1 10.7 Brownian Motion: textbook brownian motion by Mortes and peres

Idea: Probability studies "Mascroscope picture" in random systems defined by a lot of "Microscopic random effects"

Brownian Motion- Microscopic picture emerging from a particle moving in d-dim space without making "big jumps".

At any time step, the particle receives a small displacement.

If the initial position is S_0, then at time n, $S_n = S_0 + \sum_{i=1}^{n} X_i$ X_i are the independent random displacement.

The whole process $\{S_0, S_1, ...\}$ is a random walk, and when we look at the macroscopic picture we ask

1. Does S_n drift to ∞?
2. Does S_n go back to a neighbor of S_0 infinitely many time?
3. What is the speed at growth of $\{max|S_n\}_{n \leq N}$ as $N \to \infty$.

It turns out not all the features of microscopic picture contribute to the macro scopic.

If the movement of the random walk, can be neglible in an infinite amount of time ($S_A \to_{dt \to 0} S_A$), then any process $\{B_t\}$ should have

1. For time $0 \leq t_1 \leq t_2 \leq ... \leq t_k$, the r.v.s. $B(t_1), B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1})$ are independent, (independent increment)
2. the distribution of $B(th) - B(t)$ does not depend not t (stationary increment)
3. $\{B(t)\}$ has continues.

$$B(t) = \sum_{i=1}^{n} B(t_k^{(n)}) - B(t_{k-1}^{(n)}),$$
$$t = \frac{1}{n}$$

B(t) must be normal distribution (CLT).

2 Definition:

A real valued stochastic process $\{B(t)\}_{t \geq 0}$ is called a linear B.M. with start $x \in R$ if

1. $B(0) = x$
2. If $0 \leq t_1 \leq t_2 \leq ... \leq t_n$, the increments $B(t_n) - B(t_{n-1}), ..., B(t_2) - B(t_1)$ are independent.

3. $B(k+h) - B(k) \sim N(0,h)$
4. The map $t \to B(t)$ is a.s. continuous
   (For almost surely $w \in \Omega, the function\ t \to B(t,w)$ is continuous).

Remark. $If\ x = 0$, then we call $\{B(t)\}$ standard Brownian Motion. We will cover

1. construct BM
2. study the path property
3. properties of BM (reflection principle)
4. Donskov's invariant principle and skorohd Embedding

3 Properties of BM

1.
$$
\begin{aligned}
B(t) - B(0) \sim N(0,t) \\
B(t) \sim N(x,t)\ (for\ some\ b.m.\ B(t) \sim N(0,t))
\end{aligned}
$$

2. $cov(B(t), B(s)) = min\{t, s\}$
   pf: WLOG, assume X=0,

$$
\begin{aligned}
cov(B(t), B(s)) &= E(B(t)B(s)) - E(B(t))E(B(s)) \\
&= E(B(t) - B(s) + B(S))B(s))) && (\text{ Assume } s <= t) \\
&= E((B(t) - B(s))B(s)) + EB^2(s) = S = min(s,t).
\end{aligned}
$$

3. (Finite dimensional distribution) Suppose $t_1 \leq t_2 \leq ... \leq t_n$ Then $(B(t_1), B(t_2), ..., B(t_n))$ is a multivariate normal with mean vector. $(X_1, ..., X_n) \in R^n$ and covariance matrix $\sum \in R^{n \times n}$ with $\sum_{i,j} = min(t_i, t_j)$

$$
\sum = \begin{pmatrix} t_1, t_1, ..., t_1 \\ t_1, t_2, .., t_2 \\ \vdots \\ t_1 t_2, t_3, ..., t_n \end{pmatrix}
$$

proof

Recall the fact that if $X \to N(\mu, S) \in R^n$, then AX is still normal for any $A \in R^{mtimesn}$. Notice that

$$
\tilde{B} = (B(t_1), B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1}))
$$

and $B = A\tilde{B}$ for some linear transformation A.

