Classification with Naive Bayes

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Mathematical basics

Random experiment

Statistics is about the probability of events:

Example: How likely is it to have six correct numbers in the lottery?

Random experiment: Experiment (trial) with several possible outputs (throw of two dices)

Outcome: Result of an experiment (3 eyes on dice 1 and 4 eyes on dice 2)

Sample space Ω : Set of all possible outcomes

Event: Subset of the sample space (7 eyes on two dices)

Sample: Series of results in a repeated experiment

Probability distribution

Probability distribution: Function that assigns a value between 0 and 1 to each outcome, such that

$$\sum_{o\in\Omega}p(o)=1$$

The probability of a event is the sum of the probabilities of the corresponding outcomes.

Example:

probability that the number of eyes when throwing a dice is an even number

Conditional and a priori probability

Conditional probability: Probability of an event A, if the event B is known:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example: Probability that the number of points in a dice is even if the number of points is greater than 3

A priori probability P(A): Probability of event A without knowing event B

Random variables

Random variable: Function that assigns a real number to each outcome.

Example: Mapping of grades very good, good, satisfactory, sufficient, poor, insufficient to the numbers 1, 2, 3, 4, 5, 6

Example: Mapping of grades *very good, good, satisfactory, sufficient, poor, insufficient* to the numbers 0, 1 (non-pass, pass)

A random variable is called discrete if it takes only a finite number or countably infinite values.

The above examples thus describe a discrete random variables.

Probability of a value x of the random variable X:

$$P(X = x) = p(x) = P(A_x)$$

A random variable with only the values 0 and 1 is called **Bernoulli experiment**.

Joint distributions and marginal distributions

The **joint distribution** of two random variables X and Y:

$$p(x,y) = P(X = x, Y = y) = P(A_x \cap A_y)$$

The **marginal distribution** of two random variables X and Y:

$$p_X(x) = \sum_y p(x, y)$$
 $p_Y(y) = \sum_x p(x, y)$

Independence: The random variables X and Y are statistically independent if:

$$p(x,y) = p_X(x)p_Y(y)$$

Example: When throwing two dice, their numbers are statistically independent to each other.

Important rules

Chain rule: A joint probabilities can be converted into a product of conditional probabilities.

$$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1)P(A_2|A_1)...P(A_n|A_1 \cap ... \cap A_{n-1})$$
$$= \prod_{i=1}^n P(A_i|A_1 \cap ... \cap A_{i-1})$$

Theorem of Bayes: allows to "reverse" a conditional probability

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Probability estimation

$$\tilde{p}(x) = \frac{n(x)}{N}$$

The **relative frequency** n(x)/N is the number of occurrences (*counts*) n(x) an event x divided by the sample size n.

With growing sample size n, the relative frequency converges to the actual probability of an event.

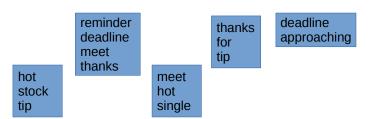
More precisely: the probability that the relative frequency differs more than ϵ from the actual probability converges to 0 for increasing sample size.

Probability estimation by relative frequency

Example:

- Random event: word occurrence is a specific word
- n(x): Number of occurrences (counts) of the word in a corpus
- N: Number of word occurrences in the corpus.

word	n(word)	$\tilde{p}(word)$
meet		
deadline		
single		

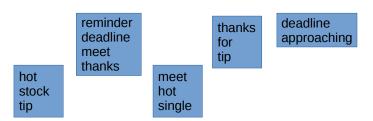


Probability estimation by relative frequency

Example:

- Random event: word occurrence is a specific word
- n(x): Number of occurrences (counts) of the word in a corpus
- N: Number of word occurrences in the corpus.

word	n(word)	$\tilde{p}(word)$
meet	2	$\frac{2}{15} \approx 0.133$
deadline	2	$\frac{2}{15} \approx 0.133$
single	1	$\frac{1}{15} \approx 0.067$



Relative frequency for conditional probabilities

$$\tilde{p}(x|y) = \frac{n(x,y)}{n_y}$$

Conditional probabilities can also be estimated from relative frequencies. n(x, y) here is the number of common occurrences of the events x and y. n_y is the number of occurrences of the event y.

