Time Series Homework 2

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1 Problem 15

Note that

$$\sum_{j=0}^{n}(1-\frac{j}{n})\mathbf{X}_{t-j}-\mathbf{Z}_t$$
$$=\sum_{j=0}^{n}(1-\frac{j}{n})(\mathbf{Z}_{t-j}-\mathbf{Z}_{t-j-1)-\mathbf{Z}_t}$$
$$=-\frac{1}{n}(\mathbf{Z}_{t-1}+\mathbf{Z}_{t-2}+\cdots+\mathbf{Z}_{t-n}-n\mathbf{Z}_{t-n-1})+((\mathbf{Z}_t-\mathbf{Z}_{t-1})+(\mathbf{Z}_{t-1}-\mathbf{Z}_{t-2})\cdots+(\mathbf{Z}_{t-n}-\mathbf{Z}_{t-n-1}))-\mathbf{Z}_t$$
$$=-\frac{1}{n}(\mathbf{Z}_{t-1}+\cdots+\mathbf{Z}_{t-n}).$$

Denote $\bar{\mathbf{Z}}=\frac{1}{n}(\mathbf{Z}_{t-1}+\cdots+\mathbf{Z}_{t-n})$, then $\bar{\mathbf{Z}}\sim N(0,\sigma^2/n)$. This implies that

$$\lim_{n\to\infty}\mathbb{E}\left(\sum_{j=0}^{n}(1-\frac{j}{n})\mathbf{X}_{t-j}-\mathbf{Z}_t\right)^2=\lim_{n\to\infty}\mathbb{E}(\bar{\mathbf{Z}})^2=0.$$

2 Problem 16

2.1 (a)

We prove by induction. By definition of non-negative definiteness of a complex-valued function, for any complex numbers $a=(a_1,...,a_n)^\top$, and any $t_1,..,t_n\in\mathbb{Z}$,

$$\sum_{j=1}^{n}\sum_{k=1}^{n}a_j\bar{a_k}K(t_j-t_k)\geq 0. \tag{*}$$

First, put $n=2$, and $a=(1,1)^\top$. Then (*) can be written as

$$2*k(0)+k(-1)+k(1)\geq 0,$$

which means $Img(k(-1)))=-Img(k(1))$. Now put $a=(1,i)^\top$, then * implies that $Re(k(-1))=Re(k(1))$, hence $k(1)=\overline{k(-1)}$.

Suppose $k(n)=\overline{k(-n)}$ for $n=1,...,(m-2)$. Now we consider when $n=m$. Put $a=(a_1,a_2)^\top$, where $a_1=(1,...,1)^\top$ is (m-1) by 1 and $a_2=(1)$. We can rewrite (*) as

$$\begin{aligned}&a_1^\top b_1a_1+a_2^\top b_3a_1+a_1^\top b_2a_2+a_2^\top b)4a_2\\&=a_1^\top b_1a_1+b_3a_1+a_1^\top b_2+b_4,\end{aligned}\tag{1}$$

where b_1 is (m-1) by (m-1) matrix such that (i,j) component is k(i-j), and $b_2 = (k(-(m-1)), \cdots, k(-1))^\top$ and $b_3 = (k(m-1), \cdots, k(1)$ and $b_4 = k(0)$. Clearly, $a_1^\top b_1 a_1 + b_4 \geq 0$ by assumption, so we only consider two terms in the middle in (1), and it gives

$$b_3 a_1 + a_1^\top b_2 = \bigg(k(1) + k(-1) + \cdots k(-(m-2)) + k(m-2)\bigg) + k(-(m-1)) + k(m-1) \geq 0.$$

By assumption, we only consider when $k(-(m-1)) + k(m-1) \geq 0$, then we have $Img(k(-(m-1))) = -Img(k((m-1)))$.

Similary, put $a = (a_1, a_2)^\top$, where $a_1 = (1, ..., 1)^\top$ is (m-1) by 1 and $a_2 = (i)$ and consider when (*) is true given assumption $k(n) = \overline{k(-n)}$ for $n = 1, ..., (m-2)$. Then we have $Re(k(-(m-1))) = -Re(k((m-1)))$. Combining the two results, we have $k(m-1) = \overline{k(-m-1)}$.

2.2 (b)

For any $a = (a_1^\top, a_2^t op)^\top$, where both a_1, a_2 are n by 1, (*) in part (a) can be written as

$$a_1^\top \mathbf{K}_1^{(n)} a_1 + a_2 \top \mathbf{K}_2^{(n)} a_1 - a_1 \top \mathbf{K}_2^{(n)} a_2 + a_2 \top \mathbf{K}_2^{(n)} a_2. \tag{2}$$

By definition of non-negative definiteness, $a_1^\top \mathbf{K}_1^{(n)} a_1 \geq 0,$ and $\mathbf{K}_2^{(n)} a_2 + a_2 \top \mathbf{K}_2^{(n)} a_2 \geq 0$. Therefore (2) ≥ 0, which proves $\mathbf{L}^{(n)}$ is non-negative definite. Also,

$$K_n = K_1 + iK_2 = K_1^\top - iK_2^\top$$

implies that K_1 is symmetric and K_2 is skew symmetric, so $L^{(n)}$ is symmetric.