4 Invariant properties

4.1 Proposition 1

Suppose $\{B(t)\}$ standard BM then $X(t) = \frac{1}{a}B(a^2 t)$ is also Standard BM for any $a > 0$

4.2 proof

1. $X(0) = \frac{1}{a}B(a^2 * 0) = \frac{1}{a}B(0) = 0$

2. Suppose $0 \leq t_1 \leq ... \leq t_n$

$$X(t_i) - X(t_{i-1}) = \frac{1}{a}\Big(B(a^2t_i) - B(a^2t_{i-1})\Big), X(t_{i-1}) - X(t_{i-2}) = \frac{1}{a}\Big(B(a^2t_{i-1}) - B(a^2t_{i-2})\Big)$$

3. $X(k+h) - X(k) = \frac{1}{a}(B(a^2(k+h)) - B(a^2(k)) \sim \frac{1}{a}N(0, a^2h) \sim N(0,h)$

eg 1

Let $a < 0 < b$ and $\{B(t)\}$ s.b.m.
Define T(a,b)= $inf\{t : B(t) = a \ or \ b\}$ ET(a,b)= $a^2\mathbb{E}inf\{t \geq 0, x(t) = 1 \ or \ \frac{b}{a}\} = a^2ET(1.\frac{b}{a})$ Suppose t_0 is the exit time for B(t) at [a,b], then $\frac{t_0}{a^2}$ is the exit time for X(t) at $[1, \frac{b}{a}]$

$$B(t_0) = a \iff X(\frac{t_0}{a^2}) = 1$$
$$B(t_0) = b \iff X(\frac{t_0}{a^2}) = \frac{b}{a}.$$

Suppose 1 we take b=-a, then

$$E(T(a,-a)) = a^2ET(1,-1)$$

2:

$$P(\{B(t)\} \ exists \ [a,b] \ at \ a) = P(\{X(t)\} \ exists \ [1, \frac{b}{a}] \ at \ 1)$$
$$= P(\{B(t)\} \ exists \ [1, \frac{b}{a}] \ at \ 1 \ )$$
$$= t(\frac{b}{a}).$$

eg.2

$\{B(t)\}$ is S.B.M. $U \perp \{B(t)\}$ and U$\sim U[0,1]$

$$\tilde{B}(t) := \begin{cases} B(t) \ t \neq U \\ 0 \ t = U \end{cases}$$

when W is fixed, $t \to \tilde{B}(t,w)$ equals B(t,w) at every $t \geq 0$ except for $t = U(w) \in [0,1]$. However $\tilde{B}(t,w)$ is not a.s. continuous therefore $\tilde{B}$ is not a brownian motion.

Proposition 4.1. *Suppose X,Y are independent N(0,1) r.v.s, then we have X+Y independent with X-Y and both $\sim N(0,2)$.*

Proof. Note that two jointly normal random variables are independent if and only if they are uncorrelated. Since X,Y are independent normal, the pair(X,Y) is normal. Since any affine transformation of a normal is also a normal, (X+Y,X-Y) is normal.

$$E[(X+Y)(X-Y)] = E[X^2 - Y^2] = 0$$

, so uncorrelated, hence independent. □

Definition 4.1. $\{B(t)\}_{t\geq 0}$ is a linear B. M.

1. $B(0) = x$
2. $$0 \leq t_1 \leq t_2...,$$
   then
   $$B(t_2) - B(t_1), B(t_3) - B(t_2), ..., B(t_n) - B(t_{n-1})$$
   are independent.
3. $$B(t+h) - B(t) \sim N(0, h)$$
4. $t \to B(t)$ is continuous a.s.

We say $\{B(t)\}$ is a standard B.M. if x=0.

