3. Problem Set

Homework 01, Due 09/21/2023.

Problem 1.

(a) Let $\{Z_t,\ t \in \mathbb{Z}\}$ be a Gaussian white noise with variance 1, and consider the time series

$$X_t = Z_t Z_{t-1}, \quad Y_t = X_t^2.$$

Find the mean, autocovariance, and autocorrelation functions of X_t and Y_t.

(b) Let X_t be a stationary Gaussian process with mean μ and autocovariance function $\gamma_X(\cdot)$. Define the nonlinear time series

$$Y_t = \exp\{X_t\}.$$

Find the mean, autocovariance, and autocorrelation functions of Y_t.

Problem 2. Suppose we have observed a time series $\{X_1, X_2, \ldots, X_n\}$ which is assumed to be stationary. Let $\hat{\gamma}(h) = \frac{1}{n}\sum_{i=1}^{n-|h|}(X_{t+|h|} - \bar{X})(X_t - \bar{X})$ be the lag-h sample autocovariance, for every $|h| < n$. Prove that the order-n sample autocovariance matrix $\hat{\Gamma}_n := [\hat{\gamma}(r-s)]_{r,s=1}^n$ is positive semi-definite.

Problem 3.

(a) Let Z_1 and Z_2 be IID $N(0,1)$ random variables, and $\omega \in (-\pi, \pi]$ be a constant. Define a stochastic process

$$Y_t = Z_1 \cos(\omega t) + Z_2 \sin(\omega t).$$

Calculate its autocovariance function and autocorrelation function. Is the time series stationary? Is it strictly stationary?

(b) From now on assume n is an even integer. Take $n = 200$ or 500, or whatever even integer you would like to choose, and set $\omega_j := (2\pi j)/n$ for $j = 0, 1, \ldots, n/2$. Now simulate $(n/2+1)$ time series of length n as described below

$$\begin{aligned}
X_{0t} &= Z_0, && \text{where } Z_0 \sim N(0,1)\\
X_{jt} &= Z_{j1}\cos(\omega_j t) + Z_{j2}\sin(\omega_j t), && \text{where } Z_{j1}, Z_{j2} \sim N(0,2), 1 \le j < n/2\\
X_{n/2,t} &= Z_{n/2}(-1)^t, && \text{where } Z_{n/2} \sim N(0,1);
\end{aligned}$$

where all the $Z_\bullet$ random variables are also assumed to be mutually independent. Finally add the $(n/2+1)$ series that you have simulated together, and call it X_t

$$X_t = \frac{1}{\sqrt{n}}\sum_{j=0}^{n/2} X_{jt}.$$

Show a time series plot (i.e. data vs time, and the data points are connected by lines as opposed to a scatterplot) of $\{X_t\}$.

(c) Simulate IID $N(0,1)$ with length n, and show the corresponding time series plot.

(d) Compare the two plots you've obtained, do they look similar? Can you explain why?

Problem 4. Which, if any, of the following functions defined on the integers is the autocovariance function of a stationary time series? Prove your results.

(a) $f(h) = (-1)^{|h|}$

(b) $f(h) = 1 + \cos\frac{\pi h}{2} + \cos\frac{\pi h}{4}$

(c) $f(h) = 1 + \cos\frac{\pi h}{2} - \cos\frac{\pi h}{4}$

(d) $$f(h) = \begin{cases} 1 & \text{if } h = 0 \\ .6 & \text{if } h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 5. Prove that $\alpha(\mathscr{G}, \mathscr{H}) \leq \frac{1}{2}\phi(\mathscr{G}, \mathscr{H})$.

Homework 02, Due on 10/03/2023.

Problem 6. Recall the definition of $Q(\cdot)$ function in the lectures. Suppose X and Y are integrable random variables, and that XY is integrable, show that

$$\mathbb{E}|XY| \leq \int_0^1 Q_X(u) Q_Y(u)\, du.$$

Problem 7. Let $\{X_i,\ i \in \mathbb{N}\}$ be a strictly stationary process, and assume $\mathbb{E}|X_0|^r \leq \infty$ for some $r > 2$. Assume the strong mixing coefficients α_k associated with $\{X_i,\ i \in \mathbb{N}\}$ satisfy

$$\alpha_k = O[k^{-r/(r-2)} (\log k)^{-\theta}].$$

Suppose we want to show that

$$\sum_{h=-\infty}^{\infty} |\gamma_X(h)| < \infty$$

using the bound of $\gamma_X(h)$ given by the Davydov's inequality. Find the condition on θ so that the preceding series does converge.

