Adv Probability

Joonwon Lee

1 Syllabus

1. General measure theory
2. Probability theory and stochastic process
3. *** Brownian motion and a little stochastic differential equation

1.1 Final presentation

1. presentation
2. do a survey
3. project

2 Chapter 1. Probability measure and integration.

Check Amir Dembo's lectures notes(stanford)

2.1 Probability space

Suppose we have a set/universe Ω. We will use 2^Ω for all the subsets of Ω. We will use F to denote a family of sets in Ω.

2.2 Definition: σ- algebra/ $\sigma - field$

We say $F \subset 2^\Omega$ is a $\sigma - algebra$ if

1. $\Omega \subset F$
2. Closed under complement
3. Closed under countable union. If $\{A_i\}$, $i \in I$, then $\bigcup_{i \in I} A_i \in F$.

Remark: $\phi \in F$ and close under intersection.

Suppose non empty F is a family of sets which is closed under union, then prove or disprove there are always exist an element $s \in \Omega$ such that s belongs to at least half of the set in F. (Frantc? , 1979)- this might be an example of project problem.

Remark: the structure of $\sigma - field$ is somewhat "intractable".

2.3 Proposition

1. check that intersection of σ -algebra is also a σ- algebra.

2. Verify that for any σ-algebras $\mathcal{H} \subset g$ and any $H \in \mathcal{H}$ the collection $\mathcal{H}^H = \{A \in g : A \bigcap H \in \mathcal{H}\}$ is a σ- algebra.

3. Show that
$$H \to \mathcal{H}^H$$
is non-increasing with respect to inclusions, and
$$\mathcal{H}^\Omega = \mathcal{H}, \mathcal{H}^\phi = g.$$

2.4 Proof of a

Suppose $\{F_i : i \in I\}$ is a collection of σ-algebras, then we will show $\bigcup F_i$ is a sigma algebra.

1. $\Omega \in F_i \to \omega \in \bigcup F_i$

2. if
$$A \in F_i \to A^c \in F_i,\ A^c \in \bigcap F_i.$$

3. if $\{A_i\}_{i \in J}$ belongs to $\bigcap F_i$ then
$$\bigcup_{i \in J} A_i \in F_i$$
for each i hence
$$\bigcup_{i \in J} A_i \in \bigcap_i F_i.$$

□

2.5 Proof of b

1. Since g is a σ algebra, $\phi \in g$, and $\phi \bigcap H = \phi$, then $\phi \in \mathcal{H}^H$

2. if $A \in \mathcal{H}^H$, then $A \in g$ and $A \bigcap H \in \mathcal{H}$. By definition, σ algebras are closed under complement,
$$A^c \in g, \tag{1}$$
$$(A \bigcap H)^c = (A^c \bigcup H^c) \in \mathcal{H},$$
and by closeness under intersection
$$(A^c \bigcup H^c) \bigcap H = A^c \bigcap H \in \mathcal{H} \tag{2}$$
by (1) and (2), $A^c \in \mathcal{H}^H$.

3. Suppose
$$A_i \in \mathcal{H}^H,\ i \in J.$$
then
$$A_i \in g, \forall i \in J,\ hence \bigcup_i A_i \in g. \tag{1}$$
Also,
$$\bigcup_i A_i \bigcap H = \bigcup_{i \in J} (A_i \bigcap H) \in \mathcal{H} \tag{2}$$
because $A_i \in \mathcal{H}^H$ implies $(A_i \bigcap H) \in \mathcal{H}$ for each i and $\mathcal{H}$ is closed under union. Then from (1) and (2), we have that
$$\bigcup_{i \in J} A_i \in \mathcal{H}^H.$$

□

2.6 Proof of c

3 Definition: Measure

A pair(ω, F) with F a σ-algebra of subsets of Ω is called a measruable space. Given a measure space, a measure μ is a any countably additive non-negative set function on this space. That is,
$$\mu : \mathcal{F} \to [0, \infty], having\ the\ properties:$$

(a) $\mu(A) \geq \mu(\phi) = 0$ for all $A \in \mathcal{F}$.

