Definition 0.1. $\{X(S)\}_{s\geq 0}$ is a martingale w.r.t. a filtration $\{F(s)\}$ if $E[|X(t)|] < \infty$ and $E[X(t)|F(s)] = X(s)$ a.s. for $t \geq s$, $\geq X(s)$ for submartingale and $\leq x(s)$ for super martingale.

Example: $\{B(t)\}_{t\geq 0}$ is a M.G. w.r.t. $\{F^+(t)\}_{t\geq 0}$.

Proof.

$$\begin{aligned}&E[B(t)|F^+(s)]\\&= E[B(t) - B(s) + B(s)|F^+(s)]\\&= E[B(t) - B(s)] + B(s) = B(s).\end{aligned}$$

□

Proposition 0.1. *Suppose $\{X(t)\}_{t\geq 0}$ is a M.G. Suppose $S \leq T$ are two stopping times. If $|X(t \wedge T)| \leq X$ for every t and $X \in L^1$. Then $E[X(T)|F(S)] = X(S)$.*

Proof. Notation: X(t) for continuous time process and X_T for discrete time process.

Now we proceed by discretization. Fix a positive integer N and consider the discrete-time MG

$$X_n = X(T \wedge n2^{-N}), S'_N = \lfloor 2^N S\rfloor + 1, T'_N = \lfloor 2^N T\rfloor + 1, g(n) = F(n2^{-N}).$$

we can check

1. X_n is a discrete M.G. w.r.t. $\{g(n)\}$.
2. $S'_N \leq T'_N$ and S'_N, T'_N are stopping time w.r.t. $\{g(n)\}$.
3. $|X_n| \leq X$.

The discrete-time optional stopping theorem gives

$$\mathbb{E}[X_{T'_N}|G_{S'_N}] = X_{S'_N}. \tag{1}$$

Note that

$$X_{T'_N} = X(T \wedge (\lfloor 2^N T\rfloor + 1)2^{-N}),\ (\lfloor 2^N T\rfloor + 1)2^{-N} \geq T,\ hence\ X_{T'_N} = X(T),$$

then (1) is equivalent to

$$\mathbb{E}[X_{T'_N}|g(S'_N)] = \mathbb{E}[X(T)|F(2^{-N}S'_N)] = X(T \wedge 2^{-N}S'_N)$$

Let $S_N := 2^{-N}S'_N \geq S$, $g(S') = F(S'2^{-N})$

$$\mathbb{E}[X_{T'_N}|g(S'_N)] = X_{S'_N} \Rightarrow E[X(T)|F(S_N)] = X(T \wedge S_N).$$

Now for every $A \in S$,

$$\int_A X_T dP = \lim_{n\to\infty} \int_A X(T \wedge S_N)dP.$$

we can take limit because above works for each n. Now, $A \in F(S) \subset F(S_N)$ and because $S_N \to S$ a.s. and by DCT

$$\begin{aligned}&= \int_A lim(X(T \wedge S_N)dP)\\&= \int_A X(T \wedge S)dP\\&= \int_A X(S)dP.\end{aligned}$$

Therefore we have that $\{X(T)\}$ is a M.G. and $T \geq S$ and $E[X(T)|F(S)] = X(S)$ a.s. by uniqueness of conditional expectation. □

Theorem 0.1. *(Doob's maximal inequality) Suppose* $\{X(t)\}_{t\geq 0}$ *is a Martingale with* $p > 1$*, then*

$$E|X(t)|^p \leq E[\underset{s\leq t}{sup}|X(s)|^p] \leq (\frac{p}{p-1})^p E|X(t)|^p.$$

(intuition is that behavior of the max can be achieved by studying an end point)

Proof. Again we partition [0,t] into 2^N pieces $X_n = X(t \wedge n2^{-N})$ with N fixed $n \in [0, 2^N]$, $g_n = F(tn \cdot 2^{-N})$. Therefore $\{X_n\}$ is a discrete-time martingale with respect to filtration g_n. Let $Y_N := \underset{0\leq n\leq 2^N}{sup}|X_n|^p$. We have

$$E[\underset{0\leq n\leq 2^N}{sup}|X_n|^p] \leq (\frac{p}{p-1})^p E|X(t)|^p.$$

We let $n \to \infty$, we $Y_N \uparrow \underset{s\leq t}{sup}|X(s)|^p \leq$ a.s. by the path-continuity of $\{X(t)\}$. Therefore $N \to \infty$ by MCT

$$E\underset{s\leq t}{sup}|X(s)|^p \leq (\frac{p}{p-1})^p E|X(t)|^p.$$

□

Theorem 0.2. *(Wald's Lemma)* $\{B(t)\}_{t\geq 0}$ *is a standard Brownian motion.* T *is a stopping time with respect to* $\{F^+(t)\}$ *such that either*

1. $ET < \infty$. *or*
2. $|B(t \wedge T)| \leq X$ *for every* t *and* $X \in L^1$*,*

then we have $E[B(T)] = 0$.

