Adv Probability

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1 Definition: Random variables

definition of measurable function:inverse image of measurable sets are measurable A mapping $x : \Omega \to S$ between two measurable spaces (Ω, F) and (S, δ) is a s-valued r.v. if

$$x^{-1}(B) := \{w : X(w) \in B\} \in F,$$

for every $B \in \delta$.

1.1 Ex: indicator variable

For every $A \in F$, define

$$I_A = \begin{cases} 0 \ ifw \notin A \\ 1 \ ifw \in A. \end{cases}$$

Then I_A is a random variable.

1.1.1 Proof:

For each $B \in \delta$, there are 4 cases,

$$X^{-1}(B) = \begin{cases} \Omega \ if \ B \supset \{0,1\} \\ \phi \ if \ B \not\supset \{0,1\} \\ A \ if \ B \ni \{1\} 0 \notin B \\ A^c \ if \ B \ni \{0\} 1 \notin B \end{cases}$$

1.2 Ex 2

$$X(w) = \sum_{n=1}^{N} c_n 1_{A_n}(w), A_n \in F$$

for each n is a random variable.

2 Proposition

For every R-valued random variable, there is a sequence $X_n(w)$ such that X_n are single r.v. and

$$X_n(w) \to X(w) \ for \ each \ w$$

2.1 Proof:

Every random variable is a limit of a simple function?

2.1.1 1

For now, assume $X(W) \geq 0$. Let $f_n(x) = (((n \cdot 1_{x>n}, \text{이부분 필요??}))) + \sum_{k=0}^{n2^{-n}-1} k2^{-n}\mathbf{1}_{[k2^{-n},(k+1)2^{-n})}(x)$.

2.1.2 detail for above

We are defining X_n in terms of X the latter is for $0 \leq X < n$ and the former is for the rest.

Define $X_n := f_n(x)$,

$$X_n(w) = n \cdot 1_{x>n} + \sum_{k=0}^{n2^{-n}-1} k2^{-n}\mathbf{1}_{X(w)\in[k2^{-n},(k+1)2^{-n})}$$

Since $X_n \uparrow X$, and $|X_n - X| \leq \frac{1}{2^n}$, if $X(w) \leq n$, then $X_n \uparrow X$

2.1.3 2

For general X, write $X = X^+ - X^-, X^- = -min(0, X)$, we have that

$$X_n^1 \uparrow X^+, X_n^1 \uparrow X^-,$$

$$X_n^1 - X_n^2 \to X^+ - X^- = X.$$

3 Definition: almost surely

X,Y defined on the same probability space. X=Y a.S. IF

$$P(\{w : X(w) \neq Y(w)\}) = 0.$$

4 Theorem

If $X : (\Omega, F) \to (S, \delta)$, $\delta = \sigma(L)$, then X is a r.v. if and only if $X^{-1}(A) \in F$ for every $A \in L.(notA \in \sigma(L)$.

4.1 Proof: $\Leftarrow$

Suppose $X^{-1}(A) \in F$ for every $A \in L$. Define

$$\hat{S} = \{B \in \delta : X^{-1}(B) \in F\}$$

Two observations:

1. $\hat{S} \supset L$ by assumption
2. $\hat{S}$ is a σ-algebra.
   (a) $S \in \hat{S}$ because
   $$X^{-1}(S) = \Omega.$$
   (b) If $A \in \hat{S}$ then $A^c \in \hat{S}$ because F is a sigma algebra , so if $X^{-1}(A) \in F$, then $X^{-1}(A)^c$ is also in F and
   $$X^{-1}(A^c) = X^{-1}(A)^c.$$
   We can check this :
   $$X^{-1}(A^c) = \{w \in \Omega : X(w) \in A^c\} = \{w \in \Omega : X(w) \in A\}^c.$$
   (c) If $A_1, ... \in \hat{S}$ then
   $$X^{-1}(\cup_{i=1}^{\infty} A_i) = \cup X^{-1}(A_i) \in F$$

Then $\hat{S} \supset \sigma(L)$(If $A \supset B, \sigma(A) \supset \sigma(B)$). This means that for every B $\in \sigma(L), X^{-1}(B) \in F$, hence X is a random variable.

Note: Subset of measurable set is not always measurable.