(b) Countable additivity: $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$, whenever $A = \bigcup A_n$ is a disjoint union of $\{A_i\}_{i=1}^{\infty} \subset \mathcal{F}$.

When in addition $\mu(\Omega) = 1$, *we call the measure* μ *a probability measure and often label it by* $\mathbf{P}$.

REMARK: In measure theory, we sometimes consider *signed measures*, whereby μ is no longer non-negative, hence its range is $[-\infty, \infty]$, and say that such measure if *finite* when its range is $\mathbb{R}$.(i.e. no set in $\mathcal{F}$ is assigned an infinite measure).

3.1 Proposition**

Let (Ω, F, P) a probability space, and A,B,A_i events in $\mathcal{F}$.
Prove the following properties of every probability measure.

(a) Monotonicity. if $A \subseteq$ B then $\mathbf{P}(A) \leq \mathbf{P}(B)$.

(b) Sub-additivity. If $A \subseteq \cup_i A_i$, then $\mathbf{P}(A) \leq \sum_i \mathbf{P}(A_i)$.

(c) Continuity from below: If $A_i \uparrow A$, that is, $A_1 \subseteq A_2 \subseteq ...$ and $\cup_i A_i = A$, then $\mathbf{P}(A_i) \uparrow \mathbf{P}(A)$.

(d) Continuity from above: If $A_i \downarrow A$, that is, $A_1 \supseteq A_2 \supseteq ...$ and $\cap_i A_i = A$, then $\mathbf{P}(A_i) \downarrow \mathbf{P}(A)$.

Proof:

$$B = A \cup (B \backslash A)$$

$$P(B) = P(A) + P(B \backslash A) \geq P(A).$$

□

For (b), and (c), we construct $\{B_i\}_{i=1}^{\infty}$ such that

1. disjoint
2. $B_i \subset A_i$
3. $\cup B_i = \cup A_i = B$

by defining $B_i = A_i - A_{i-1}$. For (d), define $B_i = A_i^c - A_{i-1}^c$

Proof b: Use countable additivity of measure on disjoint sets.

$$\mathbf{P}(A) = \mathbf{P}(\cup B_i) = \sum_i \mathbf{P}(B_i) \leq \sum_i \mathbf{P}(A_i)$$

□

Proof c: By monotonicity,

$$P(A_i) \leq P(A), \ \forall i$$

$$\{P(A_i)\}_i$$

is a increasing sequence that is bounded above, so it converges to the bound. □

Proof d:

$$
\begin{aligned}
P(A) &= P(\bigcap A_i) \\
&= 1 - P(\bigcup A_i^c) \\
&= 1 - P(A_1^c \cup \bigcup A_i^c - A_{i-1}^c) \\
&= 1 - P(A_1^c) - \sum_{i=2} P(A_i^c - A_{i-1}^c) \\
&= \lim_{n\to\infty} (1 - P(A_1^c) - \sum_{i=2}^{n} P(A_i^c - A_{i-1}^c) \\
&= \lim_{n\to\infty} (1 - P(A_1^c) - \sum_{i=2}^{n} P(A_i^c) - P(A_{i-1}^c) \qquad (1) \\
&= \lim_{n\to\infty} (1 - P(A_n^c)) \\
&= \lim_{n\to\infty} P(A_n).
\end{aligned}
$$

