Lecture Notes

# Combinatorics in the Plane

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# Contents

## 1 The Sylvester-Gallai Theorem

Every finite set of points in the plane naturally defines a set of lines – the connecting lines. A *connecting line* is a line passing through at least two points of the set. Some of the oldest as well as some of the hardest problems in combinatorial geometry ask to find point sets whose set of connecting lines are extreme in some sense.

Maybe the most famous example is the following question posed by James Joseph Sylvester (1814-1897) in 1893.

*"Is it true that any finite set of points in the plane, not all on a line, has two elements whose connecting line does not pass through a third point?"*

After several failed attempts the problem was rediscovered by Erdős 40 years later. Shortly thereafter, an affirmative answer was given by Tibor Grünwald (alias Gallai) [Gal44]. We present here the "book proof" found by Kelly.

**Theorem 1.1** (Sylvester-Gallai theorem).
*For any finite non-collinear set of points in the plane there is a line passing through exactly two of them.*

*Proof.* Consider a pair $(p, \ell)$ of a point $p$ in our set and a line $\ell$ passing through at least two points with $p$ not on $\ell$, for which the distance between $p$ and $\ell$ is minimal. We claim that $\ell$ passes through exactly two points.

Suppose not, i.e., that $\ell$ contains at least three points. Then there is a ray that is contained in $\ell$, emerges from the projection of $p$ onto $\ell$ and contains at least two points from our set, say $q$ and $r$. Assume without loss of generality that $r$ is closer to $p$ than $q$ is. Then the distance between $r$ and the line connecting $p$ and $q$ is smaller than the distance between $p$ and $\ell$. This contradicts the minimality of the pair $(p, \ell)$ and completes the proof. □

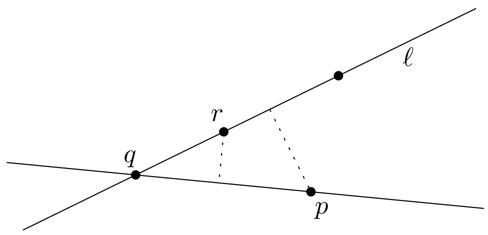


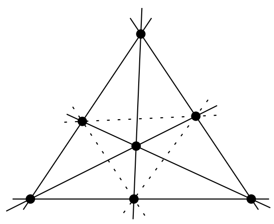

Figure 1: Illustrating the proof of the Sylvester-Gallai theorem.

For a given finite point set $P$ a line that goes through exactly two points from $P$ is called an *ordinary line.* A natural question in combinatorial geometry is to find the minimum number $ol(n)$ of ordinary lines determined by $n$ non-collinear points in the plane. The Sylvester-Gallai theorem asserts that $ol(n) \geq 1$ for all $n \geq 3$. Despite the time it took to prove this bound, people believe that the true value is much bigger.

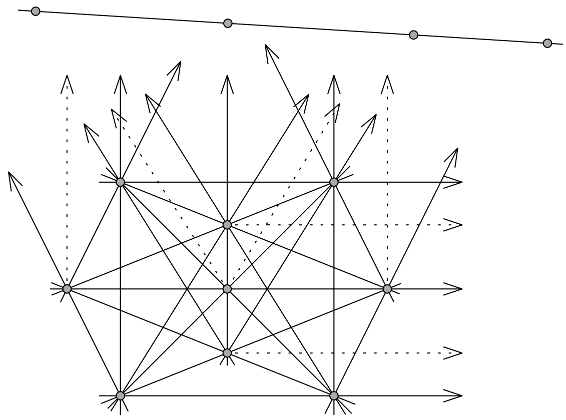
**Conjecture 1.1** (Dirac [Dir51], Motzkin).
*For every $n \neq 7, 13$, we have*

$$ol(n) \geq \lceil \frac{n}{2} \rceil.$$

Kelly and Moser [KM58] proved $ol(n) \geq \frac{3}{7}n$, with equality for $n = 7$ (see Figure 2(a)). The best known lower bound is $ol(n) \geq \frac{6}{13}n$, with equality for $n = 13$ (see Figure 2(b)), was proven by Csima and Sawyer [CS93, CS95].



(a)



(b)

Figure 2: Exceptional sets with $n = 7$ and $n = 13$ points and fewer than $\lceil \frac{n}{2} \rceil$ ordinary lines (drawn dotted). It is conjectured that such examples do not exist for $n \neq 7, 13$.

Next we prove an *upper* bound on $ol(n)$, i.e., find finite point sets defining very few ordinary lines. It is convenient to define these point sets in the real projective plane, which we define here as the projective completion of $\mathbb{R}^2$.

**Definition 1.1** (Real projective plane)**.**
*Consider $\mathbb{R}^2$ and add for each parallel class of lines one new point, called a* point at infinity*. Each point at infinity lies on every line of the corresponding parallel class and no further line from $\mathbb{R}^2$. Moreover, all points at infinity lie on a common new line, called the* line at infinity*, and no point from $\mathbb{R}^2$ lies on this line.*

The real projective plane is denoted by $\mathbb{RP}^2$ and has the following beautiful properties.

- Every two distinct points lie on a unique line.
- Every two distinct lines meet in a unique point.

The first property is also satisfied by the real plane $\mathbb{R}^2$. However, by the second property there are no parallel lines in $\mathbb{RP}^2$, which clearly exist in $\mathbb{R}^2$. In the upcoming constructions proving Theorem 1.3 we identify certain parallel classes of lines and let the corresponding points at infinity be in our constructed set. The same is already done in Figure 2(b). However, every finite sets of points and lines in $\mathbb{RP}^2$ can transformed into one in $\mathbb{R}^2$ with the same point-line incidences. For the examples in Figure 2 and Figure 3 this can be achieved by applying some small perturbations to the points. We remark that this can be done in general but omit the proof and a detailed explanation.

**Theorem 1.2.** *Every finite set $S$ of points and lines in the real projective plane is in bijection with a finite set $S'$ of points and lines in the real plane such that each of the following holds.*

- *A point and a line in $S$ are incident if and only if their images in $S'$ are incident.*
- *Two lines in $S$ are concurrent if and only if their images in $S'$ are concurrent.*
- *Three points in $S$ are collinear if and only if their images in $S'$ are collinear.*

*In particular there are no parallel lines in $S'$.* □

An easy case analysis shows $ol(3) = 3$, $ol(4) = 3$, and $ol(5) = 4$. In Theorem 1.3 below we assume $n \geq 6$.

**Theorem 1.3.** *For even $n$ we have $ol(n) \leq \frac{n}{2}$. For odd $n$ we have $ol(n) \leq 3\lfloor \frac{n}{4} \rfloor$.*

*Proof.* For even $n$, consider a regular $\frac{n}{2}$-gon $Q$ in $\mathbb{RP}^2$, which determines $\frac{n}{2}$ directions. Let $P$ be the set of corners of $Q$ together with the $\frac{n}{2}$ projective points corresponding to the directions determined by $Q$. See Figure 3(a) for an example. Then for each corner of $Q$ there is exactly one direction for which the corresponding line goes through no other corner of $Q$. Hence the number of ordinary lines is exactly $\frac{n}{2}$.

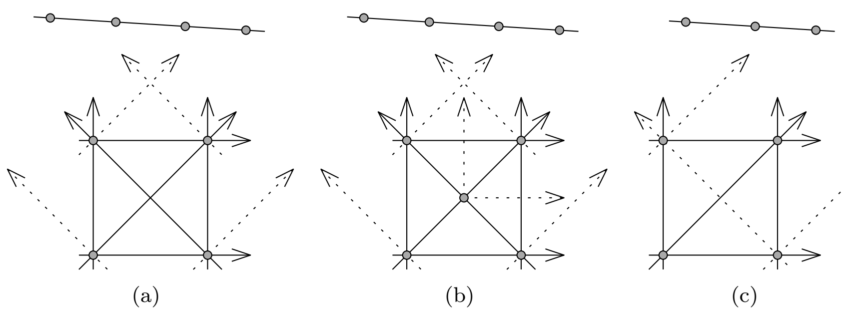



Figure 3: (a) A set of $n = 8$ points defining only $\frac{n}{2} = 4$ ordinary lines. (b) A set of $n = 9$ points defining only $3\lfloor \frac{n}{4} \rfloor = 6$ ordinary lines. (c) A set of $n = 7$ points defining only $3\lfloor \frac{n}{4} \rfloor = 3$ ordinary lines. All ordinary lines are drawn dotted.

If $n \equiv 1 \pmod 4$, we take the above construction on $n - 1$ points and add the center of the polygon $Q$ to the set. See Figure 3(b) for an example. From $n \equiv 1 \pmod 4$ follows that $Q$ has an even number of corners and hence all $\frac{n-1}{4}$ diagonals of $Q$ meet in a common point, the center. Thus the ordinary lines are given as the union of the $\frac{n-1}{2}$ ordinary lines from the construction on $n - 1$ points plus another $\frac{n-1}{4}$ ordinary lines each containing the center and one point at infinity.

Finally, if $n \equiv 3 \pmod 4$, we take the construction on $n + 1$ points and remove one of the points at infinity from it. See Figure 3(c) for an example. Again from $n \equiv 3 \pmod 4$ follows that $Q$ has an even number of corners and hence the ordinary lines in the example with $n+1$ points use only $\frac{n+1}{4}$ directions. We delete the point at infinity for such a direction. This way two ordinary lines contain now only one point and are no longer ordinary, while $\frac{n-1}{2}$ lines of that direction are now ordinary as the number of points on them drops from three to two. □

**Remark.** *On March 28, 2013 (less than three weeks ago!) Ben Green and Terence Tao [GT13] claimed to have proven the Dirac-Motzkin conjecture (Conjecture 1.1) for large $n$. Their proof also seems to give a lower bound of $3\lfloor \frac{n}{4} \rfloor$ in case of odd $n$.*

> **Problem 1.**
> Show that given any set of $n$ non-collinear points in the plane determines at least $n$ different connecting lines, i.e., lines through at least two points of the set.
>
> Show moreover that $n$ points define *exactly* $n$ connecting lines if and only if all but one of the points are collinear.

Let us continue with the Sylvester-Gallai theorem in its *dual version*. One of the most pleasing aspects of considering the real projective plane $\mathbb{RP}^2$ rather than the real Euclidean plane $\mathbb{R}^2$ is that $\mathbb{RP}^2$ comes with a very natural concept of duality between points and lines. Note that the two properties of $\mathbb{RP}^2$ below Definition 1.1 can be transformed into one-another by swapping the meaning of points and lines.

**Theorem 1.4** (Duality in real projective plane)**.**
*For every configuration $S$ of points and lines in $\mathbb{RP}^2$ we can find a* dual configuration *$S^*$ in $\mathbb{RP}^2$ with the following properties.*

- *Every point in $S$ corresponds to one line in $S^*$ and vice versa.*
- *Every line in $S$ corresponds to one point in $S^*$ and vice versa.*
- *A point and a line in $S$ are incident if and only if the corresponding line and point in $S^*$ are incident.*
- *A set of points in $S$ is collinear if and only if the corresponding lines in $S^*$ are concurrent.*
- *A set of lines in $S$ is concurrent if and only if the corresponding points in $S^*$ are collinear.*

□

See Figure 4 for an example of a configuration and one of its dual configurations. We remark that configurations are considered here as actual points and lines in the real projective plane. This makes the dual configuration not unique. Indeed, applying any "small" perturbation to one dual configuration yields another, different dual configuration. For example, the configuration in the right of Figure 4 can be modified so that the lines $C$ and $D$ are parallel and 1 is the point at infinity where these two lines meet.

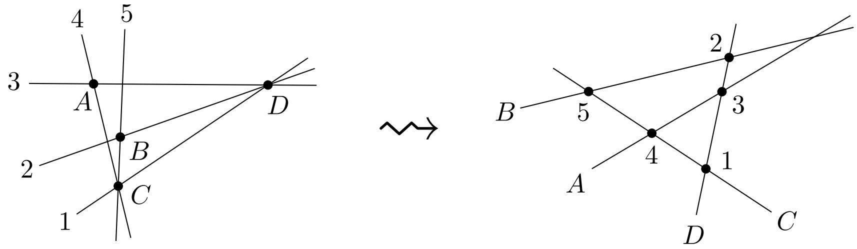



Figure 4: A configuration $S$ in $\mathbb{RP}^2$ (left) and a dual configuration $S^*$ of $S$ (right).

**Problem 2.**
Find a dual configuration of the configuration with 7 points and 9 lines in Figure 2(a).

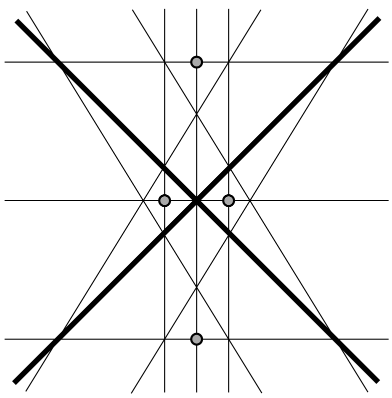


Figure 5: An arrangement of 13 lines in the real projective plane (the 13the line is the line at infinity) determining only 6 ordinary points: The 4 gray points and the 2 points where the bold lines intersect the line at infinity.

With Theorem 1.4 and Theorem 1.2 we can formulate the Sylvester-Gallai theorem (Theorem 1.1) in its dual form. Instead of finite point sets in $\mathbb{R}^2$ defining ordinary lines we now speak of *arrangements of lines* defining *ordinary points*, i.e., points contained in exactly two lines.

**Theorem 1.5** (Sylvester-Gallai theorem – dual version)**.**
*Every arrangement of finitely many lines in $\mathbb{R}^2$, not all concurrent, and not all parallel, admits an ordinary point.*

Although the dual Sylvester-Gallai theorem is equivalent to its primal version, we give an alternative proof for it, which indeed is a little stronger. For a given arrangement $\mathcal{A}$ of lines in $\mathbb{RP}^2$ we define the *vertices*, *edges* and *faces* of $\mathcal{A}$ to be the points where two lines intersect, the connected components of lines after the removal of all vertices, and the connected components of $\mathbb{RP}^2$ after the removal of all lines, respectively.

Note that in the projective plane, every line contains as many vertices as edges. Figuratively speaking, the two "ends" of a line meet at the point at infinity and hence belong to the same edge, unless the point at infinity is a vertex of $\mathcal{A}$. Similarly, in the figures some faces of $\mathcal{A}$ appear as a pair of unbounded regions on opposite sites of the arrangement, in particular such faces look as if they were disconnected. We refer to Figure 6 for an illustrative example.

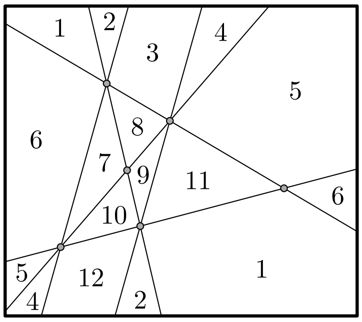



Figure 6: An arrangement of lines in $\mathbb{RP}^2$ with 7 vertices (on being a point at infinity), 18 edges and 12 faces.

The main ingredient for the proof of Theorem 1.5 is Euler's formula for arrangements of lines in the real projective plane.

**Proposition 1.1** (Euler)**.**
*If $\mathcal{A}$ is a projective arrangement of lines with $f_0$ vertices, $f_1$ edges and $f_2$ faces, then*

$$f_0 - f_1 + f_2 = 1. \tag{1}$$

**Problem 3.**
Find a proof for Proposition 1.1.

*Proof of Theorem 1.5.*
Let $\mathcal{A}$ be a fixed arrangement of lines in $\mathbb{R}^2$. We interpret $\mathcal{A}$ as an arrangement of lines in $\mathbb{RP}^2$ and thus can speak of vertices, edges and faces in $\mathcal{A}$. Let $s_i$ be the number of vertices of $\mathcal{A}$ where exactly $i$ lines meet, $i \geq 2$. Secondly, let $t_j$ be the number of faces of $\mathcal{A}$ with exactly $j$ incident edges, $j \geq 1$. With $f_0$, $f_1$ and $f_2$ denoting the number of vertices, edges and faces in $\mathcal{A}$, respectively, we have

$$\sum_{i \geq 2} s_i = f_0 \quad \text{and} \quad \sum_{j \geq 1} t_j = f_1.$$

Because every edge is incident to two faces, we have $\sum_{j \geq 1} j \cdot t_j = 2f_1$. Because a vertex in which exactly $i$ lines meet has $2i$ incidences with edges and every edge has two incidences with vertices, we have $\sum_{i \geq 2} 2i \cdot s_i = 2f_1$. Using all these equalities we calculate

$$\sum_{i \geq 2}(3-i)s_i + \sum_{j \geq 1}(3-j)t_j = 3f_0 - f_1 + 3f_2 - 2f_1 = 3(f_0 - f_1 + f_2) \overset{(1)}{=} 3.$$

Now if not all lines in $\mathcal{A}$ are concurrent, then there exist at least two vertices, which implies that $t_1 = t_2 = 0$. Thus in the leftmost term above the only positive coefficient is the one for $s_2$, and it is 1. We conclude $s_2 \geq 3$, which means that there are at least *three* ordinary points. $\square$

**Problem 4.**
For any arrangement $\mathcal{A}$ of lines in $\mathbb{R}^2$, i.e., *in the Euclidean plane*, we define the vertices, edges and faces of $\mathcal{A}$ as the points where two lines intersect, the connected components of lines after the removal of all vertices, and the connected components of $\mathbb{R}^2$ after the removal of all lines, respectively.

Consider *simple arrangements*, that is, arrangements in which no point in $\mathbb{R}^2$ belongs to more than two lines in $\mathcal{A}$, with $n$ lines. Derive and prove a formula for the total number vertices, edges and faces of $\mathcal{A}$.

# 2 The Crossing Lemma

Disclaimer: In this chapter for the first time we deal with graphs. We omit a detailed introduction into the basic terminology, such as vertices/nodes, edges, paths, cycles, trees, loops, parallel edges, degree of vertices, complete graphs, bipartite graphs, and so on. Let us just remark that all graphs considered here are finite and simple, i.e., we allow neither loops nor parallel edges.

We are interested in *topological drawings* of graphs. In particular, we want to draw vertices as points and edges as continuous curves connecting the two endpoints of the edge. We forbid edges to pass through vertices, just as we forbid two vertices to be drawn on the same spot. For simplicity we refer to topological drawings simply as drawings. Figure 7 shows drawings of $K_5$, the complete graph on 5 vertices, $K_{3,3}$, the complete bipartite graph on 3 and 3 vertices, and two drawings of the dodecahedron graph.

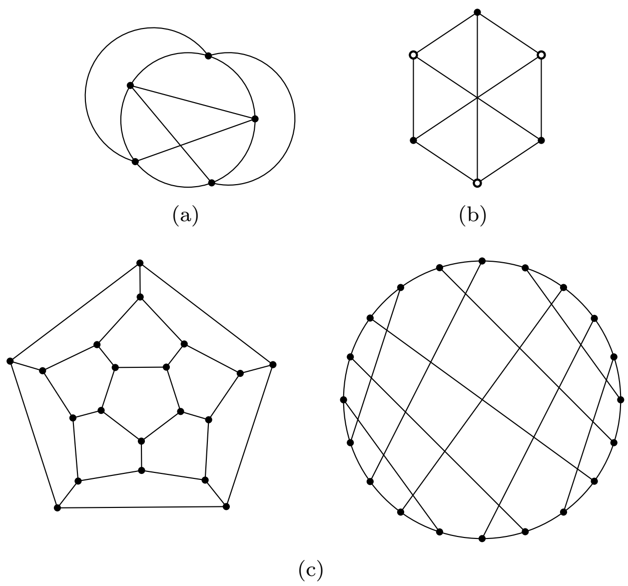



Figure 7: (a) A drawing of $K_5$. (b) A drawing of $K_{3,3}$, where the bipartition classes are given by black and white vertices, respectively. (c) Two drawings of the dodecahedron graph.

Certainly, the two drawings of the dodecahedron graph highlight different aspects of the graph. In the drawing on the left no two edges intersect except in their endpoints. Whereas the drawing on the right contains many such crossings. Finding drawings with as few crossings as possible is a very important topic in graph theory - and that not only for aesthetically reasons. However, as we will see later, understanding the matter of crossing minimization has far reaching consequences for seemingly unrelated areas, some of which we want to present here.

**Definition 2.1** (Crossing number)**.**
*A* crossing *is a point in the intersection of at least two edges but distinct from all*

*vertices. The crossing number of a drawing of a graph is the number of crossings in the drawing, where a crossing that is contained in $k$ edges is counted $\binom{k}{2}$ times.*

*The* crossing number *of $G$, denoted by $\mathrm{cr}(G)$, is the least crossing number in any drawing of $G$.*

**Problem 5.**
Show that in the definition of $\mathrm{cr}(G)$ we can safely restrict our attention to drawings with the following properties.

- No two incident edges cross.
- No pair of edges crosses more than once.
- No edge crosses itself.
- No three edges cross in a common point.

Would the same be true if we restrict in our drawings that all edges are drawn as a straight segments?

Of course, the best one can hope for is crossing number 0, i.e., a drawing in which no pair of edges cross. Such drawings are called *plane drawings*, or *plane embeddings*, and the graphs admitting such drawings are called *planar graph.* For example Figure 7(c) certifies that the dodecahedron graph is planar. We remark that neither $K_5$ nor $K_{3,3}$ are planar. Indeed both graphs have crossing number 1, even though Figure 7(b) only proves $\mathrm{cr}(K_{3,3}) \leq 3$.

We have defined drawings, and hence the crossing number, in such a way that edges can be drawn as arbitrary curves. Allowing this freedom obviously strengthens most of the results below. On the other hand, restricting the drawings of edges, e.g., as straight-line segments, or circular arcs, gives nice and interesting variants.

Let us just just briefly mention the most important notions and facts. The *rectilinear crossing number* of a graph $G$, denoted by $\overline{\mathrm{cr}}(G)$, is the minimum number of crossings in a drawing of $G$ where every edge is a straight-line segment. Fáry's Theorem [Fár48] states that for any graph $G$ we have $\mathrm{cr}(G) = 0$ if and only if $\overline{\mathrm{cr}}(G) = 0$, i.e., in case of planar graphs restricting to straight edges is no "real restriction".

However, in general $\mathrm{cr}(G)$ and $\overline{\mathrm{cr}}(G)$ can be arbitrarily far apart.

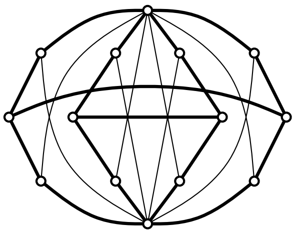


Figure 8: The graph $G$ with 8 light edges (drawn thin) and 18 heavy edges (drawn thick).

**Problem 6.**
Consider the 14-vertex graph in Figure 8; call it $G$.

- Prove that every topological drawing of $G$ in which all 8 light (drawn thin in the figure) edges are drawn as straight-line segments there exist a crossing involving a heavy edge (drawn thick). Note that heavy edges may be drawn arbitrarily!
- For any integer $c \geq 1$ consider the graph $G_c$, which arises from $G$ by introducing $c-1$ copies of every heavy edge, that is, replacing each such each by a bundle of $c$ parallel edges, and subdividing each of these heavy edges with a new vertex if degree two. Show that

$$\mathrm{cr}(G_c) \leq 4 \quad \text{and} \quad \overline{\mathrm{cr}}(G_c) \geq c.$$

Back to topological drawings. One of the first questions coming to our mind may be the following.

*"What causes a graph to have a large crossing number?"*

Intuitively, if the graph has many edges on few vertices, then there should be many crossings in every drawing of that graph. The *Crossing Lemma* given below quantifies this intuition very precisely. In particular, it gives a function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ such that every $n$-vertex $m$-edge graph has crossing number at least $f(m,n)$. Before coming to the Crossing Lemma itself, let us first prove two weaker, yet powerful, lower bounds.

We start by counting the maximum number of edges in any $n$-vertex planar graph, that is, graph with crossing number 0. It suffices to consider *maximally planar graphs* only, that is, graphs with a planar drawing for which the addition of any edge would necessarily introduce a crossing. A *face* in such a planar drawing is a connected component of the plane after the removal of all vertices and edges; The *outer face* being the one corresponding to the unbounded component. It is easy to see that in a maximally planar graph on at least 3 vertices every face is bounded by a simple (without vertex repetition) cycle of length 3 – a triangle. Hence these graphs are also called *triangulated*, while a graph is called *inner triangulated* if it admits a planar drawing in which all inner faces are triangles and the outer face is bounded by a simple cycle.

**Proposition 2.1.** *Every $n$-vertex ($n \geq 3$) inner triangulated graph has exactly $3n-3-k$ edges, where $k$ is the length of the outer face.*

*Proof.* Let $G$ be an inner triangulated graph. We fix a planar drawing of $G$ in which all inner faces are triangles. We prove the statement by induction on the number of vertices of $G$.

If $n = 3$, then $G$ itself is a triangle and the outer face has length $k = 3$. Hence $|E(G)| = 3 = 3n - 3 - k$.

So let $n > 3$. A *chord of the outer face* is an edge that is not on the outer face but connects two vertices on the outer face. Let $v$ be any vertex on the outer face that is not incident to any chord. If there is no chord, we can take any vertex from the outer face. If there exist at least one chord, consider one for which the two endpoints, say $u$ and $w$, have minimum distance along the outer face, and let $v$ be any vertex different from $u$ and $w$ on the shorter path $P$ on the outer face between $u$ and $w$. Note that $u$ and $w$ have distance at least 2 on the outer face and hence $v$ is well-defined. Since the drawing is planar, every chord starting at $v$ would have to end on $P$ again, which would contradict the minimality of the chord $uw$. The situation is illustrated in the left of Figure 9.

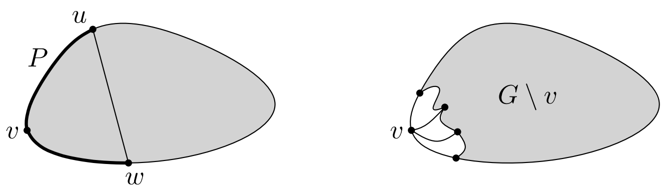



Figure 9: Illustration of the proof of Proposition 2.1.

Now consider the graph $G \setminus v$, that is the inner triangulated graph after the removal of vertex $v$ and all its incident edges. Since $v$ has no incident chord all but two neighbors of $v$ lie in the interior of $G$ and all neighbors of $v$ lie on the outer face of $G \setminus v$. See the right of Figure 9. So if $d$ denotes the degree of $v$ in $G$, then the length of the outer face of $G \setminus v$ is $k + d - 3$. Applying induction to $G \setminus v$ we obtain $|E(G \setminus v)| = 3(n-1) - 3 - (k + d - 3)$ and thus

$$|E(G)| = |E(G \setminus v)| + d = 3(n-1) - 3 - (k + d - 3) + d = 3n - 3 - k.$$

□

From Proposition 2.1 immediately implies that every $n$-vertex planar graph has at most $3n - 6$ edges. The next is a direct generalization of this fact.

**Proposition 2.2.** *Any drawing of an $n$-vertex $m$-edge graph has at least $m - 3n + 6$ crossings.*

*Proof.* Let $H$ be a maximal planar subgraph of our $n$-vertex $m$-edge graph $G$. Then every edge in $E(G) \setminus E(H)$ participates in a crossing with an edge in $E(H)$. Since by Proposition 2.1 we have $|E(H)| \leq 3n - 6$ the bound follows. □

In 1973 Erdős and Guy conjectured [EG73] that every drawing of a graph with $n$ vertices and $m$ edges has at least $cm^3/n^2$ crossing for some constant $c$. Two sets of authors independently confirmed this conjecture: Ajtai, Chvátal, Newborn and Szemerédi [ACNS82] in 1982 and Leighton [Lei84] in 1984. The statement has become very popular and is nowadays known as the Crossing Lemma.

**Theorem 2.1** (Crossing Lemma)**.**
*If $G$ is a graph with $n$ vertices and $m \geq 4n$ edges, then*

$$\mathrm{cr}(G) \geq \frac{1}{64} \frac{m^3}{n^2}.$$

*Proof.* Consider a fixed drawing of $G$. We define a random induced subgraph $\mathbf{H}$ of $G$ by taking every vertex uniformly at random with probability $p$. That $\mathbf{H}$ is induced means that it contains every edge of $G$ between any two vertices in $\mathbf{H}$.

Define $\mathbf{n} := |V(\mathbf{H})|$ and $\mathbf{m} := |E(\mathbf{H})|$. Further let $\mathbf{c}$ be the crossing number of the induced drawing of $\mathbf{H}$. Clearly we have $\mathbf{c} \geq \mathrm{cr}(\mathbf{H})$ as well as the following expectations.

$$\begin{aligned} \mathbb{E}[\mathbf{n}] &= p \cdot n && (2) \\ \mathbb{E}[\mathbf{m}] &= p^2 \cdot m && (3) \\ \mathbb{E}[\mathbf{c}] &= p^4 \cdot \mathrm{cr}(G) && (4) \end{aligned}$$

Equation (2) holds since every $v \in V(G)$ is in $\mathbf{H}$ with probability $p$. An edge $uv \in E(G)$ is in $\mathbf{H}$ if and only if both $u$ and $v$ are in $\mathbf{H}$. Hence the probability that the edge $uv$ is in $\mathbf{H}$ is $p^2$, which implies (3). For a crossing of the drawing of $G$ to be in the induced drawing of $\mathbf{H}$, the two corresponding edges must be in $\mathbf{H}$. This is the case with probability $p^2$ each, hence each crossing is in the drawing of $\mathbf{H}$ with probability $p^4$, which gives (4).

With $\mathbf{c} \geq \mathrm{cr}(\mathbf{H})$ Proposition 2.2 gives $\mathbf{c} \geq \mathbf{m} - 3\mathbf{n}$ independent of the actual subgraph $\mathbf{H}$. Hence this inequality holds also in expectation. Using the linearity of expectation we can conclude

$$\begin{aligned} \mathbb{E}[\mathbf{c}] &\geq \mathbb{E}[\mathbf{m} - 3\mathbf{n}] \\ \mathbb{E}[\mathbf{c}] &\geq \mathbb{E}[\mathbf{m}] - 3\mathbb{E}[\mathbf{n}] \\ p^4\,\mathrm{cr}(G) &\geq p^2 m - 3pn \\ \mathrm{cr}(G) &\geq \frac{m}{p^2} - \frac{3pn}{p^3}. \end{aligned}$$

Setting $p = 4n/m$ (here we need the assumption $m \geq 4n$) we obtain

$$\mathrm{cr}(G) \geq \frac{m^3}{16n^2} - \frac{3m^3}{64n^2} = \frac{1}{64}\frac{m^3}{n^2}.$$

□

The proof of the Crossing Lemma given above is attributed to Bernard Chazelle, Micha Sharir and Emo Welzl.

**Example.** *Consider the following drawing of a graph $G = G(n,k)$ for positive integers $n$ and $k$ with $k < n/2$. The vertices are (drawn as) the corners of a convex $n$-gon, denoted by $v_0, \ldots, v_{n-1}$ in clockwise order. The edges are drawn as straight-line segments between vertex $v_i$ and vertex $v_j$ whenever $|j - i| \leq k \pmod n$. See Figure 10 for an example.*

*The graph $G$ is $2k$-regular, i.e., consist of $n$ vertices and $m = kn$ edges. Consider any edge $v_iv_j$. This edge is crossed by every edge with one endpoint strictly between $v_i$ and $v_j$ in clockwise order and the other endpoint strictly between $v_i$ and $v_j$ in counterclockwise order. Thus for $l = |i - j| \mod n$ the edge $v_iv_j$ is crossed by $(l-1)2k - l(l-1)$ other edges. Hence the total number of crossings is given by*

$$\sum_{i=0}^{n-1}\sum_{l=1}^{k}(l-1)2k - l(l-1) = n \cdot (2k \cdot \frac{1}{2}k^2 - \frac{1}{6}k^3 + \mathcal{O}(k^2)) \approx \frac{1}{3}nk^3 = \frac{1}{3}\frac{m^3}{n^2}.$$

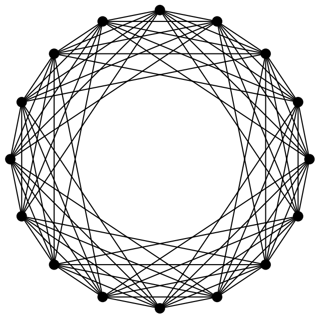


Figure 10: A drawing of $G(n,k)$ with $n = 16$ and $k = 5$.

The above example shows that the Crossing Lemma is tight up to the constant. The constant $1/64 \approx 0.015$ has been successively improved. The currently best lower bound is $1024/31827 > 0.032$ due to Pach, Radoicic, Tardos and Tóth [PRTT06]. The upper bound has been improved by Pach and Tóth [PT97] to 0.06.

Let us recall the question we started with.

*"What causes a graph to have a large crossing number?"*

The Crossing Lemma (Theorem 2.1) states that if a graph has many edges compared to its number of vertices, then it has a large crossing number. However, it is important to remark that the fraction of number of edges over number of vertices is not the only reason for a large crossing number.

**Problem 7.**
Find for every $c > 0$ and every $f > 0$ a graph $G = G(c,f)$ with the properties that

$$\frac{|E(G)|}{|V(G)|} \leq f \quad \text{and} \quad \operatorname{cr}(G) \geq c.$$

## 2.1 Applications of the Crossing Lemma

As indicated earlier the Crossing Lemma is used to prove many theorems in combinatorial geometry. (This is why it deserves to be a lemma.) We discuss here some of the most prominent applications of the Crossing Lemma. We start with an extremal incidence problem.

Consider a set of $m$ points and $n$ lines in the plane. If a point $p$ lies on a line $\ell$, then we say that $p$ and $\ell$ are *incident* and call the pair $(p, \ell)$ an *incidence*. Clearly, a single point can be incident to many lines, just like a single line can be incident to many points. On the other hand, the $m$ points and $n$ lines may define no incidence at all. We start again with a very basic question.

*"How many incidences can $m$ points and $n$ lines define at most?"*

Let us make this more formal. For a given finite point set $P$ and a finite line set $L$ we denote by $I(P, L)$ the set of all incidences $(p, \ell)$ with $p \in P$ and $\ell \in L$. For positive integers $n$ and $m$ we let $I(n, m)$ be the maximum size of $I(P, L)$ over all $n$-point sets $P$ and $m$-line sets $L$. In particular

$$I(n, m) := \max\{|I(P, L)| \mid |P| = n, |L| = m\}.$$

For example, Figure 11 shows that $I(3, 4) \geq 7$.

