Definition 0.1. (derivative) For a function f we define upper and lower right derivative

$$D^* f(t) = \underset{h \downarrow 0}{limsup} \frac{f(t+h) - f(h)}{h}$$
$$D_* f(t) = \underset{h \downarrow 0}{liminf} \frac{f(t+h) - f(h)}{h}$$

Theorem 0.1. *Fix t, almost surely B(t,w) is not differentiable at t. Moreover* $D^* B(t) = \infty, \ D_* B(t) = -\infty$.

Proof. Consider the time inversion $\{X(t)\}, X(t) = tB(1/t)$ for $t > 0$ and 0 for t=0, this is s.b.m. if original B() is s.b.m.

$$\begin{aligned} D^* X(0) &= \underset{n \to \infty}{limsup} \frac{X(\frac{1}{n}) - X(0))}{\frac{1}{n}} \\ &\geq \underset{n \to \infty}{limsup} \sqrt{n}(X(\frac{1}{n} - X(0)) \\ &= \underset{n \to \infty}{lim} \sqrt{n} X(\frac{1}{n}) \\ &= \underset{n \to \infty}{limsup} \frac{B(n)}{\sqrt{n}} = +\infty. \end{aligned}$$

Above implies that $t = 0$, then a.s. B(t,w) is not differentiable.

For fixed $t > 0$, we consider

$$\bar{X}(s) := B(s+t) - B(t)$$

is also a s.b.m.(we can directly check the 4 conditions) and s.b.m is not differentiable at 0, so $\bar{X}$ is not differentiable at 0. $\Rightarrow$ original process B is not differentiable at t. □

Theorem 0.2. *(Paley- wiener- Zygmund, stronger result) Almost surely, for every t, s.b.m. is not differentiable. Moreover, either* $D^* B(t) = +\infty$ *or* $D_* B(t) = -\infty$ *or both. In particular, this implies that almost surely, B(t) is not differentiable at every* $t \geq 0$.

Proof. We can define $E_t := \{S.B.M. \ not \ differentiable \ at \ t\}$. We know $P(E_t) = 1$ for every t, then we want to show $P(\cap_{t \geq 0} E_t) = 1$.

Suppose we have $t_0 \in [0, 1]$ such that

$$-\infty < D_* B(t) \leq D^* B(t) < \infty.$$

Then

$$\underset{h \downarrow 0}{limsup} \left| \frac{B(t+h) - B(t)}{h} \right| < \infty \tag{*}$$

because $\underset{h \downarrow 0}{limsup} - \frac{B(t+h)-B(t)}{h} = -\underset{h \downarrow 0}{liminf} \frac{B(t+h)-B(t)}{h}$. This further implies there exists some finite M such that

$$\underset{h \in [0,1]}{sup} \frac{B(t+h) - B(t)}{h} \leq M \tag{**}$$

because when h is small (*) imply (**) and if h is large, then the path continuity of B.M. guarantees (**)(otherwise if there is a point at which derivative is not finite then we don't have continuity) Then we have

$$|B(t_0 + h) - B(t_0)| \leq Mh$$

now we fix M and assume $t_o \in [\frac{k-2}{2^n, \frac{k}{2^n}}]$, then for every $1 \leq j \leq 2^n - k$,

$$\begin{aligned}
&|B(\frac{k+j}{2^n}) - B(\frac{k+j-1}{2^n})| \\
&\leq |B(\frac{k+j}{2^n}) - B(t_0)| + |B(\frac{k+j-1}{2^n}) - B(t_0)| && \text{(by triangle)} \\
&\leq \frac{M(2j+1)}{2^n}
\end{aligned}$$

