Two-part and k-Sperner families – new proofs using permutations

Péter L. Erdős [1] elp@renyi.hu

Zoltán Füredi [2] furedi@renyi.hu z-furedi@math.uiuc.edu

Gyula O.H. Katona [3] ohkatona@renyi.hu

[1,2,3] Alfréd Rényi Institute of Mathematics,
Hungarian Academy of Sciences, Budapest 1364, P.O. Box 127, Hungary
[2] Dept. of Mathematics, University of Illinois at Urbana-Champaign
Urbana, IL61801, USA

Abstract

This is a paper about the beauty of permutation method.

New and shorter proofs are given for the theorem ([5], [14]) determining all extremal two-part Sperner families and for the uniqueness of k-Sperner families of maximum size ([3]).

1 Introduction

Let X be a finite set of n elements. A family $\mathcal{F}$ is called *Sperner* (or *inclusion-free*) if $E, F \in \mathcal{F}$ implies $E \not\subset F$. The classic result of Sperner [15] states that

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} \tag{1}$$

with equality only when $\mathcal{F}$ consists either of all sets of size $\lfloor \frac{n}{2} \rfloor$ or of all sets of size $\lceil \frac{n}{2} \rceil$.

There are several generalizations and elegant proofs. However frequently the case of equality is left to the reader, since it could be rather complicated. The aim of this paper is to illustrate the strength of permutation method by presenting new shorter proofs for Sperner type theorems. We will give two proofs, one using the permutation method and another proof using **cyclic permutations**, a method developed by the senior author [8], [9] and applied successfully to Sperner theorems by the second author (see [10]).

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1.1 2-part families

Kleitman [11] and one of the present authors [7] independently observed that the statement of the Sperner theorem remains unchanged if the conditions are weakened in the following way. Let $X = X_1 \cup X_2$ be a partition of the underlying set X, $|X_i| = n_i$, $n_1 + n_2 = n$. Suppose $n_1 \geq n_2$ for the entire paper. We say that $\mathcal{F}$ is a *two-part Sperner family* if and only if $E, F \in \mathcal{F}$ $(E \neq F), E \subset F$ implies $(F - E) \not\subset X_1, X_2$. [11] and [7] proved that the size of a two-part Sperner family cannot exceed the right hand side of (1).

The family of all $\lfloor \frac{n}{2} \rfloor$-element subsets gives equality here, too. There are, however, many other optimal constructions. A family $\mathcal{F}$ is called *homogeneous* (with respect to the partition X_1, X_2) if $F \in \mathcal{F}$ implies $E \in \mathcal{F}$ for all sets satisfying $|E \cap X_1| = |F \cap X_1|$, $|E \cap X_2| = |F \cap X_2|$. A homogeneous family can be described with the set $I(\mathcal{F}) = \{(i_1, i_2) : \ |F \cap X_1| = i_1, |F \cap X_2| = i_2$ for some $F \in \mathcal{F}\}$. If $\mathcal{F}$ is a homogeneous two-part Sperner family then $I(\mathcal{F})$ cannot contain pairs with the same first or second components, resp. Consequently we have $|I(\mathcal{F})| \leq n_2 + 1$. We say that a homogeneous family $\mathcal{F}$ is *full* if $|I(\mathcal{F})| = n_2 + 1$. Then for every i_2 $(0 \leq i_2 \leq n_2)$ there is a unique $f(i_2)$ such that $(f(i_2), i_2) \in I(\mathcal{F})$. A homogeneous family is called *well-paired* if it is full and

$$\binom{n_2}{i} < \binom{n_2}{j} \quad \text{implies} \quad \binom{n_1}{f(i)} \leq \binom{n_1}{f(j)} \tag{2}$$

for every pair $1 \leq i, j \leq n_2$. The following characterization (though not in this form) was proved in [5]. Later Shahriari [14] found an alternative proof.

Theorem 1.1 *Let $\mathcal{F}$ be a two-part Sperner family with parts X_1, X_2, $|X_1| + |X_2| = n$. Then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

holds with equality if and only if $\mathcal{F}$ is a homogenous well-paired family.

We give two new, probably shorter proofs in Section 3 of the present paper.

