Applications with Groups of Order Four: The Golden Rules of Rotating

Mary Liu

April. 21, 2024

Contents

1 Abstract

This paper discusses groups with order four: their applications in real life and some properties. Mainly giving an insight into finding the golden rule for rotating/flipping between four elements and reproofing a classic theorem that there are only two groups of an order of a power of a prime number, up to isomorphism. This paper uses inspiration from (a) the paper *Group Theory in Bedroom [6]* and (b) the proof from *ProofWiki* [2].

2 Introduction

Nowadays, most of the waffle machines are double-sided. To make a waffle, you pour the waffle batter on the machine, close the top lid, flip the machine, cook for 5 minutes, flip back, and open the lid. Now we can enjoy our delicious waffle. However, the waffle makers in my hometown are different. It is more like a pan with neat grids (so waffles are just pancakes with grids in my hometown). When I cook my waffles, I need to flip them side by side by using a kitchen spatula for flipping to make sure that each side is well-cooked at the end.

While that is not a hard job, there was such a morning that I came out with this question: what if my waffle pan broke? Specifically, unlike America's typical double-sided waffle machine or my hometown's waffle pan, only half the area of the pan can be heated: if the double-sided waffle machine is 1 maker and my hometown's waffle pan is 1/2 maker, what if we only have 1/4 maker?

More technically, if we assume that the waffle machine is rectangular, without loss of generality, draw a symmetry axis parallel to the width, only the upper half (the area away from the panhandle) is working appropriately. Beforehand, for the waffle pan, I needed to take care of 2 sides, but now I need to take care of 4 sides, upper and lower parts of each side. In this situation, it seems clear that people need to rotate the waffle to a different position each time. However, I don't have a good memory of which way I flipped it last time. Beforehand I need to flip the waffle to another side whenever I think a side is ready, and if not, I can flip back and cook for the original side again. But now, how can I keep track of my motion?

3 Group Theory in Waffles

3.1 The Golden Rule

Firstly, to keep the terminology clear, let's call the double-sided machine as 1 *maker*. Similarly, we denote my hometown's waffle pan as 1/2 *maker*, and the broken waffle pan as 1/4 *maker* or just *maker*. As mentioned in the introduction, we assume the pan is rectangular and without loss of generality, only the upper half (the area away from the panhandle) is working appropriately.

We can see for a 1/2 maker the rule of cooking is trivial: We can define the game in group theory: suppose Waffle Flipping is a set, denoted as W_2, there are only two elements, relatively the front side, and the back side, denoted as f and g, in the set. It does not matter which side we call front as the other side can always be called as back. The binary operation is flipping the waffle 180 degrees with respect to the axis, denoted as $*$. It does not matter which direction or axis we choose here, but let's assume it is the axis parallel to the width clockwise.

So trivially W_2 is a group and $\langle G, *\rangle$[1] is isomorphic to $\langle \mathbb{Z}_2, +\rangle$. We could find out that there is a *golden rule* of cooking:

[1] Without further clarification, definitions and notations in this paper are the same as what referent in *Fraleigh* [4]

Arbitrary, suppose we are at f, $f * g = g$ and $g * g = f$.

Our golden rule cycles through the two sides and returns to the original side.

Definition 1. *a basic principle that should always be followed to ensure success in general or in a particular activity [7].*

In other words, golden rules are like theorems.

Now, Is there a golden rule for 1/4 maker? If we do, what is it? If we do not, why?

3.2 Under $1/4$ Maker

3.2.1 Basic Notations of $1/4$ Maker

Suppose we have a golden rule, in our 1/4 maker case, it would be an operation that, when applied repeatedly, would cycle through all four configurations and then return to the original state. Let's try to find it.

First of all, we clear up the notation:

- A waffle is a cuboid, so it can be rotated around any of three orthogonal axes. One could label the axes x, y, and z by using the Cartesian Coordinate System. But here we use the terminology of aviation which seems to be more suitable in our case. If we think of a waffle as an airplane flying toward the proper working part of the maker, then the three axes are designated *roll*, *pitch*, and *yaw* such that they are parallel to the length, width, and height respectively, denoted R, P, and Y respectively, as shown in figure 1 on top of next page.
- We define cooking waffles in 1/4 maker as K_4, K_4 is a set.
- There are 4 elements in K_4 such that the elements themselves are operations: they are the various ways of flipping the waffle.
- The rule for composite elements is as usual: perform one operation after another, denoted in multiplication format '$*$' or non-space between two or more elements.