It applies: $n_y = \sum_{x'} n(x', y)$

Relative frequency for conditional probabilities

- Random event x: Word occurrence is a certain word
- Random event y: Word occurrence is in email of a certain category,
 e.g. HAM or SPAM (HAM = "no spam")
- n(x, y): Number of word occurrences in emails of a category in the corpus

word	n(word, HAM)	$\tilde{p}(word HAM)$	n(word, SPAM)	$\tilde{p}(word SPAM)$
meet				
deadline				
single				

reminder deadline meet thanks

hot stock tip

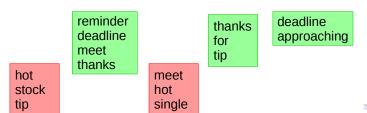
thanks for tip

thanks approaching

Relative frequency for conditional probabilities

- Random event x: Word occurrence is a certain word
- Random event y: Word occurrence is in email of a certain category,
 e.g. HAM or SPAM (HAM = "no spam")
- n(x, y): Number of word occurrences in emails of a category in the corpus

word	n(word, HAM)	$\tilde{p}(\text{word} \text{HAM})$	n(word, SPAM)	$\tilde{p}(\text{word} \text{SPAM})$
meet	1	$\frac{1}{9} \approx 0.111$	1	$\frac{1}{6} \approx 0.167$
deadline	2	$\frac{2}{9} \approx 0.222$	0	0
single	0	Ō	1	$\frac{1}{6} \approx 0.167$



Probability for word sequence

- So far we have only expressed and estimated probabilities of single words.
- How can we calculate the probabilities of whole texts (e.g. emails)?
- Application of conditional probability:

$$P(w_1, w_2, ..., w_n)$$

$$= P(w_1)P(w_2|w_1)P(w_3|w_1, w_2)...P(w_n|w_1...w_{n-1})$$

• \Rightarrow does not really solve the problem, because $P(w_n|w_1 \dots w_{n-1})$ can not be well estimated

Independence assumption: Bag of Words

- One solution: we make the statistical assumption that every word is independent of the occurrence of other words.
- This is also called bag-of-words (BOW) assumption, because the order of words becomes irrelevant.

$$P(w_1, w_2, ..., w_n)$$
= $P(w_1)P(w_2|w_1)P(w_3|w_1, w_2)...P(w_n|w_1...w_{n-1})$
= $P(w_1)P(w_2)P(w_3)...P(w_n)$
indep.

Conditional independence

- For many machine-learning algorithms, conditional independence is the central concept:
 - If the value of a random variable y is known, random variables x_1, \ldots, x_n are independent
- Middle ground between:
 - no independence
 - independence of all random variables
- In our case:

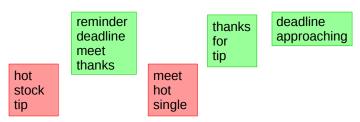
$$P(w_1, w_2, \ldots, w_n | SPAM)$$

 $= P(w_1|SPAM)P(w_2|SPAM)P(w_3|SPAM)...P(w_n|SPAM)$ cond. indep.

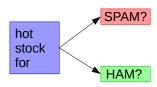
Naive Bayes Classifier

Task

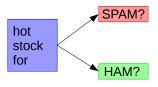
• Given a training corpus:



 Decide whether new (unseen) emails should be assigned to the category HAM or SPAM:



Decision criterion



 Given the content of the email, which category is more likely: SPAM or HAM?

• Why isn't the decision criterion:

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Bayes rule

$$P(HAM|text) = \frac{P(text|HAM) * P(HAM)}{P(text)}$$

- P(text|HAM): conditional BOW probability
- P(HAM): Prior probability that an email is assigned to the category HAM (if the content of the email is not known). Estimation:

$$\tilde{p}(HAM) = \frac{\text{number of HAM-Mails}}{\text{number of all Mails}}$$

 P(text): BOW probability of the content of the email without knowing the category.