Homogeneity type results are also true in a much more general setting (see the paper of Füredi, Griggs, Odlyzko and Shearer [6] or the joint paper of the present authors with P. Frankl [4].) In those papers it is shown, that there is a homogeneous optimal construction. Here we see that no other family can be optimal.

1.2 Families with no $k+1$-chains

In order to prove Theorem 1.1 we need another extension of the Sperner theorem which is due to Paul Erdős. A family $\mathcal{F}$ of sets is called *k-Sperner* if it contains no *chain* $F_0 \subset F_1 \subset \cdots \subset F_k$ of $k+1$ different sets. It was proved in [3] that if a family $\mathcal{F}$ of subsets of an n-element set is k-Sperner then $|\mathcal{F}|$ is at most the sum of the k largest binomial coefficients of order n. The following theorem determines the cases of equality.

This result is part of the "folklore", but we do not know any written reference for it. The proof is a direct generalization of the uniqueness proof of the original Sperner theorem, due to the second author.

Theorem 1.2 *Let $\mathcal{F}$ be a k-Sperner family of subsets of an n-element set. Then*

$$|\mathcal{F}| \leq \sum_{i=\lfloor (n-k+1)/2 \rfloor}^{\lfloor (n+k-1)/2 \rfloor} \binom{n}{i} \tag{3}$$

holds with equality if and only if $\mathcal{F}$ is the family of all sets of sizes either in the interval $[\lfloor \frac{(n-k+1)}{2} \rfloor, \lfloor \frac{(n+k-1)}{2} \rfloor]$ or in the interval $[\lceil \frac{(n-k+1)}{2} \rceil, \lceil \frac{(n+k-1)}{2} \rceil]$.

This theorem will be proved in Section 2. The upper bound in the following result is an immediate corollary. Denote by $\binom{X}{i}$ the family of all i-element subsets of X, it is called the ith *level* in X.

Theorem 1.3 *Let $\mathcal{F} = \mathcal{F}_1 \cup \ldots \cup \mathcal{F}_k$ be a disjoint union of k Sperner families of subsets of an n-element set. Then $|\mathcal{F}|$ satisfies* (3) *with equality if and only if $\mathcal{F}_i = \binom{X}{r_i}$ holds for $1 \leq i \leq k$ where $r_1, \ldots, r_k$ is a permutation of the elements of either of the interval $[\lfloor \frac{(n-k+1)}{2} \rfloor, \lfloor \frac{(n+k-1)}{2} \rfloor]$ or of the interval $[\lceil \frac{(n-k+1)}{2} \rceil, \lceil \frac{(n+k-1)}{2} \rceil]$.*

2 Uniqueness in Erdős theorem and in the generalized YBLM-inequality

First we will prove a sharper version of Paul Erdős's theorem and will characterize the cases of equality of this sharper one. $\mathcal{F}$ is called *homogeneous* if $F \in \mathcal{F}, E \subset X$ and $|E| = |F|$ imply $E \in \mathcal{F}$. If $\mathcal{F}$ is a family of subsets, $f_i(\mathcal{F})$ will denote the number of i-element members of $\mathcal{F}$.

Theorem 2.1 *Let $\mathcal{F}$ be a k-Sperner family. Then*

$$\sum_{i=0}^{n} \frac{f_i(\mathcal{F})}{\binom{n}{i}} \leq k, \tag{4}$$

with equality only when $\mathcal{F}$ is homogeneous and contains sets of k distinct sizes.

The inequality part of this theorem can be found in [4] (Theorem 5a), and is a generalization of the well-known YBLM-inequality ([16], [1], [12], [13]).

Proof. The method of cyclic permutations is used. The main point of this method is to reduce the original problem into an analogous problem on a fixed cyclic permutation.

If $\emptyset \in \mathcal{F}$ then $\mathcal{F} \setminus \{\emptyset\}$ is a $(k-1)$-Sperner family, and we can use induction on k. The case $X \in \mathcal{F}$ is similar. So from now on (in this Section) we suppose that $f_0 = f_n = 0$ and $n > k$.

Let C be a cyclic permutation of X and let $\mathcal{F}(C)$ denote the subfamily of $\mathcal{F}$ consisting of all sets forming an interval (i.e., an arc) in C. $\mathcal{F}(C)$ is said to be *homogeneous* if $F \in \mathcal{F}(C)$ implies that every interval E along C of the same size ($|E| = |F|$) is in $\mathcal{F}(C)$.

