Time Series Homework 1

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Problem 1

(a)

We assume h is an integer, it is sufficient to only consider positive integers. For X_t, $\mathbb{E}[X_t]$=0 by independence(uncorrelated multivariate normal) of Z_t. Autocovariance is as follows:

$$\gamma_x(h) = \mathbb{E}[(Z_t Z_{t-1})(Z_{t+h} Z_{t+h-1})] \begin{cases} 1 \text{ if } h = 0 \\ 0 \text{ if } h \geq 1. \end{cases}$$

Autocorrelation of X_t is

$$\rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)} = \begin{cases} 1 \text{ if } h = 0 \\ 0 \text{ if } h \geq 1. \end{cases}$$

For Y_t, mean is 1, autocovariance is

$$\gamma_y(h) = \mathbb{E}[(Z_t^2 Z_{t-1}^2 - 1)(Z_{t+h}^2 Z_{t+h-1}^2 - 1)] \begin{cases} (3 \cdot 1^4)^2 - 1 \text{ if } h = 0 \\ 2 \text{ if } h = 1 \\ 0 \text{ if } h \geq 2 \\ . \end{cases}$$

Autocorrleation function of Y_t is

$$\rho_y(h) = \frac{\gamma_y(h)}{\gamma_y(0)} = \begin{cases} 1 \text{ if } h = 0 \\ \frac{1}{4} \text{ if } h = 1 \\ 0 \text{ if } h \geq 2. \end{cases}$$

(b)

Distribution of Y_t is log normal. Using the known formula, mean is $exp(\mu + \frac{\gamma_x(0)}{2})$, where $\gamma_x(0)$ is the variance of X_t. Denote μ_2 be the mean of Y_t. Using the fact that stationary gaussian process is multivariate normal, so

$$X_t + X_{t+h} \sim N(2\mu, 2(\gamma_x(0) + \gamma_x(h))).$$

Autocovariance is

$$\gamma_y(h) = \mathbb{E}[exp(X_t + X_{t+h}) - \mu_2(exp(X_t) + exp(X_{t+h})) + \mu_2^2] = \begin{cases} exp(2 * \mu + 2\gamma_x(0)) - \mu_2^2 \; if \; h = 0, \\ exp(2 * \mu + (\gamma_x(0) + \gamma_x(h))) - \mu_2^2 \text{ if } h \neq 0. \end{cases}$$

Then autocorrelation is

$$\rho_y(h) = \begin{cases} 1 \; if \; h = 0, \\ \frac{exp(2*\mu + 2\gamma_x(0)) - \mu_2^2}{exp(2*\mu + (\gamma_x(0) + \gamma_x(h))) - \mu_2^2} \; if \; h \neq 0. \end{cases}$$

Problem 2.

Define

$$M = 0 \begin{pmatrix} 0 & \cdots & 0 & 0 & \widetilde{X}_1 & \widetilde{X}_2 & \cdots & \widetilde{X}_n \\ 0 & \cdots & 0 & \widetilde{X}_1 & \widetilde{X}_2 & \cdots & \widetilde{X}_n & 0 \\ 0 & \cdots & \widetilde{X}_1 & \widetilde{X}_2 & \cdots & \widetilde{X}_n & 0 & 0 \\ \vdots & & & & & & & \vdots \\ \widetilde{X}_1 & \widetilde{X}_2 & \cdots & \widetilde{X}_n & 0 & \cdots & 0 & \end{pmatrix},$$

where $\widetilde{X}_t = X_t - \mu$. Then we can see that

$$\widehat{\Gamma}_n = \frac{1}{n} M M^\top,$$

so $a^\top \widehat{\Gamma}_n a = \frac{1}{n}||M^\top a||^2 \geq 0$ for any non-trivial vector a.

Problem 3.

Autocovariance function is

$$\begin{aligned} \gamma_y(h) &= \mathbb{E}[(Z_1 cos(\omega t) + Z_2 sin(\omega t))(Z_1 cos(\omega(t+h)) + Z_2 sin(\omega(t+h)))] \\ &= cos(\omega t)cos(\omega(t+h) + sin(\omega t)sin(\omega(t+h)), \end{aligned}$$

and 1 if h=0. This implies that autocorrelation function is the same as auto covariance function.

Since $cosxcosy = \frac{cox(x-y)+cox(x+y)}{2}$ and $sinxsiny = \frac{cox(x-y)-cox(x+y)}{2}$, so we can rewrite

$$\gamma_y(h) = cos\omega h.$$

This is a constant, which means Y_t is stationary. Since Z_t is IID, Y_t is strictly stationary.

Problem 4.

(a)

Yes. Since f is an even function, it is sufficient to show $n \times n$ matrix $H = [(-1)^{|i-j|}]_{i,j=1}^n$ is positive semi-definite for a fixed n.

1. For a symmetric matrix, tr(H)= sum of eigen values.
2. H has rank one, and symmetric, hence it has only one non-zero eigen value.

Combining two results above, eigen values for H are $(n, 0, 0, ..., 0)$. Therefore H is positive semi-definite.

(b)

Yes. For a fixed n, define 3 matrices:

$$A = [1]_{i,j=1}^n, B = [cos\frac{\pi h}{2}]_{i,j=1}^n, C = [cos\frac{\pi h}{4}]_{i,j=1}^n.$$

From problem 3, we know that B and C are positive semi-definite and note that A is rank one with eigen values = c(n,0,...,0), therefore positive-semidefinite. These implies that for any vector a with n elements,

$$a^\top (A + B + C) a \geq 0.$$

(c)

Yes. Consider two independent stationary time series $\{X_t\}, \{Y_t\}$ with mean zeros and whose autocovariance functions are the one in (b) and $2 * cos\frac{\pi h}{2}$. Then $\{Y_t - X_t\}$ is also stationary, hence (c) must be an autocovariance function of a stationary time series.

(d)

No because if we put a=$(1, -1, 1, -1, \cdots)^\top$ with n-components and $K = [k(i-j)]_{i,j=1}^n$, then

$$a^\top K a = n - 2(n-1) * 0.6 \leq 0$$

for $n > 2 * 0.6(2 * 0.601)$.

Problem 5.

Note that

$$|P(H|G) - P(H)| = \left| \frac{p(G \cap H) - P(G)P(H)}{P(G)} \right| \geq |P(G \cap H) - P(G)P(H)|$$

because $1/P(G) \geq 1$, which proves the inequality.