Abstract of "'A Tale of 2-Spheres': How Conformal Symmetry, Chaos, and Some Elementary Algebra Led to Insights in Black Holes and Quantum Cosmology" by Chang Liu, Ph.D., Brown University, February 2022.

The unification of quantum mechanics and general relativity is the holy grail of fundamental theoretical physics. This thesis represents a small part of that endeavor, where we consider a number of problems that might be of interest to the theoretical studies of black holes and quantum cosmology. How does the study of black holes and quantum cosmology help us unify quantum mechanics and general relativity, one might ask? The answer lies in the simple observation that a black hole and our universe both share a number of key features that practically demand the simultaneous application of quantum mechanics and general relativity. They both have a singularity, where matter and energy are infinitely dense, and known laws of physics do not apply, as well as a horizon, which prevents spacetime events from making causal contact no matter how much they want to. They are also intrinsically gravitational systems, meaning that their classical (ie. non-quantum) description involves only the spacetime geometry (this is slightly less true for charged black holes, although their most important features are still gravitational). They therefore have become the primary objects to study if one wants to build a unified theory of quantum gravity, because it is in these two systems that the logical incompatibilities of quantum mechanics and general relativity become most apparent. One of the most severe incompabilities is that general relativity, being a geometric theory is intrinsically local, while quantum mechanics, being defined on a vector space, is intrinsically non-local. One promising resolution of this incompatibility is to define a "holographic" theory between a quantum mechanical model, and a (semi-)classical gravity in a spacetime with one additional spatial dimension. In other words, objects on the classical spacetime (called the bulk) are mapped into objects in the quantum mechanical model defined on its boundary. This thesis will therefore discuss three separate, but closely linked issues that all have something to do with building holographic theories between the bulk and the boundary. As these boundaries are all two-dimensional spheres, this is therefore a story of building holography on 2-spheres. Hence the title of this thesis.

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“A Tale of 2-Spheres”: How Conformal Symmetry, Chaos, and Some Elementary Algebra Led to Insights in Black Holes and Quantum Cosmology

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Publications

Conformal Wavefunctions for Graviton Amplitudes

Authors: Chang Liu (Brown U.), David A. Lowe (Brown U.)
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Abstract: The extended-BMS algebra of asymptotically flat spacetime contains an SO(3,1) subgroup that acts by conformal transformations on the celestial sphere. It is of interest to study the representations of this subgroup associated with gravitons. To reduce the equation of motion to a

Schrödinger-like equation it is necessary to impose a non-covariant gauge condition. Using these solutions, leading-order gauge invariant Weyl scalars are then computed and decomposed into families of unitary principal series representations. An invertible holographic mapping is constructed between these unitary principal series operators and massless spin-2 perturbations of flat spacetime.

Conformal Wave Expansions for Flat Space Amplitudes

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Abstract: The extended BMS algebra contains a conformal subgroup that acts on the celestial sphere as SO(3,1). It is of interest to perform mode expansions of free fields in Minkowski spacetime that realize this symmetry in a simple way. In the present work we perform such a mode expansion for massive scalar fields using the unitary principal series representations of SO(3,1) with a view to developing a holographic approach to gravity in asymptotically flat spacetime. These mode expansions are also of use in studying holography in three-dimensional de Sitter spacetime.

Holographic Map for Cosmological Horizons

Authors: Chang Liu (Brown U.), David A. Lowe (Brown U.)
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e-Print: `1912.04864 [hep-th]`
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Abstract: We propose a holographic map between Einstein gravity coupled to matter in a de Sitter background and large N quantum mechanics of a system of spins. Holography maps a spin model with a finite dimensional Hilbert space defined on a version of the stretched horizon into bulk gravitational dynamics. The full Hamiltonian of the spin model contains a non-local piece which generates chaotic dynamics, widely conjectured to be a necessary part of quantum gravity, and a local piece which recovers the perturbative spectrum in the bulk.

Notes on Scrambling in Conformal Field Theory

Authors: Chang Liu (Brown U.), David A. Lowe (Brown U.)
Date: Aug 29, 2018
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DOI: `10.1103/PhysRevD.98.126013`

Abstract: The onset of quantum chaos in quantum field theory may be studied using out-of-time-order correlators at finite temperature. Recent work argued that a timescale logarithmic in the central charge emerged in the context of two-dimensional conformal field theories, provided the intermediate channel was dominated by the Virasoro identity block. This suggests a wide class of conformal field theories exhibit a version of fast scrambling. In the present work we study this idea in more detail. We begin by clarifying to what extent correlators of wavepackets built out of superpositions of primary operators may be used to quantify quantum scrambling. Subject to certain caveats, these results concur with previous work. We then go on to study the contribution of intermediate states beyond the Virasoro identity block. We find that at late times, time-ordered correlators exhibit a familiar decoupling theorem, suppressing the contribution of higher dimension operators. However this is no longer true of the out-of-time-order correlators relevant for the discussion of quantum chaos. We compute the contributions of these conformal blocks to the relevant correlators, and find they are able to dominate in many interesting limits. Interpreting these results in the context of holographic models of quantum gravity, sheds new light on the black hole information problem by exhibiting a class of correlators where bulk effective field theory does not predict its own demise.

Two-Scale Oscillons

Authors: Chang Liu, Richard Easther (U of Auckland).

Date: Dec 21, 2016

e-Print: `1612.07228 [hep-th]`

Abstract: Oscillons are spatially stationary, quasi-periodic solutions of nonlinear field theories seen in settings ranging from granular systems, low temperature condensates and early universe cosmology. We describe a new class of oscillon in which the spatial envelope can have "off centre" maxima and pulsate on timescales much longer than the fundamental frequency. These are exact solutions of the 1-D sine-Gordon equation and we demonstrate numerically that similar solutions exist in up to three dimensions for a range of potentials. The dynamics of these solutions match key properties of oscillons that may form after cosmological inflation in string-motivated monodromy scenarios.

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2015 Software Engineer at Pharos Systems International.

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Preface

This thesis tells you all you need to know about...

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First of all I would like to thank my advisor, Prof. David Anthony Lowe, for guidance and advisory. I would also like to thank other faculty members in the high energy theory group, in particular my defense committee members and letter writers, Prof. Antal Jevicki, Prof. Marcus Spradlin, and Prof. Stephon H. Alexander, for their support. Finally, I would like to thank my friends within the department (and pity my enemies, both inside the department and elsewhere. Just kidding).

I would also like to use this opportunity to thank my weightlifting coaches, Jeremy and Tommy from PBC strength and conditioning, and Dave, Jarred, and James from the Providence Barbell Club for their help in athletic development, in particular for helping me win the men's 89kg weightclass Gold Medal in the USAW meet down at Westerly in December 2021. I also just realized that I am morally obliged to thank Steve, not just because of the shirt but also because of the very threatening look that he gave me when I mentioned that his name wasn't here. This gratitude (although in considerably less strength than the one for Steve) is also extended to the other team members of the Providence Barbell Club (at least the ones I like!).

Finally, a special thanks is awarded to Bioware Montreal for ruining the Mass Effect series forever, especially with that nonsense known as Andromeda. I have finally been able to spend less time in video games and focus on finishing my PhD.

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Notes on Scrambling in Conformal Field Theory

The onset of quantum chaos in quantum field theory may be studied using out-of-time-order correlators at finite temperature. Recent work argued that a timescale logarithmic in the central charge emerged in the context of two-dimensional conformal field theories, provided the intermediate channel was dominated by the Virasoro identity block. This suggests a wide class of conformal field theories exhibit a version of fast scrambling. In the present work we study this idea in more detail. We begin by clarifying to what extent correlators of wavepackets built out of superpositions of primary operators may be used to quantify quantum scrambling. Subject to certain caveats, these results concur with previous work. We then go on to study the contribution of intermediate states beyond the Virasoro identity block. We find that at late times, time-ordered correlators exhibit a familiar decoupling theorem, suppressing the contribution of higher dimension operators. However this is no longer true of the out-of-time-order correlators relevant for the discussion of quantum chaos. We compute the contributions of these conformal blocks to the relevant correlators, and find they are able to dominate in many interesting limits. Interpreting these results in the context of holographic models of quantum gravity, sheds new light on the black hole information problem by exhibiting a class of correlators where bulk effective field theory does not predict its own demise.

2.1 Introduction

It has been suggested that quantum theories of gravity exhibit a property known as fast scrambling, where a generic quantum state exhibits global thermalization in a timescale that is logarithmic in the system size [1]. It is interesting to explore this idea in the context of holographic theories of gravity dual to conformal field theories, where one may try to extract constraints on the class of conformal field theories with gravity duals.

One simple way to to quantify this notion of scrambling is to consider the norm (or equivalently the square) of the commutator of a pair of Hermitian operators V and W at different times. For the purposes of the present chapter, we will also consider the system at finite temperature, with inverse temperature β. This leads to a relation with out-of-time-order correlators

$$\begin{aligned} -\langle [V(0), W(t)]^2 \rangle_\beta = \langle V(0)W(t)W(t)V(0)\rangle_\beta + \langle W(t)V(0)V(0)W(t)\rangle_\beta \\ - \langle W(t)V(0)W(t)V(0)\rangle_\beta - \langle V(0)W(t)V(0)W(t)\rangle_\beta \,. \end{aligned} \tag{2.1}$$

For sufficiently late times, the first two terms are simply the time-independent disconnected diagram $\langle WW\rangle_\beta \langle VV\rangle_\beta$, while the last two terms are genuine out-of-time-order correlators. For the 2d conformal field theories of interest here, these correlators may be computed by continuing the Euclidean four-point function through the second Riemann sheet [2], as we describe in detail later. These terms vary as a function of t, unlike the disconnected terms, and from them a scrambling timescale may be extracted. In the following section we describe in more detail the dependence of this timescale on the chosen operators. Briefly, one wishes to choose operators that exhibit the longest scrambling timescale, so one may use this commutator computation as a proxy for asking that the longest timescale a generic state scrambles. There may of course exist special choices of operators with much shorter scrambling times, and likewise special choices with much longer times, such as those that commute with the Hamiltonian.

In order to study these out-of-time-order correlators at finite temperature in conformal field theory we will begin with the Euclidean theory on $S^1 \times \mathbb{R}$. The correlators in this theory may be obtained by a conformal mapping from the complex plane. The circle direction is to be periodically identified with period β and corresponds to the imaginary time direction. The spatial direction is then necessarily of infinite extent. For the purposes of the present chapter we will study four-point correlators of primary operators, as well as correlators of wavepackets of such operators. Four-point functions of primaries are expressed in the so-called conformal blocks of the theory. In general, these conformal blocks are not known beyond infinite series expansions. However there has been much progress in the literature on obtaining asymptotic expansions of these conformal blocks in a variety of limits, and we will make extensive use of these results in the following [3].

In holographic theories, the graviton mode is dual to the stress energy tensor of the CFT, which in turn is a Virasoro descendant of the identity operator. Long distance bulk physics should be dominated by the propagation of this mode, so the limit where the identity block dominates the conformal block is of particular interest. Assuming this intermediate Verma module dominates the conformal block of the four-point function [2] (as well as assuming large central charge and large external conformal weight h_w) obtained a scrambling time logarithmic in the central charge c of the CFT

$$t_* = \frac{\beta}{2\pi} \log \frac{c}{h_w} \tag{2.2}$$

suggesting (at least if the result can be continued to values h_w of order 1) that conformal field theories exhibit a version of fast scrambling.

In this chapter we will study this problem in more detail. One immediate issue is that primary operators on their own do not exhibit the timescale (2.2), but rather a thermalization timescale of order β or less. However the class of states obtained by acting on the thermal state with a primary is not necessarily a good representative of a generic state, so this is not an immediate contradiction. To proceed we fold the primary operators into wavepackets, and consider optimizing the shape of the wavepacket to obtain the longest thermalization time. When this is done, we find a timescale resembling (2.2) does indeed emerge. Next we examine the contribution of Verma modules with higher conformal weights to the four-point function. While we find the time-ordered four-point functions respect the familiar late-time decoupling theorems, and can be ignored with respect to the identity block, this is no longer true of the out-of-time-order correlators needed to compute (2.1). We compute the contributions of these higher intermediate states, and find these can indeed dominate the commutator even when all the time-ordered correlators have a sensible holographic description in terms of bulk low energy effective field theory. This implies that many of the bulk observables, defined over finite ranges of time, that one might use to probe the black hole information problem, are not accessible using low energy effective field theory. In this sense effective field theory does not predict its own demise.

2.2 Scrambling and CFT Correlators

We consider a thermal system described by a conformal field theory living on a spatial real line x with imaginary time $-it$ periodically identified with period β. We can map this spatially infinite thermal system to a CFT defined on the complex plane z via the exponential map

$$z = \exp\left(\frac{2\pi}{\beta}(x+t)\right) .$$

We are interested in computing the 4-point functions that appear in (2.1) so to this end we consider four pair-wise local operators, inserted at distinct spatial positions as in fig. 2.1. We therefore have, after conformal mapping

$$\begin{aligned} z_1 &= e^{\frac{2\pi}{\beta}x_1} \\ z_2 &= e^{\frac{2\pi}{\beta}x_2} \\ z_3 &= e^{\frac{2\pi}{\beta}(x_3+t)} \\ z_4 &= e^{\frac{2\pi}{\beta}(x_4+t)} \end{aligned}$$

where we are interested in the limit $x_1 \to x_2$, $x_3 \to x_4$ to reproduce the desired commutator.

The spacetime dependence of the conformal blocks appearing in the 4-point function will only depend on the cross-ratio $z = z_{12}z_{34}/z_{13}z_{24}$ (and $\bar{z}$) which is easily shown to be

$$z = \frac{\sinh\left(\frac{\pi}{\beta}(x_1-x_2)\right)\sinh\left(\frac{\pi}{\beta}(x_3-x_4)\right)}{\sinh\left(\frac{\pi}{\beta}(t-x_1+x_3)\right)\sinh\left(\frac{\pi}{\beta}(t-x_2+x_4)\right)} .$$

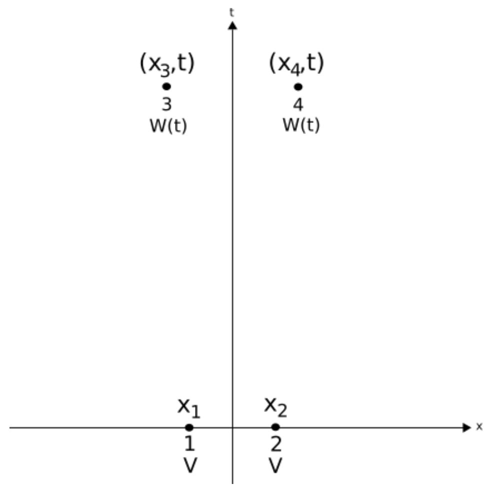

Figure 2.1: The configuration of four pair-wise identical local operators, themselves separated by large t.

As discussed in appendix A.1 we rescale the 4-point function by the coincident 2-point functions, to scale out the operator norm. The rescaled correlators then depend only on the cross-ratios as in (A.4).

As an example, let us consider the identity conformal block in a large c limit, where the V and W operators have conformal weights h_v and h_w respectively. The large c limit is to be taken with h_w/c fixed, and $h_v \ll c$ fixed. The conformal block $\mathcal{F}(z)$ in this limit is computed in [3, 4]

$$z^{2h_v}\mathcal{F}(z) \approx \left[\frac{z\alpha_w(1-z)^{(\alpha_w-1)/2}}{1-(1-z)^{\alpha_w}}\right]^{2h_v}, \tag{2.3}$$

with $\alpha_w = \sqrt{1-24h_w/c}$. The real-time out-of-time-order correlator is obtained by continuing this block to the second Riemann sheet as described in [2] and the leading contribution to the rescaled commutator is

$$z^{2h_v}\mathcal{F}(z) \approx \left[\frac{e^{-\pi i(\alpha_w-1)}z\alpha_w(1-z)^{(\alpha_w-1)/2}}{1-e^{-2\pi i\alpha_w}(1-z)^{\alpha_w}}\right]^{2h_v} \sim \left(\frac{1}{1-\frac{24\pi i h_w}{cz}}\right)^{2h_v}. \tag{2.4}$$

Let us take a limit where $\epsilon_{12} = x_1 - x_2$ and $\epsilon_{34} = x_3 - x_4$ are much smaller than β, and without loss of generality set $x_1 = 0$. The cross-ratio is then approximately

$$z \approx \frac{\pi^2}{\beta^2}\frac{\epsilon_{12}\epsilon_{34}}{\sinh^2\left(\frac{\pi}{\beta}(t+x_3)\right)}$$

provided we stay away from light-like separations where $x_3 \to -t$. As we see the conformal block on the second sheet has a simple limit as ϵ_{12} and $\epsilon_{34} \to 0$, when $z \to 0$, corresponding to the actual computation of the commutator

$$z^{2h_v}\mathcal{F}(z) \approx \left(\frac{cz}{24\pi i h_w}\right)^{2h_v}. \tag{2.5}$$

The exponential decay of this quantity indicates the commutator between V and W becomes large after a time of order

$$t = \frac{\beta}{4\pi h_v} \tag{2.6}$$

showing rapid thermalization of primary operators on a timescale much shorter than (2.2).

However the interesting physical question is whether generic states exhibit some notion of quantum scrambling on a longer timescale. To explore this question in the current context of CFT 4-point functions, we can then try to build more generic deformations of the thermal density matrix by acting with primary operators folded into wavepackets with some characteristic spatial size L. Computing the 4-point function of these wavepackets, one can attempt to vary L to maximize the convoluted amplitude, then ask what thermalization timescale emerges.

Concretely, we convolute the function (2.4) with spatial Gaussian wavepackets with width L. We will choose t, L and the x_i such that light-like singularities in z are avoided. In this regime, the resulting integral will be dominated by a saddle point value of z, and the convoluted (rescaled) conformal block may then be well approximated by simply substituting this value into (2.4). Given the simple form of (2.4), with a cusp at $z = 1$, the optimal value for L will be the one that makes z approach 1.

For simplicity let us set $x_1 + x_2 = x_3 + x_4 = 0$, and we will build Gaussian wavepackets in the variables $x_1 - x_2 = \ell_v$ and $x_3 - x_4 = \ell_w$. To fix L in terms of z, one is therefore interested in the convolution

$$z(t, L) = \frac{4}{\pi L^2} \int_0^\infty dl_v dl_w e^{-(l_v^2 + l_w^2)/L^2} \frac{\sinh\left(\frac{\pi}{\beta} l_v\right) \sinh\left(\frac{\pi}{\beta} l_w\right)}{\sinh\left(\frac{\pi}{\beta}\left(t - \frac{l_v}{2} + \frac{l_w}{2}\right)\right) \sinh\left(\frac{\pi}{\beta}\left(t + \frac{l_v}{2} - \frac{l_w}{2}\right)\right)}. \tag{2.7}$$

This formula is justified because the exponential variation of z with l_v, l_w is much more rapid than power law variation of the conformal block with z, so analyzing the convolution of z alone is sufficient to determine l_v and l_w and subsequently L. The integrand has light-like poles, however for suitable values of t and L these contributions to the smeared conformal block can be made negligible. In this limit, the integrand can be well-approximated by simply

$$z(t, L) \approx \frac{4}{\pi L^2} \int_0^\infty dl_v dl_w e^{-(l_v^2 + l_w^2)/L^2} \frac{2 \sinh\left(\frac{\pi}{\beta} l_v\right) \sinh\left(\frac{\pi}{\beta} l_w\right)}{\cosh\left(\frac{2\pi}{\beta} t\right)}.$$

This has saddle points when

$$l_v \tanh\left(\frac{l_v \pi}{\beta}\right) = \frac{\pi L^2}{2\beta}$$

and likewise for l_w. The positive solutions are to be taken corresponding to the limits of integration in (2.7). If we then ask that the resulting amplitude (2.4) is maximized in magnitude, we find that we must choose $L \sim \beta$ near $t = 0$. We choose not to change the shape of the wavepackets at time increases, and impose this condition for all values of t. At the end we find the optimal value of z is

$$z_{sad} = \text{sech}\left(\frac{2\pi}{\beta} t\right) \tag{2.8}$$

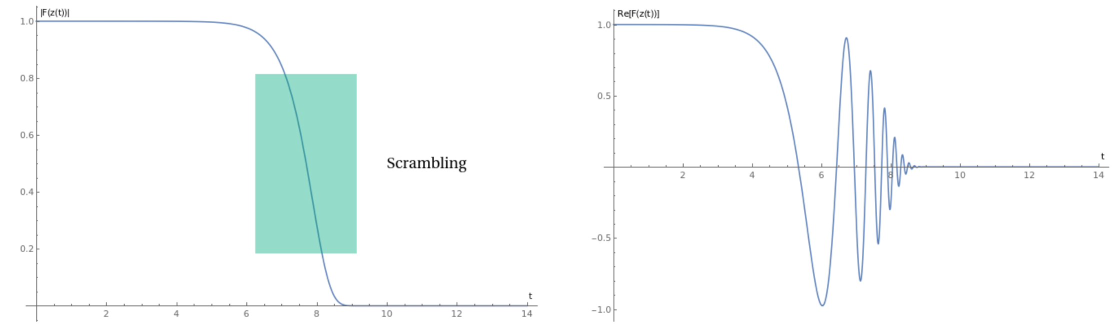

Figure 2.2: Plot of function $|z^{2h_v}\mathcal{F}(z)| = |F(z(t))| = 1/\left|1 - 12\pi i h_w \exp\left(2\pi/\beta\left(t - \log c - x\right)\right)\right|^{2h_v}$ where $c = 10^7$, $h_v = 100$, $h_w = 10$, $\beta = 2\pi$ and $x = 0$. Here $t_* = 7.7$ according to (2.10). In the right panel, a plot of $\mathrm{Re}\, F(z(t))$ is shown.

up to constant factors of order 1.