Lemma 2.1

$$|\mathcal{F}(C)| \le nk. \tag{5}$$

Here equality holds if and only if $\mathcal{F}(C)$ is homogenous and it contains k distinct sizes.

Proof of Lemma 2.1. Since $\emptyset, X \notin \mathcal{F}$ at most k sets may start at any fixed element of X along C in one direction. This establishes (5).

In the case of equality there must be exactly k intervals in $\mathcal{F}(C)$ starting from each point of C. Let $B_i(j)$ ($1 \le i \le n$, $1 \le j \le k$) denote the jth interval starting from the ith point where $|B_i(1)| < |B_i(2)| < \ldots < |B_i(k)|$ is supposed. We claim that $|B_i(j)| \le |B_{i+1}(j)|$ holds. Indeed, otherwise $B_{i+1}(1) \subset B_{i+1}(2) \subset \ldots \subset B_{i+1}(j) \subset B_i(j) \subset \ldots \subset B_i(k)$ would be a chain of intervals of length $k+1$, a contradiction. Hence we have $|B_1(j)| \le |B_2(j)| \le \ldots \le |B_n(j)| \le |B_1(j)|$ implying $|B_i(j)| = |B_{i+1}(j)|$ for all $1 \le i < n$ and $1 \le j \le k$. $\square$

Let us return to the proof of Theorem 2.1. Lemma 2.1 yields

$$\sum_C \sum_{F \in \mathcal{F}(C)} 1 = \sum_C |\mathcal{F}(C)| \le (n-1)!nk = n!k. \tag{6}$$

The number of cyclic permutations C containing a given set F as an interval is $|F|!(n - |F|)!$ (if $|F| \ne 0, n$). Hence

$$\sum_{F \in \mathcal{F}} \sum_{C : F \in \mathcal{F}(C)} 1 = \sum_{F \in \mathcal{F}} |F|!(n - |F|)! \tag{7}$$

holds. Comparing (7) and (6) we obtain (4), the inequality part of Theorem 2.1.

The formula (4) can hold with equality only when (7) and (6) are equal, that is, when (5) holds with equality for all cyclic permutations: $\mathcal{F}(C)$ is homogeneous for each C. Consider any two subsets A and B ($\subset X$) of equal cardinality. It is obvious that there is a cyclic permutation C in which they are both intervals. Therefore either $A, B \in \mathcal{F}$ or $A, B \notin \mathcal{F}$ hold, consequently $\mathcal{F}$ is also homogeneous. $\square$

We need a simple inequality, for completeness we supply a sketch of the proof, standard in Linear Programming.

Lemma 2.2 *Suppose that for integers $n \ge k \ge 1$ and non-negative reals $f_1, f_2 \ldots, f_{n-1}$ the following inequalities hold*

$$\begin{aligned} \sum_{1 \le i \le n-1} \frac{f_i}{\binom{n}{i}} &\le k \\ f_i &\le \binom{n}{i}. \end{aligned}$$

Then

$$\sum_{1\le i\le n-1} f_i \le \sum_{i=\lfloor (n-k+1)/2\rfloor}^{\lfloor (n+k-1)/2\rfloor} \binom{n}{i} := f(n,k).$$

Here equality holds if and only if
(a) *in case of* $n \not\equiv k \pmod 2$ $f_i = \binom{n}{i}$ *for* $(n-k+1)/2 \le i \le (n+k-1)/2$ *and* $f_i = 0$ *otherwise,*
(b) *in case of* $n \equiv k \pmod 2$ $f_i = \binom{n}{i}$ *for* $(n-k+2)/2 \le i \le (n+k-2)/2$ *and* $f_{(n-k)/2} + f_{(n+k)/2} = \binom{n}{(n-k)/2}$ *and* $f_i = 0$ *otherwise.*

Proof. Consider a vector $\mathbf{f} = (f_1, f_2, \ldots, f_{n-1})$ which maximizes $\sum f_i$. (The domain is compact, maximum(s) exists). For $\binom{n}{j} < \binom{n}{i}$ the inequalities $f_i < \binom{n}{i}, 0 < f_j$ lead to a contradiction, since replacing them by $f_i + \varepsilon\binom{n}{i}$ and $f_j - \varepsilon\binom{n}{j}$ keeps the constraint the lemma but increases the sum $\sum f_i$. $\square$

Proof Theorem 1.2. The constraint of Lemma 2.2 holds for the sequence $f_1(\mathcal{F}), \ldots, f_{n-1}(\mathcal{F})$ by (4) and since $f_i(\mathcal{F}) \le \binom{n}{i}$ is obvious. This implies Erdős theorem.