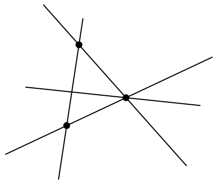


Figure 11: 3 points and 4 lines defining 7 incidences.

**Problem 8.**
Prove that $I(3, 4) = 7$ and $I(5, 6) = 14$.

Since $I(P, L) \subseteq P \times L$ we get as a first upper bound $I(n, m) \leq nm$. However, this bound is far from being tight, unless $n = 1$ or $m = 1$. The following theorem gives an upper bound on $I(n, m)$ which is asymptotically tight. It was first proven by Endre Szemerédi and William (aka Tom) Trotter in 1983 [STJ83]. The original proof was very involved and gave a much larger constant than the one presented here. The proof relying on the Crossing Lemma was found by Székely more than 10 years later [Szé97].

**Theorem 2.2** (Szemerédi-Trotter)**.**
*Let $I(n, m)$ denote the maximum number of incidences between $n$ points and $m$ lines. Then*

$$I(n, m) \leq 4n^{2/3}m^{2/3} + 4n + m.$$

*Proof.* Let $P$ be a set of $n$ points and $L$ a set of $m$ lines. We shall show that $|I(P, L)| \leq 4n^{2/3}m^{2/3} + 4n + m$. We consider the arrangement of $P$ and $L$ as a graph $G$. The vertices of $G$ are the points in $P$, i.e., $|V(G)| = |P| = n$. Each edge of $G$ is a segment of a line $\ell \in L$ between two consecutive points in $P$. In particular if there are $k$ points from $P$ on $\ell$ then there are $k - 1$ edges from $G$ contained in $\ell$. Note that without loss of generality we can assume that every line $\ell \in L$ contains at least one point $p \in P$. Thus the total number of edges is given by $|E(G)| = |I(P, L)| - m$. See Figure 12 for an example.

We want to apply the Crossing Lemma to $G$. But therefore we need $|E(G)| \geq 4|V(G)|$, which translates to $|I(P, L)| - m \geq 4n$. In case this condition fails we have $|I(P, L)| < 4n + m$, as desired. Hence we may assume that $|I(P, L)| \geq 4n + m$ so we can apply the Crossing Lemma to $G$. Bounding the crossing

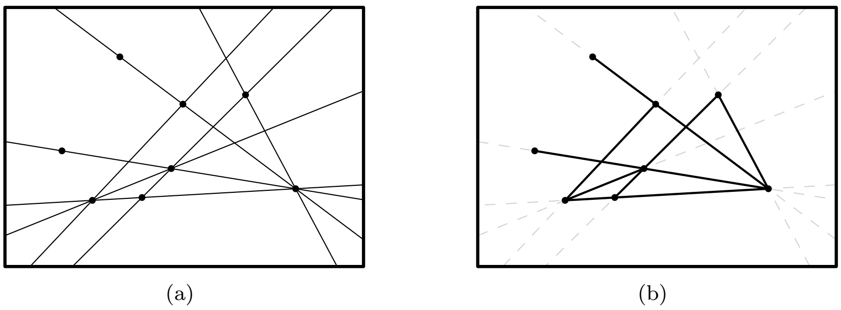


Figure 12: (a) An arrangement of a point set $P$ and a line set $L$. (b) The corresponding graph $G$ on $|P|$ vertices and $|I(P, L)| - |L|$ edges.

number of the induced drawing of $G$ very roughly by $\mathrm{cr}(G) \leq m^2$ we obtain

$$\begin{array}{lrcl} & m^2 \geq \mathrm{cr}(G) & \geq & \frac{1}{64}\frac{(|I(P,L)|-m)^3}{n^2} \\ \iff & (64m^2n^2)^{1/3} & \geq & |I(P,L)| - m \\ \iff & 4m^{2/3}n^{2/3} + m & \geq & |I(P,L)|, \end{array}$$

which proves the theorem. □

The Szemerédi-Trotter theorem is asymptotically tight. Already in 1946 it was again Paul Erdős [Erd46] who described a set of $n$ points and $m$ lines defining $\Omega(n^{2/3}m^{2/3})$ incidences. He also conjectured that his construction gives the correct order of magnitude, which is confirmed by Theorem 2.2. We sketch here the proof for $m = n$, as the general case is not much more difficult.

**Example.** *Let $n = 4k^3$ for some natural number $k$. We define*

$$P := \{p = (p_x, p_y) \mid p_x = 0, 1, 2, \ldots, 4k^2 - 1, p_y = 0, 1, 2, \ldots, k - 1\}.$$

*So $P$ is nothing else but the $4k^2 \times k$ grid. Further we define*

$$L := \{x = ay + b \mid a = 0, 1, 2, \ldots, 2k - 1, b = 0, 1, 2, \ldots, 2k^2 - 1\}.$$

*Figure 13 depicts the situation for $k = 2$.*

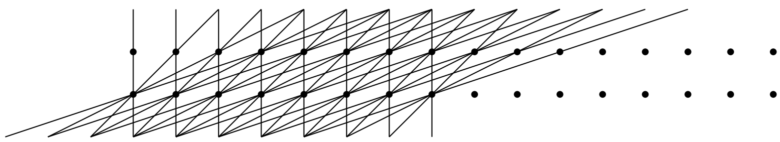


Figure 13: A set of $n = 32$ points and 32 lines with at least $\frac{1}{4^{1/3}}n^{4/3} = 64$ incidences.

*We claim that the intersection point of any line $\ell \in L$ and any horizontal line with $y$-coordinate equal to $0, 1, 2, \ldots, k - 1$ is a point from $P$. Indeed, $\ell$ meets every horizontal in a point with integer coordinates and this point $(x, y)$ satisfies $x = ay + b$. Now from $a \leq 2k - 1$ and $b \leq 2k^2 - 1$ and $y \leq k - 1$ follows*

*$x \leq 4k^2 - 1$. Similarly, from $a \geq 0$ and $b \geq 0$ and $y \geq 0$ follows $x \geq 0$. Hence $(x, y) \in P$.*

*Thus every line $\ell \in L$ is incident to at least (actually exactly) $k$ points from $P$, i.e., $|I(P, L)| \geq k|L| = k \cdot 4k^3 = \frac{1}{4^{1/3}} n^{4/3} \approx 0.63n^{4/3}$.*

**Problem 9.**
For a fixed point set $P$ and a positive integer $k$ we call a line in the plane a *big line* if it contains at least $k$ points from $P$. Let $B_k(P)$ denote the number of big lines defined by $P$ and $B_k(n)$ the maximum $B_k(P)$ over all point sets $P$ with $|P| = n$.

Prove that for every $n$ and every $k$ with $2 \leq k \leq \sqrt{n}$ we have

$$B_k(n) \leq c\frac{n^2}{k^3}$$

for some constant $c > 0$.

The next application of the Crossing Lemma makes actually use of the fact that edges are not necessarily drawn as straight-line segments. In 1946 Paul Erdős [Erd46] studied the distribution of distances defined by $n$ points in the plane. He came up with the two simple questions. Here is the first one.

*"How many distinct distances are defined by $n$ points in the plane at least?"*

We denote the minimum number of distinct distances defined by $n$ points in the plane by $D(n)$. Erdős proved that

$$c_1\sqrt{n} \leq D(n) \leq c_2\frac{n}{\sqrt{\log n}} \tag{5}$$

for some constants $c_1, c_2 > 0$. In 1997 Székely [Szé97] used a generalized version of the Crossing Lemma which deals with non-simple graphs (with more than one edge between two vertices) to improve the lower bound to $\Omega(n^{4/5})$. We do not present his proof here. The idea is similar to the proof of Theorem 2.4 below.

Anyways, the lower bound $D(n) = \Omega(n^{4/5})$ that one gets from the (generalized) Crossing Lemma falls far of Erdős's upper bound $D(n) = \mathcal{O}(\frac{n}{\sqrt{\log n}})$. In the past decade the exponent 4/5 in the lower bound has been successively improved by Solymosi and Tóth [ST01] to 6/7, by Tardos [Tar03] to $\frac{4e}{5e-1} - \epsilon \approx 0.863535$ and then by Katz and Tardos [KT04] to $\frac{48-14e}{55-16e} \approx 0.863636$. Only recently, in 2010, Larry Guth and Nets Hawk Katz [GK10] obtained a breakthrough. Combining ideas of György Elekes, Michar Sharir and others, they finally came up with the following almost tight lower bound.

**Theorem 2.3** (Guth-Katz)**.**
*Every set of $n$ points in the plane defines at least $c\frac{n}{\log n}$ distinct distances for some $c > 0$, i.e.,*

$$D(n) = \Omega(\frac{n}{\log n}).$$

Let us focus on the second question Erdős posed in 1946.

*"How often can a particular distance appear among n points in the plane?"*

It is important to note that Erdős is interested in the *most common* distance in the point set, and how often this distance can be present. Before addressing this question, let us look at the *least common* distances first. I.e., we ask the following.

*"How often can a least common distance appear among n points in the plane?"*

Let us denote by $L(n)$ the maximum number of times a least common distance appears among $n$ points. Clearly, if there are many distinct distances then one of these distances must occur only a few times. Indeed we have

$$L(n) \cdot D(n) \leq \binom{n}{2}. \tag{6}$$

Thus inequality (6) together with Theorem 2.3 implies that $L(n) \leq cn \log n$ for some constant $c > 0$. In other words, there is a distance that occurs at most $cn \log n$ times. But choosing the distance carefully we can do better.

**Proposition 2.3.** *In every set of $n$ points in the plane the maximum distance occurs at most $n$ times. In particular, we have*

$$L(n) \leq n.$$

*Proof.* Let $P$ be any finite point set. We shall show that if a point in $P$ is at maximum distance to at least three points in $P$ then there is another point in $P$ which is at maximum distance to only one point in $P$. Having this, we can iteratively remove points that are at maximum distance to only one point until every point in $P$ is at maximum distance to exactly two points, which will prove the statement.

So consider any point $p \in P$ and assume that $p$ is at maximum distance $d_{\max}$ to at least three other points $q, r, s \in P$. Clearly, the four points $p$, $q$, $r$ and $s$ lie in convex position, that is, span a quadrilateral with no reflex corner. Indeed, otherwise two of the points $q, r, s$ would have a distance greater than $d_{\max}$. Let $q$ be the point opposite to $p$ on the quadrilateral and assume that $q$ has maximum distance to another point $t \neq p$. See Figure 14 for an illustration.

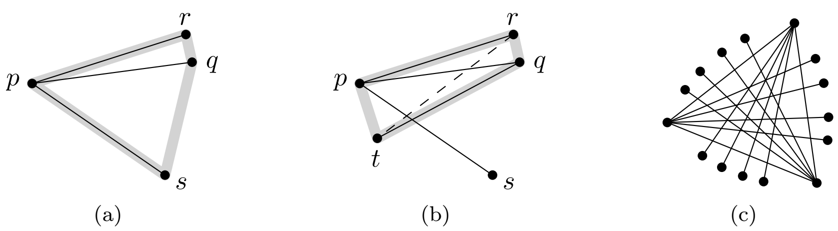



Figure 14: (a) Point $p$ is at distance $d_{\max}$ to $q, r, s$. (b) If $q$ is at distance $d_{\max}$ to some $t$, then $r, t$ or $s, t$ are at distance more than $d_{\max}$. (c) A set of $n$ points defining $n$ maximum distances.

Then $p$ and $q$ are opposite to each other in the quadrilateral spanned with $t$ and either $r$ or $s$ – say $r$. But this quadrilateral has two opposite side $\overline{pr}$ and

$\overline{qt}$ of length $d_{\max}$ implying that one of its diagonals (in fact it is $\overline{rt}$) has length strictly more than $d_{\max}$. This is a contradiction to $d_{\max}$ being the maximum distance between any two points in $P$. $\square$

It is easy to see that the upper bound $L(n) \leq n$ is best-possible. The point set in Figure 14(c) shows that the maximum distance may appear $n$ times among $n$ points. But indeed the exist sets of $n$ points in which *every* distance appears at least $n$ times. To this end consider for odd $n$, say $n = 2k+1$, a set of $n$ points at equal distance on a circle. See Figure 15 for an example. Then every point has points at $k$ distinct distances, two points for each distance. And since every point in the set "looks the same"[1] there is $k$ distances in total, each appearing exactly $n$ times.

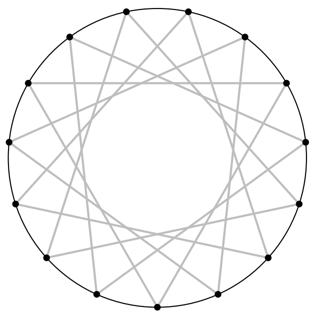


Figure 15: 15 points at equal distance around a circle and the 15 occurrences of a particular distance in gray.

Now let us turn to the Erdős' original problem, namely how often can a *particular* distance appear among $n$ points in the plane. Without loss of generality, i.e., by appropriate scaling, we can fix the particular distance to be 1. We then denote the maximum number of *unit* distances defined by $n$ points in the plane by $U(n)$. Erdős proved that

$$n^{1+c/\log\log n} \leq U(n) \leq n^{3/2} \tag{7}$$

for some positive constant $c$. He also conjectured that his lower bound on $U(n)$ in (7), as well as his upper bound on $D(n)$ in (5) are asymptotically best-possible. Remarkably, both bounds are attained by the square grid of suitable cell size.

Of course, $D(n)$ and $U(n)$ are closely related by

$$U(n) \cdot D(n) \geq \binom{n}{2}. \tag{8}$$

However, inequality (8) goes in the wrong direction in order to get upper bounds on $U(n)$ from lower bounds on $D(n)$. In fact, it is the other way around. Erdős' *upper* bound $D(n) = \mathcal{O}(n/\sqrt{\log n})$ in (5) implies the weaker *lower* bound $U(n) = \Omega(n \cdot \sqrt{\log n})$. And every *upper* bound on $U(n)$ of the form $n^{1+\epsilon}$ would immediately imply a *lower* bound on $D(n)$ of the form $(1/2)n^{1-\epsilon}$. But as of today the best known upper bound on $U(n)$ is the following application of the Crossing Lemma.

[1]We omit to introduce the formal definition of a transitive point set here.

**Theorem 2.4.** *Every set of $n$ points in the plane defines at most $8n^{4/3}$ unit distances, i.e.,*

$$U(n) \leq 8n^{4/3}.$$

*Proof.* Let $P$ be a set of $n$ points in the plane defining the maximum number of unit distances. Denoting by $U(P)$ the number of unit distances defined by $P$, we have to prove that $U(P) \leq 8n^{4/3}$.

For every point $p \in P$ we draw a circle $C_p$ centered at $p$ and with unit radius. The number of resulting point-circle incidences is exactly $2U(P)$.

**Claim.** *By the maximality of $P$ every point $p \in P$ lies on at least two circles.*

*Proof of Claim.* Clearly if some point $p \in P$ lies on no circle, then we can move it onto any circle $C_q$, $q \neq p$ increasing $U(P)$ at least by one.

So assume every point lies on at least one circle, but there is a point $p \in P$ that lies on exactly one circle. We want to move $p$ onto a crossing of some two circles $C_q, C_r$ with $p \neq q, r$. We take $q$ to the rightmost point in $P$. And $r$ to be the rightmost point on $C_q$. Without loss of generality we can assume that $p \neq q, r$. Since $r$ lies to the left of $q$ one of two points in $C_q \cap C_r$ lies to the right of $r$. Thus by the choice of $r$ this crossing is not occupied by any point in $P$ and we can place $p$ there, increasing $U(P)$ at least by one. □

Let us consider the $n$ points and $n$ circles as a topological drawing of some graph $G$ whose vertices are the points in $P$ and whose edges are (drawn as) the circular arcs between consecutive points on the circles. Going around every circle we see that the edges of $G$ corresponds one-to-one to the point-circle incidences. Thus $G$ has exactly $2U(P)$ edges.

By the above claim $G$ contains no loops (circles containing one point only). But in general $G$ is not simple. For example the point set in Figure 16 defines a pair of vertices having a triple of edges between them. However, the maximum edge multiplicity of $G$ is four since at most two unit circles contain any given pair of points. We get rid of all multiplicities by discarding at most 3/4 of the edges of $G$. Denoting the resulting graph by $G'$ and its number of edges by $m$ we get $m \geq U(P)/2$.

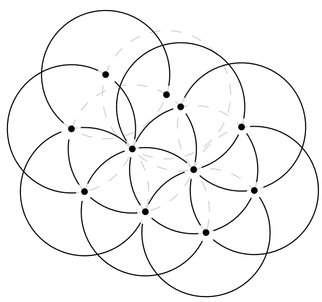


Figure 16: A point set defines a topological drawing of a graph where edges are drawn as circular arcs which are subsets of unit circles centered at the points in the set.

In case $m \leq 4n$ we immediately obtain $U(P) \leq 8n$ which is less than $8n^{4/3}$. Otherwise, if $m > 4n$, we can apply the Crossing Lemma to the drawing of $G'$ and obtain

$$\mathrm{cr}(G') \geq \frac{1}{64}\frac{m^3}{n^2} \geq \frac{1}{512}\frac{U(P)^3}{n^2}. \tag{9}$$

On the other hand any two circles can cross at most twice, which gives $\mathrm{cr}(G') \leq 2\binom{n}{2} \leq n^2$. Together with (9) we obtain $U(P)^3 \leq 512n^4$ and hence $U(P) \leq 8n^{4/3}$. □

It remains open to determine the asymptotic growth of $U(n)$. Erdős had offered \$500 for a proof or disproof of his conjecture.

**Conjecture 2.1** (Erdős [Erd46])**.**

$$U(n) = \mathcal{O}(n^{1+c/\log\log n})$$

Interestingly, the considerations of unit distances in the plane can be transferred into the notion of graphs. A *unit distance graph* is a graph that can be drawn in the plane with edges being straight-line segments of unit length. A *matchstick graph* is a graph that can be drawn in the plane with edges being straight-line segments of unit length and *without crossings*. See Figure 17 for some examples.

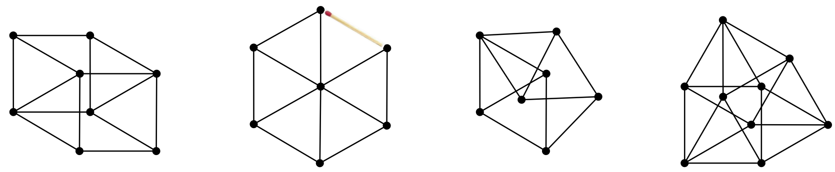


Figure 17: Four unit distance graphs, one of which is a matchstick graph.

Of course, the quantity $U(n)$ can be seen as the maximum number of edges in an $n$-vertex unit distance graph. Another interesting open question asks for the chromatic number of unit distance graphs, i.e., the minimum number of colors required to color the vertices of any unit distance graph so that vertices that share an edge receive distinct colors. The Moser spindle (the third graph in Figure 17 from the left) shows that some unit distance graphs require at least 4 colors.

An upper bound on the maximum chromatic number of unit distance graphs can be obtained by coloring the entire plane so that every two points at distance exactly 1 receive distinct colors. More precisely, we color the infinite graph $P^2 = (\mathbb{R}^2, E)$ whose vertices are the points in the plane and whose edges correspond to pairs of points at unit distance. The least number of colors in such a coloring is called the *chromatic number of the plane*, denoted by $\chi(\mathbb{R}^2)$. Determining the chromatic number of the plane was stated as a problem in 1950 by Edward Nelson and is today known as the Hadwiger-Nelson problem. Since already some unit distance graphs require 4 colors, we have $\chi(\mathbb{R}^2) \geq 4$. On the other hand Figure 18 shows a proper coloring of $P^2$ with 7 colors. Thus we have

$$4 \leq \chi(\mathbb{R}^2) \leq 7.$$

Amazingly, nobody was able to improve these bounds for 60 years now.

According to de Bruijn and Erdős [dBE51] the Hadwiger-Nelson problem, i.e., determining $\chi(\mathbb{R}^2)$, and determining the maximum chromatic number of unit distance graphs are equivalent, under the assumption of the axiom of choice.

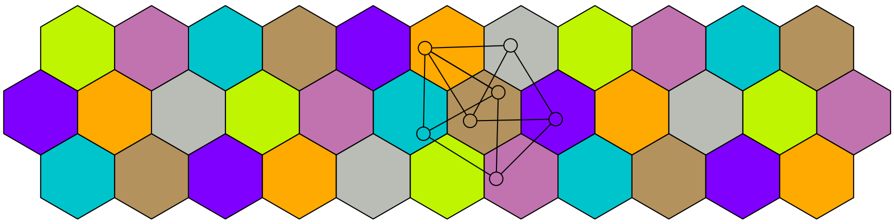


Figure 18: A coloring of the plane with 7 colors such that points at unit distance are colored differently and an induced coloring of the Moser spindle.

# 3 The Sliding Game

We begin with a quote.

> "Sam Loyd (1841–1911) was one of the greatest puzzle designers of all times. Was he a mathematician? Certainly not, but he could have become a great one."
>
> (János Pach 2009)

The puzzles that Pach is referring to are mostly chess puzzles. Indeed Sam Loyd invented over 10.000 chess puzzles, which he published in newspaper columns (the first at the age of 14(!)) and books. He spend his life developing chess strategies, producing puzzles, running music stores, and inventing and selling games. Certainly the most famous game invented by Loyd is the *Fifteen Puzzle* shown in Figure 19. It consists of 15 squares carrying the numbers 1 through 15, lying in a four-by-four box. The goal is to use the one empty space in the box to slide the squares one at a time from a given starting position to the well-ordered position in which the numbers are increasing row by row.


Figure 19: The Fifteen Puzzle invented in the 1870's by Sam Loyd.

The Fifteen Puzzle became very popular just like the Rubik's Cube a hundred years later. Actually nine out of ten people in Great Britain, the US and Europe went a little crazy trying to solve the task Sam Loyd gave to the public. He offered \$1000 for anyone who can solve the Fifteen Puzzle from the initial configuration that is obtained from the well-ordered one by swapping the square labeled 14 and 15.

**Problem 10.**
Show that no one will ever be able to claim the \$1000.

The Fifteen Puzzle has been generalized to arbitrary graphs by Kornhauser, Miller and Spirakis [KMS84]. In the *Sliding Game* one is given a graph and a number of labeled chips placed onto a subset of vertices, at most one chip at each vertex. The goal is to reach a certain final configuration, i.e., a placement of the chips, by applying a number of sliding moves. In each move a chip is send along an edge to a vertex that does not yet has a chip on it. So the Fifteen

Puzzle is equivalent to the Sliding Game with 15 labeled coins on the $4 \times 4$ square grid.

The problem of deciding whether a certain final configuration is reachable from a certain initial configuration, and if so, finding a set of few moves doing the job, has applications in memory management in distributed computing systems, as well as, motion planning, e.g., for robots.

Here we want to analyse the variant of the Sliding Game with *unlabeled chips*. Consider a given connected graph $G = (V, E)$. Let $S_1$ and $S_2$ be two $k$-element subsets of vertices of $G$, i.e., $S_1, S_2 \subseteq V$, $|S_1| = |S_2| = k$. Note that the set $S_1 \cap S_2$ may be non-empty. Imagine a chip is placed at each vertex in $S_1$ and we want to move these $k$ chips into the positions given by $S_2$. If a chip lies at a vertex $v$ and no chip lies at some vertex $w$, then a move from $v$ to $w$ is defined as sliding the chip from $v$ to $w$ along a $v, w$-path (not only an edge) in $G$ of which no intermediate vertex is occupied by a chip, if any such path exists.

**Theorem 3.1.** *In any connected $n$-vertex graph one can get from any $k$-element initial configuration ($k \leq n$) to any $k$-element final configuration in at most $k$ moves.*

*Proof.* We prove the result by induction on $k$. The induction base $k = 0$ (or $k = 1$ if preferred) is immediate. So let $k \geq 1$.

Let $S_1, S_2$ be the initial and final configuration, respectively. Let $T$ be a smallest (inclusion-minimal) tree in the graph containing all vertices in $S_1 \cup S_2$. Then every leaf of $T$ lies in $S_1 \cup S_2$. Let $v$ be a leaf.

- *Case 1:* $v \in S_1 \cap S_2$ – We remove the vertex $v$ from $T, S_1$ and $S_2$. Since $v$ is a leaf $T \setminus v$ is again a tree and hence connected. Moreover, $|S_1 \setminus v| = |S_2 \setminus v| = k - 1$. Now the result follows by applying induction to $T \setminus v$ and $S_1 \setminus v, S_2 \setminus v$.
- *Case 2:* $v \in S_1 \setminus S_2$ – Choose a path $P$ in $T$ connecting $v$ to a vertex $w \in S_2$ such that no intermediate vertex of $P$ belongs to $S_2$. Applying induction to $T \setminus v$ and $S_1 \setminus v, S_2 \setminus w$ we obtain a sequence of at most $k - 1$ moves bringing all vertices in $S_1 \setminus v$ into the positions given by $S_2 \setminus w$. Thus the path $P$ contains besides $v$ and $w$ no vertices from $S_1 \cup S_2$. In particular there is a move from $v$ to $w$.
- Case 3: $v \in S_2 \setminus S_1$ – This case is symmetric to Case 2.

We remark that the above induction not necessarily performs the optimal (minimum) number of moves. □

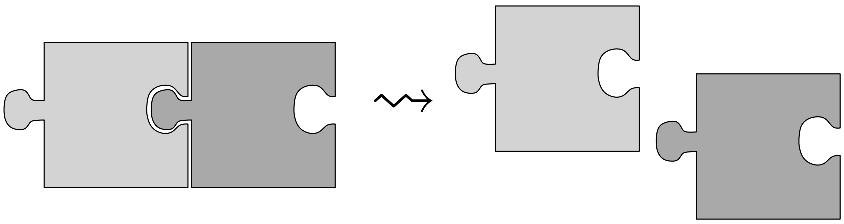
## 3.1 The Geometric Sliding Game

Now let us consider the following geometric version of the Sliding Game. Consider a set of $k$ pairwise disjoint, objects in the plane. With objects we mean "nice" subsets of $\mathbb{R}^2$, namely closed, bounded and path-connected[2] subsets such as disks ("coins") or segments ("matchsticks"). These objects need to be brought from an initial configuration $S_1$ to a final configuration $S_2$ by sliding moves. In a move we allow to take an object and slide it, possibly while rotating it in a subtle way, to another position without colliding with the other objects. In the

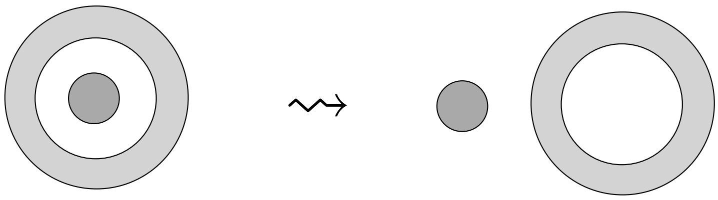
[2]The set contains a path between any two of its points.

unlabeled version the objects are congruent to each other and we do not specify which object needs to be brought to which position. In the labeled version we have general objects (every two of which may be different) and we do specify the final position of each such object.

The problem is not always feasible, that is, it may be that there exists no set of moves bringing the objects from the initial to the final configuration. This may happen in the labeled as well as the unlabeled version. Two such situations are illustrated in Figure 20.



(a)



(b)

Figure 20: Two situations where the geometric Sliding Game is infeasible.

On the other hand it is easy to see that the geometric Sliding Game is always feasible for congruent disks. Indeed, it is always feasible with convex objects (labeled or not).

**Theorem 3.2.** *Any set of $k$ convex objects in the plane can be moved from any initial configuration to any final configuration in at most $2k$ moves.*

*Proof.* Let $S_1$ and $S_2$ denote the initial and final configuration of objects, respectively. We bring the objects into the final positions in two phases, each consisting of $k$ moves. In the first phase we slide the objects, one by one, along the vertical direction far down. Indeed, these moves are pure translations along the vertical unit vector. In the second phase we slide each object into its final position given by $S_2$.

All that needs to be shown is that in any configuration at least one object can be moved vertically towards $-\infty$ without colliding the other objects. If this is possible for an object, one says that it can be *separated in the direction* $(0, -1)$.

**Lemma 3.1.** *Given any set of $k$ pairwise disjoint, convex bounded objects in the plane, there is at least one object that can be separated in the direction* $(0, -1)$.

*Proof of Lemma.* Consider for each object a leftmost and rightmost point amd shoot a vertical ray from each such point upwards. We define a walk along the objects, starting from any point on the leftmost ray as follows. On ray corresponding to leftmost points walk downwards. When reaching the end of the ray walk in counterclockwise direction around the corresponding object until encountering another ray. If this ray corresponds to some leftmost point, continue as before. Note that the walk is weakly $x$-monotone and at all times can be seen from $(0, -\infty)$. See Figure 21 for an example.

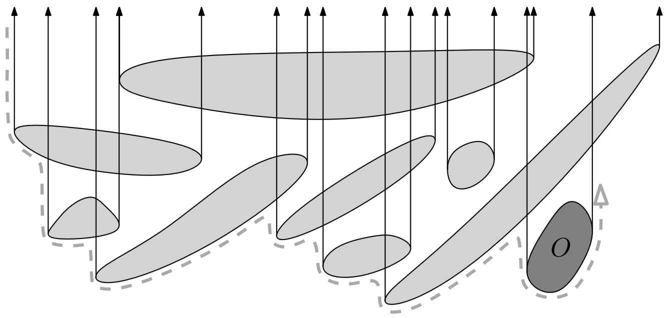



Figure 21: A set of convex objects with a ray starting from a leftmost and rightmost point of each object, and a walk along the objects. The object $O$ can be separated in the direction $(0, -1)$.

The walk will encounter a ray corresponding to a rightmost point, at the latest, when reaching the rightmost point of the rightmost object. When encountering the *first* rightmost ray the corresponding object $O$ has been traversed consecutively from a leftmost to a rightmost point in counterclockwise direction. Now this object $O$ can be separated in the direction $(0, -1)$ because the walk on $O$ can be seen from $(0, -\infty)$ and $O$ is convex. □

Iteratively applying Lemma 3.1 we obtain a separation order of $S_1$ and another separation order of $S_2$. Now in the first phase the objects are translated down according to the separation order of $S_1$, so that in their positions between the two phases no two objects can be pierced by the same horizontal line. In the second phase the objects are slided into their final position according to a *reverse* separation order of $S_2$. □

We remark that the last move in the first phase is unnecessary. Thus every set of $k$ convex objects can be reconfigured in at most $2k - 1$ moves.

**Problem 11.**
Provide an instance of the geometric Sliding Game with $k$ unlabeled convex objects which require $2k - 1$ moves for reconfiguration.

For congruent disks, the maximum number of moves needed is still unknown. Bereg, Dumitrescu and Pach [BDP05] show that for $k$ disks $\frac{3k}{2} + \mathcal{O}(\sqrt{k \log k})$ moves are always sufficient and $(1 + \frac{1}{15})k - \mathcal{O}(\sqrt{k})$ moves are sometimes necessary.

To end this chapter, we present Sam Loyd's Juggler puzzle.

The clown after juggling with the five triangular pieces of cardboard to attract attention proceeds to cut one of them into two pieces.
He then lays the six pieces upon the top of the box and shows that they will fit together and form a perfect square.
The pieces represent five right-angled triangles, say one inch high by two inches on the base, so you can readily cut five similar pieces from paper and then guess how to cut one of them so that the six pieces will form a perfect square.

# 4 Convexity

We introduce one of the most important concepts in combinatorial geometry: *Convexity.* We start by presenting the standard notions and notations, as well as, the three best-known theorems about convexity: Carathéodory's Theorem, Radon's Lemma and Helly's Theorem.

**Definition 4.1** (Convexity)**.**
*A set $X \subseteq \mathbb{R}^d$ is* convex *if for every two points $x, y \in X$ the segment $\overline{xy}$ is entirely contained in $X$. In other words, $X$ is convex if for any two points $x, y \in X$ and every real number $\lambda \in [0, 1]$ we have $\lambda x + (1 - \lambda)y \in X$.*

We refer to Figure 22 for some examples of convex and non-convex sets in the plane.

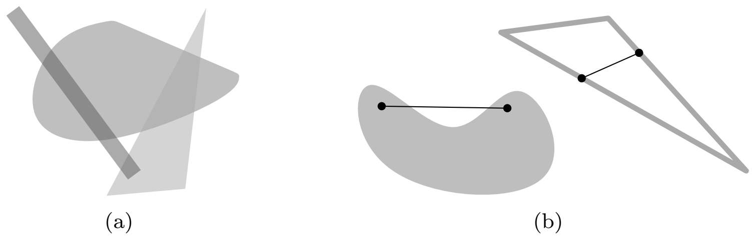


Figure 22: (a) Three convex sets. (b) Two non-convex sets, each with two points certifying non-convexity.

Note that the intersection of any (not necessarily finite) family of convex sets is again convex. However, the union of two convex sets is "most likely" not convex. For any set $X \in \mathbb{R}^d$ we define the *convex hull* of $X$, denoted by conv($X$), as the smallest (that is inclusion-minimal) convex set containing $X$. Equivalently,

$$\mathrm{conv}(X) = \bigcap_{Y \supseteq X, Y \,\mathrm{convex}} Y, \tag{10}$$

i.e., the convex hull of $X$ is the intersection of all convex sets containing $X$.

It is easy to see that the convex hull is indeed a *hull operator* (sometimes called closure operator), namely that it enjoys the following properties for all sets $X, Y \subseteq \mathbb{R}^d$.

- $\mathrm{conv}(X) \supseteq X$ (conv is extensive)
- $\mathrm{conv}(\mathrm{conv}(X)) = \mathrm{conv}(X)$ (conv is idempotent)
- $X \subseteq Y \Rightarrow \mathrm{conv}(X) \subseteq \mathrm{conv}(Y)$ (conv is monotone)

Note that by (10) we have that $\mathrm{conv}(X) = X$ if and only if $X$ is convex itself. An alternative definition of the convex hull is given by convex combinations. A point $x \in \mathbb{R}^d$ is a *convex combination of points* $x_1, \ldots, x_k \in \mathbb{R}^d$ if there exists non-negative real numbers $\lambda_1, \ldots, \lambda_k$ with $\sum_{i=1}^k \lambda_i = 1$ such that $x = \sum_{i=1}^k \lambda_i x_i$. Then for any set $X$ we have

$$\mathrm{conv}(X) = \{x \in \mathbb{R}^d \mid x \text{ is convex combination of finitely many points in } X\}.$$

That we can indeed restrict to convex combinations of very few points in $X$ (in the plane three points are already enough) is the statement known as Carathéodory's Theorem.

