Homework 3

Note. This homework is due in class on March 20, 2026. Please show detailed steps to explain how you get your solution. Simply providing a final answer may not warrant full credit.

Question 1 [43503/53903] Let $\boldsymbol{b} + \delta\boldsymbol{b}$ be a perturbation of a non-zero vector $\boldsymbol{b}$, and let $\boldsymbol{x}$ and $\delta\boldsymbol{x}$ be such that $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ and $\mathbf{A}(\boldsymbol{x} + \delta\boldsymbol{x}) = \boldsymbol{b} + \delta\boldsymbol{b}$, where $\mathbf{A}$ is a given nonsingular matrix, show that

$$\frac{\|\delta\boldsymbol{x}\|}{\|\boldsymbol{x}\|} \leq \kappa(\mathbf{A})\frac{\|\delta\boldsymbol{b}\|}{\|\boldsymbol{b}\|}.$$

Notice, here $\|\cdot\|$ denotes an induced norm and $\kappa(\mathbf{A})$ denotes the condition number of matrix $\mathbf{A}$.

Question 2 [43503/53903] Suppose $\mathbf{A}$ is a 202×202 matrix with $\|\mathbf{A}\|_2 = 100$ and $\|\mathbf{A}\|_{\mathrm{F}} = 101$. Give the sharpest possible lower bound on the 2-norm condition number $\kappa_2(\mathbf{A})$.

Question 3 [43503/53903] Given $\mathbf{A} \in \mathbb{C}^{m\times n}$ of rank n and $\boldsymbol{b} \in \mathbb{C}^m$, consider the block 2×2 system of equations

$$\begin{bmatrix} \mathbf{I} & \mathbf{A} \\ \mathbf{A}^* & \mathbf{O} \end{bmatrix} \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} \boldsymbol{b} \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{I}$ is the $m \times m$ identity matrix. Show that this system has a unique solution $(\boldsymbol{r}, \boldsymbol{x})^{\mathrm{T}}$, and that the vectors $\boldsymbol{r}$ and $\boldsymbol{x}$ are the residual and the solution of the following least squares problem

$$\begin{gathered} \text{Given } \mathbf{A} \in \mathbb{C}^{m\times n} \text{ of full rank, } m \geq n, \text{ and } \boldsymbol{b} \in \mathbb{C}^m, \\ \text{find } \boldsymbol{x} \in \mathbb{C}^n \text{ such that } \|\boldsymbol{b} - \mathbf{A}\boldsymbol{x}\|_2 \text{ is minimized.} \end{gathered}$$

Question 4 [43503/53903] Consider the problem

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 2, \\ x_1 + x_2 &= 4, \\ 3x_1 - 2x_2 + x_3 &= 1. \end{aligned}$$

(a) Write the linear system in matrix-vector form $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$. What are matrix $\mathbf{A}$ and vector $\boldsymbol{b}$?

(b) Carry out Gaussian elimination in its simplest form (without pivoting) for this question. What is the resulting LU decomposition of matrix $\mathbf{A}$?

(c) Proceed to find the solution of this linear system.

Question 5 [53903] Let $\mathbf{A} \in \mathbb{C}^{m\times m}$ $(m \geq 2)$ be nonsingular. Suppose that for each k with $1 \leq k \leq m$, the upper-left $k\times k$ block $\mathbf{A}_{1:k,\,1:k}$ is nonsingular, show that $\mathbf{A}$ has an LU factorization.