Time Series Homework 4

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1 Problem 15

Note that

$$\sum_{j=0}^{n}(1 - \frac{j}{n})\mathbf{X}_{t-j} - \mathbf{Z}_t$$

$$= \sum_{j=0}^{n}(1 - \frac{j}{n})(\mathbf{Z}_{t-j} - \mathbf{Z}_{t-j-1)} - \mathbf{Z}_t$$

$$= -\frac{1}{n}(\mathbf{Z}_{t-1} + \mathbf{Z}_{t-2} + \cdots + \mathbf{Z}_{t-n} - n\mathbf{Z}_{t-n-1}) + ((\mathbf{Z}_t - \mathbf{Z}_{t-1}) + (\mathbf{Z}_{t-1} - \mathbf{Z}_{t-2}) \cdots + (\mathbf{Z}_{t-n} - \mathbf{Z}_{t-n-1})) - \mathbf{Z}_t$$

$$= -\frac{1}{n}(\mathbf{Z}_{t-1} + \cdots + \mathbf{Z}_{t-n}).$$

Denote $\bar{\mathbf{Z}} = \frac{1}{n}(\mathbf{Z}_{t-1} + \cdots + \mathbf{Z}_{t-n})$, then $\bar{\mathbf{Z}} \sim N(0, \sigma^2/n)$. This implies that

$$\lim_{n\to\infty} \mathbb{E}\left(\sum_{j=0}^{n}(1 - \frac{j}{n})\mathbf{X}_{t-j} - \mathbf{Z}_t\right)^2 = \lim_{n\to\infty} \mathbb{E}(\bar{\mathbf{Z}})^2 = 0.$$

2 Problem 16

2.1 (a)

We prove by induction. By definition of non-negative definiteness of a complex-valued function, for any complex numbers $a = (a_1, ..., a_n)^\top$, and any $t_1, .., t_n \in \mathbb{Z}$,

$$\sum_{j=1}^{n}\sum_{k=1}^{n} a_j \bar{a}_k K(t_j - t_k) \geq 0. \tag{*}$$

First, put $n = 2$, and $a = (1, 1)^\top$. Then (*) can be written as

$$2 * k(0) + k(-1) + k(1) \geq 0,$$

which means $Img(k(-1))) = -Img(k(1))$. Now put $a = (1, i)^\top$, then * implies that $Re(k(-1)) = Re(k(1))$, hence $k(1) = \overline{k(-1)}$.

Suppose $k(n) = \overline{k(-n)}$ for $n = 1, ..., (m - 2)$. Now we consider when $n = m$. Put $a = (a_1, a_2)^\top$, where $a_1 = (1, ..., 1)^\top$ is (m-1) by 1 and $a_2 = (1)$. We can rewrite (*) as

$$a_1^\top b_1 a_1 + a_2^\top b_3 a_1 + a_1^\top b_2 a_2 + a_2^\top b)4a_2$$
$$= a_1^\top b_1 a_1 + b_3 a_1 + a_1^\top b_2 + b_4, \tag{1}$$

where b_1 is (m-1) by (m-1) matrix such that (i,j) component is k(i-j), and $b_2 = (k(-(m-1)), \cdots, k(-1))^\top$ and $b_3 = (k(m-1), \cdots, k(1)$ and $b_4 = k(0)$. Clearly, $a_1^\top b_1 a_1 + b_4 \geq 0$ by assumption, so we only consider two terms in the middle in (1), and it gives

$$b_3 a_1 + a_1^\top b_2 = \left(k(1) + k(-1) + \cdots k(-(m-2)) + k(m-2) \right) + k(-(m-1)) + k(m-1) \geq 0.$$

By assumption, we only consider when $k(-(m-1)) + k(m-1) \geq 0$, then we have $Img(k(-(m-1))) = -Img(k((m-1)))$.

Similary, put $a = (a_1, a_2)^\top$, where $a_1 = (1, ..., 1)^\top$ is (m-1) by 1 and $a_2 = (i)$ and consider when (*) is true given assumption $k(n) = \overline{k(-n)}$ for $n = 1, ..., (m-2)$. Then we have $Re(k(-(m-1))) = -Re(k((m-1)))$. Combining the two results, we have $k(m-1) = \overline{k(-m-1)}$.