Decision criterion

$$\Leftarrow$$

$$\Leftrightarrow$$

$$\frac{P(\textit{HAM}|\textit{text})}{P(\textit{SPAM}|\textit{text})} > 1$$



Decision criterion

Email is HAM

$$\Leftrightarrow P(HAM|text) > P(SPAM|text)$$

$$\Leftrightarrow \frac{P(HAM|text)}{P(SPAM|text)} > 1$$

 \Leftrightarrow

$$\frac{\frac{1}{P(\text{text})}P(\text{text}|\text{HAM})*P(\text{HAM})}{\frac{1}{P(\text{text})}P(\text{text}|\text{SPAM})*P(\text{SPAM})} > 1$$

What is a decision rule for more than two categories?

Example (preliminary)

reminder deadline meet thanks

thanks for tip deadline approaching

hot stock tip meet hot single

- $\tilde{p}(HAM) = \frac{3}{5}$
- $\tilde{p}(SPAM) = \frac{2}{5}$
- p(hot stock for | HAM)

$$= \tilde{p}(\mathsf{hot}|HAM)\tilde{p}(\mathsf{stock}|HAM)\tilde{p}(\mathsf{for}|HAM) = ...$$

p(hot stock for | SPAM)

$$= \tilde{p}(\mathsf{hot}|SPAM)\tilde{p}(\mathsf{stock}|SPAM)\tilde{p}(\mathsf{for}|SPAM) = ...$$

• ...

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Example (preliminary)

reminder deadline meet thanks

thanks for tip deadline approaching

hot stock tip meet hot single

- $\tilde{p}(HAM) = \frac{3}{5}$
- $\tilde{p}(SPAM) = \frac{2}{5}$
- p(hot stock for | HAM)

$$= \tilde{p}(\mathsf{hot}|HAM)\tilde{p}(\mathsf{stock}|HAM)\tilde{p}(\mathsf{for}|HAM) = \frac{0\cdot 0\cdot 1}{9\cdot 9\cdot 9} = 0$$

p(hot stock for | SPAM)

$$= \tilde{p}(\mathsf{hot}|SPAM)p(\mathsf{stock}|SPAM)\tilde{p}(\mathsf{for}|SPAM) = \frac{2 \cdot 1 \cdot 0}{6 \cdot 6 \cdot 6} = 0$$

• Problem: Decision criterion is not defined $\left(\frac{0}{0}\right)_{\square}$

Add-1 smoothing

Add-1 smooting (Laplace smoothing)

$$\widetilde{p}(w) = \frac{n(w) + 1}{N + V}$$

(V = number of possible words; N = number of tokens)

- ... is optimal if the uniform distribution is most likely is what is rarely the case in text corpora \Rightarrow Zipf's distribution
- ... therefore, overestimates the probability of unseen words.

Add- λ smoothing

reduces the amount of smoothing

$\mathbf{Add}\text{-}\lambda\ \mathbf{smoothing}$

$$\tilde{p}(w) = \frac{n(w) + \lambda}{N + V\lambda}$$

Add- λ smoothing for conditional probabilities

$$\tilde{p}(w|y) = \frac{n(w,y) + \lambda}{n_v + V\lambda}$$

Example (with Add-1 smoothing)

reminder deadline meet thanks

thanks for tip deadline approaching

hot stock tip meet single

- $\tilde{p}(HAM) = \frac{3}{5}$, $\tilde{p}(SPAM) = \frac{2}{5}$
- ullet Vocabulary contains V=10 different words
- $p(\text{hot stock for}|HAM) = \tilde{p}(\text{hot}|HAM)\tilde{p}(\text{stock}|HAM)\tilde{p}(\text{for}|HAM)$

$$=\frac{(0+1)\cdot(0+1)\cdot(1+1)}{(9+10)\cdot(9+10)\cdot(9+10)}\approx 0.00029$$

• $p(\text{hot stock for}|SPAM) = \tilde{p}(\text{hot}|SPAM)\tilde{p}(\text{stock}|SPAM)\tilde{p}(\text{for}|SPAM)$

$$=\frac{(2+1)\cdot(1+1)\cdot(0+1)}{(6+10)\cdot(6+10)\cdot(6+10)}\approx 0.00146$$

