title: "Homework 3" author: "Joonwon Lee" date: "2023-10-05" output: pdf_document —

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Problem 10.

(a) To remove a constant term 0.3 in (i),(ii), and (iii), define a new process $\{X_t : X_t = r_t - 1/3\}_{t \geq 1}$.

AR(3) Assuming $\{X_t\}$ is a mean zero process, for AR(3),

$\gamma(0) - 0.8\gamma(1) + 0.5\gamma(2) + 0.2\gamma(3) = \sigma^2.$
$\gamma(1) - 0.8\gamma(0) + 0.5\gamma(1) + 0.2\gamma(2) = 0.$
$\gamma(2) - 0.8\gamma(1) + 0.5\gamma(0) + 0.2\gamma(1) = 0.$
$\gamma(3) - 0.8\gamma(2) + 0.5\gamma(1) + 0.2\gamma(0) = 0.$

Solving linear equations above gives ACFs

```
fun1 = function(x1,x2,x3){
  out = 0.8*x1 -0.5*x2 -0.2*x3
  return(out)
}

mat_ar3= matrix(c(1,-0.8,0.5,0.2, -0.8, 1.5, 0.2,0, 0.5,-0.6,1,0, 0.2,0.5,-0.8,1 )  ,4,4,byrow=TRUE)
gamma0_3 = (solve(mat_ar3)%*%c(4,0,0,0))
# cat("acf0 to acf3\n: ", gamma0_3/gamma0_3[1])
cat("acf0 to acf3 : ", gamma0_3/gamma0_3[1], "\n")
```

```
## acf0 to acf3 :  1 0.5555556 -0.1666667 -0.6111111
```

```
gamma4to12 = rep(0,9)
cur_list = gamma0_3[4:2]
for (i in 1:9){
  gamma4to12[i] = fun1(cur_list[1],cur_list[2],cur_list[3])
  a=sprintf("acf%d is %f", i+3, gamma4to12[i]/gamma0_3[1])
  print(a)
  cur_list = c(gamma4to12[i], cur_list[1],cur_list[2])
}
```

```
## [1] "acf4 is -0.516667"
## [1] "acf5 is -0.074444"
## [1] "acf6 is 0.321000"
## [1] "acf7 is 0.397356"
## [1] "acf8 is 0.172273"
## [1] "acf9 is -0.125059"
## [1] "acf10 is -0.265655"
## [1] "acf11 is -0.184449"
## [1] "acf12 is 0.010280"
```

MA(3) Now let's look at MA(3). By putting $\sigma^2 = 4$,

$\gamma(0) = 4 + 0.8^2 * 4 + 0.5^2 * 4 + 0.2^2 * 4 = 7.72$
$\gamma(1) = 0.8 * 4 - 0.5 * 0.8 * 4 + (-0.2) * (-0.5) * 4 = 2$
$\gamma(2) = -0.5 * 4 + (-0.2) * 0.8 * 4 = -2.64$
$\gamma(3) = -0.2 * 4 = -0.8 \quad \gamma(k) = 0,\ k \geq 4.$

Which gives acfs:

acf(0)= 1, acf(1) = 0.259, acf(2) = -0.342, acf(3) = -0.104

ARMA(3,2) Lastly ARMA(3,2). ϕ, θ have no common zeros, and $\phi(z) \neq 0$ for all $|z| < 1$, hence by the Theorem 3.11.a,
$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$, and $\psi(z)\phi(z) = \theta(z)$.
This implies that $\gamma(h) = \sum_{j=0}^{\infty} \psi_{j+h}\psi_j\sigma^2$.

Also it can be seen that
$\psi_j - \sum_{0 \leq k \leq j} \phi_k \psi_{j-k} = \theta_k,\ 0 \leq j < max(p, q+1)$
$\psi_j - \sum_{0 \leq k \leq p} \phi_k \psi_{j-k} = \theta_k,\ j \geq max(p, q+1)$.

$\psi_0 = \theta_0 = 1$
$\psi_1 = \theta_1 + \phi_1 = 1.3$
$\psi_2 = \theta_2 + \psi_0\phi_2 + \psi_1\phi_1 = 0.84$
$\psi_n = \phi_1\psi_{n-1} + \phi_2\psi_{n-2} + \phi_3\psi_{n-3},\ n \geq max(3, 2+1)$.