Theorem 4.1. *(wiener 1923) Standard Brownian motion exists.*

Proof. we will first construct B.M on c [0,1]. Idea: Construct on dyadic points. We define

$$D_n = \{\frac{k}{2^n} : 0 \leq k \leq 2^n\}$$

$D := \cup_{n=1}^{\infty} D_n = \{\frac{k}{2^n} | for\ some\ n\ and\ k \leq 2^n.\}$ dense enough for R.

$$D_0 = \{0, 1\},$$
$$D_1 = \{0, \frac{1}{2}, 1\}$$
$$D_2 = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$$

Let (Ω, F, P) be a probability space that we can define $\{Z_t\}_{t\in D}$ of iid N(0,1) on it. Let $B(0) := 0, B(1) := Z_1$. For each $d \in D_n$, we define B(d) $d \in D_n$ such that

1. For every $r < s < t$ in D_n the r.v.
   $$B(t) - B(s) \perp B(s) - B(r),$$
   and $B(t) - B(s) \sim N(0, t-s)$.
2. (B(d), $d \in D_n$) and ($Z_t := t \in D \backslash D_n$) are independent.

We have already done this for n=0. Suppose we have already defined $B(d)$ for $d \in D_{n-1}$, then for $d \in D_n \backslash D_{n-1}$, it means $d = \frac{2k+1}{2^n}$ and let

$$B(d) = \frac{B(d - 2^{-n}) + B(d + 2^{-n})}{2} + \frac{Z_d}{2^{(n+1)/2}} \tag{1}$$

interpolation of left and right neighborhood and add an independent noise.

$$\Rightarrow B(d + 2^{-n}) - B(d) \perp B(d) - B(d - 2^{-n}) \sim N(0, \frac{1}{2^n})$$

Note that both neighbors are on the previous stage.

$$B(d) - B(d - 2^{-n}) = \frac{B(d + 2^{-n}) - B(d - 2^{-n})}{2} + \frac{Z_d}{2^{(n+1)/2}},$$

$$\sim \frac{N(0,\frac{1}{2^{n-1}})}{2} + \frac{N(0,1)}{2^{(n+1)/2}} \sim N(0,\frac{1}{2^{n+1}}) + N(0,\frac{1}{2^{n+1}}) \sim N(0,\frac{1}{2^n})$$

$$B(d+2^{-n}) - B(d) = \frac{B(d+2^{-n}) - B(d-2^{-n})}{2} - \frac{Z_d}{2^{(n+1)/2}},$$

□

We define

$$F_0(t) = \begin{cases} Z_1, \ t=1 \\ 0, \ t=0 \\ linear \ inbetween \end{cases},$$

$$F_n(t) = \begin{cases} Z_t 2^{-(n+1)/2}, \ t \in D_n \backslash D_{n-1} \\ 0, \ t \in D_{n-1} \\ linear \ inbetween \end{cases},$$

we will define $B(t) := \sum_{i=0}^{\infty} F_i(t)$. Then for every n and and every $d \in D_n$, we have that

$$B(d) = \sum_{i=1}^{n} F_i(d) = \sum_{i=1}^{\infty} F_i(d), \qquad \text{(definition 2)}$$

because when $i > n$, then $d \in D_n \subset D_i$, so $d \in D_{i-1}, F_i(d) = 0$. To check (1),

$$B(d) = \frac{B(d-2^{-n}) + B(d+2^{-n})}{2} + \frac{Z_d}{2^{(n+1)/2}} \qquad (1)$$

suppose $d \in D_n \backslash D_{n-1}$,

$$\begin{aligned}
\sum^{n} F_i(d) &= \sum^{n-1} F_i(d) + \frac{Z_d}{2^{-(n+1)/2}} \\
&= \sum_{i=0}^{n-1} \frac{F_i(d-2^{-n}) + F_i(d+2^{-n})}{2} + \frac{Z_d}{2^{(n+1)/2}} && (2) \\
&= \frac{B(d-2^{-n}) + B(d+2^{-n})}{2} + \frac{Z_d}{2^{(n+1)/2}} && \text{(by induction of matching def 2 and 1)} \\
&= B(d)
\end{aligned}$$

, where in (2), if f is a linear function on [a,b], then

$$f(\frac{a+b}{2}) = \frac{f(a)+f(b)}{2},$$

and $d - 2^{-n} \in D_{n-1}$, hence $F_i(d-2^{-n}) = 0$ for $i < n-1$, and $F_{n-1}(d-2^{-n}) =$ similarly, we assumed definition 2 holds upto n-1 then we showed definition 2 holds for n.