• $\frac{P(\text{text}|\text{HAM})*P(\text{HAM})}{P(\text{text}|SPAM)*P(SPAM)} = \frac{0.00029 \cdot 0.6}{0.00146 \cdot 0.4} \approx 0.298 \Rightarrow \text{Category?}$

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Example (with Add-1 smoothing)

reminder deadline meet thanks thanks for tip deadline approaching

hot stock tip meet hot single

• $\tilde{p}(HAM) = \frac{3}{5}, \ \tilde{p}(SPAM) = \frac{2}{5}$

- Vocabulary contains v = 10 different words
- $p(\text{hot stock for}|HAM) = \tilde{p}(\text{hot}|HAM)\tilde{p}(\text{stock}|HAM)\tilde{p}(\text{for}|HAM)$

$$=\frac{(0+1)\cdot(0+1)\cdot(1+1)}{(9+10)\cdot(9+10)}\approx 0.00029$$

• $p(\text{hot stock for}|SPAM) = \tilde{p}(\text{hot}|SPAM)\tilde{p}(\text{stock}|SPAM)\tilde{p}(\text{for}|SPAM)$

$$=\frac{(2+1)\cdot(1+1)\cdot(0+1)}{(6+10)\cdot(6+10)\cdot(6+10)}\approx 0.00146$$

• $\frac{P(\text{text}|\text{HAM})*P(\text{HAM})}{P(\text{text}|\text{SPAM})*P(\text{SPAM})} = \frac{0.00029 \cdot 0.6}{0.00146 \cdot 0.4} \approx 0.298 < 1 \Rightarrow \text{Email is spam}$

Calculating with logarithms

- When multiplying many small probabilities (for example, all words in a long text), the result can quickly approach 0 and may not be represented correctly.
- That's why you always avoid the multiplication of probabilities.
- Instead, use the sum of the logarithmized probabilities.
- $\log(a \cdot b \cdot c \cdot \dots) = \log(a) + \log(b) + \log(c) + \dots$
- Example:

Calculating with logarithms

- When multiplying many small probabilities (for example, all words in a long text), the result can quickly approach 0 and may not be represented correctly.
- That's why you always avoid the multiplication of probabilities.
- Instead, use the sum of the logarithmized probabilities.
- $\log(a \cdot b \cdot c \cdot \dots) = \log(a) + \log(b) + \log(c) + \dots$
- Example:

$$0.0001 * 0.001 * 0.00001 * 0.01 = 0.00000000000001$$

$$\log_{10}(0.0001*0.001*0.0001*0.01) = -4 + (-3) + (-5) + (-2) = -14$$

• $\log(\frac{a}{b}) = ?$



Decision rule with logarithms

- The logarithm is increasing monotonically, i.e. we can apply it to inequalities on both sides.
- The decision rule is now:

$$P(HAM|text) > P(SPAM|text)$$
 \Leftrightarrow
 $\log P(HAM|text) > \log P(SPAM|text)$
 \Leftrightarrow

Decision rule with logarithms

 The logarithm is increasing monotonically, i.e. we can apply it to inequalities on both sides.

P(HAM|text) > P(SPAM|text)

 $\log P(HAM|text) > \log P(SPAM|text)$

• The decision rule is now:

$$\Leftrightarrow \log P(HAM|text) - \log P(SPAM|text) > 0$$

$$\Leftrightarrow \log P(text|HAM) + \log P(HAM) - \log P(text|SPAM) - \log P(SPAM) > 0$$

- The quotient of the probabilities of two complementary events is also called Odds.
- The logarithm of this quotient is called Log-Odds.

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Naive Bayes with other distributions

- Depending on the problem, the distribution P(X|category) can take various forms.
- In the case just discussed, the distribution is the multinomial distribution (probability that with text of length n exactly the observed numbers of words occur)
- If the observed values are real values (e.g., values from a sensor), one can e.g. use Gaussian Distributions.
 - ⇒ Smoothing is also important for real-valued features (for example, what variance should be assumed for a category with little data?)
- For machine-learning software (such as Scikit-learn), you can choose the type of distribution as a hyper parameter.

