Time Series Homework 4

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Problem 15

Note that

$$\sum_{j=0}^{n}(1-\frac{j}{n})\mathbf{X}_{t-j}-\mathbf{Z}_t$$
$$=\sum_{j=0}^{n}(1-\frac{j}{n})(\mathbf{Z}_{t-j}-\mathbf{Z}_{t-j-1)-\mathbf{Z}_t}$$
$$=-\frac{1}{n}(\mathbf{Z}_{t-1}+\mathbf{Z}_{t-2}+\cdots+\mathbf{Z}_{t-n}-n\mathbf{Z}_{t-n-1})+((\mathbf{Z}_t-\mathbf{Z}_{t-1})+(\mathbf{Z}_{t-1}-\mathbf{Z}_{t-2})\cdots+(\mathbf{Z}_{t-n}-\mathbf{Z}_{t-n-1}))-\mathbf{Z}_t$$
$$=-\frac{1}{n}(\mathbf{Z}_{t-1}+\cdots+\mathbf{Z}_{t-n}).$$

Denote $\bar{\mathbf{Z}}=\frac{1}{n}(\mathbf{Z}_{t-1}+\cdots+\mathbf{Z}_{t-n})$. This implies that

$$\lim_{n\to\infty}\mathbb{E}\left(\sum_{j=0}^{n}(1-\frac{j}{n})\mathbf{X}_{t-j}-\mathbf{Z}_t\right)^2=\lim_{n\to\infty}\mathbb{E}(\bar{\mathbf{Z}})^2=0.$$

Convergence in L2 implies there exists a sub-sequence converging almost surely, hence we caan write

$$\mathbf{Z}_t=\sum_{j=0}^{\infty}\psi_j\mathbf{X}_{t-j},$$

therefore

$$\mathbf{Z}_t\in\bar{\text{sp}}\{\mathbf{X}_j:-\infty<j<=t\}.$$

Problem 16

(a)

We prove by induction. By definition of non-negative definiteness of a complex-valued function, for any complex numbers $a=(a_1,...,a_n)^\top$, and any $t_1,..,t_n\in\mathbb{Z}$,

$$\sum_{j=1}^{n}\sum_{k=1}^{n}a_j\bar{a_k}K(t_j-t_k)\geq 0. \qquad (*)$$

First, put $n=2$, and $a=(1,1)^\top$. Then (*) can be written as

$$2*k(0)+k(-1)+k(1)\geq 0,$$

which means $Img(k(-1)))=-Img(k(1))$. Now put $a=(1,i)^\top$, then * implies that $Re(k(-1))=Re(k(1))$, hence $k(1)=\overline{k(-1)}$.

Suppose $k(n) = \overline{k(-n)}$ for $n = 1, ..., (m-2)$. Now we consider when $n = m$. Put $a = (a_1, a_2)^\top$, where $a_1 = (1, ..., 1)^\top$ is (m-1) by 1 and $a_2 = (1)$. We can rewrite (*) as

$$\begin{aligned} &a_1^\top b_1 a_1 + a_2^\top b_3 a_1 + a_1^\top b_2 a_2 + a_2^\top b)4a_2 \\ &= a_1^\top b_1 a_1 + b_3 a_1 + a_1^\top b_2 + b_4, \end{aligned} \tag{1}$$

where b_1 is (m-1) by (m-1) matrix such that (i,j) component is k(i-j), and $b_2 = (k(-(m-1)), \cdots, k(-1))^\top$ and $b_3 = (k(m-1), \cdots, k(1)$ and $b_4 = k(0)$. Clearly, $a_1^\top b_1 a_1 + b_4 \geq 0$ by assumption, so we only consider two terms in the middle in (1), and it gives

$$b_3 a_1 + a_1^\top b_2 = \Big(k(1) + k(-1) + \cdots k(-(m-2)) + k(m-2)\Big) + k(-(m-1)) + k(m-1) \geq 0.$$

By assumption, we only consider when $k(-(m-1)) + k(m-1) \geq 0$, then we have $Img(k(-(m-1))) = -Img(k((m-1)))$.