We can have equality in this theorem only when (4) holds with equality. Then Theorem 2.1 implies that $\mathcal{F}$ is homogeneous and consists of k distinct sizes. $\square$

Proof Theorem 1.3. The inequality part is trivial, since $\mathcal{F}$ is a k-Sperner family. It is clear from the previous proof that the equality implies equality in (4). Since $\mathcal{F}_i$ $(1 \le i \le k)$ is a Sperner family, (4) holds for $\mathcal{F}_i$ with $k = 1$. Hence (4) with $k = 1$ must hold with equality for each $\mathcal{F}_i$. Therefore $\mathcal{F}_i = \binom{X}{r_i}$ for some r_i. Since $\mathcal{F}_i$ are disjoint, r_i's must be different, $\mathcal{F}$ is a union of k distinct levels. The maximality of $|\mathcal{F}|$ implies that these k levels must be the k middle ones. $\square$

2.1 Uniqueness in Erdős theorem using intervals

Here we give another proof for Theorem 1.2.

Let $\mathcal{F}$ be a k-Sperner family on the n-element underlying set $X = [n]$. We may suppose that $\emptyset, X \notin \mathcal{F}$ because these cases can easily be reduced to the general case. As in the classical proofs, consider a **permutation** π of X. The initial segments of π, i.e., the sets of the form $\{\pi(1), \pi(2), \ldots, \pi(i)\}_{1\le i<n}$ form a **chain** $\mathcal{C}(\pi)$ of length $n-1$. The k-Sperner property of $\mathcal{F}$ implies that $\mathcal{C}(\pi)$ contains at most k members of $\mathcal{F}$, so we have

$$\sum_{F: F\in\mathcal{F}, F\in\mathcal{C}(\pi)} \binom{n}{|F|} \le \sum k \text{ largest binomial coefficients } := f(n,k). \tag{8}$$

Add this up for all the $n!$ permutations.

$$\sum_{\pi} \sum_{F\in\mathcal{F}, F\in\mathcal{C}(\pi)} \binom{n}{|F|} \le n! f(n,k).$$

Here the left hand side can be determined exactly.

$$\sum_{F:F\in\mathcal{F}}\ \sum_{\pi:F\in\mathcal{C}(\pi)}\binom{n}{|F|}=\sum_{F}|F|!(n-|F|)!\binom{n}{|F|}=n!|\mathcal{F}|.$$

This gives $|\mathcal{F}|\le f(n,k)$.

If $|\mathcal{F}|=f(n,k)$ then equality holds in (8) for every π, so the sizes of the members of $\mathcal{F}$ in $\mathcal{C}(\pi)$ form a middle interval of length k. In case of $n\not\equiv k \pmod 2$ this middle interval is unique, we get that $\mathcal{F}$ is homogeneous, it consists of all sets of sizes at least $(n-k+1)/2$ and at most $(n+k-1)/2$. In case of $n\not\equiv k \pmod 2$ there are two possibilities for a middle interval, so $f_i=\binom{n}{i}$ for $(n-k+2)/2\le i\le(n+k-2)/2$ and $f_{(n-k)/2}+f_{(n+k)/2}=\binom{n}{(n-k)/2}$ and $f_i=0$ otherwise. We also obtain that for $|F'|=(n-k)/2$, $|F''|=(n+k)/2$, $F'\subset F''$ one and only one of $\{F',F''\}$ belongs to $\mathcal{F}$. Suppose that there exists an $F\in\mathcal{F}$, $|F|=(n-k)/2$. We claim that $f_{(n-k)/2}=\binom{n}{(n-k)/2}$ and then $f_{(n+k)/2}=0$, and are done.

