# Advanced Algorithm Homework 4

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**Problem 1.**

*proof:*

1.

$$\mathrm{E}[Z_i] = 1 - 2p$$

$$\mathrm{E}[S_i | Z_1, Z_2, \ldots, Z_{i-1}] = \mathrm{E}[S_{i-1} + Z_i + 2p - 1 | Z_1, Z_2, \ldots, Z_{i-1}] = S_{i-1}$$

2.

$$\mathrm{E}\left[\left(\frac{p}{1-p}\right)^{Z_t}\right] = p(\frac{1-p}{p}) + (1-p)(\frac{p}{1-p}) = 1$$

$$\mathrm{E}[P_t | Z_1, Z_2, \ldots, Z_{t-1}] = \mathrm{E}\left[P_{t-1}\left(\frac{p}{1-p}\right)^{Z_t} \mid Z_1, Z_2, \ldots, Z_{t-1}\right] = P_{t-1} \cdot \mathrm{E}\left[\left(\frac{p}{1-p}\right)^{Z_t}\right] = P_{t-1}$$

3. Note that $S_t = X_t + (2p-1)t$ and $P_t$ is a martingale, let $p_a = \Pr[X_\tau = a]$, than

$$\mathrm{E}[S_\tau] = p_a \mathrm{E}[-a + (2p-1)\tau] + (1 - p_a)\mathrm{E}[b + (2p-1)\tau] = 0$$

$$\mathrm{E}[P_\tau] = p_a \left(\frac{p}{1-p}\right)^{-a} + (1 - p_a)\left(\frac{p}{1-p}\right)^{b} = 1$$

Hence,

$$p_a = \frac{1 - (\frac{p}{1-p})^b}{(\frac{1-p}{p})^a - (\frac{p}{1-p})^b}$$

$$\mathrm{E}[\tau] = \frac{ap_a - b(1 - p_a)}{2p - 1}$$

In special, when $p = \frac{1}{2}$, we know $\mathrm{E}[\tau] = ab$ in class.

□

**Problem 2.**

*proof:*

1. Note that we can calculate the expectation of longest common sub-sequence when $n = 2, 3$ by enumerating. We get that $\mathrm{E}_2[X] = \frac{9}{8}$, and $\mathrm{E}_3[X] = \frac{29}{16}$.

   When $n$ is even, we can split two strings to $\frac{n}{2}$ segments with length of 2, then calculate the sum of LCS of corresponding segments as lowerbound. Hence, $\mathrm{E}[X] \geq \frac{9}{16}n$.

   When $n$ is odd, we split teo strings to $\frac{n-1}{2}$ segments, first string's length is 3, other string's length is 2. Then we do the same get that $\mathrm{E}[X] \geq \frac{n-3}{2}\frac{9}{8} + \frac{29}{16}$. Hence, we can let $c_1 = \frac{9}{16}$.

   Now we compute the upper bound, let $t = \lambda n$,

$$\begin{aligned}
\mathrm{E}[X] &\leq n\Pr[X \geq t] + t\Pr[X < t] \\
&= t + (n-t)\Pr[X \geq t] \\
&\leq t + (n-t)\frac{\binom{n}{t}^2 2^{2(n-t)}}{2^{2n}} \\
&= t + (n-t)2^{-t}\binom{n}{t}^2
\end{aligned}$$

   then we using Stirling's formula, $c > 1$ is a constant,

$$\begin{aligned}
\binom{n}{\lambda n} &= \frac{n!}{(\lambda n)!(1-\lambda n)!} \\
&\leq c\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\right)\left(\sqrt{2\pi\lambda n}\left(\frac{\lambda n}{e}\right)^{\lambda n}\right)^{-1}\left(\sqrt{2\pi(1-\lambda)n}\left(\frac{(1-\lambda)n}{e}\right)^{(1-\lambda)n}\right)^{-1} \\
&= c\frac{((1-\lambda)^{1-\lambda}\lambda^\lambda)^{-n}}{\sqrt{2\pi\lambda(1-\lambda)n}}
\end{aligned}$$

   Hence

$$(n-t)2^{-t}\binom{n}{t}^2 \leq c^2\frac{(((1-\lambda)^{1-\lambda}\lambda^\lambda)^2 2^{-\lambda})^{-n}}{2\pi\lambda}$$

   If we choose $\lambda = 0.91$, then $((1-\lambda)^{1-\lambda}\lambda^\lambda)^2 2^{-\lambda} = 1.026 > 1$. It means $(n-t)2^{-t}\binom{n}{t}^2 \to 0$ when $n \to \infty$. So let $c_2 = 0.92$, $\mathrm{E}[X] \leq 0.92n$.