We can write the general formula as
$\psi_n = \sum_{i=1}^{k} \sum_{j=0}^{r_i-1} \alpha_{ij} n^j \xi_i^{-n},\ n \geq max(p, q+1) - p$ where ξ_i are distinct zeros of $\phi(z)$. We can check the roots are $0.68 + 0.91i, 0.68 - 0.91i, -3.87$. r_i is the multiplicity of ξ_i. Then we can rewrite the formula as

$\psi_n = \sum_{i=1}^{3} \alpha_{i0}\xi_i^{-n}$. Using the known facts that $\psi_0 = 1, \psi_1 = 1.3, \psi_2 = 0.84, \xi_1 = 0.68 + 0.91i, \xi_2 = 0.68 - 0.91i, \xi_3 = -3.87$, we solve linear equations to get
$c(\alpha_{10}, \alpha_{20}, \alpha_{30}) = c(0.393 + 0.668i, 0.393 - 0.668i, 0.213)$.

```
library(rootSolve)
# function(x) {1-0.8*x+0.5*x^2+0.2*x^3}
roots = polyroot(c(1, -0.8,0.5,0.2))
a = roots[1]^{-1}; b=roots[2]^{-1}; c= roots[3]^{-1}
mat = matrix(c(1,1,1, a,b,c,a^2,b^2,c^2 ),3,3, byrow=TRUE)
a_list = solve(mat)%*% c(1,1.3,0.84)
a_list = round(a_list,3)
cat("a10: ", a_list[1],
    "\na20: ", a_list[2],
    "\na30: ", a_list[3])
```

```
## a10:  0.393+0.668i
## a20:  0.393-0.668i
## a30:  0.213+0i
```

Now, we compute $\psi_n\psi_{n+h}$ and get
$\gamma(h) = 4\sum_{n=0}^{\infty} \psi_n\psi_{n+h}$.

```r
a10 = a_list[1]
a20 = a_list[2]
a30 = a_list[3]
a = roots[1]; b= roots[2]; c= roots[3]
gamma_k = function(h){
  sigma = 4
  g1 = a10^2/(1-a^{-2})*a^{-h}
  g2 = a10*a20/(1-(a*b)^{-1})*b^{-h}
  g3 = a10*a30/(1-(a*c)^{-1})*c^{-h}
  g4 = a20^2/(1-b^{-2})*b^{-h}
  g5 = a20*a10/(1-(a*b)^{-1})*a^{-h}
  g6 = a20*a30/(1-(b*c)^{-1})*c^{-h}
  g7 = a30^2/(1-c^{-2})*c^{-h}
  g8 = a30*a10/(1-(a*c)^{-1})*a^{-h}
  g9 = a30*a20/(1-(b*c)^{-1})*b^{-h}
  out = sigma*(g1+g2+g3+g4+g5+g6+g7+g8+g9)
  return(Re(out))
}
for( i in 1:13){
  result=sprintf("acf%d is %0.3f", (i-1) , gamma_k((i-1))/gamma_k(0))
  print(result)
}
```

```
## [1] "acf0 is 1.000"
## [1] "acf1 is 0.647"
## [1] "acf2 is -0.058"
## [1] "acf3 is -0.570"
## [1] "acf4 is -0.556"
## [1] "acf5 is -0.148"
## [1] "acf6 is 0.274"
## [1] "acf7 is 0.404"
## [1] "acf8 is 0.216"
## [1] "acf9 is -0.084"
## [1] "acf10 is -0.256"
## [1] "acf11 is -0.206"
## [1] "acf12 is -0.020"
```

