Note

1 Conditional expectation

$E[X|F_0]$ is a random variable Y satisfying

1. For every $A \in F_0$
$$\int_A YdP = \int_A XdP$$
2. ***** $Y \in F_0$, so $\sigma(Y) \subset F_0$ or equivalently, $Y^{-1}(B) \in F_0$, for every Borel set in R.

2 Filtration

Filtration is a family of σ algebras $\{F_i\}_{i\in I}$ such that $F_i \subseteq F_j,\ if\ i \leq j$. I can be

1. $\{0, 1, 2, ..., n\}$
2. N
3. $Z^{\leq 0} = \{0, -1, -2, ...,\}$
4. $R^{\geq 0}$ for brownian motion.

3 Stochastic Process

A stochastic process is a set of random variables such that $\{X_i\}_{i\in I}$ is said to be adaptive to the filtration $\{F_i\}$, if $X_i \in F_i$ for each $i \in I$. For example, given a sequence of random variables, we can make $F_n = \sigma(X_1, ..., X_n)$. Then we have a filtration and a stochastic process.

4 Martingale discrete

Suppose we have a set of X adpated to $\{F_n\}$ then the stochastic process is a martingale if

1. $EX_n < \infty$
2. $E[X_{n+1}|F_n] = X_n$.

5 Stopping time

A random variable T is called a stopping time if $\{T \leq n\} \in F_n$ for each n. Stopping time only depends on the information up to present.

6 MG convergence theorem

Suppose $\{X_n\}$ is a sub MG and $sup_{n\geq 1}EX_n^{<}\infty$ then X_n converges almost surely to some X with $E|X| < \infty$.

6.1 Interpretation

If a sequence of r.v.s converges(if it has a limit) than for any $a < b$ the sequence cross over a and b finitely many time. If it does not have a limit then

$$a_1 = limsupX_n, a_2 = liminfX_n,$$

then we can take $a_1 < a < b < a_2$.

7 Crossover

For $a < b$, let

$$\begin{aligned}
&N_0 = -1\\
&N_{2k-1} = inf\{m > N_{2k-2} : X_m \leq a\}\\
&N_{2k} = inf\{m > N_{2k-1} : X_m \geq b\}\\
&U_n[a, b] = sup\{k : N_{2k} \leq n\}
\end{aligned}$$

Then

$$EU_n[a, b] \leq \frac{E[(X_n - a)^+]}{b - a}$$

8 Backward Martingale

Let $\{X_n\}_{n\leq 0}^{\infty} = \{X_0, X_{-1}, ...\}$ be adapted to $\{F_n\}_{n\leq 0}^{\infty} = \{F_0, F_{-1}, ...\}$ such that $F_{-2} \subset F_{-1} \subset F_0$. Then The stochastic process is a backward martingale if

$$E[X_{n+1}|F_n] = X_n$$

9 Theorem

Suppose $\{X_n\}_{n\leq 0}$ is a backward MG then $X_n \to X$ a.s. and in L^1. So this provides very strong convergence property.

In particular,

$$X = E[X_0|F_{-\infty}]\ a.s.(will\ not\ \ prove\ this), F_{-\infty} = \bigcap_{n=-1}^{-\infty} F_n.$$

9.1 proof:

Fix a finite $n < 0$. The sequence $\{X_{-n}, ..., X_0$ can be written as $Y_1, Y_2, ..., Y_n$ which is then an ordinary MG with respect to corresponding filtration set. Note that

$$EU_n[a, b] \leq \frac{E[(X_0 - a)^+]}{b - a}$$

this does not depend on n, by taking n goes to ∞, we have that

$$EU_\infty[a, b] < \infty,$$

hence $EU_x[a,b] < \infty$ a.s. Therefore limit exists. A difference compared to MG is that we don't know the tail.

10 Backward MG $X_n \to E[X_0|F_{-\infty}]$ a.s. and in L1.

10.1 Proof:

We have shown $X_n \to X$(for some r.v. X) a.s. as $n \to -\infty$. We will assume the convergence in L1 (check Durrett chapter 4.7). Now we will show X is equal to $E[X_0|F_{-\infty}]$.