It is known that for standard normal,

$$P(|Z_d| \geq x) \leq e^{\frac{-x^2/2}{x\sqrt{2\pi}}},$$

for $c > 1$,

$$P(|Z_d| \geq c\sqrt{n}) \leq e^{\frac{-c^2 n}{2}}$$

moreover, if $c > 1$ and large n,

$$\begin{aligned}
&\sum_{n=0}^{\infty} P(there\ is\ a\ d \in D_n\ with\ |Z_d| \geq c\sqrt{n}) \\
&\leq \sum_{n=0}^{\infty} (2^n + 1)e^{-c^2 n/2} \qquad (1) \\
&= \sum_{n=0}^{\infty} e^{nlog2 - c^2 n/2} + \sum_{n=0}^{\infty} e^{-c^2 n/2} \\
&\leq \sum_{n=0}^{\infty} e^{(-c^2 + log2)n} + \sum_{n=0}^{\infty} e^{-c^2 n/2} \qquad (*)
\end{aligned}$$

Note that in (1), there are $2^n + 1$ elements in D_n, so apply union bound. If $c > \sqrt{log2}$, we will have $* < \infty$. By B-C lemma,

$$P(|Z_d| \geq c\sqrt{n}\ for\ some\ d \in D_n\ \text{i.o.}) = 0$$

This means that with probability 1, there is a finite $N(w)$ s.t. when $n \geq N(w)$

$$|Z_d(w)| \leq c\sqrt{n}\ for\ every\ d \in D_n$$

Hence w.p.1, for all $n \geq N(w)$,

$$||F_n||_\infty \leq \frac{c\sqrt{n}}{2^{(n+1)/2}}$$

$\Rightarrow \sum_{n=0}^{\infty} F_n(t)$ converges uniformly.
$\Rightarrow \sum_{n=0}^{\infty} F_n(t) = B(t)$ each F_n is continuous, so B is continuous almost everywhere on D and below we expend to [0,1] by taking limit t_{ik} on D to t_i in [0,1]

For $0 \leq t_1 \leq t_2, ... \leq t_n \leq 1$ we can use $t_{i,k} \uparrow t_i$ for each i, $t_{i,k} \in D$ and $t_{1,k} \leq t_{2,k}...$ We know

$$\begin{aligned}
&B(t_2) - B(t_1), B(t_3) - B(t_2), ..., B(t_n) - B(t_{n-1}) \\
&= lim_{k\to\infty}(B(t_{2,k}) - B(t_{1,k}), B(t_{3,k}) - B(t_{2,k}), ..., B(t_{n,k}) - B(t_{n-1,k}))\ a.s.
\end{aligned}$$

We know RHS $\sim N(t_{2,k} - t_{1,k}) \times ...\quad N(0, t_2 - t_1) \times N(0, t_3 - t_2) \times ...N(0, t_n - t_{n-1})$ a B.M. on $[0, 1]$ can be constructed a.s.

Corollary 4.1. *A Brownian motion on $[0, \infty)$ can be constructed a.s. "Construct $B_1, B_2, ...$"" of independent in $[0, 1]$ B.M. then we define*

$$B(t) = B_{\lfloor t \rfloor}(t - \lfloor t \rfloor) + \sum_{i=0}^{\lfloor t \rfloor - 1} B_i(1).$$

4.3 Gaussian tails (sharp when X is large)

$\frac{x}{x^2+1}\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \leq P(N(0,1) > X) \leq \frac{1}{X}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

Theorem 4.2. *Continuity of B.M. For almost every $w \in \Omega$, the function $B(w,t)$ is continuous on $R^{\geq 0}$. (Compact subset of a set, on which f is continuous, gives that f is uniformly continuous on that compact subset.) Therefore $B(w,t)$ is uniformly continuous on [0,1]. We will show a more stronger result than uniform continuity, i.e.*

$$\sup_{t\in[0,1-h]} |B(t+h) - B(t)| \sim \sqrt{hlog(\frac{1}{h})}\ for\ small\ enough\ h(w).$$

Theorem 4.3. *There exists $c > 0$ such that almost surely, for every small $h(w)$ and $0 \leq t \leq 1-h$, we have*

$$\sup_{t\in[0,1-h]} |B(t+h) - B(t)| \leq c\sqrt{hlog(\frac{1}{h})}$$

Remark: h may depend on w.