Let us now return to the example of the identity conformal block continued to the second Riemann sheet as considered in [2]. In this case, the saddle point approximation to the (rescaled) convoluted block function is for sufficiently late times

$$z^{2h_v}\mathcal{F}(z) \approx \left(\frac{1}{1 - \frac{12\pi i h_w}{c} e^{\frac{2\pi}{\beta}(t-x)}}\right)^{2h_v} \tag{2.9}$$

where we have restored dependence on the spatial separation x of the centers of the wavepackets, and inserted the saddle point approximation value for z (2.8) for $t \gg \beta$. It is helpful to plot this for sample parameters as in fig. 2.2. As $t - x$ increases from 0 to

$$t_* = \frac{\beta}{2\pi} \log \frac{c\sqrt{\log 2}}{12\pi h_v^{1/2} h_w} \tag{2.10}$$

the conformal block decreases in magnitude by a factor of about 1/2. This thermalization time may be viewed as a proxy for the true scrambling time of the system, and shows the distinctive appearance of the logarithm of the system size. The formula is valid for $0 < h_v \ll c$, but ideally one would want to argue this formula continues to hold as h_w becomes of order 1. Unfortunately it is not yet possible to prove this. We note fig. 2.2 also shows in the late-time limit the asymptotic form (2.5) is applicable and the timescale for variation is the much shorter time (2.6).

The correlator of the wavepackets is given by (2.9) provided one steers clear of the light-cone singularities in (2.7) which render the approximation (2.8) invalid. This is a signature that even the wavepackets of primaries are not ideal representatives of a generic state, and retain regions of spacetime where thermalization has not yet occurred, outside the light-cone of the wavepacket. Nevertheless for the present purposes, the reduced state inside the light-cone appears to be well-thermalized according to the correlators, so this procedure should yield a good measure of the global scrambling time. Again it remains to be seen whether (2.10) holds in the case of most physical interest where h_w is of order 1.

2.3 Higher Weight Intermediate States

We now turn our attention to the contribution of higher weight intermediate states to the out-of-time order correlators, and will find the surprising result that these may dominate over the identity block in the late-time limit. Again we will assume we are taking $c \gg 1$ with h_w/c fixed and $h_v \ll c$ fixed. In addition we will generalize from the identity block to an intermediate channel with conformal weight h_p fixed as $c \to \infty$.

Our starting point is the formula for the conformal block at next-to-leading order in this large c expansion of [4]

$$\mathcal{F}(z) = \mathcal{F}_0(z) \left(\frac{1-(1-z)^{\alpha_w}}{\alpha_w} \right)^{h_p} {}_2F_1\left(h_p, h_p, 2h_p, 1-(1-z)^{\alpha_w}\right)$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ is the Gauss hypergeometric function. To continue this expression to the second sheet we use the hypergeometric function identity [5]

$$\frac{\Gamma(h)^2}{\Gamma(2h)} {}_2F_1(h,h;2h;w) = \left(\sum_{k=0}^{\infty} \frac{2\,(h)_k^2\,(\psi(k+1)-\psi(h+k))}{k!^2} (1-w)^k \right) - \log(1-w)\, {}_2F_1(h,h,1;1-w)$$

valid for $|1-w| < 1$, where $(h)_k$ is the Pochhammer symbol, and $\psi(a)$ is the digamma function. Continuing to the second sheet we then obtain

$$\begin{aligned} \mathcal{F}_{II}(z) = \mathcal{F}_{0,II}(z) \left(\frac{1-e^{-i2\pi\alpha_w}(1-z)^{\alpha_w}}{\alpha_w} \right)^{h_p} \Big({}_2F_1\left(h_p, h_p, 2h_p, 1-e^{-i2\pi\alpha_w}(1-z)^{\alpha_w}\right) \\ + 2\pi i \alpha_w \frac{\Gamma(2h_p)}{\Gamma(h_p)^2} {}_2F_1(h_p, h_p, 1; e^{-i2\pi\alpha_w}(1-z)^{\alpha_w}) \Big) . \end{aligned} \tag{2.11}$$

Expanding for small h_w/c and $z \ll 1$ leads to

$$\mathcal{F}_{II}(z) \sim \mathcal{F}_{0,II}(z) \left(\frac{z - \frac{\pi i h_w}{6c}}{\alpha_w} \right)^{h_p} \left(1 + i\tan(\pi h_p) - 2\pi^2 i z^{1-2h_p} \frac{\Gamma(2h_p)}{\Gamma(2-2h_p)\Gamma(h_p)^4 \sin(2\pi h_p)} \right) .$$

This ends up being dominated by the last term in the third factor, and in fact grows at late times. Even at early times (z near 1) the last term in (2.11) dominates over the other term in the third factor for $h_p > 1$. The second factor in (2.11) rapidly approaches a constant much smaller than 1.

The upshot is the identity block dominates for a finite period of time, however after

$$t_* \approx \frac{\beta}{4\pi} \log\left(\frac{c}{h_w} \right)$$

the higher weight intermediate states take over. This late time sum over intermediate states apparently diverges when considered term by term. This would lead one to conclude the commutator grows initially while dominated by the identity block, but then may again decrease at later times, indicating a lack of true scrambling in the conformal field theory.

One possible way to avoid this conclusion is to demand an infinite tower of higher weight intermediate primaries, such that the apparently divergent sum might be resummed to a finite answer.

However in the following section we find contributions for $h_p \gg c$ are actually suppressed. We conclude that even a sparse spectrum of intermediate primaries with weights $1 < h_p \ll c$ are sufficient to destroy or drastically modify the onset of quantum chaos. In light of our previous discussion, this may simply mean such smeared primaries are still not good representatives of generic states, and instead one would need to consider commutators of much more general operators to see the correct timescale for global thermalization, or quantum scrambling. Alternatively, it may happen that only operators dual to black hole states efficiently scramble, and these must be reflected in a choice of external operators that do not couple at all (or only very weakly) to higher weight primaries, such that the identity block may dominate the out-of-time order correlators.

For conformal field theories with holographic anti-de Sitter gravity duals, the implication of the higher intermediate channels is that the bulk effective field theory breaks down when it is used to compute out-of-time-ordered correlators at finite time. On the other hand, there is no indication of such a breakdown when time-ordered CFT correlators are computed (see also [6, 7]), which correspond to the boundary S-matrix of the bulk theory. To see this we simply note that as higher dimensional operators in CFT_2 correspond to interactions of increasing mass scale in AdS_3, domination of all intermediate channels with dimension $h_p \geq 1$ means that there would be a dual set of an infinite sequence of interactions in the gravitation theory in AdS_3. If these high scale interactions affect the infrared physics of the theory, then the standard decoupling theorems of effective field theory such [8] break down.

Now the usual measurements we perform can be well-approximated by transition amplitudes, built out of time-ordered correlators which may be computed as within effective field theory. It is only the particular set of observables corresponding to out-of-time-order correlators, or norms of commutators that exhibit this peculiar behavior. For the black hole information problem this would seem to imply that contrary to expectations, commutators that measure limits on the causal propagation of information are indeed observables sensitive to the ultra-violet structure of the theory, as long hinted at in perturbative string theory computations [9, 10].

2.4 Intermediate Channels with $h_p \gg c$

So far we have only considered intermediate channels with fixed $h_p \ll c$. It is also instructive to perform the same analysis for intermediate channels with $h_p \gg c$ where the limit is $h_p \to \infty$ with c/h_p, h_v/h_p and h_w/h_p fixed and small. For this we consider equation (16) in [11],

$$\mathcal{F}(z) = (16q)^{h_p - \frac{c}{24}}\, z^{\frac{c}{24} - 2h_v} (1-z)^{\frac{c}{24} - (h_v + h_w)} \theta_3(q)^{\frac{c}{2} - 8(h_v + h_w)} H(c, h_p, h_i, q) \tag{2.12}$$

where the nome $q = e^{i\pi\tau}$ is related to the cross-ratio z by

$$\tau = i\frac{K'(z)}{K(z)} = i\frac{K(1-z)}{K(z)}$$

where $K(z)$ is the complete elliptic integral with parameter[1] z. Here H is a function that is $1 + O(1/h_p)$ and

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \, . \tag{2.13}$$

Eq. (2.12) has a branch cut at $z = 1$ from the $1 - z$ factor which will lead to the same analytic behavior for the intermediate case $h_p \ll c$, which we have previously considered. To see this we expand the nome q around $z = 0$ to obtain

$$q = e^{i\pi\tau} = \frac{z}{16} + \frac{z^2}{32} + \cdots \, .$$

As $\theta_3(q)$ is regular near $q = 0$, we see that on the principal sheet $\mathcal{F}(z)$ goes to zero as $z \to 0$. Therefore the heavy intermediate channels are perfectly suppressed on the first Riemann sheet. Crossing the branch cut $z = 1$ from above, the complete elliptic function $K(z)$ picks up an additional imaginary part [12]:

$$\lim_{\epsilon \to 0^+} K(z + i\epsilon) = K(z) + 2iK(1 - z) \, .$$

Analyticity implies that on the second Riemann sheet the nome is now

$$q = \exp\left[-\frac{\pi K(1-z)}{K(z) + 2iK(1-z)}\right] = \exp\left[-\frac{\pi}{\frac{K(z)}{K(1-z)} + 2i}\right] .$$

To expand this expression near $z = 0$, we use

$$\frac{K(z)}{K(1-z)} \approx \frac{\pi}{4\log 2 - \log z} + \mathcal{O}\left(\frac{z}{\log^2 z}\right)$$

so that

$$q \approx e^{\frac{i\pi}{2} + \frac{\pi^2}{4\log z}} \, . \tag{2.14}$$

We then need to expand (2.13) near $q = i$. The expansion near $q = 1$ is

$$|\theta_3(q)| \approx \left|\frac{\sqrt{\pi}}{\sqrt{1-q}}\right|$$

but we can obtain the expansion near $q = i$ by using the relation

$$|\theta_3(q)| = \left|\frac{\sqrt{\pi}}{\sqrt{\log q}}\theta_3\left(e^{\frac{\pi^2}{\log q}}\right)\right|$$

and substituting in (2.14) to give $\theta_3(q)$ near $q = i$ as

$$|\theta_3(q)| \approx \left|\frac{\sqrt{-2\log z}}{\sqrt{\pi}}\right| . \tag{2.15}$$

Assembling the various factors, we find again a dramatic enhancement of the higher weight channel on the second Riemann sheet arising from the behavior (2.15), compared to the behavior on the principal sheet. However when we compare to the $h_p = 0$ expression of the previous section, the $z^{c/24}$ factor of (2.12) dominates for small z so we conclude they do not dominate versus the identity channel (again modulo restrictions on the operator couplings C_p of (A.2)).

[1] We clarify that in most mathematical literature, the complete elliptic integral K is defined with the modulus k as the argument. Our z is related to k through $z = k^2$. It is also common for many mathematicians to use the symbol m for our z.

2.5 Conclusions

In this chapter we discussed the issue of smearing local operators in a thermal CFT and its connection with quantum scrambling. We pointed out that the correct scrambling time should be identified with operators that maximize the timescale of variation of the out-of-time ordered correlator, which may occur well before the asymptotic late-time limit. We then examined a somewhat independent issue, that the higher intermediate states with $0 < h_p \ll c$ can have large out-of-time ordered correlators. We discussed the implications of this statement, which is that in the AdS_3 gravity dual the UV dynamics and IR dynamics is no longer decoupled when these observables are computed. This lack of decoupling appears even when the usual time-ordered correlators, or transition amplitudes satisfy the standard decoupling lore. When applied to scattering in AdS_3 black hole backgrounds this implies that the commutators that lead one to conclude information is lost semiclassically, are in fact not computable without a full specification of the ultraviolet physics of the theory. The ordinary bulk effective field theory does not predict its own demise when computing these observables.

As for the appearance of a scrambling time of the form (2.2) we have found a variant of this expression (2.10), valid when the identity block dominates. The expression involves a term of the form $\beta/2\pi \log c$, but other significant terms are also present. If other intermediate primaries appear, with conformal weights fixed in a large c limit, they will dominate the late-time behavior and may completely spoil thermalization. It will be very interesting to extend the range of validity of these expressions to determine whether there exist a class of 2d conformal field theories that may be viewed as fast scramblers at finite temperature.

Chapter 3

Holographic Map for Cosmological Horizons

We propose a holographic map between Einstein gravity coupled to matter in a de Sitter background and large N quantum mechanics of a system of spins. Holography maps a spin model with a finite dimensional Hilbert space defined on a version of the stretched horizon into bulk gravitational dynamics. The full Hamiltonian of the spin model contains a non-local piece which generates chaotic dynamics, widely conjectured to be a necessary part of quantum gravity, and a local piece which recovers the perturbative spectrum in the bulk.

3.1 Introduction

Previous work has argued for a unitary, holographic description of black hole dynamics via certain spin models [13, 14] defined on the stretched horizon [15] of the black hole. These spin models have the common feature that non-local interactions generate chaotic dynamics, widely conjectured to be an integral part of a full quantum mechanical description of gravity [1]. In this chapter we argue that a similar approach works for the cosmological horizon in de Sitter spacetime, given that the static patch metric has a similar form to the metric in Schwarzschild coordinates. To this end we will give an explicit prescription to map perturbative bulk fields to a quantum mechanical operator defined in the holographic spin model. This map will then allow us to construct a local Hamiltonian that reproduces the classical energy of a perturbation around de Sitter spacetime. We argue that this local Hamiltonian, together with the non-local long-range interaction necessary to generate chaotic dynamics, can potentially be a viable description of de Sitter quantum gravity.

Before we begin, we will review relevant facts of the de Sitter space-time to establish our convention of notations. We mostly follow the conventions in Ref. [16]. Throughout the chapter we will restrict our discussion to $(1 + 3)$-dimensional space-time entirely, although the methodology presented can in principle be applied to higher (or lower) dimensions. Our metric signatures are

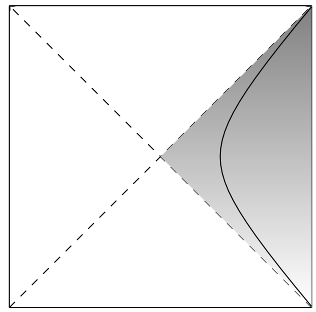

Figure 3.1: Penrose diagram for de-Sitter spacetime, where shaded region is covered by the static coordinates. The stretched horizon (solid curve inside the static patch) is defined as a hypersurface at fixed r.

always mostly positive, i.e. $(-++\cdots)$.

Static coordinates cover only one triangular region in the Penrose diagram (see Fig. 3.1)

$$\mathrm{d}s^2 = -(1-\frac{r^2}{\ell^2})\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1-\frac{r^2}{\ell^2}} + r^2\,\mathrm{d}\Omega^2 \tag{3.1}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2$ is the line element on the unit 2-sphere S^2 and ℓ is the radius of curvature of the de Sitter spacetime.

3.2 Mode Functions in de-Sitter

3.2.1 Static Patch

Our goal is to construct a spin model which reproduces the bulk spectrum in a de Sitter background. To proceed we first review some standard results concerning mode functions in static de Sitter. For simplicity, we will treat the massless minimally coupled scalar, and follow the point of view of [17] that with a cutoff to exclude the zero mode, the system can be quantized around the Bunch-Davies (or Euclidean) vacuum state. Had we been interested in the system including the zero-mode, this quantization preserving de Sitter isometries would be inadmissible [18]. This quantization of the massless minimally coupled scalar around the Bunch-Davies vacuum is closely related to that of perturbative gravitons, as explained in [19, 20]. The results may then be straightforwardly applied to other bulk modes once this case is understood.