Claim: W_4 is a group.

Proof. $\mathcal{G}_1$: **Associativity**. For all $f, g, h \in K_4$, we have

$$(fg)h = f(gh)$$

which is true based on Figure 2 shown next page.

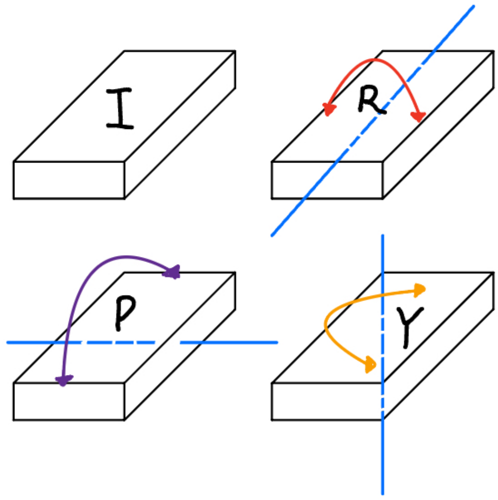

Figure 1: identity, roll, pitch, and yaw

$\mathcal{G}_2$: **Identity element**. There is an element $e \in K_4$, doing nothing, i.e., performing no flipping action, such that for all $f \in K_4$,

$$ef = f = fe$$

$\mathcal{G}_3$: **Inverse**. For all $f \in W_4$, there is an element $f' \in K_4$ such that

$$ff' = e = f'f$$

as each of the three basic rotations is its own inverse. Flipping the waffle a half-turn around any of the axes twice, we leave at the original configuration. Notice that e, doing nothing, is its own inverse.

Closure. For any $f, g \in W_4$,

$$fg \in K_4$$

as shown in Figure 2. The table gives the result of all possible pair-wise combinations of the four operations I, R, P, and Y. Every such combination is equivalent to one of the fundamental operations. □

3.2.2 Finding a Golden Rule for 1/4 Maker

The fact of *closure* of group K_4 is what dooms the search for a golden rule. Suppose we have a golden rule that involves an n-step sequence of roll, pitch, and yaw turns, since we already know that none of the single operations yields a cycle through all four states of the waffle, no pair of operations can be composed to form a golden rule. For the n-step sequence, we can replace the first two of these actions with a single turn, creating an equivalent procedure with $n-1$ steps. Continuing the process $n-1$ times, we eventually reduce the entire sequence to a single symmetry operation, which cannot be a golden rule.

Let's try adding another element/operation to K_4 that is not the 4 symmetry operations, as with the quarter turns in Figure 3 on the next page.

		second operation			
		I	R	P	Y
First operation	I	I	R	P	Y
	R	R	I	Y	P
	P	P	Y	I	R
	Y	Y	P	R	I

Figure 2: table for K_4

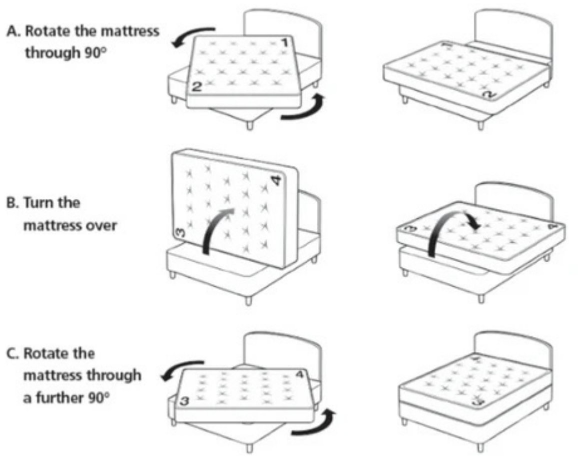

Figure 3: quarter turns included

It is an image from Clencraft's Official Website to suggest people how to flip a mattress. As flipping mattress has identical elements/operations with flipping waffles in 1/4 maker, this image is being used to represent what the possible elements/operations we can add to K_4.

However, it is not working either. Any move that qualifies as a waffle flip has to begin and end with the waffle in one of the four canonical positions on the 1/4 maker. In between, one can add infinitely many different operations by changing the angle of the waffle with respect to any axis, but when we put the waffle down, only the net effect of our gyrations can be observed. And again, the multiplication table for the group says that all our manipulations, no matter how acrobatic, can be replaced by a single symmetry operation, either I, R, P or Y.

