} C_v \cos\theta_t \\ 0 & \epsilon_{str} C_v \sin\theta_t \\ 0 & (1-\epsilon_{str}) C_r \end{bmatrix} \begin{bmatrix} \tau_{L,t} \\ \tau_{R,t} \end{bmatrix} \\ x_{t+1} &= A x_t + B_t u_t \end{aligned}$$

6. Kalmen Filter Procedure

6.1. Kalman Gain Update.

(1) Initial Error Covariance $P_{1|0} = P_0, t = 1$
(2) Compute Gain: $K_t = P_{t|t-1} H_t^T [H_t^T P_{t|t-1} H_t^T + R_t]^{-1}$
(3) Update error covariance
    (a) $P_t = (I - K_t H_t) P_{t|t-1}$
    (b) $P_{t+1|t} = A_t P A_t^T + G_t Q_t G_t^T$
(4) t = t+1
(5) Go back to (2) until stop condition

where

$$\begin{aligned} R_t &= E[v_t v_t^T] \\ Q_t &= E[w_t w_t^T] \end{aligned}$$

6.2. State Estimation.

(1) Initialize estimated state $\hat{x}_0$
(2) Collect new measurement y_t
(3) Update State Estimate with new measurement, with updated Kalman gain from above
    (a) $\hat{x}_{t|t-1} = A_{t-1} \hat{x}_{t-1}$
    (b) $\hat{x}_t = \hat{x}_{t|t-1} + K_t (y_t - H_t \hat{x}_{t|t-1})$
(4) t = t+1
(5) Go back to (2) until stop condition