**Theorem 4.1** (Carathéodory's Theorem)**.**
*Let $X \subseteq \mathbb{R}^d$. Then each point of* $\text{conv}(X)$ *is a convex combination of at most $d+1$ points of $X$.*

In the plane Theorem 4.1 says, that the convex hull of any set $X$ is equal to the union of all triangles with endpoints in $X$. If $X$ is finite, one can even restrict to a small subset of triangles by *triangulating the points in $X$.* We won't go into more details here and just refer to Figure 23 for an example. In general there is many ways to triangulate the points in $X$, some of which we will consider later.

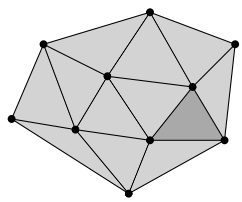


Figure 23: Triangulating a finite point set $X$ with 10 points. The convex hull of $X$ is the union of the 12 triangles with non-intersecting interiors.

**Problem 12.**
Prove Carathéodory's Theorem. You may use Radon's Lemma from below.

We continue with the second basic theorem about convex sets after Carathéodory's Theorem.

**Theorem 4.2** (Radon's Lemma)**.**
*Every set $P$ of $d+2$ points in $\mathbb{R}^d$ contains two disjoint subsets $P_1, P_2$ such that*

$$\text{conv}(P_1) \cap \text{conv}(P_2) \neq \emptyset.$$

*Proof.* The $d+2$ points $p_1, \ldots, p_{d+2}$ in $P$ are affinely dependent, i.e., there exists $\lambda_1, \ldots, \lambda_{d+2}$, not all zero, such that

$$\sum_{i=1}^{d+2} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{d+2} \lambda_i p_i = 0.$$

Thus we have

$$0 = \sum_{i=1}^{d+2} \lambda_i p_i = \sum_{i:\lambda_i>0} \lambda_i p_i - \sum_{i:\lambda_i<0} (-\lambda_i) p_i.$$

Moreover, since $\sum_{i=1}^{d+2} \lambda_i = 0$ we have $\sum_{i:\lambda_i>0} \lambda_i = \sum_{i:\lambda_i<0} (-\lambda_i) = \Lambda$. Together we obtain a point $x$ which is a convex combination of $P_1 = \{p_i : \lambda_i > 0\}$ as well as a convex combination of $P_2 = \{p_i : \lambda_i < 0\}$:

$$x = \sum_{i:\lambda_i>0} \frac{\lambda_i}{\Lambda} p_i \quad \text{and} \quad x = \sum_{i:\lambda_i<0} \frac{-\lambda_i}{\Lambda} p_i$$

Clearly, $P_1$ and $P_2$ are both non-empty and disjoint, which proves the statement. $\square$

In the plane, Radon's Lemma amounts for simply checking the only two possible situations, which are depicted in Figure 24.

Figure 24: The only two combinatorial different configurations of four points in the plane and two disjoint subsets (black points and white points respectively) with intersecting convex hulls.

The third, and probably most famous, combinatorial result about convex sets is Helly's Theorem.

**Theorem 4.3** (Helly's Theorem).
*Let $\mathcal{C}$ be a finite set of convex sets in $\mathbb{R}^d$. If any $d+1$ of these sets have a non-empty intersection, then all the sets have a non-empty intersection.*

*Proof.* We proceed by induction on $n = |\mathcal{C}|$. The case $n \leq d+1$ is immediate, so assume that $n \geq d+2$ and consider the sets $X_1, \ldots, X_n$ in $\mathcal{C}$.

For every $i = 1, \ldots, n$ the sets in $\mathcal{C} \setminus X_i$ satisfy the assumptions of Helly's Theorem and hence we conclude by induction that all these sets have a non-empty intersection. We fix a point $p_i \in \bigcap_{j \neq i} X_j$ arbitrarily. This gives an $n$-element point set $P = \{p_1, \ldots, p_n\}$ in $\mathbb{R}^d$ with $n \geq d+2$. By Radon's Lemma (Theorem 4.2) there exist disjoint subsets $P_1, P_2$ of $P$ such that $\operatorname{conv}(P_1) \cap \operatorname{conv}(P_2) \neq \emptyset$.

We pick a point $x$ in this intersection and claim that $x \in X_i$ for all $i = 1, \ldots, n$. $\square$

Helly's Theorem is no longer true for collections $\mathcal{C}$ of infinitely many convex sets. Already in $\mathbb{R}^1$, i.e., on the real line, there are sets of infinitely many intervals such that any finite subset of these have a non-empty intersection but there is no point contained in each and every interval. For example consider $\mathcal{C} = \{(0, 1/n) \mid n \in \mathbb{N}\}$ or $\mathcal{C} = \{[n, \infty) \mid n \in \mathbb{N}\}$. However, for *compact* (meaning bounded and closed) convex sets, Helly's theorem remains true even if $\mathcal{C}$ consists of infinitely many sets.

Inspired by Helly's Theorem (Theorem 4.3) we make the following definitions. Let $P$ be a *hereditary property of sets in* $\mathbb{R}^d$, meaning that if a family $\mathcal{F}$ has property $P$ then so has every $\mathcal{F}' \subseteq \mathcal{F}$. Examples for hereditary properties are

- having a non-empty intersection,
- containing a set from a certain class of sets in the common intersection,
- being contained in an affine subspace of dimension $d-1$,
- being pairwise disjoint.

**Definition 4.2** (Helly number).
*A family $\mathcal{C}$ of sets in $\mathbb{R}^d$ is said to have* Helly number $k$ with respect to a hereditary property $P$ *if $k$ is the smallest positive integer for which the following is true for every finite subfamily $\mathcal{F} \subseteq \mathcal{C}$:*

*If every subset $A \subseteq \mathcal{F}$ of size $|A| \leq k$ has property $P$, then so has $\mathcal{F}$.*

So Helly's Theorem says that the family of all convex sets in $\mathbb{R}^d$ has Helly number $d + 1$.

**Problem 13.**
Determine the Helly number of family $\mathcal{C}$ and property $P$ in each of the following cases:

- $\mathcal{C}$ is the family of all axis-aligned boxes in $\mathbb{R}^2$ and $P$ is "having a non-empty intersection".
- $\mathcal{C}$ is the family of all convex sets in $\mathbb{R}^2$ and $P$ is "having a translated copy of a fixed convex set $X \neq \emptyset$ in the intersection".
- $\mathcal{C}$ is the family of all convex sets in $\mathbb{R}^2$ and $P$ is "having some ray in the intersection".
- $\mathcal{C}$ is the family of all closed convex sets in $\mathbb{R}^2$ and $P$ is "the intersection fits between two parallel lines at distance 1".

## 4.1 Sets of Constant Width

The following is an interesting and equally important question. For example, think of the reconstruction of 3-dimensional objects from 2-dimensional scans such as X-rays or Magnetic Resonance Imaging.

*"Can a convex $d$-dimensional object be recovered from all its $(d-1)$-dimensional projections?"*

Let us consider the above question in the plane (for $d = 2$). So let $X \subseteq \mathbb{R}^2$ be a convex set. For any $a, b \in \mathbb{R}$ the projection of $X$ onto the line $\ell(a, b) = \{(x, y) \in \mathbb{R}^2 \mid ax + by = 0\}$ is a (open, closed, or half-open) segment denoted by $X|_{\ell(a,b)}$. For bounded $X$ the *width* of $X$ in direction $(a, b)$ is the length of the segment $X|_{\ell(a,b)}$. See Figure 25 for an example.

**Problem 14.**
Show that if $X \subseteq \mathbb{R}^2$ is a compact convex set and $p \in X$ is any point, then there exists $a, b \in \mathbb{R}$ and a segment $s_{a,b}(p)$ with the following properties.

- $s_{a,b}(p)$ is parallel to $\ell(a, b)$.
- $p$ is contained in $s_{a,b}(p)$ and $s_{a,b}(p)$ is contained in $X$.
- The length of $s_{a,b}(p)$ is the width of $X$ in direction $(a, b)$.

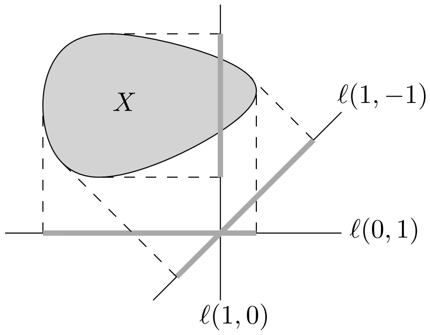



Figure 25: A bounded convex set $X$ and its projections to the lines $\ell(1,-1)$, $\ell(0,1)$ and $\ell(1,0)$.

For $d = 2$ the above question translates into the following. Can a bounded convex set $X$ in the plane be recovered from all its widths? Clearly, if $X$ is not closed, then its closure defines the same widths. But (maybe surprisingly) the answer for closed sets is also *NO*. Even in the most simple case, when all the widths are the same, there is infinitely many sets defining these widths.

**Definition 4.3** (Set of Constant Width)**.**
*A compact (meaning bounded and closed) convex set $X$ in the plane is a* set of constant width $w$ *if for every $a, b \in \mathbb{R}$ the width of $X$ in direction $(a, b)$ equals $w$.*

Since by appropriate scaling we can assume that $w = 1$ we often omit to specify the width $w$ and simply call those sets sets of constant width. Figure 26 shows some examples of sets of constant width. Such sets can be constructed as follows.

**Reuleaux polygons:** Start with a regular $n$-gon $P$ with $n$ odd. Draw a circular arc between any two consecutive points of $P$ with center being the point opposite on $P$. The convex hull of all these arcs is a set of constant width. Figure 26(a) shows Reuleaux polygons for $n = 3$ and $n = 5$.

**Based on a triangle:** Consider three distinct lines $\ell_1, \ell_2, \ell_3$, not all concurrent, no two being parallel. For every crossing point $p_{ij} = \ell_i \cap \ell_j$ draw a circular arc with center $p_{ij}$ between the rays of $\ell_i$ and $\ell_j$ containing another crossing point and a second circular arc between the rays of $\ell_i$ and $\ell_j$ *not* containing any other crossing point. Choose the radii so that these six circular arcs meet with their endpoints in a cyclic way. Depending on the triangle you may choose one or more radii to be 0. The convex hull of the six circular arcs is a set of constant width. See the top and middle of Figure 26(c) for two examples.

**Based on a convex curve:** Consider two opposite points on a $w \times w$ square, one on the left and one on the right. Draw a convex curve $\gamma$ between these points so that $\gamma$ is tangent to the bottom side of the square and at all points the curvature of $\gamma$ is at least the curvature of a circle of radius $w$. Next, consider all segments of length $w$ standing orthogonally on $\gamma$

pointing up. The union of all these segments is a set of constant width $w$. See the bottom of Figure 26(c) for an example.

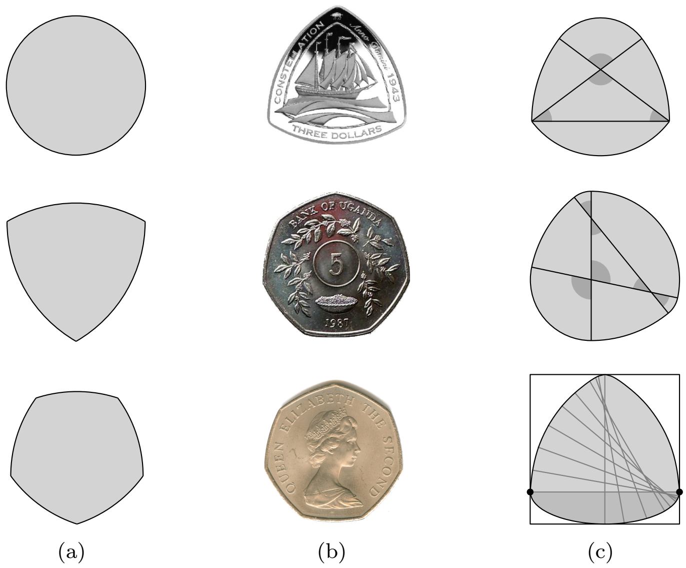



Figure 26: (a) Three sets of constant width: The ball (top), the Reuleaux triangle (middle) and the Reuleaux pentagon (bottom). (b) Some examples of non-circular coins. (c) Constructing a set of constant width on basis of an isosceles triangle (top), a general triangle (middle) and a half-ellipse inscribed in a square (bottom).

In particular the Reuleaux polygons are used to design coins, see for example Figure 26(b), because coins are recognized by coin-operated machines only based on their width, weight and/or engraving at the rim. Non-circular coins are introduced amongst other reasons to save raw materials. The set of constant width $w$ that has the smallest area was determined by Blaschke and Lebesgue in the 1910's.

**Theorem 4.4** (Blaschke [Bla15], Lebesgue [Leb14]).
*Among all sets of constant width $w$, the Reuleaux triangle minimizes the area.*

Note that no two distinct sets of constant width $w$ are contained in another. And it is not visible to the naked eye that the area of the Reuleaux triangle is less than the area of the ball. Indeed, as of today the 3-dimensional set of constant width with minimum volume is still unknown.

Another 50 years before the Blaschke-Lebesgue Theorem, in 1860, the 21-years old Frenchman Emile Barbier proved that the perimeter of all sets of constant width $w$ is the same as that of the circle of diameter $w$.

**Theorem 4.5** (Barbier [Bar60]).
*Every set of constant width $w$ has perimeter $\pi w$.*

To prove Barbier's Theorem let us first introduce *Buffon's needle problem.* In 1777 Georges Louis Leclerc, Comte de Buffon, asked the following question.

*"Suppose you drop a needle on ruled paper, where the needle is shorter than the distance between the lines on the paper. What is the probability that the needle comes to lie in a position where it crosses one of the lines?"*

The needle problem can be solved by evaluating a suitable integral, which would also solve the problem for a long needle. However, Emile Barbier's proof uses a very elegant method in probabilistic geometry.

**Theorem 4.6.** *If a needle of length $\ell$ is dropped on ruled paper with distance $d \geq \ell$ between the lines, then the probability that the needle comes to lie in a position where it crosses one of the lines is exactly*

$$p = \frac{2}{\pi}\frac{\ell}{d}.$$

*Proof.* Clearly if we drop a needle of length $\ell$, no matter the distance between the lines, then the expected number of lines that it crosses is

$$E(\ell) = \sum_{i \geq 0} i \cdot p_i,$$

where $p_i$ denotes the probability that the needle crosses exactly $i$ lines. Now if we write $\ell = x + y$ then we get

$$E(x + y) = E(x) + E(y), \tag{11}$$

because each crossing of the needle is produced with probability 1 *either* by the front part of length $x$ *or* by the back part of length $y$. From (11) we can derive $E(nx) = nE(x)$ for all $n \in \mathbb{N}$, which implies $mE(\frac{n}{m}x) = E(m\frac{n}{m}x) = E(nx) = nE(x)$. Thus for all rational number $r = \frac{n}{m}$ holds $E(rx) = rE(x)$. Because $E(x)$ is monotone, we conclude that

$$E(x) = cx \text{ for all } x \geq 0 \text{ and some constant } c = E(1). \tag{12}$$

Now note that (11) did not use the fact that the front part and the back part are aligned to one longer straight segment. Indeed the same holds if front and back are glued with their ends in any angle. And the same holds if the needle is a general polygonal line with any number of straight segments. If its total length is $\ell$, then the expected number of crossings with the lines is exactly $E(\ell)$ and (12) still holds.

Finally we may even consider a curved needle by approximating it with polygonal lines of the same length but with more and more segments. In the limit, $E(\ell)$ still gives the same number for any arbitrary curve of length $\ell$. In particular, we can consider a needle that is a perfect circle of diameter $d$. Such a needle has length $\pi d$ and, more importantly, if it is dropped it produces exactly two crossings no matter where it comes to lie. We conclude

$$E(\pi d) = 2 \quad \text{and thus} \quad c = \frac{2}{\pi d}.$$

Together with $E(\ell) = p_1$, provided $\ell \leq d$, this proves the theorem. □

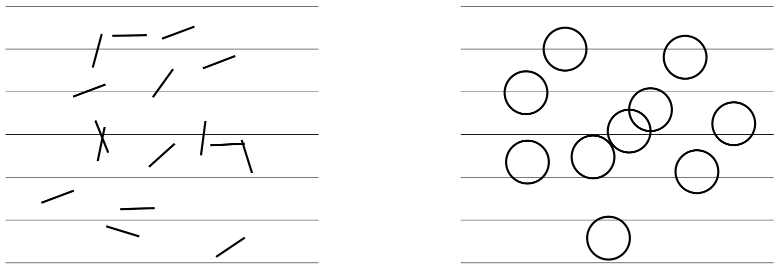


Figure 27: Dropping needles and circles on ruled paper at random.

*Proof of Barbier's Theorem.* Consider any set $X$ of constant width $w$. To prove that $X$ has perimeter $\pi w$, it is now sufficient to note that in the above proof a needle that forms the boundary of $X$ produces exactly two crossings with equally spaced lines at distance $w$. Thus the length $\ell$ of the needle, which is also the perimeter of $X$, satisfies

$$\frac{2}{\pi}\frac{\ell}{w} = E(\ell) = 2.$$

In other words $\ell = \pi w$. $\square$

For more on sets of constant width, including how to drill a square hole with the Reuleaux triangle and 3-dimensional sets of constant width, we refer to the short survey of Kawohl [Kaw09].

Next let us focus on more discrete problems with convexity. In particular, we are interested in finite point sets $X$ in $\mathbb{R}^d$; of course with strong preference for the case $d = 2$. Given a finite point set $X \subset \mathbb{R}^d$ its convex hull $\mathrm{conv}(X)$ is called a *polytope*. We will always assume that $\mathrm{conv}(X)$ is full-dimensional, i.e., it is not contained in any $(d-1)$-dimensional hyperplane. Let us fix some notations. We refer to Figure 28 for examples illustrating these definitions.

- A point $p \in \mathrm{conv}(X)$ is called a *corner* of $\mathrm{conv}(X)$ if

  $$p \notin \mathrm{conv}(X \setminus p).$$

  In particular, the set of all corners of $\mathrm{conv}(X)$ is a subset $\bar{X}$ of $X$.

- A finite set $X \subset \mathbb{R}^d$ is said to lie *in convex position* if

  $$\text{every } x \in X \text{ is a corner of } \mathrm{conv}(X).$$

  Equivalently, $X$ is the set of all corners of some polytope in $\mathbb{R}^d$.

- A $(d-1)$-dimensional hyperplane $h$ is called a *supporting hyperplane* of $\mathrm{conv}(X)$ if there is a closed half-space $H$ defined by $h$ such that

  $$\mathrm{conv}(X) \subseteq H \quad \text{and} \quad |h \cap \bar{X}| \geq d.$$

  Equivalently, $h \cap \mathrm{conv}(X)$ is a $(d-1)$-dimensional set contained in the boundary of $\mathrm{conv}(X)$.

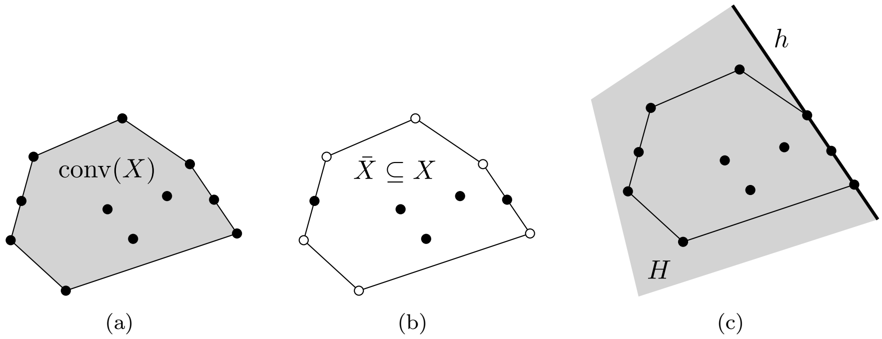



Figure 28: (a) The convex hull of finitely many points is a polytope. (b) The set of corners of conv($X$) is a subset $\bar{X}$ of $X$. (c) A supporting hyperplane $h$ contains at least $d$ corners of conv($X$).

In the plane, i.e., when $d = 2$, polytopes are also called *convex polygons*, or simply *polygons* when convexity is given from the context. For a convex polygon $P = \text{conv}(X)$ the supporting hyperplanes are simply lines and the intersection of a supporting line $\ell$ with $P$ is called an *edge* of $P$. Clearly, every convex polygon has as many edges as corners.

**Problem 15.**
Consider a compact convex set $X$ in the plane, a number $\alpha \in (0, \pi]$ and the locus $L_\alpha$ of all points that can see $X$ with an $\alpha$ aperture angle. See Figure 29 for an illustration.

a) For which values of $\alpha$ is $L_\alpha$ the boundary of some convex set?

b) Describe the shape of $L_{\pi/2}$ in case $X$ is a convex polygon.

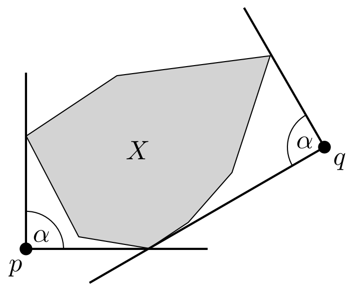



Figure 29: A compact convex set $X$ in the plane and two points $p$ and $q$ that see $X$ with an $\alpha$ aperture angle with $\alpha = \pi/2$.

**Problem 16.**
Consider a convex polygon $P$ in the plane and a number $t$ strictly greater than the area of $P$. For each point $q \notin P$ let $P(q)$ be the polygon $\mathrm{conv}(P \cup \{q\})$.

a) Prove that the locus $L_t$ of all points $q$ for which the area of $P(q)$ equals $t$ is the boundary of a convex polygon enclosing $P$.

b) Prove that if $P$ has $n$ corners, then $L_t$ has between $n$ and $2n$ corners. When does $L_t$ have less than $2n$ corners?

## 4.2 Centerpoints

Let $X$ be a finite set of points in the plane. Let us think of which points of $\mathbb{R}^d$ are very "central" in $X$ or lie "deep" within $X$. Intuitively, a point is deep within $X$ if it cannot be reached from the outside (say $\mathbb{R}^d \setminus \mathrm{conv}(X)$) without passing through many (say half or one third) other points of $X$. We present two attempts to formalize this intuition. In especially, we investigate the concepts of centerpoints of $X$; once with respect to half-spaces and once with respect to quadrants.

**Definition 4.4** (Half-Space Centerpoint).
*A* half-space centerpoint *of a set $X$ of $n$ distinct points in $\mathbb{R}^d$ is a point $x \in \mathbb{R}^d$ for which each closed half-space that contains $x$ contains at least $\frac{n}{d+1}$ points from $X$.*

For $d = 1$ the set $X$ is just a set of $n$ real numbers $\{r_1, \ldots, r_n\}$ and a half-space centerpoint corresponds to a number $r$ that is less than or equal to half of the numbers in $X$ and greater than or equal to half of the numbers in $X$. Indeed, in this case we can always pick $r$ from the set $X$. If $|X| = n$ is odd, then we even have to pick such $r$ from $X$. Such a half-space centerpoint (number) of a 1-dimensional finite set (set of numbers) is better known as the *median* of $X$.

For $d > 1$ there are cases in which no half-space centerpoint belongs to $X$. For example, consider $X$ to be any set of $n$ points on a circle $C \subset \mathbb{R}^d$. Then for every point $x \in X$ we can find a closed half-space $H$, e.g., one that lies tangent on $C$ at $x$, that contains $x$ but no second point from $X$. See Figure 30(a) for an illustration.

The fact that every finite point set in $\mathbb{R}^d$ has a half-space centerpoint was proven by Rado in 1946 [Rad46] and is today known as the *Centerpoint Theorem.* Since we will need it later, we include a "moreover" statement for the 2-dimensional case.

**Theorem 4.7** (Centerpoint Theorem).
*For every finite point set $X$ in $\mathbb{R}^d$ there exists a half-space centerpoint $p$.*

*Moreover, if $d = 2$ then $p$ can be chosen to be a point of $X$ or as the intersection of two segments with endpoints in $X$.*

*Proof.* Let $X$ be any set of $n$ points in $\mathbb{R}^d$. First note that $x \in \mathbb{R}^d$ is a half-space centerpoint if and only if $x$ lies in every open half-space $H$ with $|X \cap H| > \frac{d}{d+1}n$. Let $\mathcal{H}_X = \{H : H \text{ open half-space with } |X \cap H| > \frac{d}{d+1}n\}$. Hence, we

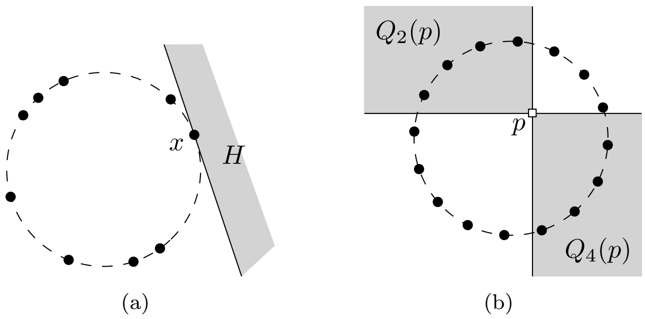



Figure 30: (a) If $X$ is any set of $n$ points on a circle, then no point of $X$ is an $\alpha$-half-space centerpoint for $\alpha > 1/n$. (b) If $X$ is a set of $n$ equidistant points on a circle, then no point of $\mathbb{R}^2$ is an $\alpha$-quadrant centerpoint for $\alpha > 1/4 + 1/n$.

have to show that the intersection of all these half-spaces is non-empty, i.e., $\bigcap_{H \in \mathcal{H}_X} H \neq \emptyset$. However, we cannot use Helly's Theorem directly since we have infinitely many half-spaces all of which are open and unbounded.

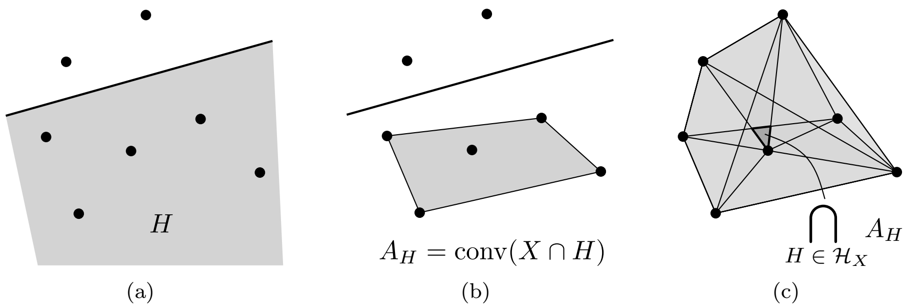



Figure 31: (a) A half-space $H$ containing more than $\frac{d}{d+1}n$ points from a set $X$ of $n$ points. (b) The corresponding convex compact set $A_H = \mathrm{conv}(X \cap H)$. (c) The intersection of all $A_H$ with $H \in \mathcal{H}_X$ is a convex polytope.

Instead, we consider for each half-space $H \in \mathcal{H}_X$ the set $A_H = \mathrm{conv}(X \cap H)$, which is convex and compact. See Figure 31(a)-(b) for an illustration. Moreover, there is only finitely many of them. Every $A_H$ contains more than $\frac{d}{d+1}n$ points from $X$, so the intersection of any $d+1$ of these $A_H$ contains at least one point of $X$, i.e., is non-empty. By Helly's Theorem (Theorem 4.3) there exists a point $x$ such that

$$x \in \bigcap_{H \in \mathcal{H}_X} A_H.$$

With $A_H \subset H$ for each $H \in \mathcal{H}_X$ we conclude $x \in \bigcap_{H \in \mathcal{H}_X} H$, i.e., $x$ is a half-space centerpoint of $X$.

Finally observe that the intersection of all $A_H$ with $H \in \mathcal{H}_X$ is a convex polytope because there are finitely many $A_H$ each of which is a convex polytope. Choosing in case $d = 2$ the half-space centerpoint $p$ to be a corner of this polytope (it is a polygon now) we see that either $p \in X$ or $p$ is the intersection of two edges from distinct $A_H$. Since every edge of every $A_H$ is a segment with endpoints in $X$ we get that in the latter case $p$ is the intersection of two such segments. See Figure 31(c) for an example. $\square$

The fraction $\frac{1}{d+1}$ in the definition of a half-space centerpoint is the maximal number for which the Centerpoint Theorem (Theorem 4.7) still holds. Indeed, every set $X$ of $d+1$ affinely independent points in $\mathbb{R}^d$ has no "$\alpha$-half-space centerpoint" for $\alpha > \frac{1}{d+1}$.

Recall that Radon's Lemma (Theorem 4.2) states that every set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two sets such that the convex hulls of these sets intersect, meaning that they have a non-empty intersection. Now suppose we want to partition our point set not only into two sets but $r$ sets whose $r$ convex hulls mutually intersect. Let us call such a partition a *good $r$-partition*. For example, Figure 32(a) shows two good 3-partitions of the same set of seven points in the plane. It is not difficult to see that if we increase the number of points (formally $d+2$ in Radon's Lemma) to some high enough number than such a good $r$-partition always exists.

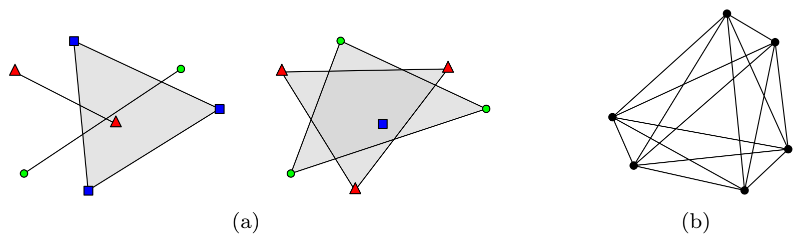


Figure 32: (a) Two good 3-partitions of the same set of seven points in the plane. (b) A set of six points in the plane with no good 3-partition.

It is also not very hard to come up with examples of sets of $2d+2$ points in $\mathbb{R}^d$ that have no good 3-partition. Indeed, Figure 32(b) shows a set $X$ of six points in the plane with no good 3-partition. Since the points in $X$ are in convex position each set in any good $r$-partition must contain at least two points and hence all sets in a good 3-partition must consist of exactly two points. However, no three of the fifteen segments with endpoints in $X$ intersect in a common interior point, which implies that no good 3-partition exists.

In general one requires more than $(r-1)(d+1)$ points in $\mathbb{R}^d$ in order to guarantee the existence of a good $r$-partition. That $(r-1)(d+1)+1$ points are indeed always sufficient for a good $r$-partition was proven by Tverberg in 1966 [Tve66]. He also published a better proof in 1981 [Tve81] and yet another one together with Vrećica in 1993 [TV93].

**Theorem 4.8** (Tverberg's Theorem)**.**
*Every set $X$ of $(r-1)(d+1)+1$ points in $\mathbb{R}^d$ can be partitioned into $r$ disjoint non-empty sets $X_1, \ldots, X_r$ such that*

$$\operatorname{conv}(X_1) \cap \cdots \operatorname{conv}(X_r) \neq \emptyset.$$

Note that setting $r=2$ we obtain Radon's Lemma (Theorem 4.2). We omit a proof of Tverberg's Theorem in full generality here. Instead let us present a short and elegant proof for the case $d=2$.

*Proof of Tverberg's Theorem in $d = 2$ dimensions.*
We have to show that every set $X$ of $n = 3r - 2$ points in the plane can be partitioned into $r$ subsets $X_1, \ldots, X_r$ such that there is a point $p$ in the convex hull of each $X_i$, $i = 1, \ldots, r$.

Somehow strangely, we first identify $p$ and afterwards the sets $X_1, \ldots, X_r$. We choose $p$ to any half-space centerpoint for $X$ with the additional property that either $p \in X$ or $p = \mathrm{conv}(x_1, x_2) \cap \mathrm{conv}(x_3, x_4)$ for some $x_1, x_2, x_3, x_4 \in X$. The existence of such a point $p$ is guaranteed by the Centerpoint Theorem (Theorem 4.7). In the first case, i.e., when $p \in X$, we define $X_r = \{p\}$. In the second case we define $X_r = \{x_1, x_2\}$ and $X_{r-1} = \{x_3, x_4\}$. In either case we are left with a set $X'$ of $3k$ points from $X$ that are yet to be partitioned into $k$ sets, with $k \in \{r - 1, r - 2\}$.

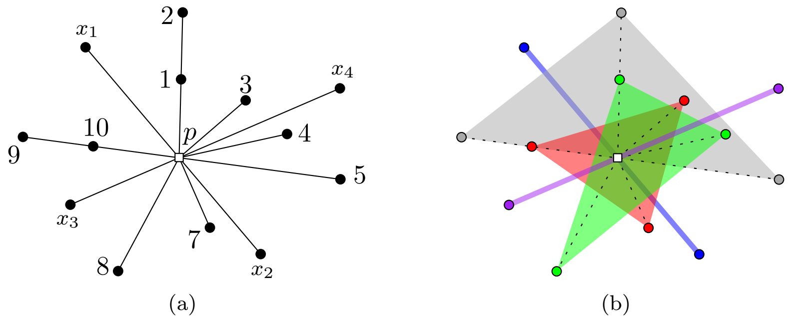



Figure 33: (a) A set $X$ of $n = 3r - 2 = 13$ points in the plane, a half-space centerpoint $p$ of $X$ of the form $p = \mathrm{conv}(x_1, x_2) \cap \mathrm{conv}(x_3, x_4)$, and a numbering of the points in $X' = X \setminus \{x_1, x_2, x_3, x_4\}$ according to their cyclic order around $p$. (b) An $r$-good partition of $X$.