Similary, put $a = (a_1, a_2)^\top$, where $a_1 = (1, ..., 1)^\top$ is (m-1) by 1 and $a_2 = (i)$ and consider when (*) is true given assumption $k(n) = \overline{k(-n)}$ for $n = 1, ..., (m-2)$. Then we have $Re(k(-(m-1))) = -Re(k((m-1)))$. Combining the two results, we have $k(m-1) = \overline{k(-m-1)}$.

(b)

For any $a + ib$, where both a, b are n by 1, (*) in part (a) can be written as

$$\begin{aligned} &(a+ib)^\top [K_1 + iK_2](a - ib) \\ &= \Big[a^\top K_1 a - b^\top K_2 a + a^\top K_2 b + b^\top K_1 b\Big] + i\Big[a^\top K_2 a + b^\top K_1 a - a^\top K_1 b + b^\top K_2 b\Big] \geq 0. \end{aligned}$$

This implies that $\Big[a^\top K_1 a - b^\top K_2 a + a^\top K_2 b + b^\top K_1 b\Big] \geq 0$. Hence

$$\begin{bmatrix} a^\top & b^\top \end{bmatrix} \begin{bmatrix} K_1^{(n)} & K_2^{(n)} \\ -K_2^{(n)} & K_1^{(n)} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \Big[a^\top K_1 a - b^\top K_2 a + a^\top K_2 b + b^\top K_1 b\Big] \geq 0,$$

which means $L^{(n)}$ is non-negative definite. Also imaginary part in autocovariance function $k_2(\cdot)$ is odd, hence this implies the matrix K_2 is skew symmetric. Therefore $L^{(n)}$ is syemmtric.

(c)

Note that

$$\begin{aligned} &E(W_n W_n^\top) \\ &\qquad = E((Y_n + iZ_n)(Y_n + iZ_n)^\top) \\ &\qquad = E(Y_n Y_n^\top + Z_n Z_n^\top) + iE(Z_n Y_n^\top - Y_n Z_n^\top) \\ &\qquad = \frac{1}{2}(K_1^{(n)} + K_1^{(n)}) + i * \frac{1}{2}(K_2^{(n)} - (-K_2^{(n)})) \\ &\qquad = K_1^{(n)} + iK_2^{(n)} = K^{(n)}. \end{aligned}$$

(d)

Let $F_{t_1,...,t_n}$ be a cumulative distribution of $\Big((Y_1, Z_1), ..., (Y_n, Z_n)\Big)$,a probability measure on $(\mathbf{R}^2)^k$.

1. Since the distribution is normal, for all permutations π of $\{1, ..., k\}$ and measurable sets $F_i \in \mathbf{R}^2$,

$$F_{t_{\pi(1)},...,t_{\pi(n)}}(F_{\pi(1)} \times \cdots \times F_{\pi(n)}) = F_{t_1,...,t_n}(F_1 \times \cdots \times F_n).$$

2. For all measurable sets $F_i \in \mathbf{R}^2, m \in \mathbb{N}$,

$$F_{t_1,...,t_n}(F_1 \times \cdots \times F_n) = F_{t_1,...,t_n,...,t_{k+m}}(F_1 \times \cdots \times F_n \times \underbrace{R^2 \times \cdots R^2}_{m}),$$

because marginalization of multivariate normal is still a normal.

With 1. and 2. satisfied, the Kolmogorov extension theorem (also known as Kolmogorov existence theorem, Daniell-Kolmogorov theorem), there exists a probability space $(\Omega, F, \mathbb{P})$ and a stochastic process $X : T \times \Omega \to R^2$ such that

$$F_{t_1,...,t_n}(F_1 \times \cdots \times F_n) = \mathbb{P}(X_{t_1} \in F_1, ..., X_{t_n} \in F_n)$$

for any finite finite collection of times $(t_1, ..., t_n)$, which proves the existence of $\{Y_t\}, \{Z_t\}$. The theorem says a suitably "consistent" collection of finite-dimensional distributions will define a stochastic process.