(b) Simulation

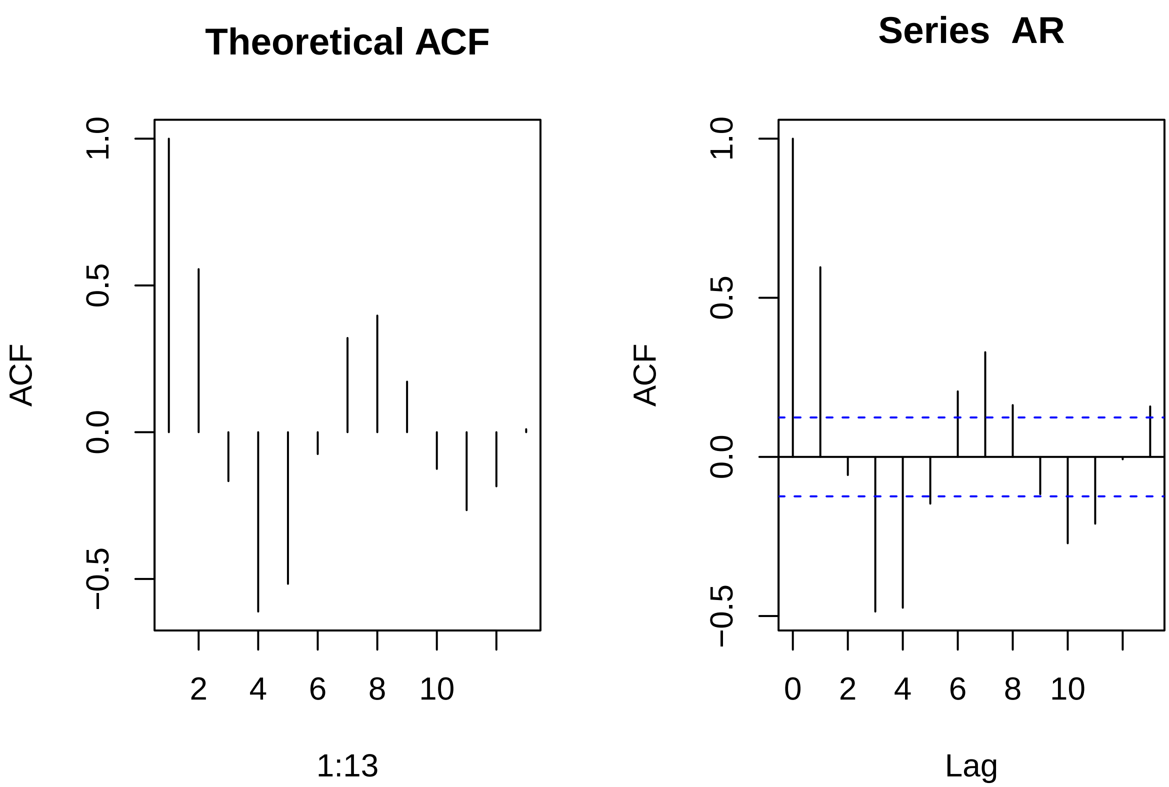
```r
par(mfrow=c(1,3))
AR <- arima.sim(model = list(order = c(3, 0, 0),               ar = c(.8,-0.5,-0.2)), n = 250)

MA <- arima.sim(model = list(
              ma = c(.8,-0.5,-0.2)), n = 250)

ARMA <- arima.sim(model = list(order = c(3, 0, 3),
                ar = c(.8,-0.5,-0.2),
              ma = c(.8,-0.5,-0.2)), n = 250)
```

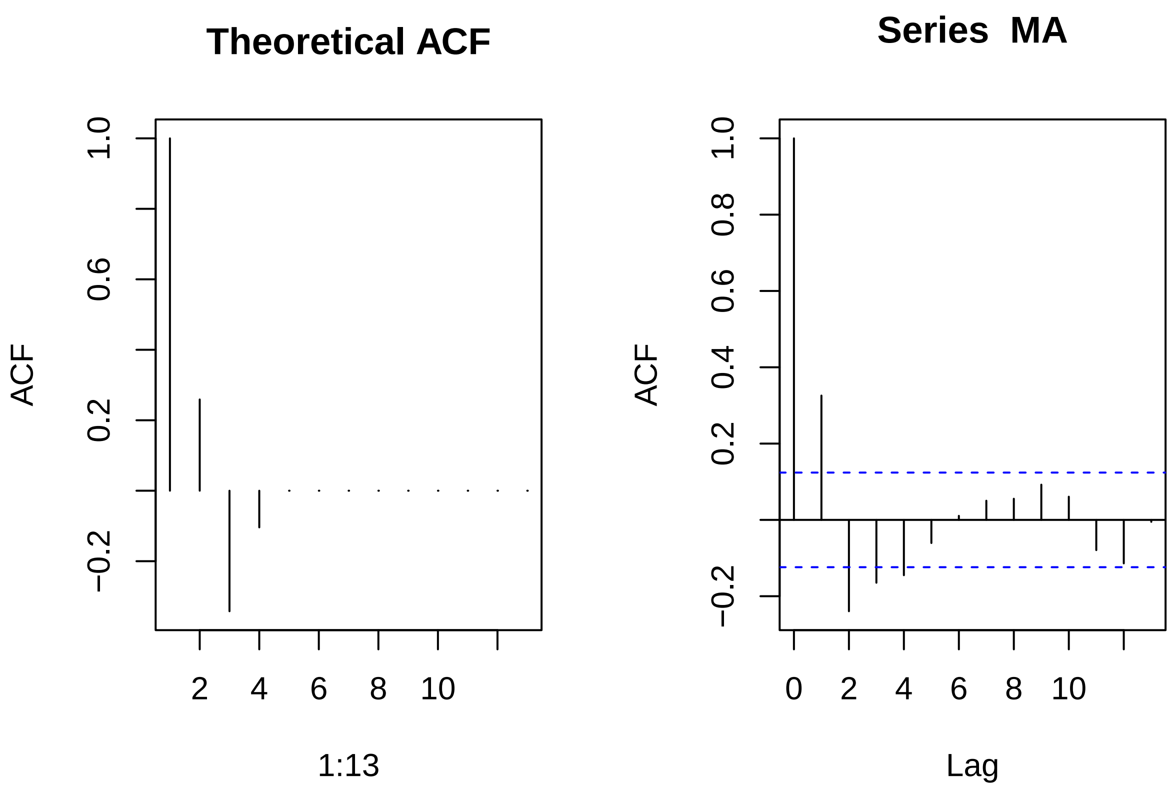
(c) Comparisons Theoretical acf vs Simulation result

```
par(mfrow=c(1,2))
AR3_th= c(gamma0_3, gamma4to12)/gamma0_3[1]
plot(1:13, AR3_th, type = "h", ylab = "ACF", col = "black", main = "Theoretical ACF")
acf1 = acf(AR,13)$acf
```