Let

$$\Omega_{n,k} = \{|B(\frac{k+j}{2^n}) - B(\frac{k+j-1}{2^n})| \leq \frac{M(2j+1)}{2^n} \ for\ j = 1, 2, 3\}$$

note that above is intersection of independent events,

$$\begin{aligned}
P(\Omega_{n,k}) &\leq \prod_{j=1}^{3} P(|B(\frac{k+j}{2^n}) - B(\frac{k+j-1}{2^n})| \leq \frac{M(7)}{2^n}) \\
&= P(N(0, \frac{1}{2^n}) \leq \frac{7M}{2^n})^3 && = P(N(0,1) \leq \frac{7M}{2^{n/2}})^3 \\
(&\leq \frac{1}{\sqrt{2\pi}} \cdot 2 \times \frac{7M}{2^{n/2}})^3 = \frac{cM^3}{2^{1.5n}} && \text{(check integration of two symmetric width)}
\end{aligned}$$

then we have that by B-C

$$P(\cup_{k=1}^{2^n-3} \Omega_{n,k}\ happen\ for\ i.o.) = 0$$

$\Rightarrow$

$$\begin{aligned}
&P(there\ is\ t_0 \in [0,1]\ with\ \sup_{h \in 0,1]} \frac{|B(t_0+h) - B(t_0)|}{h} \leq M) \\
&\leq P(\cup_{k=1}^{2^n-3} \Omega_{n,k}\ happen\ for\ i.o.) = 0
\end{aligned}$$

this proves that there is no derivative almost everywhere.

(heuristic)

$$f'(x_0+h) - f(x_0) \approx f'(x_0) \cdot h\ but$$
$$|B(x+h) - B(x)| \sim \sqrt{hlog(1/h)} > h.$$

$\square$

Definition 0.2. A function $f : [0, \infty) \to R$ is said to be α- Holder continuous at $x \geq 0$ if there exists $\epsilon > 0$ and $c > 0$ such that if $|y - x| < \epsilon$, then

$$|f(x) - f(y)| \leq c|x - y|^\alpha.$$

If f is α- Holder continuous at every point x, then we say f is locally α-Holder continuity every where.

Corollary 0.1. *If $\alpha < \frac{1}{2}$, then almost surely, B.M. is locally Holder continuous.*

Proof. We know for fixed x,

$$|B(x+h) - B(x)| \leq c\sqrt{hlog(1/h)} \leq ch^\alpha\ (since\ \alpha < 1/2)$$

also because $\sqrt{log(1/h)} < 1$ for $h \in (0, 1]$ $\square$

Definition 0.3. A right continuous function f:$[0, t] \to R$ is of bounded variation if

$$V_f^{(1)}(t) := \underset{\{0=t_0<t_1\ldots\leq t_k=t\}}{sup} \sum_{j=1}^{k} |f(t_j) - f(t_{j-1})| < \infty.$$

If $V_f^{(1)}(t) = \infty$ then we say f is unbounded variation. Suppose $f \in c^1[0, t]$, then $|f'| \leq M$ then

$$V_f^{(1)}(t) \leq \sum_{j=1}^{k} M(t_j - t_{j-1}) = M.T < \infty$$

In fact $V_f^{(1)}(t) = \int_0^t |f')ds$ if f is smooth enough.

analysis: f is of bounded variation if and only if f can be written as difference of increasing functions. If f is non decreasing then is of bounded variance since

$$V_f^{(1)}(t) = f(t) - f(0).$$

Facts

1. If $\{X_n\}$ is a martingale, then

$$E(\sum_{n=1}^{k}(X_n - X_{n-1})^2)$$
$$= E(x_n^2 - X_{n-1}^2)$$
$$= EX_k^2 - EX_0^2 \geq 0$$

because $EX_nX_{n-1} = EX_{n-1}^2$ by using tower law. Observe that later term has larger 2nd moment.