Consider an arbitrary pair $x\in F$ and $y\in X\setminus F$. We claim that $F\setminus\{x\}\cup\{y\}\in\mathcal{F}$. Indeed, consider a permutation π where $F\setminus\{x\}$, F and $F\cup\{y\}$ are initial segments, and let π' be a permutation obtained from π be exchanging the places of x and y. The largest member of $\mathcal{F}$ in $\mathcal{C}(\pi)$ has $(n+k-2)/2$ elements, so the same is true for $\mathcal{C}(\pi')$. Since the sizes of the members of $\mathcal{C}(\pi')\cap\mathcal{F}$ form a middle interval the smallest member has $(n-k)/2$ elements. This smallest member is $F\setminus\{x\}\cup\{y\}$.

Call two $(n-k)/2$-element sets, F_1 and F_2 neighbors is $|F_1\cap F_2|=|F_1|-1$. Then the above property of the extremal $\mathcal{F}$ can be formulated as it contains all neighbors of F whenever $F\in\mathcal{F}$. It follows that in that case it contains the second, third, etc. neighbors, so $\mathcal{F}$ contains the whole $(n-k)/2$'th level. □

3 Two-part Sperner families

In the method of cyclic permutations a given problem on subsets is reduced to intervals in a cyclic permutation of the underlying set. In the present proof the problem will be reduced to a family of certain mixed objects, pairs (A,B) where A is a subset of X_1 and B is an interval along a fixed cyclic permutation of X_2. Therefore the method can be called the *mixcyc* method.

Let C_2 be a cyclic permutation of X_2 and $\mathcal{F}$ a family of subsets of X. Then $\mathcal{F}(C_2)$ will denote those members of $\mathcal{F}$ for which $F\cap X_2$ is an interval along C_2.

Introduce the notation

$$t(j)=\begin{cases} n_2 & \text{if}\quad j=0,n_2\\ 1 & \text{if}\quad 1\le j\le n_2-1\end{cases}.$$

The double sum

$$\sum_{\substack{(C_2,F)\\ F\in\mathcal{F}(C_2)}} t(|F\cap X_2|)\binom{n_2}{|F\cap X_2|}. \tag{9}$$

will be evaluated in two different ways. First

$$\sum_{F\in\mathcal{F}} \qquad \sum_{C_2:\ F\in\mathcal{F}(C_2)} t(|F\cap X_2|)\binom{n_2}{|F\cap X_2|} =$$
$$\sum_{F\in\mathcal{F}} t(|F\cap X_2|)\binom{n_2}{|F\cap X_2|} \sum_{C_2:\ F\in\mathcal{F}(C_2)} 1.$$

Here

$$\sum_{C_2:\ F\in\mathcal{F}(C_2)} 1 = \begin{cases} (n_2-1)! & \text{if} \quad F\cap X_2=\emptyset \text{ or } X_2 \\ |F\cap X_2|!\,(n-|F\cap X_2|)! & \text{if} \quad \text{otherwise} \end{cases} .$$

Therefore

$$(9) = \sum_{F\in\mathcal{F}(C_2)} n_2! = |\mathcal{F}|n_2!.$$

On the other hand (9) is equal to

$$\sum_{C_2} \quad \sum_{F\in\mathcal{F}(C_2)} t\left(|F\cap X_2|\right)\binom{n_2}{|F\cap X_2|}. \tag{10}$$

Introduce the notation

$$w(i) = t(i)\binom{n_2}{i}, \qquad i=0,...,n_2,$$

and let $(j_0, j_1, ..., j_{n_2})$ be one of the permutations of $(0, 1, ..., n_2)$ satisfying $w(j_0) \geq w(j_1) \geq \ldots \geq w(j_{n_2}) = n_2$. There are four cases of w with value n_2. Suppose that j_{n_2-1} and j_{n_2} are chosen to be 0 and n_2. Now fix a cyclic permutation $C_2 = (c_1, ..., c_n)$ of X_2 and decompose its intervals into n_2 *chains* of intervals: define

$$\mathcal{L}_1 = \{\emptyset, \{c_1\}, \{c_1, c_2\}, ..., \{c_1, c_2, ..., c_{n_2-1}\}, \{c_1, ..., c_{n_2}\}\},$$

while for $i = 2, ..., n_2$ let

$$\mathcal{L}_i = \{\{c_i\}, \{c_i, c_{i+1}\}, ..., \{c_i, c_{i+1}, ..., c_{n_2}, c_1, ..., c_{i-3}\}, \{c_i, ..., c_{i-2}\}\}.$$