We begin by considering modes in the static patch (3.1). The equation of motion for the free scalar field $\Phi(t, r, \theta, \phi)$ is

$$\frac{\partial_t^2\Phi}{1-r^2/\ell^2} - \frac{\partial_r[r^2(1-r^2/\ell^2)\partial_r\Phi]}{r^2} - \frac{\partial_\theta(\sin\theta\partial_\theta\Phi)}{r^2\sin\theta} - \frac{\partial_\varphi^2\Phi}{r^2\sin^2\theta} = 0\,.$$

Separating variables, we have

$$\Phi_{\omega lm}(t, r, \theta, \varphi) = A_{\omega l} e^{-i\omega t} f_{\omega l}(r) Y_{lm}(\theta, \varphi)\,,$$

where $f_{\omega l}(r)$ satisfies

$$(1-r^2/\ell^2)f''_{\omega l}(r)+\frac{2(1-2r^2/\ell^2)}{r}f'_{\omega l}(r)+\left(\frac{\omega^2}{1-r^2/\ell^2}-\frac{l(l+1)}{r^2}\right)f_{\omega l}(r)=0\,.$$

We pick the set of solutions that are regular at $r=0$ and find

$$f_{\omega l}(r)=\frac{(r/\ell)^l}{\ell}(1-r^2/\ell^2)^{i\omega\ell/2}{}_2\mathrm{F}_1\left(\frac{l+i\omega\ell}{2},\frac{l+i\omega\ell+3}{2};l+\frac{3}{2};\frac{r^2}{\ell^2}\right)\,.$$

We fix the normalization constant $A_{\omega l}$ by computing the Klein-Gordon norm. This is defined on a spacelike surface Σ by

$$\langle f,g\rangle=-i\int_\Sigma \mathrm{d}\Sigma\, n^\lambda\,(f\partial_\lambda g^*-g^*\partial_\lambda f)$$

where n^λ is a timelike unit vector normal to Σ. Evaluating this on a constant t slice gives

$$\langle f,g\rangle=-i\int(f\partial_t g^\star-g^\star\partial_t f)\frac{r^2\,\mathrm{d}r\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi}{1-r^2/\ell^2}\,. \tag{3.2}$$

Computing the mode normalization then gives

$$\langle\Phi_{\omega lm},\Phi_{\omega' l'm'}\rangle=A_{\omega l}A^\star_{\omega' l}(\omega+\omega')\delta_{ll'}\delta_{mm'}\int_0^\ell\frac{f_{\omega l}(r)f^\star_{\omega' l}(r)\,r^2\,\mathrm{d}r}{1-r^2/\ell^2}\,.$$

Using the equation of motion for $f_{\omega l}(r)$ we have

$$[r^2(1-r^2/\ell^2)f'_{\omega l}(r)]'\,f^\star_{\omega' l}(r)+\left(\frac{\omega^2r^2}{1-r^2/\ell^2}-l(l+1)\right)f_{\omega l}(r)f^\star_{\omega' l}(r)=0$$

and likewise

$$[r^2(1-r^2/\ell^2)f^{\star\prime}_{\omega' l}(r)]'\,f_{\omega l}(r)+\left(\frac{\omega'^2r^2}{1-r^2/\ell^2}-l(l+1)\right)f^\star_{\omega' l}(r)f_{\omega l}(r)=0\,.$$

Subtracting we have

$$\frac{(\omega^2-\omega'^2)r^2}{1-r^2/\ell^2}f_{\omega l}(r)f^\star_{\omega' l}(r)=[r^2(1-r^2/\ell^2)f^{\star\prime}_{\omega' l}(r)]'\,f_{\omega l}(r)-[r^2(1-r^2/\ell^2)f'_{\omega l}(r)]'\,f^\star_{\omega' l}(r)\,,$$

and integrating by parts, we have

$$\int_0^\ell\frac{(\omega^2-\omega'^2)r^2}{1-r^2/\ell^2}f_{\omega l}(r)f^\star_{\omega' l}(r)=r^2(1-r^2/\ell^2)f^{\star\prime}_{\omega' l}(r)\,f_{\omega l}(r)-r^2(1-r^2/\ell^2)f'_{\omega l}(r)\,f^\star_{\omega' l}(r)\Big|_0^\ell\,.$$

This gives

$$\int_0^\ell\frac{r^2\,\mathrm{d}r}{1-r^2/\ell^2}f_{\omega l}(r)f^\star_{\omega' l}(r)=\frac{\ell^2}{\omega^2-\omega'^2}\lim_{r\to\ell}(1-r^2/\ell^2)\left[f^{\star\prime}_{\omega' l}(r)\,f_{\omega l}(r)-f'_{\omega l}(r)\,f^\star_{\omega' l}(r)\right]\,.$$

Using the hypergeometric identity near $z=1$

$${}_2\mathrm{F}_1(a,b,c,z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}$$

we can expand $f_{\omega l}(r)$ near $r=\ell$ to give

$$\ell f_{\omega l}(r) \approx \frac{\Gamma(l+\frac{3}{2})\Gamma(-i\ell\omega)}{\Gamma\left(\frac{l-i\omega\ell}{2}\right)\Gamma\left(\frac{l-i\omega l+3}{2}\right)}(1-r^2/\ell^2)^{\frac{i\ell\omega}{2}} + \frac{\Gamma(l+\frac{3}{2})\Gamma(i\ell\omega)}{\Gamma\left(\frac{l+i\omega\ell}{2}\right)\Gamma\left(\frac{l+i\omega l+3}{2}\right)}(1-r^2/\ell^2)^{-\frac{i\ell\omega}{2}} .$$

Letting

$$B_{\omega l} = \frac{\Gamma(l+\frac{3}{2})\Gamma(i\ell\omega)}{\Gamma\left(\frac{l+i\omega\ell}{2}\right)\Gamma\left(\frac{l+i\omega l+3}{2}\right)}$$

we see that we have

$$\ell f_{\omega l}(r) \approx B^{\star}_{\omega l}(1-r^2/\ell^2)^{\frac{i\ell\omega}{2}} + B_{\omega l}(1-r^2/\ell^2)^{-\frac{i\ell\omega}{2}}$$

and

$$f'_{\omega l}(r) \approx \frac{-i\omega}{1-r^2/\ell^2}\left[B^{\star}_{\omega l}(1-r^2/\ell^2)^{\frac{i\ell\omega}{2}} - B_{\omega l}(1-r^2/\ell^2)^{-\frac{i\ell\omega}{2}}\right] .$$

Multiplying $f_{\omega l}(r)$ and $f'_{\omega l}(r)$ and dropping terms that are rapidly oscillating as $|\omega-\omega'|>0$ and $r\to\ell$, we find that

$$\int_0^{\ell} \frac{r^2\,\mathrm{d}r}{1-r^2/\ell^2} f_{\omega l}(r) f^{\star}_{\omega' l}(r) = \lim_{r\to\ell} \frac{2|B_{\omega l}|^2}{\omega-\omega'} \sin\left[\frac{(\omega-\omega')\,\ell}{2}\log\left(\frac{1}{1-r^2/\ell^2}\right)\right] \tag{3.3}$$

Using

$$\lim_{C\to\infty}\frac{\sin Cx}{x} = \pi\delta(x)$$

we have

$$\int_0^{\ell} \frac{r^2\,\mathrm{d}r}{1-r^2/\ell^2} f_{\omega l}(r) f^{\star}_{\omega' l}(r) = 2\pi|B_{\omega l}|^2\delta(\omega-\omega')$$

or

$$\langle \Phi_{\omega l m}, \Phi_{\omega' l' m'}\rangle = 4\pi|A_{\omega l}|^2|B_{\omega l}|^2\delta_{ll'}\delta_{mm'}\omega\delta(\omega-\omega') . \tag{3.4}$$

If we normalize according to

$$\langle \Phi_{\omega l m}, \Phi_{\omega' l' m'}\rangle = \delta_{ll'}\delta_{mm'}\omega\delta(\omega-\omega') ,$$

we need to pick $A_{\omega l}$ such that

$$|A_{\omega l}|^2 = \frac{1}{4\pi|B_{\omega l}|^2} .$$

3.2.2 Flat Slicing

Now let us consider the analogous problem for modes in the flat slicing. The metric takes the form

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + e^{2\tau/\ell}\mathrm{d}\vec{x}^2 = -\mathrm{d}\tau^2 + e^{2\tau/\ell}(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\theta^2 + \rho^2\sin^2\theta\mathrm{d}\varphi^2)$$

where $\tau\in(-\infty,+\infty)$ and $\rho\in(0,+\infty)$. The wave equation for the massless minimally coupled scalar is given by

$$\partial_\tau^2\phi + \frac{3}{\ell}\partial_\tau\phi - e^{-2\tau/\ell}\Delta\phi = 0$$

where $\Delta\phi$ is the usual spatial Laplacian operator

$$\Delta\phi = \partial_\rho^2\phi + \frac{2}{\rho}\partial_\rho\phi + \frac{\partial_\theta^2\phi}{\rho^2} + \frac{\partial_\theta\phi}{\rho^2\tan\theta} + \frac{\partial_\varphi^2\phi}{\rho^2\sin^2\theta} .$$

Separating variables, we use the ansatz

$$\phi_{klm}(\tau,\rho,\theta,\varphi) = T_k(\tau)R_{kl}(\rho)Y_{lm}(\theta,\varphi)$$

where $T_k(\tau)$ satisfies

$$T''(\tau) + \frac{3}{\ell}T'(\tau) + k^2 e^{-2\tau/\ell}T(\tau) = 0$$

and $R_{kl}(\rho)$ satisfies

$$R''(\rho) + \frac{2}{\rho}R'(\rho) + \left(k^2 - \frac{l(l+1)}{\rho^2}\right)R(\rho) = 0\,.$$

Assuming regularity at $\rho = 0$ we can solve for $R(\rho)$

$$R_{kl}(\rho) = C_{kl}\, j_l(k\rho)$$

where $j_l(x)$ is the spherical Bessel function of first kind $j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+\frac{1}{2}}(x)$. We will determine the normalization constant C_{kl} later.

The equation for $T(\tau)$ can be solved to give

$$T_k(\tau) = c_1 e^{ik\ell e^{-\tau/\ell}}\left(1 - ik\ell e^{-\tau/\ell}\right) + c_2 e^{-ik\ell e^{-\tau/\ell}}\left(1 + ik\ell e^{-\tau/\ell}\right)$$

or, in terms of the conformal time $\eta = -\ell e^{-\tau/\ell}$

$$T_k(\tau) = c_1 e^{-ik\eta}\left(1 + ik\eta\right) + c_2 e^{ik\eta}\left(1 - ik\eta\right)\,.$$

We assume Bunch-Davies vacuum and therefore pick the special solution

$$T_k(\eta) = e^{-ik\eta}\left(1 + ik\eta\right)$$

and absorb the normalization constant into C_{kl}, which we will fix now. The mode functions are normalized according to the Klein-Gordon norm

$$\langle f, g\rangle = -i\int_\Sigma \left(f\partial_\mu g^\star - g^\star\partial_\mu f\right) n^\mu \sqrt{\gamma}\,\mathrm{d}^3x\,.$$

Here Σ is a spacelike hypersurface with unit norm n^μ and $\sqrt{\gamma}$ is the spatial volume element. We pick the $\tau = 0$ timeslice in the flat slicing since the metric at $\tau = 0$ is conveniently the Minkowski metric. In addition, we have $\partial_\tau = \partial_\eta$ on the $\tau = 0$ timeslice. We therefore have

$$\langle f, g\rangle = -i\int \left(f\partial_\tau g^\star - g^\star\partial_\tau f\right)\rho^2\,\mathrm{d}\rho\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi\,.$$

We use this to fix the normalization factor C_{kl}. We have, for the ϕ_{klm} modes on $\tau = 0$

$$\partial_\tau\phi_{klm} = -C_{kl}\,k^2\ell\, e^{ik\ell} j_l(k\rho)\,Y_{lm}(\theta,\varphi)\,.$$

Therefore

$$\langle\phi_{klm}, \phi_{k'l'm'}\rangle = i\ell C_{kl}C^\star_{k'l'}\,\delta_{ll'}\delta_{mm'}e^{i(k-k')\ell}\int[(1-ik\ell)k'^2 - (1+ik'\ell)k^2]j_l(k\rho)j_l(k'\rho)\rho^2\,\mathrm{d}\rho\,.$$

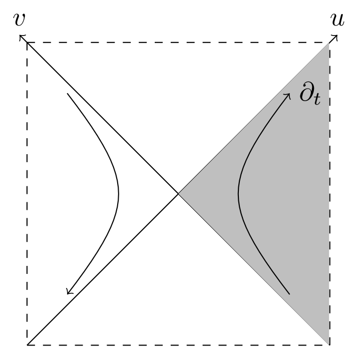

Figure 3.2: Penrose diagram for the de-Sitter spacetime. Flat slicing modes cover the right upper half of the diagram and are matched to static patch modes on the line $u = 0$. Positive frequency modes in the Bunch-Davies vacuum are then analytic in the lower-half-complex $v-$plane.

Using the orthogonality of spherical Bessel functions

$$\int_0^\infty \rho^2 j_l(u\rho) j_l(v\rho)\, \mathrm{d}\rho = \frac{\pi}{2u^2}\delta(u-v)$$

we have

$$\langle \phi_{klm}, \phi_{k'l'm'} \rangle = \ell^3\, \pi k\, |C_{kl}|^2\, \delta_{ll'}\delta_{mm'}\delta(k\ell - k'\ell)\,.$$

We therefore find

$$C_{kl} = \frac{1}{\sqrt{\pi k \ell}}\frac{1}{\ell}\,,$$

and the full solution is therefore

$$\phi_{klm}(\tau, \rho, \theta, \varphi) = \frac{1}{\ell\sqrt{\pi k \ell}}\, e^{-ik\eta}(1 + ik\eta)\, j_l(k\rho)\, Y_{lm}(\theta, \varphi)\,.$$

3.2.3 Matching Modes Across the Cosmological Horizon

Following [21], modes in the flat slicing may be viewed as modes entangled across the left and right static patches. In Kruskal coordinates in the right static patch $u = e^{x^-/\ell}$ and $v = -e^{-x^+/\ell}$ (other choices of sign generate the other patches in static coordinates) where $x^\pm = t \pm r^\star$ and

$$r^\star = \frac{\ell}{2}\log\frac{1 + r/\ell}{1 - r/\ell} \approx \frac{\ell}{2}\log\frac{2}{1 - r/\ell}$$

we have

$$\Phi_{\omega lm} \approx A_{\omega l}(B^\star_{\omega l}2^{i\omega\ell}|v|^{i\omega\ell} + B_{\omega l}2^{-i\omega\ell}|u|^{-i\omega\ell})Y_{lm}(\theta, \varphi)$$

near the cosmological horizon.

We now define a mode function $\Phi^+_{\omega lm}$ which is non-zero on the right quadrant, and a $\Phi^-_{\omega lm}$ which is non-zero on the left quadrant. We require that the linear combination

$$\bar{\Phi}_{\omega lm} = c\,\Phi^+_{\omega lm} + d\,\Phi^-_{\omega lm}$$

be analytic in the lower half $v-$plane on the past horizon on the right (and the past horizon on the left), i.e. the surface $u = 0$. On the right quadrant we can rewrite the function $(-v)^{i\omega\ell} = (e^{i\pi}v)^{i\omega\ell} = e^{-\pi\omega\ell}v^{i\omega\ell}$. Therefore, the linear combination $\Phi^+_{\omega lm} + e^{-\pi\omega\ell}\Phi^-_{\omega lm}$ is analytic in the lower-half-complex $v-$plane at $u = 0$ (see fig. 3.2) corresponding to a combination of positive frequency flat-slicing modes. The properly normalized mode is

$$\bar{\Phi}_{\omega lm} = \frac{1}{\sqrt{2\sinh(\pi\omega\ell)}}\left(e^{\pi\omega\ell/2}\Phi^+_{\omega lm} + e^{-\pi\omega\ell/2}\Phi^-_{\omega lm}\right) . \tag{3.5}$$

This mode is analytic in the lower-half v-plane for either choice of sign for ω, so should be identified with a linear combination of positive k flat-slicing modes.

We can now expand a quantum field operator $\hat{\Phi}$ in terms of normal modes

$$\begin{aligned}
\hat{\Phi} &= \int_0^\infty dk \sum_{lm} \phi_{klm} a_{klm} + \text{h.c.}\,(\text{flat slicing}) \\
&= \int_0^\infty d\omega \sum_{lm} \Phi^+_{\omega lm} b^+_{\omega lm} + \Phi^-_{\omega lm} b^{-\dagger}_{\omega lm} + \text{h.c.}\,(\text{static patches}) \\
&= \int_{-\infty}^\infty d\omega \sum_{lm} \bar{\Phi}_{\omega lm} c_{\omega lm} + \text{h.c.}\,(\text{entangled static patches}) .
\end{aligned}$$

We define the Bunch-Davies vacuum $|0\rangle$ to be annihilated by all a_{klm} with $k > 0$

$$a_{klm}|0\rangle = 0$$

which coincides with

$$c_{\omega lm}\,|0\rangle = 0$$

for $-\infty < \omega < \infty$. We similarly define the static patch vacuum $|\Omega\rangle$ to be annihilated by all $b^\pm_{\omega lm}$ with $\omega > 0$

$$b^\pm_{\omega lm}|\Omega\rangle = 0 .$$

From the relation we obtained between $\Phi^\pm$ and $\bar{\Phi}$ we obtain the relations between $b^\pm_{\omega lm}$ and $c_{\omega lm}$

$$c_{\omega lm} = \begin{cases} \frac{1}{\sqrt{2\sinh\pi\omega\ell}}\left(e^{\pi\omega\ell/2}b^+_{\omega lm} + e^{-\pi\omega\ell/2}b^{-\dagger}_{\omega lm}\right) & \omega > 0 \\ \frac{1}{\sqrt{2\sinh\pi\omega\ell}}\left(e^{\pi\omega\ell/2}b^{+\dagger}_{\omega lm} + e^{-\pi\omega\ell/2}b^-_{\omega lm}\right) & \omega < 0 . \end{cases}$$

As usual, the vacuum state $|0\rangle$ becomes a thermal density matrix in the right static patch when the modes $b^-_{\omega lm}$ are traced over.

We now need to compute the overlap of modes $\Phi_{\omega lm}$ with $\phi_{\omega lm}$ to construct the Bogoliubov transformation. In order to perform the integrals, we need the coordinate transformation between (t, r) and (η, ρ). We define null coordinates in the flat slicing

$$U = \frac{\eta - \rho}{2} \qquad V = \frac{\eta + \rho}{2}$$

leading to the metric. In the flat slicing using the (η, ρ) coordinates the metric is

$$\mathrm{d}s^2 = \frac{\ell^2}{\eta^2}(-\mathrm{d}\eta^2 + \mathrm{d}\rho^2 + \rho^2\mathrm{d}\Omega^2)$$

which in terms of (U, V) coordinates becomes

$$\mathrm{d}s^2 = \frac{\ell^2}{(U+V)^2}(-4\mathrm{d}U\mathrm{d}V + (V-U)^2\mathrm{d}\Omega^2)\,.$$

In the static patch, we define null coordinates (u, v)

$$u = e^{x^-/\ell}\,, v = -e^{-x^+/\ell}$$

and one verifies that the metric is

$$\mathrm{d}s^2 = \frac{\ell^2}{(1-uv)^2}(-4\mathrm{d}u\mathrm{d}v + (1+uv)^2\mathrm{d}\Omega^2)\,.$$

We see that the relation between (u, v) and (U, V) is simply

$$U = -\frac{\ell}{u} \qquad V = \ell v$$

which gives

$$\eta = -\frac{\ell}{u} + \ell v \qquad \rho = \frac{\ell}{u} + \ell v\,.$$

The flat slicing mode function therefore becomes

$$\phi_{klm} = \frac{1}{\ell}\frac{1}{\sqrt{\pi k\ell}}\,e^{ik\ell(\frac{1}{u}-v)}\left[1 - ik\ell\left(\frac{1}{u} - v\right)\right]\frac{\sin(k\ell(v+1/u) - l\pi/2)}{k\ell(v+1/u)}\,Y_{lm}(\theta,\varphi)$$

where we have used the behavior of spherical Bessel function at infinity

$$\lim_{\rho\to\infty} j_l(k\rho) = \frac{\sin(k\rho - l\pi/2)}{k\rho}\,.$$

Near the past horizon on the left patch, $u \to 0$ with v fixed, the flat slice mode becomes

$$\phi_{klm} = \frac{1}{\ell}\frac{-i}{\sqrt{\pi k\ell}}\,e^{ik\ell(\frac{1}{u}-v)}\sin(k\ell(v+1/u) - l\pi/2)\,Y_{lm}(\theta,\varphi)\,.$$

Using the identity $\sin z = (e^{iz} - e^{-iz})/2i$ we can rewrite the flat slice mode function as

$$\begin{aligned}\phi_{klm} &= \frac{1}{\ell}\frac{-1}{2\sqrt{\pi k\ell}}\,e^{ik\ell(\frac{1}{u}-v)}(i^{-l}e^{ik\ell(v+1/u)} - i^l e^{-ik\ell(v+1/u)})Y_{lm}(\theta,\varphi)\\ &= \frac{1}{\ell}\frac{1}{2\sqrt{\pi k\ell}}(i^l e^{-2ik\ell v} - i^{-l}e^{2ik\ell/u})Y_{lm}(\theta,\varphi) \sim \frac{1}{\ell}\frac{i^l}{2\sqrt{\pi k\ell}}e^{-2ik\ell v}Y_{lm}(\theta,\varphi)\,.\end{aligned}$$

We have shown above that near the past horizon the static patch mode is

$$\Phi_{\omega lm} = A_{\omega l}B^\star_{\omega l}(-v)^{i\omega\ell}2^{i\omega\ell}Y_{lm}(\theta,\varphi)\,.$$

On the past horizon the Klein-Gordon norm is

$$\langle f, g\rangle = -i\ell^2\int \mathrm{d}\Omega\mathrm{d}v\,(f\partial_v g^\star - g^\star\partial_v f) = i\ell^2\int_{-\infty}^0 \mathrm{d}\Omega\left(2\int g^\star\partial_v f\,\mathrm{d}v - g^\star f\Big|_0^{-\infty}\right)\,.$$

Since we have

$$\partial_v\Phi_{\omega lm} = i\omega\ell v^{-1}\Phi_{\omega lm}$$

adding a small imaginary part to ω and k to dampen the oscillation of $(-v)^{i\omega\ell}$ and e^{-ikv} we have, since the boundary term vanishes,

$$\langle \Phi_{\omega lm}, \phi_{klm} \rangle = \frac{(-i)^l}{\sqrt{\pi k\ell}} A_{\omega l} B^\star_{\omega l} 2^{i\omega\ell} \int_0^{+\infty} e^{-2ik\ell v} v^{i\omega\ell - 1}\, \mathrm{d}v$$

where we replaced $v \to -v$. This can be evaluated to give

$$\langle \Phi_{\omega lm}, \phi_{klm} \rangle = \frac{(-i)^l}{\sqrt{\pi k\ell}} A_{\omega l} B^\star_{\omega l} (ik\ell)^{-i\omega\ell} \Gamma(i\omega\ell)\,.$$

By choosing the phase of $A_{\omega l}$ appropriately such that

$$A_{\omega l} B^\star_{\omega l} = \frac{1}{\sqrt{4\pi}}$$

we finally have

$$\langle \Phi_{\omega lm}, \phi_{klm} \rangle = \frac{(-i)^l}{\sqrt{k\ell}} \frac{(ik\ell)^{-i\omega\ell}\Gamma(i\omega\ell)}{2\pi}\,.$$

With the Bogoliubov transformation at hand, we can now map modes in the flat slicing to entangled modes in the left and right static patch. These in turn define modes in the upper quadrant of the static slicing by continuation.