(1)은 포함 관계 때문에 성립함. □

4 Construction of a probability measure

4.1 Discrete case

When the sample space Ω is countable, we can choose $F \subset 2^{\Omega}$. We assign a probability $p_\omega \geq 0$ such that

$$\sum_{\omega\in\Omega} p_\omega = 1, P(A) = \sum_{\omega\in A} P_\omega$$

4.2 Example1 : Finite case

$$p_\omega = \frac{1}{|\Omega|}.$$

4.3 Example 2:

$$\Omega = \{0, 1, 2, ...\}, \quad p_\omega = \frac{e^{-\lambda}\lambda^k}{k!}$$

for $\lambda > 0$. c.f. law of small numbers(binomial to poisson approximation)

4.4 Example 3:Uncountable

If we put $p(\{\omega\}) > 0$ for uncountably infinite many then we will have

$$p(\Omega) = \infty.$$

4.4.1 Definition: non-atomic

We say $(\Omega, \mathcal{F}, P)$ is non-atomic if $p(A) > 0$ implies existence of $B \in F$ such that $0 < P(B) < P(A)$

4.5 Exercise 1.1.8

Suppose P is non-atomic and $A \in F, P(A) > 0$. Implication is that if $A = \Omega$, then for any $0 < a < 1$, there exists B such that

$$P(B) = a.$$

(a) Show that for every $\epsilon > 0$, we have $B \subseteq A$ such that $0 < P(B) < \epsilon$.

(b) Prove that if $0 < a < P(A)$ then there exists $B \subset A$ with P(B)=a. Hint:

Fix $\epsilon_n \downarrow 0$ and define inductively numbers x_n and sets $G_n \in F$ with $H_0 = \phi$, $H_n = \cup_{k<n} G)k, x_n = sup\{P(G) : G \subseteq A \backslash H_n, P(H_n \cup G) \leq a\}$ and $G_n \subseteq A \backslash H_n$ such that $p(H_n \bigcup G_n) \leq a$ and $P(G_n) \geq (1 - \epsilon_n)x)n$. Consider $B = \cup)kG_k$.

Proof a: By definition, we have B such that

$$0 < P(B) < P(A),$$

then one of P(B), $P(A \backslash B) \leq \frac{P(A)}{2}$. Then we can choose a sequence $\{B_k\}$ such that

$$P(B_1) < \frac{P(A)}{2}$$

$$P(B_2) < \frac{P(A)}{2^2}.$$

□

Proof b: □

5 Definition: Borel sigma algebra

If Ω is topological space, the borel σ-algebra is defined as

$$\sigma(\{O : O \; is \; open \; in \; \Omega\}).$$

5.1 Definition: Countably generated sigma algebra

Let F be sigma algebra. If f can be generated by countably many sets, we say F is countably generated.

5.2 Proposition. The Borel sigma algebra on R is countably generated

Every open set $O \subset R$ can be written as

$$0 = (\cup_{i=1}^{\infty}(a_i, b_i), a_i, b_i \in R)$$

Every open interval (q_1, q_2 are rational)

$$(a, b) = \cup_{b \geq q_2 \geq q_1 \geq a}(q_1, q_2),$$

then we claim

$$\tilde{B} = \sigma(\{(q_1, q_2) : q_1 \leq q_2 \in O\})$$

Obviously, $\tilde{B} \subset B$ but also $\tilde{B}$ contains all open intervals

$$(a, b) \in \tilde{B} \to O \in \tilde{B}$$

then

$$B = \sigma(\{O\}) \subset \tilde{B}.$$

5.3 Below p is a probability measure

1. measure maps a σ field to $[0, \infty)$, and [0,1] for probability measure.
2. additive under countable disjoint unions.

5.4 Algebra and sigma algebra

Algebra is closed under pairwise unions, then by induction, it is closed under finite union because induction won't work for infinity. σ-algebra requires stronger condition that it is closed under countable union.

6 Thm: Caratheodory's extension

Let $F_o \subset 2^{\Omega}$ be an algebra(difference is that it is closed in pairwise union), let

$$P : F_0 \to [0, 1] \tag{1}$$

be a finite additive function with $P(\Omega) = 1$, and for disjoint $\{A_i\}_{i=1}^{\infty} \in F_0$, if $\cup_{i=1}^{\infty} A_i \in F_0$, then $P(\cup A_i) = \sum_{i=1}^{\infty} P(A_i)$. Then P can be extended to a probability measure on $\sigma(F_0)$.

