CS477 Combinatorics: Homework 13

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Problem 1. Read the proof of Theorem 11.13 in the notes and do the following questions.
(a) Let $\mathcal{H}$ be the Fano plane on [7], describe the hyperedges of $\mathcal{H}$ as $e_1, e_2, \ldots, e_7$.
(b) Pick an outcome $((R, B, R, B, R, B, R), (Y, Y, Y, N, N, N, Y), \sigma)$, where σ is the identity permutation. Find two edges e_i and e_j such that $B_{i,j}$ happens.
(c) Let

$$A = \{a_1, a_2, \ldots, a_{r-1}, b_1, b_2, \ldots, b_{r-1}, v\}.$$

Consider a random permutation σ on A, let $i(\sigma)$ be the number of i's such that a_i is before v, $j(\sigma)$ be the number of i's such that b_i is before v. Prove that

$$\mathsf{E}_\sigma[(1+p)^i(1-p)^j] \le 1.$$

Solution. (a)

$$\begin{aligned}
e_1 &= \{1, 2, 3\} \\
e_2 &= \{1, 6, 7\} \\
e_3 &= \{1, 4, 5\} \\
e_4 &= \{2, 4, 6\} \\
e_5 &= \{2, 5, 7\} \\
e_6 &= \{3, 4, 7\} \\
e_7 &= \{3, 5, 6\}
\end{aligned}$$

(b) $i = 1, j = 4$

(c)

$$\begin{aligned}
&\mathsf{E}_\sigma((1+p)^i(1-p)^j) \\
=&\sum_{i=0}^{r-1}\sum_{j=0}^{r-1}\Pr(i,j)(1-p)^i(1+p)^j \\
=&\sum_{k=0}^{2r-2}\sum_{i+j=k}\Pr(i,j)(1-p)^i(1+p)^j
\end{aligned}$$

其中 $\Pr(i,j) = \binom{r-1}{i}\binom{r-1}{j}\frac{(i+j)!(2r-2-i-j)!}{(2r-1)!}$, 也可以写成

$$\Pr(i,j) = \frac{1}{2r-1}\binom{r-1}{i}\binom{r-1}{j}\Big/\binom{2r-2}{i+j}$$

下面我们证明，当 $i+j=k$ 为一定值的时候，有

$$\sum_{i+j=k}\Pr(i,j)(1-p)^i(1+p)^j \le \frac{1}{2r-1}$$

即

$$\sum_{i+j=k}\binom{r-1}{i}\binom{r-1}{j}(1-p)^i(1+p)^j \le \binom{2r-2}{k}$$

可以发现左边为多项式

$$(1+(1-p)x)^{r-1}(1+(1+p)x)^{r-1} = (1+2x+(1-p^2)x^2)^{r-1}$$

中 x^k 的系数, 据此我们可以得到系数的另一种表示

$$\sum_{a+2b=k,a+b<r}\binom{r-1}{a,b,r-1-a-b}2^a(1-p^2)^b$$

不难发现每一项都是在 $[0,1]$ 上单调递减的，因此我们验证 $p=0$ 的时候发现是满足的。由于 k 只有 $2r-1$ 种取值，所以 $\mathsf{E}_\sigma((1+p)^i(1-p)^j) \le 1$

$\square$

Problem 2. Let G be a graph on n vertices, and there are no two cycles in G with the same length. Prove that the number of edges in G is at most $n + O(\sqrt{n})$.

Solution. 考虑一个符合要求的图 G, 我们将图上的每条边以 $\frac{1}{\sqrt{n}}$ 的概率删掉，那么考虑图上剩下的圈的个数期望

$$E(X) \leq \sum_{i=3}^{n}(1-\frac{1}{\sqrt{n}})^i = \sqrt{n}(1-\frac{1}{\sqrt{n}})^3(1-(1-\frac{1}{\sqrt{n}})^n)$$

所以 $E(X) \in O(\sqrt{n})$

再考虑图上删去的边数 $E(Y) = \frac{m}{\sqrt{n}}$, 由于剩下的边减去圈数后最多只有 $n-1$ 条边，所以 $m \leq n-1+E(X+Y)$, 可以发现 $m \in n+O(\sqrt{n})$. □

Problem 3. Let P be a set of 10 points in the two dimensional plane. Prove that there exists 10 unit circles so that their interiors are disjoint and together they cover all the points in P.

Solution. 考虑对平面进行随机六边形密铺，六边形的大小为正好可以装入一个单位圆的大小，那么圆和六边形的面积比为 $\frac{\pi}{2\sqrt{3}} > 0.9$。因此这样所有的圆期望可以覆盖到的点数大于 9, 因此一定存在一种方案可以覆盖 10 个点。而一个点最多被一个圆覆盖，所以存在这样的十个圆。 □