Therefore, there is no golden rule with cooking the waffle while using the 1/4 maker.

4 Group Theory in Tires

Not all swappings are like waffle flipping. For example, we can find a golden rule in rotating the tires of a typical four-wheeled vehicle:

Always rotate a quarter turn clockwise.

In other words, move the right front tire to the right rear, the right rear tire to the left rear, etc. Counterclockwise would work just as well. In either case, repeating the process 4 times would return the tires to their original positions with each one having visited all four corners. Why is rotating the 4 tires so different from flipping 4 configurations of waffle?

4.1 Klein 4-Group and Cyclic 4-Group

The answer is rotating tires are governed by a different set from K_4, denoted as $\langle C_4\rangle$. Clearly, $\langle C_4\rangle$ is a cyclic group that is generated by the quarter-turn rotation, assuming it is clockwise, denoted as Q. $\langle C_4\rangle$ has an order of 4 with $0, 90, 180$, and $270(Q^3)$ degrees rotations as all the elements, denoted as I, Q, Q^2, and Q^3 respectively. $\langle C_4\rangle$ is called a ***Cyclic 4-group***.

Definition 2. *The cyclic group of order 4 is defined as a group with four elements e, a, a^2, a^3, where $a^n a^m = a^{n+m}$ and the exponent $n+m$ is reduced modulo 4. In other words, it is the cyclic group whose order is four [2].*

which can also be viewed as

- The quotient group of the group of integers by the subgroup comprising multiples of 4.
- The multiplicative subgroup of the nonzero complex numbers under multiplication, generated by i, the square root of -1.
- The group of rotational symmetries of the square, i.e., $\langle C_4\rangle$ in our tire rotation case.

		second operation			
		I	Q	Q^2	Q^3
first operation	I	I	Q	Q^2	Q^3
	Q	Q	Q^2	Q^3	I
	Q^2	Q^2	Q^3	I	Q
	Q^3	Q^3	I	Q	Q^2

Figure 4: table for $\langle C_4 \rangle$

Hence the notation Q, Q^2, Q^3 for their respective degree makes intuitive sense.

Notice that $\langle C_4 \rangle$ is isomorphic to $\langle \mathbb{Z}_4, +_4 \rangle$ by *Theorem* 6.10 in Fraleigh below [4]:

Theorem 3. *Let G be a cyclic group with generator a. If the order of G is infinite, then G is isomorphic to $\langle \mathbb{Z}, + \rangle$. If G has finite order n, then G is isomorphic to $\langle \mathbb{Z}_n, +_n \rangle$.*

And our waffle flipping group, K_4, is a ***Klein 4-group*** as the table in Figure 2 shows.

Definition 4. ***Klein 4-group*** *is an abelian group with four elements, in which each element is self-inverse (composing it with itself produces the identity) and in which composing any two of the three non-identity elements produces the third one [5].*

So for further convenience, we use $\mathbb{Z}_4$ to denote all Cyclic 4-groups, and K_4 to denote all Klein 4-groups.

4.2 Property of Group with Order of a Power of a Prime Number

Is there any other flipping or rotation that involves four elements but is different but is different from both K_4 or $\mathbb{Z}_4$, i.e., not isometric to either K_4 or $\mathbb{Z}_4$?

The answer is K_4 and $\mathbb{Z}_4$ are the only two possibilities for a group of order 4. They are abelian groups, specifically, are abelian of prime power order. In $\mathbb{Z}$, 4 is the first natural number such that there are non-isomorphic groups of that order. While completing Figures 2 and 4, the multiplication table for K_4 and $\mathbb{Z}_4$ respectively can show this classification, it follows more generally from *the classification of groups of prime-square order* or

We first proof a lemma that will be used for proofing the next lemma.

Lemma 5. *Let G be a group, $Z(G)$ be the center of G. The elements of $Z(G)$ form singleton conjugacy classes, and the elements of $G/Z(G)$ belong to multi-element conjugacy classes. [3]*

Proof. Let C_a be the conjugacy class of a in G. Let $a \in Z(G)$, then

$$\forall x \in G, xa = ax$$

which implies that

$$xax^{-1} = a$$

Hence $C_a = \{a\}$. □

Now we proof a lemma that will be used for proofing our theorem.

Lemma 6. *Let G be a group whose order is the power of a prime. Then the center of G is non-trivial.*

$$\forall G : |G| = P^r : P \text{ is a prime, } r \in \mathbf{N}_{>0} : Z(G) \neq \{e\}$$

Proof. G is either abelian or not.