3.3 Holographic Map

Our goal is to build a holographic version of the bulk theory that might be viewed as living on the so-called "stretched horizon" [15]. The essence of the black hole membrane paradigm is that to an external observer outside the horizon, the black hole horizon behaves more or less like a hydrodynamic membrane with properties such as resistance and viscosity. Quantum mechanically, the stretched horizon acts as a mirror [22] which scrambles and reflects information sent into it. Our viewpoint in this chapter is that since the static patch of de-Sitter has essentially the same mathematical form as the Schwarzschild metric, one ought to be able to construct a similar stretched horizon theory for static de-Sitter. We further assume that the horizon entropy of de Sitter is to be matched with the logarithm of the Hilbert space dimension. Thus, the stretched horizon theory will be a finite dimensional quantum mechanical system.

Usually the stretched horizon is defined as the constant r surface such that the local temperature measured by a fiducial observer (constant r) is the Planck temperature. When redshifted down to $r = 0$, this will match the Hawking temperature. In other words, one usually defines $r_\star$ such that

$$\frac{1}{2\pi\ell_p}\sqrt{1 - r_\star^2/\ell^2} = T_{\mathrm{H}} = \frac{1}{2\pi\ell}\,, \tag{3.6}$$

where ℓ_p is the Planck length.

As it stands, the bulk Hilbert space is infinite dimensional, labelled by the oscillators $c_{\omega lm}$ where $\omega \in \mathbb{R}$, and the angular momentum ranges up to infinity. Each oscillator mode creates a mode entangled across both static patches, with a stress-energy tensor non-singular on the future cosmological horizon on the right patch.

As a first step, we can discretize the sphere in coordinate space. There are many possibilities for such a discretization, and the details will not be too important for us, except to note that such a discretization will produce an effective cutoff l_{max} on the angular momentum.

Likewise, it is necessary to discretize the frequency ω which can in turn be viewed as a radial quantum number. This discretization may then be viewed as a kind of regulator for the radial coordinate. In order to produce a useful effective field theory with such a cutoff we choose a finite set of frequencies in the range

$$\frac{1}{\ell_p n_{UV}} > |\omega| \geq \frac{\pi}{\ell \log\left(\ell/\ell_p\right)} . \tag{3.7}$$

For simplicity we can take the $\omega' s$ to be evenly spaced in this range, with spacing $\pi/\ell \log\left(\ell/\ell_p\right)$. This corresponds to $n_{rad} = 2\ell \log(\ell/\ell_p)/\pi \ell_p n_{UV}$ radial points in the static patch. In particular, this number is conserved with time. Here $n_{UV} > 1$ is a factor introduced to parameterize the ultraviolet cutoff. We will see momentarily why the log factor appears.

An issue we immediately face is regulating the continuum mode normalization (3.4) to the discrete case. To do this we replace the upper limit on the radial integral in (3.2) by $\ell \to \ell - \epsilon$. Then the result of the integral (3.3) may be replaced by

$$\int_0^{\ell-\epsilon} \frac{r^2\,\mathrm{d}r}{1-r^2/\ell^2} f_{\omega l}(r) f^{\star}_{\omega' l}(r) = \frac{2|B_{\omega l}|^2}{\omega-\omega'} \sin\left[\frac{(\omega-\omega')\,\ell}{2}\log\left(\frac{1}{1-r^2/\ell^2}\right)\right]\Bigg|_{r=\ell-\epsilon} .$$

Keeping in mind $\omega - \omega' = \pi n/\ell \log\left(\ell/\ell_p\right)$ for some integer n we choose

$$\log\left(\frac{1}{1-r^2/\ell^2}\right)\Bigg|_{r=\ell-\epsilon} = 2\log\left(\ell/\ell_p\right) \tag{3.8}$$

which fixes r_* according to (3.6),

$$\int_0^{\ell-\epsilon} \frac{r^2\,\mathrm{d}r}{1-r^2/\ell^2} f_{\omega l}(r) f^{\star}_{\omega' l}(r) = 2\ell \log\left(\frac{\ell}{\ell_p}\right) |B_{\omega l}|^2 \delta_{\omega,\omega'}$$

up to rapidly oscillating terms. This unusual relation between a short distance cutoff and an infrared cutoff is typical in holographic models.

Finally, each harmonic oscillator mode $c_{\omega l m}$ produces an infinite dimensional Hilbert space. To regulate these Hilbert subspaces, we use the Holstein-Primakoff map [23] to replace $c_{\omega l m}$ by spin operators, introducing the parameter $s_{max} \gg 1$

$$s_+^{\omega l m} = \sqrt{2 s_{max}} \sqrt{1 - \frac{c^{\dagger}_{\omega l m} c_{\omega l m}}{2 s_{max}}}\, c_{\omega l m},\ s_-^{\omega l m} = \sqrt{2 s_{max}}\, c^{\dagger}_{\omega l m} \sqrt{1 - \frac{c^{\dagger}_{\omega l m} c_{\omega l m}}{2 s_{max}}},\ s_z^{\omega l m} = s - c^{\dagger}_{\omega l m} c_{\omega l m} .$$

For states near the ground state, we can approximate $\sqrt{1 - \frac{c^{\dagger}_{\omega l m} c_{\omega l m}}{2 s_{max}}}$ by 1.

This regularization of the Hilbert space then allows us to write the energy in the scalar field at quadratic order as a spin model

$$H_0 = \sum_{\{\omega\}} \sum_{l=0}^{l_{max}} \sum_{m=-l}^{l} \omega \left(c^{\dagger}_{\omega l m} c_{\omega l m} + c_{\omega l m} c^{\dagger}_{\omega l m}\right) .$$

The dimension of the Hilbert space, for large l_{max} is $(2s_{max}+1)^{l_{max}^2 n_{rad}} = e^{S_{BH}}$, identified with the Bekenstein-Hawking entropy of the cosmological horizon $S_{BH} = \pi\ell^2/\ell_p^2 \equiv N$. If we follow the arguments of [24], we identify

$$l_{max}^2 n_{rad} \log s_{max} \sim N$$

and a natural choice would be to scale $n_{rad} \sim l_{max} \sim N^{1/3}$, dropping subleading log factors for simplicity in a large N limit. This leads to a short distance cutoff length of order $\ell_p N^{1/6}$ in all directions (and a choice $n_{UV} \sim N^{1/6}$). We note if our present universe was replaced by a pure de Sitter region with the same Hubble parameter, we would find $N \approx 10^{120}$ and $\ell_p N^{1/6}$ would correspond to a GeV UV cutoff.

So far, we have simply regulated the scalar field theory at the level of free field theory and found a holographic dual that reproduces that. The holographic dual can be viewed as living on an S^2 with the discrete parameter ω labelling different variables at each point on the sphere. This construction is guaranteed to reproduce the bulk correlators of free scalar field theory with this particular regulator.

The ground state of the Hamiltonian corresponds to the Bunch-Davies vacuum state, and the Hamiltonian is diagonal in modes that are entangled between the left and right "patches". Tracing over one set leads to an approximately thermal density matrix in the other, subject to the regulator on mode number imposed by finite s_{max}. The excitations of this model will lead to stress-energy tensors regular on the cosmological horizon, avoiding the firewall conundrum.

In general, we also expect to have to add perturbative interactions to this model, which will typically be suppressed by powers of N relative to the quadratic term. One might hope to follow a construction paralleling HKLL [25, 26] to reproduce perturbative field theory in the bulk.

Such a theory might be satisfactory for de Sitter spacetime. Once the initial state corresponding to Bunch-Davies is specified on the past horizon of the right static patch (and its continuation onto the left static patch) it evolves according to the standard rules of quantum mechanics. The future cosmological horizon would essentially behave like a remnant, becoming entangled with the degrees of freedom in the left patch. A priori this poses no issues for the information problem, because the cosmological horizon in de Sitter is eternal.

Motivated by the physics of black hole horizons, it is interesting to explore what happens when this model is supplemented by an additional nonlocal term as studied in [13, 14, 27] which is thought to generate chaotic dynamics over sufficiently long timescales. In the black hole case, the timescale associated with quantum scrambling is linked to the timescale the horizon can retain quantum information, before emitting it to the region outside the black hole. In the de Sitter case, we view the static patch as analogous to the black hole interior and are mostly interested in developing the holographic map on timescales shorter than this scrambling time. We may then study the decoherence of local observables built using the holographic map described above, when supplemented by chaotic interactions.

The full Hamiltonian includes a non-local piece and a local piece, where the non-local piece is given by

$$H_{\rm nl} = \sum_{ijkl} J_{ijkl} s_i s_j s_k s_l \tag{3.9}$$

Here the coupling J_{ijkl} is drawn randomly from a Gaussian distribution with zero mean (tensor indices are suppressed). We do not have in mind averaging over this coupling, but rather work with a fixed set of J_{ijkl} as needed to generate chaotic dynamics. We impose the condition that the variance of the non-local Hamiltonian $\mathrm{var}(H_{\rm nl}) = 1$. This forces the width of the Gaussians to scale like $1/N^2$, due to the following analysis

$$1 = \langle H_{nl}^2 \rangle \sim J^2 \left\langle \sum_{i_1 \cdots i_8} s_{i_1} \cdots s_{i_8} \right\rangle \sim J^2 N^4 \tag{3.10}$$

where in the last step we have used the fact that on average $\langle s_i s_j \rangle = \delta_{ij}$. We note this unusual scaling with N is designed to reproduce the Bekenstein-Hawking entropy via microstate counting for fixed N as opposed to the more conventional large N limit where $\langle H_{nl}^2 \rangle \sim N$, which would widen the spectrum to much larger energies.

Our proposal for the full Hamiltonian is then

$$H = H_0 + T_H H_{nl} \tag{3.11}$$

and the chaotic term may then be treated as a small perturbation for short enough time intervals, where it will shift energies at leading order by terms of order T_H.

One may then study how local perturbations of the thermal state decohere when this term is included. Following the analysis of [27] we expect the timescale of such decoherence to be

$$t_{dec} = \beta \log N \tag{3.12}$$

This resembles the scrambling time, however the interpretation here is somewhat different. With the scaling (3.10) the global scrambling time is expected to be

$$t_{scr} = \beta N^{1/2} \log N \gg t_{dec}$$

if the bounds derived in [28] happen to be saturated. However, the local decoherence time is the quantity of most relevance in deciding when the holographic map derived above breaks down. A similar breakdown of the bulk description via effective field theory in a black hole interior was noted in [29, 13, 14].

Given that a local operator will evolve to a highly non-local operator in the time (3.12), rather than simply undergoing the free propagation governed by the term H_0, our holographic map based on the mode functions (3.5) will break down after this timescale. In the case of applying this to a pure de Sitter region with ℓ matched to our present cosmological horizon, this would imply a breakdown in the local laws of physics after a timescale of order 4000 billion years due to quantum gravity effects. It would be very interesting to devise experiments sensitive to this local decoherence. While the shifts in energy levels are tiny, of order $10^{-33} eV$ the nonlocal character of the decoherence opens the door to more sensitive experiments.

One might wonder whether such a holographic description is ruled out for primordial inflation. In that case, one can try to embed "small" de Sitter models into a much larger Hilbert space, which is needed to describe the late-time phase of cosmology. Holographic bounds with these considerations in mind were considered in [30, 31]. The decoherence times in this case can be made much longer than the timescale associated with primordial inflation.

It should also be noted that once a local basis of operators has decohered, for example in Heisenberg picture

$$c_{\omega lm}(t) = e^{iHt} c_{\omega lm}(0) e^{-iHt}$$

with $t > t_{dec}$ one may simply do a change of basis by the unitary transformation $e^{iHt_{dec}}$ to return to another local basis

$$\tilde{c}_{\omega lm}(t) = e^{-iHt_{dec}} c_{\omega lm}(t) e^{iHt_{dec}}$$

therefore, in some basis, one always retains an approximately local description of spacetime physics. This realizes the proposal of [29] in a concrete model, when adapted to de Sitter spacetime.

3.4 Conclusions

Now that we have a detailed proposal for the stretched horizon theory of the de Sitter cosmological horizon, we can try to adapt the method to black holes. A key step in the development of the holographic map was the assumption of regularity of the modes on the pole of the static patch. This eliminated the non-normalizable modes and allowed us to make a one-one map from frequency/radial quantum number space to mode functions (3.5). For black holes in asymptotically flat space, one would need to perform a similar restriction, which might be accomplished by placing a mirror around the black hole to prevent evaporation. In practice, as we have learned over the years, the best substitute for this procedure is simply to introduce a negative cosmological constant which has the same effect and can be handled much more precisely. Thus, we expect the present considerations will apply largely unchanged to a large black hole in anti-de Sitter spacetime which does not evaporate. In this way, we can use the present construction to derive a holographic map for the interior of such a black hole. One might then hope to derive the spin model directly from the conformal field theory description available in that case. Note here we have in mind realizing the black hole in a single conformal field theory representing, perhaps, a large black hole formed by collapse, rather than the tensor product conformal field theories describing wormholes.

Turning this argument around, we then expect the much more interesting case of the evaporating black hole in asymptotically flat space, or a small black hole in asymptotically anti de Sitter space will involve important extra ingredients. The coupling between this stretched horizon theory and some larger holographic theory describing the asymptotic region will need to be specified. Nevertheless, for timescales shorter than t_{dec} we expect to be able to apply the considerations of the present chapter, which is sufficient to extend the holographic map to black hole interiors.

In the case of anti-de Sitter/conformal field theory duality, it is often suggested one has control of the holographic map all the way to the stretched horizon. In that case one has a fixed local basis extending from asymptotic infinity down to the stretched horizon. The present picture implies the coupling between the exterior and the stretched horizon eventually become highly nonlocal, contaminating the exterior physics with non-local effects. Indeed, nonlocal interactions akin to (3.9) must emerge from the correspondence in a smooth way as one approaches the stretched horizon. This has the profound consequence that nonlocal scrambling effects might be detected outside large black holes, if sufficiently long timescales can be probed to overcome the T_H suppression factor in (3.11). Indeed, such effects are probed in current gravitational wave experiments [32]. For example, for black hole mergers with masses of order a solar mass, one must probe around 100 light crossing times to access the timescale (3.12). As these experiments become more precise it will be very interesting to look for signs of violations of the equivalence principle. For example, one might look for anomalies in the late time ringing profile following black hole merger.

Finally, we end with a comment on an interesting numerological coincidence of this holographic model. We noted above, that if we replace our present cosmology with a de Sitter horizon with size around 14 billion light years, an unacceptably small ultraviolet cutoff emerges on bulk effectively field theory of about 1 GeV. This may simply be a signal that a more precise holographic model would produce a bulk cutoff in a much more subtle way. However, for now, let us instead explore the possibility that the current observable entropy $S \approx 10^{88}$ which arises largely from cosmic microwave background photons, might be equated with a late-time de Sitter entropy. Interestingly, this predicts the cosmic acceleration must increase versus the previous possibility, a feature also noted in the Hubble tension experiments, and the ultraviolet cutoff that emerges is the more experimentally interesting value of 100 TeV. This raises the possibility that holographic physics might appear in collider experiments at experimentally accessible scales. Unfortunately, the model also predicts the horizon size must shrink to of order 10^4 m to reach the late time de Sitter phase, so we are presently far off from the phase, and it is not clear how much to trust the ultraviolet cutoff result. Nevertheless, the model was designed so a freely falling observer will use a cutoff with fixed proper spatial resolution, so there is reason to be optimistic.

Chapter 4

Conformal Wave Expansions for Flat Space Amplitudes

The extended BMS algebra contains a conformal subgroup that acts on the celestial sphere as $SO(3,1)$. It is of interest to perform mode expansions of free fields in Minkowski spacetime that realize this symmetry in a simple way. In the present work we perform such a mode expansion for massive scalar fields using the unitary principal series representations of $SO(3,1)$ with a view to developing a holographic approach to gravity in asymptotically flat spacetime. These mode expansions are also of use in studying holography in three-dimensional de Sitter spacetime.

4.1 Introduction

There has been considerable interest recently in constructing holographic theories between flat 4d Minkowski spacetime and a 2d boundary celestial sphere conformal field theory [33, 34, 35, 36]. Central to this mission is the construction of conformally covariant wavefunctions that form unitary representations of the $\mathrm{SO}(d,1)$ group. These wavefunctions are defined on a d-dimensional de Sitter spacetime dS_d, on which $\mathrm{SO}(d,1)$ acts naturally through an embedding of dS_d as a submanifold of a $(d+1)$-dimensional Minkowski spacetime.

As part of the program to realize the dS/CFT correspondence [37], numerous papers have previously constructed unitary principal series representations of the $\mathrm{SO}(2,1)$ group on two-dimensional de Sitter spacetimes [38, 39], as well as q-deformed versions of the principal series on the three-dimensional de Sitter spacetime [40]. In this chapter we construct the unitary principal series representation of the $\mathrm{SO}(3,1)$ group on the three-dimensional de Sitter spacetime. We also compute the uplifted version of these wavefunctions on the ambient four-dimensional Minkowski spacetime. Finally, we comment on relevant previous work [41, 42, 43] and discuss how our results fit within the program to develop holographic approaches to gravity in de Sitter and asymptotically Minkowski spacetime.

The sections are organized as follows: we first establish our coordinate systems and fix our notations in section 4.2. We then construct massive scalar mode functions on both the three-dimensional de Sitter spacetime dS_3 and the four-dimensional Minkowski spacetime M_4, in section 4.3 and 4.4, respectively. We then show in section 4.6 that these mode functions form a unitary principal series representation of the SO(3, 1) group. We note that previous work [41, 42, 43] uses modes that form non-unitary highest weight representations of $SO(3,1)$. Finally we comment in section 4.8 on how our mode functions can serve as a conformal basis to develop holographic formulations of gravity in asymptotically de Sitter and Minkowski spacetimes.

4.2 Conformal Coordinates

Before we begin our discussion of the representation theory of SO(3, 1) in de Sitter and Minkowski spacetimes, we would like to present the coordinate systems we employ and fix notation. A schematic plot of the relevant hypersurfaces can be found in Fig. 4.1.