2. Lemma: If X,Z independent symmetric r.v. in L2, then

$$E[(X+Z)^2|X^2+Z^2] = X^2 + Z^2$$

Proof. It suffices to show $E[XZ|X^2 + Z^2] = 0$. By symmetry,

$$E[(X+Z)^2|X^2+Z^2] = E[X-Z)^2|X^2+(-Z)^2]$$

□

3. B.M. is locally Holder continuous with $\alpha < 1/2$.

Theorem 0.3. *(Quadratic variation) Suppose fix $t > 0$ and suppose we have a sequence of partitions such that*

$$p_n := 0 = t_0^{(n)} \leq t_1^{(n)} \cdots t_{k(n)}^{(n)} = t$$

which are nested and

$$\nabla_n := \underset{1\leq j\leq k(n)}{max}\{t_j^{(n)} - t_{j-1}^{(n)}\} \to 0$$

(so we are considering thinner partitions), then

$$\underset{n\to\infty}{lim} \sum_{j=1}^{k(n)} \left(B(t_j^{(n)}) - B(t_{j-1}^{(n)})\right)^2 = t \text{ a.s. and in } L_1.$$

Which further implies that B.M. is of unbounded variation.

First we prove quadratic variation converges a.s.

Proof. We define

$$X_{-n} = \sum_{j=1}^{k(n)} \left(B(t_j^{(n)}) - B(t_{j-1}^{(n)}) \right)^2,$$

define

$$g_{-n} = \sigma(X_{-n}, X_{-(n+1)}, ...).$$

Therefore $X_{-n} \in g_{-n}$. We want to show

$$E[X_{-n+1}|g_{-n}] = X_{-n}.$$

W.L.O.G, we assume P_{n+1} only adds one more point to P_n. Suppose at step (n+1), the added point is $s \in (t_1, t_2)$.

$$X_{-(n+1)} - X_{-n} = (B(S) - B(t_1))^2 + (B(t_2) - B(S))^2 - (B(t_2) - B(t_1))^2$$

let $X = B(S) - B(t_1), Z = B(t_2) - B(S)$ since we have $E[(X+Z)^2|X^2+Z^2] = X^2+Z^2$, X_{-n} is a backward M.G.

$$\begin{aligned} E[X_{-(n+1)} - X_{-n}|g_{-n-1}] &= X_{-(n+1)} - E[X_{-n}|g_{-n-1}] \\ &= 0(RHS). \end{aligned}$$

We know that $X_{-n} \to X$ a.s. and in L1,

$$EX = E[X_{-n}]$$

since $\{B(0) = 0, ..., B(t_{k(n)}^{(n)}\}$ is a discrete time MG,

$$E[X_{-n}] = E[B(t)^2 - B(0)^2] = t - 0 = t = EX$$

By Fatou's lemma

$$\begin{aligned} Var(X) &\leq \underset{n\to\infty}{limif} var\left(\sum_{j=1}^{k(n)} (B(t_j^{(n)} - B(t_{j-1}^{(n)})^2 \right) \\ &= \underset{n\to\infty}{limif} \sum_{j=1}^{k(n)} var\left(B(t_j^{(n)}) - B(t_{j-1}^{(n)}) \right)^2 \\ &= \underset{n\to\infty}{limif} 3 \sum_{j=1}^{k(n)} (t_j - t_{j-1})^2 \qquad (1) \\ &\leq \underset{n\to\infty}{limif} 3\nabla_n \sum_{j=1}^{k(n)} (t_j - t_{j-1}) \to 0. \end{aligned}$$

So X_{-n} converges to X in L_1, EX=t, but it degenerates, hence X_{-n} converges to t a.s.

Above (1) use the fact that $B(t_j^{(n)}) - B(t_{j-1}^{(n)}) \sim N(0, t_j - t_{j-1})^2$, so

$$Var[(\sqrt{t_j - t_{j-1}}N(0,1))^2] = (t_j - t_{j-1})^2 Var(N^2(0,1)) = 3(t_j - t_{j-1})^2.$$

□

Now we prove B.M. is of unbounded variation.

