Time Series Homework 2

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Problem 1

For real valued random variables X, and Y, denote mixing coefficient $\alpha(X,Y)$ as

$$\alpha = \sup_{(x,y)\in R^2} |P(X > x, Y > y) - P(X > x)P(Y > y)|.$$

Now we see that

$$\begin{aligned}
&E|XY| \\
&= E|\int_0^\infty (I_{X>x} - I_{X<-x})dx \int_0^\infty (I_{Y>y} - I_{Y<-y})dy| \\
&\le E \int_0^\infty I_{|X|>x}dx \int_0^\infty I_{|Y|>y}dy \\
&= \int_0^\infty \int_0^\infty E[I_{|X|>x}I_{|Y|>y}]dxdy. && \text{(Fubini's theorem)}
\end{aligned}$$

We want to show

$$E[I_{|X|>x}I_{|Y|>y}] \le \min\{1, P(|X|>x), P(|Y|>y)\}. \tag{1}$$

This is true by Holder's inequality:

$$\int I_{|X|>x}I_{|Y|>y} \le \int |I_{|X|>x}| \int \max |I_{|Y|>y}| = P(|X|>x).$$

and we can change the order of X, and Y to bound by $P(|Y|>y)$. We can show

$$\begin{aligned}
&\min\{1, P(|X|>x), P(|Y|>y)\} \\
&= \int_0^1 I_{u<P(|X|>x)}I_{u<P(|Y|>y)}du \\
&= \int_0^1 I_{x<Q_x(u)}I_{y<Q_y(u)}du.
\end{aligned}$$

Combining above all together, we have that

$$\begin{aligned}
&E|XY| \\
&\le \int_0^\infty \int_0^\infty \int_0^1 I_{x<Q_x(u)}I_{y<Q_y(u)}dudxdy \\
&= \int_0^1 Q_X(u)Q_Y(u)du && \text{(Fubini's theorem.)}
\end{aligned}$$

Problem 2

The sequence of strong mixing coefficients $\{\alpha_n, n \geq 0\}$ of $\{X_i, i \in N\}$ is defined by

$$\alpha_0 = 1/2 \text{ and } \alpha_n = \sup_{|i-j|\geq n} \alpha(\sigma(X_i), \sigma(X_j)) \ for \ n > 0.$$

By definition, it is non-increasing and right continuous. For $x \geq 0$, define $\alpha(x) = \alpha_{\lfloor x \rfloor}$.

$$\alpha^{-1}(u) = \inf\{k \in N : \alpha_k \leq u\} = \sum_{i=0}^{\infty} I_{u<\alpha_i}.. \tag{count}$$

Now by Holder's inequality

$$M_{2,\alpha}(Q_k) \leq \left(\int_0^1 [\alpha^{-1}(u)]^{r/(r-2)} du \right)^{1-2/r} \left(\int_0^1 Q_k^{2*\frac{r}{2}}(u) du \right)^{2/r}.$$

The second integral on r.h.s. is equal to $||X_k||_r^2$ because $Q_k(U)$ has the same distribution as X_k. S

Since α is non-increasing right continuous,

$$x \geq \alpha^{-1}(u) \text{ iff } \alpha(x) \geq u.$$

This implies that

$$\begin{aligned}
&\int_0^1 [\alpha^{-1}(u)]^{r/(r-2)} du \\
&= \int_0^1 \int_0^\infty I_{x \leq [\alpha^{-1}(u)]^{r/(r-2)}} dx du \\
&= \int_0^\infty \int_0^1 I_{u \leq \alpha(x)^{1-2/r}} du dx \\
&= \int_0^\infty \alpha(x)^{1-2/r} dx.
\end{aligned}$$

When $1 \leq x^{1-2/r} < i+1$, $i^{r/(r-2)} \leq x < (i+1)^{r/(r-2)}$ and $\alpha(x^{1-2/r}) = \alpha_i$. Then

$$\begin{aligned}
&\int_0^\infty \alpha(x)^{1-2/r} dx \\
&= \sum_{i=0}^{\infty} ((i+1)^{r/(r-2)} - i^{r/(r-2)}) \alpha_i \\
&\leq \frac{r}{r-2} \sum_{i \geq 0} (i+1)^{2/(r-2)} \alpha_i. && \text{(by MVT)}
\end{aligned}$$

Combining above,

$$M_{2,\alpha}(Q_k) \leq (\frac{r}{r-2})^{1-2/r} \left(\sum (i+1)^{2/(r-2)} \alpha_i \right)^{1-2/r} ||X_k||_r^2.$$

In particular, DMR conditions holds if

$$\sum_{i \geq 0} (i+1)^{2/(r-2)} \alpha_i < \infty.$$

If $\alpha_i = O[i^{-r/(r-2)} (log i)^{-\theta}]$, then

$$(i+1)^{2/(r-2)} \alpha_i = O[(1/i)^{r/(r-2)} \frac{1}{(log i)^\theta}] \leq O[\frac{1}{i(log i)^\theta}]$$

converges when $\theta > 1$ because
$$\int \frac{1}{x(logx)^\theta} dx = \int 1/u^\theta du < \infty.$$
when $\theta > 1$

1 Problem 3.

Norm convergence of a sequence $\{X_n\}$ of elements of L^2 to the limit X means
$$||X_n - X||^2 = E|X_n - X|^2 \to 0$$
as $n \to \infty$. This is called mean-square convergence.

The space of complex-valued random variable X with finite second moment is a complex Hilbert space. To show mean-square convergence of $S_m := \sum_{j=1}^m \theta^j X_{n+1-j}$ it is sufficient to show the sum is Cauchy.

$$\begin{aligned} ||S_l - S_m||^2 &= E \sum_{m+1 \le i \le l} \sum_{m+1 \le j \le l} E\theta^8 X_{n-(i-1)\theta^j X_{n-(j-1)}} \\ &= \sum_{m+1 \le i \le l} \sum_{m+1 \le j \le l} \theta^{i+j}(\mu^2 + \gamma(j-i)). \end{aligned}$$

Since $|\theta| < 1$, we can find a large integer N such that if $m, l \ge N$, then
$$\theta^{i+j}(\mu^2 + \gamma(j-i)) < \epsilon/(l-m)^2$$
for any pre-specified ϵ so that $\{S_i, i \ge 1\}$ is a Cauchy sequence.

2 Problem 4.

2.1 a

Note that in a separable Hilbert space, $\{x_1, ...\}$ are complete orthonormal bases.
$$P_{\bar{sp}\{x_1,\ldots,x_n\}}(x) = \sum_{i=1}^n \langle x, x_i \rangle x_i.$$
as n goes to infinity, it converges to x because
$$P_{\bar{sp}\{x_1,\ldots,x_\infty\}}(x) = \sum_{i=1}^\infty \langle x, x_i \rangle x_i = x$$
as shown in the lecture.

2.2 b

Random variables in a stationary process has finite second moment, so it is a L2 space which is a Hilbert space.

Let S_r be a linear space of $X_{n-1}, ..., X_{n-r}$, and $\widehat{S}_r$ be a projection of X_n onto S_r such that
$$E(X_n - \widehat{S}_r)S_r = 0.$$
If we put $Y_r := X_n - \widehat{S}_r$, then
$$E(X - \widehat{S}_r)S_r = cov(Y_r, S_r).$$

By continuity of inner product in Hilbert space,

$$cov(Y_r, S_r) \rightarrow cov(Y, S),$$

where $Y := X_n - S$, S is a linear space of $X_{n-j}, 1 \leq j \leq \infty$.