We start with the 4d flat Minkowski spacetime labeled by coordinates (x^0, x^1, x^2, x^3) with the following metric

$$\mathrm{d}s^2 = -(\mathrm{d}x^0)^2 + (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + (\mathrm{d}x^3)^2\,.$$

We embed our 3D de Sitter spacetime as a hypersurface within the 4d Minkowski spacetime. To do so we first switch to hyperbolic coordinates $(t, \rho, \theta, \varphi)$ with $-\infty < t < \infty$, $\rho > 0$, $0 \le \theta < \pi$ and $0 \le \varphi < 2\pi$, defined by

$$\begin{aligned}
x^0 &= \rho \sinh(t/\rho) \\
x^1 &= \rho \cos\theta \cosh(t/\rho) \\
x^2 &= \rho \sin\theta \cos\varphi \cosh(t/\rho) \\
x^3 &= \rho \sin\theta \sin\varphi \cosh(t/\rho)\,.
\end{aligned} \tag{4.1}$$

Note that these coordinates only cover the region of the Minkowski spacetime defined by points x^μ where $x \cdot x > 0$. The metric then becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{2t\,\mathrm{d}\rho\,\mathrm{d}t}{\rho} + \left(1 - \frac{t^2}{\rho^2}\right)\mathrm{d}\rho^2 + \rho^2\cosh^2\left(\frac{t}{\rho}\right)\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\,.$$

The d'Alembertian in this coordinate system looks rather complicated. We therefore perform the following change of variables

$$\eta = \frac{t}{\rho}$$

and the metric takes the following simpler form

$$\mathrm{d}s^2 = -\rho^2\mathrm{d}\eta^2 + \mathrm{d}\rho^2 + \rho^2\cosh^2\eta\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right)\,.$$

A 3D de Sitter spacetime can then be embedded into this 4d Minkowski spacetime as a hypersurface of constant $\rho = \ell$, where ℓ is the de Sitter length scale which we will set to unity in what

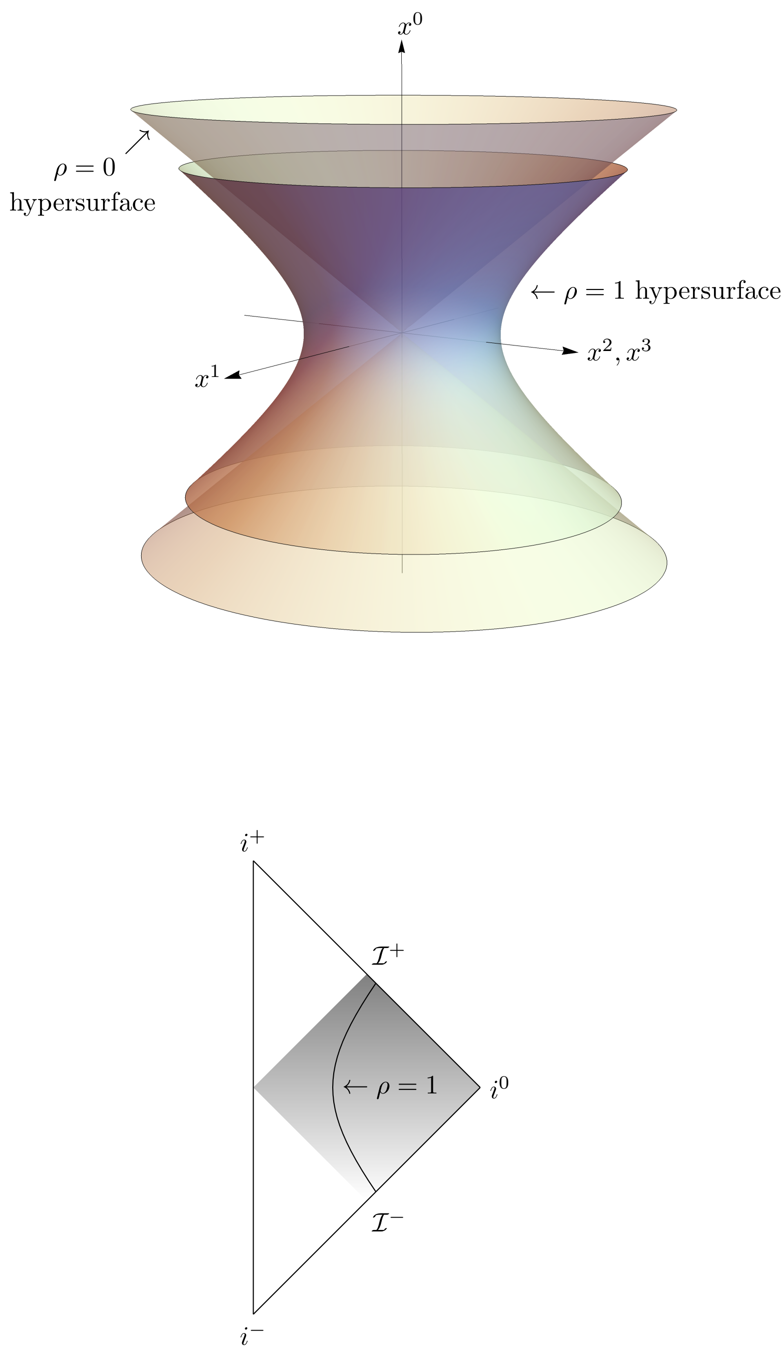

Figure 4.1: Minkowski spacetime may be divided up into radial (ρ) slices isometric to 3D de Sitter spacetime as shown in the top panel. The bottom panel shows the Penrose diagram of Minkowski spacetime. The shaded region is the region bounded by $\rho = 0$ and $\rho = \infty$. In particular, $i^{\pm}$ are excluded from this region.

follows. On the 3D de Sitter spacetime the induced metric is simply

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \cosh^2 t \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) .$$

Through a stereographic projection we can parameterize the 2-sphere covered by coordinates (θ, φ) with a complex variable z, and obtain the Fubini-Study metric on the 2-sphere:

$$\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2 = \frac{4\,\mathrm{d}z\,\mathrm{d}\bar{z}}{(1+|z|^2)^2} .$$

This allows us to rewrite the 3D de Sitter metric as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \cosh^2 t\,\frac{4\,\mathrm{d}z\,\mathrm{d}\bar{z}}{(1+|z|^2)^2} \tag{4.2}$$

and the 4d Minkowski metric in hyperbolic coordinates as

$$\mathrm{d}s^2 = -\rho^2\mathrm{d}\eta^2 + \mathrm{d}\rho^2 + \rho^2\cosh^2\eta\,\frac{4\,\mathrm{d}z\,\mathrm{d}\bar{z}}{(1+|z|^2)^2} .$$

4.3 3D de-Sitter Mode Functions

We start from the metric on the 3D de Sitter spacetime (4.2). The isometry group of the 3D de Sitter spacetime is $\mathrm{SO}(3,1)$ which has 6 generators with real coefficients. The first step in building a unitary representation of $\mathrm{SO}(3,1)$ on 3D de Sitter is to solve the scalar field equation with mass μ

$$(\Delta - \mu^2)\phi(t, z, \bar{z}) = 0$$

where the d'Alembertian is defined in general as

$$\Delta\phi = \frac{1}{\sqrt{|g|}}\partial_i(g^{ij}\sqrt{|g|}\partial_j\phi) .$$

Here and in what follows we use Latin indices when we are referring to the 3D de Sitter submanifold and use Greek indices when we are dealing with the ambient 4d Minkowski spacetime. The d'Alembertian is computed to be

$$\Delta\phi = \left[-\partial_t^2 - 2(\tanh t)\partial_t + (\mathrm{sech}^2\, t)(1+|z|^2)^2\partial_z\partial_{\bar{z}}\right]\phi .$$

The massive scalar field equation therefore is

$$\left(-\frac{\partial^2}{\partial t^2} - 2\tanh t\frac{\partial}{\partial t} + \mathrm{sech}^2 t\,(1+|z|^2)^2\frac{\partial^2}{\partial z\partial\bar{z}} - \mu^2\right)\phi(t, z, \bar{z}) = 0 .$$

We can perform separation of variables as usual and write a mode function as

$$\phi_{lm}(t, z, \bar{z}) = \phi_l(t)\,Y^l_m(z, \bar{z})$$

where m is a j_3 eigenvalue. The Y^l_m are the standard spherical harmonics [44] in $(z, \bar{z})$ coordinates. They satisfy the eigenvalue equation

$$(1+|z|^2)^2\frac{\partial^2}{\partial z\partial\bar{z}}Y^l_m = -l(l+1)Y^l_m .$$

Explicitly, we have

$$Y^l_m(\theta,\varphi)=\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}P^m_l(\cos\theta)e^{im\varphi}$$

where

$$P^m_l(x)=\frac{(1+x)^{m/2}}{(1-x)^{m/2}}\sum_{k=0}^{l}\frac{\left(\frac{1-x}{2}\right)^k(-l)_k(l+1)_k}{k!\,\Gamma(k-m+1)}\,.$$

The coordinate transformation linking (θ,φ) and $(z,\bar{z})$ is

$$(\sin\theta\, e^{i\varphi},\cos\theta)=\left(\frac{2z}{1+|z|^2},\frac{1-|z|^2}{1+|z|^2}\right)\,.$$

It is easy to see that the spherical harmonics $Y^m_l(z,\bar{z})$ can be built out of homogeneous polynomials of the following three variables

$$F_z=\frac{2z}{1+|z|^2}\qquad F_{\bar{z}}=\frac{2\bar{z}}{1+|z|^2}\qquad F_t=\frac{1-|z|^2}{1+|z|^2}\,.$$

We will later see that these three variables will appear in (4.10). One is then left to solve

$$\left(\frac{\partial^2}{\partial t^2}+2\tanh t\frac{\partial}{\partial t}+l(l+1)\mathrm{sech}^2 t+\mu^2\right)\phi_l(t)=0\,.$$

This has the following two linearly independent solutions

$$\phi_{l,1}(t)=\mathrm{sech}t\,P_l^{\sqrt{1-\mu^2}}(\tanh t)\,,\quad \phi_{l,2}(t)=\mathrm{sech}t\,Q_l^{\sqrt{1-\mu^2}}(\tanh t)\tag{4.3}$$

where P and Q are the associated Legendre functions of first and second kind, respectively. We expect to get a unitary representation corresponding to a general complex linear combination, one of which will be the Euclidean vacuum (see Section 4.5 and Appendix B). Other combinations will generate modes around an α-vacuum [45, 46, 47].

4.4 Uplifting onto 4D Minkowski

To uplift the 3D de Sitter mode functions onto the 4d Minkowski spacetime we consider the scalar field equation in 4d with mass M

$$(\Box-M^2)\Phi(\eta,\rho,z,\bar{z})=0\,.$$

The d'Alembertian is computed to be

$$\Box=-\frac{1}{\rho^2}(\partial_\eta^2+2\tanh\eta\,\partial_\eta)+3\frac{\partial_\rho}{\rho}+\partial_\rho^2+\frac{\mathrm{sech}^2\,\eta\,(1+|z|^2)^2\,\partial_z\partial_{\bar{z}}}{\rho^2}\,.$$

Separating variables, the mode functions can be written as

$$\Phi_{plm}(\eta,\rho,z,\bar{z})=\phi_{pl}(\eta)\,\psi_p(\rho)\,Y^l_m(z,\bar{z})$$

where $l = 0, 1, \cdots, m = -l, \cdots, l$ and the range of the real parameter p will be discussed later. We find the differential equation for ϕ_{pl} to be identical to the 3D de Sitter modes, except that we replace μ with a real parameter p,

$$\left(\partial_\eta^2 + 2(\tanh\eta)\partial_\eta + l(l+1)\,\mathrm{sech}^2\,\eta + p^2\right)\phi_{pl}(\eta) = 0$$

which has solutions (4.3) with the replacement $t \to \eta$ and $\mu \to p$.

The differential equation for ψ_p therefore becomes

$$\left(\partial_\rho^2 + \frac{3}{\rho}\partial_\rho + \frac{p^2}{\rho^2} - M^2\right)\psi_p(\rho) = 0\,.$$

This has two independent solutions

$$\psi_{p,1} = \frac{I_{\sqrt{1-p^2}}(M\rho)}{\rho}\,, \qquad \psi_{p,2} = \frac{K_{\sqrt{1-p^2}}(M\rho)}{\rho} \tag{4.4}$$

where I_α and K_α are modified Bessel functions of first and second kind, respectively.

4.5 Klein-Gordon Norm and Orthonormality Conditions

Before presenting the explicit form of our unitary principal series representation we would like to first establish the orthonormalizability of the 4d Minkowski mode functions. The mode functions are normalized with respect to the Klein-Gordon norm, the most general form of which is [48]

$$\langle f, g\rangle = -i\int_\Sigma \mathrm{d}\Sigma\, n^\lambda (f\partial_\lambda g^\star - g^\star \partial_\lambda f)\,. \tag{4.5}$$

Here Σ is a spacelike surface, n^λ is a timelike unit vector field normal to Σ and $d\Sigma$ is the volume element in Σ. This norm is time-independent, and in principle can be evaluated on any Cauchy slice. Here, for convenience we choose the $\eta = \eta_0$ slice, where η_0 is an arbitrary constant. On the 4d Minkowski spacetime with coordinate system $(\eta, \rho, \theta, \varphi)$ this then becomes

$$\langle f, g\rangle = -i\int_\rho\int_{S^2} (f\partial_\eta g^\star - g^\star\partial_\eta f)|_{\eta=\eta_0}\,(\cosh^2\eta_0)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\Omega$$

where $\mathrm{d}\Omega$ is the area element of the unit 2-sphere S^2. Given that our mode functions are separable $\Phi_{plm}(\eta, \rho, z, \bar{z}) = \phi_{pl}(\eta)\psi_p(\rho)Y^l_m(z, \bar{z})$ we can further evaluate this norm to obtain

$$\begin{aligned}\langle \Phi_{plm}, \Phi_{p'l'm'}\rangle = &-i\,\left(\phi_{pl}(\eta_0)\dot{\phi}^\star_{p'l'}(\eta_0) - \phi^\star_{p'l'}(\eta_0)\dot{\phi}_{pl}(\eta_0)\right)\,(\cosh^2\eta_0)\\ &\times\int_0^\infty \psi_p(\rho)\psi^\star_{p'}(\rho)\,\rho\,\mathrm{d}\rho\int_{S^2} Y^l_m(z,\bar{z})Y^{\star l'}_{m'}(z,\bar{z})\,\mathrm{d}\Omega\end{aligned}$$

where $\dot{\phi}(\eta) = \partial_\eta\phi$. The orthonormality of the spherical harmonics allows us to conclude that

$$\int_{S^2} Y^l_m(z,\bar{z})Y^{\star l'}_{m'}(z,\bar{z})\,\mathrm{d}\Omega = \delta_{ll'}\delta_{mm'}\,.$$

For the ρ integral, note that multiplying the differential equation satisfied by ψ_p with $\psi^\star_{p'}$ gives

$$\psi''_p\psi^\star_{p'} + \frac{3}{\rho}\psi'_p\psi^\star_{p'} + \left(\frac{p^2}{\rho^2} - M^2\right)\psi_p\psi^\star_{p'} = 0$$

where $\psi' = \partial_\rho\psi$. Likewise swapping $\psi_p \leftrightarrow \psi^*_{p'}$ and subtracting we have

$$-\frac{p^2 - p'^2}{\rho^2}\psi_p\psi^\star_{p'} = \psi''_p\psi^\star_{p'} - \psi^{\star''}_{p'}\psi_p + \frac{3}{\rho}\psi'_p\psi^\star_{p'} - \frac{3}{\rho}\psi^{\star'}_{p'}\psi_p\,.$$

Integrating, we have

$$-(p^2 - p'^2)\int_0^\infty \psi_p\psi^\star_{p'}\,\rho\,\mathrm{d}\rho = \int_0^\infty \mathrm{d}\rho\,[\rho^3(\psi''_p\psi^\star_{p'} - \psi^{\star''}_{p'}\psi_p) + 3\rho^2(\psi'_p\psi^\star_{p'} - \psi^{\star'}_{p'}\psi_p)]\,.$$

The integral on the right hand side can be integrated by parts to yield

$$-(p^2 - p'^2)\int_0^\infty \psi_p\psi^\star_{p'}\,\rho\,\mathrm{d}\rho = [\rho^3(\psi'_p\psi^\star_{p'} - \psi^{\star'}_{p'}\psi_p)]\Big|_0^\infty\,. \tag{4.6}$$

The modified Bessel functions have the following mirror symmetry

$$I^\star_\alpha(z) = I_{\alpha^\star}(z^\star) \qquad K^\star_\alpha(z) = K_{\alpha^\star}(z^\star)\,.$$

At $z = +\infty$, $I_\alpha(z) \sim e^z$ which increases exponentially while $K_\alpha(z) \sim e^{-z}$ which decreases exponentially. We therefore discard the $\psi_{p,1}$ set of solutions as these modes are not normalizable and study the (4.4) solutions $\psi_{p,2}$. We begin by considering the case $p^2 > 1$ and take the branch $\sqrt{1-p^2} = i\sqrt{p^2-1}$. Let us define $\alpha = \sqrt{p^2-1}$ and henceforth we will drop the 2 subscript on $\psi_{p,2}$. Near $z = 0$, the expansion

$$K_\nu(z) = 2^{\nu-1}\Gamma(\nu)z^{-\nu} + 2^{-\nu-1}\Gamma(-\nu)z^\nu + \cdots$$

allows us to evaluate the surface term to give

$$\begin{aligned}&\lim_{\rho\to 0}\rho^3\left(\psi'_p\psi^*_{p'} - \psi^{*'}_{p'}\psi_p\right) = \\ &\quad\lim_{\rho\to 0} i(\alpha+\alpha')2^{-2}\left(\Gamma(i\alpha')\Gamma(-i\alpha)(M\rho/2)^{i(\alpha-\alpha')} - \Gamma(i\alpha)\Gamma(-i\alpha')(M\rho/2)^{-i(\alpha-\alpha')}\right)\end{aligned} \tag{4.7}$$

treating the rapidly oscillating terms in α as vanishing in the sense of a distribution. For $\rho \to 0$, this is proportional to a sinc representation of the Dirac delta function

$$\lim_{\rho\to 0}\rho^3\left(\psi'_p\psi^*_{p'} - \psi^{*'}_{p'}\psi_p\right) = -\frac{\alpha\Gamma(i\alpha)\Gamma(-i\alpha)}{2}\sin\left[(\alpha-\alpha')\log(M\rho/2)\right]$$

which gives

$$\int_0^\infty \psi_p\psi^\star_{p'}\,\rho\,\mathrm{d}\rho = \frac{\Gamma(i\alpha)\Gamma(-i\alpha)}{4}\lim_{C\to\infty}\frac{\sin[C(\alpha-\alpha')]}{\alpha-\alpha'}$$

where we have set $C = \log(M\rho/2)$. Using the following identity

$$\lim_{C\to\infty}\frac{\sin(Cx)}{x} = \pi\delta(x)$$

we then have

$$\int_0^\infty \psi_p \psi_{p'}^\star \, \rho \, \mathrm{d}\rho = \frac{\pi\Gamma(i\alpha)\Gamma(-i\alpha)}{4}\delta(\alpha-\alpha') = \frac{\pi^2}{4p\sinh\left(\pi\sqrt{p^2-1}\right)}\delta\left(p-p'\right) . \tag{4.8}$$

For $0 \le p^2 < 1$, instead of oscillating terms in eq. 4.7 we have power-law divergencies, and the mode functions ψ_p in this case again are not normalizable.

To conclude, the normalizable radial mode functions arise from $\psi_{p,2}$ in (4.4) with $p^2 > 1$. The p-dependent prefactor in (4.8) may then be absorbed into the normalization of these functions. We assume this has been done, and in the interest of notational clarity we will from now use ψ_p to denote the normalized radial mode functions. The normalized radial mode functions in this case will satisfy the following

$$\int_0^\infty \psi_p \psi_{p'}^\star \, \rho \, \mathrm{d}\rho = \delta(p-p') . \tag{4.9}$$

For the η dependence, we form the following linear combination of $\phi_{pl,1}$ and $\phi_{pl,2}$ to obtain

$$\phi_{pl} = \frac{i\pi}{2}\phi_{pl,1} + \phi_{pl,2}$$

which, as we will show in Appendix B, are the (unnormalized) modes corresponding to the 4d Minkowski vacuum. Using

$$\frac{\partial P_\nu^\mu}{\partial z} = \frac{\nu z P_\nu^\mu(z) - (\mu+\nu)P_{\nu-1}^\mu(z)}{z^2-1} \qquad \frac{\partial Q_\nu^\mu}{\partial z} = \frac{\nu z Q_\nu^\mu(z) - (\mu+\nu)Q_{\nu-1}^\mu(z)}{z^2-1}$$

and

$$P_\nu^\mu(0) = \frac{\pi^{1/2}2^\mu}{\Gamma\left(\frac{1-\mu-\nu}{2}\right)\Gamma\left(\frac{\nu-\mu}{2}+1\right)} \qquad Q_\nu^\mu(0) = -\frac{\pi^{3/2}2^{\mu-1}\tan\frac{\pi(\mu+\nu)}{2}}{\Gamma\left(\frac{1-\mu-\nu}{2}\right)\Gamma\left(\frac{\nu-\mu}{2}+1\right)}$$

we can evaluate the η-dependent part of the Klein-Gordon norm to obtain

$$-i[\phi_{pl}(\eta_0)\phi_{pl}^{\star'}(\eta_0) - \phi_{pl}^\star(\eta_0)\phi_{pl}'(\eta_0)](\cosh^2\eta_0) = \pi e^{-\pi\sqrt{p^2-1}}$$

which does not depend on η_0 due to conservation of the Klein-Gordon norm. We can therefore normalize the modes ϕ_{pl} by replacing

$$\phi_{pl} \to \frac{e^{\frac{\pi}{2}\sqrt{p^2-1}}}{\pi^{1/2}}\phi_{pl} .$$

In what follows we will assume that this has been done and in the interest of notational simplicity we will use ϕ_{pl} to denote the normalized modes.