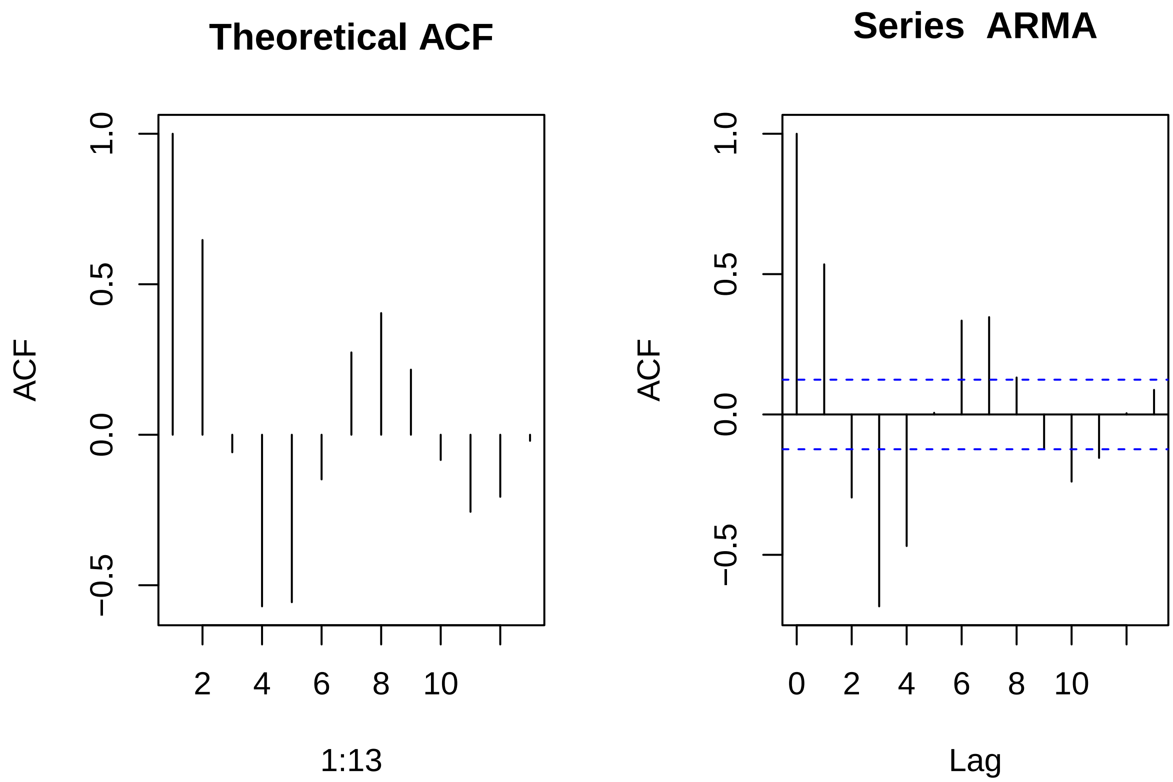
AR(3)

```
par(mfrow=c(1,2))
MA.th = c(1,0.259,-0.342,-0.104, rep(0,9))
plot(1:13, MA.th, type = "h", ylab = "ACF", col = "black", main = "Theoretical ACF")
acf2 = acf(MA,13)$acf
```



MA(3)

```
par(mfrow=c(1,2))
ARMA.th = rep(0,13)
for( i in 1:13){
  ARMA.th[i]= gamma_k(i-1)/gamma_k(0)
}
plot(1:13, ARMA.th, type = "h", ylab = "ACF", col = "black", main = "Theoretical ACF")
acf3 = acf(ARMA,13)$acf
```


Theoretical ACF
ACF
1.0
0.5
0.0
−0.5
2
4
6
8
10
1:13
Series ARMA
ACF
1.0
0.5
0.0
−0.5
0
2
4
6
8
10
Lag

ARMA(3,2)