5 Definition σ-algebra generated by r.v.s

Given a r.v.
$$X :\rightarrow (\Omega, F) \rightarrow (R, B),$$
we define the σ-algebra generated by X to be one of the followings(equivalent):

1. The smalleset σ-algebra such that
   $$X :\rightarrow (\Omega, \sigma(X)) \rightarrow (R, B),$$
   is measurable.
2. $$\sigma(\{x^{-1}(B) : B \in \mathcal{B}_R\}).$$
3. $$\sigma(\{x^{-1}(-\infty, x)) : x \in R\}).$$

6 Distribution

of a real valued r.v. X denoted by $p_x()$ is a probability measure on (R,B) such that
$$p_x(B) = p(X^{-1}(B)).$$

6.1 Definition: The distribution function

F_x is a function $R \to [0,1]$

$$F_x(a) := p_x((-\infty, a]).$$

7 Proposition:

If X,Y have the same distribution function $F_x = F_y$, then $p_x = p_y$(as a measure on (R,B))

7.1 Proof:

We define $\delta = \{A \in B : p_x(A) = p_y(A)\}$ then

1. $\delta \supset \{(-\infty, x]\}$
2. δ is a sigma algebra.

8 Theorem

A function F is distribution function of some r.v. X iff

1. F is non-decreasing
2. $\underset{X\to\infty}{lim} F(X) = 1$, $\underset{X\to-\infty}{lim} F(X) = 0.$
3. F is right continuous
$$\underset{y\downarrow x}{lim} F(y) = F(x).$$

For proof $\Rightarrow$, use (592 note: proof use continuity of probability measure) for right continuous.

8.1 Proof: $\Leftarrow$

Use Skorohod's representation. Idea: $F^{-1}(U[0,1]) \sim F]$

$$Let\ (\Omega, F, P) = ((0,1), B_{(0,1]}, U)$$

We define
$$X^{-1}(w) = sup\{y : F(y) < w\}$$

나머지 쓰다가 안 씀. 정 필요하면 노트 볼 것.

9 Integration and convergence theorem

Let (Ω, F, μ) be a measure space fixed through out this lecture. LEt $f : \Omega \to R^* = R \cup \{+\infty, -\infty\}$. Our goal is to define

$$\int_\Omega f(w) d\mu(w).$$

1. Indicator if
$$f(w) = 1_A(w), \int_\Omega f(w) d\mu(w) := \mu(A).$$

2. Simple, non negative function (SF^+)
$$f = \sum_{i=1}^n a_i 1_{A_i} \geq 0$$
for disjoint A_i, then
$$\int_\Omega f(w) d\mu(w) := \sum_{i=1}^n a_i \mu(A_i).$$

3. non negative function $f : \Omega \to [0, \infty]$. Let $SF^+(f)$ be all the simple functions g such that $g \leq f$.
$$Def: \int_\Omega f(w) d\mu(w) = \sup_{g \in SF^+(f)} \int g d\mu.$$

4. General measurable function f, f= $f^+ - f^-$, f is called integrable if
$$\int f^+ d\mu < \infty, \int f^- d\mu < \infty.$$
and if f is integrable, then
$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu.$$

9.1 Definition: Integration of a set

$$\int_S f d\mu := \int_\Omega f 1_s d\mu$$

Remark: When μ is a probability measure,

$$\int f d\mu$$

is called the expectation of f under μ, written as $E_\mu(f)$.

9.2 Proposition 1.

The second step ② assigns a unique value to each $\phi \in SF^+$. Further

1. $$\int \phi d\mu = \int \psi d\mu \ if \ \mu(\{w : \psi(w) \neq \phi(w)\}) = 0.$$

2. linearity

3. monotonicity: if $\phi \leq \psi$ for every w, then
$$\int \phi d\mu \leq \int \psi d\mu.$$

4. Note that form of simple function is not unique because for example, 1*[0.5,1]+ 1*[1,1.5]= 1*[0.5,1.5]. Hence we can write
$$\phi(s) = \sum_{l=1}^{n} c_l I_{A_l}(s) = \sum_{k=1}^{n} d_K I_{B_k}(s),$$
we can further write $\phi(s)$ as $\sum_{j=1}^{t} e_j I_{c_j}(s)$, where c_j is a subset of some A_l, and some B_k, then
$$\sum_{j=1}^{t} e_j I_{c_j}(s) = \sum_{l=1}^{n} c_l I_{A_l}(s) = \sum_{k=1}^{n} d_K I_{B_k}(s),$$
$$\sum_{j=1}^{t} e_j \mu(c_j) = \sum_{l=1}^{n} c_l \mu(A_l) = \sum_{k=1}^{n} d_K \mu(B_k).$$

9.2.1 Proof B:

We can choose representation of $\phi, \psi \in SF^+, \phi = \sum a_i 1_{A_i}, \psi = \sum b_i 1_{A_i}$, then
$$\phi + \psi = \sum (a_i + b_i) 1_{A_i},$$
$$\int (\phi + \psi) d\mu = \sum (a_i + b_i)\mu(A_i) = \sum a_i \mu(A_i) + \sum b_i \mu(A_i).$$
Similarly we can show monotonicity.

