Advanced Algorithm Homework 3

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Problem 1.

proof: Let $X = \sum_{i=1}^{n} p_i b_i(t)$ be the effect of all noise. If $b(t) = -1$, the probability that receiver makes an error is

$$\begin{aligned}
\Pr(s(t) \geq 0) &= \Pr\left(X \geq 1\right) \\
&= \Pr(\mathrm{e^{tX}} \geq \mathrm{e^{t}}) \\
&\leq \frac{\mathrm{E}(e^{tX})}{e^t} = e^{-t} \prod_{i=1}^{n} \mathrm{E}(e^{tp_i b_i}) \\
&= e^{-t} \prod_{i=1}^{n} \frac{1}{2}(e^{tp_i} + e^{-tp_i}) \\
&\leq e^{-t} \prod_{i=1}^{n} e^{t^2 p_i^2/2} \\
&= \exp(\frac{t^2}{2} \sum_{i=1}^{n} p_i^2 - t)
\end{aligned}$$

So we let $t = \frac{1}{\sum_{i=1}^{n} p_i^2}$, so $\Pr(s(t) \geq 0) \leq \exp(-\frac{1}{2\sum_{i=1}^{n} p_i^2})$, and if $b(t) = +1$ is symmetrical.

$\square$

Problem 2.

proof: Consider we repeating the algorithm $q(n)$ times. We define the X_i is the i-th output, let $\Pr(X_i = 1) = \frac{1}{2} + \frac{1}{p(n)}$, $\Pr(X_i = 0) = \frac{1}{2} - \frac{1}{p(n)}$, $X = \sum_i X_i$ Then the error probability is

$$\Pr\left(X \leq \frac{q(n)}{2}\right) = \Pr\left(X \leq \mu\left(1 - \frac{2}{p(n)+2}\right)\right) \quad \mu = q(n)\left(\frac{1}{2} + \frac{1}{p(n)}\right)$$

Note that $0 < \frac{2}{p(n)+2} < 1$, so we can use the Multiplicative Chernoff bound

$$\Pr\left(X \leq \mu\left(1 - \frac{2}{p(n)+2}\right)\right) \leq \exp\left(-\frac{\mu}{2}\left(\frac{2}{p(n)+2}\right)^2\right) = \exp\left(-\frac{q(n)}{p(n)(p(n)+2)}\right)$$

We can simply set $q(n) = np(n)(p(n)+2)$, the error probability is less that 2^{-n}. $\square$

Problem 3.

proof:

(1) We can expand the chernoff bound by Taylor expansion

$$\frac{\mathrm{E}[e^{t|X|}]}{e^{t\delta}} = \frac{\sum_{i=0}^{\infty} \mathrm{E}\left(\frac{t^i|X|^i}{i!}\right)}{\sum_{i=0}^{\infty} \frac{(t\delta)^i}{i!}}$$

and let

$$b = \inf_k \frac{\mathrm{E}[|X|^k]}{\delta^k}$$

then we have for any i,

$$\frac{\mathrm{E}\left(\frac{t^i|X|^i}{i!}\right)}{\frac{(t\delta)^i}{i!}} = \frac{\mathrm{E}[|X|^i]}{\delta^i} \geq b$$

Hence, we get $\frac{\mathrm{E}[e^{t|X|}]}{e^{t\delta}} \geq b$ by the suger water inequality, such k that the kth moment bound is stronger than the Chernoff bound can be found.

(2) To find the specific k is not easy, we need to solve a complex transcendental equation to get it. Although the chernoff bound is not tighter than k-momnet bound, it is convenient to calculate. $\square$

Problem 4.

proof: We have known that $R(T) \leq \sum_{i=1}^{k}(L\Delta_i + T\Delta_i \exp(-\frac{L\Delta_i^2}{2}))$ in class. In this problem,

$$R(T) \leq L\Delta_2 + T\Delta_2 \exp\left(-\frac{L\Delta_2^2}{2}\right)$$

let $f(\Delta_2, L) = L\Delta_2 + T\Delta_2 \exp\left(-\frac{L\Delta_2^2}{2}\right)$, then we just need to calculate

$$\max_{\Delta_2} \min_{L} f(\Delta_2, L)$$

Compute the min point

$$\frac{\partial f}{\partial L} = \Delta_2 - \frac{T\Delta_2^3}{2} \exp\left(-\frac{L\Delta_2^2}{2}\right) \quad L^* = \frac{2}{\Delta_2^2} \ln \frac{T\Delta_2}{2}$$

Then let $g(\Delta_2) = \frac{2}{\Delta^2} \ln \frac{T\Delta_2^2}{2} + \frac{2}{\Delta_2}$, we need find the max point,

$$\frac{\partial g}{\partial \Delta_2} = -\frac{2}{\Delta_2^2} \ln \frac{T\Delta_2^2}{2} + \frac{2}{\Delta_2^2} \quad \Delta_2^* = \sqrt{\frac{2e}{T}}.$$

Finally we get $g(\sqrt{\frac{2e}{T}}) = 2\sqrt{\frac{2T}{e}} = O(\sqrt{T})$.

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