To summarize, for $p^2 > 1$ we have constructed mode functions of the 4d Klein-Gordon equation corresponding to the Minkowski vacuum, which when restricted to the de Sitter slice $\rho = 1$ correspond to the Euclidean vacuum of the 3D de Sitter spacetime. These modes $\Phi_{plm}(\eta,\rho,z,\bar{z}) = \phi_{pl}(\eta)\psi_p(\rho)Y_m^l(z,\bar{z})$ are normalized with respect to to the Klein-Gordon norm (4.5) with the following orthonormality condition

$$\langle \Phi_{plm}, \Phi_{p'l'm'}\rangle = \delta(p-p')\delta_{ll'}\delta_{mm'} .$$

For $0 \le p^2 < 1$, the radial mode functions are not normalizable.

4.6 Unitary Principal Series Representation

We are now in a position to build the unitary principal series representation of $\mathrm{SO}(3,1)$ which acts on the $\mathrm{dS}_3/\mathrm{M}_4$ mode functions. Since the action of $\mathrm{SO}(3,1)$ on M_4 leaves the radial coordinate ρ invariant, the actions is identical on both dS_3 mode functions and on M_4 mode functions. For simplicity of presentation we will focus on dS_3 modes, but all equations in this section carry over to the M_4 modes trivially.

On the past and future null infinities $\mathcal{I}^{\pm}$ of the 3D de Sitter spacetime, the Killing vectors can be written as conformal Killing vectors of the spatial 2-sphere

$$L_n = -z^{n+1}\frac{\partial}{\partial z},\, \bar{L}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}}$$

where $n = 0, \pm 1$. However one should use caution in applying this formula. It is correct when acting on the metric, or massless scalars, but as we will see there will be additional terms that must be added depending on the class of functions or fields considered. General complex combinations of these vectors will not preserve the desired reality conditions, so we will need to be careful to construct the correct 6 independent generators that will appear with real coefficients.

To extend these into the bulk of de Sitter it is helpful to arrange them into an $\mathrm{SO}(3)$ corresponding to the isometries of the spatial slices. With the convention $g = \exp(i\theta J)$ we find the generators:

$$\begin{aligned}
J_1 &= \frac{i}{2}\left(L_{-1} + L_1 + \bar{L}_{-1} + \bar{L}_1\right)\\
J_2 &= \frac{1}{2}\left(L_{-1} - L_1 - \bar{L}_{-1} + \bar{L}_1\right)\\
J_3 &= L_0 - \bar{L}_0\,.
\end{aligned}$$

These immediately extend into the bulk without time dependent contributions. This also allows us to read off the conjugation condition to be imposed on the generators. Since the J_k are Hermitian we require

$$L_n^{\dagger} = -\bar{L}_n, \qquad \bar{L}_n^{\dagger} = -L_n\,.$$

The time-dependent Killing vectors take the form

$$K = F\frac{\partial}{\partial t} + \frac{1}{2}\left(1 + z\bar{z}\right)^2 \tanh t\; (\partial_{\bar{z}}F)\,\frac{\partial}{\partial z} + \frac{1}{2}\left(1 + z\bar{z}\right)^2 \tanh t\; (\partial_z F)\,\frac{\partial}{\partial \bar{z}}$$

where F is one of the three solutions

$$F_z = \frac{2z}{1+|z|^2} \qquad F_{\bar{z}} = \frac{2\bar{z}}{1+|z|^2} \qquad F_t = \frac{1-|z|^2}{1+|z|^2}\,. \tag{4.10}$$

At $\mathcal{I}^{+}$ these reduce to

$$\begin{aligned}
\tilde{K}_1 &= L_1 - \bar{L}_{-1}\\
\tilde{K}_2 &= \bar{L}_1 - L_{-1}\\
\tilde{K}_3 &= L_0 + \bar{L}_0
\end{aligned}$$

when acting on the metric. It is convenient to assemble these into Hermitian linear combinations:

$$K_1 = \frac{1}{2}\left(L_1 - \bar{L}_{-1} - \bar{L}_1 + L_{-1}\right)$$
$$K_2 = \frac{i}{2}\left(\bar{L}_1 - L_{-1} + L_1 - \bar{L}_{-1}\right)$$
$$K_3 = -i\left(L_0 + \bar{L}_0\right)$$

which at a general spacetime point become

$$\begin{aligned} K_1 &= \frac{1}{2}\left(\frac{2(z-\bar{z})}{1+|z|^2}\frac{\partial}{\partial t} - \tanh t\left(\left(z^2+1\right)\frac{\partial}{\partial z} - \left(\bar{z}^2+1\right)\frac{\partial}{\partial \bar{z}}\right)\right) \\ K_2 &= \frac{i}{2}\left(\frac{2(z+\bar{z})}{1+|z|^2}\frac{\partial}{\partial t} - \tanh t\left(\left(z^2-1\right)\frac{\partial}{\partial z} + \left(\bar{z}^2-1\right)\frac{\partial}{\partial \bar{z}}\right)\right) \\ K_3 &= -i\left(\frac{1-|z|^2}{1+|z|^2}\frac{\partial}{\partial t} - \tanh t\left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right)\right) . \end{aligned} \tag{4.11}$$

Note the generators satisfy the canonical Lorentz algebra

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$
$$[J_i, K_j] = i\epsilon_{ijk}K_k$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k .$$

On the three-dimensional de Sitter mode functions (4.3) the generators (4.11) take the simplified form

$$\begin{aligned} K_1 &= \frac{1}{2}\left(\frac{2(z-\bar{z})}{1+|z|^2}(2h_+-2) - \left(\left(z^2+1\right)\frac{\partial}{\partial z} - \left(\bar{z}^2+1\right)\frac{\partial}{\partial \bar{z}}\right)\right) \\ K_2 &= \frac{i}{2}\left(\frac{2(z+\bar{z})}{1+|z|^2}(2h_+-2) - \left(\left(z^2-1\right)\frac{\partial}{\partial z} + \left(\bar{z}^2-1\right)\frac{\partial}{\partial \bar{z}}\right)\right) \\ K_3 &= -i\left(\frac{1-|z|^2}{1+|z|^2}(2h_+-2) - \left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right)\right) \end{aligned} \tag{4.12}$$

where we define

$$2h_\pm = \frac{d-1}{2} \pm \sqrt{\left(\frac{d-1}{2}\right)^2 - \mu^2} = 1 \pm \sqrt{1-\mu^2} .$$

To check this we note that the group of rotations SO(3) acts straightforwardly in the basis (4.3). The rotation generators also rotate the K_i amongst themselves, so we can focus on the action of say K_3 on the mode functions (4.3). It is then straightforward to check that

$$\frac{\partial}{\partial t}\phi_{l=0} = 2(h_+-1)\phi_{l=1}$$

which holds for the solutions $\phi_{l,1}$ and $\phi_{l,2}$ of (4.3) independently and determines the prefactors that appear in (4.12). Likewise, when acting on the four-dimensional Minkowski modes the generators take the exact same expression with μ replaced by p.

To confirm the generators match the principal series we will use the representation of SO(3, 1) on functions $L^2(S^2)$. Note this differs from the more common representation on functions $L^2(\mathbb{C})$, which would be applicable to the flat slicing of de Sitter. Likewise there is a representation on $L^2(H^2)$, though we won't need that here. The upshot of these different realizations of the principal series is that the extra terms in (4.12) take completely different forms.

To realize the representation we consider the cone C^3_+ embedded in 4d Minkowski spacetime, where

$$x_1^2 + x_2^2 + x_3^2 - x_0^2 = 0\,.$$

We then consider the slice through the cone where $x_0 = 1$. This slice is an S^2 which may be parameterized by coordinates z above using the Fubini-Study metric. The cone maps into itself under $\mathrm{SO}(3,1)$. The principal series may be defined as functions on the slice that behave as [49]

$$(T^\sigma(g)f)(z) = \alpha(z,g)^\sigma f\left(\frac{g^{-1}\cdot z}{\alpha(z,g)}\right) \tag{4.13}$$

where g is a $\mathrm{SO}(3,1)$ group element, and $\alpha(z,g)$ is defined to be the rescaling factor needed to return $g^{-1}\cdot x^\mu$ to the slice $x_0 = 1$. The action of $\mathrm{SO}(3,1)$ on z is the usual fraction linear transformation, but the factor α depends on which realization of the principal series we are considering. To write the action of $\mathrm{SL}(2,\mathbb{C})$ on the Minkowski coordinates it is helpful to use the familiar representation

$$x^\mu = \frac{1}{2}\mathrm{Tr}\left(M\sigma^\mu\right), \qquad M = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \qquad \sigma^\mu = \left(\mathbb{1}, \sigma^i\right)$$

where σ^i are the Pauli matrices. Then an $\mathrm{SL}(2,\mathbb{C})$ transformation acts as

$$M' = SMS^\dagger\,.$$

To rescale back to the slice $x^0 = 1$ we rescale to

$$\tilde{M'} = \frac{M'}{\frac{1}{2}\mathrm{Tr}\left(M'\right)}\,.$$

Finally, the coordinates on the 2-sphere $x^0 = 1$ are matched with the Fubini-Study coordinates via

$$\begin{aligned} x^1 + ix^2 &= \frac{2z}{1+|z|^2} \\ x^1 - ix^2 &= \frac{2\bar{z}}{1+|z|^2} \\ x^3 &= \frac{1-|z|^2}{1+|z|^2}\,. \end{aligned}$$

From these equations we can read off the factor α and determine the action of the z coordinate. For

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the fractional linear transformation is

$$z' = \frac{dz + c}{bz + a}\,.$$

Since we know the group of rotations acts in a straightforward way on the spherical harmonics, it suffice to consider one of the boost generators to check the matching of the generators. To do this we Taylor expand (4.13) for a group element of the form $g = \exp(ik_3\epsilon)$. Plugging in the above relations gives

$$K_3 = -i\left(\frac{1-|z|^2}{1+|z|^2}\sigma - \left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right)\right) .$$

We therefore identify the scalar field representation with a principal series representation where $\sigma = -2h_-$. We have $\sigma = -1 + i\sqrt{\mu^2 - 1}$ in the notation of [49]. We note the equivalence of the representations under the replacement $h_+ \to h_-$ and $2h_+ - 2 = -2h_-$. For the principal series, the inner product is simply the usual integral over the 2-sphere in Fubini-Study coordinates which matches the Klein-Gordon norm up to a constant factor. For $0 < \mu^2 < 1$ we have the complementary series representations and the above results extend straightforwardly to that case.

4.7 Relation to Previous Work

Here we show the mode functions computed in Ref. [42] form a non-unitary highest-weight representation, and therefore do not produce a unitary principal series representation of $SO(3,1)$. To do so we first show that the generators of the special conformal transformations annihilate a mode function corresponding to the highest weight of the representation. The mode functions (eq. 2.19 of [42]) are parameterized by the tuple $(\Delta, \vec{w})$ where Δ is in general a complex number and $\vec{w} = (w_x, w_y) \in \mathbb{R}^2$. These mode functions are

$$\phi^{\pm}_{\Delta}(X^\mu; \vec{w}) = \frac{4\pi}{(im)} \frac{(\sqrt{-X^2})^{\Delta-1}}{(-q(\vec{w})\cdot X \mp i\epsilon)^{\Delta}} K_{\Delta-1}(m\sqrt{X^2}) . \tag{4.14}$$

Here X^μ are the usual flat coordinates of the 4d Minkowski spacetime M_4. The $q^\mu(\vec{w})$ is the following 4-vector

$$q^\mu(\vec{w}) = (1 + |\vec{w}|^2, 2\vec{w}, 1 - |\vec{w}|^2) .$$

The conformal group $\mathrm{SO}(3,1)$ acts on the space of scalar functions defined on $\mathrm{M}_4 \times \mathbb{R}^2$ by acting on M_4 with the usual Lorentz transformation and on $\mathbb{R}^2$ with the 2D conformal transformations (2D translations, 2D rotations, dilatations and special conformal transformations):

$$\phi_\Delta(X^\mu; \vec{w}) \to \phi_\Delta(\Lambda^\mu{}_\nu X^\nu; \vec{w}'(\vec{w})) .$$

Here $\Lambda^\mu{}_\nu$ is the Lorentz transformation corresponding to the $\mathrm{SO}(3,1)$ group element, and the $\vec{w}'(\vec{w})$ is the conformal transformation corresponding to the $\mathrm{SO}(3,1)$ element. For special conformal transformation, it is

$$\vec{w}' = \frac{\vec{w} + |\vec{w}|^2\vec{b}}{1 + 2\vec{b}\cdot\vec{w} + |\vec{b}|^2|\vec{w}|^2} .$$

These are labeled by a vector $\vec{b} \in \mathbb{R}^2$.

Consider an infinitesimal group element near the identity

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + (\delta\Lambda)^\mu{}_\nu$$

this has the corresponding infinitesimal transformation on $\mathbb{R}^2$

$$\vec{w}' = \vec{w} + \delta\vec{w}\,.$$

In particular, for the special conformal transformation parameterized by an infinitesimal $\vec{b}$ we have the infinitesimal transformation

$$\vec{w}' = \vec{w} - 2(\vec{w}\cdot\vec{b})\vec{w} + |\vec{w}|^2\vec{b}\,.$$

Following the conventions of [42] this corresponds to the infinitesimal Lorentz transformation where

$$(\delta\omega)^0{}_i = (\delta\omega)^i{}_0 = (\delta\omega)^3{}_i = -(\delta\omega)^i{}_3 = b_i$$

with all other components being zero. Here $i = x, y$ labels the indices of $\mathbb{R}^2$. In other words we have

$$\begin{bmatrix} X'^0 \\ X'^i \\ X'^3 \end{bmatrix} = \begin{bmatrix} X^0 \\ X^i \\ X^3 \end{bmatrix} + \begin{bmatrix} b_i X^i \\ b^i(X^0 - X^3) \\ b_i X^i \end{bmatrix}\,.$$

Taylor-expanding $\phi_\Delta(\Lambda^\mu{}_\nu X^\nu; \vec{w}'(\vec{w}))$ and substituting the expressions for $\delta\omega$ and $\delta\vec{w}$ above, we have

$$\begin{aligned} \phi_\Delta(\Lambda^\mu{}_\nu X^\nu; \vec{w}'(\vec{w})) &= \phi_\Delta(X^\mu + (\delta\omega)^\mu{}_\nu X^\nu; \vec{w} + \delta\vec{w}) \\ &= \phi_\Delta(X^\nu; \vec{w}) + (\delta\omega)^\mu{}_\nu X^\nu \left(\frac{\partial}{\partial X^\mu}\phi_\Delta\right) + \delta\vec{w}\cdot\left(\frac{\partial}{\partial\vec{w}}\phi_\Delta\right) \\ &= \phi_\Delta(X^\nu; \vec{w}) + b_i X^i \left(\frac{\partial}{\partial X^0} + \frac{\partial}{\partial X^3}\right)\phi_\Delta + b^i(X^0 - X^3)\left(\frac{\partial}{\partial X^i}\phi_\Delta\right) \\ &\quad + \left(-2(\vec{w}\cdot\vec{b})\vec{w} + |\vec{w}|^2\vec{b}\right)\cdot\left(\frac{\partial}{\partial\vec{w}}\phi_\Delta\right)\,. \end{aligned}$$

For infinitesimal $\vec{b}$ this evaluates to

$$\delta\phi_\Delta = \frac{8\pi\Delta}{im}\vec{b}\cdot\vec{w}\frac{\left(\sqrt{-X^2}\right)^{\Delta-1}}{(-q(\vec{w})\cdot X \mp i\epsilon)^\Delta} K_{\Delta-1}\left(m\sqrt{X^2}\right)\,. \tag{4.15}$$

In particular, the special conformal transformations annihilate the mode functions when the weight $\vec{w} = (0,0)$. This implies [50] that the mode functions (4.14) form a highest-weight representation of $SO(3,1)$, and since $SO(3,1)$ has only non-unitary highest-weight representations, these mode functions cannot form a unitary principal series.

4.8 Discussion

The results of the previous sections lead to a proposal for a holographic mapping between the 3D de Sitter modes and conformal operators on a two-sphere. Likewise, the construction may be lifted to 4D Minkowski spacetime.

4.8.1 Holographic Mapping Between 3D de-Sitter and a Euclidean 2-Sphere

For the case of a flat slicing, it was possible to define a bulk-to-boundary map and its inverse map [51] via a construction reminiscent of the LSZ reduction formula in asymptotically flat spacetime [52]. That construction does not extend immediately to the case of the sphere slicing, but we will see the Klein-Gordon inner product defined above can still be used to extract a natural set of operators living on the 2-sphere.

We work with a scalar bulk field of mass μ

$$\phi(t,z,\bar{z}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}\phi_{lm}(t,z,\bar{z}) + a_{lm}^{\dagger}\phi_{lm}^{\dagger}(t,z,\bar{z}) \tag{4.16}$$

where ϕ_{lm} is defined using the Euclidean vacuum modes (B.1) and a_{lm} and $a_{lm}^{\dagger}$ are standard creation/annihilation operators. Let us consider a late-time sphere at $t = T$ and build the following inner product using the Klein-Gordon inner product in 3D de Sitter spacetime

$$\mathcal{O}_{lm} = \langle \phi_{lm}(t), \phi(t,z,\bar{z})\rangle_{t=T}$$

which identifies $\mathcal{O}_{lm} = a_{lm}$. Likewise one can define $\mathcal{O}_{lm}^{\dagger} = a_{lm}^{\dagger}$. One may view this mapping as a holographic map, with l, m being dual variables to the coordinates on the 2-sphere $z, \bar{z}$. The formula (4.16) can then be viewed as the inverse mapping reconstructing the bulk field in terms of boundary operators. The boundary operators will obey their usual commutation relations, and at the same time will transform as a representation of the unitary principal series as described above. All the above is established at the level of free field theory. Once interactions are included it seems difficult to view the resulting boundary theory as any kind of conventional field theory [53].

4.8.2 Holographic Mapping Between Celestial Sphere and 4D Minkowski

This procedure can be extended to the 4D Minkowski case. However we will now have a continuous spectrum of allowed conformal weights Δ corresponding to the continuous spectrum of radial quantum numbers p. For a scalar field we can use the orthogonality of the radial mode functions (4.9) to project onto a particular p eigenvalue and then follow the procedure of the previous subsection to build a boundary operator. The mode expansion of the bulk field is now

$$\Phi(\eta,\rho,z,\bar{z}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{1}^{\infty} dp\left(a_{plm}\Phi_{plm}(\eta,\rho,z,\bar{z}) + a_{plm}^{\dagger}\Phi_{plm}^{\dagger}(\eta,\rho,z,\bar{z})\right)$$

where a_{plm} and $a_{plm}^{\dagger}$ are annihilation and creation operators. Again we can define an operator on the celestial 2-sphere by constructing

$$\mathcal{O}_{\Delta_p,lm} = \langle \Phi_{plm}(\eta,\rho,z,\bar{z}), \Phi(\eta,\rho,z,\bar{z})\rangle = a_{plm}$$

where the operator transforms as a unitary principal series representation parameterized by Δ_p. Likewise one may define a conjugate operator $\mathcal{O}_{\Delta_p,lm}^{\dagger} = a_{plm}^{\dagger}$. We can therefore interpret this as a holographic map between the bulk 4D Minkowski spacetime and the boundary 2D celestial sphere. One ends up with a continuous family of boundary operators labelled by the radial quantum number p. As above, it is not clear how the construction extends to interacting theories.

