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1 Derivation

Setting

I used the following Matern function where $\mathcal{K}_v(\cdot)$ is a modified Bessel function of second kind.

$$M_{v,\beta}(r) = \sigma^2 \frac{2^{1-v}}{\Gamma(v)} \left(\frac{r}{\beta}\right)^v \mathcal{K}_v\left(\frac{r}{\beta}\right).$$

Empirical semi variogram at lag $k\delta$ can be written as

$$\widehat{\gamma}(k\delta) = \frac{1}{2(m-k)} \sum_{j=1}^{m-k} \left\{ \mathbf{Z}(j\delta + k\delta) - \mathbf{Z}(j\delta) \right\}^2.$$

2 Relative standard error of semi veriogram

We can use (1) to obtain covariance function. Take square root of covariance gives standard deviation. Then divide this by $\mathbb{E}\widehat{\gamma}(k\delta)$ as below gives relative standard error.

3 Correlation and Covariance function of semi veriogram

Then correlation between $\widehat{\gamma}(k\delta)$ and $\widehat{\gamma}(k'\delta)$ is

$$cor(\widehat{\gamma}(k\delta), \widehat{\gamma}(k'\delta)) = \frac{cov(\widehat{\gamma}(k\delta), \widehat{\gamma}(k'\delta))}{\sqrt{var(\widehat{\gamma}(k\delta))}\sqrt{var(\widehat{\gamma}(k'\delta))}}.$$

Now let's look at $cov(\widehat{\gamma}(k\delta), \widehat{\gamma}(k'\delta))$,

$$cov(\widehat{\gamma}(k\delta), \widehat{\gamma}(k'\delta)) = \mathbb{E}\widehat{\gamma}(k\delta)\widehat{\gamma}(k'\delta) - \mathbb{E}\widehat{\gamma}(k\delta)\mathbb{E}\widehat{\gamma}(k'\delta). \tag{1}$$

Observe that

$$\begin{aligned}
&\mathbb{E}\widehat{\gamma}(k\delta) \\
&\quad = \frac{1}{2(m-k)} \sum_{j=1}^{m-k} \mathbb{E}\left[\mathbf{Z}(j\delta + k\delta)^2 - 2\mathbf{Z}(j\delta + k\delta)\mathbf{Z}(j\delta) + \mathbf{Z}(j\delta)^2 \right] \\
&\quad = \frac{1}{2(m-k)} * (m-k) * 2[C(0) - C(k\delta)] \\
&\quad = C(0) - C(k\delta),
\end{aligned}$$

where $C(\cdot)$ is a true covariance.

Similarly, $\mathbb{E}\widehat{\gamma}(k'\delta) = C(0) - C(k'\delta)$, therefore

$$\mathbb{E}\widehat{\gamma}(k\delta)\mathbb{E}\widehat{\gamma}(k'\delta) = C(0)^2 - \left(C(k\delta) + C(k'\delta) \right) C(0) + C(k\delta)C(k'\delta). \tag{2}$$

It is sufficient to calculate $\mathbb{E}\widehat{\gamma}(k\delta)\widehat{\gamma}(k'\delta)$ to obtain the correlation in (1). Let

$$\begin{aligned} A_j &= \mathbf{Z}(j\delta + k\delta)^2 - 2\mathbf{Z}(j\delta + k\delta)\mathbf{Z}(j\delta) + \mathbf{Z}(j\delta)^2 \\ B_s &= \mathbf{Z}(s\delta + k'\delta)^2 - 2\mathbf{Z}(s\delta + k'\delta)\mathbf{Z}(s\delta) + \mathbf{Z}(s\delta)^2, \end{aligned}$$

then we have that

$$\mathbb{E}\widehat{\gamma}(k\delta)\widehat{\gamma}(k'\delta) = \frac{1}{4(m-k)(m-k')} \sum_{j=1}^{m-k} \sum_{s=1}^{m-k'} \mathbb{E} A_j B_s, \tag{3}$$

$$\begin{aligned} A_j B_s = & \\ & + \mathbf{Z}(j\delta + k\delta)^2 \mathbf{Z}(s\delta + k'\delta)^2 \\ & - 2\Big[\mathbf{Z}(j\delta + k\delta)^2 \mathbf{Z}(s\delta + k'\delta)\mathbf{Z}(s\delta)\Big] \\ & + \mathbf{Z}(j\delta + k\delta)^2 \mathbf{Z}(s\delta)^2 \\ \\ & - 2\Big[\mathbf{Z}(s\delta + k'\delta)^2 \mathbf{Z}(j\delta + k\delta)\mathbf{Z}(j\delta)\Big] \\ & + 4\Big[\mathbf{Z}(j\delta + k\delta)\mathbf{Z}(j\delta)\mathbf{Z}(s\delta + k'\delta)\mathbf{Z}(s\delta)\Big] \\ & - 2\Big[\mathbf{Z}(s\delta)^2 \mathbf{Z}(j\delta + k\delta)\mathbf{Z}(j\delta)\Big] \\ \\ & + \mathbf{Z}(j\delta)^2 \mathbf{Z}(s\delta + k'\delta)^2 \\ & - 2\Big[\mathbf{Z}(j\delta)^2 \mathbf{Z}(s\delta + k'\delta)\mathbf{Z}(s\delta)\Big] \\ & + \mathbf{Z}(j\delta)^2 \mathbf{Z}(s\delta)^2. \end{aligned}$$

Defining $cov(Z(j), Z(k) = \sigma_{jk})$. Assuming weak stationarity, the central fourth moments of jointly Gaussian random variables are

$$\begin{aligned} & \mathbb{E}(\mathbf{Z}(j) - \mu_j)(\mathbf{Z}(k) - \mu_k)(\mathbf{Z}(l) - \mu_l)(\mathbf{Z}(m) - \mu_m) \\ & = \sigma_{jk}\sigma_{lm} + \sigma_{jl}\sigma_{km} + \sigma_{jm}\sigma_{kl} \\ & = C(j-k)C(l-m) + C(j-l)C(k-m) + C(j-m)C(k-l). \end{aligned}$$

Applying this to above $A_j B_s$,

$$
\begin{aligned}
&\mathbb{E}A_jB_s \\
&\qquad + C(0)^2 + 2 * C((j+k-s-k')\delta)^2 \\
&\qquad - 2\Big[C(0)C(k'\delta) + 2 * C((j+k-s-k')\delta)C((j+k-s)\delta)\Big] \\
&\qquad + C(0)^2 + 2 * C((j+k-s)\delta)^2 \\
&\\
&\qquad - 2\Big[C(0)C(k\delta) + 2 * C((j+k-s-k')\delta)C((j-s-k')\delta)\Big] \\
&\qquad + 4\Big[C((k)\delta)C(k')\delta) + C((j+k-s-k')\delta)C((j-s)\delta) + C((j+k-s)\delta)C((j-s-k')\delta)\Big] \\
&\qquad - 2\Big[C(0)C(k\delta) + 2 * C((j+k-s)\delta)C((j-s)\delta)\Big] \\
&\\
&\qquad + C(0)^2 + 2 * C((j-s-k')\delta)^2 \\
&\qquad - 2\Big[C(0)C(k'\delta) + 2 * C((j-s-k')\delta)C((j-s)\delta)\Big] \\
&\qquad + C(0)^2 + 2 * C((j-s)\delta)^2.
\end{aligned}
$$

In this way, we can compute (3), and combining with (2) gives (1).