Recall

Lp convergence implies p-th moment convergence. For L1, we can see that for $A \in F_n$.

$$E[|X1_A - X_n 1_A|] \leq E|X - X_n| \to 0.$$

10.2 Part 1 $X_{-\infty} = E[X|F_{-\infty}]$

For every $A \in F_{-\infty}$,

$$E[X1_A] = \lim_{n\to-\infty} EX_n 1_A \qquad \text{(by L1 convergence.)}$$

We will use the fact that

$$E[X_0|F_n] = X_n$$

by $E[X_0|F_{-1}] = X_{-1}$, $E[X_0|F_{-2}] = X_{-2}$.

For every $A \in F_{-\infty} \subset F_{-n}$, and

$$\begin{aligned} E[X_0 1_A] &= E[E[X_0 1_A|F_{-n}]] \\ &= E[E[X_0|F_{-n}] \cdot 1_A], \end{aligned}$$

which implies that

$$\begin{aligned} &E[X_n 1_A] = E[E[X_0|F_{-n}]1_A] = E[X_0 1_A] \\ &limE[X_n 1_A] = E[X1_A] = E[X_0 1_A] = E[X_0 1_A] = E[E[X_0|F_{-\infty}]1_A]. \end{aligned} \tag{1}$$

10.3 part 2

We will show $x \in F_{-\infty}$. In other words, $x \in F_k$ for every $k < 0$. (we know intersection of sigma algebra is a sigma algebra, so $F_{-\infty}$ is a sigma algebra in the filtration and we want to show x or $x_{-\infty}$ is filterated to that σ algebra).

$$\begin{aligned} X(w) &= limX_n(w) \\ &= lim_{n\to-\infty} X_{k+n}(w) \in F_k \Rightarrow X \in \cap_{k<0} F_k = F_{-\infty}. \end{aligned}$$

11 Corollary

If we have $F_n \downarrow F_{-\infty}$ and Y is a random variable in (Ω, F, P), then

$$E[Y|F_n] \to E[Y|F_{-\infty}]$$

pf: $\{E[Y|F_n]\}$ is a backward M.G. (we can easily check this by tower property)

12 Example 1

$\{\xi_n\}$ iid in L_1 on (Ω, F, P) (this condition is stronger than last pf condition of independence.) then

$$\frac{\sum \xi_k}{n} \to_{a.s.} E[\xi]$$

12.1 proof:

Let $F_{-n} = \sigma(S_n, S_{n+1}, ...) = \sigma(S_n, \xi_{n+1}, \xi_{n+2}, ...)$, then $\frac{S_n}{n} \in F_{-n}$.

$$\begin{aligned} E[\frac{S_n}{n}|F_{-(n+1)}] \\ &= E[\frac{S_n}{n}|S_{n+1}, \xi_{n+2}, ...] \\ &= E[\frac{S_{n+1} - \xi_{n+1}}{n}|same] \\ &= E[\frac{S_{n+1}}{n}|same] - E[\frac{\xi_{n+1}}{n}|same] \\ &= \frac{S_{n+1}}{n} - \frac{1}{n}E[\xi_{n+1}|S_{n+1}] = \frac{S_{n+1}}{n} - \frac{S_{n+1}}{n(n+1)} = \frac{S_{n+1}}{n+1}. \end{aligned}$$

because by iid

$$E[\xi_1|S_{n+1}] = E[\xi_2|S_{n+1}]....,$$
$$\sum E[\xi_i|S_{n+1}] = E[S_{n+1}|S_{n+1}] = S_{n+1}.$$

So we have that $\frac{S_n}{n}$ is a B.W. MG. So, we have the result.