2.3 (c)

Note that

$$\begin{aligned}
&E(W_n W_n^\top) \\
&\qquad = E((Y_n + iZ_n)(Y_n + iZ_n)^\top) \\
&\qquad = E(Y_n Y_n^\top + Z_n Z_n^\top) + iE(Z_n Y_n^\top - Y_n Z_n^\top) \\
&\qquad = \frac{1}{2}(K_1^{(n)} + K_1^{(n)}) + i * \frac{1}{2}(K_2^{(n)} - (-K_2^{(n)})) \\
&\qquad = K_1^{(n)} + iK_2^{(n)} = K^{(n)}.
\end{aligned}$$

2.4 (d)

Let $F_{t_1,...,t_n}$ be a cumulative distribution of $\bigg((Y_1, Z_1), ..., (Y_n, Z_n)\bigg)$,a probability measure on $(\mathbf{R}^2)^k$.

1. Since the distribution is normal, for all permutations π of $\{1, ..., k\}$ and measurable sets $F_i \in \mathbf{R}^2$,

$$F_{t_{\pi(1)},...,t_{\pi(n)}}(F_{\pi(1)} \times \cdots \times F_{\pi(n)}) = F_{t_1,...,t_n}(F_1 \times \cdots \times F_n).$$

2. For all measurable sets $F_i \in \mathbf{R}^2, m \in \mathbb{N}$,

$$F_{t_1,...,t_n}(F_1 \times \cdots \times F_n) = F_{t_1,...,t_n,...,t_{k+m}}(F_1 \times \cdots \times F_n \times \underbrace{R^2 \times \cdots R^2}_{m}),$$

because marginalization of multivariate normal is still a normal.

With 1. and 2. satisfied, the Kolmogorov extension theorem (also known as Kolmogorov existence theorem, Daniell-Kolmogorov theorem), there exists a probability space $(\Omega, F, \mathbb{P})$ and a stochastic process $X : T\times\Omega \to R^2$ such that

$$F_{t_1,...,t_n}(F_1 \times \cdots \times F_n) = \mathbb{P}(X_{t_1} \in F_1, ..., X_{t_n} \in F_n)$$

for any finite finite collection of times $(t_1, ..., t_n)$, which proves the existence of $\{Y_t\}, \{Z_t\}$. The theorem says a suitably "consistent" collection of finite-dimensional distributions will define a stochastic process.

2.5 (e)

1. $$E|X_t|^2 = EY_t^2 + EZ_t^2 = \sigma_y^2 + \sigma_z^2 < \infty.$$

2. $$EX_t = EY_t + i * EZ_t = 0$$

   is independent of t.

3. $$\gamma_x(h) = EX_{t+h}\bar{X}_t = EY_{t+h}Y_t + EZ_{t+h}Z_t + i * EZ_{t+h}Y_t - i * EZ_tY_{t+h} = K_1(h) + K_2(h)$$

   is independent of t.

3 Problem 17

3.1 (a)

take derivative of both RHS and LHS with respect to t, then

$$i\sum_{j=1}^{n} a_j\lambda_j e^{it\lambda_j} = 0,$$

which means $\sum_{j=1}^{n} a_j\lambda_j e^{it\lambda_j} = 0$ for all $t \in Z$. Then it must hold that

$$a_j\lambda_j e^{it\lambda_j} = 0$$

for all $1 \leq j \leq n$. Since λ_j can take 0 at most 1, so at least (n-1) elements in $\{a_1, ..., a_n\}$ must be 0. If there is one non-zero in $\{a_1, ..., a_n\}$, then we don't satisfy the assumption of the problem so we must have $a_1 = \cdots = a_n = 0$.

3.2 (b)

$\Rightarrow$ We can rewrite X_t as

$$\begin{aligned} &X_t \\ &= (A_1e^{it\lambda_1} + A_{n-1}e^{it\lambda_{n-1}}) + (\cdots) + \cdots A_ne^{it\lambda_n}. \end{aligned} \tag{1}$$

Since A_n is real, it is sufficient to show $(A_je^{it\lambda_j} + A_{n-j}e^{it\lambda_{n-j}})$ is real. Let $A_j = a + bi, a, b \in R$, and denote $\theta := t\lambda_j$, then

$$\begin{aligned} &(A_je^{it\lambda_j} + A_{n-j}e^{it\lambda_{n-j}}) \\ &= (a + bi)(cos\theta + isin\theta) + (a - bi)(cos\theta - isin\theta) = \\ &C + i(bcos\theta + asin\theta - bcos\theta - asin\theta), C \in R, \end{aligned}$$

which proves that sum of pairs in (1) are real, hence X_t is real valued.

$\Leftarrow$

Note that $\lambda_n = \pi$ but λ_j cannot take $-\pi$, so λ_n must be real.

Now, $\{\lambda_1, ..., \lambda_{n-1}\}$ must be paired in a way that λ_j, λ_k in a pair satisfies $\lambda_j - \lambda_k = 0$. But considering the condition $-\pi < \lambda_1 < \lambda_2 < \cdots < \lambda_n = \pi$, we must have $\lambda_j = -\lambda_{n-j}$, so that

$$exp(it\lambda_j) + exp(-it\lambda_j) = 2cos(t\lambda_j) \tag{1}$$

is real because $sin(-\theta) = -sin\theta$. Then we must have $A_j = \bar{A}_{n-j}$.