(e)

1.

$$E|X_t|^2 = EY_t^2 + EZ_t^2 = \sigma_y^2 + \sigma_z^2 < \infty.$$

2.

$$EX_t = EY_t + i * EZ_t = 0$$

is independent of t.

3.

$$\gamma_x(h) = EX_{t+h}\bar{X}_t = EY_{t+h}Y_t + EZ_{t+h}Z_t + i * EZ_{t+h}Y_t - i * EZ_tY_{t+h} = K_1(h) + K_2(h)$$

is independent of t.

Problem 17

(a)

Consider $t = 0, ..., n-1$.

$$V := \begin{bmatrix} e^{i0\lambda_1} & e^{i0\lambda_2} & \cdots & e^{i0\lambda_n} \\ e^{i1\lambda_1} & e^{i1\lambda_2} & \cdots & e^{i1\lambda_n} \\ \vdots & \vdots & \vdots & \vdots \\ e^{i(n-1)\lambda_1} & e^{i(n-1)\lambda_2} & \cdots & e^{i(n-1)\lambda_n} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{1}$$

This is a Vandermonde matrix. Determinant is

$$\prod_{0 \le i < j \le n} (V_{1,j} - V_{1,i}) \neq 0$$

because $\lambda_1 < \cdots < \lambda_n$. Hence L.H.S. in (1) is invertible, which means $a_1, ..., a_n = 0$.

(b)

$\Rightarrow$ We can rewrite X_t as

$$
\begin{aligned}
&X_t \\
&\quad = (A_1 e^{it\lambda_1} + A_{n-1} e^{it\lambda_{n-1}}) + (\cdots) + \cdots A_n e^{it\lambda_n}.
\end{aligned} \tag{1}
$$

Since A_n is real, it is sufficient to show $(A_j e^{it\lambda_j} + A_{n-j} e^{it\lambda_{n-j}})$ is real. Let $A_j = a + bi, a, b \in R$, and denote $\theta := t\lambda_j$, then

$$
\begin{aligned}
&(A_j e^{it\lambda_j} + A_{n-j} e^{it\lambda_{n-j}}) \\
&\qquad = (a+bi)(cos\theta + isin\theta) + (a-bi)(cos\theta - isin\theta) = \\
&\qquad C + i(bcos\theta + asin\theta - bcos\theta - asin\theta), C \in R,
\end{aligned}
$$

which proves that sum of pairs in (1) are real, hence X_t is real valued.

$\Leftarrow$

Assume $\sum_{j=1}^{n} A_j e^{it\lambda_j}$ is real. W.L.O.G., we can assume $\lambda_n = \pi, \lambda_i = -\lambda_{n-i}$. We can do this by adding terms $\lambda_{n+1}, \cdots, \lambda_{2n}$ with 0 coefficients. Observe that

$$
\sum_{j=1}^{n} \overline{A}_j e^{-it\lambda_j} = \sum_{j=1}^{n} A_j e^{it\lambda_j},
$$

then we have that

$$
\begin{aligned}
\sum_{j=1}^{n} \overline{A_j} e^{-it\lambda_j} &= \sum_{j=1}^{n-1} \overline{A_j} e^{it\lambda_{n-j}} + \overline{A_n} e^{it\lambda_n} \\
&= \sum_{k=1}^{n-1} \overline{A}_{n-k} e^{it\lambda_k} + \overline{A}_n e^{it\lambda_n} = \sum_{k=1}^{n} \overline{A}_{n-k} e^{it\lambda_k}.
\end{aligned}
$$

Lastly,

$$
\sum_{j=1}^{n} \Big(A_j - \overline{A}_{n-j} \Big) e^{it\lambda_j} = 0 \iff A_j = \overline{A}_{n-j},\ 1 \le j \le n.
$$