2. $X = \mathrm{LCS}(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$. Note that if we flip any $x_i$ or $y_i$, the $X$ will change at most 1. So that $f$ is 1-Lipschitz. By McDiarmid inequality,

$$\Pr(|X - \mathrm{E}[X]| \geq \lambda) \leq 2\exp(-\frac{\lambda^2}{n})$$

□

**Problem 3.**

*proof:* Let $X_i$ be the indicator of the $i$-th ball is red, $S_n = \sum_{i=1}^n X_i$. Consider we choose a permutation with size $r+g$, then the first $n$ number is ball we selected. Then the probability of any ball color is red is $\frac{r}{r+g}$. So $\mathrm{E}[S_n] = \frac{nr}{r+g}$ We construct a Doob martingale $Z_i = \mathrm{E}[S_n|X_1, X_2, \ldots, X_i]$. We know that

$$Z_i = S_i + (n-i)\frac{r - S_i}{r+g-i} \text{ and } Z_{i-1} = S_{i-1} + (n-i+1)\frac{r - S_{i-1}}{r+g-i+1}$$

We can compute $\delta_i = |Z_i - Z_{i-1}|$,

- If $S_i = S_{i-1}$, note that $g + S_i \geq i$, so

$$\delta_i \leq \frac{r - S_i}{r+g-i} \leq 1$$

- If $S_i = S_{i-1} + 1$,

$$\delta_i = 1 + \frac{-(n-i) - (r - S_{i-1})}{r+g-i+1} \leq 1$$

  Hence, $|Z_i - Z_{i-1}| \leq 1$, By Azuma-Hoeffding Ineqality, we get

$$\Pr[|Z_n - Z_0| \geq \lambda] = \Pr[|S_n - \frac{nr}{r+g}| \geq \lambda] \leq 2\exp(\frac{-\lambda^2}{2n})$$

  □

**Problem 4.**

*proof:*

1. Using the Linear of Expectation,

$$\mathrm{E}[X] = \frac{n\binom{\binom{n-1}{2}}{cn}}{\binom{\binom{n}{2}}{cn}}$$

2. Consider we get the random graph in such way, we uniform choose a edge, if the edge is not in graph, then add it. Repeat the process until the graph has $N$ edges. Let $Z_i$ be $i$-th new edge. $Y$ be the number of isolated vertices. We construct the Doob martingale $X_i = \mathrm{E}[Y|Z_1, Z_2, \ldots, Z_i]$. If $|X_i - X_{i-1}| \leq 2$. By Corollary 12.5,

$$\Pr[|X - \mathrm{E}[X]| \geq 2\lambda\sqrt{cn}] \leq 2e^{-\lambda^2/2}$$

Then we need to proof $|X_i - X_{i-1}| \leq 2$, we note that for any $e \notin \{Z_1, Z_2, \dots, Z_{i-1}\}$

$$\mathrm{E}[X|Z_1, Z_2, \dots, Z_{i-1}, Z_i = e] = \mathrm{E}[X|Z_1, Z_2, \dots, Z_{i-1}, e \in \{Z_{i+1}, \dots, Z_N\}]$$

Hence, we have

$$\begin{aligned}
&\mathrm{E}[X|Z_1, Z_2, \dots, Z_{i-1}] \\
=&\sum_{Z_i = e, \dots, Z_N} \Pr[Z_i, \dots, Z_N|Z_1, \dots, Z_{i-1}]X + \\
&\sum_{Z_i, \dots, Z_N, e \notin \{Z_i, \dots, Z_N\}} \Pr[Z_i, \dots, Z_N|Z_1, \dots, Z_{i-1}]X \\
=&\sum_{Z_i, \dots, Z_N, e \notin \{Z_{i+1}, \dots, Z_N\}} \Pr[Z_i, \dots, Z_N|Z_1, \dots, Z_{i-1}]X \\
=&\mathrm{E}[X|Z_1, \dots, Z_{i-1}, e \notin \{Z_{i+1}, \dots, Z_N\}]
\end{aligned}$$

And we know that a edge may eliminate at most two isolated vertices.

$$X(Z_1, Z_2, \dots, e, \dots, Z_N) - X(Z_1, Z_2, \dots, Z_N) \leq 2$$

Therefore

$$\begin{aligned}
&|X_i - X_{i-1}| \\
=&|\mathrm{E}[X|Z_1, \dots, Z_i = e] - \mathrm{E}[X|Z_1, \dots, Z_{i-1}, e \notin \{Z_{i+1}, \dots, Z_N\}]| \\
\leq&2
\end{aligned}$$

□

**Problem 5.**

*proof:* Let $X_1$ be a r.v.

$$\Pr(X_1 = +1) = \Pr(X_1 = -1) = \frac{1}{2}$$

then let $X_i = X_1$, $f(X_1, X_2, \dots, X_n) = \sum X_i$, and $f$ is 2-Lipschitz. Note that

$$Z_0 = \mathrm{E}[f(X_1, X_2, \dots, X_n)] = 0, \quad Z_1 = \mathrm{E}[f(X_1, X_2, \dots, X_n)|X_1] = nX_1$$

so we have $|Z_1 - Z_0| = n|X_1| = n > 2$. □