2.2 (b)

For any $a = (a_1^\top, a_2^t op)^\top$, where both a_1, a_2 are n by 1, (*) in part (a) can be written as

$$a_1^\top \mathbf{K}_1^{(n)} a_1 + a_2 \top \mathbf{K}_2^{(n)} a_1 - a_1 \top \mathbf{K}_2^{(n)} a_2 + a_2 \top \mathbf{K}_2^{(n)} a_2. \tag{2}$$

By definition of non-negative definiteness, $a_1^\top \mathbf{K}_1^{(n)} a_1 \geq 0$, and $\mathbf{K}_2^{(n)} a_2 + a_2 \top \mathbf{K}_2^{(n)} a_2 \geq 0$. Therefore $(2) \geq 0$, which proves $\mathbf{L}^{(n)}$ is non-negative definite. Also,

$$K_n = K_1 + iK_2 = K_1^\top - iK_2^\top$$

implies that K_1 is symmetric and K_2 is skew symmetric, so $L^{(n)}$ is symmetric.

2.3 (c)

Note that

$$
\begin{aligned}
E(W_n W_n^\top) \\
&= E((Y_n + iZ_n)(Y_n + iZ_n)^\top) \\
&= E(Y_n Y_n^\top + Z_n Z_n^\top) + iE(Z_n Y_n^\top - Y_n Z_n^\top) \\
&= \frac{1}{2}(K_1^{(n)} + K_1^{(n)}) + i * \frac{1}{2}(K_2^{(n)} - (-K_2^{(n)})) \\
&= K_1^{(n)} + iK_2^{(n)} = K^{(n)}.
\end{aligned}
$$

2.4 (d)

Let $F_{t_1, ..., t_n}$ be a cumulative distribution of $\left((Y_1, Z_1), ..., (Y_n, Z_n) \right)$,a probability measure on $(\mathbf{R}^2)^k$.

1. Since the distribution is normal, for all permutations π of $\{1, ..., k\}$ and measurable sets $F_i \in \mathbf{R}^2$,

$$F_{t_{\pi(1)}, ..., t_{\pi(n)}}(F_{\pi(1)} \times \cdots \times F_{\pi(n)}) = F_{t_1, ..., t_n}(F_1 \times \cdots \times F_n).$$

2. For all measurable sets $F_i \in \mathbf{R}^2, m \in \mathbb{N}$,

$$F_{t_1, ..., t_n}(F_1 \times \cdots \times F_n) = F_{t_1, ..., t_n, ..., t_{k+m}}(F_1 \times \cdots \times F_n \times \underbrace{R^2 \times \cdots R^2}_{m}),$$

because marginalization of multivariate normal is still a normal.

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With 1. and 2. satisfied, the Kolmogorov extension theorem (also known as Kolmogorov existence theorem, Daniell-Kolmogorov theorem), there exists a probability space $(\Omega, F, \mathbb{P})$ and a stochastic process $X : T \times \Omega \to R^2$ such that

$$F_{t_1,\ldots,t_n}(F_1 \times \cdots \times F_n) = \mathbb{P}(X_{t_1} \in F_1, \ldots, X_{t_n} \in F_n)$$

for any finite finite collection of times $(t_1, \ldots, t_n)$, which proves the existence of $\{Y_t\}, \{Z_t\}$. The theorem says a suitably "consistent" collection of finite-dimensional distributions will define a stochastic process.

2.5 (e)

1.

$$E|X_t|^2 = EY_t^2 + EZ_t^2 = \sigma_y^2 + \sigma_z^2 < \infty.$$

2.

$$EX_t = EY_t + i * EZ_t = 0$$

is independent of t.

3.

$$\gamma_x(h) = EX_{t+h}\bar{X}_t = EY_{t+h}Y_t + EZ_{t+h}Z_t + i * EZ_{t+h}Y_t - i * EZ_tY_{t+h} = K_1(h) + K_2(h)$$

is independent of t.

3   Problem 17

3.1   (a)

take derivative of both RHS and LHS with respect to t, then

$$i \sum_{j=1}^n a_j \lambda_j e^{it\lambda_j} = 0,$$

which means $\sum_{j=1}^n a_j \lambda_j e^{it\lambda_j} = 0$ for all $t \in Z$. Then it must hold that

$$a_j \lambda_j e^{it\lambda_j} = 0$$

for all $1 \le j \le n$. Since λ_j can take 0 at most 1, so at least (n-1) elements in $\{a_1, \ldots, a_n\}$ must be 0. If there is one non-zero in $\{a_1, \ldots, a_n\}$, then we don't satisfy the assumption of the problem so we must have $a_1 = \cdots = a_n = 0$.