But

$X_{-\infty}(w) = lim \frac{S_n}{n}(w) = f(\xi_1(w), ...), f is invariant under any finite permutation, so by H-S 01 law implies that f is a constant.$

We know

$$\begin{aligned} X_{-\infty} &= E[\xi_1|F_{-\infty}] \\ EX_{-\infty} &= E[E[\xi_1|F_{-\infty}]] \\ &= E\xi_1 \end{aligned}$$

So $EX = E\xi_1$ a.s. ()

13 Example Ballot theorem

$\{\xi_j\}$ iid non-negative integer valued r.v. $S_k = \xi_1 + \xi_2 + ... + \xi_k$. Let $G = \{S_j < j \ for \ 1 \le j \le n\}$ then $P(G|S_n) = (1 - \frac{S_n}{n})^+$. Suppose ξ are $\{0, 2\}$ (I think it should be $\{0, 1, 2\}$. Otherwise, how can we have $S_3 = 3$? if we can move by even numbers?) - valued r.v. each with 0.5 then the event G stands for the event that $\{A \ leads \ B \ throughout \ the \ voting\}$.

13.1 proof:

1. If $S_n \geq n$, then $P(G|S_n) = 0$ So it suffices to consider the case $S_n < n$.

2. Consider $S_n < n$ define $X_{-j} = \frac{S_j}{j}$(for j: $1 \leq j \leq n-1$,)
which are backward martingale wrt $F_{-j} = \sigma(S_j, ..., S_n)$. Let

$$T = inf\{(-1 \geq k \geq -n)k \geq -n|X_k \geq 1\} \ and$$

(Notice that T is bounded above and bounded below, so that we can apply Wald's lemma below) if the set is empty (if there is no crossing for $1 \leq j \leq n-1$) , $T = -1$. We can check T is a stopping time, T stands for the time "Last crossing" or -1.

We claim (under $S_n < n$) $X_T = 1$ a.s. on G^c(which means $S_i \geq i$ for some $1 \leq i < n$) and 0 a.s. on G(crossing never happens so T=-1 and $S_1 = 0$.

   (a) If the crossing never happens, then $X_T = X_{-1} = S_1 = 0$ S_1 is integer does not cross 0.($S_1 < 1$)

   (b) Suppose the last crossing happen at $1 \leq j < n(S_j \geq j)$, then $S_j = j$ otherwise $S_{j+1} \geq j+1$ which is a contradiction(this means j is not the time for last crossing), then

$$X_T = X_{-j} = 1,$$

   (remember that we are only considering the case $S_n < n$ from above, so we are not considering the case L-C happens at j=n) . We know $X_{-j} = \frac{S_j}{j}$ is a BW MG w.r.t. $F_{-j} = \sigma(S_j, S_{j+1}, ..., S_n)$ and we now know $X_T = 1_{G^c}$

$$p(G^c|S_n) = E[1_{G^c}|S_n] = E[X_T|S_n] = E[X_T|F_{-n}]$$

   Now by Wald's lemma(T is bounded almost surely), $E[X_T|F_{-n}] = X_{-n} = \frac{S_n}{n}$, where $X_{-n}, ..., X_0$ is a MG w.r.t. $F_{-n}, ..., F_0$. Therefore,

$$p(G|S_n) = 1 - p(G^c|S_n) = 1 - \frac{S_n}{n}$$

   if $S_n < n$.

Wald's Lemma

$\{F_n\}$ is filtration $\{Z_n\}$ is M.G. adapted to $\{F_n\}$. T is a stopping time adapted to $\{F_n\}$. Suppose there exists $N > 0$, s.t. $T \leq N$ a.s., then $E[Z_T] = E[Z_1]$.

Proof:

$$\begin{aligned}
E[Z_T] &= E[\sum_{i=1}^{n} Z_i 1(T = i)] \\
&= \sum_{i=1}^{n} E[Z_i * 1(T = i)] \\
***** &= \sum E[Z_n 1(T = i)] \qquad (1) \\
&= E[Z_n]
\end{aligned}$$

For the equation in (1), we use the conditional expectation property and the fact that $i \leq n$.

$$*****E[Z_n 1_{T=i}] = E[E[Z_n|F_i] \cdot 1_{T=i}] = E[Z_i \cdot 1_{T=i}]$$