Next we number the points in $X'$ from 1 to $3k$ according to their cyclic order around $p$ – say clockwise around $p$. In case two or more points of $X'$ lie on the same ray emanating from $p$ we give these points consecutive numbers in any way. See Figure 33(a) for an example. For $i = 1, \ldots, k$ we define the set $X_i$ to be the triple of points from $X'$ whose numbers equal $i + 1$ modulo $k$. We claim that for every $i = 1, \ldots, k$ the convex hull of $X_i$ contains the point $p$, which will prove the claim. We refer to Figure 33(b) for an example.

So assume for the sake of contradiction that $p \notin \mathrm{conv}(X_i)$ for some $i \in \{1, \ldots, k\}$. Then $p$ and $\mathrm{conv}(X_i)$ can be separated by some line $\ell$, i.e., one half-space $H$ defined by $\ell$ contains $p$ and no point from $X_i$. (In case $k = r - 2$ we may choose without loss of generality $\ell$ to be not parallel to the segments $x_1x_2$ and $x_3x_4$.) Since $p$ is a half-space centerpoint of $X$ the half-space $H$ contains at least $\lceil \frac{n}{3} \rceil = \lceil \frac{3r-2}{3} \rceil = r$ points from $X$, at most $r - k$ of which are not in $X'$. Thus $H$ contains at least $k$ points from $X'$ and hence at least one of these points has a number equal to $i$ modulo $k$ – a contradiction to the assumption that the $H \cap X_i = \emptyset$. $\square$

Next we define another variant of a centerpoint, this time with respect to quadrants. We focus on the case $d = 2$ first. For a point $p = (p_x, p_y) \in \mathbb{R}^2$ the *four* *quadrants centered at p* are defined as the sets

$$\begin{aligned} Q_1(p) &= \{(x,y)\in\mathbb{R}^2 \mid p_x \le x \text{ and } p_y \le y\},\\ Q_2(p) &= \{(x,y)\in\mathbb{R}^2 \mid p_x \ge x \text{ and } p_y \le y\},\\ Q_3(p) &= \{(x,y)\in\mathbb{R}^2 \mid p_x \ge x \text{ and } p_y \ge y\},\\ Q_4(p) &= \{(x,y)\in\mathbb{R}^2 \mid p_x \le x \text{ and } p_y \ge y\}. \end{aligned}$$

In particular, each quadrant $Q_i(p)$ is closed, axis-aligned and has apex $p$, $i = 1,2,3,4$. The pairs of quadrants $\{Q_1(p), Q_3(p)\}$ as well as $\{Q_2(p), Q_4(p)\}$ are called *opposite quadrants* at $p$. In order to get points that are indeed central for the set $X$ we seek points that contain many points of $X$ in both quadrants of at least one pair of opposite quadrants. For convenience, we define these quadrant centerpoints for general reals $\alpha \in [0,1]$.

**Definition 4.5** (Quadrant Centerpoint)**.**
*Let $X$ be a finite set of $n$ distinct points in the plane and $\alpha \in [0,1]$. An $\alpha$-*quadrant centerpoint *is a point $x \in \mathbb{R}^2$ for which there exist two opposite quadrants $Q_i(x), Q_{i+2}(x)$ at $x$ ($i \in \{1,2\}$) each of which contains at least $\alpha n$ points from $X$.*

Similarly to Theorem 4.7 we want to determine the maximum $\alpha$ for which every set $X$ of $n$ points in the plane has an $\alpha$-quadrant centerpoint. Note that without loss of generality we can assume that no two points have the same $x$- or $y$-coordinate. This is because we consider closed quadrants and moving one of two axes-aligned points $p, q$ slightly in diagonal direction just decreases the number of incidences $q \in Q_i(p)$ and $p \in Q_i(q)$ for $i = 1,2,3,4$. Indeed we may assume that the $n$ points of $X$ lie on the vertices of an $n \times n$ square grid such that no two lie in the same column or row.

By considering $n$ equidistant points on a circle (see Figure 30(b)) we see that no point of the plane is an $\alpha$-quadrant centerpoint with $\alpha > 1/4 + 1/n$. Hence the maximum $\alpha$ we seek is at most $1/4$ and it turns out that this value is already the truth.

**Proposition 4.1.** *For every finite point set in $\mathbb{R}^2$ there exists a $\frac{1}{4}$-quadrant centerpoint. This is best-possible.*

*Proof.* Fix $X$ to be any set of $n$ points in the plane. Let $\ell_v$ be a vertical line such that both half-spaces defined by $\ell_v$ contain at least $\lceil \frac{n}{2} \rceil$ points of $X$. Further let $\ell_h$ be a horizontal line such that both half-spaces defined by $\ell_h$ contain at least $\lceil \frac{n}{2} \rceil$ points of $X$. Note that if $n$ is odd, then $\ell_v$ and $\ell_h$ contain some point of $X$ and that it may be the same point. See Figure 34 for an illustration.

We claim that the point $p = \ell_v \cap \ell_h$ is a $\frac{1}{4}$-quadrant centerpoint of $X$. By the definition of $\ell_v$ and $\ell_h$ we have

$$\begin{aligned} |Q_1(p)\cap X| + |Q_4(p)\cap X| \ge n/2,\\ |Q_2(p)\cap X| + |Q_3(p)\cap X| \ge n/2 \end{aligned}$$

and

$$\begin{aligned} |Q_1(p)\cap X| + |Q_2(p)\cap X| \ge n/2,\\ |Q_3(p)\cap X| + |Q_4(p)\cap X| \ge n/2. \end{aligned}$$

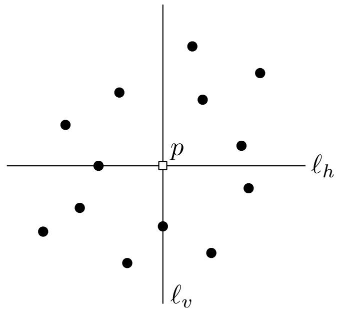



Figure 34: A set $X$ of $n = 13$ points and the lines $\ell_h$ and $\ell_v$ whose half-spaces contain $\lceil \frac{n}{2} \rceil = 7$ points each. The point $p$ where $\ell_h$ and $\ell_v$ intersect is a 1/4-quadrant centerpoint of $X$.

If each of $|Q_i(p) \cap X|$ $(i = 1, 2, 3, 4)$ is at least $n/4$ we are done by choosing two opposite quadrants at $p$ arbitrarily. Hence we may assume by symmetry that $|Q_1(p) \cap X| < n/4$. But this implies $|Q_2(p) \cap X| > n/4$ and $|Q_4(p) \cap X| > n/4$, i.e., $p$ is a $\frac{1}{4}$-quadrant centerpoint of $X$. $\square$

The 1/4-quadrant centerpoint constructed in the proof of Proposition 4.1 may or may not be contained in $X$. Moreover, there is examples where every 1/4-quadrant centerpoint is not in $X$. However, we do not lose to much if we restrict our attention to points in $X$. As we pointed out earlier, this is in sharp contrast to half-space centerpoints.

**Theorem 4.9** (Brönnimann, Lenchner, Pach [BLP07]).
*Every finite point set $X$ in the plane contains a point $x \in X$ that is a 1/8-quadrant centerpoint of $X$.*

*If the points in $X$ are in convex position then there exists $x \in X$ that is a 1/4-quadrant centerpoint of $X$.*

We present here the shorter proof taken from [ABDF+11]. It derives Theorem 4.9 as an immediate consequence of the following lemma.

**Lemma 4.1.** *Every set $X$ of $n$ points in the plane contains a point $x \in X$ such that*

$$\min(|Q_1(x) \cap X|, |Q_3(x) \cap X|) + \min(|Q_2(x) \cap X|, |Q_4(x) \cap X|) \geq \frac{n}{4}.$$

*Proof.* Consider the sets $X_T$, $X_B$, $X_L$ and $X_R$ of the first $\lfloor n/4 \rfloor$ points of $X$ from the top, the bottom, the left and the right, respectively. Each set $X_i$ with $i \in \{T, B, L, R\}$ is associated with a horizontal or vertical line $\ell_i$ that contains a point of $X_i$ and separates the set $X_i$ from its complement $X \setminus X_i$. See Figure 35 for an illustrative example.

It follows that there is a point $x \in X$ in the intersection of the four "larger" half-spaces defined by $\ell_T$, $\ell_B$, $\ell_L$ and $\ell_R$, namely the closed half-spaces containing $\lceil \frac{3}{4} n \rceil$ points of $X$ each. Denoting this intersection by $R$ we claim that every point $x \in R \cap X$ satisfies the claimed inequality.

Let $x$ be such a point and assume that $\min(|Q_1(x) \cap X|, |Q_3(x) \cap X|) = |Q_1(x) \cap X| = k$. Since $X_T$ is contained in $Q_1(x) \cup Q_2(x)$ it follows that $|Q_2(x) \cap$

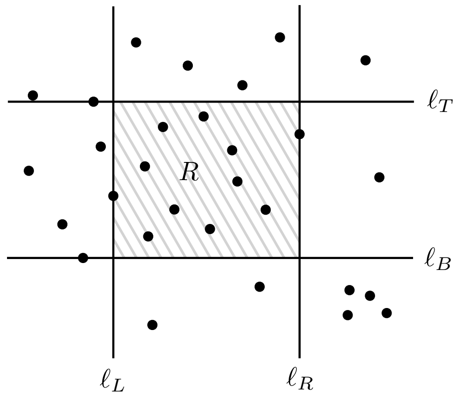



Figure 35: A set $X$ of $n = 29$ points, the lines $\ell_T$, $\ell_B$, $\ell_L$, $\ell_R$, and the region $R$.

$X| \geq n/4 - k$. Considering $X_R$ we obtain $|Q_4(x) \cap X| \geq n/4 - k$. Consequently,

$$\min(|Q_1(x) \cap X|, |Q_3(x) \cap X|) = k$$

and

$$\min(|Q_2(x) \cap X|, |Q_4(x) \cap X|) \geq n/4 - k,$$

which proves the lemma. □

*Proof of Theorem 4.9.*
From Lemma 4.1 we get a point $x \in X$ with

$$\min(|Q_1(x) \cap X|, |Q_3(x) \cap X|) + \min(|Q_2(x) \cap X|, |Q_4(x) \cap X|) \geq n/4.$$

For part one of the theorem it is enough to observe that at least one of the minima is at least $n/8$.

For the second part note that if $X$ is in convex position one of the four open quadrants centered at $x$ contains no point from $X$. Therefore, one of the two minima is zero and thus the other minimum is at least $n/4$. □

Clearly, Figure 30(b) shows that the second part of Theorem 4.9 is best-possible. For the first part consider a set $X$ of $n = 8k$ points ($k \in \mathbb{N}$) that comes in 8 sets of $k$ collinear points each as illustrated in Figure 36. It is straight-forward to check that $X$ does not contain an $\alpha$-quadrant centerpoint for any $\alpha > 1/8$.

The restriction to axes-aligned quadrants in the definition of $\alpha$-quadrant centerpoints may seem to be artificial. It surely makes sense to define a *general $\alpha$-quadrant centerpoint* of a set $X$ of $n$ points in the plane as a point $p \in \mathbb{R}^2$ such that there exists a pair of opposite quadrants centered at $p$ defined by two perpendicular lines through $p$, each if which contains at least $\alpha n$ points of $X$.

Indeed, Figure 30(b) shows that Proposition 4.1 remains still true and best-possible, even when considering general $\alpha$-quadrant centerpoints. On the other hand, the first part of Theorem 4.9 can be improved slightly. In [BDPZ10] Ben-Dan, Pinchasi and Ziv show that every finite point set $X$ in the plane contains a point $x \in X$ which is a general $\alpha$-quadrant centerpoint for $X$ with $\alpha = \frac{1}{8} + \frac{1}{8 \cdot 39}$.

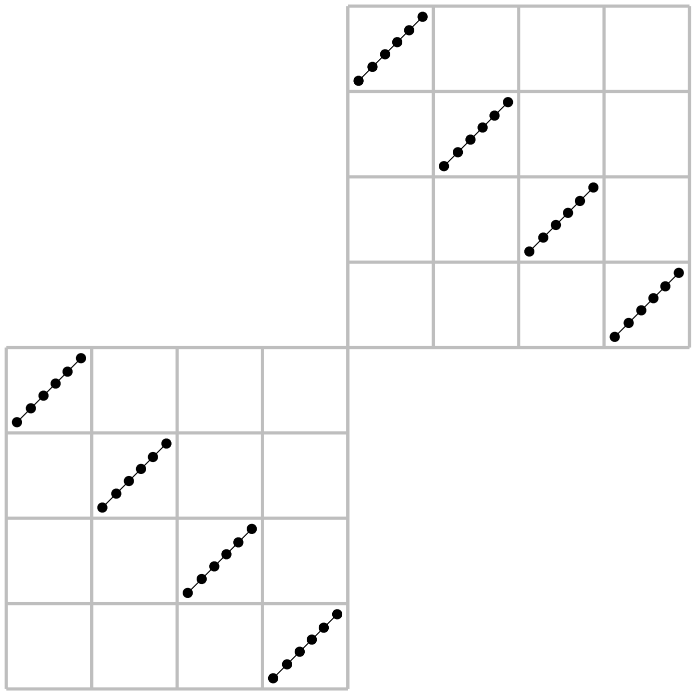


Figure 36: A set $X$ of $n = 8 \cdot 6$ points in the plane none of which is an $\alpha$-quadrant centerpoint for $\alpha > 1/8$.

The best known upper bound is $1/7$, obtained by suitably placing $n/7$ points on each of 7 rays emanating from the origin with an angle of $2\pi/7$ between consecutive rays.

**Problem 17.**
(Research Problem)

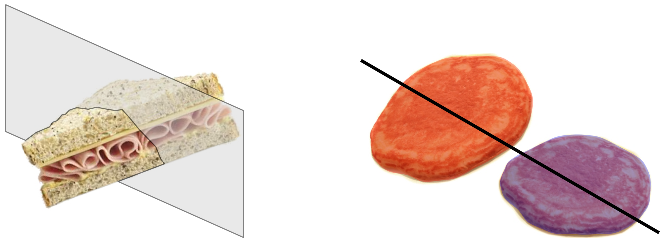
Prove that there exists some $\alpha > 1/8$ such that every finite point set $X$ in the plane contains a point $x \in X$ that is a general(!) $\alpha$-quadrant centerpoint of $X$.

For other interesting variant of quadrant centerpoints, including the higher-dimensional cases, we refer to [ABDF$^+$11].

In the next section, we consider *two* finite point sets $X, Y$ in $\mathbb{R}^2$ (in general it would be $d$ sets in $\mathbb{R}^d$). We will try to find a line (in general it would be a $(d-1)$-dimensional hyperplane) rather than a point that is "central" to both point sets. We consider finite point sets only, even though most of what follows holds in a much more general setting, as will be mentioned later.

## 4.3 Ham Sandwiches

Although not closely related to convexity, let us next consider the *Ham Sandwich Theorem.* Consider a ham sandwich in $\mathbb{R}^3$, see Figure 37(a) for an example, i.e., some bread $B \subseteq \mathbb{R}^3$, some ham $H \subseteq \mathbb{R}^3$ and some cheese $C \subseteq \mathbb{R}^3$. A *ham sandwich cut* is a cut of the ham sandwich with a straight motion of a knife so that the bread as well as the ham and the cheese are cut into halves. In $\mathbb{R}^2$ we consider two pancakes, a red one and a blue one, and call a straight cut that halves either pancake a *pancake cut*. See Figure 37(b) for an example.



(a) (b)

Figure 37: (a) A ham sandwich cut. (b) A pancake cut.

More formally a compact $d$-dimensional set in $X \subset \mathbb{R}^d$ is said to be *bisected* by a $(d-1)$-dimensional hyperplane $h$ if both half-spaces defined by $h$ contain half of the $d$-dimensional volume of $X$. The Ham Sandwich Theorem then states that given any $d$ such compact $d$-dimensional sets in $\mathbb{R}^d$ there exists a hyperplane that bisects each of these sets at the same time.

**Theorem 4.10** (Ham Sandwich Theorem)**.**
*Any $d$ compact $d$-dimensional sets in $\mathbb{R}^d$ can be simultaneously bisected by a $(d-1)$-dimensional hyperplane.*

**Problem 18.**
Consider 50 apples and 50 bananas being packed in 50 boxes in some way. For example, there may be empty boxes, boxes containing 2 apples and 42 bananas, etc.

a) Prove that one can always pick at most 26 boxes such that these boxes contain in total at least 25 apples and 25 bananas.

b) How do you have to pack the apples and bananas into the boxes so that picking 50 boxes is not enough?

c) What about 50 apples, 50 bananas and 50 coconuts in 50 boxes?

If $d = 2$ the Ham Sandwich Theorem is also known as the *Pancake Theorem*. It states that any two compact sets (or pancakes) in the plane can be simultaneously bisected by some line $\ell$. This means that both sets have half its area on either side of $\ell$. We again refer to Figure 37(b) for an example.

The Ham Sandwich Theorem is usually proven as a consequence of the *Borsuk-Ulam Theorem* from algebraic topology. Here we consider the discrete version of the Ham Sandwich Theorem, which deals with $d$ finite point sets in $\mathbb{R}^d$. Instead of the area as the underlying measure we now take the counting measure, which is simply the number of points in each set. A $(d-1)$-dimensional hyperplane $h$ is said to *bisect* a finite set $X \subset \mathbb{R}^d$ if each open half-space defined by $h$ contains at most $\lfloor |X|/2 \rfloor$ points from $X$. Note that if $|X|$ is odd and $h$ bisects $X$, then at least one point of $X$ lies on $h$.

**Theorem 4.11** (Discrete Ham Sandwich Theorem).
*Any $d$ finite sets in $\mathbb{R}^d$ can be simultaneously bisected by a $(d-1)$-dimensional hyperplane.*

*Proof of the Discrete Ham Sandwich Theorem in $d = 2$ dimensions.*
We have to show that for any finite point set $X$ and any finite point set $Y$ in the plane there exists a line $\ell$ such that each open half-space defined by $\ell$ contains at most $\lfloor \frac{|X|}{2} \rfloor$ points of $X$ and at most $\lfloor \frac{|Y|}{2} \rfloor$ points of $Y$.

We start with some line $\ell$ that bisects $X$. By rotating the plane we can assume that without loss of generality that no two points in $X$ have the same $x$-coordinate. Then we choose $\ell$ to be a vertical line through some point $p_0 \in X$ that has exactly $\lfloor \frac{|X|}{2} \rfloor$ points of $X$ to the left.

Now we start rotating $\ell$ in counterclockwise direction around $p_0$. We will abuse notation and always refer to the rotated line again as $\ell$, although these are different lines in $\mathbb{R}^2$. We rotate until $\ell$ hits some point in $X \setminus p_0$. Clearly, one open half-space defined by $\ell$ (the "left" one) contains at most $\lfloor \frac{|X|}{2} \rfloor$ points of $x$ and the other open half-space (the "right" one) contains at most $\lfloor \frac{|X|}{2} \rfloor - 1$ points of $X$. See Figure 38 for an example[3].

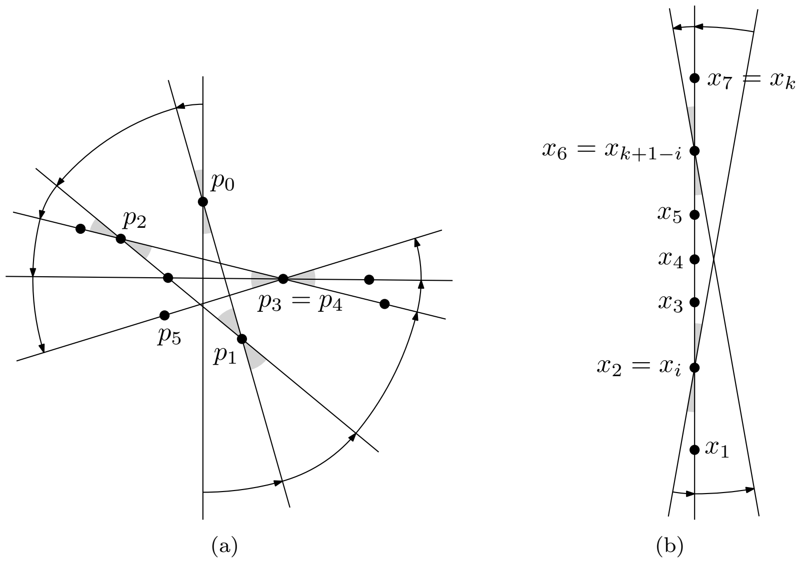



Figure 38: (a) Rotating a line such that at all times it bisects a point set $X$ of 9 points in the plane. (b) Rotating across a set of $k = 7$ collinear points of $X$.

Note that $\ell$ may have hit more than one point from $X$. Let $x_1, \ldots, x_k$ be the points in $\ell \cap X$ indexed as they appear along $\ell$. We have $k \geq 2$ and $p_0 = x_i$ for some $i \in \{1, \ldots, k\}$. We now consider the point $p_1 = x_{k+1-i}$ and start rotating

[3] Bonus question: What kind of set will the arrowed circular arcs in Figure 38(a) enclose after a full rotation of the line?

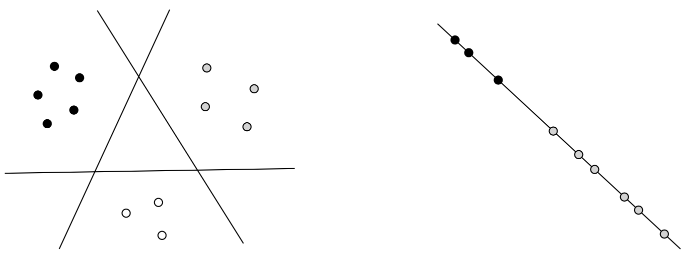
$\ell$ in counterclockwise direction around $p_1$ instead of $p_0$. Note that $p_0 = p_1$ in case $i = (k+1)/2$. Considering the line $\ell$ shortly before hitting the second point in $X$ and shortly after starting the rotation around $p_1$ we see that all points $x_j$ with $j = i+1, \ldots, k+1-i$ remain on the same side of $\ell$ while all points $x_j$ with $j < i$ or $j > k+1-i$ switch the side of $\ell$. Since the number of points switching from one side of $\ell$ to the other equals the number of points changing sides in opposite direction $\ell$ again bisects $X$.

We continue rotating $\ell$ around $p_1$ instead of $p_0$. This way we get a sequence $p_0, p_1, p_2, \ldots$ of points around which $\ell$ is rotated. See Figure 38(a) for an illustrating example. At all times $\ell$ contains at least one point of $X$ and bisects $X$.

Now consider the time that $\ell$ is vertical again, i.e., after a rotation of $\pi$. Clearly, $\ell$ bisects $X$ and contains some point of $X$. If $|X|$ is odd, this point must be $p_0$, which means that $\ell$ is again in its initial position. If $|X|$ is even, $\ell$ could be parallel but not identical to its initial position. In that case no point of $X$ lies strictly between the final and the initial position of $\ell$ and we continuously slide $\ell$ from its final onto its initial position without passing through any point if $X$.

In either case, we end up with $\ell$ being in the same position as in the very beginning. But the sides of $\ell$ are swapped! Hence if initially the line $\ell$ had $m$ points of $Y$ on its first side and $|Y| - m$ points of $Y$ on its second side, after a rotation of $\pi$ (and a possible shift) it has $|Y| - m$ points of $Y$ on its first side and $m$ points of $Y$ on its second side. Thus at some intermediate step, $\ell$ must have had at most $\lfloor \frac{|Y|}{2} \rfloor$ points on each side and at this point in time $\ell$ was bisecting $X$ and $Y$ simultaneously. $\square$

It is easy to verify that the Ham Sandwich Theorem is best possible in terms of the number of sets that can be bisected simultaneously. Already in the discrete 2-dimensional version there exist three sets that cannot be simultaneously bisected by a single line $\ell$. Consider for example three sets each of which can be separated from the other two by some line, see Figure 39(a). Secondly, there is situations in which there is only one Ham Sandwich cut for two sets $X$ and $Y$, e.g., when all the points in $X \cup Y$ are collinear, see Figure 39(b).



(a) (b)

Figure 39: (a) Three sets in the plane that cannot simultaneously be bisected by a single line. (b) Two sets in the plane for which there exists only one Ham Sandwich cut.

The proof of the Discrete Ham Sandwich Theorem (Theorem 4.11) above uses a sweep line argument that is very common in combinatorial problems on planar point sets. Suppose we have to show the existence of a line $\ell$ that satisfies two properties $P_1$ and $P_2$. In the above case $P_1$ was "bisecting $X$" while $P_2$ was "bisecting $Y$". The general idea is to start with a line $\ell$ that has property $P_1$ and then rotate $\ell$ in discrete steps, maintaining property $P_1$ for $\ell$ at all times. If we can also prove that after a full rotation of $\ell$ at some intermediate step it necessarily satisfies property $P_2$, then we are done. In particular, we have proven the existence of a line $\ell$ satisfying properties $P_1$ and $P_2$ simultaneously.

**Problem 19.**
Prove that for every finite point set $X$ in the plane there exists a point $p \in \mathbb{R}^2$ and a pair of perpendicular lines $\ell_1, \ell_2$ through $p$ such that each of the four quadrants centered at $p$ defined by $\ell_1$ and $\ell_2$ contains at least $\lfloor |X|/4 \rfloor$ points of $X$.

Sweep line arguments can also be used in non-discrete settings. However, the line $\ell$ does not necessarily rotate in discrete steps around finitely many points in the plane. For example Figure 40 shows the trace of the point around which a line rotates while bisecting the area of an equilateral triangle at all times[4]. In this case one would rather prove that for all directions $(a, b)$ there exists a line $\ell$ parallel to $\ell(a, b)$ satisfying property $P_1$. But even if this line changes continuously in the direction $(a, b)$ the trace of points around which $\ell$ is rotated during such a continuous sweep can be a very complicated curve.

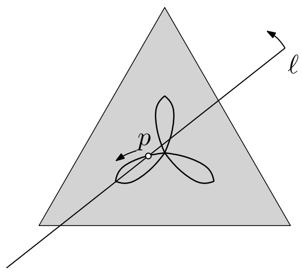



Figure 40: A line $\ell$ is continuously swept over an equilateral triangle so as to halve the area of the triangle at all times. The trace of the rotation point $p$ is indicated.

**Problem 20.**
Let $Q$ be some convex polygon.

- Prove that for every point $p \in Q$ there is a line $\ell_p$ that contains $p$ and bisects the area of $Q$.
- Prove that there is a line $\ell$ that simultaneously bisects the area of $Q$ and the perimeter of $Q$.

[4]The author would like to thank Sebastian Ziesche for his help in computing the trace.

## 4.4 Convex Independent Subsets

In this section we consider finite sets $X$ of points in the plane in *general position*, meaning that no three points in $X$ are collinear. We are interested in finding a subset of many points in $X$ that lie in convex position, i.e., form the corners of some convex polygon. This problem has already been considered 80 years ago and still leads to interesting new results these days.

Let us start with an easy lemma.

**Lemma 4.2.** *Every set of* 5 *points in the plane in general position contains a convex quadruple.*

*Proof.* Let $X$ be a set of 5 points in the plane in general position. If $\text{conv}(X)$ has at least four corners, then any set of four corners form a convex quadruple. Otherwise, $\text{conv}(X)$ is a triangle and two points $x_1, x_2 \in X$ lie interior to it. Since no three points are collinear the line through $x_1$ and $x_2$ separates one corner of $\text{conv}(X)$ from the other two, called $x_3$ and $x_4$. Then $\{x_1, x_2, x_3, x_4\} \subset X$ is a convex quadruple. See Figure 41 for an illustration. $\square$

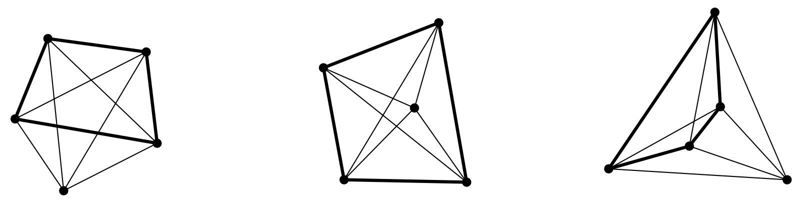


Figure 41: All combinatorially different sets of 5 points in the plane in general position and a convex quadruple (indicated by thick edges) in each of them.

> **Problem 21.**
> Find a set of 8 points in the plane in general position which does not contain any convex pentagon.

By Lemma 4.2 all sets of five or more points in the plane contain a convex independent subset of size 4. In general we would like to answer the following question.

*"Does every sufficiently large point set in the plane contain a convex independent subset of prescribed size?"*

The answer to above question is 'YES' as proven by Erdős and Szekeres in 1935.

**Theorem 4.12** (Erdős-Szekeres [ES35])**.**
*For every $k \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that every set of $n \geq N$ points in the plane in general position contains a convex $k$-gon.*

> **Problem 22.**
> How large convex $k$-gons can you find among the points of the $n \times n$ square grid? (It is enough to give the asymptotic growth.)

This result (Theorem 4.12) is considered to be one of the first *Ramsey-type results* in history, where a Ramsey-type question is loosely speaking the following.

*"Does every sufficiently large structure of a certain type contain a "regular" substructure of prescribed size?"*

Let us briefly mention the original Ramsey-type question considered by Frank Plumpton Ramsey in 1928 (published in 1930 [Ram30]). Ramsey was a young (he was born in 1903 in Cambridge) very talented mathematician, philosopher and economist. After having mastered German, he moved to Vienna at the age of 19, where he became a close friend of Ludwig Wittgenstein. Ramsey translated Wittgenstein's texts from German into English and later convinced him to return to Cambridge where Wittgenstein once studied. Back in Cambridge in 1924 Ramsey became Wittgenstein's supervisor and provided him with financial support. Ramsey died at the age of 26 from chronic liver problems.

In mathematics Ramsey was mostly interested in logic, especially in first-order logic and decidability problems. What is nowadays known as Ramsey's Theorem was actually just a lemma to prove decidability of a special class of first-order logic. Based on this Alonzo Church later showed undecidability of the decision problem in first-order logic in general, known as Church's Theorem [Chu36], answering Hilbert's problem in the negative.

The setting of our question above that has been considered by Ramsey is a very general one. To this end we consider the *complete $r$-uniform hypergraph* on $N$ vertices $K_N^r$, that is, the vertices of $K_N^r$ are the elements of some $N$-element set $V$ and the edges of $K_N^r$ are all $r$-element subsets of $V$. For example Figure 42 depicts the complete hypergraphs $K_6^2$ (which is the same as $K_6$) and $K_5^3$.

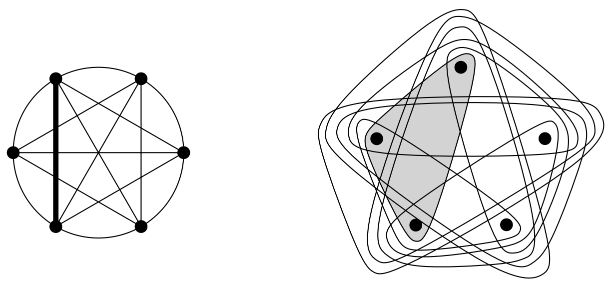


Figure 42: The complete 2-uniform hypergraph on 6 vertices $K_6^2 = K_6$ and the complete 3-uniform hypergraph on 5 vertices $K_5^3$, each with one of its edges highlighted.

Now in Ramsey's Theorem the "arbitrary" but sufficiently large structure is a coloring of the edges of some large $r$-uniform hypergraph with a bounded number of colors. The size of such a structure is the number of vertices. And as "regular" substructure we consider complete $r$-uniform sub-hypergraphs whose edges are colored all the same, that is monochromatically. Ramsey's theorem states that if we want to find a monochromatic $K_n^r$ inside *every* $K_N^r$ which is colored with $c$ colors, then we just have to make $N$ large enough (depending on $n$, $r$ and $c$) and are guaranteed to find it.

**Theorem 4.13** (Ramsey's Theorem [Ram30]).
*For every natural numbers $n$, $r$ and $c$ there exists an $N \in \mathbb{N}$ such that in every coloring of the edges of $K_N^r$ with at most $c$ colors we can find a subset of $n$ vertices all of which induced edges are colored the same, i.e., a monochromatic copy of $K_n^r$ in $K_N^r$.*

Given $n$, $r$ and $c$ the minimum $N$ for which Ramsey's Theorem holds is called the *Ramsey number* $R_c^r(n)$. So Ramsey's Theorem asserts that $R_c^r(n)$ exists, i.e., is finite, for all $n, r, c$. However, the upper bounds on $R_c^r(n)$ are enormous. For example, a first greedy approach provides a bound that goes up by the Ackermann hierarchy. Of course better upper bounds are known, such as

$$R_2^r(n) \leq 2^{\binom{s}{r-1}+1} \quad \text{where } s = R_2^{r-1}(n-1)+1,$$

which was proven by Erdős and Rado [ER56]. Note that this is roughly only a tower of 2's of height $r$. Indeed, we cannot hope for much better upper bounds. From the so-called *Stepping-up Lemma* of Erdős and Hajnal (see e.g. [GRS80]) it follows that

$$R_2^r(n) \geq t_{r-2}(C_n) \quad \text{for some constant } C_n \text{ depending on } n,$$

where the tower function $t_i(x)$ is defined as $t_1(x) = x$ and $t_{i+1}(x) = 2^{t_i}(x)$ for $i \geq 2$. For further results on Ramsey numbers and a good introduction into Ramsey theory we refer to the book of Graham, Rothschild and Spencer [GRS80].

Finally, let us prove the Erdős-Szekeres Theorem (Theorem 4.12). Indeed we present three proofs, the first two are based on Ramsey numbers and thus give very huge bounds on $N$, the third is an induction that is making more use of geometry. For convenience let us repeat the statement.

**Theorem 4.12** (Erdős-Szekeres Theorem [ES35]).
*For every $k \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that every set of $n \geq N$ points in the plane in general position contains a convex $k$-gon.*

*First proof using Ramsey's Theorem.*
The case $k \leq 3$ is immediate and the case $k = 4$ is covered by Lemma 4.2.

So let $k \geq 5$ and $X$ be any set of $N$ points in the plane in general position. We consider $X$ as the vertex set of $K_N^4$. We define a 2-coloring of the edges of $K_N^4$ by coloring a quadruple red if the four points are in convex position and blue otherwise. By Ramsey's Theorem (Theorem 4.13) if $N \geq R_2^4(k)$ then there exists a set $Y \subseteq X$ of $k$ points for which all quadruples contained in $Y$ have the same color $c$.