Proof. By Holder continuity, for every fixed ω, we can find n large enough such that

$$|B(a) - B(b) \leq |a - b|^{\alpha}$$

for every $a, b \in [0, t]$ with $|a - b| \leq \nabla(n)$. Then

$$\sum_{j=1}^{k(n)} \left(B(t_j^{(n)}) - B(t_{j-1}^{(n)}) \right)^2 \leq \sum_{j=1}^{k(n)} \nabla(n)^{\alpha} |B(t_j^{(n)}) - B(t_{j-1}^{(n)})|,$$

$$\sum_{j=1}^{k(n)} |B(t_j^{(n)}) - B(t_{j-1}^{(n)})| \geq \nabla(n)^{-\alpha} \sum_{j=1}^{k(n)} \left(B(t_j^{(n)}) - B(t_{j-1}^{(n)}) \right)^2$$

where quadratic variation goes to t and $\nabla(n)^{-\alpha}$ goes to ∞, so B.M. has infinite variation. □

Brownian motion as a Markov process

Markov property: If we have stochastic process $\{X(t)\}$ and if "predicting" $X(t')$ based on $\{X(S)\}_0^t$ is the same as $X(t)$, then we say the process has Markov property. Moreover a process is called (time-homogeneous) Markov process if it starts refresh at any fixed S. (like B.M.)

Theorem 0.4. *(Markov Property) Suppose $\{B(t)\}$ is a B.M. started at x, then $\{B(t+s) - B(s)\}$ is again a B.M. started at origin and is independent of $\{B(u)\}_{u=0}^s$.*

Proof. Definition check. □

Definition 0.4. 1. Filtration on (Ω, F, P) is a family $\{F(t)|t \geq 0\}$ of σ- algebra such that

$$F(S) \subset F(t) \subset F \ for\ every\ s \leq t.$$

2. (Ω, F, P) with filtration $\{F(t)\}$ is called a filtered probability space. $\{X(t)\}_{t \geq 0}$ is adapted to $\{F(t)\}$ if $X(t) \in F(t)$ for every $t \geq 0$. Suppose we have a B.M. $\{B(t)\}$ on (Ω, F, P), then

$$F^0(t) = \sigma(B(S)\ 0 \leq s \leq t)$$

is a filtration. We know

$$B(t+s) - B(s) \perp\!\!\!\perp \{B(u), u \in [0, s]\} = F^0(s).$$

We can also define

$$F^+(S) = \cap_{t>s} F^0(t),\ F^+(s) \supset F^0(s). \tag{right limit}$$

Theorem 0.5. *Fix S, we know $\{B(t+s) - B(s)\}$ is independent with $F^+(s)$.*

Proof. By continuity,

$$B(t+s) - B(s) = \underset{n \to \infty}{lim} B(S_n + t) - B(S_n)$$

for $S_n \in Q$ and $S_n \downarrow S$.

$$B(S_n + t) - B(S_n) \perp\!\!\!\perp F^+(S)\ for\ each\ n \tag{markov property}$$

we can choose $s_0 \in (s, s_n)$, $F^+(s) \subset F^0(s_0) \subset F^+(s_0)$ and LHS $\perp F^+(S_0)$, so $B(t+s) - B(s) \perp F^+(s)$ by taking limit(use right continuity of F^+) to both sides. Similarly for any fixed $t_1, t_2, ..., t_n$

$$\{B(t_i + s)\}_{i=1}^n \perp F^+(s).$$

learn a trick here using rational number, limit and squeeze the sandwich. □

Theorem 0.6. *(Blumenthal's 0-1 law) Let $X \in R$ and $A \in F^+(0)$, then we have $P_x(A) = \{0,1\}$. (P_x means the B.M. starts at X)*

Proof. We apply previous theorem for s=0,

$$\{B(t)\}_{t\geq 0} \perp F^+(0)$$

1. $A \in F^+(0)$
2. $A \in \{B(t)\}_{t\geq 0} \Rightarrow A \perp A \Rightarrow P(A) = \{0,1\}$

B.M. is a continuous process and filteration is continuos as well □

Theorem 0.7. *Suppose $\{B(t)\}$ is S.B.M. let*

$$\tau = inf\{t > 0, B(t) > 0\}$$
$$\sigma = inf\{t > 0, B(t) = 0\}$$

then

$$P(\tau = 0) = p(\sigma = 0) = 1$$

(then we show $P(\tau = 0)$ is either 0 or 1 then show it cannot be 0.