6.1 Define outer measure

For every $A \subset \Omega$, $P^*(A) = \underset{\{A_n\} \in F_0, \bigcup A_n \supset A}{inf} \sum_{n=1}^{\infty} P(A_n)$.

Now we claim

1. $p^*(\phi) = 0$
2. $p^*(A) \geq 0 \ \forall A$
3. if $A \subset B$, $p^*(A) \leq p^*(B)$, monotonicity.
4. if $A \subset \sum_n A_n$, $p^*(A) \leq \sum_n p^*(A_n)$,
5. $p^* = p$ for every $A \in F_o$.

6.2 Proof:

1. Since $\Omega \in F_0$, $p^*(\Omega) = p(\omega) = 1$, so we have the result.
2. 2,3 is obvious.
3. 4: for every A_n, choose an F_0 covering $\{A_n^k\} \in F_0$ such that
$$\sum_{k=1}^{\infty} p(A_n^K) \leq p^*(A_n) + \epsilon/(2^n).$$
then $\bigcup_{k,n} A_n^k$ forms a covering of A
$$p^*(A) \overset{def\ of\ outer\ measure}{\leq} \sum_{n,k} p(A_n^K) \leq \sum_n p^*(A_n) + \epsilon/(2^n) = \sum_n p^*(A_n) + \epsilon.$$
Since ϵ is arbitrary,
$$p^*(A) \leq \sum p^*(A_n).$$
4. 5 is also obvious because we can choose $\{A_n\} = \{B_n\}, B_1 = A, B_k = \phi, k \geq 2$.

Now we define
$$g := \{A \subset \Omega : p^*(E \bigcup A) + p^*(E \bigcup A^c) = p^*(E) \ \forall E \subset \Omega\}.$$

We want to argue $g \supset \sigma(F_0)$ and and $p^*|_g$ is an extension to p. We will call the sets in g, measurable sets.
$$p^*(E \bigcup A) + p^*(E \bigcup A^c) \geq p^*(E)$$
is automatic by monotonicity of outer measure, so we only need to check $\leq$ to verify if a set is in g.

6.3 Lemma 1: g is an algebra

1. $\phi \in g$.
2. If $A \in g,\ A^c \in g$. Obvious my symmetry.
3. If $A \in g, B \in g$. For every $E \in \Omega$,

$$\begin{aligned} p^*(E) &= p^*(E \bigcap A) + p^*(E \bigcap A^c) \\ &= p^*(E \bigcap A \bigcap B) + p^*(E \bigcap A \bigcap B^c) \\ &+ p^*(E \bigcap A^c \bigcap B) + p^*(E \bigcap A^c \bigcap B^c). \end{aligned} \tag{1}$$

Note that

$$E \bigcap (A \bigcap B)^c = E \bigcap (A \bigcap B),$$

$$E \backslash \left(E \bigcap (A \bigcap B)^c \right) = (E \bigcap A \bigcap B^c) \bigcup (E \bigcap A^c \bigcap B) \bigcup E \bigcap A^c \bigcap B^c) \tag{2}$$

so

$$p^*(E) \geq p^* \left(E \bigcap (A \bigcap B) \right) + p^* \left(E \bigcap (A \bigcap B)^c \right),$$

so $A \cup B \in g$, hence g is an algebra.