Note that Lemma 4.2 together with $|Y| = k$ implies that this color $c$ can not be blue, because every set of 5 or more points contains a convex quadruple. Thus every quadruple of points in $Y$ is in convex position. If $Y$ would not be convex independent, then by Carathéodory's Theorem (Theorem 4.1) there is a point of $Y$ in the convex hull of three other points of $Y$. Since this would be a quadruple in non-convex position, we conclude that $Y$ is a convex $k$-gon. $\square$

*Second proof using Ramsey's Theorem.*
Let $X$ be any set of $N$ points in the plane in general position. We number the points in $X$ by $x_1, \ldots, x_N$ in any order. Now we consider $X$ to be the vertex

set of $K_N^3$ and color a triple $\{x_i, x_j, x_k\}$ with $i < j < k$ red if going from $x_i$ to $x_j$ via $x_k$ is a left turn and blue if it is a right turn.

By Ramsey's Theorem (Theorem 4.13) if $N \geq R_2^3(k)$ then there exists a set $Y \subseteq X$ of $k$ points for which all triples contained in $Y$ have the same color $c$, without loss of generality red. Again by Carathéodory's Theorem (Theorem 4.1) if $Y$ is not a convex $k$-gon there is a quadruple of points in $Y$ in non-convex position.

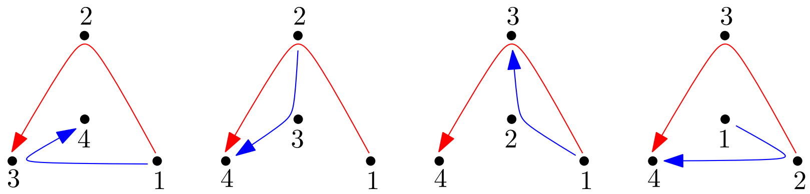



Figure 43: Any numbering of four points in non-convex position with distinct numbers gives at least one left turn $i \to j \to k$ with $i < j < k$ and one right turn $i \to j \to k$ with $i < j < k$.

But it is impossible to number such a quadruple such that all triples are left turns. See Figure 43 for a complete case distinction. Thus, $Y$ is a convex $k$-gon as desired. □

*Third proof using induction.*
Let $X$ be any set of $N$ points in the plane in general position. First, let us rotate the plane so that no two points in $X$ have the same $x$-coordinate. Then the points of every subset $Y$ of $X$ can be connected by segments from left to right such that the union of these segments is the graph of some piecewise linear function. We call a set $Y$ of size $k$ a *$k$-cup* if this function is a convex function, i.e., going through the points in $Y$ from left to right we do only left turns, and a *$k$-cap* if the function is concave, i.e., we do only right turns when walking along the function graph left-to-right. See Figure 44 for an illustrative example.

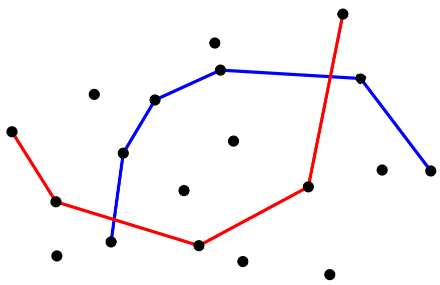


Figure 44: A set of 19 points in the plane, a 5-cup in red and a 6-cap in blue.

Clearly, each $k$-cup and each $k$-cap is a convex $k$-gon. Hence it suffices to prove that if $N$ is large enough then $X$ contains some $k$-cup or $k$-cap. We define $f(k, \ell)$ to be the smallest number $N$ such that every set of $N$ points in the plane in general position contains a $k$-cup or an $\ell$-cap.

**Lemma 4.3.** *For every $k, \ell \in \mathbb{N}$ we have*

$$f(k,\ell) \leq \binom{k+\ell-4}{k-2} + 1.$$

*Proof of Lemma 4.3.* We proceed by induction on $k$ and $\ell$. For $k \leq 2$ or $\ell \leq 2$ the statement clearly holds. Thus let $k, l \geq 3$ and consider any set $X$ of $N = \binom{k+\ell-4}{k-2} + 1$ points in general position. Thus $N = \binom{(k-1)+\ell-4}{(k-1)-2} + \binom{k+(\ell-1)-4}{k-2} + 1$, which by induction hypothesis gives

$$N \geq f(k-1,\ell) + f(k,\ell-1) - 1.$$

Suppose that $X$ contains no $\ell$-cap. Let $A \subset X$ be the set of those points that are a rightmost point of some $(k-1)$-cup in $X$. Since $X \setminus A$ contains no $(k-1)$-cup and no $\ell$-cap we have $|X \setminus A| < f(k-1,\ell)$ and hence

$$|A| \geq N - f(k-1,\ell) + 1 = f(k,\ell-1).$$

Thus $A$ contains a $k$-cup, in which case we are done, or an $(\ell-1)$-cap $Y$, each point of which is a rightmost point of some $(k-1)$-cup in $X$. In particular, there is a $(k-1)$-cup $Z$ whose rightmost point is the leftmost point $p$ of $Y$. The situation is depicted in Figure 45.

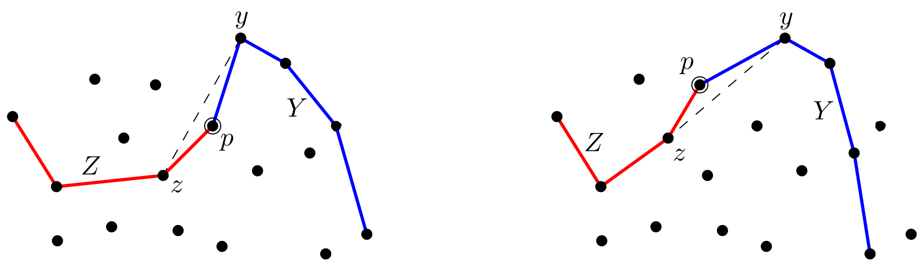



Figure 45: Gluing a $(k-1)$-cup $Z$ and an $(\ell-1)$-cap $Y$ necessarily gives a $k$-cup (left case) or an $\ell$-cap (right case).

Now we look at the predecessor $z$ of $p$ in $Z$, $p$ itself, and the successor $y$ of $p$ in $Y$. Going from $z$ to $y$ via $p$ is either a left turn or a right turn. In the former case $Z \cup y$ is a $k$-cup and in the latter case $Y \cup z$ is an $\ell$-cap, which proves the lemma. $\square$

From Lemma 4.3 immediately follows that if $N \geq f(k,k)$ then the $N$-element point set $X$ contains a convex $k$-gon, which proves the Erdős-Szekeres Theorem. $\square$

**Problem 23.**
Prove a $d$-dimensional Erdős-Szekeres Theorem, i.e., that for all natural numbers $k$ and $d$ there exists $N \in \mathbb{N}$ such that every set of $N$ points in $\mathbb{R}^d$ in general position (no $d+1$ points lie on a common $(d-1)$-dimensional hyperplane) contains $k$ points in convex position.

Of the three proofs given above the last one implies the best bound on $N$, namely $N \leq \binom{2k-4}{k-2} + 1$. Even though the proof of Erdős-Szekeres from 1935 gives something stronger then a convex $k$-gon, namely a $k$-cup or $k$-cap, the derived bound on $N$ was only recently improved by Tóth and Valtr in 1998 and only by roughly a factor of 2 [TV98].

More precisely, let us define $ES(k)$ to be the minimum $n$ such that every set of $n$ points in the plane in general position contains a convex $k$-gon. Equivalently, $ES(k) - 1$ is the maximum size of a point set in the plane in general position *without* any convex $k$-gon. Then it is known that

$$2^{k-2} + 1 \leq ES(k) \leq \binom{2k-5}{k-2} + 2.$$

The upper bound is due to Tóth and Valtr [TV98] and the lower bound is a construction due to Erdős and Szekeres [ES60]. The so-called "Happy Ending Problem" (named so because it led to the marriage of George Szekeres and Esther Klein) asks whether the lower bound is the actual truth as conjectured in [ES35] and confirmed for $k = 2, 3, 4, 5, 6$.

Before we come to the next problem, recall that $f(k, \ell)$ denotes the minimum $n$ such that every set of $n$ points in the plane in general position contain a $k$-cup or an $\ell$-cap. In Lemma 4.3 we have shown that $f(k, \ell) \leq \binom{k+\ell-4}{k-2} + 1$. Interestingly, this bound on $f(k, \ell)$ is tight.

**Proposition 4.2.** *For all natural numbers $k, \ell \geq 1$ there exists a set $X_{k,\ell}$ of $\binom{k+\ell-4}{k-2}$ points in the plane in general position with no $k$-cup and no $\ell$-cap.*

*Proof.* We construct the set $X_{k,\ell}$ by induction on $k$ and $\ell$. For $k \leq 2$ or $\ell \leq 2$ we define $X_{k,\ell}$ to be a single point. Having defined $X_{k,\ell-1}$ and $X_{k-1,\ell}$ we simply place these two sets suitably next to each other. More precisely, we place $X_{k,\ell-1}$ anywhere and $X_{k-1,\ell}$ to the left of $X_{k,\ell-1}$ so that

- $X_{k-1,\ell}$ lies completely below all connecting lines of $X_{k,\ell-1}$ of positive slope, and
- $X_{k,\ell-1}$ lies completely above all connecting lines of $X_{k-1,\ell}$ of positive slope.

We refer to Figure 46 for an example.

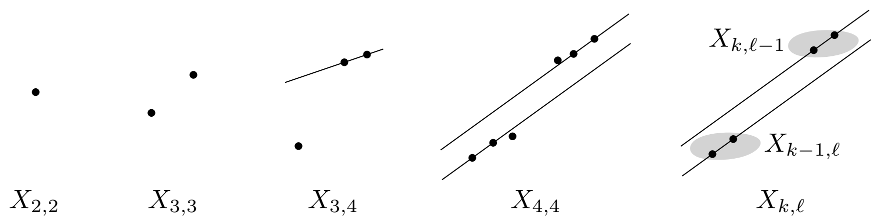



Figure 46: Putting a set $X_{k,\ell-1}$ with no $k$-cup and no $(\ell-1)$-cap next to a set $X_{k-1,\ell}$ with no $(k-1)$-cup and no $\ell$-cap so as to obtain a set $X_{k,\ell}$ with no $k$-cup and no $\ell$-cap.

By construction each cup $Z$ in $X_{k,\ell}$ is either completely contained in $X_{k,\ell-1}$ or contains at most one point of $X_{k,\ell-1}$. By induction hypothesis we have in the former case $|Z| < k$ and in the latter case $|Z| < k - 1 + 1 = k$. In other words, $X_{k,\ell}$ contains no $k$-cup. Symmetrically, every cap $Y$ in $X_{k,\ell}$ is either completely contained in $X_{k-1,\ell}$ or contains at most one point of $X_{k-1,\ell}$ and we get by induction hypothesis that $|Y| < \ell$ in the former case and $|Y| < \ell - 1 + 1 = \ell$ in the latter case. In particular $X_{k,\ell}$ contains no $\ell$-cap, which proves the claim. $\square$

## 4.5 Empty Convex Independent Subsets

In the preceding section we have been looking for large convex independent subsets in any big enough set $X$ of points in the plane. Here we consider a very prominent variant of this problem. Namely, what if we look for large convex independent subsets in $X$ whose convex hull does not contain any other point of $X$? Let us call a subset $Y$ of $k$ points from $X$ a *$k$-hole of $X$* if it is a convex independent set with $\mathrm{conv}(Y) \cap X = Y$. See Figure 47 for some examples. Then the following question is immediate.

*"Does every sufficiently large finite point set in the plane contain a hole of prescribed size?"*

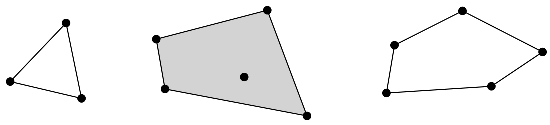


Figure 47: A 3-hole on the left, a 5-hole on the right, and a convex 4-gon that is no 4-hole in the middle.

Recall that Lemma 4.2 asserts that every set of five points in the plane contains a convex quadruple. Indeed, by looking at Figure 41 and choosing a different convex 4-gon the middle case, we see that every such set even contains a 4-hole.

So let us prove that 5-holes always exist in $X$, as long as $X$ contains sufficiently many points.

**Proposition 4.3.** *Every set of $ES(6) = 17$ points in the plane in the general position contains a 5-hole.*

*Proof.* Per definitionem every set $X$ of $ES(6)$ points in the plane in general position contains a convex 6-gon $Y \subset X$. Let us refer to the interior of $\mathrm{conv}(Y)$ as simply the *interior of $Y$*. We proceed by distinguishing three cases.

**Case 1** – *(The interior of $Y$ contains at most one point from $X$.)* If there is no such point then any 5-element subset of $Y$ is a 5-hole. If there is a point $p \in X$ in the interior of $Y$, then consider any point $q \in Y$ and the line $\ell$ connecting $p$ and $q$. Clearly on one side of $\ell$ there is at least three points of $Y$. together with $p$ and $q$ these three points form a 5-hole. See Figure 48(a) for an illustrating example.

**Case 2** – *(The connecting line of two points of $X$ in the interior of $Y$ leaves four or more points of $Y$ on one side.)* Let $p$ and $q$ be the two points of $X$ and $\ell$ their connecting line. Taking $p, q$ and four points of $Y$ from the same side of $\ell$ gives another convex 6-gon with area smaller than $\text{conv}(Y)$. Hence repeating this step (or starting with a convex 6-gon of minimal area) we can reduce the problem to the other two cases. See Figure 48(b) for an illustrating example.

**Case 3** – *(The connecting line of any two points of $X$ in the interior of $Y$ bisects $Y$.)* Consider the convex hull $C$ of all the points of $X$ in the interior of $Y$. Let $p$ and $q$ be two consecutive points on $C$ and $\ell$ be their connecting line. Then one side of $\ell$ contains three points of $Y$ and no point of $X$ in the interior of $Y$. Together with $p$ and $q$ these three points form a 5-hole, as illustrated in Figure 48(c).

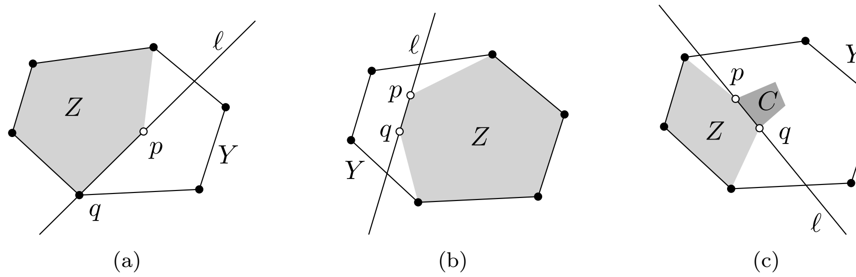



Figure 48: (a) A convex 6-gon $Y$ with exactly one point $p$ inside gives rise to a 5-hole $Z$. (b) If a convex 6-gon $Y$ is not split evenly by the line connecting $\ell$ of two points $p$ and $q$ inside $Y$, then this gives rise to a convex 6-gon $Z$ of smaller area. (c) If a convex 6-gon $Y$ is split evenly by any line $\ell$ of any two points in the interior of $Y$, then this gives rise to a 5-hole $Z$.

Since the above is a complete case distinction and in each case we have identified a 5-hole or a convex 6-gon of smaller area, the statement is proven. $\square$

Although not difficult, the proof of Proposition 4.3 is significantly more involved then for example the proof of Lemma 4.2, which asserts that every set of five points in the plane in general position contains a 4-hole. One might think that it is even harder to answer our question above, i.e., whether we always find a $k$-hole in any set of $n$ points in general position provided $n$ is big enough.

Indeed, let us do one more step and consider the question whether any $n$-element point set, for large $n$, contains a 6-hole. Apparently this has been asked by Erdős in 1978 [Erd78] and just recently been answered in the affirmative by Nicolás [Nic07] and independently by Gerken [Ger08], as well as by Koshelev [Kos09]. The bound $n \geq ES(9)$ stated below is due to Koshelev and up to today best-known.

**Theorem 4.14** (Six-Hole Theorem [Nic07, Ger08, Kos09]).
*Every set of $ES(9)$ points in general position contains a 6-hole.*

Plugging in the best known upper bound on $ES(9)$ we obtain that every set of at least 463 points in the plane contains a 6-hole. On the other hand Figure 49 shows a set of 29 points with no 6-hole, which was found by a computer search [Ove02]. The exact minimum number of points that always enforce a 6-hole is not known.

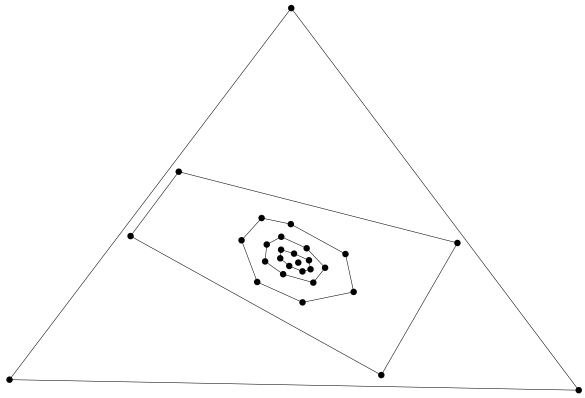


Figure 49: A set of 29 points with no 6-hole. This example is due to Overmars [Ove02].

We do not present a full proof of Theorem 4.14 here. But all known proofs proceed as follows.

- Consider any set $X$ of $ES(n)$ points in the plane for some suitable $n$. Hence $X$ contains some convex $n$-gon.
- Take such a convex $n$-gon $C \subset X$ of minimum area.
- Partition $C$ into layers $L_1, L_2, L_3, \ldots$ by defining $L_1$ to be the set of corners of $\operatorname{conv}(C)$ and for $i > 1$ define $L_i$ to be the set of corners of $\operatorname{conv}(C \setminus (L_1 \cup \cdots \cup L_{i-1}))$. See Figure 50 for an example.
- Prove Lemma 4.4 below.

**Lemma 4.4.** *If $X$ has no 6-hole and $|C| \geq 7$, then $L_4 = \emptyset$.*

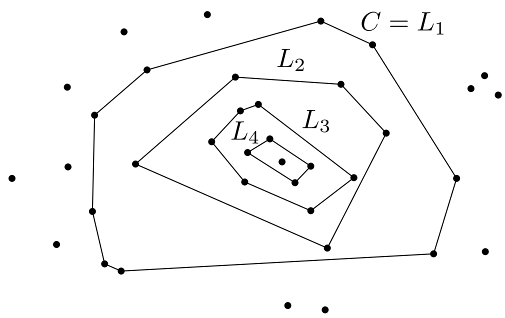



Figure 50: Partitioning $\operatorname{conv}(C) \cap X$ into layers $L_1, L_2, \ldots$ by iteratively taking the convex hull.

*Proof that Theorem 4.14 is implied by Lemma 4.4.*
By Lemma 4.4 we shall assume that $L_4 = \emptyset$ and there is no 6-hole, and conclude that $|C|$ is bounded above by 215. Hence any set of $ES(216)$ points necessarily contains a 6-hole. Slightly more careful analysis leads to better bounds on $|C|$.

So if $L_4 = \emptyset$ and there is no 6-hole then clearly $|L_3| \leq 5$. Moreover, the convex hull of every 6 consecutive points on $L_2$ contains some point of $L_3$, since otherwise they would form a 6-hole. Thus $|L_2| \leq 6|L_3| + 5 \leq 35$. Similarly, the convex hull of every 6 consecutive points of $L_1 = C$ contains some point of $L_2$ and thus $|C| \leq 6|L_2| + 5 \leq 215$. $\square$

Let us finally answer the initial question of this section, namely whether for every $k$ there exists an $n$ such that every set of $n$ points in general position contains a $k$-hole. We have seen that for $k = 3, 4, 5, 6$ the answer is 'YES'. However, for $k \geq 7$ the answer is 'NO' as noted first by Horton already in 1983 [Hor83].

**Theorem 4.15** (Seven-Hole Theorem [Hor83])**.**
*There exist arbitrarily large finite sets in the plane in general position with no 7-hole.*

The construction presented here is due to Valtr [Val92] and somehow similar to the one in Proposition 4.2. Indeed we start with a set with only one point and then define larger sets as the disjoint union of two previously defined sets $A$ and $B$ with a suitable choice of the relative position of the two sets to each other.

**Definition 4.6.** *A set $H$ is a* Horton set *if $|H| \leq 1$ or $H$ is the disjoint union of two Horton sets $A$ and $B$, i.e., $H = A \dot{\cup} B$, satisfying conditions (i)–(iv) below.*

*(i) The points in $H = A \dot{\cup} B$ are equidistant with respect to their $x$-coordinates.*

*(ii) With increasing $x$-coordinates the points in $A$ and $B$ alternate.*

*(iii) Every line connecting two points in $A$ lies completely above $B$.*

*(iv) Every line connecting two points in $B$ lies completely below $A$.*

Note that (ii) implies that $|A|$ and $|B|$ differ by at most 1.

**Lemma 4.5.** *For every $n \geq 1$, there exists a Horton set $H$ on exactly $n$ points.*

*Proof.* One easily sees that if $H$ is a Horton set on $n$ points, then removing the $k$ points with largest $x$-coordinate from $H$ gives a Horton set on $n - k$ points for every $k = 0, 1, \ldots, n-1$. Hence it suffices to prove the existence of a Horton set $H^{(k)}$ on $2^k$ points for every $k \in \mathbb{N}$.

We start by defining the set $H^{(0)}$ to consist of just one point. Then we shall define the set $H^{(k)}$ for $k \geq 1$ as the disjoint union of two copies $A, B$ of $H^{(k-1)}$. We scale $A$ and $B$ so that in either set the $x$-coordinates of any two consecutive points differ by exactly 2. Then we place $A$ arbitrarily and $B$ so that the leftmost point of $B$ lies one unit to the lefts of the leftmost point of $A$. This ensures that (i) and (ii) holds for $H^{(k)}$. Moreover, we place $B$ very far below $A$ so that also (iii) and (iv) hold.

We refer to Figure 51 for some illustrating examples. $\square$

*Proof of Seven-Hole Theorem (Theorem 4.15).*
We shall show that no Horton set contains a 7-hole. To this end we first show that for every 4-cup $C$ in a Horton set $H$ there is a fifth point $p \in H \setminus C$ whose $x$-coordinate lies between the leftmost and rightmost point in $C$.

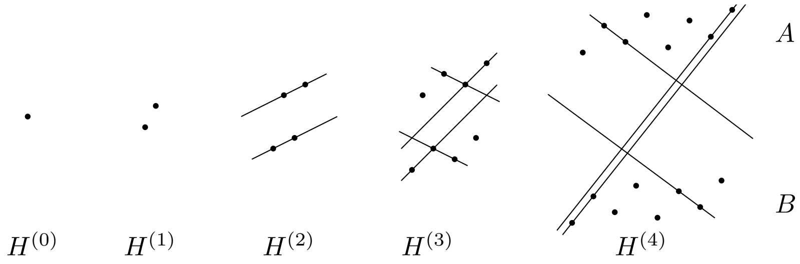



Figure 51: Horton sets $H^{(k)}$ on $2^k$ points for $k = 0, 1, 2, 3, 4$.

So let $C$ be any 4-cup in the Horton set $H$. We write $H$ as $H = A \dot{\cup} B$ for Horton sets $A$ and $B$ and may assume without loss of generality that $C \cap A \neq \emptyset$ and $C \cap B \neq \emptyset$. Otherwise, if $C \subset A$ or $C \subset B$, we consider the Horton set $A$, respectively $B$, find the desired point $p$ in there and conclude that $p$ is also good for $H$.

Now note that (iii) means that the upper half-space of any two points in $A$ is disjoint from $B$, see Figure 52(a). This implies that going through $C$ in left-to-right order we do not see the patterns $B - A - B$ and $A - A$. Hence at least one pair of two consecutive points on $C$ is contained in $B$. But then it follows from (ii) that between these two points from $B$ there is a point $p$ from $A$, as desired. We refer to Figure 52(b) for an illustration.

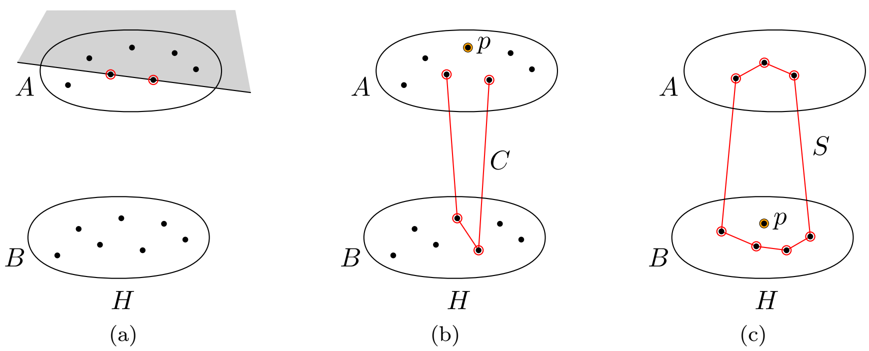



Figure 52: (a) By (iii) any two points in $A$ define a half-space disjoint from $B$. (b) A 4-cup $C$ and a point $p \in H \setminus C$ with $x$-coordinate strictly between the leftmost and rightmost point in $C$. (c) A convex 7-gon $S$ in a Horton set $H = A \dot{\cup} B$ and a point $p \in H$ inside $S$.

Analogous arguments show that for every 4-cap $C$ in a Horton set $H$ there is a point $p \in H \setminus C$ whose $x$-coordinate is strictly between the leftmost and rightmost point of $C$.

Finally assume that $H = A \dot{\cup} B$ contains some 7-hole $S$. Again we may assume without loss of generality that $S \not\subseteq A$ and $S \not\subseteq B$. Since $|S| = 7$ either $A$ or $B$ contains at least four points of $S$, say $B$. Because $A$ is far above $B$, $S \cap B$ is a cup of size at least 4. But for every 4-cup $C$ in the Horton set $B$ there is a point $p \in B \setminus C$ whose $x$-coordinate is between the leftmost and rightmost point in $C$. That this point $p$ is not inside the hole $S$ means that either the line connecting $p$ with the leftmost or the rightmost point of $C$ cuts through $A$.

This is a contradiction to (iv) and hence no such 7-hole may exist. □

Let us summarize the results presented in this and the preceding section with the following table. We refer to Figure 53 for a set of 9 points in general position not containing any 5-hole.

| $k$ | 3 | 4 | 5 | 6 | $\geq 7$ |
|---|---|---|---|---|---|
| # points enforcing a convex $k$-gon | 3 | 5 | 9 | 17 | $\geq 2^{k-2}+1$<br>$\leq \binom{2k-5}{k-2}+2$ |
| # points enforcing a $k$-hole | 3 | 5 | 10 | $\geq 30$<br>$\leq 463$ | $\infty$ |

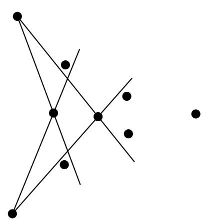


Figure 53: A set of 9 points in the plane in general position with no 5-hole.

# 5 Coloring Problems

In this chapter we consider coloring problems that arise from geometric settings in the plane. All the coloring problems we consider here fit into the following framework.

**Definition 5.1** (The $(c, k)$-Coloring Problem)**.**
*We are given a finite set $X$, a family $\mathcal{F}$ of subsets of $X$ and two natural numbers $c$ and $k$. We want to color the elements of $X$ with $c$ colors so that every subset $Y \in \mathcal{F}$ contains points of at least $k$ different colors.*

*For us a* $c$-coloring *of $X$ is a mapping $\phi : X \to [c]$, so the color of some element $x \in X$ is the number $\phi(x)$. We then call a $c$-coloring of $X$* $k$-good with respect to $\mathcal{F}$, *or simply* $k$-good *if $\mathcal{F}$ is clear from the context, if*

$$|\phi(Y)| \geq k \qquad \text{for every } Y \in \mathcal{F}.$$

*Hence the $(c, k)$-coloring problem asks for a $k$-good $c$-coloring of $X$ with respect to $\mathcal{F}$.*

Let us refer to Figure 54 for an example.

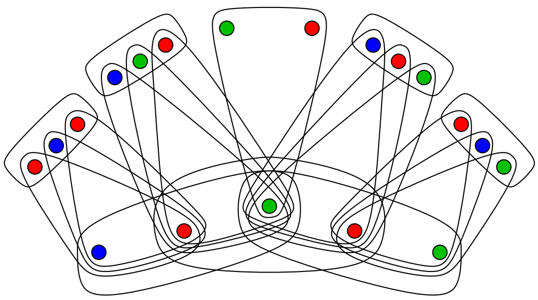


Figure 54: A 2-good 3-coloring of a set of 19 elements with respect to a family of 20 sets, each of cardinality three.

**Problem 24.**
Determine all pairs $(c, k)$ for which the example in Figure 54 admits a $k$-good $c$-coloring.

Everybody is probably familiar with the *proper graph coloring problem* in which one wants to color the vertices of a graph such that any two vertices that are joint by an edge receive distinct colors. Of course, assigning every color at most once gives a proper coloring. So the problem becomes only interesting if we restrict ourselves to make use of at most $c$ different colors. Translated into our framework, a proper coloring is just a 2-good coloring of $X$ with respect to $\mathcal{F}$, where $X$ denotes the vertex set of the graph and $\mathcal{F}$ its edge set, and the proper $c$-coloring problem for graphs is nothing other than the $(c, 2)$-coloring problem.

As already mentioned at the end of Chapter 2 the *chromatic number of a graph* is the minimum $c$ for which there exists a proper $c$-coloring of that graph.

Of course, the tuple $(X, \mathcal{F})$ can be seen as the vertex set and edge set of some hypergraph. And sometimes we refer to it as a hypergraph, but sometimes do not when it appears to be misleading in the current setting. There is many ways to define a chromatic number of a hypergraph. We use the following one.

**Definition 5.2** ($k$-Chromatic Number of $(X, \mathcal{F})$)**.**
*The* $k$-chromatic number *of* $(X, \mathcal{F})$ *is denoted by* $\chi_k(X, \mathcal{F})$ *and defined as the minimum* $c$ *for which there exists a* $k$*-good* $c$*-coloring of* $X$ *with respect to* $\mathcal{F}$*.*

Note that if $(X, \mathcal{F})$ is a graph, i.e., every set in $\mathcal{F}$ has size two, then the 2-chromatic number of $(X, \mathcal{F})$ is just the ordinary chromatic number of that graph.

Let us further define a special hypergraph which appears frequently in the literature. It is defined based on any rooted tree $T$, i.e., a tree with a distinguished vertex $v_0$ called the *root of* $T$. For any vertex $v$ in $T$ the neighbors of $v$ at larger distance to $v_0$ than $v$ are called the *children of* $v$.

**Definition 5.3.** *For any rooted tree* $T$ *the hypergraph* $H(T)$ *has vertex set* $V(T)$ *and edge set* $\mathcal{F}(T)$ *with* $Y \in \mathcal{F}(T)$ *if and only if*

- $Y$ *is the set of children of some vertex*
- *or* $Y$ *is the set of vertices on a path from the root to a leaf.*

An example of the hypergraph $H(T)$ defined on bases of a rooted tree $T$ is provided in Figure 55.

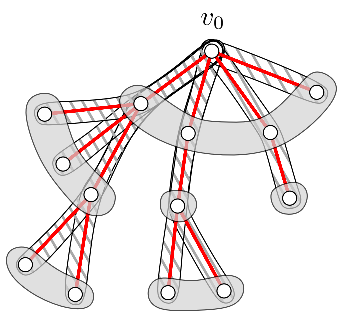



Figure 55: A rooted tree $T$ in red with root $v_0$ and the hypergraph $H(T)$. Edges corresponding to sets of siblings are highlighted in gray, those corresponding to path from the root to some leaf are striped.

A rooted tree of particular interest is the *full rooted* $m$*-ary tree* $T_m$ *of depth* $m-1$. That is, $T_m$ has a root $v_0$, every vertex $v$ at distance at most $m-2$ from $v_0$ (counted by number of edges) has exactly $m$ children, and every vertex at distance $m-1$ is a leaf of $T_m$. We remark that 2-ary and 3-ary trees are also called binary trees and ternary trees, respectively. Note that often a rooted $m$-ary tree is called full if the root has degree $m$ and all other vertices have degree either $m+1$ or 1. However, we additionally require here that all leaves have the same distance to the root. Figure 56 depicts $T_4$, the full rooted 4-ary tree of depth 4.

**Lemma 5.1.** *For every rooted tree* $T$ *we have* $\chi_2(H(T)) \in \{3, \infty\}$*.*

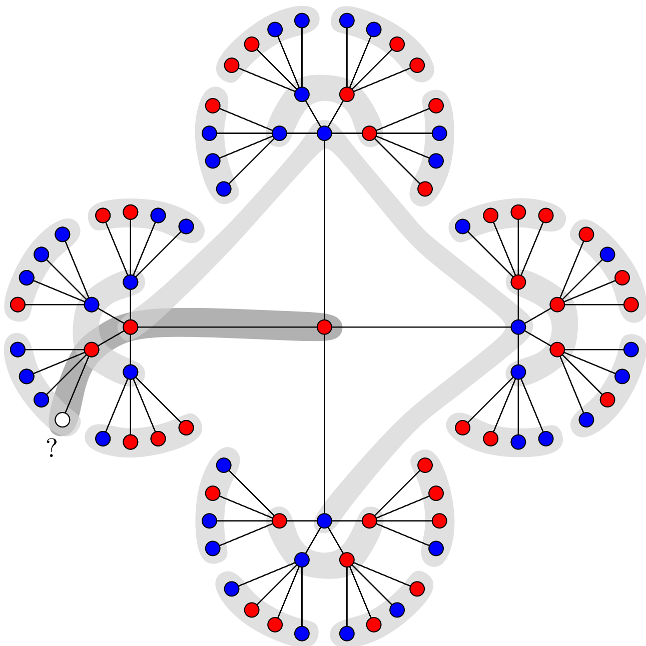



Figure 56: The full rooted 4-ary tree $T_4$ of depth 4 with the root in the center. The hypergraph $H(T_4)$ that is defined based on $T_4$ consists of all the sets (hyperedges) indicated in light gray, as well as all the $4^{4-1} = 64$ sets corresponding to a path between the root and some leaf. One such set is highlighted in dark gray.

