Time Series Homework 5

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Problem 21

Suppose to the contrary that there exists a stationary solution. Then we can write

$$f_x(\lambda)|\phi(e^{-i\lambda})|^2 = |\theta(e^{-i\lambda})|^2 \cdot \frac{\sigma^2}{2\pi}. \tag{1}$$

Note that if there exists $z*$ such that $\phi(z*) = 0,\ |z*| = 1$, then we can write $|z*| = e^{i\lambda*}$, where x is a real value. This means

$$|\phi(e^{-i\lambda})|^2 = |1 - e^{-i\lambda*}e^{i\lambda}|^2|1 - a_j^{-1}e^{i\lambda}| \cdots,$$

where a_j's are roots of ϕ. If we consider continuous frequency times, as $\lambda \to \lambda*$, LHS on (1) becomes 0 which means $\theta()$ must have the same root $e^{i\lambda*}$, which is a contradiction to the assumption that there are no common zeroes.

Problem 22

$\Gamma_{n,X}, \Gamma_{n,Y}$ are valid autocoavriance matrices, so we can write klth component of $\Gamma_{n,Y} - \Gamma_{n,X}$ as

$$\int_{-\pi}^{\pi} e^{i(k-l)\lambda}[f_y(\lambda) - f_x(\lambda)]d\lambda.$$

Then for any vector $a = [a_1, ..., a_n]^\top$, it can be seen that

$$\begin{aligned}
&a^\top(\Gamma_{n,Y} - \Gamma_{n,X})\bar{a} \\
&= \sum_{kl} a_k \bar{a}_l [\Gamma_Y - \Gamma_X]_{kl} \\
&= \int |\sum_r a_r e^{ir\lambda}|^2 \cdot (f_y(\lambda) - f_x(\lambda)d\lambda \geq 0.
\end{aligned}$$

Problem 23

Let $Y_t = X_t - X_{t-d}$, then

$$\gamma_y(h) = 2\gamma_x(h) - \gamma_x(d+h) - \gamma_x(h-d).$$

Now observe that

$$\gamma_x(h) = \int_{[-\pi,-\pi/6)} e^{ih\lambda}d\lambda + 2\pi \cdot e^{-\pi/6ih} + \int_{(-\pi/6,\pi/6)} e^{ih\lambda}d\lambda + 2\pi \cdot e^{ih\pi/6} + \int_{(\pi/6,\pi]} e^{ih\lambda}d\lambda = 4\pi(cos(\pi h/6)).$$

Then we can see

$$\gamma_y(h) = 8\pi cos(\pi h/6) - (4\pi cos(\pi(h+d)/6) + 4\pi cos(\pi(h-d)/6)).$$

Since cos(a+b)+cos(a-b) = 2cos(a) cos(b),

$$\gamma_y(h) = 8\pi cos(\pi h/6) - (8\pi cos(\pi h/6)cos(\pi d/6)).$$

If $|d| = 12k,\ k \geq 0$ then $cos(\pi d/6)) = 1$, and $\gamma_y(h) = 0$. This means γ_y is square summable and by stationarity, Theorem 4.3.2. implies that it has a spectral density.

Problem 24

Please see the R markdown pages.

Problem 25

a

$$f_n(\lambda) = |1 - e^{-i\lambda}|^{2n} f(\lambda).$$

b

$$g_n(\lambda)/g_n(\pi) = \frac{(1 - e^{-i\lambda})^{2n}}{2^{2n}} \cdot \frac{f(\lambda)}{f(\pi)},$$

because $e^{-i\pi} = -1$. Since $(1 - e^{-i\lambda}) < 2$ for $\lambda \in [0, \pi)$, $g_n(\lambda)/g_n(\pi)$ converges to 0.

c

(b) suggests that for large values of n, $g_n(\lambda)$ will be dominated by the frequency π. As n goes large, most of the variation will be concentrated at the frequency π.

d

(Plots are displayed on a different page.) Yes. As n goes large, it can be seen that oscillations primarily occur at the frequency π.