Problem 18

If autocovariance function is real, it is an even function,

$$
\begin{aligned}
\gamma(h) &= \int_{-\pi}^{\pi} e^{ihx} dF(x) \\
&= \int_{-\pi}^{\pi} e^{-ihx} dF(x) = \gamma(-h)
\end{aligned}
$$

The second equation implies that

$$
\int_{-\pi}^{\pi} -e^{ihx} dF(-x) = \int_{-\pi}^{\pi} e^{-ihx} dF(x) = \int_{-\pi}^{\pi} e^{ihx} dF(x)
$$

which means $dF(\lambda) = -dF(-\lambda)$. Then we have

$$
\begin{aligned}
&F(-\lambda^-) \\
&\qquad = \int_{-\pi^+}^{-\lambda^-} dF(x) = \int_{-\pi^+}^{-\lambda^-} -1 \cdot dF(-x) \\
&\qquad = \int_{\lambda^+}^{\pi^-} dF(x).
\end{aligned}
$$

Therefore we have

$$F(\lambda) + F(-\lambda^-) = F(\pi^-).$$

Problem 19

(a)

MA(1):

$$f_x(\lambda) = \frac{2}{2\pi}(1 \pm 1.8\cos\lambda + 0.81)$$

HW4_MA1.png

(b)

AR(1):

$$f_x(\lambda) = \frac{3}{2\pi}(1 \pm 1.8\cos\lambda + 0.81)^{-1}$$

HW4_AR1.png

Problem 20

Let X_t be a mean zero complex valued stochastic process and Γ_m be its m-degree autocovariance matrix. Suppose n is the smallest positive integer such that Γ_n is singular then there exists $a = (1, a_1, ..., a_{n-1})$ (we can normalise 'a' to make the first element 1, and this means the solution is non-trivial) such that $\Gamma_n a = 0$, hence $a^\top \Gamma_n a = 0$.

Now define

$$Y_t := (1 + a_1 B + a_2 B^2 + \cdots a_{n-1} B^{n-1}) X_t.$$

We can observe that $var(Y_t) = a^\top \Gamma_n a = 0$. So we have that $Y_t = 0$.

Next, let

$$a(B) = (1 + a_1 B + a_2 B^2 + \cdots a_{n-1} B^{n-1}) = (1 - b_1^{-1} B) \cdots (1 - b_{n-1}^{-1} B),$$

where b_j are the roots of $a(B)$. Then

$$F_Y(\lambda) = \int_{-\pi}^{\lambda} |a(e^{-i\nu})|^2 dF_X(\nu) = 0.$$

We can show $|b_j| = 1, \ \forall \ 1 \leq j \leq n-1$. Otherwise, there exists b_1 s.t. $|b_1| \neq 1$, then we can define

$$W_t := (1 - b_2^{-1} B) \cdots (1 - b_{n-1}^{-1} B) X_t,$$

and

$$Y_t = (1 - b_1^{-1} B) W_t.$$

Then we have that

$$F_Y(\lambda) = \int_{-\pi}^{\lambda} |1 - b_1^{-1} e^{-i\nu})|^2 dF_W(\nu) > 0$$

for $|b_1| \neq 1$ and

$$F_Y(\lambda) = 0 \iff F_W(\nu) = 0$$

which implies that there is an $n' < n$ such that $\Gamma_{n'}$ is singular, contradiction to the assumption.

Therefore $|b_j| = 1, \ \forall \ 1 \leq j \leq n-1$, then we can write $b_j = e^{i\lambda_j}$. Observe that

$$F_Y(\lambda) = \int_{-\pi}^{\lambda} |1 - b_1^{-1} e^{-i\lambda}|^2 dF_W(\lambda) = 0$$

is valid only when $\lambda = \lambda_j$, which means X_t has a point mass at $\lambda = \lambda_j$.