6.4 Lemma 2.

If $A_1, A_2, ...$ disjoint sets in g, then

$$p^*(E \cap (\cup_n A_n)) = \sum_n p^*(E \cap A_n).$$

6.4.1 Finite case

For n=1, it is clear.

$$B_k = \cup_{i=1}^k A_i$$

$$\begin{aligned} p^*(E \cap B_n) &= p^*(E \cap B_n \cap B_{n-1}) + p^*(E \cap B_n \cap B_{n-1}^c) \\ &= p^*(E \cap B_{n-1}) + p^*(E \cap A_n) \end{aligned}$$

6.4.2 Infinite case

For each n,

$$p^*(E \cap (\cup_{i=1}^\infty A_i)) \geq p^*(E \cap (\cup_{i=1}^n A_i) = \sum_{i=1} p^*(E \cap_{i=1}^n A_i)$$

then $n \to \infty$ and we are done. $\leq$ is done by monotonivity.

6.4.3 Countably subadditive

$$f(x+y) \leq f(x) + f(y).$$

6.4.4 lebesgue measure

The restriction of the set function outer measure to the class of measurable sets is called Lebesgue measure.(Outer measure is defined for any set) Also, the collection of lebesgue measurable sets is a σ algebra.

6.4.5 Borel measurable

Inverse image of open set(every open and closed is a borel set) is a borel set in domain.

6.5 Lemma 3

The class g is a σ- algebra, and p^* is countably additive when restricting on g(not necessarily disjoint), g is a measurable set, and outer measure is countably additive on g.

6.6 Show g is a sigma algebra

We are using lemma 2 for the proof, but also we can use the fact that countable union of measurable set is a measurable set.

Note we have shown g is an algebra in lemma 1.

Suppose $A_1, ..., A_n, ...$ disjoint g-sets, let $B = \cup A_i, B_n = \cup_{i=1}^n A_i$. For any E $\in \Omega$,

$$\begin{aligned} p^*(E) &= p^*(E \cap B_n) + p^*(E \cap B_n^c) \\ &= \sum_{i=1}^n p^*(E \cap A_i) + p^*(E \cap B_n^c) \\ &\geq \sum_{i=1}^n p^*(E \cap A_i) + p^*(E \cap B^c) \end{aligned}$$

as $n \to \infty$

$$p^*(E) \geq p^*(E \cap B) + p^*(E \cap B^c)$$

The other direction done by monotonicity.

For general $A_1, ... \in g$, not disjoint, let

$$C_1 = A_1, C_2 = A_2 - A_1, C_3 = A_3 - A_2 - A_1 ..., D_n = \cup_{i=1}^n C_i,$$

then

$$\begin{aligned}
p^*(E) &= p^*(E \cap D_n) + p^*(E \cap D_n^c) \\
&= \sum_{i=1}^{n} p^*(E \cap C_i) + p^*(E \cap D_n^c) \\
&\geq \sum_{i=1}^{n} p^*(E \cap C_i) + p^*(E \cap D^c)
\end{aligned}$$

as $n \to \infty$

$$p^*(E) \geq p^*(E \cap D) + p^*(E \cap D^c)$$

The other direction done by monotonicity. Hence outer measure is closed under countable union.

6.7 Show outer measure is countably additive on g

For additivity, take $E = D = B$. We directly use lemma 2.

$$\begin{aligned}
&p^*(E \cap (\cup_n C_n)) = p^*(E \cap (\cup_n A_n)) = \sum_n p^*(E \cap A_n) \\
&p^*(D) = \sum_n^{\infty} p^*(A_n)
\end{aligned}$$

7 Lemma 4.

Recall that F_0 is an algebra.

7.1 Statement:

$g \supset \sigma(F_0)$.