Problem 8. Show that if $\{X_t,\ t \in \mathbb{Z}\}$ is stationary and $|\theta| < 1$, then for each n, $\sum_{j=1}^m \theta^j X_{n+1-j}$ converges in mean square as $m \to \infty$.

Problem 9. (a) Suppose $\mathscr{H}$ is a separable Hilbert space and $\mathscr{H} = \overline{\mathrm{sp}}\{x_i,\ i = 1, 2, \infty\}$.Let x be an element of $\mathscr{H}$. Show that

$$P_{\overline{\mathrm{sp}}\{x_1, x_2, \ldots, x_n\}}(x) \to x \quad \text{as } n \to \infty.$$

(b) Suppose $\{X_t,\ t \in \mathbb{Z}\}$ is a stationary process. Show that

$$P_{\overline{\mathrm{sp}}\{X_{n-j},\ 1 \leq j \leq \infty\}}(X_n) = \lim_{r \to \infty} P_{\overline{\mathrm{sp}}\{X_{n-1}, X_{n-2}, \ldots, X_{n-r}\}}(X_n).$$

Homework 03, Due on 10/12/2023.

Problem 10. Consider the following ARMA processes.

(i) AR(3): $r_t = 0.3 + 0.8r_{t-1} - .5r_{t-2} - .2r_{t-3} + a_t$.
(ii) MA(3): $r_t = 0.3 + a_t + 0.8a_{t-1} - .5a_{t-2} - .2a_{t-3}$.
(iii) ARMA(3,2): $r_t = 0.3 + 0.8r_{t-1} - .5r_{t-2} - .2r_{t-3} + a_t + 0.5a_{t-1} + 0.3a_{t-2}$.

Assume all a_t are i.i.d $N(0, 4)$. For each of the three preceding process, do the following:

(a) Calculate the ACF up to lag 12. [Hint. You may need to read Section 3.3 before trying (iii).]
(b) Simulate a series of length $T = 250$, give the time series plot.
(c) Compare the true ACF plot (plot what you obtained in Part (a)) with the sample ACF plot (use the `R` function `acf()`).

Problem 11. Consider the AR(1) process $X_t = 2X_{t-1} + Z_t$, where $Z_t \sim \mathrm{WN}(0, \sigma^2)$. Define

$$Z_t^* := .25Z_t - \frac{3}{4}\sum_{j=1}^{\infty} 2^{-j} Z_{t+j}.$$

(a) Express the unique stationary solution X_t in terms of Z_t.
(b) Prove that $\{Z_t^*\}$ is a white noise. What is its variance?
(c) Prove that $X_t = .5X_{t-1} + Z_t^*$.

Problem 12. Suppose that $\{X_t\}$ and $\{Y_t\}$ are two zero-mean stationary processes with the same autocovariance function, and that Y_t is an ARMA(p, q) process.

(a) If $\phi_1, \ldots, \phi_p$ are the AR coefficients for Y_t, define $W_t := X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p}$. Show that $\{W_t\}$ has an autocovariance function which is zero for lags $|h| > q$.
(b) Apply Proposition 3.2.1 to $\{W_t\}$ to conclude that $\{X_t\}$ is also an ARMA(p, q) process.

Problem 13. Read Proposition 5.1.1 and its proof (a very nice one!) before you work on this problem. Suppose there are n observations $X_1, X_2, \ldots, X_n$ of a stationary time series. Define

$$\hat{\gamma}(h) = \begin{cases} n^{-1}\sum_{t=1}^{n-|h|}(X_{t+h} - \bar{X})(X_t - \bar{X}) & \text{if } |h| < n, \\ 0 & \text{if } |h| > n. \end{cases}$$

Note that although the sample autocovariannces are usually only defined for lags $|h| < n$, here $\hat{\gamma}(\cdot)$ is defined as a function on all integers, where it takes value 0 when $|h| \geq n$.

(a) Show that the function $\hat{\gamma}(\cdot)$ is non-negative definite.
(b) There is nothing you need to do for this part. But observe that (i) by Theorem 1.5.1, there exists some stationary process $\{Y_t\}$ of which $\hat{\gamma}(\cdot)$ is the autocovariance function; and (ii) from Proposition 3.2.1 it then follows that $\{Y_t\}$ is an MA$(n-1)$ process.
(c) Prove that if $\hat{\gamma}(0) > 0$, then $\hat{\Gamma}_n$ is non-singular. (In the last Homework, you showed that $\hat{\Gamma}_n$ is non-negative definite, and now you know that it is also strictly positive-definite unless the n observations are all equal.)