3.2   (b)

$\Rightarrow$ We can rewrite X_t as

$$X_t$$
$$= (A_1 e^{it\lambda_1} + A_{n-1} e^{it\lambda_{n-1}}) + (\cdots) + \cdots A_n e^{it\lambda_n}. \tag{1}$$

Since A_n is real, it is sufficient to show $(A_j e^{it\lambda_j} + A_{n-j} e^{it\lambda_{n-j}})$ is real. Let $A_j = a + bi, a, b \in R$, and denote $\theta := t\lambda_j$, then

$$(A_j e^{it\lambda_j} + A_{n-j} e^{it\lambda_{n-j}})$$
$$= (a + bi)(cos\theta + isin\theta) + (a - bi)(cos\theta - isin\theta) =$$
$$C + i(bcos\theta + asin\theta - bcos\theta - asin\theta), C \in R,$$

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which proves that sum of pairs in (1) are real, hence X_t is real valued.

$\Leftarrow$

Note that $\lambda_n = \pi$ but λ_j cannot take $-\pi$, so λ_n must be real.

Now, $\{\lambda_1, ..., \lambda_{n-1}\}$ must be paired in a way that λ_j, λ_k in a pair satisfies $\lambda_j - \lambda_k = 0$. But considering the condition $-\pi < \lambda_1 < \lambda_2 < \cdots < \lambda_n = \pi$, we must have $\lambda_j = -\lambda_{n-j}$, so that

$$exp(it\lambda_j) + exp(-it\lambda_j) = 2cos(t\lambda_j) \tag{1}$$

is real because $sin(-\theta) = -sin\theta$. Then we must have $A_j = \bar{A}_{n-j}$.

4   Problem 18

If it has a spectral density, then spectral density f() is even function. Then we have

$$F(-\lambda^-)$$
$$= \int_{-\pi^+}^{-\lambda^{-1}} = \int_{\lambda_+}^{\pi^-} f(-x)dx$$
$$= \int_{\lambda^+}^{\pi^-} f(x)dx.$$

Therefore we have
$$F(\lambda) + F(-\lambda^-) = F(\pi^-).$$

5   Problem 19

Please see the last pages.

6   Problem 20

Suppose there exists some n such that Γ_{n-1} is not singular but Γ_n is singular. Then there exists linear dependencies and there exist some α such that $\gamma(j + (n-1)) = \alpha\gamma(j)$. Assuming spectral density exists.

$$f_x(\lambda) = \gamma(0) + \frac{1}{\pi}\left(\frac{\sum_{j=1}^{n-2} \gamma(j) * cos(\lambda * j)}{1 - \alpha}\right)$$
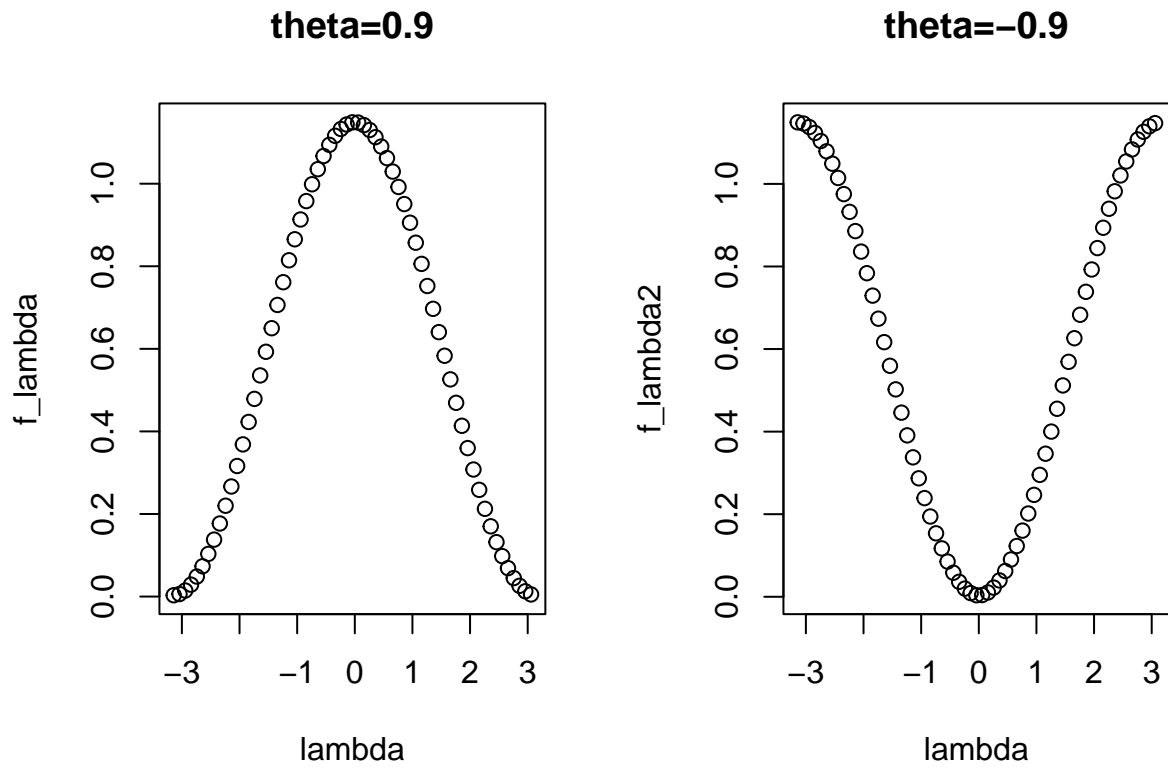
TS_HW_4