<u>*Case 1*</u>: Suppose G is abelian, by definition of an abelian group,

$$Z(G) = G$$

So that G is itself non-trivial. Also, as prime group is cyclic and a cyclic group is abelian, this will always be the case for $r = 1$.

<u>*Case 2*</u>: Suppose G is non-abelian. Thus $Z(G) \neq G$, implies $G/Z \neq \emptyset$. Let $C_{x_1}, C_{x_2}, ..., C_{x_n}$ be the *conjugacy classes* such that $G/Z(G)$ is partitioned. By Lemma 5, all of these will have more than one element.

As the *Conjugacy Class Equation* [1] shows

$$|Z(G)| = |G| - \sum_{j=1}^{n} |C_{x_j}|$$

As the number of conjugates is the number of cosets of centralizer:

$$|C_{x_j}|/|G|$$

Let $N_G(x)$ be the normalizer of x in G (normalizer as defined in *Fraleigh* [4]). Then,

$$\forall j : 1 \geq j \geq n, [G : N_G(x_j)] > 1$$

which implies

$$P/[G : N_G(x_j)]$$

Since $P/|G|$, it follows that

$$P/|Z(G)|$$

and the resuly follows. □

Theorem 7. *Let P be a prime number. Then there are only two groups of order P^2 up to isomorphism:*

- *Cyclic group of prime-square order, i.e. the cyclic group of order P^2, denoted as $\mathbb{Z}_{P^2}$.*
- *Elementary abelian group of prime-square order, i.e., the elementary abelian ggroup of order P^2, denoted as E_{p^2} and equal to $\mathbb{Z}_p \times \mathbb{Z}_p$ (when $P = 2$, it is more commonly called the Klein 4-group).*

Both of these are abelian groups.

Proof. Suppose P is a prime number, G is a group with order p^2. Let Z be the center of G, denoted as $Z(G)$. Our proof includes two parts:

1. Show that G is abelian.
2. Show that there are precisely two possibilities of groups up to isomorphism.

To show that G is abelian, our goal is to show that $Z(G) = G$.

By the previous lemma, $Z(G)$ is nontrivial. As center is normal, $G/Z(G)$ exists. By Lagrange Theorem, if $|Z(G)| = P$,

$$|G/Z(G)| = \frac{P^2}{P} = P$$

. Hence $G/Z(G)$is a cyclic group. As cyclic over central implies abelian, G is an abelian group. By Lagrange Theorem again, $|Z(G)|$ is either $1, P$, or P^2. But the previous steps rule out 1 and P, so $|Z(G)| = P^2$. Hence $Z(G) = G$, and therefore G is abelian.

According to the Fundamental Theorem of Finitely Generated Abelian Groups, G must be a direct product of cyclic groups, giving precisely the two possibilities in the theorem.

□

5 Conclusion

Therefore as $4 = 2^2$ and 2 is a prime, the only two groups of order 4 up to isomorphism are the Klein 4-group and the Cyclic 4-group. As we discussed in previous sections, if a group is isomorphic to K_4, there does not exist a golden rule that cycles through all elements and then returns to the original state; if a group is isomorphic to $\mathbb{Z}_4$, rotating in a quarter turn in one direction is the golden rule.

References

[1] "Conjugacy Class Equation." *ProofWiki*, 13 June 2009, www.proofwiki.org/wiki/Conjugacy_Class_Equation.

[2] "Cyclic Group:Z4." *Groupprops*, groupprops.subwiki.org/wiki/Cyclic_group:Z4. Accessed 17 Apr. 2024.

[3] "Definition:Singleton." *ProofWiki*, www.proofwiki.org/wiki/Definition:Singleton. Accessed 20 Apr. 2024.

[4] Fraleigh, John B. *A First Course in Abstract Algebra*. Seventh Edition ed., Pearson, 2014.

[5] "Klein Four-Group." *Wikipedia*, Wikimedia Foundation, 30 Mar. 2024, en.wikipedia.org/wiki/Klein_four-group.

[6] Hayes, Brian. "Group theory in the bedroom." *American Scientist*, vol. 93, no. 5, 2005, p. 395, https://doi.org/10.1511/2005.55.3465.

[7] "The Home of Language Data." Oxford Languages, languages.oup.com/. Accessed 14 Apr. 2024.