Time Series Homework 3

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Problem 11.

(a)

$$\begin{aligned}
X_t &= 0.5X_{t+1} - 0.5Z_{t+1} \\
&= 0.5(X_{t+2} - 0.5Z_{t+2}) - 0.5Z_{t+1} \\
&\vdots \\
&= -\sum_{k=0}^{\infty}(0.5)^{k+1}Z_{t+1+k},
\end{aligned}$$

since stationary process X_t has a finite variance so it goes away as k goes to infinity.

(b)

mean

$$\begin{aligned}
E[Z_t^*] &= 0.25EZ_t - E\frac{3}{4}\lim_{n\to\infty}\sum_{j=1}^{n}2^{-j}Z_{t+j} \\
&= 0.25EZ_t - \lim_{n\to\infty}E[\sum_{j=1}^{n}2^{-j}Z_{t+j}] = 0,
\end{aligned}$$

where we use dominated convergence theorem(DCT) to interchange lim and expectation.

variance, h=0

$$\begin{aligned}
Ez*_t^2 &= 0.25^2EZ_t^2 + E9/16\lim_{n\to\infty}\sum_{j=1}^{n}Z_{t+j}^2(1/4)^j \\
&= 0.25^2\sigma^2 + 9/16 * \lim_{n\to\infty}\sigma^2\sum_{j=1}^{n}(1/4)^j \qquad \text{(DCT)} \\
&= \sigma^2(0.25^2 + \frac{3}{16}).
\end{aligned}$$

covariance, $h > 1$

$$\begin{aligned}
EZ_t^*Z_{t+h}^* &= -3/42^{-h}E[Z_{t+h}^2] + 9/16\sum_{j=1}^{\infty}2^{-2j-h}EZ_{t+h+j}^2 \qquad \text{(DCT)} \\
&= -3/42^{-h}\sigma^2 + 3/16 * 2^{-h}\sigma^2 = -9/16 * 2^{-h}\sigma^2.
\end{aligned}$$

(c)

Using the formula in (a), we can write RHS $0.5X_{t-1} + Z_t^*$ as

$$\begin{aligned} 0.5X_{t-1} + Z_t^* &= -0.5\sum_{k=0}^{\infty}(0.5)^{K+1}Z_{t+k} + 1/4Z_t - 3/4\sum_{k=0}^{\infty}(0.5)^{k+1}Z_{t+k+1} \\ &= -1/4Z_t - 1/4\sum_{k=0}^{\infty}(0.5)^{k+1}Z_{t+k+1} + 1/4Z_t - 3/4\sum_{k=0}^{\infty}(0.5)^{k+1}Z_{t+k+1} \\ &= -\sum_{k=0}^{\infty}(0.5)^{K+1}Z_{t+k+1}, \end{aligned}$$

which is the X_t.

Problem 12

(a)

$$\gamma_w(h) = E\phi(B)X_t\phi(B)X_{t+h} = \phi(B)^2\gamma_x(h) = \phi(B)^2\gamma_y(h),$$

where $\gamma_y(h) = 0$ for $q > 0$ because $\{Y_t\}$ is ARMA(p,q) process. To see this, let $M_t = \phi(B)Y_t$, then $\{M_t\}$ is M(q) process, which means $\gamma_M(h) = 0,\ \forall h > q$. Then

$$E(Y_t\cdots)(Y_{t+h}\cdots) = 0,\ \forall h > q,$$

which also implies $EY_tY_{t+h} = 0$ for $h > q$.

(b)

From (a), $\{W_t\}$ is zero-mean stationary process with autocovariance $\gamma(h) = 0$ for $|h| > q$ and $\gamma(q) \neq 0$.By proposition 3.3.1, $\{W_t\}$ is a MA(q) process, which means $\{X_t\}$ is ARMA(p,q) process.