Unknown words in the test data

- It may be that words appear in the test data that did not occur in the training data.
- The possible values of the random variable were chosen based on the training data, i.e. the probability of the new words is not defined.
- Two common solutions:
 - ▶ Words that do not occur in the training data are ignored (⇒ Test documents are getting shorter)
 - Words that are rarely in the training data (for example, 1-2 times) or do not occur at all, (in training and testing) are replaced with a placeholder <UNK>.
- Advantages and disadvantages of the two methods?

Implementation

Training or test instance

In our case:

- Features = words (tokens)
- Label
 - Binary classification: HAM (True) vs SPAM (False)
 - Multi-Class Classification (Exercise Sheet): String for category("work", "social", "promotions", "spam", ...)

```
class DataInstance:
```

```
def __init__(self, feature_counts, label):
    self.feature_counts = feature_counts
    self.label = label
#...
```

Training or test set

- Amount of possible feature values is e.g. important for smoothing.
- Sanity-check: What accuracy would result from predicting the most common category?
- Some learning algorithms require several training iterations between which the training set should be re-permuted (mixed).

```
class Dataset:
    def __init__(self, instance_list, feature_set):
        self.instance_list = instance_list
        self.feature_set = feature_set
    def most_frequent_sense_accuracy(self):
        # ...
    def shuffle(self):
        # ...
```

Classifier

What information do we need to create the Naive-Bayes model?

• ...

Classifier

What information do we need to create the Naive-Bayes model?

- For the estimation of P(w|HAM) or P(w|SPAM)
 - n(w, HAM) or n(w, SPAM):
 One dictionary for each category, which maps each word to its frequency in the respective category.
 - n_{HAM} or n_{SPAM}:
 The number of word occurrences per category
 (can be summed up from the values of the dictionaries)
 - lacktriangle For smoothing: Parameters λ and size of the vocabulary V
- For the estimation of P(HAM) or P(SPAM)
 - In each case the number of training emails per category.

Classifier: constructor

```
def __init__(self, positive_word_to_count, negative_word_to_count,\
        positive_counts, negative_counts, vocabsize, smoothing):
    # n(word, HAM) and n(word, SPAM)
    self.positive_word_to_count = positive_word_to_count
    self.negative_word_to_count = negative_word_to_count
    # n HAM and n SPAM
    self.positive_total_wordcount = \
        sum(positive_word_to_count.values())
    self.negative_total_wordcount = \
        sum(negative_word_to_count.values())
    self.vocabsize = vocabsize
    # P(HAM) and P(SPAM)
    self.positive_prior = \
        positive_counts / (positive_counts + negative_counts)
    self.negative_prior = \
        negative_counts / (positive_counts + negative_counts)
```

self.smoothing = smoothing

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Classifier: Overview

```
class NaiveBayesWithLaplaceClassifier:
    def log_probability(self, word, is_positive_label):
        # ...
    def log_odds(self, feature_counts):
        # ...
    def prediction(self, feature_counts):
        # ...
    def prediction_accuracy(self, dataset):
        # ...
    def log_odds_for_word(self, word):
        # ...
    def features_for_class(self, is_positive_class, topn=10):
        # ...
```

Calculation of P(w|HAM) or P(w|SPAM)

Probability estimation...

- ... smoothed
- ... is returned logarithmized

```
def log_probability(self, word, is_positive_label):
    if is_positive_label:
        wordcount = self.positive_word_to_count.get(word, 0)
        total = self.positive_total_wordcount
    else:
        wordcount = self.negative_word_to_count.get(word, 0)
        total = self.negative_total_wordcount
    return math.log(wordcount + self.smoothing) \
        - math.log(total + self.smoothing * self.vocabsize)
```

Calculation of Log-Odds

• What is calculated in the two sums?

