Problem 1.

In the last assignment, you developed a code for solving the elliptic PDE

$$\nabla \cdot \Gamma \nabla \phi = 0 \tag{1}$$

In this assignment, you'll add another term to Eq. (1) to represent the advective transport of the scalar quantity ϕ. This equation will take the form

$$\nabla \cdot \vec{u}\phi = \nabla \cdot \Gamma \nabla \phi \tag{2}$$

where $\vec{u} = \vec{u}(x, y)$ is a prescribed velocity field on the domain Ω. Note the similarity of the term $\nabla \cdot \vec{u}\phi$ in Eq. (2) to the underlined advective term in the Navier-Stokes momentum equation:

$$\frac{\partial \rho \vec{u}}{\partial t} + \underline{\nabla \cdot \rho \vec{u}\vec{u}} = \mu \nabla^2 \vec{u} - \nabla p \tag{3}$$

For problem 1, develop a code (building on your previous assignment) that can solve a discretized form of Eq. (2). The code should be able to switch between two different advection schemes:

1. Central difference
2. 1st order upwind

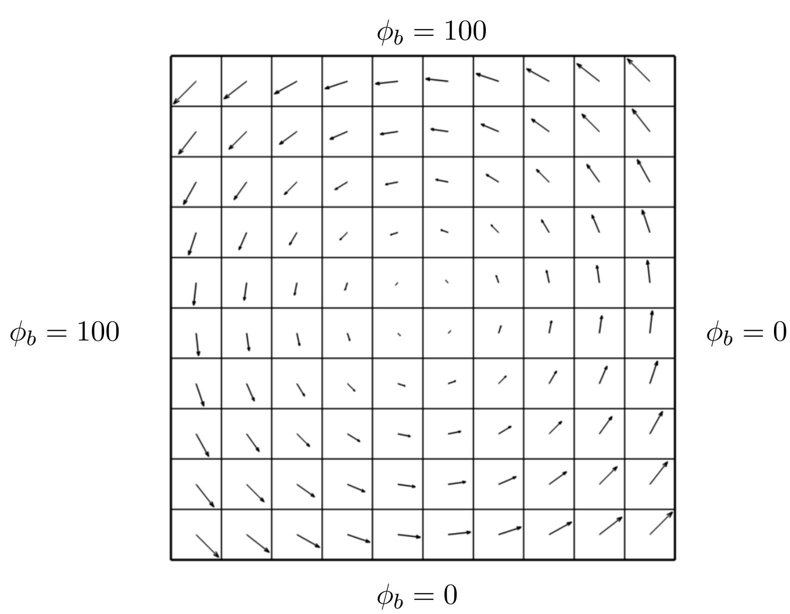

Figure 1: Schematic of the discretized domain, with the rotational velocity field.

The following parameters will be used:

- $L_x = 1$
- $L_y = 1$
- $\Gamma = 0$, $\Gamma = 5$

- $\vec{u}_x = 2$, $\vec{u}_x = -r\sin\theta$
- $\vec{u}_y = 2$, $\vec{u}_y = r\cos\theta$
- $\phi = \phi_b \; \forall \; x, y \in \partial\Omega$, e.g. all boundaries are fixed (Dirichlet)

Note that $r = \sqrt{(x - L_x/2)^2 + (y - L_y/2)^2}$ and $\theta = \text{arctan2}\,\frac{y-L_y/2}{x-L_x/2}$, assuming you have defined your domain on the interval $x, y \in [0, 1]$.

For the report, please include the following:

1. Describe the discretization for Eq. (2). For the discretization of the diffusion term, you can simply refer to your previous report. You should include a description for the central difference scheme and the 1st order upwind scheme (1 point).
2. With $\Gamma = 0$, test the central difference and upwind schemes for $\vec{u}_x = 2$ and $\vec{u}_y = 2$. Provide detail comparison similar to Fig. 5.15 in your text book. Are there any stability issues? If so, explain why (3 points).
3. Repeat the previous step with $\Gamma = 5$. If there were stability issues it the previous step, did they persist? Explain (2 points).
4. Create contour plots for a rotational velocity field with $\vec{u}_x = -r\sin\theta$ and $\vec{u}_y = r\cos\theta$ and $\Gamma = 5$. Repeat the order of convergence test from assignment 2 (only for a uniform mesh), and compare the results for central difference difference and first order upwind. Comment on your findings. Use grids with cell resolutions of 80×80, 160×160 and 320×320 (use finer if you wish). Remember from Assignment #2: the vertices of the fine grid cells will sit on the cell centres of the coarser grids, as per Fig. 2. As such, suitable interpolation method should be used. Note that bilinear interpolation is second order, so it shouldn't affect the order of convergence. Other methods of interpolation are fine as well (4 points).

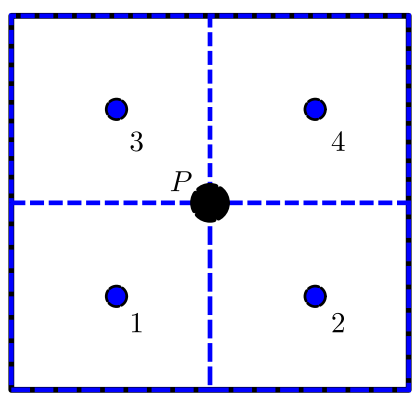

Figure 2: Fine cells (dashed blue line) overlayed on a coarse cell (solid black). The cell centres of the fine mesh should be interpolated to the cell centre of the coarse mesh using a suitable interpolation method such as bilinear interpolation should be used.

Bonus

Implement your choice of either the QUICK scheme for a piecewise linear reconstruction with a suitable slope limiter (e.g. Van Leer). Estimate the order of convergence. How does it compare with the upwind scheme? (+1.5 points).