Advanced Time Series HW5 PR24

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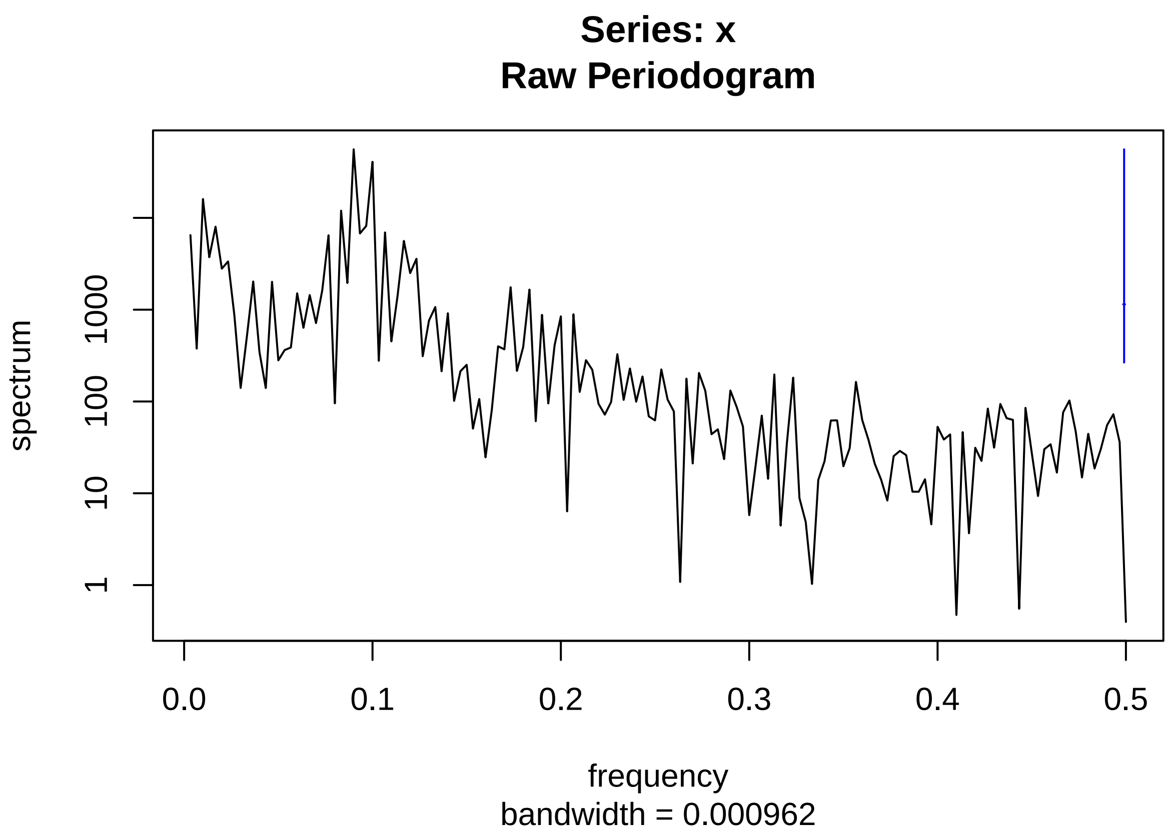
Problem 24

```
data("sunspot.year")
X = sunspot.year
Y = X - 49.13
```

```
# Define the ARIMA model
# Calculate the spectral density
spectrum <- spectrum(Y)
```



```r
max_freq <- spectrum$freq[which.max(spectrum$spec)]

# Calculate the corresponding period
period <- 1 / max_freq

cat("Frequency at Maximum Spectral Density:", max_freq, "Hz\n")
```

```
## Frequency at Maximum Spectral Density: 0.09 Hz
```

```r
cat("Corresponding Period:", period, "time units")
```

```
## Corresponding Period: 11.11111 time units
```