Chapter 5

Conformal Wavefunctions for Graviton Amplitudes

Building on the results of the previous chapter, it is of interest to study the representations of the $\mathrm{SO}(3,1)$ group associated with gravitons. To reduce the equation of motion to a Schrödinger-like equation it is necessary to impose a non-covariant gauge condition. Using these solutions, leading-order gauge invariant Weyl scalars are then computed and decomposed into families of unitary principal series representations. An invertible holographic mapping is constructed between these unitary principal series operators and massless spin-2 perturbations of flat spacetime.

5.1 Introduction

In the previous chapter, massive scalar fields in 4D Minkowski spacetime were decomposed into modes on 3D de-Sitter spacetime slices where they form unitary principal series representations of $\mathrm{SO}(3,1)$. This study was motivated by the program of [41] where the goal is to formulate gravity in asymptotically flat spacetime as a theory on the celestial sphere with conformal symmetry. In this chapter we extend this construction to massless spin-2 particles, or gravitons, in 4D Minkowski spacetime. To this end, we consider linearized gravitational waves living in flat 4D background spacetime with the standard spherical coordinates. The background metric is simply

$$g_{\mu\nu} = \mathrm{diag}(-1, 1, r^2, r^2 \sin^2\theta)$$

Following the notations in [55], from now on indices $a, b, c, \ldots$ refer to the "orbit" spacetime labeled by the coordinates (t, r), and $i, j, k, \ldots$ refer to the 2-sphere labeled by (θ, φ). In other words, we write the background metric as

$$g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = g_{ab}\, \mathrm{d}y^a \mathrm{d}y^b + r^2\, \mathrm{d}\sigma^2$$

where $g_{ab} = \mathrm{diag}(-1, 1)$ and $\mathrm{d}\sigma^2 = \gamma_{ij}\, \mathrm{d}z^i \mathrm{d}z^j = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2$.

For Minkowski spacetime in four dimensions, the gravitational perturbations can be expanded in terms of both the scalar and the vector spherical harmonics defined on the 2-sphere. These are also known as the “even” and “odd” waves in [56], and will automatically have the desired transformation properties under the rotation group SO(3). However, as we shall see, the specific gauge conditions that we will choose in what follows are not Lorentz covariant, and therefore these metric perturbations do not transform as tensors under the full SO(3, 1) group. To remedy this, we consider the Newman-Penrose formalism [57] of general relativity and construct leading-order gauge-invariant scalars known as the Weyl scalars. These scalars can then be mapped onto the celestial sphere through a generalization of the method described in the previous work [54]. This generalization involves performing a spectral decomposition into radial eigenvalues using the Meijer K-transform [58, 59]. We therefore obtain an invertible holographic map between graviton fields on the flat Minkowski background spacetime and conformal operators on the celestial sphere. We stress that this procedure is defined for tree-level amplitudes and it remains to be seen whether an interacting holographic theory can be defined independently of the four-dimensional gravitational description.

5.2 Graviton Wavefunctions

5.2.1 Scalar Perturbations

Refs. [60, 61] present a general formalism which expresses the metric perturbation $h_{\mu\nu}$ in terms of a master function ϕ. For the scalar perturbation, a gauge choice allows us to express the metric perturbation $h_{\mu\nu}$, expanded in terms of the spherical harmonics Y_{lm}, as

$$h_{ab} = f_{ab} Y_{lm}, \quad h_{ai} = 0, \quad h_{ij} = f\,\gamma_{ij} Y_{lm} \tag{5.1}$$

where f_{ab} and f are functions that are related to a master function $\phi(t, r)$

$$\begin{aligned} f &= \frac{l(l+1)r\phi}{2} + r^2 \partial_r \phi \\ f_{ab} &= \partial_a \partial_b (r\phi) - \frac{g_{ab}}{2} \Box (r\phi) \\ &= r\partial_a\partial_b\phi + \partial_a r\,\partial_b\phi + \partial_b r\,\partial_a\phi - \frac{g_{ab}}{2}(r\Box\phi + 2\partial_r\phi)\,. \end{aligned} \tag{5.2}$$

Here $\partial_a r \partial_b \phi$ stands for $(\partial_a r)(\partial_b \phi)$ and $\Box$ is the d’Alembertian on the orbit spacetime. The components of f_{ab} are

$$\begin{aligned} f_{tt} = f_{rr} &= r\partial_t^2\phi + \partial_r\phi + \frac{l(l+1)}{2r}\phi \\ f_{tr} &= r\partial_t\partial_r\phi + \partial_t\phi\,. \end{aligned}$$

The master function $\phi(t, r)$ satisfies the following master equation

$$\Box\phi - \frac{l(l+1)}{r^2}\phi = 0$$

which can be solved to yield a basis of mode functions

$$\phi_{\omega l}(t,r) = e^{-i\omega t}\sqrt{r}(c_1 J_{l+\frac{1}{2}}(\omega r) + c_2 Y_{l+\frac{1}{2}}(\omega r))$$

for $\omega \neq 0$. Here J_n is the Bessel function of first kind and Y_n is the Bessel function of second kind. We demand that the mode functions be regular at the origin, and therefore discard the second set of solutions. We therefore find the (unnormalized) modes for the master function

$$\phi_{\omega l}(t,r) = e^{-i\omega t}\sqrt{r} J_{l+\frac{1}{2}}(\omega r)\,.$$

For $\omega \neq 0$ this agrees with the master equation found in Ref. [62].

For $\omega = 0$ we have the special time-independent solution of the master equation

$$\phi_{0l}(t,r) = c_1 r^{l+1} + c_2 r^{-l}\,.$$

Discarding solutions that are divergent at $r \to \infty$ we find the basis of functions for the $\omega = 0$ modes

$$\phi_{0l}(t,r) = r^{-l}\,.$$

Physically this represents a time-independent spacetime perturbation that is rotating at a constant angular momentum (for $l \neq 0$. For $l = 0$ the metric perturbation is zero), similar to the eternal Kerr black hole which appears to a distant observer as having a total angular momentum. Substituting the master function into eq. 5.1 we find the scalar metric perturbation for $\omega = 0$

$$h_{\mu\nu} = Y_{lm} \begin{bmatrix} \frac{l(l-1)}{2} r^{-l-1} & 0 & 0 & 0 \\ 0 & \frac{l(l-1)}{2} r^{-l-1} & 0 & 0 \\ 0 & 0 & \frac{l(l-1)}{2} r^{1-l} & 0 \\ 0 & 0 & 0 & \frac{l(l-1)}{2} r^{1-l} \sin^2\theta \end{bmatrix}.$$

5.2.2 Vector Perturbations

We define the vector spherical harmonics as a vector field $Y_i^{(lm)}$ on the unit 2-sphere satisfying

$$[\Delta_2 + (l(l+1) - 1)]Y_i^{(lm)} = 0$$

with $D^i Y^{(lm)}{}_i = 0$. Here Δ_2 and D_i are the Laplace operator and the covariant derivative on the unit 2-sphere, respectively. In terms of the scalar spherical harmonics Y_{lm} we find

$$Y_i^{(lm)}(\theta,\varphi) = \frac{\epsilon_{ij}}{\sqrt{l(l+1)}} \partial^j Y_{lm}(\theta,\varphi)$$

Here indices are raised and lowered using the metric γ_{ij} on the unit 2-sphere, and ϵ^{ij} is the Levi-Civita tensor on the 2-sphere defined by $\epsilon_{\theta\varphi} = \sqrt{|\det\gamma_{ij}|} = \sin\theta$. Given suitable gauge choice [56], the metric perturbation can then be expanded in terms of $Y_i^{(lm)}$ as follows

$$h_{ab} = 0, \quad h_{ai} = f_a Y_i^{(lm)}, \quad h_{ij} = 0 \tag{5.3}$$

where f^a is related to the master function $\phi(t,r)$ above via

$$f^a = \epsilon^{ab}\partial_b(r\phi)\,.$$

Here ϵ_{ab} is the Levi-Civita tensor on the two-dimensional orbit spacetime defined by $\epsilon^{tr} = +1$. For $\omega \neq 0$ this agrees with the so-called odd waves in Ref. [56]. For $\omega = 0$ substituting the master function $\phi = r^{-l}$ into eq. 5.3 we find the metric perturbation

$$h_{\mu\nu} = \sqrt{\frac{l}{l+1}}r^{-l-1}\begin{bmatrix} 0 & 0 & \frac{1}{\sin\theta}\partial_\varphi Y_{lm} & -\sin\theta\,\partial_\theta Y_{lm} \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sin\theta}\partial_\varphi Y_{lm} & 0 & 0 & 0 \\ -\sin\theta\,\partial_\theta Y_{lm} & 0 & 0 & 0 \end{bmatrix}.$$

5.3 Klein-Gordon Inner Product

The metric perturbation expressions constructed above are yet to be normalized. Following [61] we define the inner product between two metric perturbations $h_{\mu\nu}$ and $h'_{\mu\nu}$ as

$$\langle h, h'\rangle = -i\int_\Sigma \mathrm{d}\Sigma\, n_\lambda(h^\star_{\mu\nu}p'^{\lambda\mu\nu} - h'_{\mu\nu}p^{\star\lambda\mu\nu})$$

where Σ is a Cauchy surface and n^λ is the future-directed unit vector field normal to Σ. Here $p^{\lambda\mu\nu}$ is the conjugate momentum current

$$p^{\lambda\mu\nu} = g^{\lambda\nu}\nabla_\kappa h^{\kappa\nu} + g^{\lambda\nu}\nabla_\kappa h^{\kappa\mu} - \nabla^\lambda h^{\mu\nu} + g^{\mu\nu}(\nabla^\lambda h - \nabla^\kappa h^\lambda{}_\kappa) - \frac{g^{\lambda\nu}\nabla^\mu h + g^{\lambda\mu}\nabla^\nu h}{2}$$

Here all indices are raised and lowered with respect the the background metric $g_{\mu\nu}$ and ∇ is the covariant derivative of the background metric.

5.3.1 Scalar Perturbations

For the scalar perturbation Ref. [61] eq. 88 has shown that for $h^{\omega lm}_{\mu\nu}$ and $h^{\omega' l' m'}_{\mu\nu}$, derived from the master function $\phi_{\omega l}$ and $\phi_{\omega' l'}$ via eq. 5.1 respectively, the conserved inner product is

$$\langle h^{\omega lm}, h^{\omega' l' m'}\rangle = -i\int_0^{+\infty}\mathrm{d}r\,\delta_{ll'}\delta_{mm'}\,J^0$$

where the orbit spacetime current J^a is given by

$$J^a = \frac{4}{r}\partial^c r(f^{\star ab}f'_{bc} - f^{ab}f'^\star_{bc}) - (f^{\star bc}\partial^a f'_{bc} - f'^{bc}\partial^a f^\star_{bc})$$

Here f_{ab} is related to the master function $\phi_{\omega l}$ and f'_{ab} is related to the master function $\phi_{\omega' l'}$ via eq. 5.2. Substituting eq. 5.2 and following the derivations in eqs. 90–100 and Appendix B of Ref. [61] we find

$$J^0 = -\frac{l(l-1)(l+1)(l+2)}{2}(\phi^\star_{\omega l}\partial_t\phi_{\omega' l'} - \phi_{\omega' l'}\partial_t\phi^\star_{\omega l})\,.$$

For $\phi_{\omega l} = e^{-i\omega t}\sqrt{r}J_{l+\frac{1}{2}}(\omega r)$ we have

$$i\,(\phi^\star_{\omega l}\partial_t\phi_{\omega' l} - \phi_{\omega' l}\partial_t\phi^\star_{\omega l})|_{t=0} = (\omega+\omega')rJ_{l+\frac{1}{2}}(r\omega)J_{l+\frac{1}{2}}(r\omega')\,.$$

Using

$$\int_0^\infty rJ_\alpha(\omega r)J_\alpha(\omega' r)\,\mathrm{d}r = \frac{\delta(w-w')}{\omega}$$

we find

$$\langle h^{\omega lm}, h^{\omega' l' m'}\rangle = l(l-1)(l+1)(l+2)\delta(\omega-\omega')\delta_{ll'}\delta_{mm'}\,.$$

We see that in order to normalize the metric perturbations to have

$$\langle h^{\omega lm}, h^{\omega' l' m'}\rangle = \delta(\omega-\omega')\delta_{ll'}\delta_{mm'}$$

we will perform a change of variable $h_{\mu\nu} \to [l(l-1)(l+1)(l+2)]^{-1/2}h_{\mu\nu}$. In the interest of notational clarity we will assume this has been done and will continue to use $h_{\mu\nu}$ to denote the normalized scalar metric perturbation.

5.3.2 Vector Perturbations

For the vector perturbations Ref. [61] eq. A10 has shown that for $h^V_{\mu\nu}$ and ${h'}^V_{\mu\nu}$ given by master functions $\phi_{\omega lm}$ and $\phi_{\omega' l' m'}$ (via eq. 5.3) we have

$$\langle h^V, {h'}^V\rangle = -i\,\delta_{ll'}\delta_{mm'}(l-1)(l+2)\int_0^{+\infty}\mathrm{d}r\,(\phi^\star_{\omega l}\partial_t\phi_{\omega' l'} - \phi_{\omega' l'}\partial_t\phi^\star_{\omega l})\,.$$

Following the calculations of the previous section we find

$$\langle h^V, \tilde{h}^V\rangle = 2(l-1)(l+2)\delta(\omega-\omega')\delta_{ll'}\delta_{mm'}\,.$$

We see that a change of variable $h^V_{\mu\nu} \to [2(l-1)(l+2)]^{-1/2}h_{\mu\nu}$ will allow us to normalize the vector metric perturbations to have

$$\langle (h^V)^{\omega lm}, (h^V)^{\omega' l' m'}\rangle = \delta(\omega-\omega')\delta_{ll'}\delta_{mm'}\,.$$

5.4 Gauge Invariant Observables

The solutions we have described above depend on the choice of gauge (5.1). Since our graviton modes are expanded in terms of scalar or vector spherical harmonics, we expect that they transform under the rotation group SO(3) in the usual way. Indeed, denoting $(h^S)^{\omega lm}_{\mu\nu}$ by $|\omega lm\rangle$, we find that

$$L_3\,|\omega lm\rangle = m\,|\omega lm\rangle$$

and

$$(L_1 \pm iL_2)\,|\omega lm\rangle = \sqrt{(l\mp m)(l\pm m+1)}\,|\omega l, m\pm 1\rangle$$

where the rotation operators L_i act on the graviton modes as

$$L_i h_{\mu\nu} = -i \left[\epsilon_{ijk} x^j \partial^k h_{\mu\nu} + (\omega_{ij})_\mu{}^\lambda h_{\lambda\nu} + (\omega_{ij})_\nu{}^\lambda h_{\mu\lambda} \right]$$

with

$$(\omega_{\mu\nu})_\alpha{}^\beta = \eta_{\mu\alpha} \delta^\beta_\nu - \eta_{\nu\alpha} \delta^\beta_\mu \,.$$

However, since the gauge conditions (eqs. 5.1,5.3) for either the scalar and vector perturbations are not Lorentz covariant, we do not expect that these graviton modes transform as representations of the full Lorentz group SO$(3,1)$. Indeed, acting with the boost operators

$$K_i h_{\mu\nu} = -i \left[(x^0 \partial^i - x^i \partial^0) h_{\mu\nu} + (\omega_{0i})_\mu{}^\lambda h_{\lambda\nu} + (\omega_{0i})_\nu{}^\lambda h_{\mu\lambda} \right]$$

does not produce the correct Lorentz algebra. In order to restore the correct Lorentz algebra, one needs to perform additional gauge transformations after a boost to restore the gauge conditions (eqs. 5.1,5.3). The exact gauge transformations required are non-trivial and do not have a closed expression as far as we know.

Alternatively, one could consider the transverse-traceless gauge that is indeed Lorentz covariant and rewrite our graviton modes in this gauge, and attempt to quantize such a theory with a procedure similar to the Gupta-Bleuler formalism of electromagnetism. However this procedure does not provide a complete gauge fixing, and one is still left with a Hilbert space containing zero-norm states. Neither of these approaches will be completely satisfactory in producing a conformal description of the graviton modes as gauge invariant operators on the celestial sphere.

In the context of AdS/CFT the usual procedure would be to adopt Fefferman-Graham coordinates where one can simply identify components of the metric expansion around spatial infinity with a boundary stress-energy tensor. The boundary stress-energy tensor then provides a complete description of the boundary data for gravitational waves.

Finding an analogous set of variables in the case of asymptotically flat spacetime is a somewhat more thorny problem. One approach is simply to choose a Bondi metric near null infinity [63, 64] and describe the gravitational waves using the asymptotic variables that appear there. Another approach, used in the numerical study of gravitational waves from time-dependent collapsing/colliding objects, is to instead pick a distinguished tetrad and compute the so-called Weyl scalars (see for example [65, 66]. An infinitesimal gauge transformation of the Riemann tensor is

$$\delta R^\alpha_{\mu\nu\beta} = \mathcal{L}_\xi R^\alpha_{\mu\nu\beta}$$

where ξ parameterizes the diffeomorphism and $\mathcal{L}_\xi$ is the Lie derivative. Since the Lie derivative is linear in $R^\alpha_{\mu\nu\beta}$ and linear in ξ this will vanish at leading order in ξ if the Riemann tensor is computed at linear order in the perturbation around flat spacetime. So the Weyl scalars, which amount to picking particular components of the Riemann tensor in this context, will be a set of gauge invariant observables at leading order.

We are therefore led to the consideration of the so-called spin-coefficient formalism [67] of general relativity, a special example of which is known as the Newman-Penrose formalism [57]. Here one picks a null tetrad satisfying

$$l_\mu l^\mu = n_\mu n^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = 0$$

normalized so that

$$l_\mu n^\mu = -1, \qquad m_\mu \bar{m}^\mu = 1$$

with other cross contractions between two vectors vanishing. For our specific purpose we will pick the limit of the Kinnersley tetrad [68]

$$\begin{aligned} l^\mu &= (1,1,0,0) \ , & n^\mu &= \left(\frac{1}{2}, -\frac{1}{2}, 0, 0\right) \ , \\ m^\mu &= \frac{1}{\sqrt{2}r}\left(0,0,1,\frac{i}{\sin\theta}\right) \ , & \bar{m}^\mu &= \frac{1}{\sqrt{2}r}\left(0,0,1,-\frac{i}{\sin\theta}\right) \ . \end{aligned} \tag{5.4}$$

The five Weyl scalars Ψ_i for $i = 0, \ldots, 4$ are built out of the Weyl tensor $C_{\alpha\beta\gamma\delta}$ of the full spacetime as

$$\Psi_0 = C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma m^\delta \tag{5.5}$$

$$\Psi_1 = C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta l^\gamma m^\delta \tag{5.6}$$

$$\Psi_2 = C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta \bar{m}^\gamma n^\delta \tag{5.7}$$

$$\Psi_3 = C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar{m}^\gamma n^\delta \tag{5.8}$$

$$\Psi_4 = C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta n^\gamma m^\delta \, . \tag{5.9}$$

Here we expand $C_{\alpha\beta\gamma\delta}$ to first order in $h_{\mu\nu}$. Since the full spacetime satisfies the vacuum Einstein equation, we find that the Weyl tensor $C_{\alpha\beta\gamma\delta}$ is equal to the Riemann tensor $R_{\alpha\beta\gamma\delta}$. The Weyl scalars, gauge invariant under infinitesimal gauge transformations, may then be expressed in terms of the graviton wavefunctions of the previous section. Our strategy will then be to decompose these gauge invariant scalars into representations of the conformal group.