*Proof.* Let $T$ be any rooted tree. Note that if some non-leaf vertex of $T$ has only one child, then there is no 2-good coloring of $H(T)$ and hence $\chi_2(H(T)) = \infty$. So, for the remainder of this proof we assume that $T$ is a fixed rooted tree and every non-leaf vertex of $T$ has at least two children.

First we define a 2-good 3-coloring of $H = H(T)$. To this end color the vertex of $H$ corresponding to the root of $T$ in color 1 and the remaining vertices of $H$ with colors 2 and 3 so that no set of children is monochromatic. Clearly this 3-coloring of $H$ is 2-good, and hence $\chi_2(H) \leq 3$.

To prove $\chi_2(H) \geq 3$, we assume for the sake of contradiction that $H$ has a 2-good 2-coloring $\phi$. Considering any non-leaf vertex $v$ we note that not all children of $v$ have the same color. Since there is only two colors, there is some child of $v$ of color $\phi(v)$. Hence we can start at the root $v_0$ and make our way towards some leaf by always going to the child of color $\phi(v_0)$. But then for the corresponding hyperedge $Y$ we have

$$|\phi^{-1}(Y)| = |\{\phi(v_0)\}| = 1,$$

a contradiction to the 2-goodness of $\phi$. We refer again to Figure 56 for an illustration. $\square$

## 5.1 Coloring Points with respect to Ranges

We are particularly interested in hypergraphs that arise from geometric settings, i.e., are defined geometrically. The examples presented in this section consist of a finite set $X$ of points in the plane and a possibly infinite collection of subsets of the plane, so-called *ranges*. The ranges we consider here are lines, strips and wedges.

Any given point set $X$ and any collection of ranges defines a family $\mathcal{F}$ of subsets of $X$ by

$$Y \in \mathcal{F} \qquad \text{if and only if} \qquad \exists_{\text{range } R} \, : \, Y = R \cap X.$$

When a range $R$ gives rise to a set $Y \subseteq X$ by $Y = X \cap R$, then we sometimes say that $R$ *captures the points in* $Y$. We start by investigating the following question.

*"For which ranges does the tuple* $(X, \mathcal{F})$ *have small* 2*-chromatic number?"*

Let us fix any finite point set $X$ and consider lines in the plane as ranges. Of course we cannot consider all lines in the plane, since many lines would contain only one point from $X$ and hence give rise to 1-element sets in $\mathcal{F}$, which rules out the existence of any 2-good coloring.

Hence we restrict only to certain lines, for example all connecting lines of $X$. Recall that a line $\ell$ is a connecting line of $X$ if $|\ell \cap X| \geq 2$. Hence in this case the family $\mathcal{F}$ of subsets of $X$ is defined by

$$Y \in \mathcal{F} \qquad \text{if and only if} \qquad Y = \ell \cap X \text{ for some line } \ell \text{ with } |\ell \cap X| \geq 2.$$

Now we are interested in the 2-chromatic number of the the tuple $(X, \mathcal{F})$ obtained this way. That is, we want to color the points in $X$ with as few colors as possible so that every connecting line contains points of at least two distinct colors. Clearly, if all points in $X$ are collinear then two colors suffice. Thus we have $\chi_2(X, \mathcal{F}) = 2$ if $X$ is collinear, but it turns out that $\chi_2(X, \mathcal{F}) \geq 3$ otherwise.

**Theorem 5.1** (2-Colored Sylvester-Gallai Theorem)**.** *For every finite non-collinear* 2*-colored set of points in the plane there is a connecting line containing only points of the same color.*

Recall that the Sylvester-Gallai Theorem (Theorem 1.1) states that for every finite non-collinear set of points in the plane there is a connecting line containing exactly two of these points, called an ordinary line. Theorem 5.1 is often called the 2-colored variant of the Sylvester-Gallai Theorem, probably because it guarantees the existence of a special connecting line in every (2-colored) finite point set, unless all points are collinear. However, Theorem 5.1 does *not* imply the original Sylvester-Gallai Theorem, and statements of 2-colored point sets that might be considered as natural strengthening of Sylvester-Gallai's Theorem are simply false.

**Problem 25.**
Construct a finite non-collinear 2-colored set of points for which

a) every ordinary line contains a point from either color.

b) every ordinary line contains two points of the same color.

Anyways, Theorem 5.1 is due to Motzkin [Mot67] and has a beautiful proof that reminds very much of Kelly's proof for Theorem 1.1.

*Proof of Theorem 5.1.*
Let $X$ be a finite 2-colored set of points, not all on a line. Say some points are colored red and some blue. We consider $X$ and all its connecting lines as a configuration $\mathcal{C}$ in the real projective plane $\mathbb{RP}^2$. Then by Theorem 1.4 the dual $\mathcal{C}^*$ of this configuration is a set of lines and points in $\mathbb{RP}^2$, one line for each point in $X$ and one point for each connecting line of $X$, such that a point is on a line in $\mathcal{C}$ if and only if the corresponding line contains the corresponding point in $\mathcal{C}^*$.

The 2-coloring of $X$ transfers into a 2-coloring of the lines in the dual. We now have to prove that there is a point in the dual for which all lines containing this point have the same color.

If every point is contained in only two lines, i.e., no three lines meet in a point, then the intersection of any two lines of the same color (which exists since there are at least three lines) is a point we seek.

So let $p$ be any point which lies on at least three lines, not all of which have the same color. Consider three lines at $p$, without loss of generality two red lines and one blue line. Since not all lines are concurrent there is another line not containing $p$. We choose such a line $\ell$ which is blue. Indeed, if there is only red lines not containing $p$ then the intersection of any such red line with a red line through $p$ is a point we seek.

So $\ell$ is a blue line not containing $p$. We denote the intersections of $\ell$ with the two red lines at $p$ by $q_1$ and $q_2$, and the intersection of $\ell$ with the blue line at $p$ by $p'$. We refer to Figure 57 for an illustration.

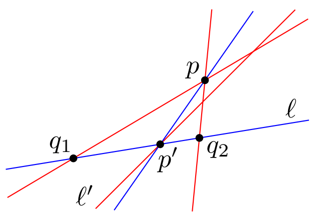



Figure 57: The situation in the proof of the 2-colored Sylvester-Gallai Theorem (Theorem 5.1).

The triangle $pq_1q_2$ is bounded by three lines, one of color $i$ and the other two of color $j$ ($i \neq j$). Moreover there is a third line that A) has color $i$, B) is concurrent with the two lines of color $j$, and C) intersects the side of the triangle of color $i$. Now if $p'$ is not a point we seek, there is a red line $\ell'$ containing $p'$ and one of the triangles $pp'q_1$ or $pp'q_2$ has all the properties mentioned for $pq_1q_2$, but strictly smaller area than $pq_1q_2$. Hence iterating this process (or starting with a minimum triangle with these properties) finally leads to a point which is contained only in lines of the same color. ☐

If $\mathcal{F}$ is the family of subsets of $X$ captured by some connecting line of $X$, then Theorem 5.1 states that $\chi_2(X, \mathcal{F}) \geq 3$, unless all points in $X$ are collinear. Clearly, if $|X| \geq 2$ and all points are collinear, then $\chi_2(X, \mathcal{F}) = 2$. And considering a set $X$ of $n$ points in general position, we easily see that $(X, \mathcal{F})$ is the complete graph on $n$ vertices and hence $\chi_2(X, \mathcal{F}) = \chi(K_n) = n$.

**Problem 26.**
Prove that if $X$ is any finite point set in the plane and $Z$ is a maximal subset of $X$ in general position, i.e., no three points of $Z$ are on a line, then

$$\chi_2(X, \mathcal{F}) \leq |Z|,$$

where $\mathcal{F}$ is the family of subsets of $X$ captured by some connecting line of $X$. Moreover, find for every $m$ and every $n \geq m + 1$ an $n$-element point set $X$ whose maximum subset $Z$ of points in general position has size $m$ and for which $\chi_2(X, \mathcal{F}) < |Z|$.

One might think that it is impossible to 2-color a set $X$ of non-collinear points such that all connecting lines contain both colors, because there are many ordinary lines, i.e., lines containing only two points from $X$. This intuition makes us consider only a subset of connecting lines as ranges. While a connecting line of $X$ is a line $\ell$ satisfying $|\ell \cap X| \geq 2$, recall from Chapter 2 that for a fixed number $m$ a line $\ell$ is called *$m$-big* if it contains at least $m$ points from $X$, i.e., $|\ell \cap X| \geq m$. We then want to answer the following question.

*"Can one 2-color every finite point set $X$ so that every $m$-big line of $X$ contains points of either color?"*

Clearly, taking $m > |X|$ every 2-coloring will work. But can we find one number $m$ that works for all sets $X$? Theorem 5.1 tells us that every set $X$ of non-collinear points requires $m \geq 3$. And for some sets $m = 3$ is already enough. For example, Figure 58 shows a finite point set with a 2-good 2-coloring with respect to its 3-big lines.

Before we answer the question above, let us introduce the $m$-big idea for general ranges.

**Definition 5.4.** *For a finite point set $X$ and a natural number $m$, a range $R$ is called* $m$-big *if $|X \cap R| \geq m$. The family of subsets of $X$ captured by $m$-big ranges is denoted by $\mathcal{F}_m$.*

Clearly, for every point set $X$, every type of ranges and for all $m \geq m'$ we have $\mathcal{F}_m \subseteq \mathcal{F}_{m'}$ and hence

$$\text{for all } k \text{ and all } m \geq m' : \qquad \chi_k(X, \mathcal{F}_m) \leq \chi_k(X, \mathcal{F}_{m'}).$$

The question above can now be rephrased and simply asks whether there exists some $m$ so that every finite point set $X$ admits a 2-good 2-coloring with respect to $m$-big lines, i.e., whether $\chi_2(X, \mathcal{F}_m) \leq 2$. However, as proven by Pach, Tardos and Tóth in 2007 [PTT07] this is not the case.

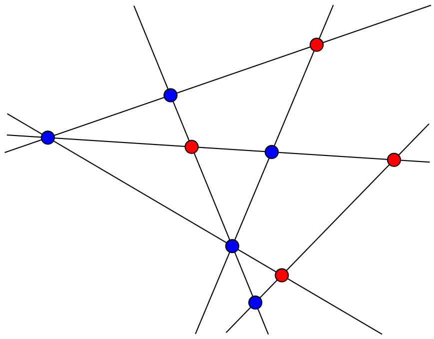


Figure 58: A set of 9 points with a 2-good 2-coloring with respect to its 3-big lines.

**Theorem 5.2.** *For every natural numbers $m$ and $c$ there exists a finite point set $X$ with $\chi_2(X, \mathcal{F}_m) > c$, where $\mathcal{F}_m$ denotes the set of all connecting lines of $X$ containing at least $m$ points of $X$.*

Since we need it in the proof of Theorem 5.2, and because it is interesting in its own right, we state the Hales-Jewett Theorem first [HJ63]. Let $H_m^d$ denote the set of integer points of the $d$-dimensional cube of side length $m$, i.e.,

$$H_m^d = \{(x_1, \ldots, x_d) \mid x_i \in \{1, \ldots, m\} \ \forall i = 1, \ldots, d\}.$$

A subset $\ell$ of $H_m^d$ is called a *combinatorial line* if there exists a non-empty set $\emptyset \neq I \subseteq \{1, \ldots, d\}$ and a number $y_i \in \{1, \ldots, m\}$ for every $i \notin I$ such that

$$\ell = \{x^\alpha \mid \alpha = 1, \ldots, m\}, \text{ where}$$
$$x_i^\alpha = \begin{cases} y_i & \text{if } i \notin I \\ \alpha & \text{if } i \in I \end{cases}$$

It is easily seen that every combinatorial line of $H_m^d$ is a $m$-element subset of $H_m^d$ that is captured by some line in $\mathbb{R}^d$. (Note that no line captures more than $m$ points.) However, not every $m$-element subset of $H_m^d$ that is captured by a line in $\mathbb{R}^d$ is a combinatorial line. See Figure 59 for an example.

**Theorem 5.3** (Hales-Jewett Theorem)**.**
*For every natural numbers $m$ and $c$ there exists a number $d$ such that every $c$-coloring of $H_m^d$ contains a monochromatic combinatorial line.*

The Hales-Jewett Theorem is considered to be the most applicable theorem from Ramsey theory. It implies several classic theorems from this area, such as Van der Waerden's Theorem [vdW27] and Gallai-Witt Theorem [Rad45, Wit52].

*Proof of Theorem 5.2.* Let $m$ and $c$ be fixed. We consider $H_m^d$ with the $d$ guaranteed by the Hales-Jewett Theorem. Further we consider all $m$-big lines of $H_m^d$, i.e., the set $\mathcal{L}_m$ of all lines in $\mathbb{R}^d$ containing $m$ points from $H_m^d$. Clearly, $\mathcal{L}_m$ is a strict superset of the set of lines in $\mathbb{R}^d$ capturing some combinatorial line of $H_m^d$. Now by the Hales-Jewett Theorem (Theorem 5.3) every $c$-coloring $\phi$ of $H_m^d$ contains a monochromatic combinatorial line, i.e.,

$$|\phi(Y)| = 1 \qquad \text{for some } \ell \in \mathcal{L}_m \text{ with } Y = H_m^d \cap \ell.$$

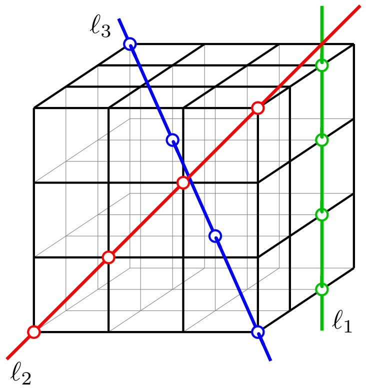



Figure 59: The point set $H_4^3$. i.e., the integer points of the 3-dimensional hypercube with side length 4, and three big lines $\ell_1, \ell_2, \ell_3$ capturing exactly $m = 4$ points of $H_4^3$. The sets $\ell_1 \cap H_4^3$ and $\ell_2 \cap H_4^3$ are combinatorial lines, while the set $\ell_3 \cap H_4^3$ is not.

In particular, there is no 2-good $c$-coloring of $H_m^d$ with respect to the family of all $m$-big lines. In order to get a 2-dimensional point set, we simply project $H_m^d$ onto some 2-dimensional subspace of $\mathbb{R}^d$ so that no two points of $H_m^d$ are mapped to the same point and no new collinearities are created. $\square$

Considering finite points sets $X$ and the family $\mathcal{F}_m$ of subsets of $X$ captured by $m$-big lines, then by Theorem 5.2 there is no upper bound on $\chi_2(X, \mathcal{F}_m)$ for fixed $m$ that is independent of $X$. Next, let us consider another set of ranges, namely strips. A *strip* is a subset of the plane spanned between two parallel lines. In what follows, we consider open strips only.

So fix a finite point set $X$, we obtain a family $\mathcal{F}$ of subsets of $X$ by

$$Y \in \mathcal{F} \qquad \text{if and only if} \qquad Y = S \cap X \text{ for some strip } S \text{ with } |S \cap X| \geq m.$$

In particular, our ranges are $m$-big strips. Recall that Theorem 5.2 asserts that there is no general upper bound on the 2-chromatic number of a finite point set $X$ with respect to $m$-big lines. Of course, every such line can be widened to a narrow strip and hence there is no bound on the 2-chromatic number with respect to $m$-big strips neither. However, let us present another, different proof for strips.

**Definition 5.5.** *For a class $\mathcal{C}$ of ranges, we say that a pair $(X, \mathcal{F})$ of a finite set $X$ and a family $\mathcal{F}$ of subsets of $X$ has a* realization *with $\mathcal{C}$ if there is a point set $P(X)$ of size $|X|$ and a set of ranges from $\mathcal{C}$ such that for every subset $Y \subseteq X$ we have*

$$Y \in \mathcal{F} \qquad \textit{if and only if} \qquad Y = P(X) \cap R \textit{ for some range } R.$$

Clearly, if we find a realization of some a hypergraph $(X, \mathcal{F})$ with our class $\mathcal{C}$ of ranges, then the maximum 2-chromatic number of a finite point set with respect to these ranges is at most $\chi_2(X, \mathcal{F})$. In particular, if we find a realization with $\mathcal{C}$ of the hypergraph $H(T)$ for some rooted tree $T$, then by Lemma 5.1 not all finite point sets admit a 2-good 2-coloring with respect to $\mathcal{C}$. This argument has for example been used by Pach *et al.* [PTT07] as follows.

**Theorem 5.4.** *For every rooted tree $T$, the hypergraph $H(T)$ has a realization with open strips.*

*Proof.* We prove the statement by induction on the number of vertices in the rooted tree $T$. If $T$ has only one vertex, the root, then it suffices to define a one-element point set $P(X)$. This point can clearly be captured by some open strip.

If $T$ contains at least two vertices, we consider any non-leaf vertex $v$ at maximum distance to the root. In particular, all children of $v$ are leaves in $T$. We remove the children of $v$ from $T$ and apply induction to the tree $T'$ obtained this way. We get a realization of $H(T')$ with open strips, i.e., a set of $|V(T')|$ points, one for each vertex in $T'$ such that every subset of $|V(T')|$ corresponding to the set of children of some vertex or the set of vertices on some path from the root to a leaf can be captured by some open strip.

Let $p(v)$ be the point for vertex $v$, which is a leaf in $T'$. Consider the open strip $S$ corresponding to the path from the root $v_0$ to $v$ in $T'$. Thus $S$ captures some subset of points, including the points for $v_0$ and $v$. Since $S$ is open we can slightly rotate it changing the subset of captured points. This way we introduce as many distinct strips, each a slight rotation of the others, as their are children at $v$ in $T$. Then there exists another strip $S'$ that intersects all these strips, does not intersect the intersection of any two strips, and does not contain any point corresponding to a vertex. We introduce one new point for each child of $v$ in the intersection of $S'$ with the strip we introduced for that vertex.

It is not hard to see that this way we have constructed a realization of $H(T)$ with open strips. □

**Corollary 5.1.** *For every natural number $m$ there exists a finite point set $X$ with 2-chromatic number at least 3 with respect to open $m$-big strips.*

*Proof.* Take $T_m$, the full rooted $m$-ary tree of depth $m-1$ and the hypergraph $H(T_m)$. By Theorem 5.4 $H(T_m)$ has a realization with open strips. Since every hyperedge of $H(T_m)$ has size $m$, $H(T_m)$ has a realization with open strips capturing at least $m$ points of $X$ each. Now the result follows from Lemma 5.1, which asserts that $\chi_2(H(T_m)) = 3$. □

After several results asserting the non-existence of some colorings (Theorem 5.1, Theorem 5.2, and Theorem 5.4) let us finally present two cases in which we can find $k$-good $c$-coloring with small $c$ and $k$ for certain classes of $m$-big ranges. Here we focus on $k$-good $k$-colorings, which are of particular interest in combinatorial geometry.

In the first case, we consider ranges that are wedges. A *wedge with apex $p$* is a connected component of the plane after the removal of two distinct rays starting at $p$. Recently, it has been shown that if we consider wedges with only two distinct apices as ranges, then for any point set we have $\chi_k((X, \mathcal{F}_m)) = k$, provided $m \geq 2k$ [ACC+11].

**Theorem 5.5.** *Every finite set of points admits a $k$-good $k$-coloring with respect to $2k$-big wedges with two distinct apices.*

*Proof.* Fix $k \geq 2$ and let $X$ be a finite set of $n$ points and $p_1, p_2$ the two apices of wedges. For $i = 1, 2$ consider the circular ordering $\sigma_i$ of the points in $X$ around

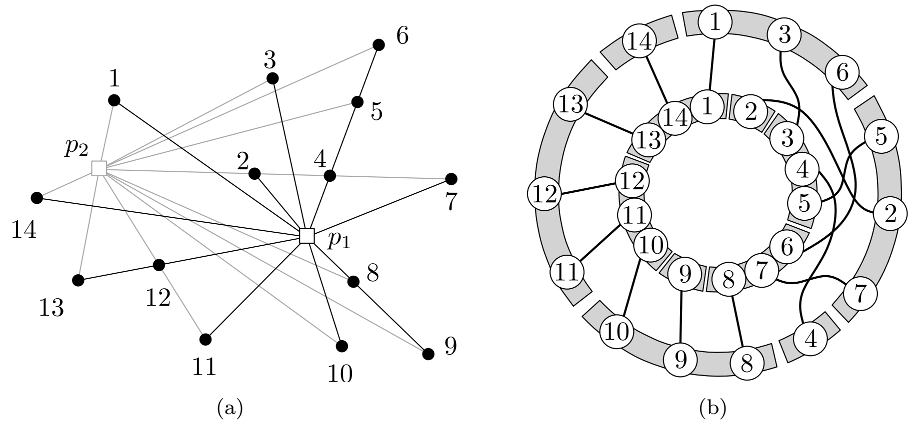



Figure 60: (a) A set of 14 points in the plane numbered along a circular ordering $\sigma_1$ around a point $p_1$. A circular ordering $\sigma_2$ around $p_2$ is given by $(\ldots, 1, 3, 6, 5, 2, 7, 4, 8, 9, 10, 11, 12, 13, 14, 1, \ldots)$. (b) Grouping $\sigma_1$ and $\sigma_2$ into blocks of size $k = 3$ and size 1. Elements in $\sigma_1$ and $\sigma_2$ that correspond to the same point in $X$ are connected.

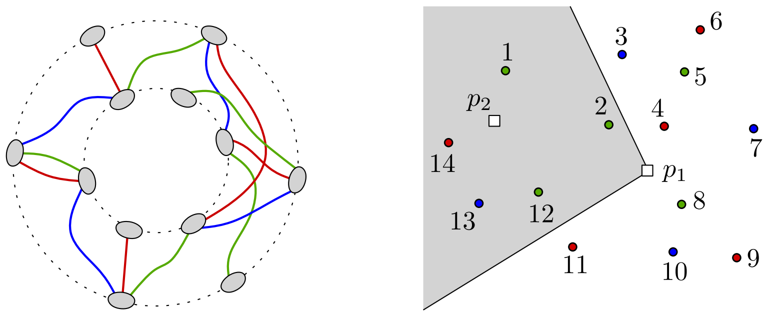
apex $p_i$. In case two or more points of $X$ lie on the same ray emerging from $p_i$ these points are ordered arbitrarily in $\sigma_i$. See Figure 60(a) for an example.

We group the points in $\sigma_1$, as well as in $\sigma_2$, into $\lfloor n/k \rfloor$ groups of $k$ consecutive points each and $n \bmod k$ groups of 1 point each, so that in no circular two 1-element groups are consecutive. This is possible, since $\lfloor n/k \rfloor \geq n \bmod k$ provided $n \geq k^2$. See Figure 60(b) for an example.

Next we consider the $2(\lfloor n/k \rfloor + n \bmod k)$ blocks as vertices of a multigraph $G$. For each point $p$ in $X$ we draw an edge between the block in $\sigma_1$ containing $p$ and the block in $\sigma_2$ containing $p$. Note that this way we may draw several edges between the same pair of blocks. Since every block consists of either 1 or $k$ points, every vertex in $G$ has degree either 1 or $k$. Now by Vizing's Theorem [Viz64] the edges of $G$ can be colored with $k$ colors so that any two edges that share a vertex receive distinct colors. See Figure 61(a) for the continued example.

Since edges of $G$ correspond 1-to-1 to points in $X$ we get a $k$-coloring of $X$. Moreover, since every block of size $k$ contains all the $k$ colors and between any two consecutive $k$-blocks there is at most one further point, we get that every set of $2k$ consecutive points in $\sigma_1$, as well as $\sigma_2$ contain all the $k$ colors. In other words every wedge with apex $p_1$ or $p_2$ that captures at least $2k$ points of $X$ contains at least one point of each color, see Figure 61(b). This concludes the proof. □

Next we consider ranges that are $x$-monotonously decreasing curves in the plane, which we call *decreasing curves* for short. That is, a set $Y$ of points is captured by some decreasing curve $\gamma$ if and only if the points in $Y$ have distinct $x$-coordinates and going through the points in order of increasing $x$-coordinates the $y$-coordinates are strictly decreasing. See Figure 62 for an illustration. Note that without loss of generality we can restrict ourselves to continuous decreasing curves.




(a) (b)

Figure 61: (a) Blocks are contracted to vertices of a bipartite graph of maximum degree $k = 3$, whose edges are colored with $k = 3$ colors. (b) The coloring is transferred to the point set $X$.

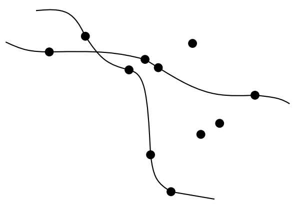


Figure 62: A finite point set and two decreasing curves capturing four points each.

For convenience let us call a subset $Y$ of $X$ that is captured by some decreasing curve a *decreasing subset*. It is easily seen that every subset of a decreasing subset is decreasing again. Hence, if a decreasing subset of $n$ points admits a $k$-good $c$-coloring with respect to $m$-big decreasing curves, then $m \cdot c \geq n \cdot k$. Looking at $k$-good $k$-colorings we get $m \geq n$. In particular, a first necessary condition for every set $X$ to have a $k$-good $k$-coloring with respect to $m$-big decreasing curves is that

(1) every decreasing subset of $X$ has size at most $m$.

Clearly, if every subset that is captured by a range shall contain at least $k$ different colors, then every subset should contain at least $k$ elements. Thus another obviously necessary condition for the existence of any $k$-good coloring is that

(2) all ranges are at least $m$-big with $m \geq k$.

With (1) and (2) we have two obviously necessary conditions that need to be satisfied in order that a finite set $X$ admits a $k$-good $k$-coloring with respect to $m$-big decreasing curves. Interestingly, these conditions are already sufficient.

**Theorem 5.6.** *Every finite point set $X$ in which every decreasing subset has size at most $m$ (i.e., (1) is satisfied) admits a $k$-good $k$-coloring with respect to $m$-big decreasing curves, provided $m \geq k$ (i.e., (2) is satisfied).*

*Proof.* Let $k$ and $m \geq k$ be fixed. Consider any point set $X$ that satisfies (1). We shall find a $k$-good $k$-coloring of $X$ with respect to its $m$-big decreasing curves. Clearly, it is enough to consider the case $m = k$, since for all $m' \geq m$ we have $\chi_k((X, \mathcal{F}_{m'})) \leq \chi_k((X, \mathcal{F}_m))$.

We shall prove the existence of a $k$-good $k$-coloring $\phi_k$ of $(X, \mathcal{F}_k)$ for every point set $X$ satisfying (1) with $m = k$ by induction on $k$. If $k = 1$, then (1) we seek a 1-good 1-coloring, which simply amounts to coloring all the points in $X$ the same.

So let $k \geq 2$. Consider any set $Y \in \mathcal{F}_k$, i.e., any decreasing subset of $X$ of size $m = k$. No two points in $Y$ have the same $x$-coordinate and thus there is a unique rightmost point in $Y$, i.e., the point with maximum $x$-coordinate. We define $Z$ to be the set of all rightmost points of all sets $Y \in \mathcal{F}_k$. If we remove $Z$ from $X$ we get a point set $X' = X \setminus Z$ in which every decreasing subset has size at most $k - 1$. Hence by induction there exists a $(k-1)$-good $(k-1)$-coloring $\phi_{k-1}$ of $X'$. Extending $\phi_{k-1}$ to the larger set $X$ by assigning a new $k$-th color to all points in $Z$, we obtain a desired $k$-good $k$-coloring $\phi_k$. $\square$

**Problem 27.**
Let $m \geq 2$ and $X$ be a point set in the plane in which all decreasing subsets have size at most $m$. Find a finite superset $\bar{X} \supseteq X$, such that all decreasing subsets of $\bar{X}$ have size at most $m$ and every decreasing set $Y \subseteq X$ is contained in a decreasing subset of $\bar{X}$ of size exactly $m$.

## 5.2 The Dual Coloring Problem

In the $(c, k)$-coloring problem considered in Section 5.1 we have colored points in a finite point set $X$ with respect to ranges of a certain type. In this section we introduce the dual problem, in which we color a finite set of ranges of a certain type with respect to points.

We start by defining the *dual of a hypergraph* $H = (X, \mathcal{F})$. This is a hypergraph $H^* = (X^*, \mathcal{F}^*)$ whose vertices are in bijection with the hyperedges of $H$, $X^* \equiv \mathcal{F}$, and whose hyperedges are in bijection with the vertices of $H$, $\mathcal{F}^* \equiv X$. More precisely, for every hyperedge $Y$ of $H$ there is a vertex $Y$ in $H^*$, for every vertex $x$ of $H$ there is a hyperedge $x$ in $H^*$, and $Y \in X^*$ is contained in $x \in \mathcal{F}^*$ if and only if $x \in X$ is contained in $Y \in \mathcal{F}$. In other words

$$Y \in x \text{ in } H^* \qquad \Leftrightarrow \qquad x \in Y \text{ in } H.$$

So when taking the dual $H^*$ of a hypergraph $H$, we just swap the meanings of vertices and hyperedges and reverse the containment relations. See Figure 63 for an example.

Dualizing a hypergraph is very similar to dualizing an arrangement of points and lines in projective plane (c.f. Theorem 1.4) where we exchange the meanings of points and lines. Indeed the dual configuration of a configuration of points and lines in $\mathbb{RP}^2$ obtained in Theorem 1.4 is just a realization of the dual hypergraph defined by the first configuration. This fact is illustrated in Figure 64.

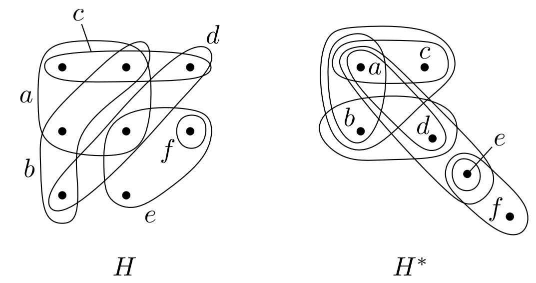



Figure 63: A hypergraph $H$ with 7 vertices and 6 hyperedges and its dual hypergraph $H^*$ with 6 vertices and 7 hyperedges.

**Problem 28.**
Show that for every rooted tree $T$ the *dual* hypergraph of $H(T)$ admits a realization with open strips.

Now assume the hypergraph $H = (X, \mathcal{F})$ arises from a geometric setting where $X$ denotes a point set and $\mathcal{F}$ the family of subsets of $X$ captured by a range from a given collection of ranges. Clearly, we can pick for every set $Y \in \mathcal{F}$ one representative range $R$ that captures this set $Y$. For every set $Y$ there could be infinitely many ranges $R$ that capture $Y$, but for every two distinct sets $Y, Y'$ every representative range for $Y$ is distinct from every representative range for $Y'$. This way $H = (X, \mathcal{F})$ can be completely described by $|X|$ points and $|\mathcal{F}|$ representative ranges.

Now the dual hypergraph $H^*$ of $H$ can be interpreted as follows. The vertices of $H^*$ are the finite set of representative ranges and a subset $Y$ of ranges form a hyperedge in $H^*$ if there is a point $x$ in $X$ that is contained in precisely the ranges in $Y$. In this case we say that a set $Y$ of ranges is *pierced* by the point $x$. We can now think of each point in $X$ as a representative point for a set $Y$ of ranges. Clearly, there can be infinitely many points in the plane that pierce exactly the set $Y$. See Figure 65 for some illustration.

In the general dual setting we consider a hypergraph $H^*$ whose vertex set is a finite set of ranges and whose hyperedges are defined with respect to a certain (most of the times infinite) collection of points in the plane. Analogous to $m$-big ranges, we now consider $m$-deep points.

**Definition 5.6.** *For a finite set $X^*$ of ranges and a natural number $m$, a point $p$ is called* $m$-deep *if $|\{R \in X^* \mid p \in R\}| \geq m$. The family of subsets of $X^*$ pierced by $m$-deep points in the plane is denoted by $\mathcal{F}^*_m$.*

The dual $(c, k)$-coloring problem then asks for a $k$-good $c$-coloring of a set $X^*$ of ranges with respect to $m$-deep points. In particular, we want to color the ranges in $X^*$ with $c$ colors such that every set of ranges that is pierced by an $m$-deep point contains at least $k$ distinct colors.

For simplicity we refer to the original $(c, k)$-coloring problem, where the points are colored with respect to ranges, as the *primal problem*. While the $(c, k)$-coloring problem in which ranges are colored with respect to points is called the *dual problem*. The following table summarizes the different notions in the primal and the dual setting.