Applying the classifier, test accuracy

- Prediction
 - ▶ Apply the model to the feature counts of a test instance
 - Prediction of a category (HAM/True or SPAM/False) according to the decision rule

```
def prediction(self, feature_counts):
    # ...
```

- Calculation of test accuracy
 - First, prediction for all instances of the dataset
 - Then compare with the correct category label

```
def prediction_accuracy(self, dataset):
    # ...
```

- Extension: Classifier distinguishes n different categories $(n \ge 2)$
- ⇒ Exercise sheet
- Decision rule: choose category c^* , that maximizes probability $p(c^*|text)$.

$$c^* = \arg\max_{c} p(c|text)$$

- $arg max_x f(x)$ selects a value x (from the definition set) for which the function value f(x) is maximal.
- By applying the calculation rules, the conditional independence assumption, and our estimation method (Laplace):

$$c^* = rg \max_{c} p(c) p(text|c)$$

$$= rg \max_{c} \log[p(c)] + \sum_{w \in text} \log[\tilde{p}(w|c)]$$

• Decision rule: choose category c^* , that maximizes probability $p(c^*|text)$.

$$c^* = \arg\max_{c} p(c|text)$$

• Does the following implication apply?

$$c^* = rg \max_{c} p(c|text) \Rightarrow rac{p(c^*|text)}{1 - p(c^*|text)} \geq 1$$

• Does the following implication apply?

$$rac{
ho(c^*|text)}{1-
ho(c^*|text)}>1\Rightarrow c^*=rg\max_c
ho(c|text)$$



Does the following implication apply?

$$c^* = rg \max_{c}
ho(c|text) \Rightarrow rac{
ho(c^*|text)}{1 -
ho(c^*|text)} \geq 1$$

No. For 3 or more categories, the most likely category may be WK $p(c^*|text) < 0.5$ and the Odds are < 1.

Does the following implication apply?

$$\frac{p(c^*|text)}{1 - p(c^*|text)} > 1 \Rightarrow c^* = \arg\max_{c} p(c|text)$$

Yes. If the most likely category odds has > 1, the WK $p(c^*|text) > 0.5$, and all other categories must have a smaller WK.

Multi-classes Naive Bayes: Implementation

- To calculate the values $\tilde{p}(w|c)$, we need the word frequencies per class n(w,c)Solution: Dictionary (str,str) \rightarrow int
- For the priors p(c) we need the number of instances per class: $\mathsf{str} \to \mathsf{int}$
- and the vocabulary size and the smoothing parameter

```
class NaiveBayesClassifier:
```

```
def __init__(self, word_and_category_to_count, \
    category_to_num_instances, vocabsize, smoothing):
    # ...
```

Log odds per word

- \Rightarrow Exercise sheet
 - The log odds for a category c can also be calculated for only one word (instead of one whole document).
 - Begin with $\log \frac{p(c|w)}{1-p(c|w)}$ and apply the calculation rules

$$\log \frac{p(c|w)}{1 - p(c|w)} = \dots$$

$$= \log[\tilde{p}(w|c)] + \log[p(c)] - \log[\sum_{c' \neq c} \tilde{p}(w|c')p(c')]$$

- The log odds per word indicate how strongly a word indicates the respective category
- You can then sort all words based on their log odds, and get an idea of what the model has learned (i.e., what's important for the model)

Train and evaluate a classifier

In order to train and evaluate a classifier, we need 3 datasets:

- Training data: On this data, the model estimates its parameters automatically (e.g., word probabilities and category priors).
- 2 Development data: On this data, various model architectures and hyper-parameters¹ can be compared. What, for example in our case?
- Test data: An estimate of how well the model works on further unseen data can be obtained on this data, after the development data finally determined a model architecture.
 - Depending on the nature of the data (domain etc), this estimate can deviate very much from reality.
 - The estimate may also be very unreliable depending on the amount of test data (\Rightarrow significance tests).
 - Why can not we use the performance on the development data?

¹Parameters that are not automatically learned.

Summary

- Probability calculation
 - ► Theorem of Bayes
 - Conditional independence
- Naive Bayes classificator
 - Decision rule, and "flip" the formula by using theorem of Bayes
 - Smoothing the probabilities
 - Log-Odds
- Questions?