7.2 Proof:

For $A \in F_0$, and $E \cap \Omega$, we have an F_0 cover $\{A_n\}$ of E such that

$$p^*(E) \geq \sum_n p(A_n) - \epsilon$$

by definition of outer measure.(infimum cover)

Let $B_n = A_n \cap A$ and $C_n = A_n \cap A^c$, so $\{B_n\}$ is F_0 covering of $E \cap A, \{C_n\}$ is F_0 covering of $E \cap A^c$.

$$
\begin{aligned}
p^*(E \cap A) + p^*(E \cap A^c) &\le \sum_n p(A_n \cap A) + \sum_n p(A_n \cap A^c) \\
&= \sum_n p(A_n) \le p^*(E) + \epsilon
\end{aligned}
$$

then epsilon was arbitrary, so

$$p^*(E \cap A) + p^*(E \cap A^c) \le p^*(E).$$

$\square$

Combining all the lemmas, we know p^* is countably addtiive on measurable set g(lemma3), $p^*\Big|_{F_0} = p$(by definition of outer measure) and $g \supset \sigma(F_0)$(lemma4), so p^* is an extension of p on $g \supset \sigma(F_0)$.

7.3 Question

1. Is g larger than $\sigma(F_0)$
2. Is the extension unique?
3. Sometimes g can be larger than $\sigma(F_0)$
4. A measure space $\Omega, F, \mu)$ is called complete if the subset of zero-measurable set is also measure.
5. g is the completion of $\sigma(F_0)$. B(R) is not complete, Lebesgue measurable sets(preimage of open set is measurable). are complete.

8 Uniqueness and $\pi - \lambda$ theorem.

[alternative title goes here] Def: A π system is a collection of sets closed under finite intersection, a λ-system $\mathcal{L}$ is a system which

1. $\Omega \in \mathcal{L}$
2. Close under complementation
3. if $A_n \uparrow A$ and $A_n \in \mathcal{L}$ then $A \in \mathcal{L}$ (closed under monotone increasing limits.)

9 Proposition: A collection F is a σ-algebra if and only if both π and λ system

$\Rightarrow$ easy Suppose F is a sigma algebra.

1. $\Omega \in F$ so condition 1 for λ system
2. closed under complementation and closed under countable union, countable collection
3. $A = \cup A_i$, then we have the result.

$\Leftarrow$ If $A, B \in F$

$$A \cup B = \Omega \backslash (A^c \cap B^c) \in F, (Closed under finite union)$$

then

$$\cup_{i=1}^{\infty} A_i = \lim_{n \to \infty} \cup_{i=1}^{n} A_i \in F$$

if $A_i \in F$. (limit to above)

10 Dykin's $\pi - \lambda$ theorem

If $P \subseteq L$ with P π sys, L $\lambda - sys$ then $\sigma(P) \subseteq L$.

11 If μ_1, μ_2 agree on π-system P and $\mu_1(\Omega) = \mu_2(\Omega) < \infty$

then $\mu_1 = \mu_2$ on $\sigma(P)$.

11.1 Proof:

$\mathcal{L} = \{A \in \sigma(P) : \mu_1(A) = \mu_2(A)\}$ then $L \supset P$, also $\Omega \in L$, a closed under complement.

A closed under monotone increasing sets $\Rightarrow$ L is a λ- sys. We can set a partial sum and make it monotone.

Since an algebra is automatically a π sysmtem, then extension is unique. Finally, we see an example of constructing Lebesgue measure on [0,1).

If two measure agree on algebra, then extension of measure is also the same.

12 Example

Consider sets of finite disjoint union of intervals

$$B_0 = \{A = \cup_{k=1}^{n}[a_k, b_k) : 0 \leq a_1 \leq b_1 \leq a_2 ... \leq b_n \leq 1\}$$

We define

$$a : B_0 \to [0, 1] \ a.s.$$

$$\cup_{i=1}^{n}[a_k, b_k) \to \sum_{k=1}^{n} b_k - a_k$$

We can check B_0 is an algebra(empty set in is B_0 and closed under complementation, and closed under finite union. Hence λ is finitely additive on B_0.

Then by CAATHEODORY extension a can be uniquely extended to $\sigma(B_0)$. (Borel sigma algebra in [0,1).

Note: Usually bigger algebra is complete for Borel algebra. Sometimes σ algebra is intractable but algebra is tractable.