Problem 14.

(a) Consider a MA(∞) process $X_t = \sum_{j=0}^{\infty}\psi_j Z_{t-j}$, where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, and $\sum_{j=0}^{\infty}|\psi_j| < \infty$. Show that the autocovariance function $\gamma(\cdot)$ of $\{X_t\}$ satisfies $\sum_{h=-\infty}^{\infty}|\gamma(h)| < \infty$.
(b) Let $\{X_t\}$ be a causal ARMA process with autocovariance function $\gamma(\cdot)$. Show that there exist a constant $C > 0$ and another constant $s \in (0, 1)$ such that $|\gamma(h)| \leq Cs^{|h|}$ for all $h \in \mathbb{Z}$, and hence $\sum_h |\gamma(h)| < \infty$.

Homework 04, Due on 10/26/2023.

Problem 15. The process $X_t = Z_t - Z_{t-1}$, where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, is not invertible according to Definition 3.1.4. Show however that $Z_t \in \overline{\mathrm{sp}}\{X_j, -\infty < j \le t\}$ by considering the mean square limit of the sequence $\sum_{j=0}^{n}(1 - j/n)X_{t-j}$ as $n \to \infty$.

Problem 16. Assume that $K(\cdot)$ is a complex-valued function defined on $\mathbb{Z}$, and that $K(\cdot)$ is non-negative definite.

(a) Prove that $K(\cdot)$ is Hermitian, *i.e.* $K(h) = \overline{K(-h)}$.

(b) Let $K_1(\cdot)$ and $K_2(\cdot)$ be the real and imaginary part of $K(\cdot)$, *i.e.* $K(h) = K_1(h) + iK_2(h)$ for all $h \in \mathbb{Z}$. According to Part (a), we know that $K_1(\cdot)$ is even and $K_2(\cdot)$ is odd. For any positive integer n, define the $(2n) \times (2n)$ matrix

$$L^{(n)} = \frac{1}{2}\begin{pmatrix} K_1^{(n)} & -K_2^{(n)} \\ K_2^{(n)} & K_1^{(n)} \end{pmatrix}, \quad \text{where } K_1^{(n)} := [K_1(j-k)]_{j,k=1}^{n} \text{ and } K_2^{(n)} := [K_2(j-k)]_{j,k=1}^{n}.$$

Prove that $L^{(n)}$ is symmetric and non-negative definite. [Hint. Here you need to use the non-negative definiteness of $K(\cdot)$.]

(c) Let $(Y_1, \dots, Y_n, Z_1, \dots, Z_n)'$ be a random vector which has a multivariate normal distribution with mean zero and covariance matrix $L^{(n)}$. Define $W_t = Y_t + iZ_t$ for $1 \le t \le n$. Show that the covariance matrix of $(W_1, \dots, W_n)'$ is given by $K^{(n)} := [K(j-k)]_{j,k=1}^{n}$.

(d) Apply the Kolmogorov's Existence Theorem to deduce that there exist a bivariate mean zero Gaussian process $(Y_t, Z_t)'$ such that

$$\mathbb{E}(Y_{t+h}Y_t) = \mathbb{E}(Z_{t+h}Z_t) = \tfrac{1}{2}K_1(h)$$
$$\mathbb{E}(Z_{t+h}Y_t) = -\mathbb{E}(Y_{t+h}Z_t) = \tfrac{1}{2}K_2(h).$$

(e) Show that $\{X_t = Y_t + iZ_t,\ t \in \mathbb{Z}\}$ is a complex-valued process with autocovariance function $K(\cdot)$.

Problem 17. Consider n frequencies $-\pi < \lambda_1 < \lambda_2 < \cdots < \lambda_n = \pi$.

(a) Let $a_1, a_2, \dots, a_n$ be complex numbers. Prove that if

$$\sum_{j=1}^{n} a_j e^{it\lambda_j} = 0 \quad \text{for all } t \in \mathbb{Z}$$

then it must hold that $a_1 = a_2 = \cdots = a_n = 0$.