Problem 13

(a)

By definition, $\widehat{\gamma}(\cdot)$ is non-negative definite if and only if

$$\sum_{i,j=1}^{n} a_i\widehat{\gamma}(t_i - t_j)a_j \geq 0,$$

for any $a = (a_1, ..., a_n)^\top$ and $\{t_1, ..., t_n\} \subset Z$ for given n. We can rewrite

$$\begin{aligned} &\sum_{i,j=1}^{n} a_i\widehat{\gamma}(t_i - t_j)a_j \\ &= a^\top\Gamma a, \end{aligned}$$

where Γ is a $n \times n$ matrix with $\Gamma_{ij} = \widehat{\gamma}(|i - j|)$. Then

$$a^\top\Gamma a = var(a^\top Z_t) \geq 0,\ Z_t = (X_1 - EX_1, ..., X_n - EX_n)^\top.$$

Therefore Γ is non-negative definite, which means $\widehat{\gamma}(h)$ is non-negative definite.

(c)

Assume $\widehat{\gamma}(h) > 0$ and Γ_n is singular for some fixed n. This means the matrix is not a full rank and there exits $1 \leq r \leq n$ and $(a_1, ..., a_r)$ such that

$$X_{r+1} = \sum_{i=1}^{r} a_i X_i.$$

W.L.O.G, we can assume $EX_t = 0$, then

$$X_{r+h} = \sum_{i=1}^{r} a_i X_{i+h-1},$$

which means there exists $(a_1^{(n)}, ..., a_r^{(n)})$ such that

$$X_n = a^{(n)^\top} X_r.$$

Now,

$$\gamma(0) = a^{(n)^\top} \Gamma_r a^{(n)} = a^{(n)^\top} P \Lambda P^\top a^{(n)}.$$

But Γ_r is positive definite with eigenvalues $\lambda_1 \leq \cdots \leq \lambda_r$, hence

$$\gamma(0) \geq \lambda_1 \sum_{i=1}^{r} (a^{(n)})^2.$$

On the other hand, $\gamma(0) = Cov(X_n, a_1^{(n)} X_1 + \cdots a_r^{(n)} X_r)$, so

$$\gamma(0) \leq \sum_{j=1}^{r} |a_j^{(n)}| |\gamma(n-j)|. \tag{(*)}$$

If we take limit($n \to \infty$)) to both sides in (*), then RHS goes to zero which is a contradiction because $\gamma(0) > 0$. Therefore, Γ_n is non-singular.

Problem 14

(a)

Since $\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$, the summability of absolute autocovariance can be shown by using absolute summable coefficients:

$$\begin{aligned}
&\sum_{h=-\infty}^{\infty} |\gamma(h)| \\
&\leq \sum_{h=-\infty}^{\infty} \sum_{j=0}^{\infty} |\psi_j| |\psi_{j+h}| \\
&= \sum_{j=0}^{\infty} |\psi_j| \left(\sum_{h=-\infty}^{\infty} |\psi_{j+h}| \right) && \text{((Fubini-Tonelli))} \\
&= C * \sum_{j=0}^{\infty} |\psi_j| < \infty.
\end{aligned}$$

(b)

When $h > max(p, q+1)$, acf behaves like AR(p),

$$\psi_j = \psi_{j-1}\phi_1 + \cdots \psi_{j-p}\phi_p.$$

This implies that

$$\gamma(h) - \phi_1\gamma(h-1) - \cdots - \phi_p\gamma(h-p) = 0.$$

By using Yule-Walker equations, this has a solution of the form

$$\gamma(h) = A_1\alpha_1^h + \cdots + A_p\alpha_p^h \leq C * max\{\alpha_1, ..., \alpha_p\}^h,$$

where $\alpha_1, ..., \alpha_p$ are the roots of $x^p - \phi_1 x^{p-1} - \cdots - \phi_p = 0$, with $|\alpha_i| < 1$.

Therefore $\gamma(h)$ exponentially decays.