Proof. Suppose (ii) holds, then by the previous proposition with S=0, Then

$$E[B(T)|F(0)] = B(0) \ a.s.$$

Suppose (i) holds, define

$$M_k = \underset{0\leq t\leq 1}{max}|B(t+k) - B(k)|, \ M = \sum_{k=1}^{\lceil T\rceil} M_k(my\ control\ function)$$

and note that $|B(t \wedge T)| \leq M$.(by triangle inequality for every t) Then

$$\begin{aligned}
EM =& E[\sum_{k=1}^{\lceil T\rceil} M_k] \\
&= E[\sum_{k=1}^{\infty} 1_{T>k-1} M_k] \\
&= \sum_{k=1}^{\infty} E[1_{T>k-1} M_k] \\
&= \sum_{k=1}^{\infty} P(T > k-1) EM_k \\
&= E[M_1] \sum_{k=1}^{\infty} P(T > k-1) \\
&= E[M_1] E(T+1) < \infty
\end{aligned}$$

because $M_1 < \infty$ (continuity in a compact set) and E(T+1) by assumption (1). Above, $\{T \leq k-1\}^c \in F(k-1)$ and $M_k \in F(k)$.($\{T > k\} \in F(k)$. □

Lemma 0.1. *$\{B^2(t) - t\}$ is a M.G. Suppose X_i iid mean zero and variance 1 then $S_n^2 - n$ is a martingale.*

$$\begin{aligned}
&E[B^2(t) - t|F(s)] \\
&= E[(B(t) - B(s))^2] + B(s)^2 - t \\
&= t - s + B^2(s) - t = B^2(S) - s.
\end{aligned}$$

Theorem 0.3. *(continuous time Optional stopping lemma) Suppose $\{X(t)\}_{t\geq 0}$ is a continuous MG W.R.T. F, and T is a stopping time w.r.t. F. If the process $\{X(T \wedge t)\}_{t\geq 0}$ is dominated by an integrable RV X, then*

$$\mathbb{E}[X(T)] = X(0),$$

almost surely.

Theorem 0.4. *(Wald's lemma 2) Suppose T is a stopping time and $ET < \infty$. then $E(B^2(T) - T) = 0$.)*

Proof. Define $T_n := inf\{t \geq 0||B(t)| = n\}$. Then

$$B^2(t \wedge T \wedge T_n) - t \wedge T \wedge T_n$$

is dominated by $n^2 + T$. Also it is a martingale by Lemma1 0.1., we can apply walds lemma 1(thm about continuous martingale), (actually we apply Optional stopping theorem for continuous martingale)

$$E(B^2(T \wedge T_n)) = E[T \wedge T_n] \tag{0}$$

when n goes to infinity, $E[T \wedge T_n) \uparrow E(T)$ by MCT

$$E(B^2(T \wedge T_n)) \leq E(B^2(T))$$

$$limsup E(B^2(T \wedge T_n)) \leq E(B^2(T)) \tag{1}$$

Also, by Fatou

$$E(B^2(T)) \leq liminf E(B^2(T \wedge T_n)) \tag{2}$$

then by taking limit to above (0)

$$ET = E(B^2(T)).$$

□

Theorem 0.5. *(Gambler's ruin) $T = inf\{t : B(t) \in \{a, b\})$ where $a < 0 < b$.*

1. $P(B(T) = a) = \frac{b}{|a|+b}$
2. $ET = |a|b$

Proof. We need to observe that $ET < \infty$ bounded by geometric as in discrete case.

$$E(B(T)) = 0 = aP(B(T) = a) + b(1 - P(B(T) = a))) \to \; P(B(T) = a) = \frac{b}{|a| + b}$$

$$\begin{aligned}
&E(B^2(T)) = ET \\
&\qquad a^2 \frac{b}{|a| + b} + b^2(1 - \frac{b}{|a| + b}) = |a|b.
\end{aligned}$$

□

Definition 0.2. A martingale $\{X_n\}$ is binary splitting if whenever the event

$$A(X_0, X_1, ..., X_n) = \{X_= x_0, ..., X_n = x_n\}$$

has positive probability, X_{n+1} given A is supported on at most two values.

