1 Stochastic integral

Goal: Define $\int_0^\infty f(s)dB(s)$ where f is potentially random.

Review

Fix t and consider partition $0 \leq t_0^{(n)} \leq ... \leq t_{k(n)}^{(n)}$, where $k(n) = t$.

Construct the stochastic integral. Traditionally people define $\int_0^\infty f(s)dg(s)$ using Lebesgue-Stieltjes integration if g is of bounded variation. If g is monotone ($g : R \to R$) then g induces a measure on R. $B_G([S,T]) := g(t) - g(s)$. then we define

$$\int_0^\infty f(s)dg(s) := \int_0^\infty f dB_g$$

If g is bounded variation, then we can find $g = g_1 - g_2$ where both are monotone, then

$$\int_0^\infty f(s)dg(s) := \int_0^\infty f d dg_1 - \int_0^\infty f d dg_2$$

If we want to define stochastic integral, we need to specify a reasonable class of integrals.

Definition 1.1. $\{X(t,w)\}$ is progressively measurable if for each fixed t, $x : [0,t] \times \Omega \to R$ is measurable w.r.t. $B([0,t]) \times F(t)$

Lemma 1.1. *Any adapted process that is left or right continuous is p.m.*

We can define p.m. step process H_n and we define

$$\int_0^\infty H(S)dB(S) = \underset{lim}{n \to \infty} \int_0^\infty H_n(S)dB(S)$$

in L2 sense.

Theorem 1.1. *Suppose $\{H_n\}$ is a step p.m. process s.t.*

$$||H_n - H||_2 \to 0,$$

then

$$\int_0^\infty H(S)dB(S) := \underset{lim}{n \to \infty} \int_0^\infty H_n(S)dB(S)$$

exists in the L^2 sense and is independent of the choice of $\{H_n\}$ mean while

$$E[(\int_0^\infty H(S)dB(S))^2] = E[(\int_0^\infty H^2(S)dB(S))]$$