Proof. Recall that when $c > \sqrt{2log2}$ then for a.s. every w, there is N=$N(w)$ such that when $n > N$

$$||F_n||_\infty \leq c\sqrt{n}2^{-n/2},$$

($F_n(t)$ is piecewise linear, so the largest derivative is $\frac{||F_n||_\infty - 0}{1/2^n}$, so

$$||F_n'||_\infty \leq 2^n||F_n||_\infty \leq c\sqrt{n}2^{n/2}$$

then not fixing h,

$$\begin{aligned}
|B(t+h) - B(t)| &\leq \sum_{n=0}^{\infty} ||F_n(t+h) - F_n(t)|| \\
&\leq \sum_{n=0}^{\infty} ||F_n'||_\infty h && \text{(by MVT)} \\
&+ \sum 2||F_n|_\infty. && (*)
\end{aligned}$$

Suppose $l > N$,

$$* \leq \sum_{n=0}^{N} ||F_n'||_\infty h \ (1) + 2ch \sum_{n=N+1}^{l} \sqrt{n}2^{\frac{n}{2}} \ (2) + 2c\sum_{l+1}^{\infty} \sqrt{n}2^{-\frac{n}{2}} \ (3).$$

Now, we choose h=h(w) so small such that

$$(1) \leq \sqrt{hlog\frac{1}{h}} \ (easy)$$

and l defined as $2^{-l} < h \leq 2^{-l+1}$ ($l \approx log_2(\frac{1}{h})$) is greater than N,then

$$\begin{aligned}
&1 \leq \sqrt{hlog\frac{1}{h}} \\
&2 \leq c_1\sqrt{l}2^{l/2} \leq \tilde{c}_1\sqrt{log(1/h)}\sqrt{h^{-1}}h = \tilde{c}_1\sqrt{log(1/h)}\sqrt{h} \\
&3 \leq c_2\sqrt{l}2^{-l/2} \leq \tilde{c}_2\sqrt{log(1/h)}\sqrt{h}
\end{aligned}$$

$\sqrt{l} > \sqrt{log(1/h)}$ but we use the fact that their difference is no larger than 2. So we can make up that difference. □

Theorem 4.4. *For every $c < \sqrt{2}$ almost surely for every $\epsilon > 0$, there is $h \in (0, \xi)$ and $t \in [0, 1-h]$ such that*

$$|B(t+h) - B(t)| > c\sqrt{hlog\frac{1}{h}}.$$

Proof. Let $A_{k,n} = \{B(k+1)e^{-n} - B(ke^{-h}) > c\sqrt{n}e^{-n/2}\}$

$$P(A_{k,n}) = P(N(0, e^{-n}) > c\sqrt{n}e^{-n/2}) = P(N(0,1) > c\sqrt{n}) \geq \frac{c\sqrt{n}}{c^2n+1}\frac{1}{\sqrt{2\pi}}e^{-c^2n/2},$$

therefore

$$e^n P(A_{k,n}) \geq \frac{c\sqrt{n}}{c^2 n+1}\frac{1}{\sqrt{2\pi}}e^{(1-c^2/2)n} \to \infty \; for c < \sqrt{2}$$

then using $1-x \leq e^{-x}$,

$$\begin{aligned} P(\cap_{k=0}^{\lfloor e^n-1 \rfloor} A_{k,n}^c) &= \prod_{k=0}^{\lfloor e^n-1 \rfloor} P(A_{k,n}^c) \\ &\prod_{k=0}^{\lfloor e^n-1 \rfloor} (1-P(A_{k,n})) \\ &= (1-P(A_{0,n}))^{e^n} \leq e^{-e^n P(A_{0,n})} \to 0 \end{aligned}$$