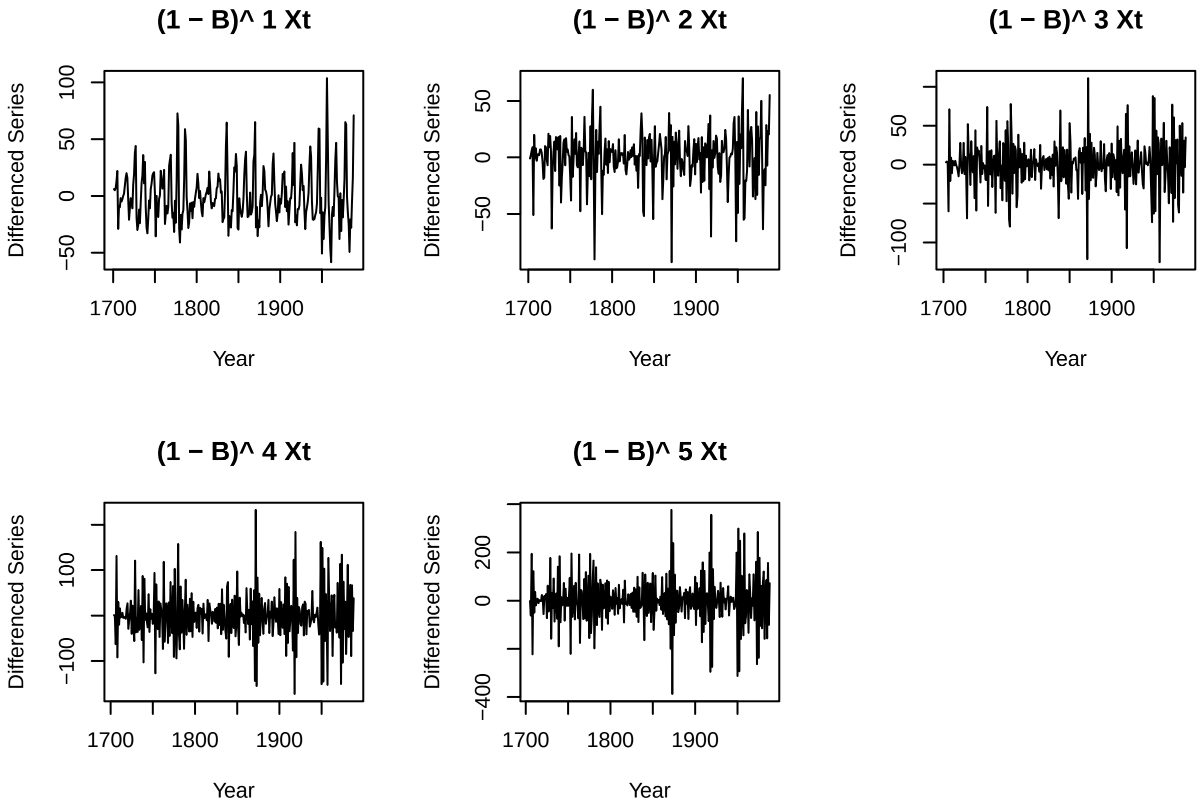
Problem 25. (d)

```r
n_values <- c(1, 2, 3, 4, 5)

# Create a plot for each n value
par(mfrow = c(2, 3))  # Arrange plots in a 2x3 grid

for (n in n_values) {
  # Calculate the differenced series for the given n
  diff_series <- diff(Y, differences = n)

  # Plot the differenced series
  plot(diff_series, main = paste("(1 - B)^", n, "Xt"), xlab = "Year", ylab = "Differenced Series")
}
```


(1 − B)^ 1 Xt
Differenced Series
100
50
0
−50
1700
1800
1900
Year
(1 − B)^ 2 Xt
Differenced Series
50
0
−50
1700
1800
1900
Year
(1 − B)^ 3 Xt
Differenced Series
50
0
−100
1700
1800
1900
Year
(1 − B)^ 4 Xt
Differenced Series
100
−100
1700
1800
1900
Year
(1 − B)^ 5 Xt
Differenced Series
200
0
−400
1700
1800
1900
Year