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Problem 19

By theorem 4.4.2,

$$f_x(\lambda) = \frac{\sigma^2 |\theta(e^{-i\lambda})|^2}{2\pi |\phi(e^{-i\lambda})|^2}, \quad -\pi \leq \lambda \leq \pi.$$

(a)

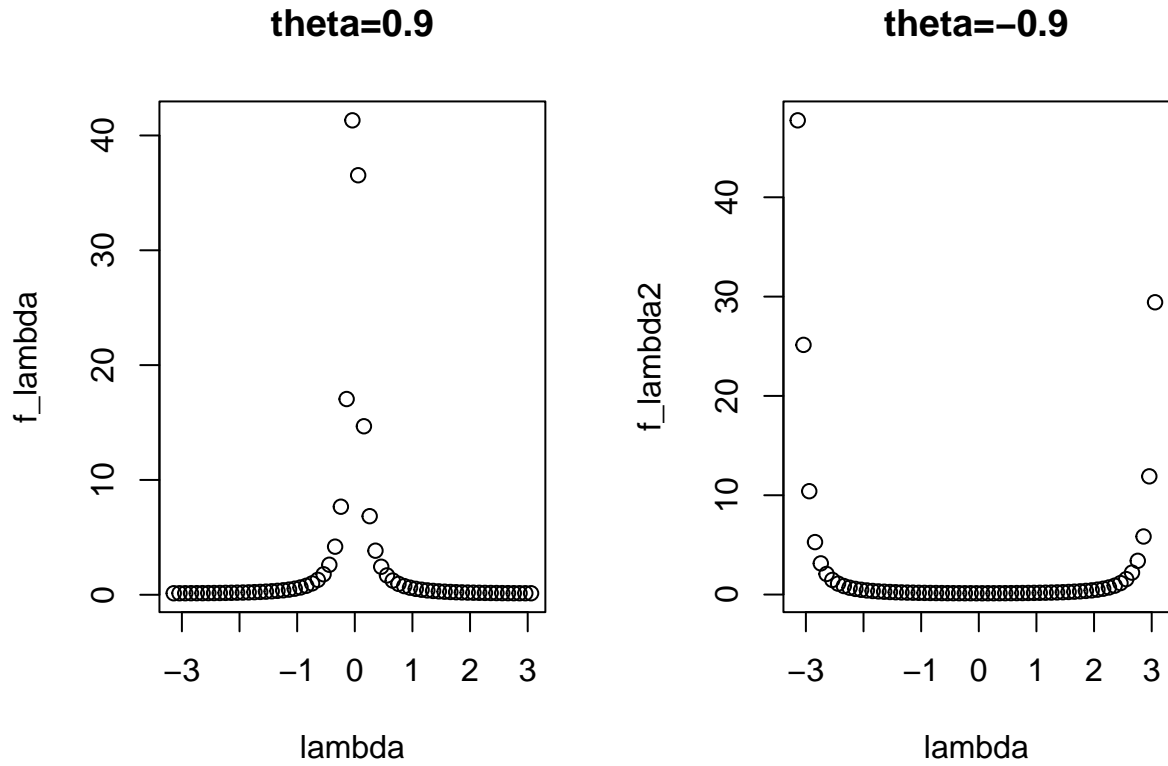$$f_x(\lambda) = \sigma^2/(2*\pi) * (1 + 2\theta \cos\lambda + \theta^2).$$

```
par(mfrow=c(1,2))
theta = 0.9
sigma_sq = 2
lambda = seq(-pi,pi, 0.1)
f_lambda = sigma_sq/(2* pi) * (1+ 2 * theta * cos(lambda) + theta^2)
plot(lambda,f_lambda, main="theta=0.9")
theta = -0.9
f_lambda2 = sigma_sq/(2* pi) * (1+ 2 * theta * cos(lambda) + theta^2)
plot(lambda,f_lambda2, main="theta=-0.9")
```