Note that for a particular value of l, m it would be possible to find modification of the tetrad, at leading order in the perturbation, such as picking l along a principal null direction, that would make some of the Weyl scalars vanish. However if make pick the tetrad independently of the perturbation, all the Weyl scalars will typically be non-vanishing, and will provide a basis for gauge invariant observables (at leading order).

A straightforward calculation leads to the expressions for the Weyl scalars evaluated on the scalar and vector perturbations in appendices A and B respectively. These expressions involve radial derivations of the master function $\phi(t,r)$ and angular derivatives of the spherical harmonics. For us, the main point is that the Weyl scalars do not satisfy a simple wave equation in 4D spacetime. In order to use the methods of [54] we next must perform a spectral decomposition in the radial direction to allow us to use that basis functions as a complete basis for the Weyl scalars.

5.5 Holographic Mapping to the Celestial Sphere

The Weyl scalars encode all the information of the gravitational perturbations. Our strategy will be to proceed in two steps: first consider a fixed radius 3D de Sitter slice of flat spacetime, and use the Plancherel (or completeness) theorem for the unitary principle series representations to decompose a general function on such a slice into irreducible representations; next we allow for a general radial variation of such functions, effectively decomposing a general solution into solutions of the 4D massive scalar wave equations with a continuous spectrum of masses.

The starting point is the unitary principal series mode functions that we have computed in [54]

$$\Phi_{pMlm}(\eta, \rho, z, \bar{z}) = \phi_{pl}(\eta)\, \psi_{pM}(\rho)\, Y^l_m(z, \bar{z})$$

where

$$\phi_{pl}(\eta) = \operatorname{sech}\eta \left[\frac{i\pi}{2} P_l^{i\sqrt{p^2-1}}(\tanh\eta) + Q_l^{i\sqrt{p^2-1}}(\tanh\eta)\right]$$

and

$$\psi_{pM}(\rho) = \frac{K_{i\sqrt{p^2-1}}(M\rho)}{\rho}\,.$$

Here M is the 4D scalar mass, l, m are the usual angular momentum quantum numbers, and p labels the unitary principal series representation and also behaves as a radial quantum number. $K_\nu(x)$ is the modified Bessel function of second kind, and $(\eta, \rho, z, \bar{z})$ are the hyperbolic coordinates on Minkowski spacetime with metric

$$\mathrm{d}s^2 = -\rho^2 \mathrm{d}\eta^2 + \mathrm{d}\rho^2 + \rho^2 \cosh^2\eta\, \frac{4\,\mathrm{d}z\,\mathrm{d}\bar{z}}{(1+|z|^2)^2}\,.$$

We will apply the main result of Ref. [54], which states that these modes Φ_{pMlm} form a unitary principal series representation of $\mathrm{SO}(3,1)$. This allows us to apply the Plancherel theorem [69] and map the Weyl scalars into sets of conformal operators defined on the celestial sphere. We now discuss this procedure in detail.

To motivate the full 4D map, let us first consider the simpler case where one is to construct a holographic map on the 3D de-Sitter slice (which we take to be the $\rho = 1$ hypersurface) of 4D Minkowski. On the 3D de-Sitter slice, the mode functions Φ_{pMlm} above reduce to the mode functions $\phi_{plm}(\eta, z, \bar{z}) = \phi_{pl}(\eta) Y_{lm}(z, \bar{z})$. We can then use these modes as basis for the analogue of Fourier transform, and maps a scalar function $f(\eta, z, \bar{z})$ into "Fourier" coefficients labeled by $\hat{f}(p, l, m)$

$$\hat{f}(p, l, m) = \int_{-\infty}^{+\infty} \cosh^2\eta\, \mathrm{d}\eta \int_{\mathbb{C}} \frac{4\mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2}\, \bar{\phi}_{pl}(\eta)\, \bar{Y}_{lm}(z, \bar{z})\, f(\eta, z, \bar{z})\,.$$

The Plancherel theorem [69] guarantees that this map is unitary. Using eq. 10.40 of Ref. [69], we find the inverse map

$$f(\eta, z, \bar{z}) = \int_1^{\infty} 2p \tanh\left(\frac{\pi}{2}\sqrt{p^2-1}\right) \mathrm{d}p \sum_{lm} \phi_{pl}(\eta)\, Y_{lm}(z, \bar{z})\, \hat{f}(p, l, m)\,.$$

In particular, the integral measure of the inverse map follows from setting $v = \sqrt{p^2 - 1}$ in the expression $v \tanh(\pi v/2)\,\mathrm{d}v$ of eq. 10.40 in Ref. [69], and multiplying it by two, since both $p \in (1, +\infty)$ and $p \in (-\infty, -1)$ map to $v \in (0, +\infty)$.

In order to include the radial variation of the scalars, we note the fact that the following Meijer K-transform of order ν, defined for a function $f(x)$ as

$$\hat{f}(y) = \int_0^\infty f(x) K_\nu(xy)\,(xy)^{1/2}\,\mathrm{d}x$$

has the following inverse transform

$$f(x) = \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} \hat{f}(y) I_\nu(xy)\,(xy)^{1/2}\,\mathrm{d}y$$

where c is an arbitrary real number and $I_\nu(x)$ is the modified Bessel function of the first kind. This is developed in a series of papers [58, 59, 70, 71] and summarized in Chapter X of Ref. [72], which we simply quote without going into further details of the proof. This gives us a unitary transformation for the radial component, from which we obtain the full 4D forward and inverse map: given a scalar function $\Psi(\eta, \rho, z, \bar{z})$, we define the following analogue of the Fourier transform on four-dimensional Minkowski spacetime

$$\hat{\Psi}(p, M, l, m) = \int_{-\infty}^{+\infty} d\eta\, \bar{\phi}_{pl}(\eta) \cosh^2\eta \int_0^\infty d\rho K_{i\sqrt{p^2-1}}(M\rho)(M\rho)^{1/2} \int_{\mathbb{C}} \frac{4\mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2} \bar{Y}_{lm}(z,\bar{z})\,\Psi(\eta, \rho, z, \bar{z}) \tag{5.10}$$

that maps any scalar function $\Psi(\eta, \rho, z, \bar{z})$ to "Fourier" coefficients $\hat{\Psi}$ labeled by (p, M, l, m), where $p > 1$, $M > 0$, $l \geq 0$, and $-l \leq m \leq l$. We note this ρ integral is indeed convergent. This transformation is invertible, with the inverse map given by

$$\begin{aligned}\Psi(\eta, \rho, z, \bar{z}) = \frac{1}{\pi i} \int_1^{+\infty} 2p \tanh\left(\frac{\pi}{2}\sqrt{p^2-1}\right) \mathrm{d}p \int_{-i\infty}^{+i\infty} I_{i\sqrt{p^2-1}}(M\rho)(M\rho)^{1/2}\,\mathrm{d}M \\ \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \phi_{pl}(\eta)\, Y_{lm}(z,\bar{z}) \hat{\Psi}(p, M, l, m)\end{aligned} \tag{5.11}$$

Here $I_\nu(x)$ is the modified Bessel function of the first kind.

We are now in a position to apply the forward and inverse map described above to each of the Weyl scalars of the gravitational modes computed earlier. This allows us to identify two families of celestial sphere operators that encode the 5 complex Weyl scalars for each variety of perturbation (scalar and vector) which we can label as $\hat{\Psi}^{S,\alpha}_{pMlm}$ and $\hat{\Psi}^{V,\alpha}_{pMlm}$. Here S, V refer to scalar and vector, and $\alpha = 0, \cdots, 4$ labels the 5 Weyl scalars. The l, m angular momentum space is conjugate to the 2-sphere coordinate space $z, \bar{z}$ so in this sense we obtain a holographic mapping of the gravitational modes to the celestial sphere.

These celestial sphere operators will automatically have the desired conformal transformation properties as shown in [54]. We therefore find that the procedure described above allows us to build a celestial sphere description of the gravitational modes living in the Minkowski bulk spacetime at leading order. To further illustrate this point we summarize the flat spacetime holographic map that we have constructed above in the following section.

5.6 Discussion

What we have constructed above is a holographic map between gravitons in 4D Minkowski background and conformal operators in the 2D boundary of 4D Minkowski known as the celestial sphere. The forward map, going from a metric perturbation $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ in 4D to conformal operators $\{\hat{\Psi}^{\alpha}_{pMlm}\}$ in 2D celestial sphere, is the following

1. Compute the Riemann tensor $R_{\alpha\beta\gamma\delta}$ to first order in $h_{\mu\nu}$. Since the full spacetime satisfies the vacuum Einstein equation, the Weyl tensor $C_{\alpha\beta\gamma\delta}$ is equal to the Riemann tensor $R_{\alpha\beta\gamma\delta}$

2. From the Weyl tensor, compute the five Weyl scalars $\{\Psi^{\alpha}\}$ using eq. 5.4 and eqs. 5.5–5.9. Here $\alpha = 0, \ldots, 4$.

3. Apply the forward map (eq. 5.10) to the five Weyl scalars, and obtain five conformal operators $\{\hat{\Psi}^{\alpha}_{pMlm}\}$ on the celestial sphere, labeled by four quantum numbers p, M, l, and m.

Likewise, the inverse map, starting from the complete set of conformal operators $\{\hat{\Psi}^{\alpha}_{pMlm}\}$, is to apply eq. 5.11 to obtain the five (complex) Weyl scalars $\{\Psi^{\alpha}(\eta, \rho, z, \bar{z})\}$. These five complex Weyl scalars encode the ten independent components of the Weyl tensor, which fully specify the metric perturbation $h_{\mu\nu}$, up to coordinate (gauge) choices. In particular, one can apply the holographic map above to the case of individual gravitational wave modes (there are two types, scalar and vector perturbations, see eqs. 5.1–5.2 and eq. 5.3) and obtain 2D conformal operators that describe 4D gravitons in flat spacetime. The significance of all these formalisms developed in this chapter is precisely to allow us to write down a simple procedure that holographically map gravitons living in flat 4D spacetime, to 2D conformal operators living on the celestial sphere.

We stress that this simple procedure amounts to use kinematic information to organize the modes in a convenient way. With interactions included, one can still use this procedure to map bulk dynamics to the celestial sphere. The important question then is whether the dynamics has any useful description incorporating the conformal symmetry of the celestial sphere as in the program advocated in [41]. In the case of AdS/CFT the analogous answer was the holographic theory was simpler than the gravity theory, being a quantum field theory with conformal symmetry. In the case of asymptotically flat spacetime it remains unclear whether the celestial sphere theory is a quantum field theory. It remains a logical possibility that the 4D gravitational description will be the simplest way to describe dynamics of the theory.

Appendix A

Supplementary Materials for Chapter 2

A.1 Correlators and Conformal Blocks

Conformal blocks are usually written in terms of 4-point functions after a global $SL(2,C)$ conformal transformation has sent generic points in the complex plane to the values $0, z, 1, \infty$. Here we briefly unpack the relation between these conformal blocks and 4-point functions for general z_i.

A canonical form for the 4-point function at general z_i in the complex plane is [73]

$$\left\langle \prod_{i=1}^{4} \mathcal{O}_i(z_i) \right\rangle = f(z,\bar{z}) \prod_{i<j} z_{ij}^{-(h_i+h_j)+h/3} \prod_{i<j} \bar{z}^{-(\bar{h}_i+\bar{h}_j)+\bar{h}/3} \tag{A.1}$$

where $z_{ij} = z_i - z_j$, the cross-ratio $z = z_{12}z_{34}/z_{13}z_{24}$ and $h = \sum_i h_i$. The conformal block on the other hand is usually defined [74] for the special choice $z_i = 0, z, 1, \infty$. To define the correlator as the point z_4 moves to infinity we must rescale by a factor of $z_4^{2h_w}$

$$\lim_{z_4 \to \infty} z_4^{2h_w} \bar{z}_4^{2\bar{h_v}} \left\langle \prod_{i=1}^{4} \mathcal{O}_i(z_i) \right\rangle = \sum_p C_{12p} C_{34p} \mathcal{F}(p,z) \bar{\mathcal{F}}(p,\bar{z})\,. \tag{A.2}$$

Comparing the two formulae yields

$$\begin{aligned} \lim_{z_4 \to \infty} z_4^{2h_w} \bar{z}_4^{2\bar{h_v}} \left\langle \prod_{i=1}^{4} \mathcal{O}_i(z_i) \right\rangle \Bigg|_{z_1=0, z_3=1, z_2=z} &= f(z,\bar{z})\,(1-z)^{h/3-h_2-h_3}\; z^{h/3-h_1-h_2} \\ &\quad \times (1-\bar{z})^{\bar{h}/3-\bar{h}_2-\bar{h}_3}\; \bar{z}^{\bar{h}/3-\bar{h}_1-\bar{h}_2} \\ &= \sum_p C_{12p} C_{34p} \mathcal{F}(p,z) \bar{\mathcal{F}}(p,\bar{z}) \end{aligned} \tag{A.3}$$

and we see the canonical form of the 4-point function involves a nontrivial rescaling of the conformal block by a function of the cross-ratio.

Later when we study the commutator of two operators, V and W as a function of time, it will be convenient to factor out the norm of the operators. To accomplish this we compute

$$\frac{\langle V(z_1)V(z_2)W(z_3)W(z_4)\rangle}{\langle V(z_1)V(z_2)\rangle\,\langle W(z_3)W(z_4)\rangle} = z^{2h_v}\,\bar{z}^{2\bar{h}_v}\sum_p C_{12p}C_{34p}\mathcal{F}(p,z)\bar{\mathcal{F}}(p,\bar{z}) \tag{A.4}$$

using (A.1) and (A.3). Now the expression for general z_i is a function only of the cross-ratios. Finally we note that in performing a coordinate transformation to a different coordinate system, each correlator of primaries transforms by

$$\left\langle \prod_i \mathcal{O}(x_i)\right\rangle = \prod_i \left(\frac{\partial z}{\partial x}\right)^{h_i}_{z=z_i}\left(\frac{\partial \bar{z}}{\partial \bar{x}}\right)^{\bar{h}_i}_{\bar{z}=\bar{z}_i}\left\langle \prod_i \mathcal{O}(z_i)\right\rangle$$

and these factors cancel in the expression (A.4).

Appendix B

Supplementary Materials for Chapter 4

B.1 Euclidean Vacuum

The closed form expression of the positive-frequency Euclidean modes has been computed in Ref. [75], eq. 3.37, which we reproduce below in the interest of being self-contained. Translating the notations of Ref. [75] to our notations, the time-dependent component of the (unnormalized) Euclidean modes of the 3D de Sitter spacetime is

$$\phi_l^E(t) = (\cosh^l t)\, e^{(l+1+i\sqrt{\mu^2-1})t}\, {}_2\mathrm{F}_1(l+1, l+1+i\sqrt{\mu^2-1}; 2l+2; 1+e^{2t})\,. \tag{B.1}$$

The identities [76] and [77] may be used provided we continue t to the complex plane. It is then straightforward to verify that the linear combination of modes in (4.3)

$$\phi_l^E(t) \propto \frac{i\pi}{2}\phi_{l,1}(t) + \phi_{l,2}(t)$$

are the (unnormalized) positive-frequency modes corresponding to the Euclidean vacuum of the 3D de Sitter spacetime upon continuing t to real values. These when uplifted to the 4d Minkowski spacetime will correspond to the Minkowski vacuum, since both are distinguished by the fact that their Wightman function has a Hadamard singularity [78]. The normalization factor is determined in Section 4.5.

Appendix C

Supplementary Materials for Chapter 5

C.1 Weyl Scalars for Scalar Perturbation

The Weyl scalars are evaluated at linear order for the scalar perturbation mode $h^{\omega lm}_{\mu\nu}$ defined above. Here we use $\phi' = \frac{\partial}{\partial r}\phi$.

$$
\begin{aligned}
\Psi_0 =& -\frac{[l^2+l-2r\omega(r\omega+i)]\phi + 2r(1-ir\omega)\phi'}{4r^3}\Bigg[\sqrt{(l-m)(l+m+1)(l+m+2)}e^{-i2\varphi}Y_l^{m+2} \\
&+(m-1)m\tan^2\left(\frac{\theta}{2}\right)Y_l^m - 2me^{-i\varphi}\tan\left(\frac{\theta}{2}\right)\sqrt{(l-m)(l+m+1)}Y_l^{m+1}\Bigg] \\
\Psi_1 =& \frac{e^{-i\varphi}}{4\sqrt{2}r^3}\left\{\left(l^2+l-2\right)r\phi' + 2r^2(1-ir\omega)\phi'' + i\left[\left(l^2+l+2\right)r\omega + 2il(l+1) - 2r^3\omega^3 - 2ir^2\omega^2\right]\phi\right\} \\
&\left[\sqrt{(l-m)(l+m+1)}Y_l^{m+1} - me^{i\varphi}\tan\left(\frac{\theta}{2}\right)Y_l^m\right] \\
\Psi_2 =& -\frac{1}{8r^3}\left[2r^3\phi''' + l(l+1)r^2\phi'' - 2r\left(l^2+l-r^2\omega^2\right)\phi' + l(l+1)\left(r^2\omega^2+2\right)\phi\right]Y_l^m \\
\Psi_3 =& \frac{e^{-i\varphi}}{8\sqrt{2}r^3}\left\{\left[i\left(l^2+l+2\right)r\omega + 2l(l+1) - 2ir^3\omega^3 - 2r^2\omega^2\right]\phi - \left(l^2+l-2\right)r\phi' - 2r^2(1+ir\omega)\phi''\right\} \\
&\left[\sqrt{(l-m)(l+m+1)}Y_l^{m+1} + me^{i\varphi}\cot\left(\frac{\theta}{2}\right)Y_l^m\right] \\
\Psi_4 =& -\frac{e^{-i2\varphi}}{16r^3}\left\{\left[l^2+l-2r\omega(r\omega-i)\right]\phi + 2r(1+ir\omega)\phi'\right\}\Bigg[\sqrt{(l-m)(l+m+1)(l+m+2)}Y_l^{m+2} \\
&+2me^{i\varphi}\cot\left(\frac{\theta}{2}\right)\sqrt{(l-m)(l+m+1)}Y_l^{m+1} + (m-1)me^{2i\varphi}\cot^2\left(\frac{\theta}{2}\right)Y_l^m\Bigg].
\end{aligned}
$$

C.2 Weyl Scalars for Vector Perturbation

The Weyl scalars are evaluated at linear order for the vector perturbation mode $h_{\mu\nu}^{\omega lm}$ defined above. Here we use $\phi' = \frac{\partial}{\partial r}\phi$. For brevity of presentation we follow [56] and present the $m = 0$ modes. The $m \neq 0$ modes can always be obtained by acting with an SO(3) generator.

$$
\begin{aligned}
\Psi_0 =& \frac{\sqrt{(l-1)(l+2)}e^{-i2\varphi}}{2r^2}\left[ir\phi'' + 2(r\omega + i)\phi' + \omega\phi(2 - ir\omega)\right] Y_l^2 \\
\Psi_1 =& -\frac{e^{-i\varphi}}{2\sqrt{2}r^2}\left\{i\left[\left(r^2\omega^2 - 2\right)\phi' + r^2\phi''' + (2 - ir\omega)r\phi''\right] + \omega\phi(r\omega - 1 + i)(r\omega + 1 + i)\right\} Y_l^1 \\
\Psi_2 =& -\frac{ie^{-i2\varphi}\left[r^2\phi'' + \phi\left(r^2\omega^2 - 2\right)\right]}{4r^3}\left[\sqrt{(l-1)(l+2)}Y_l^2 + 2e^{i\varphi}\cot(\theta)Y_l^1\right] \\
\Psi_3 =& \frac{e^{-i\varphi}}{4\sqrt{2}r^2}\left\{i\left[\left(r^2\omega^2 - 2\right)\phi' + r^2\phi''' + (2 + ir\omega)r\phi''\right] + \omega\phi[2 + r\omega(-r\omega + 2i)]\right\} Y_l^1 \\
\Psi_4 =& \frac{\sqrt{(l-1)(l+2)}e^{-i2\varphi}}{8r^2}\left[ir\phi'' + (-2r\omega + 2i)\phi' - \omega\phi(2 + ir\omega)\right] Y_l^2 \,.
\end{aligned}
$$

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