Lemma 0.2. *(Dubin's embedding) Let X is a random variable with finite 2nd moment. Then we have a binary splitting MG $\{X_n\}_{n=1}^{\infty}$ such that $X_n \to X$ in L^2 and a.s.*

Proof. We define $\{X_n\}$ and $\{g_n\}$ in the following recursive way.

$$g_0 = \{\phi, \Omega\} \; and \; X_0 = EX.$$

$$\xi_0 = \begin{cases} 1 \; if \;\; X \geq X_0 \\ -1 \; if \;\; X < X_0. \end{cases}$$

For every $n > 0$ we define $g_n = \sigma(\xi_0, ..., \xi_{n-1})$ and $X_n = E[X|g_n]$ and $\xi_n = \begin{cases} 1 \; if \;\; X \geq X_n \\ -1 \; if \;\; X < X_n. \end{cases}$. $g_1 = \sigma(\xi_0) = \sigma(\{\xi_0 = 1\}, \{\xi_0 = -1\})$

$$X_1 = E[X|X \geq EX]1_{X \geq EX} + E[X|X < EX]1_{X < EX}.$$

Check

1. $\{X_n\}$ is a MG. ($E[X_n|g_{n-1}] = X_{n-1}$ by the law small wins and by definition of conditional expectation, $X_n \in g_n$)
2. $\{X_n\}$ is a binary splitting converging to X.

(key observation is that by using tower, $EXX_n = E[E[XX_n|g_n]] = EX_n^2$. So crossterm goes to 0 in the following)

□

Theorem 0.6. *(Skorohkod embedding) Suppose $\{B(t)\}$ is a S.B.M. and that X is a RV with mean zero and finite 2nd moment. Then there exists a stopping time T with respect to $\{F^+(t)\}$ such that B(T) has the law of X ($B(T) \overset{d}{=} X$) AND $\mathbb{E}[T] = \mathbb{E}[X^2]$.*

https://people.math.wisc.edu/ roch/grad-prob/gradprob-notes29.pdf

Theorem 0.7. *(converge in distribution in metric space) Suppose (E,P) is a metric space and A is a σ-algebra of E. Suppose X_{01} and X are E-valued random variable then $X_n \to_d X$ if $g: E \to R$, we have*

$$\lim_{n \to \infty} \mathbb{E}[g(X_n)] = E[g(X)].$$

Remark: If E=R, ρ = Euclidean distance, then this is equivalent to converge in distribution in usual sense. We will view the B.M. on [0,1] as a C[0,1](continuous function on [0,1] to R)-valued r.v.

B(t,w) as $\Omega \to C[0,1]$ and $w \to B(\cdot, w)$ from $[0,1]R$. $\rho(f,g) = ||f - g||_\infty$. Suppose $\{X_n\}$ a set of E-valued r.v.s then $X_n \to_d X \iff limsupp(X_n \in K) \leq P(X \in k)$ for every closed set k in E.

Theorem 0.8. *Given mean 0 variance 1 $\{X_n\}$ we look at $S_n = \sum X_i$, then define interpolate linearly $S_t = S_{\lfloor t \rfloor} + (t - \lfloor t \rfloor)(S_{\lfloor t \rfloor + 1} - S_{\lfloor t \rfloor})$. Then we can define $S_n^*(t) = \frac{S(nt)}{\sqrt{n}}$, $t \in [0,1]$. Then $\{S_n^*\} \to_d \{B(t), t \in [0,1]\}$ converge in the sense of viewing both LHS and RHS as C[0,1]-valued r.v.s. In a special case, put t=1*

$$\frac{S_n}{\sqrt{n}} \to_d B(1)$$

this is a functional clt.

Proof. See the notes. ☐

Theorem 0.9. $\{X_k\}$ *iid with* $EX'_k = 0, var(X_k) = 1$ *Let* $M_n = max\{S_k, k \leq n\}$. *Then*

$$limP(M_n \geq X\sqrt{n}) = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/2} dy.$$

$(P(B(n)) > X\sqrt{n})$.