theta=0.9   **theta=−0.9**

(b)

$$f_x(\lambda) = \sigma^2/(2*\pi) * (1 - 2\phi cos\lambda + \phi^2)^{-1}.$$

```r
par(mfrow=c(1,2))
phi = 0.9
sigma_sq = 3
lambda = seq(-pi,pi, 0.1)
f_lambda = sigma_sq/(2* pi) * (1 - 2 * phi * cos(lambda) + phi^2)^{-1}
plot(lambda,f_lambda, main="theta=0.9")
phi = -0.9
f_lambda2 = sigma_sq/(2* pi) * (1 - 2 * phi * cos(lambda) + phi^2)^{-1}
plot(lambda,f_lambda2, main="theta=-0.9")
```

theta=0.9 theta=−0.9

(c)

For AR(3),

$$f_x(\lambda) = \frac{\sigma^2}{2\pi} \Bigg(2cos\lambda * (-\phi_1 + \phi_1\phi_2 + \phi_2\phi_3) + 2cos2\lambda * (-\phi_2 + \phi_1\phi_3) + 2cos(3\lambda) * (-\phi_3) + 1 + \phi_1^2 + \phi_2^2 + \phi_3^2 \Bigg)^{-1}$$

and for MA(3),

$$f_x(\lambda) = \frac{\sigma^2}{2\pi} \Bigg(2cos\lambda * (\theta_1 + \theta_1\theta_2 + \theta_2\theta_3) + 2cos2\lambda * (\theta_2 + \theta_1\theta_3) + 2cos(3\lambda)\theta_3 + 1 + \theta_1^2 + \theta_2^2 + \theta_3^2 \Bigg)$$

lastly for ARMA(3,2)

$$f_x(\lambda) = \frac{\sigma^2}{2\pi} \Bigg(2cos\lambda*(-\phi_1+\phi_1\phi_2+\phi_2\phi_3)+2cos2\lambda*(-\phi_2+\phi_1\phi_3)+2cos(3\lambda)*(-\phi_3)+1+\phi_1^2+\phi_2^2+\phi_3^2 \Bigg)^{-1}*(2cos(\lambda)*(\theta_1+\theta_1\theta_2)+2*c$$

```r
library(rootSolve)
fun <- function(x) {1-0.8*x+0.5*x^2+0.2*x^3}
roots = polyroot(c(1, -0.8,0.5,0.2))

par(mfrow=c(1,3))
theta = c(0.8, -0.5, -0.2)
sigma_sq = 4
lambda = seq(-pi,pi, 0.1)
```

3

```
tmp2 =     ( 2*cos(lambda)*(-theta[1]+theta[1]*theta[2]+theta[2]*theta[3])+
              2*cos(2*lambda) *(-theta[2]+theta[1]*theta[3]) + 2*cos(3*lambda)*(-theta[3]) +
              1 + theta[1]^2+theta[2]^2+theta[3]^2)
f_lambda2 = sigma_sq/(2* pi) * tmp2^{-1}
plot(lambda,f_lambda2, main="AR(3)")

tmp1 =     ( 2*cos(lambda)*(theta[1]+theta[1]*theta[2]+theta[2]*theta[3])+
              2*cos(2*lambda) *(theta[2]+theta[1]*theta[3]) + 2*cos(3*lambda)*(theta[3]) +
              1 + theta[1]^2+theta[2]^2+theta[3]^2)
f_lambda = sigma_sq/(2* pi) * tmp1

plot(lambda,f_lambda, main="MA(3)")

theta2 = c(0.5,0.3)
tmp3 = 2* cos(lambda) * (theta2[1]+ theta2[1]*theta2[2]) + 2*cos(2*lambda)*theta2[2]+ 1+ +theta2[2]^2+t
f_lambda3 = sigma_sq/(2* pi) *tmp2^{-1} * tmp3
plot(lambda,f_lambda3, main="ARMA(3,2)")
```