$$\{\tau = 0\} = \cap_n \{there\ is\ 0 < \epsilon < \frac{1}{n}\ s.t.\ B(\epsilon) > 0\}$$

so the intersection is also $\in F^+(0)$. Then by above Theorem 0.6,

$$P(\tau = 0) = \{0,1\}$$

Now,

$$P(\tau \leq t) = P(B(s) \leq 0\ for\ every\ s \in [0,t]) \geq P(B(t) \leq 0) = \frac{1}{2},$$

which means $P(\tau \leq t) \geq 1/2$, then $P(\tau = 0) = 1$.

We can also define

$$\tau = \inf\{t > 0, B(t) < 0\},$$

then $P(\tau = 0) = 1$. This implies that $P(\sigma = 0) = 1$ by continuity of brownian path.

We can also define the tail σ-field

$$\mathcal{T} = \cap_{t\geq 0} g(t) \Big(= \cap_{t\geq s} g(t),\ finite\ time\ invariant\Big),\ where\ g(t) = \sigma(\{B(s)|s \geq t\}).$$

Theorem 0.8. *For $A \in \mathcal{T}$, then $p(A) = 0$ or 1.*

Proof. Since $x(t) = t \cdot B(\frac{1}{t})$ is also a B.M., $\sigma(x(t)) = \sigma(B(\frac{1}{t}))$, therefore $A \in \mathcal{T}^B$ is the same as $A \in \cap_{t\geq 0}\sigma(\{B(s)|s \geq t\}) = \cap_{t>0}\sigma(\{B(s)|s \geq t\})(finite\ invariant)$ and

$$\begin{aligned} \cap_{t>0}\sigma(\{B(s)|s \geq t\}) &= \cap_{t>0}\sigma(\{x(u)|u \leq \frac{1}{t}\}) \\ &= F^+(0) \end{aligned}$$

hence P(A) = 0 or 1. □

Given a filtration $\{F(t)\}_{t\geq 0}$ a r.v. T is called a stopping time if the event $\{T \leq t\} \in F(t)$ for every t.

Proposition 0.1. *1. Constant t is always a stopping time*

2. *Suppose $T_n \uparrow T$ and each T_n is a stopping time, then T is a stopping time.*

$$\{T \leq t\} \cap_{n=1}^{\infty} \{T_n \leq t\} \in F_t$$

3. *Suppose T is a stopping time. Let*

$$T_n = (m+1)2^{-n} \; if \; T \in [m2^{-n}, (m+1)2^{-n})$$

then each T_n is a stopping time and $T_n \downarrow T$.

4. *Every stopping time w.r.t. $F^0(t)$ is also a stopping time w.r.t. $F^+(t)$.*

$$\{T \leq t\} \in F^0(t) \subset F^+(t). \qquad \text{(Right continuity)}$$

5. *The first hitting time of a Brownian motion to a closed set H is a stopping time w.r.t. $F^0(t)$.*

6. *The first hitting time of a Brownian motion to an open set H is a stopping time w.r.t. $F^+(t)$.*

$$T = inf\{t : B(t) \in H\}$$

$$\{T \leq t\} = \{\exists t_0 \leq t, \text{ such that } B(t_0) \in H\} \bigcup \{\text{there is a sequence } t_n > t \text{ s.t. } t_n \to t \text{ and } B(t_n) \in H\} \in F^+(t)$$

when H is a closed set, Suppose we have $t_n > t$ and $t_n \to t$ and $B(t_n) \in H$. Then we know $\lim\limits_{n\to\infty} B(t_n) = B(t) \in H$ because H is closed.

In the future, we use $F^+(t)$ as the filtration for $\{B(t)\}$ by default.