9.3 non negative function integrable

Let F^+ be the set of non-negative function, then we can prove all the previous porperties: uniqueness, linearity, monotonicity.

1.
$$\int f d\mu = sup \int g d\mu$$
is well defined.

2. monotonicty: if $0 \leq f \leq g$, $SF^+(g) \supset SF^+(f)$
$$\int f d\mu = \sup_{h \in SF^+(f)} \int h d\mu \leq \sup_{h \in SF^+(g)} \int h d\mu = \int g d\mu.$$

9.4 Proposiiton

The integral $\int f d\mu$ assigns a unique value to each non-negative or integrable f, further

1.
$$\int f d\mu = \int g d\mu \ if \ f = g \ a.e.$$

9.5 Standard machine

(a) Choose indicator function

(b) Extend to simple function

(c) $\int$ non negative function

(d) extend to general measurable function by decomposition.

9.6 EX

Let $f : R \to R$ be Lebesgue integrable, take any $a \in R$ define $g(x) = f(x + a)$, prove
$$\int_e g(x) d\lambda(x) = \int_R f(x) d\lambda(x).$$

9.7 Lemma for MCT

If $S \in SF^+$ then we can define
$$v(s) = \int_s S d\mu = \int_\Omega S(w) 1_s(w) d\mu(w).$$

Then V is a measure on (Ω, F)

9.7.1 Proof:

We can represent S as $S = \sum a_i 1_{A_i}$, by definition $v(\phi) = 0$, hence it suffices to show additivity. Suppose $S_1, ...$ disjoint sets with $\cup_{i=1}^{\infty} S_i = S$

$$\begin{aligned} V(s) &= \int S 1_s d\mu \\ &= \int \sum a_i 1_{A_i} 1_s d\mu \\ &= \int \sum a_i 1_{A_i \cap s} d\mu \\ &= \sum a_i \int 1_{A_i \cap s} d\mu \\ &= \sum a_i \mu(A_i \cap s) \qquad \text{(b)} \\ &= \sum_{i=1}^{\infty} a_i \sum_{j=1}^{\infty} \mu(A_i \cap s_j) \qquad \text{(a)} \\ &= \sum_j \int_{s_j} S d\mu = \sum_{j=1}^{\infty} V(s_j) \end{aligned}$$

Where (a) use the fact that μ is a measure so apply countable additivity for disjoint $S_j's$. For (b), review the definition of integral of indicator function.

9.8 Proof of MCT

For one direction, $\int f \geq \int f_n$, so $\int f \geq lim \int f_n$.

9.8.1 Different direction

To show $lim \int f_n \geq \int f$, it is suffices to show

$$lim \int f_n \geq a \int f, \; a \in (0, 1).$$

Fix a and let

$$S_n := \{w \in \Omega | af(w) \leq f_n(w).\}$$

This is the key part, observe that $S_n \uparrow \Omega$.

$$\begin{aligned} &\int_{\Omega} f d\mu \\ &= V(\Omega) = limV(S_n) \\ &= lim \int_{S_n} f d\mu \end{aligned}$$

Since $af \le f_n$ on S_n,

$$aV(S_n) = \int_{S_n} af \le \int_{S_n} f_n \le \int_{\Omega} f_n.$$

Let $n \to \infty$

$$aV(\Omega) = a \int f d\mu \le lim \int_{\Omega} f_n,$$

so

$$\int f \le lim \int_{\Omega} f_n.$$

9.9 Fatou's lemma

For non negative functions

$$\int liminf f_n d\mu \le liminf \int f_n d\mu.$$

9.9.1 Proof:

Let $g_n = \underset{m \ge n}{inf} f_n, g_n \uparrow \underset{n \to \infty}{lim} inf f_n$ By MCT,

$$\int liminf f_n = lim \int \inf f_n d\mu = lim \int g_n d\mu = liminf \int g_n d\mu \le liminf \int f_n d\mu.$$

We use the $\int g_n d\mu \le \int f_n d\mu$ in last inequality.