Consider the subsum

$$\sum_{(F\cap X_2)\in\mathcal{L}_1} t(|F\cap X_2|)\binom{n_2}{|F\cap X_2|} = \sum_{i=0}^{n_2} |\mathcal{F}(j_i)|\, w(j_i), \tag{11}$$

where $\mathcal{F}(j)$ is defined by

$$\mathcal{F}(j) = \{F\cap X_1 : F\in\mathcal{F}, |F\cap X_2| = j \text{ and } F\cap X_2\in\mathcal{L}_1.\}$$

It is easy to see that the family $\mathcal{F}(j)$ is Sperner for every j, and that $\mathcal{F}(j_k) \cap \mathcal{F}(j_l) = \emptyset$ holds when $k \neq l$. Formula (11) can be written as

$$\begin{aligned}
(11) \quad = \quad & \Big(|\mathcal{F}(j_0)| + ... + |\mathcal{F}(j_{n_2})|\Big) w(j_{n_2}) + \\
+ \quad & \Big(|\mathcal{F}(j_0)| + ... + |\mathcal{F}(j_{n_2-1})|\Big)\Big(w(j_{n_2-1}) - w(j_{n_2})\Big) + \\
+ \quad & ... + \Big(|\mathcal{F}(j_0)| + |\mathcal{F}(j_1)|\Big)\Big(w(j_1) - w(j_2)\Big) \\
+ \quad & |\mathcal{F}(j_0)|\Big(w(j_0) - w(j_1)\Big). \qquad (12)
\end{aligned}$$

By the Erdős theorem the total size of k pairwise disjoint Sperner families on X_1, cannot exceed the k largest levels. Therefore if $m(i) = \binom{n_1}{i}$ and $(l_0, l_1, ..., l_{n_1})$ is one of the permutations of $(0, 1, ..., n_1)$ satisfying $m(l_0) \geq m(l_1) \geq \ldots \geq m(l_{n_1})$ then

$$\begin{aligned}
(12) \quad \leq \quad & \Big(m(l_0) + m(l_1) + ... + m(l_{n_2})\Big) w(j_{n_2}) + \\
+ \quad & \Big(m(l_0) + m(l_1) + ... + m(l_{n_2-1})\Big)\Big(w(j_{n_2-1}) - w(j_{n_2})\Big) + ... \\
+ \quad & \Big(m(l_0) + m(l_1)\Big)\Big(w(j_1) - w(j_2)\Big) + m(l_0)\Big(w(j_0) - w(j_1)\Big) \\
= \quad & \sum_{i=0}^{n_2} m(l_i) w(j_i). \qquad (13)
\end{aligned}$$

The same estimations can be applied for the other $n_2 - 1$ chains $\mathcal{L}_k$, $(k = 2, ..., n_2)$:

$$\sum_{F \cap X_2 \in \mathcal{L}_k} t(|F \cap X_2|) \binom{n_2}{|F \cap X_2|} \leq \sum_{i=0}^{n_2-2} m(l_i) w(j_i).$$

Using the fact, that the number of cyclic permutations C_2 is $(n_2 - 1)!$, and putting together the previous inequalities, we obtain

$$\begin{aligned}
(10) \quad \leq \quad & \sum_{C_2} \left(n_2 \sum_{i=0}^{n_2-2} m(l_i) w(j_i) + m(l_{n_2-1}) w(j_{n_2-1}) + m(l_{n_2}) w(j_{n_2}) \right) \\
= \quad & n_2! \sum_{i=0}^{n_2} \binom{n_1}{l_i} \binom{n_2}{j_i} = n_2! \sum_{i=0}^{n_2} \binom{n_1}{\lceil \frac{n_1+n_2}{2} \rceil + i} \binom{n_2}{i} \\
= \quad & n_2! \sum_{i=0}^{n_2} \binom{n_1}{\lfloor \frac{n_1+n_2}{2} \rfloor - i} \binom{n_2}{i} = \binom{n}{\lfloor \frac{n}{2} \rfloor}. \qquad (14)
\end{aligned}$$

$(9) = (10) \leq (14)$ finishes the proof of the two-part Sperner theorem.