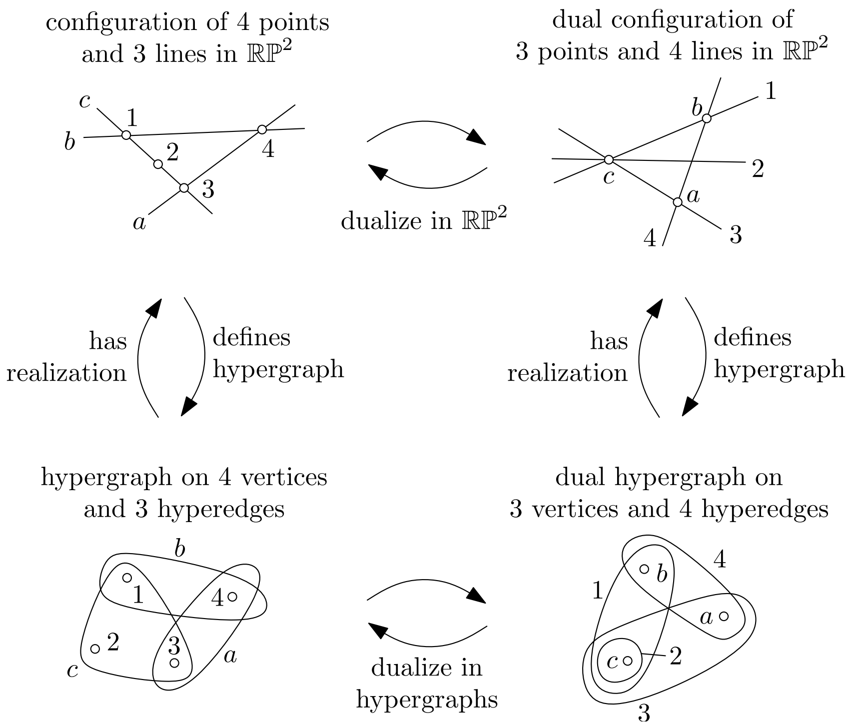



Figure 64: Dualization of point-line configurations in $\mathbb{RP}^2$ is a special case of hypergraph dualization.

| | **Primal** | **Dual** |
|---|---|---|
| **General Hypergraphs** | vertex $x \in X$ | hyperedge $x \in \mathcal{F}^*$ |
| | hyperedge $Y \in \mathcal{F}$ | vertex $Y \in X^*$ |
| | $x \in Y$ | $Y \in x$ |
| **Geometric Hypergraphs** | point $x \in X$ | range $R \in X^*$ |
| | $m$-big range $R$ | $m$-deep point $p$ |
| | $R$ captures $Y \in \mathcal{F}_m$ | $p$ pierces $Y \in \mathcal{F}_m^*$ |
| **$(\mathbf{c}, \mathbf{k})$-Coloring Problem** | color points in $X$ with $c$ colors | color ranges in $X^*$ with $c$ colors |
| | such that every $R$ captures $k$ colors | such that every point $p$ pierces $k$ colors |

As already mentioned before, we are particularly interested in $k$-good $k$-

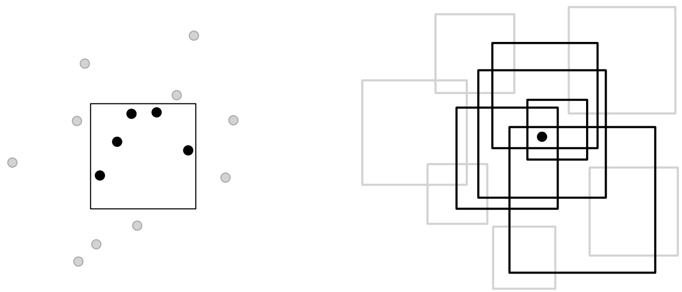


Figure 65: Five points captured by a square and five squares pierced by a point.

colorings of geometric hypergraphs in the primal as well as in the dual setting. That is, we color the elements (points in the primal problem, ranges in the dual problem) with $k$ colors so that every set in $\mathcal{F}$, respectively $\mathcal{F}^*$, (points captured by a range in the primal, ranges pierced by a point in the dual) contains elements of each of the $k$ colors. Such sets are sometimes also called *colorful*.

**Definition 5.7.** *For a fixed type $\mathcal{T}$ of ranges we define*

- *$m(k)$ to be the minimum $m$ such that every finite point set $X$ in the plane admits a $k$-good $k$-coloring with respect to $m$-big ranges of type $\mathcal{T}$, and*
- *$m^*(k)$ to be the minimum $m$ such that every finite set $X^*$ of ranges of type $\mathcal{T}$ admits a $k$-good $k$-coloring with respect to $m$-deep points in the plane.*

Given any $k$-good $k$-coloring with respect to $m$-big ranges in the primal setting or with respect to $m$-deep points in the dual setting, we can obtain a $(k-1)$-good $(k-1)$-coloring with respect to the same $m$ by simply uniting two colors into one. Thus for every type of ranges the functions $m(k)$ and $m^*(k)$ are monotonously non-decreasing in $k$. Thus if $m(k) = \infty$ for some $k \in \mathbb{N}$ then $m(k') = \infty$ for all $k' \geq k$.

Curiously, we do not know whether there is a type of ranges such that $m(2)$ is finite but $m(k)$ is infinite for some large $k$. And the same question is open in the dual setting, i.e., whether there is a type of ranges such $m^*(2) < \infty$ but $m^*(k) = \infty$ for some $k$. For example, in the next section we discuss a type of ranges for which was known that $m^*(2) = 3$ but open whether $m^*(3) < \infty$. (Today we know for this particular type of ranges that $m^*(k) < \infty$ for all $k$.)

We can derive the following bounds on $m(k)$ and $m^*(k)$ from our results in the previous section (Section 5.1).

| type of range | $m(k), m^*(k)$ | reference |
|---|---|---|
| lines | $m(2) = \infty$ | Theorem 5.2 |
| lines | $m^*(2) = \infty$ | Theorem 5.2 and Theorem 1.4 |
| strips | $m(2) = \infty$ | Theorem 5.4 |
| strips | $m^*(2) = \infty$ | Problem 28 |
| wedges with two apices | $m(k) \leq 2k$ | Theorem 5.5 |
| decreasing curves | $m(k) = k$ | Theorem 5.6 |

**Remark.** *It is important to note that the value $m(k) = k$ for decreasing curves actually holds only if we restrict ourselves to point sets $X$ in which no $k + 1$ points lie on a decreasing curve.*

Before we continue, let us briefly mention the original motivation for the $(c, k)$-coloring problem of geometric hypergraphs. In the 1980's János Pach and others considered $k$-fold coverings of the plane with certain type of bodies, such as unit discs. A $k$-fold covering is an infinite collection of subsets, all begin translates of another, such that every point in the plane is covered at least $k$ times, i.e., is contained in at least $k$ such subsets. (Usually those coverings are assumed to be locally finite, that is, every point in the plane is contained in only finitely many subsets.)

To investigate the minimum density of such a $k$-fold covering, the question arose whether every $k$-fold splits into roughly $k$ disjoint coverings, or about $k/l$ disjoint $l$-fold coverings for some $l < k$. Note that the former question simply asks whether $m^*(k) < \infty$ with respect to ranges that are translates of a fixed set, while the latter asks whether there is a dual $l$-good $k/l$-coloring.

In 1986 [Pac86] János Pach showed that if the ranges are translates of a fixed convex polygon $P$ then $m^*(k) < \infty$. However, his bound was doubly-exponential in $k$, and has only been improved recently by Pach and Tóth [PT09] to $m^*(k) \in \mathcal{O}(k^2)$. Both bounds involve a constant factor $c_P$ that depend on the number of corners of the convex polygon $P$, which tends to infinity as $P$ tends to a smooth object. The only "result" for smooth ranges is in an unpublished manuscript by Mani-Levitska and Pach [MLP87] in which they show that every 33-fold covering with unit discs decomposes into two coverings. However, this manuscript appears to be lost and the statement has never been verified by others. The convexity is indeed crucial, since if regions are translates of a fixed concave quadrilateral, then $m^*(2) = \infty$ [PTT07].

The next theorem shows that in all of the above cases that are stated in the dual version, the same bounds hold also for the primal problem.

**Theorem 5.7.** *If we consider ranges that are translates of a fixed body $S$ in the plane then we have*

$$m(k) = m^*(k) \qquad \text{for every } k \in \mathbb{N}.$$

*Proof.* Let $S$ be a fixed body in the plane and $X$ be any finite point set. We fix for every set $Y \in \mathcal{F}_m$, i.e., any subset $Y$ of $X$ that is captured by some

$m$-big range, one representative range $S_Y$ with $X \cap S_Y = Y$. Clearly, $S_Y$ is a translated copy of $S$, i.e., there exists some $p_Y \in \mathbb{R}^2$ such that

$$S_Y = S + p_Y = \{p \in \mathbb{R}^2 \mid p = s + p_Y \text{ for some } s \in S\}.$$

The hypergraph $H = (X, \mathcal{F}_m)$ is now given by a finite set $X$ of points and a finite set $\mathcal{S}$ of translates of $S$, one for each set in $\mathcal{F}_m$, such that a vertex $x$ is contained in a hyperedge $Y$ if and only if the point $x$ is contained in the range $S_Y$.

Next we shall prove that the dual hypergraph $H^* = (X^*, \mathcal{F}_m^*)$ has a realization with translates of $S$, too. To this end, we define for every $x \in X$

$$S_x \stackrel{\text{def}}{=} x - S = \{p \in \mathbb{R}^2 \mid p = x - s \text{ for some } s \in S\}.$$

That is, $S_x$ is a translated copy of $S$ after a rotation of $\pi$. We refer to Figure 66 for an illustration.

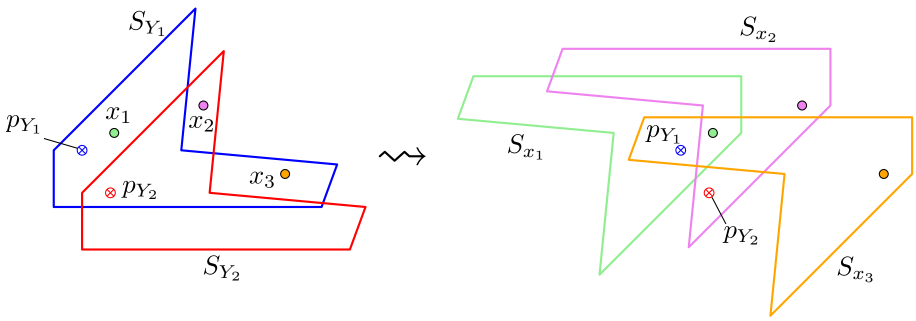



Figure 66: Two representative ranges $S_{Y_1}$ and $S_{Y_2}$ and three points $x_1, x_2, x_3$ are transformed into two points $p_{Y_1}$ and $p_{Y_2}$ and three representative ranges $S_{x_1}, S_{x_2}, S_{x_3}$.

Now we claim that $H^* = (X^*, \mathcal{F}_m^*)$ has a representation with translates of $S$, which is given by

- $X^* = \{p_Y \mid Y \in \mathcal{F}_m\}$ and
- $\mathcal{F}_m^* = \{Y_x \subseteq X^* \mid Y_x = X^* \cap S_x \text{ for some } x \in X\}$.

Indeed, we have for every $Y \in \mathcal{F}_m$ and every $x \in X$ that

$$\begin{aligned} x \in Y \;&\Leftrightarrow\; x \in S_Y \;\Leftrightarrow\; x = p_Y + s \text{ for some } s \in S \\ &\Leftrightarrow\; p_Y = x - s \text{ for some } s \in S \;\Leftrightarrow\; p_Y \in S_x \;\Leftrightarrow\; p_Y \in Y_x. \end{aligned}$$

Because both $H$ and $H^*$ admit a realization by the same type of ranges, it follows that the $(c, k)$-coloring problem for $H = (X, \mathcal{F}_m)$ is equivalent to the dual $(c, k)$-coloring problem for $H^* = (X^*, \mathcal{F}_m^*)$. In particular, this implies $m(k) = m^*(k)$ for every $k \in \mathbb{N}$. □

All the considerations above involve only *translates of regions*. But the "big question" of Pach that apparently is open for almost 30 years now concerns *homothetic regions*.

**Question 5.1** (Pach 1986)**.**
*Is it true that for every convex polygon $P$ in the plane, if we consider ranges that are homothetic copies of $P$, then $m^*(k) < \infty$ for every $k \in \mathbb{N}$?*

If $P$ is smooth, rather than a polygon, then the answer is 'NO', because for every tree $T$ the hypergraph $H(T)$ admits a dual realization with homothetic discs [PTT07].

## 5.3 Online Coloring Problems

In this section we present a set of techniques that in recent years proved to be very useful to bound $m(k)$ and $m^*(k)$ for several types of ranges: *online colorings*.

Actually, every combinatorial optimization problem that given a certain finite input seeks for a certain solution that maximizes or minimizes a certain value also has a meaningful online variant. Let us motivate this with a small example.

**Example** (The Online Hanging Problem)**.**
*When at home the washing machine is finished, the set $C$ of wet clothes needs to be put onto a laundry rack for drying. The rack has some number $j$ of drying lines, each of length 1. Each piece of clothing $x$ has a width $w(x)$ and must be assigned to a consecutive subset of length $w(x)$ of one of the lines so that distinct pieces of clothing are assigned to disjoint subsets of drying lines. We refer to Figure 67 for an illustration.*

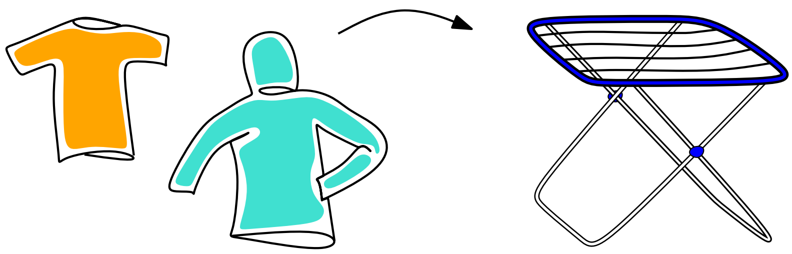


Figure 67: A set $C$ of two pieces of clothes and a laundry rack with $j = 4$ drying lines.

*Because we might decide for another run of the washing machine, the* (offline) hanging problem *is now to find a placement for each piece of clothes in $C$ on the laundry rack such that it occupies the least number of drying lines.*

*In case my wife and I are both at home we usually share the work load of hanging out the laundry. Then my wife has the pile $C$ of wet clothes, takes one piece at a time, unfolds it and hands it over to me. I then take the piece of clothes $x$ and assign it to a consecutive subset of width $w(x)$ of one of the drying lines. This lets us finish very quickly. However, in order not to lose much time we have to obey the following two restrictions.*

**No forecast.** *I cannot inspect the pile of clothes yet to come. That is, I neither know how many clothes nor what kind of clothes are yet to come. I do not even know whether the current piece of clothes in my hand is the last one.*

**No reconfiguration.** *I cannot reassign any piece of clothes. As soon as I decided to place a piece onto a certain subset, this decision is irrevocable.*

*The problem I am facing now is called the* online hanging problem. *That is, I shall assign each piece of clothes once I have it in my hands to some suitable subset of a line, under the restriction that I have no information about the clothes yet to come and I may not reconfigure any piece of clothes that already hangs. And I shall do it in such a way that I use as few lines as possible. Indeed, I want to minimize the difference between the number of lines I used in the end and the number of lines that would carry the set $C$ of clothes.*

*For example, let us consider the following situation. I have $j = 2$ racks at my disposal and my wife guarantees me the laundry can fit onto these two lines. (This is a piece of information that is usually not provided!) Then the following happens.*

1) *My wife starts by giving me a T-shirt $x_1$ of width $w(x_1) = 0.4$ and I put it onto the first drying line $\ell_1$ at the very left. (It is easily seen that placing pieces of clothes always left-aligned is not disadvantageous.)*

2) *Then my wife gives me another T-shirt $x_2$ of width $w(x_2) = 0.4$ and I can choose to place it also on $\ell_1$ or on the second line $\ell_2$.*

3) *I decide to put $x_2$ on $\ell_2$. Then my wife gives me a blanket $x_3$ of width $w(x_3) = 1$ and I cannot place it onto the rack, although all three pieces could fit on it. This is illustrated in Figure 68.*

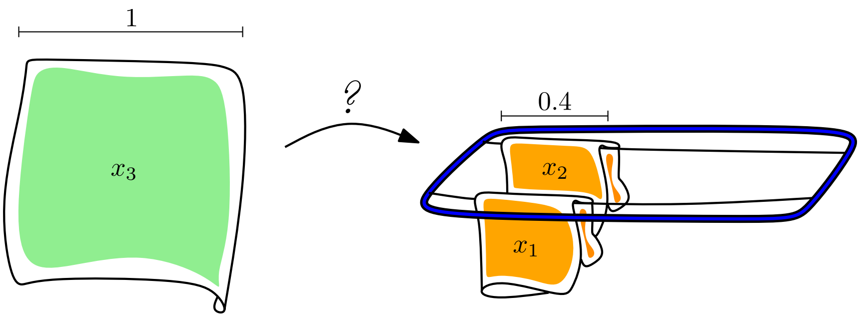



Figure 68: Placing T-shirts $x_1$ and $x_2$ on different drying lines is bad if next comes a large blanket.

*It seems that I should have put $x_2$ also on $\ell_1$.*

3') *I decide to put $x_2$ also on $\ell_1$. Then my wife gives me a sweater $x_3$ of width $w(x_3) = 0.6$ and another sweater $x_4$ of the same width $w(x_4) = 0.6$. I can place only $x_3$ on the rack, although all four pieces could fit on it. This is illustrated in Figure 69.*

*Apparently, when my wife and I hang the laundry online, then we are a lot quicker, but might need a larger laundry rack.*

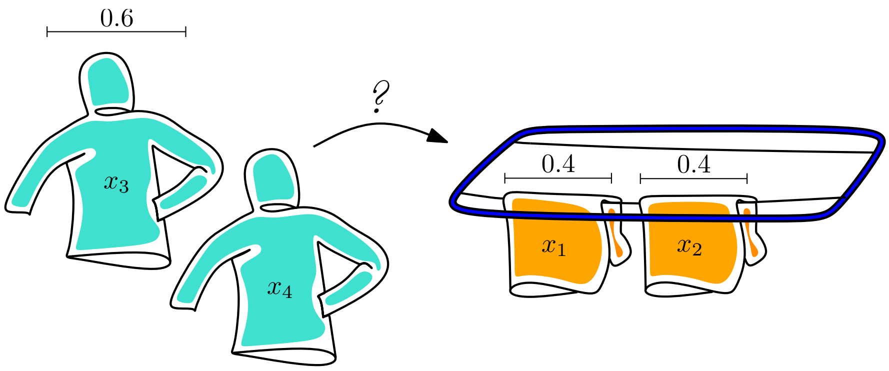



Figure 69: Placing T-shirts $x_1$ and $x_2$ on the same drying line is bad if next comes two medium size sweaters.

Here we want to consider online versions of the $(c, k)$-coloring problem introduced in Definition 5.1 at the very beginning of this chapter. In particular, a hypergraph $H = (X, \mathcal{F})$ is *presented in rounds* (instead of given as input) as follows.

- In each round exactly one new vertex is presented.
- Every hyperedge $Y$ is presented together with its vertex that is presented last.

So in each round there is exactly one new vertex, but there could be any number of new hyperedges, ranging from none to arbitrarily many. In the online $(c, k)$-coloring problem we then want to do the following.

**(O1)** Color every vertex immediately when it appears.

**(O2)** Use at most $c$ colors.

**(O3)** Never recolor any vertex.

**(O4)** Ensure that every hyperedge contains vertices of at least $k$ different colors.

Before we actually define the *online* $(c, k)$*-coloring problem* and the *online* $k$*-chromatic number*, let us already present a first result due to Gyárfás and Lehel [GL88]. It states that the online proper coloring problem is basically hopeless, even for very simple graphs, namely trees.

**Theorem 5.8.** *There is no constant $c$ such that every tree can be properly colored online with at most $c$ colors.*

*Proof.* We shall define for every $c \in \mathbb{N}$ an adversarial strategy $\mathcal{S}(c)$ that no matter how vertices are colored eventually produces a tree that contains vertices of at least $c$ different colors. We remark that we present a strategy and not a tree. The tree produced by $\mathcal{S}(c)$ will depend on the intermediate coloring of vertices.

We prove the existence of $\mathcal{S}(c)$ by induction on $c$. In the base case $c = 1$ it suffices to present a single vertex. So let us assume that $c \geq 2$. By induction there exists a strategy $\mathcal{S}(c-1)$ that eventually produces a tree containing at least $c-1$ different colors. We present the strategy $\mathcal{S}(c-1)$ $c-1$ times in a row, which gives us $c-1$ trees $T_1, \ldots, T_{c-1}$, each containing at least $c-1$ different colors.

Now we distinguish two cases.

*Case 1: There are at least $c$ different colors in $T_1 \cup \ldots \cup T_{c-1}$.* Then we simply present one more vertex $v$ with exactly one edge into each $T_i$ for $i = 1, \ldots, c-1$. This gives a tree $T$ that, independent of the color assigned to $v$, contains vertices of at least $c$ different colors.

*Case 2: Each $T_i$ contains the same set of $c-1$ colors for $i = 1, \ldots, c-1$.* Without loss of generality the $c-1$ colors are $1, \ldots, c-1$. Then we choose for each $i \in \{1, \ldots, c-1\}$ a vertex $v_i$ from $T_i$ of color $i$. We present one new vertex $v$ with an edge to each $v_i$, $i = 1, \ldots, c-1$. Clearly, $v$ must receive a new color and we have forced $c$ colors in total. We refer to Figure 70 for an illustration.

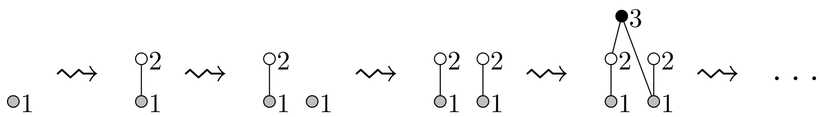



Figure 70: How the first steps in the adversarial strategy to force arbitrarily many colors in a tree may look like.

This completes the proof. □

Under the light of Theorem 5.8 the usefulness of coloring a (hyper)graph online may seem questionable. Of course, online problems in general are very important in practice, e.g., when it comes to assignments of jobs to machines inside some never-ending mechanism, such as the internet. However, as we shall see later, at least for geometric hypergraphs online colorings also imply strong results in the offline settings. And sometimes the best known offline bounds are actually generated by online considerations.

In the online setting it no longer makes sense to talk about one fixed hypergraph and something like its ”online chromatic number“. Because, if it is only one hypergraph $H$ and admits some $k$-good $c$-coloring $\phi$ then we could simply color according to $\phi$. And if $H$ has no $k$-good $c$-coloring, then of course presenting $H$ online does not make it better. The idea of online coloring problem is that we do not know which hypergraph $H$ is going to be presented. All that we know is that $H$ is a hypergraph from a certain *class of hypergraphs*, such as, trees, the hypergraphs $H(T)$ for any rooted tree $T$, or the hypergraphs that admit a realization with points in the plane and subsets of those points captured by a certain type of ranges.

Note that, while we defined the (offline) $(c,k)$-coloring problem for one particular fixed hypergraph, most of our investigations actually also considered a certain class of hypergraphs.

Let us make this a little more formal.

**Definition 5.8** (Online $(c, k)$-coloring problem).
*Let $\mathcal{C}$ be a class of hypergraphs. An* online $k$-good $c$-coloring *of $\mathcal{C}$ is a class $\Phi$ of exactly one $k$-good $c$-coloring $\phi$ for each $H \in \mathcal{C}$ and each ordering of the vertices in $H$, such that the following is true.*

- *Whenever the vertex orderings of two hypergraph $H, H' \in \mathcal{C}$ coincide in the first $n$ positions for some $n$, then the corresponding colorings $\phi$ and $\phi'$ coincide on these first $n$ vertices, too.*

*The* online $k$-chromatic number *of $\mathcal{C}$ is then defined as*

$$\chi_k^{ON}(\mathcal{C}) \stackrel{def}{=} \min\{c \in \mathbb{N} \mid \text{ there exists an online } k\text{-good } c\text{-coloring of } \mathcal{C}\}.$$

With this definition Theorem 5.8 can be conveniently rephrased as

$$\chi_k^{\mathrm{ON}}(\mathcal{T}) = \infty \quad \forall k \geq 2,$$

where $\mathcal{T}$ denotes the class of all trees.

Next we consider a different class of hypergraphs and show that its online $k$-chromatic number is indeed finite for every $k$. Let $\mathcal{C}_m$ be the class of hypergraphs that admit a realization with $m$-big decreasing curves, such that every decreasing curve captures at most $m$ points. In particular, we consider exactly the class of hypergraphs from Theorem 5.6, in which we showed that for the case $m = k$ every such hypergraph $H \in \mathcal{C}_{m=k}$ admits a $k$-good $k$-coloring, that is $\chi_k(H) = k$.

It is easy to see that there is no *online* $k$-good $k$-coloring of $\mathcal{C}_k$. Hence we have to either increase the number of colors, or decrease the level of goodness. We take the first option and want to find the minimum $c$ for which there exists an online $k$-good $c$-coloring of $\mathcal{C}_k$. In particular, we want to determine $\chi_k^{\mathrm{ON}}(\mathcal{C}_k)$.

**Theorem 5.9.** *For every $k \geq 1$, every set of points in the plane with no $k+1$ points on a decreasing curve can be colored online with at most $\binom{k+1}{2}$ colors such that every decreasing subset of size $k$ contains $k$ different colors.*

*Proof.* We are presented points in the plane one after another and we shall color each point as it appears with one of at most $\binom{k+1}{2}$ colors, such that whenever two points are captured by some decreasing curve they receive distinct colors. So let $p$ be the point that just appeared and is yet to be colored, and $X$ be the set of points presented so far. We define two auxiliary numbers for $p$. We say that a decreasing subset $Y$ *starts at* $s \in Y$ and *ends at* $t \in Y$ if $s$ is the leftmost point in $Y$ and $t$ is the rightmost point in $Y$.

$$L(p) \stackrel{def}{=} \text{size of the largest decreasing subset in } X \cup p \text{ ending at } p.$$
$$R(p) \stackrel{def}{=} \text{size of the largest decreasing subset in } X \cup p \text{ starting at } p.$$

Now we color the point $p$ with the ordered tuple $(L(p), R(p))$. Since the union of any decreasing subset ending at $p$ and any decreasing set starting at $p$ is a decreasing set again, we conclude that $L(p) + R(p) \leq k + 1$. Hence the total number of colors we use is at most

$$\sum_{L(p)=1}^{k} \sum_{R(p)=1}^{k+1-L(p)} 1 = \sum_{L(p)=1}^{k} (k + 1 - L(p)) = \sum_{L(p)=1}^{k} L(p) = \binom{k+1}{2}.$$

It remains to show that if two points $p$ and $q$ are captured by some decreasing curve, then these points receive distinct colors. So consider such $p$ and $q$ and assume without loss of generality that $q$ appeared prior to $p$. The point $q$ received the color $(L(q), R(q))$ with respect to the point set $X$ that was presented prior to $q$. Now consider the moment that $p$ is presented and consider the point set $Y$ that is already presented at this time. Clearly, $Y \supset X$ and hence every decreasing subset in $X$ is also a decreasing subset in $Y$.

By assumption $p$ and $q$ are on a decreasing curve, that is, $p$ is presented either to the top-left of $q$ or to the bottom-right of $q$. In the first case the decreasing subset $R \subseteq X$ of size $R(q)$ can be extended by $p$ to a decreasing subset of $Y$ of size at least $R(q) + 1$ starting at $p$. In particular we have $R(p) \geq R(q) + 1$ and the colors of $p$ and $q$ are different. Similarly, if $p$ is presented to the bottom-right of $q$ then the decreasing subset $L \subseteq X$ of size $L(q)$ can be extended by $p$ to a decreasing subset of $Y$ ending at $p$. Hence $L(p) \geq L(q) + 1$ and the colors of $p$ and $q$ are not the same. $\square$

Using the notation from Definition 5.8 we can state Theorem 5.9 as

$$\chi_k^{\mathrm{ON}}(\mathcal{C}_k) \leq \binom{k+1}{2}.$$

Next, we show that this upper bound is indeed tight for every $k \geq 1$.

**Theorem 5.10.** *For every $k \geq 1$ there is no online $k$-good $c$-coloring of $\mathcal{C}_k$ with $c < \binom{k+1}{2}$. In particular with Theorem 5.9 we have*

$$\chi_k^{ON}(\mathcal{C}_k) = \binom{k+1}{2}.$$

*Proof.* For a fixed $k$ we shall provide an adversarial strategy $\mathcal{S}_k$ to present points in the plane with no $k+1$ on a decreasing curve such that whenever the points are colored online so that any decreasing subset of size $k$ contains $k$ different colors, then this uses at least $\binom{k+1}{2}$ colors. So we fix any online $k$-good $c$-coloring $\Phi$ of $\mathcal{C}_k$ and shall prove that $c \geq \binom{k+1}{2}$.

**Claim.** *Whenever two of the presented points lie on some decreasing curve then these points are colored differently by $\Phi$.*

*Proof of Claim.* If two such points $p$ and $q$ would get the same color, then we could simply present more points to complete some decreasing subset $Y$ of size $k$ containing $p$ and $q$. (Note that this is always possible without creating decreasing subsets of size more than $k$.) Then $Y$ would contain at most $k-1$ colors, contradicting the fact that $\Phi$ is $k$-good. $\square$

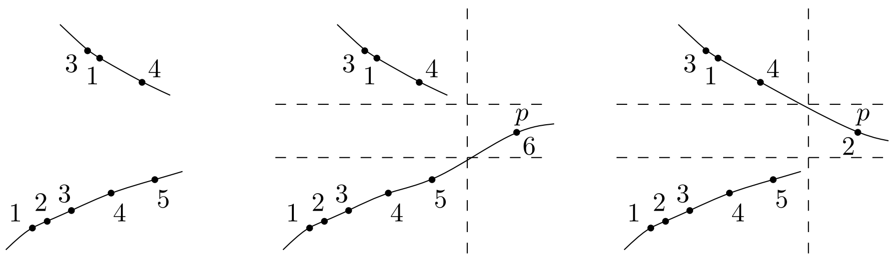
The adversarial strategy $\mathcal{S}_k$ consists of certain "building blocks“. In each such block we present points in such a way that they satisfy a certain set of properties, which we call $X(i)$ for some natural number $i$, no matter how these are colored with respect to $\Phi$. A point set $X$ is said to have property $X(i)$ for some $i \in \mathbb{N}$ if the following holds.

I) The $i$ lowest points $p_1, \ldots, p_i$ in $X$ (those with smallest $y$-coordinate) are all colored differently by $\Phi$ and can be captured by an *increasing* curve.

II) The points in $X \setminus \{p_1, \ldots, p_i\}$ can be captured by a *decreasing* curve. Moreover, each point in $X \setminus \{p_1, \ldots, p_i\}$ has the same color as some $p_j$, $j \in \{1, \ldots, i\}$.

Note that by the Claim above, all points in $X \setminus \{p_1, \ldots, p_i\}$ are colored differently and hence from II) it follows that $|X| \leq 2i$.

We show that for each $i \in \mathbb{N}$ we can present a point set that is forced to have property $X(i)$ by induction on $i$. For $i = 1$ it clearly suffices to present a single point. For $i \geq 2$ we first present a point set $X$ that has property $X(i-1)$ and denote the $i-1$ lowest points of $X$ by $p_1, \ldots, p_{i-1}$. Then we present the next point $p$ to the right of all points in $X$ and horizontally between $\{p_1, \ldots, p_{i-1}\}$ and $X \setminus \{p_1, \ldots, p_{i-1}\}$, i.e., $p$ is above $p_1, \ldots, p_{i-1}$ and below all points in $X \setminus \{p_1, \ldots, p_{i-1}\}$. See Figure 71 for an illustration.




$X$ satisfies property $X(i-1)$ | $X \cup p$ satisfies property $X(i)$ | $X \cup p$ satisfies property $X(i-1)$

Figure 71: Starting with a point set $X$ with property $X(i-1)$ (left) we produce a point set with property $X(i)$ (middle) or a larger point set with property $X(i-1)$ (right).

Now $p$ receives a color $c$ by $\Phi$. If $c$ is a new color, that is, a color that is not present in $X$ so far, then $X \cup p$ satisfies property $X(i)$ and we are done. See the middle of Figure 71. Otherwise, if some point in $X$ already has color $c$ then by II) it must be a color that appears among $p_1, \ldots, p_{i-1}$. In particular, $X \cup p$ has again property $X(i-1)$ as illustrated in the right of Figure 71. Since every point set with property $X(i-1)$ has at most $2(i-1)$ points, it follows that at latest after the $(2i-1)$-th point is presented we have a set that has property $X(i)$.

Finally, we use the building blocks $X(i)$, $i \in N$, to define the adversarial strategy $\mathcal{S}_k$ as follows. We refer to Figure 72 for an illustration.

- Start by presenting a set $X_k$ having property $X(k)$. Denote the set of the $k$ lowest points in $X_k$ by $L_k$.
- For $i \geq 1$, present a set $X_{k-i}$ having property $X(k-i)$ to the left of $X_{k-i+1}$, above $L_{k-i+1}$ and below $X_{k-i+1} \setminus L_{k-i+1}$. Denote the set of the $k-i$ lowest points in $X_{k-i}$ by $L_{k-i}$.

By I) we have that each $L_i$ contains exactly $i$ colors. And by the Claim above these colors are different for different $L_i$. Thus the total number of colors

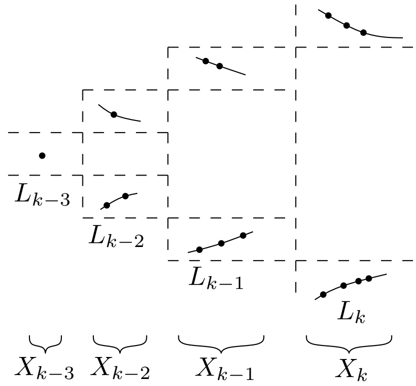



Figure 72: An example point set that could arise from the adversarial strategy $\mathcal{S}_k$ with $k = 4$.

in $X = X_k \cup X_{k-1} \cup \cdots \cup X_1$ is

$$\sum_{i=k}^{1} |L_i| = \sum_{i=k}^{1} i = \binom{k+1}{2}.$$

It remains to show that no $k+1$ points of $X$ lie on some decreasing curve. So let $Y$ be any decreasing subset of $X$. From the construction follows that if $Y \cap (X_i \setminus L_i) \neq \emptyset$ for some $i$ then $Y \cap X_j = \emptyset$ for all $j < i$ and $Y \cap (X_j \setminus L_j) = \emptyset$ for all $j > i$. Moreover, by I) we have that $|Y \cap L_j| \leq 1$ for all $j = 1, \ldots, k$. Doing the counting we get

$$|Y| = |Y \cap (X_i \setminus L_i)| + \sum_{j=i}^{k} |Y \cap L_j| \leq (i-1) + (k-i+1) = k,$$

which concludes the proof. $\square$

Next we introduce another type of ranges, which will be the subject of our investigations for the remainder of this chapter: *bottomless rectangles*.

A bottomless rectangle is an axis-aligned rectangle in the plane whose bottom edge is at $-\infty$. Given any finite point set $X$ and a natural number $m$ we are interested in the subsets $Y$ of $X$ captured by some $m$-big bottomless rectangle. As usual, the family of those subsets is denoted by $\mathcal{F}_m$. See Figure 73(a) for an illustrating example.