(b) Let $A_1, A_2, \dots A_n$ be complex random variables, and define $X_t = \sum_{j=1}^{n} A_j e^{it\lambda_j}$. Show that the process $\{X_t,\ t \in \mathbb{Z}\}$ is real-valued if and only if $\lambda_j = -\lambda_{n-j}$ and $A_j = \bar{A}_{n-j}$ for $1 \le j < n$, and A_n is real.

Problem 18. Prove that if $\gamma(\cdot)$ is real, then its spectral distribution $F(\cdot)$ is symmetric in the sense

$$F(\lambda) = F(\pi^-) - F(-\lambda^-), \quad -\pi < \lambda < \pi.$$

Problem 19. Give an expression and a plot for the spectral density of each of the following processes. [Try to plot many more for fun!]

(a) MA(1). $X_t = Z_t \pm 0.9Z_{t-1}$, where $\{Z_t\} \sim \mathrm{WN}(0, 2)$.

(b) AR(1). $X_t = \pm 0.9X_{t-1} + Z_t$, where $\{Z_t\} \sim \mathrm{WN}(0, 3)$.

(c) Each of the processes in Problem 10.

Problem 20. Suppose $\gamma(\cdot)$ is a real-valued autocovariance function such that $\gamma(0) > 0$, and the covariance matrix Γ_n is singular for some $n > 1$. Find out the spectral distribution of $\gamma(\cdot)$.

Homework 05, Due on 11/09/2023.

Problem 21. Show that if $\phi(\cdot)$ and $\theta(\cdot)$ have no common zeros, and if $\phi(z) = 0$ for some $|z| = 1$, then the ARMA equations

$$\phi(B)X_t = \theta(B)Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$$

have no stationary solution. [Hint. Assume there is one, then use the relationship between the spectral distributions of $\{X_t\}$ and $\{Z_t\}$ to derive a contradiction. Also think how you would do it without using spectral distributions.]

Problem 22. Let $\{X_t\}$ and $\{Y_t\}$ be two stationary mean zero processes with spectral densities $f_X(\cdot)$ and $f_Y(\cdot)$. If $f_X(\cdot) \leq f_Y(\cdot)$ for all $\lambda \in [-\pi, \pi]$, show that $\Gamma_{n,Y} - \Gamma_{n,X}$ is a non-negative definite matrix, where $\Gamma_{n,X}$ and $\Gamma_{n,Y}$ are the covariance matrices of $(X_1, \ldots, X_n)'$ and $(Y_1, \ldots, Y_n)'$ respectively.

Problem 23. Assume $\{X_t\}$ is a mean zero stationary process with the spectral distribution function

$$F_X(\lambda) = \begin{cases} \pi + \lambda, & -\pi \leq \lambda < -\pi/6, \\ 3\pi + \lambda, & -\pi/6 \leq \lambda < \pi/6, \\ 5\pi + \lambda, & \pi/6 \leq \lambda \leq \pi. \end{cases}$$

For which value d does the differenced process $\Delta_d X_t := X_t - X_{t-d}$ have a spectral density?

Problem 24. Please install the `R` package `datasets`, and use `data(sunspot.year)` to load the Wölfer sunspot numbers from 1700 to 1988. Let $\{X_t\}$ denote the original data, and $\{Y_t\}$ denote the mean-corrected series, $Y_t = X_t - 49.13$. The following AR(2) model for $\{Y_t\}$ is obtained

$$Y_t = 1.389Y_{t-1} - .691Y_{t-2} + Z_t, \quad \{Z_t\} \sim \mathrm{WN}(0, 273.6).$$

Determine and plot the spectral density of the fitted model and find the frequency at which it achieves its maximum value. What is the corresponding period?

Problem 25. Let $\{X_t\}$ be any stationary time series with continuous and symmetric spectral density f such that $0 \leq f(\lambda) \leq K$ and $f(\pi) = f(-\pi) \neq 0$. Let $f_n(\cdot)$ denote the spectral density of the differenced series $\{(1 - B)^n X_t\}$.

(a) Give an expression for $f_n(\lambda)$.

(b) Let $g_n(\cdot)$ be the normalized version of $f(\cdot)$ which integrates to one, i.e.

$$g_n(\lambda) = \left(\int_{-\pi}^{\pi} f(\nu)\, d\nu \right)^{-1} f_n(\lambda).$$

Show that $g_n(\lambda)/g_n(\pi) \to 0$ as $n \to \infty$ for each $\lambda \in [0, \pi)$.