Proposition 0.2. *($F^+(t)$ properties)*

1. *Right-continuity.*

Proof.

$$lim_{\epsilon\downarrow 0} F^+(t+\epsilon) = \cap_{\epsilon>0} F^+(t+\epsilon)$$
$$F^+(t) = \cap_{S>0} F^0(t+s) = \cap_{\epsilon>0} F^+(t+\epsilon).$$

(trick: $\cap_{S>0} F^0(t+s) = \cap_{S>t} F^0(s)$) In contrast, $F^0(t)$ is not right continuous.

$$lim F^0(t+\epsilon) = F^+(t).$$

□

2. *Suppose T satisfy $\{T < t\} \in F(t)$ and F(t) is right continuous, then T is a stopping time w.r.t. F(t)*

$$\{T \leq t\} = \bigcap_{k\in N^+} \{T < t + \frac{1}{k}\} \bigcap_{k\in N^+} F(t + \frac{1}{k}) = F(t).$$

Theorem 0.9. *(Strong Markov Property) Suppose T is a.s. finite stopping time, then $\{B(T+t) - B(T)\}$ is a s.b.m. independent of $F^+(T)$, where $F^+(T) = \{A | A \cap \{T \leq t\} \in F(t)\}$,a σ-algebra generated by T.*

Proposition 0.3. *If $\{B(T)\}$ is a Brownian motion, then*

1. *(symmetry) $\{-B(T)\}$ is a Brownian motion.*

2. *(Invariance under scaling) For every $\lambda > 0$, $\{\frac{1}{\lambda}B(T)\}$ is a Brownian motion.*

3. *(simple markov property) $\{B(T+s) - B(s)\}$ is a Brownian motion.*

4. *Time reversal $\{B(T) - B(T-t)\}$ is a Brownian motion.*

Theorem 0.10. *(Refelection principle) Suppose T is a stopping time a.s. finite and $\{B(t)\}$ is a standard brownian motion, then*

$$B^*(t) = B(t)1_{t \le T} + \{2B(T) - B(t)\}1_{t > T}$$

is also a Brownian motion.

if M=(x,y), then the reflection of M with respect to the line passing through (0,a) and parallel to the X-axis is $M^* = (X, 2a - y)$(draw a picture).

Proof. $\{B(t+T) - B(T)|t \ge 0\}(*)$ is a standard B.M. independent with $F^+(T)$. $\{-B(t+T) + B(T)\}$(**) is also a B.M. independent with $F^+(T)$. If we glue (*), which depends on ω as well, to $\{B(t)|0 \le t \le T\}$ then we have a new Brownian motion which is just $\{B(t)\}$. If we glue (**) to $\{B(t)|0 \le t \le T\}$ then we have B^*, a standard Brownian motion. ◻

Application of reflection property:

Define $M(t) = \underset{0 \le s \le t}{max} B(s)$, where B(s) is a.s. finite is well defined in a compact set. Then

$$P(M(t) > a) = 2P(B(t) > a) = P(|B(t)| > a) \le 2 * \frac{\sqrt{t}}{a}\frac{1}{\sqrt{2\pi}}e^{-\frac{a^2}{2t}} \qquad \text{(gaussian tail)}$$

for all $a > 0$.

Proof. Let $T_a = inf\{t|B(t) = a\}$, then

$$P(M(t) > a) = P(B(t) > a) + P(B(t) \le a,\ there\ exists\ s\ such\ that\ B(S) > a)(C).$$

Then we can show $\{B(t) \le a,\ there\ exists\ s\ such\ that\ B(S) > a\} = \{B^{*T_a}(t) > a\}(D)$. $\rightarrow$ Suppose C happens then we know $T_a \le s$ for some $s \in [0, t]$ and $B^{*T_a}(t) > a$. $\leftarrow$ Similarly if D happens $\Rightarrow T_a \le t$, $B(t) < a$ (before reflection where reflection $> a$).So

$$P(M(t) > a) = P(B(t) > a) + P(B^*(t) > a).$$

◻

Omit the proof of claim that $B(t), B^*(t)$ have the same distribution. Check notes.