Therefore, for every $\xi > 0$

$$P(|B(t+h)-B(t)| \leq c\sqrt{hlog\frac{1}{h}} \; for \; every \; h \in (0,\xi), t \in [0,1-h]) \leq P(\cap A_{k,n}^c) \to 0 \tag{1}$$

No proof for correcting c. (1) implies lower bound because if we take $h = e^{-n}$, consider $t_i = \frac{c}{2^n}$, □

5 Brownian motion is nowhere differentiable

We use two fact

1. Invariance property: If $\{B(t)\}$ is standard brownian motion, then $X(t) := \frac{1}{a}B(a^2t)$ is a standard brownian motion.
2. If $\{A_i\}_{i=1}^{\infty}$ and $P(A_i) = 1$ for each i, then $P(\cap_{i=1}^{\infty} A_i) = 1$

$$\begin{aligned} P(\cap_{i=1}^{\infty} A_i)^c &= P(\cup_{i=1}^{\infty} A_i^c \\ &\leq \sum_{i=1}^{\infty} P(A_i^c) \\ &= \sum_{i=1}^{\infty} 0 = 0. \end{aligned}$$

5.1 Example: uncountable set

Consider the uniform measure on [0,1], define $A_s = [0,1]\backslash\{s\}$ $(s \in [0,1])$, then $P(A_s) = 1$ for every s but $P \cap_{s\in[0,1]} A_s = P(\phi) = 0$.

Theorem 5.1. *Time inversion: If $\{B(t)\}$ is a standard brownian motion,*

$$X(t) = \begin{cases} 0 \; t=0 \\ tB(\frac{1}{t}) \; t>0 \end{cases} \quad is \; also \; a \; s.b.m.$$

Proof. We argue for $0 \leq t_1 \leq t_2 \leq ... \leq t_n$,

$$(B(t_1),...,B(t_n)) \underset{d}{=} (x(t_1),...,x(t_n)),$$

which will imply $x(t)$ has stationary independent increment, and

$$x(t)-x(s) \sim N(0,t-s) \; for \; t>s.$$

Since both are normal, we check the covariance structure

$$\begin{aligned}
&cov(X(t_i), X(t_j)) \\
&\qquad = cov(t_i B(\frac{1}{t_i}), t_j B(\frac{1}{t_j})) \\
&\qquad = t_i t_j cov(B(\frac{1}{t_i}), B(\frac{1}{t_j})) \\
&\qquad = min\{t_i, t_j\} = cov(B(t_i), B(t_j)).
\end{aligned}$$

Now we want to show almost surely, X(t) is continuous in $(0, \infty)$, then we just need to check

$$\lim_{t \to 0^+} X(t) = 0 \ or \ \lim_{t \to 0^+} tB(1/t) = 0.$$

Check yourself that

$$\{\lim_{t \to 0^+} X(t) = 0\} = \cap_{n=1}^{\infty} \cup_{m=1}^{\infty} \cap_{q \in Q \in [0, \frac{1}{m}]} \{|x(q)| \leq \frac{1}{n}\}.$$

Think of m as δ condition and n as ϵ condition,

$$P(\cap_{n=1}^{\infty} \cup_{m=1}^{\infty} \cap_{q \in Q \in [0, \frac{1}{m}]} \{|x(q)| \leq \frac{1}{n}\})$$
$$\lim_{n \to \infty} P(\cup_{m=1}^{\infty} \cap_{q \in Q \in [0, \frac{1}{m}]} \{|x(q)| \leq \frac{1}{n}\})$$

$$\begin{aligned}
P(\cup_{m=1}^{\infty} \cap_{q \in Q \in [0, \frac{1}{m}]} \{|x(q)| \leq \frac{1}{n}\}) &= P(\cup_{m=1}^{\infty} \cap_{q \in Q \in [0, \frac{1}{m}]} \{|B(q)| \leq \frac{1}{n}\}) \\
&= 1
\end{aligned}$$

because $\{B(t)\}$ is a.s. continuous at point 0, which menas

$$P(X_t \ continuous \ at \ 0) = 1.$$

$\square$

Corollary 5.1. $\lim_{t \to \infty} \frac{B(t)}{t} = 0$ *a.s. (LLN for B.M.)*

$$\lim_{t \to \infty} \frac{B(t)}{t} = \lim_{s \to 0^+} SB(\frac{1}{S}).$$

Theorem 5.2. *Almost surely, for all* $0 < a < b < \infty$*, b(T) is not monotone on [a,b].*