(c) What does (b) suggest regarding the behaviour of the sample paths of $\{(1 - B)^n X_t\}$ for large values of n?

(d) Plot $\{(1-B)^n X_t\}$ for $n = 1, 2, 3, 4, 5$, where X_t are the sunspot numbers. Do the realizations exhibit the behaviour expected from (c)?

Homework 06, Due 11/28/2023.

Problem 26. Suppose $X_t = \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + Z_t$ is a causal AR(p) process, where $\{Z_t\} \sim \mathrm{WN}(0, \sigma^2)$, and let $\{\psi_n,\, n \geq 0\}$ be its memeory function. Show that

$$\left\| X_{n+h} - \mathcal{P}_{\overline{\mathrm{sp}}\{X_1,\ldots,X_n\}} X_{n+h} \right\|^2 = \sigma^2 \sum_{j=0}^{h-1} \psi_j^2, \quad \text{when } n \geq p.$$

Problem 27. Suppose that

$$X_t = A\cos(\pi t/3) + B\sin(\pi t/3) + Z_t + .5Z_{t-1}, \qquad t \in \mathbb{Z},$$

where $\{Z_t\} \sim \mathrm{WN}(0,1)$, A and B are uncorrelated random variables with mean 0 and variance 4, and $E(AZ_t) = E(BZ_t) = 0$ for all $t \in \mathbb{Z}$.

(a) Find the best linear predictor of X_{t+1} based on X_t and X_{t-1}.

(b) What is the mean squared error of the best linear predictor of X_{t+1} based on $\{X_j,\, j \leq t\}$?

(c) Show how A and B can be predicted by $\{X_j,\, j \leq n\}$.

Problem 28. Suppose $(\Omega, \mathscr{F}, P)$ is a probability space, and $T : \Omega \to \Omega$ is a $\mathscr{F}/\mathscr{F}$ measurable mapping which is also measure preserving. The mapping T is called *mixing* if

$$P(A \cap T^{-n}B) \to P(A)P(B)$$

for all $A, B \in \mathscr{F}$.

(a) Show that mixing implies ergodicity.

(b) Show that T is mixing if the preceding equation holds for all A and B in a π-system which generates $\mathscr{F}$.

(c) Show that the Bernoulli shift is mixing.

(d) Take $\Omega = \{a, b, c, d, e\}$, $\mathscr{F} = 2^{\Omega}$ be the power sets, and P be the uniform distribution over Ω. Let T be a cyclic permutation which maps $a \mapsto b \mapsto c \mapsto d \mapsto e \mapsto a$. Show that T is ergodic but not mixing.

Problem 29. Recall the product space $\mathbb{R}^{\mathbb{Z}}$ with the coordinate σ-field $\mathscr{R}^{\mathbb{Z}}$. Also recall that a stochastic process $\{X_t,\ t \in \mathbb{Z}\}$ on the probability space $(\Omega, \mathscr{F}, P)$ defines a map $\xi : \Omega \to \mathbb{R}^{\infty}$. Show that if the X_t are IID random variables, then the shift operator on $\mathbb{R}^{\mathbb{Z}}$ is ergodic under $P\xi^{-1}$.

Homework 07, Due on 12/12/2023.

Problem 30. Consider the linear process $\{X_t,\, t \in \mathbb{Z}\}$ defined as $X_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j}$, where ϵ_t are IID mean zero random variables, and a_j, $j \geq 0$ are real numbers such that $\sum_{j=0}^{\infty} a_j^2 < \infty$. Let $S_n := X_0 + X_1 + \cdots + X_{n-1}$. Recall the two sufficient conditions for the CLT of $S_n/\sqrt{n}$:

$$\text{Condition 1:} \quad \sum_{j=0}^{\infty} \sum_{k=j+1}^{\infty} a_k \epsilon_{j-k} \text{ converges in } \mathcal{L}^2.$$

$$\text{Condition 2:} \quad \sum_{j=0}^{\infty} |a_j| < \infty.$$

(a) Find an example such that Condition 1 holds, but Condition 2 fails.
(b) Find an example such that Condition 2 holds, but Condition 1 fails.

Problem 31. Suppose the process $\{X_t,\, t \in \mathbb{Z}\}$ is defined as $X_t = g(\epsilon_t, \epsilon_{t-1}, \ldots)$, where ϵ_t are IID and g is a measurable function on $(\mathbb{R}^{\mathbb{Z}}, \mathscr{R}^{\mathbb{Z}})$. Suppose