Time Series Homework 6

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Problem 26

Denote $\mathcal{P}_{\overline{sp}\{X_1,...,X_n\}} = P_n$, and $\widehat{X}_n = P_n X_n$. Since the process is causal,

$$X_{n+h} = \sum_{j}^{\infty} \psi_j Z_{n+h-j}$$
$$\widehat{X}_{n+h} = \sum_{j=h}^{\infty} \psi_j Z_{n+h-j},$$

where last equation holds because Z_{n+h-j} is orthogonal to $\overline{sp}\{X_1, ..., X_n\}$ when $j = 1, ..., (j-1)$, and $Z_{n+h-j} = X_{n+h-j} ... - \phi_p X_{n+h-j-p}$, hence projection of Z_{n+h-j} onto $\overline{sp}\{X_1, ..., X_n\}$ is the same when $n > p$. Then

$$||X_{n+h} - \widehat{X}_{n+h}||^2 = ||\sum_{j=0}^{h-1} \psi_j Z_{n+h-j}||^2$$
$$= \sum \psi_j^2 ||Z_{n+h-j}||^2 = \sigma^2 \sum \psi_j^2.$$

Problem 27

(a)

First compute covariance functions using the formula cos(A)cos(B) = 1/2 [cos(A-B) + cos(A+B)] and sin(A)sin(B) = [cos(A-B) - cos(A+B)].

$$\gamma(h) = \begin{cases} 4 \cdot cos(\pi h/3) & if\ |h| > 1, \\ 2 & if\ |h| = 1, \\ 4 & if\ h = 0. \end{cases}$$

The best linear predictor is $\Gamma^{-1}\gamma$, that is,

$$2 \cdot \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ \sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 2.09 \\ -0.17. \end{pmatrix}$$

(b)

(c)

Problem 28

(a)

Let A be an arbitrary invariant set.

$$P(A \cap T^{-n}A) \to P(A)^2$$

implies P(A) = 0 or 1.

(b)

Note that the π system generate $\mathcal{F}$. This means that for any given B $\in \mathcal{F}$, there exists a countable family of sets in π system such that union of those set is B. So we can extend the equation that holds in π system to $\mathcal{F}$.

(c)

It is sufficient to show the equation holds in π system. Let C be a class of all cylinders in F. Clearly, this is a π system. Note that if two cylinders A, and B are disjoint, then $P(A \cap B) = P(A)P(B)$ because outcomes in Bernoulli scheme are independent. For any rank k cylinder A and rank P cylinder B, we have $A \cap T^{-n}B \to \emptyset$. Lastly, Bernoulli shifting is measure preserving, hence the equation holds in the class of cylinders, which is a π system that generates F.

(d)

There is not invariant set, so there is no non-trivial invariant set, hence T is ergodic. This is not mixing because the sequence $p(A \cap T^{-n}B)$ does not converge.

Problem 29

Let T be a map shifting a stochastic process to the left by one. Define a set A:

$$A := [w \in \Omega | (\cdots, X_{-1}(w), X_0(w), X_1(w), \cdots) \in [0,1]^{\mathbf{Z}}].$$

Then I.I.D. assumption implies that $P\xi^{-1}(T^{-1}A) = P\xi^{-1}(A)$, hence A is an invariant set. Suppose A is a non trivial invariant set with $0 < P\xi^{-1}(A) < 1$. This means that $0 < P\xi^{-1}(X_0(w) \in [0,1]^Z) < 1$,but this is a contradiction because

$$P\xi^{-1}(A) = \lim_{n\to\infty} \prod_i^n P\xi^{-1}(X_i(w) \in [0,1]^Z),$$

which is 0 if $0 < P\xi^{-1}(X_0(w) \in [0,1]^Z) < 1$. Therefore $P\xi^{-1}(X_0(w) \in [0,1]^Z)$ must be 0 or 1, and T is ergodic.