Proof.

$$\begin{aligned}
&p(B(t) \ monotone \ [a, b]) \\
&\qquad \leq P(B(\frac{a+b}{2}) - B(a) \ has \ the \ same \ sign \ as \ B(b) - B(\frac{a+b}{2})) = 1/2
\end{aligned}$$

because both are mean 0 independent normal. We cut [a,b] into n pieces

$$a = t_0 < t_1 = a + \frac{b-a}{n} < t_2 = a + \frac{2(b-a)}{n} < ... < t_n = b$$

then

$$p(B(t)\ monotone\ [a,b]) \leq P(B(t_{i+1} - B(t_i)\ has\ the\ same\ sign\ for\ every\ i) = \frac{1}{2^n}.$$

$$\begin{aligned} p(B(t)\ not\ monotone\ [a,b]) = 1 \\ & P(\cap_{a \leq a < b} B(t) not monotone on [a,b]) \\ & = P(\cap_{0 \leq p < q \in Q} B(t) not monotone on [p,q]) = 1 \end{aligned}$$

note that we need last line holds only on countable set. This means B.M. is not smooth. □

Proposition 5.1. *Almost surely,*

$$\underset{n\to\infty}{limsup} \frac{B(n)}{\sqrt{n}} = \infty \tag{1}$$

$$\underset{n\to\infty}{liminf} \frac{B(n)}{\sqrt{n}} = -\infty \tag{2}$$

Proof. Recall that law of iterated logarithms, and given $X_1, ...$are independent mean 0 and variance 1,

$$P(\underset{n\to\infty}{limsup} \frac{S_n}{\sqrt{2nlognlogn}} = 1) = 1.$$

Note that

$$B(n) = \sum_{i=0}^{n-1} B(i+1) - B(i),\ let\ Y_i := B(i+1) - B(i)$$

then

$$\underset{n\to\infty}{limsup} \frac{B(n)}{\sqrt{2nlognlogn}} = 1 \Rightarrow \underset{n\to\infty}{limsup} \frac{B(n)}{\sqrt{n}} = \infty.$$

The proof for (2), we use LIL on $-B(n) = \sum_{i=0}^{n}(-Y_i)$, then

$$\underset{n\to\infty}{limsup} \frac{-B(n)}{\sqrt{2nlognlogn}} = 1 = -\underset{n\to\infty}{liminf} \frac{B(n)}{\sqrt{n}} = \infty.$$

□

Remark: (different proof for (1)). Proving (1) is the same as showing for every $\epsilon > 0$

$$P(B(n) > c\sqrt{n}\ i.o.) = 1$$

The event above is exchangable in a sense that it is invariant after any permutation which permutes finitely many eventually, thenyl by Hewitt-savage 0-1 law, $P(B(n) > c\sqrt{n}\ i.o.)$=1 or 0.

$$\begin{aligned} P(B(n) > c\sqrt{n}\ i.o.) &= P(\cap_{k=1}^{\infty} \cup_{n \geq k} B_n > c\sqrt{n}) \\ &= \lim_{k\to\infty} P(\cup_{n\geq k} B_n > c\sqrt{n}) \\ &\geq \underset{k\to\infty}{limsup} P(B_k > c\sqrt{k}) \\ &= \underset{k\to\infty}{limsup} P(N(0,1) > c) \\ &> 0. \end{aligned}$$

Note the tricks 1: $\cap\cup$ in limsup notation can be replaced with $lim \cup$. 2:

$$\frac{B_k}{\sqrt{k}} \sim N(0,1)$$