Let us consider another class of hypergraph which is denoted by $\mathcal{I}_m$ for some natural number $m$ and defined as follows. A hypergraph $H = (X, \mathcal{F})$ is in $\mathcal{I}_m$ ($\mathcal{I}$ stands for interval) if every hyperedge contains exactly $m$ vertices and for each vertex $v$ in $X$ there exist two real numbers $x_v$ and $t_v$ such that the following is true.

$$Y \in \mathcal{F} \iff \exists \ell, r, t \in \mathbb{R} \text{ such that } Y = \{v \in X \mid t_v \leq t \text{ and } \ell \leq x_v \leq r\}.$$

The numbers are interpreted as follows. Every vertex $v$ corresponds to a point on the real line together with an *insertion time* $t_v$, i.e., $v$ is an ordered

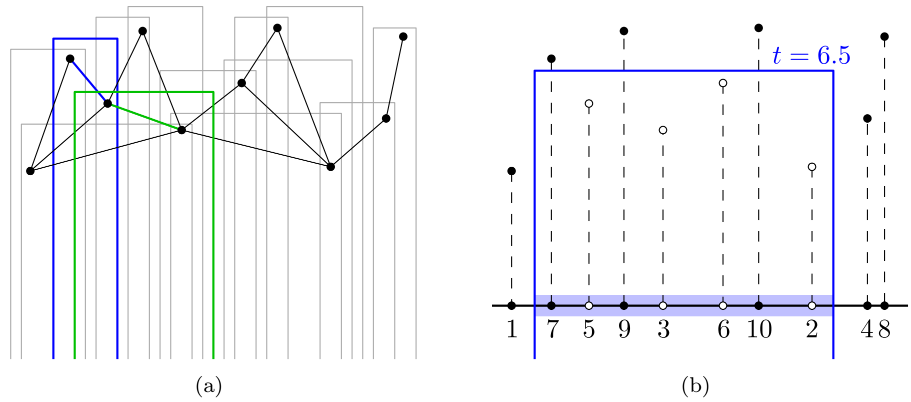



Figure 73: (a) A finite point set $X$ on 10 points in the plane, the (hyper)graph $H = (X, \mathcal{F}_2)$ and one 2-big bottomless rectangle for each (hyper)edge in $H$. (b) The point set $X$ seen as points on the real line with insertion times and a bottomless rectangle seen as an interval with a time $t$. The points captured by the bottomless rectangles and interval at time $t$ are highlighted, respectively.

tuple of two real numbers $x_v$ and $t_v$. Time starts at $-\infty$ and increases steadily. Then the points show up on the initially empty real line one-by-one. Indeed, the point for $v$ shows up at its insertion time $t_v$ and never disappears again. If at some time $t$ an interval $I = [\ell, r] \subset \mathbb{R}$ contains the subset $Y$ of points then we say that interval $I$ captures the points in $Y$. Here a point can be captured only if it already appeared by time $t$, that is, if its insertion time is at most $t$. See Figure 73(b) for an example.

**Lemma 5.2.** *For every $m \in \mathbb{N}$ every hypergraph $H = (X, \mathcal{F}_m)$ with respect to $m$-big bottomless rectangles is a hypergraph in $\mathcal{I}_m$ and vice versa.*

*In particular, for all natural numbers $m$ and $k$ and every hypergraph $H = (X, \mathcal{F}_m)$ with respect to $m$-big bottomless rectangles we have*

$$\chi_k(H) \leq \chi_k^{ON}(\mathcal{I}_m).$$

*Proof.* A bijection $\sigma$ between the model with points in the plane and bottomless rectangles and the model with points on the real line with insertion times and intervals with times is given as follows.

| | | |
|---|---|---|
| point $(x, t) \in \mathbb{R}^2$ | $\xrightarrow{\sigma}$ | point $x \in \mathbb{R}$ with insertion time $t$ |
| bottomless rectangle $R = [\ell, r] \times (-\infty, t]$ | $\xrightarrow{\sigma}$ | interval $I = [\ell, r]$ with time $t$ |

It is straightforward to check that a set $Y$ of points in the plane can be captured by some bottomless rectangle if and only if the corresponding interval $I$ at time $t$ captures the points on the real line that correspond to $Y$. We refer to Figure 73(b) for an illustration.

Hence we can transfer the offline setting of points in the plane with respect to bottomless rectangles to the online setting of points on a line with respect to intervals. Now if we want to color the points in $X$ in the plane, we can instead *online* color the corresponding points that appear on the real line. This can be seen as sweeping the point set $X$ with a horizontal line starting at $-\infty$ and moving up, and coloring every point of $X$ when it is reached by the sweepline *without considering the points above the sweepline.* In particular, when we decide which color we want to give the point $p \in X$ then this depends only on the points below $p$ and the way we already colored these points and is *independent* of how the points above $p$ are arranged.

Hence coloring $\mathcal{I}_m$ online can only be more difficult then coloring the point set $X$ with respect to $m$-big bottomless rectangles offline, which establishes the claimed inequality. $\square$

We want to use the inequality in Lemma 5.2 to get an upper bound on the $k$-chromatic number of any point set $X$ in the plane with respect to $m$-big bottomless rectangles. Let us first consider the case $m = k$, just like we did for ranges that are decreasing curves in Theorem 5.9.

**Theorem 5.11.** *For all $k \in \mathbb{N}$ we have*

$$\chi_k^{ON}(\mathcal{I}_k) \leq 2k - 1.$$

*Proof.* At the arrival of a new point $p$ denote by $(\ell_1, \ldots, \ell_{k-1})$ and $(r_1, \ldots, r_{k-1})$ the $k-1$ points to its left and to its right, respectively. Together these $2k-2$ points have at most $2k-2$ colors. Thus, there is at least one of the $2k-1$ colors unused among these points. It is easy to see that if we color $p$ with that color, then all intervals capturing $k$ points contain $k$ different colors. $\square$

With the second part of Lemma 5.2 we can immediately draw the following corollary.

**Corollary 5.2.** *For every $k \in \mathbb{N}$ and every hypergraph $H = (X, \mathcal{F}_k)$ with respect to $k$-big bottomless rectangles we have*

$$\chi_k(H) \leq 2k - 1.$$

Perhaps surprisingly, the easy-to-prove bound in Theorem 5.11 is best possible. Even stronger, the bound in Corollary 5.2 is best possible.

**Theorem 5.12.** *For every $k \in \mathbb{N}$ there is a point set $X_k$ in the plane such that $\chi_k((X, \mathcal{F}_k)) = 2k-1$. In particular with Lemma 5.2 and Theorem 5.11 we have*

$$\max\{\chi_k((X, \mathcal{F}_k)) \mid X \textit{ point set in the plane}\} = \chi_k^{ON}(\mathcal{I}_k) = 2k - 1.$$

*Proof.* We consider the point set $X_k$ consisting of $k$ points of the form $\{(i, 2i) \mid 0 \leq i \leq k-1\}$ and $k-1$ points of the form $\{(2k-i, 2i-1) \mid 1 \leq i \leq k-1\}$. See Figure 74 for an illustration.

It is easy to see that every pair of points in such a point set is in some $k$-big bottomless rectangle. Thus if every $k$-big bottomless contains $k$ different colors, then no two points of $X_k$ can be colored the same, which gives $\chi_k(X_k) = |X_k| = 2k-1$. $\square$

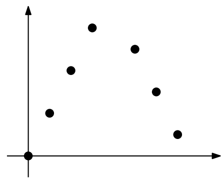


Figure 74: A point set $X_k$ with $\chi_k(X_k) = |X_k| = 2k - 1$ for $k = 4$.

We have seen that if we want to color points in the plane with respect to bottomless rectangles, then it can be very useful to simply sweep the points bottom to top and color points as they appear without considering the points above the sweepline. In particular, Theorem 5.12 states that we can find a $k$-good $c$-coloring with respect to $m$-big bottomless rectangles and small $c$ this way, provided $m = k$. However, for different values of $m$ the situation can change drastically. For example if we want to bound

$$m(k) = \min\{m \in \mathbb{N} \mid \forall X \subset \mathbb{R}^2 : \chi_k((X, \mathcal{F}_m)) = k\}.$$

For convenience let us define similarly to $m(k)$ the following quantity.

$$m^{\mathrm{ON}}(k) \stackrel{\mathrm{def}}{=} \min\{m \in \mathbb{N} \mid \chi_k^{\mathrm{ON}}(\mathcal{I}_m) = k\}.$$

Clearly, we have by Lemma 5.2 for all natural number $k$ that

$$m(k) \leq m^{\mathrm{ON}}(k). \tag{13}$$

However, inequality (13) turns out to be a *very* poor estimate.

**Lemma 5.3.** *For every $m \geq 2$ we have $\chi_2^{ON}(\mathcal{I}_m) \geq 3$. In particular,*

$$m^{ON}(k) = \infty \quad \textit{for every } k \in \mathbb{N}.$$

*Proof.* We fix $m \in \mathbb{N}$ and shall show that there is no online 2-good 2-coloring $\Phi$ of $\mathcal{I}_m$. That is, we shall describe an adversarial strategy to present points on the real line so that no matter how every point is colored by $\Phi$ (either red or blue) we can force a set of $m$ consecutive points that are all colored the same.

Indeed, the strategy is easy: Always present the next point to the right of all red points and to the left of all blue points. It is easily seen that as soon as $2m-1$ points are presented the $m$ leftmost points are all red or the $m$ rightmost points are all blue. □

We will see next that $m(k) < \infty$ for every $k \in \mathbb{N}$, which makes inequality (13) basically useless. However, we remark that for a different type of range (other than bottomless rectangles) the analogous inequality could be very valueable. To prove that $m(k) < \infty$ we consider a new variant of the online $(c, k)$-coloring problem, which is actually just a slight weakening of the properties **(O1)**–**(O4)** that have to be fulfilled by every online $k$-good $c$-coloring $\Phi$.

## 5.4 Semi-Online Coloring Problems

Recall that loosely speaking every online $k$-good $c$-coloring has to satisfy the following for every hypergraph that is presented vertex by vertex.

**(O1)** Color every vertex immediately when it appears.

**(O2)** Use at most $c$ colors.

**(O3)** Never recolor any vertex.

**(O4)** Ensure that every hyperedge contains vertices of at least $k$ different colors.

A *semi-online $k$-good $c$-coloring* also satisfies **(O2)**–**(O4)** but it does not have to satisfy **(O1)**. This means that a vertex can be left uncolored when it appears. For completeness we then have to add the possibility of coloring vertices that were left uncolored on their arrival.

**(O5)** At any time any uncolored vertex can be assigned any color.

More formally we define a *partial* $k$-good $c$-coloring of a hypergraph $H = (X, \mathcal{F})$ to be coloring of a subset of vertices of $H$ with at most $c$ colors such that every hyperedge $Y \in \mathcal{F}$ contains vertices of at least $k$ distinct colors. Then the semi-online $(c, k)$-coloring problem is the same as the online $(c, k)$-coloring problem where we simply replace every appearence of "coloring" by "partial coloring".

**Definition 5.9** (Semi-online $(c, k)$-coloring problem).
*Let $\mathcal{C}$ be a class of hypergraphs. A* semi-online $k$-good $c$-coloring *of $\mathcal{C}$ is a class $\Phi$ of exactly one partial $k$-good $c$-coloring $\phi$ for each $H \in \mathcal{C}$ and each ordering of the vertices in $H$, such that the following is true.*

- *Whenever the vertex orderings of two hypergraph $H, H' \in \mathcal{C}$ coincide in the first $n$ positions for some $n$, then the corresponding partial colorings $\phi$ and $\phi'$ coincide on these first $n$ vertices, too.*

*The* semi-online $k$-chromatic number of $\mathcal{C}$ *is then defined as*

$$\chi_k^{S\text{-}ON}(\mathcal{C}) \stackrel{def}{=} \min\{c \in \mathbb{N} \mid \text{ there exists a semi-online } k\text{-good } c\text{-coloring of } \mathcal{C}\}.$$

We get the following relations between the three versions of the $(c, k)$-coloring problem that we considered here: offline, online and semi-online.

**Lemma 5.4.** *For every class $\mathcal{C}$ of hypergraphs and every natural number $k$ we have*

$$\max\{\chi_k(H) \mid H \in \mathcal{C}\} \leq \chi_k^{S\text{-}ON}(\mathcal{C}) \leq \chi_k^{ON}(\mathcal{C}).$$

*Proof.* The first inequality holds since presenting a hypergraph $H \in \mathcal{C}$ with one vertex at a time cannot make it easier to find a $k$-good $c$-coloring of $H$. The second inequality follows as every online $k$-good $c$-coloring of $\mathcal{C}$ is also a semi-online $k$-good $c$-coloring of $\mathcal{C}$. $\square$

Let us come back to the class of hypergraphs induced by finite point sets in the plane and $m$-big bottomless rectangles and the class $\mathcal{I}_m$ induced by points appearing on a line and $m$-big intervals at any time. Similarly to $m(k)$ and $m^{\mathrm{ON}}(k)$ we can now define

$$m^{\text{S-ON}}(k) \stackrel{\text{def}}{=} \min\{m \in \mathbb{N} \mid \chi_k^{\text{S-ON}}(\mathcal{I}_m) = k\}$$

and obtain as a consequence of Lemma 5.2 and Lemma 5.4 that for every $k \in \mathbb{N}$ we have

$$m(k) \le m^{\text{S-ON}}(k) \le m^{\text{ON}}(k). \tag{14}$$

By Lemma 5.3 we have that $m^{\text{ON}}(k) = \infty$ for every $k$. However, in the semi-online setting we can do much better.

**Theorem 5.13.** *For every $k \ge 2$ we have*

$$m^{S\text{-}ON}(k) \le 3k - 2.$$

*Proof.* Points appear on the real line and we need to color them with $k$ colors in a semi-online way so that at all times every occurrence $3k-2$ consecutive points contain at least one point from each of the $k$ colors.

We define a *gap for color $i$* as a maximal set of consecutive points containing no point of color $i$, that is, either between two successive occurrences of color $i$, or before the first occurrence (first gap), or after the last occurrence (last gap), or all the points if no point has color $i$. A *gap* is simply a gap for color $i$, for some $1 \le i \le k$. Observe that $k$-goodness with respect to $(3k-2)$-big intervals is equivalent to the fact that at all times all gaps have size at most $3k-3$.

We propose a semi-online $k$-coloring that leaves the first $k-1$ points uncolored and from then on maintains the following invariant:

- The size of every gap is between $k-1$ points (included) and $3k-3$ points (included).

This invariant is vacuous when there are only $k-1$ (uncolored) points. Now, suppose that the invariants hold for an intermediate set of points and consider a new point that appears on the line. Clearly, the lower bound on the size of gaps cannot be violated in the extended set as no gaps decrease in size. However, there may arise some gaps of size $3k-2$ violating the upper bound. If not then the invariant holds for the extended set and the semi-online coloring does not color any point in this step.

Suppose there are some gaps of size $3k-2$, consider one of them, say a gap of color $i$, and denote the points in the gap in their natural ordering on the line from left to right as $(\ell_1, \ldots, \ell_{k-1}, m_1, \ldots, m_k, r_1, \ldots r_{k-1})$. Now, color $i$ does not appear among these points. The lower bound in our invariant yields that none of the $k-1$ remaining colors (those different from $i$) appears twice among $m_1, \ldots, m_k$. Thus, there is some $m_j$, which is uncolored and we color it with color $i$. This splits the large gap for color $i$ into two smaller gaps. Moreover, since there are $k-1$ $\ell$-points and $k-1$ $r$-points these two smaller gaps have size at least $k-1$, i.e., the lower bound in the invariant is maintained for all gaps. We refer to Figure 75 for an illustration.

The algorithm repeats that process until all gaps are of size at most $3k-3$. Note that the treatment of one such gap requires only the validity of the lower bound in the invariant and thus these gaps can be treated one after another. $\square$

With the first inequality in (14) we conclude the following.

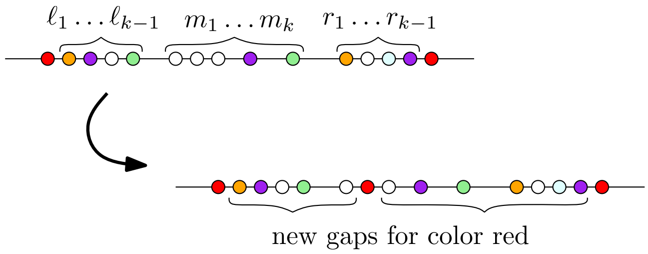



Figure 75: A gap for color red of size $3k-2$ that is split into two gaps of size at least $k-1$ each. Here $k=5$.

**Corollary 5.3.** *For every $k \in \mathbb{N}$ every finite set of points in the plane can be colored with $k$ colors such that every $m$-big bottomless rectangle contains all $k$ colors, provided $m \geq 3k-2$. In other words,*

$$m(k) \leq 3k-2.$$

Perhaps surprisingly, Corollary 5.2 is the best bound on $m(k)$ that we know of. We do not even know whether there is a number $k$ where $m(k)$ and $m^{\text{S-ON}}(k)$ disagree. Anyways, we can conclude that the first inequality in (14) is much better than the second. In especially, consider semi-online colorings instead of online colorings allowed us to prove much better results for the offline setting. Next we will show that it is not always the case that semi-online considerations give good results at all.

To this end, recall that in the dual $(c,k)$-coloring problem with respect to bottomless rectangles we are given a finite set $X^*$ of bottomless rectangles and want to color them with at most $c$ colors such that whenever $m$ bottomless rectangles are pierced by some point of the plane among these $m$ rectangles we see at least $k$ different colors. Moreover, in case $c=k$, i.e., for $k$-good $k$-colorings, we have defined

$$m^*(k) = \min\{m \in \mathbb{N} \mid \max_{X^* \text{ set of bottomless rectangles}} \chi_k((X^*, \mathcal{F}^*_m)) = k\}.$$

Again, we can try to prove that $m(k) < \infty$ by considering an appropiate on-line or semi-online coloring problem. Just like in the primal setting we consider a horizontal sweep line, but this time we start *above* all rectangles and move the sweepline *down*. When we hit a bottomless rectangles this corresponds to an interval appearing on the real line. And a set $Y$ of bottomless rectangles is pierced by some point in the plane if and only if there is a time and a point on the real line that is contained in exactly the intervals corresponding to the rectangles in $Y$.

Hence we define $\mathcal{I}^*_m$ to be the class of hypergraphs that admit the following representation. Every vertex in a hypergraph $H \in \mathcal{I}^*_m$ corresponds to an interval on the real line together with a number, its *insertion time*. And an $m$-set $Y$ of intervals form a hyperedge if and only if there is a point $p \in \mathbb{R}$ and a number $t$ such that $Y$ is exactly the set of intervals that contain $p$ and have insertion time at most $t$.

If we define

$$m^{*\mathrm{ON}}(k) = \min\{n \in \mathbb{N} \mid \chi_k^{\mathrm{ON}}(\mathcal{I}_m^*) = k\},$$
$$m^{*\mathrm{S\text{-}ON}}(k) = \min\{n \in \mathbb{N} \mid \chi_k^{\mathrm{S\text{-}ON}}(\mathcal{I}_m^*) = k\},$$

then analogously to (14) we obtain the following chain of inequalities for every natural number $k$.

$$m^*(k) \leq m^{*\mathrm{S\text{-}ON}}(k) \leq m^{*\mathrm{ON}}(k) \tag{15}$$

It can be shown that $m^*(k) \in \mathcal{O}(k^6)$ for every $k \in \mathbb{N}$. However, both inequalities in (15) are useless because of the next theorem.

**Theorem 5.14.** *For every $m \in \mathbb{N}$ we have*

$$\chi_2^{S\text{-}ON}(\mathcal{I}_m^*) = \infty.$$

*In particular, $m^{*S\text{-}ON}(2) = \infty$.*

*Proof.* For all natural numbers $m$ and $c$ we shall provide an adversarial strategy to present intervals on the real line such that these intervals cannot be colored in semi-online fashion with at most $c$ colors so that every $m$-deep point is contained in intervals of two distinct colors. Here a point of the real line is $m$-deep if it is contained in exactly $m$ so far presented intervals.

We fix $c$ the number of colors. We shall define for every $m$ and $n$ an adversarial strategy $\mathcal{S}(m,n)$ for presenting intervals such that the following is true:

(i) Every semi-online 2-good $c$-coloring $\Phi$ of $\mathcal{I}_m^*$ colors the intervals in $\mathcal{S}(m,n)$ such that there are $c$ points $p_1, \ldots, p_c$ and for $i = 1, \ldots, c$ point $p_i$ is covered by exactly $t_i$ intervals, all of which have color $i$, and

(ii) $t_1 + \ldots + t_c \geq n$.

Clearly, if for some semi-online $c$-coloring $\Phi$ there is eventually a point that is at least $m$-deep and all intervals containing it have the same color, then $\Phi$ is not 2-good with respect to $m$-deep points. Thus if $S(m, cm)$ exists and satisfies (i) and (ii), then there is no semi-online 2-good $c$-coloring of $\mathcal{I}_m^*$, which proves the theorem.

We prove the existence of $\mathcal{S}(m,n)$ by double-induction on $m$ and $n$. Strategies $\mathcal{S}(m,0)$ are vacuous as (i) and (ii) for $n = 0$ hold for the empty set of intervals and any set of $c$ distinct points $p_1, \ldots, p_c$. We define $\mathcal{S}(m,n)$, for $n > 0$, once we have defined $\mathcal{S}(m-1, c(m-1))$ and $\mathcal{S}(m, n-1)$.

Before continuing let us present the following useful claim.

**Claim.** *Given a set $Y$ of intervals already presented and any $I' \subset I \in Y$ with $I' \cap J = \emptyset$ for all $J \in Y \setminus I$. If $\mathcal{S}(m-1, c(m-1))$ exists we can present the intervals of $\mathcal{S}(m-1, c(m-1))$ inside $I'$ forcing any semi-online 2-good $c$-coloring of $\mathcal{I}_m^*$ to color $I$.*

*Proof.* We present the intervals for $\mathcal{S}(m-1, c(m-1))$ completely inside $I'$. If the coloring does not color $I$ then it can be seen as a 2-good $c$-coloring with respect to $(m-1)$-deep points executed against $\mathcal{S}(m-1, c(m-1))$. We already know that no such coloring exists and therefore every 2-good $c$-coloring of $\mathcal{I}_m^*$ has to color interval $I$. $\square$

Now, we are ready to define $S(m,n)$ for $n > 0$. First present two families of intervals, both realizing strategy $S(m,n-1)$, disjointly next to each other. By (i) there exist two sets of $c$ points each, $p_1, p_2, \ldots, p_c$ and $p'_1, \ldots, p'_c$, and non-negative integers $t_1, \ldots, t_c, t'_1, \ldots, t'_c$ such that $p_i$ is $t_i$-covered and all its intervals are colored with $i$ and also $p'_i$ is $t'_i$-covered and all its intervals are colored with $i$, for every $i = 1, \ldots, c$. Moreover, by (ii) we have $t_1 + \ldots + t_c \geq n$ and $t'_1 + \ldots + t'_c \geq n$.

If there exists some $i \in \{1, \ldots, c\}$ with $t_i \neq t'_i$ then the sequence of maxima $m_i = \max(t_i, t'_i)$ satisfies $m_1 + \ldots + m_c \geq n + 1$. Thus, taking for each $i \in \{1, \ldots, c\}$ the point from $\{p_i, p'_i\}$ that corresponds to the larger value of $t_i, t'_i$, we obtain a set of $c$ points satisfying (i) and (ii).

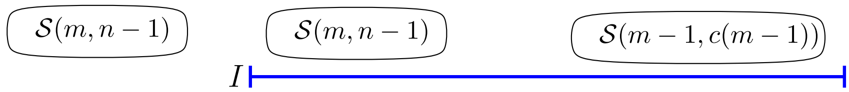



Figure 76: Defining strategy $\mathcal{S}(m,n)$ once $\mathcal{S}(m-1,c(m-1))$ and $\mathcal{S}(m,n-1)$ are defined, in the case $t_i = t'_i$ for all $i \in \{1, \ldots, c\}$.

Hence we assume without loss of generality that $t_i = t'_i$ for all $i \in \{1, \ldots, c\}$. Then we present one additional interval $I$ that contains all the points $p'_1, \ldots, p'_c$ but none of the points $p_1, \ldots, p_c$. Moreover, $I$ is chosen big enough so that there exists some $I' \subset I$ that is disjoint from all the other intervals presented so far. We present the intervals realizing strategy $S(m-1, c(m-1))$ inside $I'$, forcing $I$ to be colored, see Figure 76. Let $j$ be the color of $I$. Then $p'_j$ is now contained in exactly $t'_j + 1$ intervals all of which are colored with $j$. Thus $(\{p_1, \ldots, p_c\} \setminus p_j) \cup p'_j$ is a set of $c$ points satisfying (i) and (ii), which concludes the proof. $\square$

## References


[ABDF$^+$11] Roel Apfelbaum, Itay Ben-Dan, Stefan Felsner, Tillmann Miltzow, Rom Pinchasi, Torsten Ueckerdt, and Ran Ziv. Points with large quadrant depth. *Journal of Computational Geometry*, 2(1):128–143, 2011.

[ACC$^+$11] Greg Aloupis, Jean Cardinal, Sébastien Collette, Shinji Imahori, Matias Korman, Stefan Langerman, Oded Schwartz, Shakhar Smorodinsky, and Perouz Taslakian. Colorful strips. *Graphs and combinatorics*, 27(3):327–339, 2011.

[ACNS82] Miklós Ajtai, Vašek Chvátal, Monroe M Newborn, and Endre Szemerédi. Crossing-free subgraphs. *North-Holland Mathematics Studies*, 60:9–12, 1982.

[Bar60] Emile Barbier. Note sur le probleme de l'aiguille et le jeu du joint couvert. *J. math. pures appl*, 5(273-286):96–102, 1860.

[BDP05] Sergey Bereg, Adrian Dumitrescu, and János Pach. Sliding disks in the plane. In Jin Akiyama, Mikio Kano, and Xuehou Tan, editors, *Discrete and Computational Geometry*, volume 3742 of *Lecture Notes in Computer Science*, pages 37–47. Springer Berlin Heidelberg, 2005. ISBN 978-3-540-30467-8. URL `http://dx.doi.org/10.1007/11589440_4`.

[BDPZ10] Itay Ben-Dan, Rom Pinchasi, and Ran Ziv. On a problem about quadrant-depth. *Computational Geometry*, 43(6):587–592, 2010.

[Bla15] Wilhelm Blaschke. Konvexe Bereiche gegebener konstanter Breite und kleinsten Inhalts. *Mathematische Annalen*, 76(4):504–513, 1915.

[BLP07] Hervé Brönnimann, Jonathan Lenchner, and János Pach. Opposite-quadrant depth in the plane. *Graphs and Combinatorics*, 23 (1):145–152, 2007.

[Chu36] Alonzo Church. An unsolvable problem of elementary number theory. *American journal of mathematics*, 58(2):345–363, 1936.

[CS93] Judit Csima and ET Sawyer. There exist 6n/13 ordinary points. *Discrete & Computational Geometry*, 9(1):187–202, 1993.

[CS95] J Csima and ET Sawyer. The 6n/13 theorem revisited. *Graph Theory, Cobinatorics, and Applications*, 1:235–249, 1995.

[dBE51] Nicolaas Govert de Bruijn and Paul Erdős. A colour problem for infinite graphs and a problem in the theory of relations. *Nederl. Akad. Wetensch. Proc. Ser. A*, 54:371–373, 1951.

[Dir51] Gabriel A Dirac. Collinearity properties of sets of points. *The Quarterly Journal of Mathematics*, 2(1):221–227, 1951.

[EG73] Paul Erdős and Richard K Guy. Crossing number problems. *The American Mathematical Monthly*, 80(1):52–58, 1973.

[ER56] Paul Erdős and Richard Rado. A partition calculus in set theory. *Bulletin of the American Mathematical Society*, 62(5):427–489, 1956.

[Erd46] Paul Erdős. On sets of distances of n points. *The American Mathematical Monthly*, 53(5):248–250, 1946.

[Erd78] Paul Erdős. Some applications of graph theory and combinatorial methods to number theory and geometry. *Algebraic Methods in Graph Theory, vols. I, II, Szeged*, pages 137–148, 1978.

[ES35] Paul Erdős and George Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935.

[ES60] Paul Erdős and George Szekeres. On some extremum problems in elementary geometry. In *Annales Univ. Sci. Budapest*, pages 3–4, 1960.

[Fár48] István Fáry. On straight-line representation of planar graphs. *Acta Univ. Szeged. Sect. Sci. Math.*, 11:229–233, 1948.

[Gal44] Tibor Gallai. Solution to problem number 4065. *American Mathematical Monthly*, 51:169–171, 1944.

[Ger08] Tobias Gerken. Empty convex hexagons in planar point sets. *Discrete & Computational Geometry*, 39(1-3):239–272, 2008.

[GK10] Larry Guth and Nets Hawk Katz. On the Erdős distinct distance problem in the plane. *ArXiv e-prints*, November 2010. URL `http://arxiv.org/abs/1011.4105`.

[GL88] András Gyárfás and Jenö Lehel. On-line and first fit colorings of graphs. *Journal of Graph theory*, 12(2):217–227, 1988.

[GRS80] Ronald L Graham, Bruce L Rothschild, and Joel H Spencer. *Ramsey theory*, volume 2. Wiley New York, 1980.

[GT13] Ben Green and Terence Tao. On sets defining few ordinary lines. *ArXiv e-prints*, March 2013. URL `http://arxiv.org/abs/1208.4714`.

[HJ63] Alfred W Hales and Robert I Jewett. Regularity and positional games. *Transactions of the American Mathematical Society*, 106 (2):222–229, 1963.

[Hor83] Joseph D Horton. Sets with no empty convex 7-gons. *Canadian Mathematical Bulletin*, 26(4):482, 1983.

[Kaw09] Bernd Kawohl. Convex sets of constant width. page 4, 2009. URL `http://www.pims.math.ca/files/Kawohl-Lecture3-GA09.pdf`.

[KM58] Leroy M Kelly and William OJ Moser. On the number of ordinary lines determined by n points. *Canad. J. Math*, 10:210–219, 1958.

[KMS84] Daniel Kornhauser, Gary Miller, and Paul Spirakis. Coordinating pebble motion on graphs, the diameter of permutation groups, and applications. In *25th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 241–250. IEEE, 1984.

[Kos09] Vitalii Anatol'evich Koshelev. On Erdős-Szekeres problem for empty hexagons in the plane. *Modelirovanie i Analiz Informatsionnykh Sistem [Modeling and Analysis of Information Systems]*, 16(2):22–74, 2009.

[KT04] Nets Hawk Katz and Gábor Tardos. A new entropy inequality for the Erdős distance problem. *Contemporary Mathematics*, 342: 119–126, 2004.

[Leb14] Henri Lebesgue. Sur le problème des isopérimètres et sur les domaines de largeur constante. *Bull. Soc. Math. France, CR*, 42: 72–76, 1914.

[Lei84] Frank Thomson Leighton. New lower bound techniques for VLSI. *Mathematical Systems Theory*, 17(1):47–70, 1984.

[MLP87] P. Mani-Levitska and János Pach. Decomposition problems for multiple coverings with unit balls. 1987.

[Mot67] Theodore S Motzkin. Nonmixed connecting lines. *Notices of the American Mathematical Society*, 14:837, 1967.

[Nic07] Carlos M Nicolás. The empty hexagon theorem. *Discrete & Computational Geometry*, 38(2):389–397, 2007.

[Ove02] Mark Overmars. Finding sets of points without empty convex 6-gons. *Discrete & Computational Geometry*, 29(1):153–158, 2002.

[Pac86] János Pach. Covering the plane with convex polygons. *Discrete & Computational Geometry*, 1(1):73–81, 1986.

[PRTT06] János Pach, Rados Radoicic, Gábor Tardos, and Géza Tóth. Improving the crossing lemma by finding more crossings in sparse graphs. *Discrete & Computational Geometry*, 36(4):527–552, 2006.

[PT97] János Pach and Géza Tóth. Graphs drawn with few crossings per edge. *Combinatorica*, 17(3):427–439, 1997.

[PT09] János Pach and Géza Tóth. Decomposition of multiple coverings into many parts. *Comput. Geom. Theory Appl.*, 42(2):127–133, 2009.

[PTT07] János Pach, Gábor Tardos, and Géza Tóth. Indecomposable coverings. In *Discrete Geometry, Combinatorics and Graph Theory*, pages 135–148. Springer, 2007.

[Rad45] Richard Rado. Note on combinatorial analysis. *Proceedings of the London Mathematical Society*, 2(1):122–160, 1945.

[Rad46] Richard Rado. A theorem on general measure. *Journal of the London Mathematical Society*, 1(4):291–300, 1946.

[Ram30] Frank Plumpton Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, 2(1):264–286, 1930.

[ST01] József Solymosi and Csaba D Tóth. Distinct distances in the plane. *Discrete & Computational Geometry*, 25(4):629–634, 2001.

[STJ83] Endre Szemerédi and William T Trotter Jr. Extremal problems in discrete geometry. *Combinatorica*, 3(3-4):381–392, 1983.

[Szé97] László A Székely. Crossing numbers and hard Erdős problems in discrete geometry. *Combinatorics, Probability, and Computing*, 6: 353–358, 1997.

[Tar03] Gábor Tardos. On distinct sums and distinct distances. *Advances in Mathematics*, 180(1):275–289, 2003.

[TV93] Helge Tverberg and Siniša Vrecica. On generalizations of Radon's theorem and the ham sandwich theorem. *European Journal of Combinatorics*, 14(259-264):9, 1993.

[TV98] Géza Tóth and Pavel Valtr. Note on the Erdős-Szekeres theorem. *Discrete & Computational Geometry*, 19(3):457–459, 1998.

[Tve66] Helge Tverberg. A generalization of Radon's theorem. *Journal of the London Mathematical Society*, 41(1):123–128, 1966.

[Tve81] Helge Tverberg. A generalization of Radon's theorem II. *Bull. Austral. Math. Soc*, 24(3):321–325, 1981.

[Val92] Pavel Valtr. Convex independent sets and 7-holes in restricted planar point sets. *Discrete & Computational Geometry*, 7(1):135–152, 1992.

[vdW27] Bartel Leendert van der Waerden. Beweis einer baudetschen vermutung. *Nieuw Arch. Wisk*, 15(2):212–216, 1927.

[Viz64] Vadim G Vizing. On an estimate of the chromatic class of a p-graph. *Diskretnyĭ Analiz*, 3(7):25–30, 1964.

[Wit52] Ernst Witt. Ein kombinatorischer satz der elementargeometrie. *Mathematische Nachrichten*, 6(5):261–262, 1952.