In order to prove the equality part of Theorem 1.1 we only have to check carefully the cases of equality in the above proof of the two-part Sperner theorem.

Define

$$\mathcal{F}_1(B) = \{A : \ A \subset X_1, A \cup B \in \mathcal{F}\} \quad \text{for } B \subset X_2.$$

If $\mathcal{F}$ is a family satisfying equality in the Erdős theorem, then there must be equality in (13), that is,

$$|\mathcal{F}(j_0)| + |\mathcal{F}(j_1)| + ... + |\mathcal{F}(j_r)| = m(l_0) + m(l_1) + ... + m(l_r) \tag{15}$$

holds whenever $w(j_r) - w(j_{r+1}) > 0$ (where $w(j_{n_2+1}) = 0$). It is obvious that every second of these differences is zero, the other ones are positive. If n_2 is even then $w(j_0) - w(j_1)$ is positive, $w(j_1) - w(j_2)$ is zero, $w(j_2) - w(j_3)$ is positive, and so on. On the other hand, if n_2 is odd then this sequence starts with a zero. We should not forget however that there are some irregularities at the end. Firstly, the last coefficient $w(j_{n_2})$ is always positive, secondly, it is preceded by 3 zeros. This implies, by Theorem 1.3, that in the case of even n_2 $\mathcal{F}(j_0)$ must be one of the (one or two) largest levels in X_1, $\mathcal{F}(j_0), \mathcal{F}(j_1), \mathcal{F}(j_2)$ must be 3 largest levels, and so on. Hence $\mathcal{F}(j_1)$ and $\mathcal{F}(j_2)$ are two levels next in size. The same holds for $\mathcal{F}(j_{2s+1})$ and $\mathcal{F}(j_{2s+2})$ for $0 \le s \le \frac{n_2-6}{2}$. If n_2 is odd then $\mathcal{F}(j_0)$ and $\mathcal{F}(j_1)$ are two largest levels, $\mathcal{F}(j_2)$ and $\mathcal{F}(j_3)$ are next two levels, ... In general $\mathcal{F}(j_{2s})$ and $\mathcal{F}(j_{2s+1})$ $(0 \le s \le \frac{n_2-5}{2})$ is a pair of the $(2s+1)$st and $(2s+2)$th largest levels.

Therefore $\mathcal{F}(j_0), \dots, \mathcal{F}(j_{n_2})$ are $n_2 + 1$ largest levels in X_1. If $\mathcal{F}(j_u) = \binom{X_1}{w}$ then we write $f^*(j_u) = w$. We need to check the ordering determined by (2). If n_2 is even then the left hand side of (2),

$$\binom{n_2}{j_u} < \binom{n_2}{j_v} \ (u < n_2 - 3)$$

holds if and only if $v \le u$ and u is not an even integer following $v + 1$. Then

$$\binom{n_1}{f^*(j_u)} \le \binom{n_1}{f^*(j_v)} \tag{16}$$

is obvious. The case when n_2 is odd is analogous. That is, the order follows (2) up to $n_2 - 4$. But (16) also holds when $u = n_2 - 3, n_2 - 2, n_2 - 1, n_2$ and $n_2 - 3 < v$. An important consequence is that $f^*(j_v)$ cannot be $\lfloor \frac{n_1-n_2}{2} \rfloor$ or $\lceil \frac{n_1+n_2}{2} \rceil$ when $n_2 - 3 < v$.

The above ideas are valid for all cyclic permutations of X_2, therefore $\mathcal{F}_1(B)$ is defined for all $B \subset X_2$ and it is a full level $\binom{X_1}{j}$ for some $j = j(B)(\lfloor \frac{n_1-n_2}{2} \rfloor \le j \le \lceil \frac{n_1+n_2}{2} \rceil)$.

We have to show that $\mathcal{F}_1(B)$ depends only on the size of B, that is, $|B_1| = |B_2|$ implies $\mathcal{F}_1(B_1) = \mathcal{F}_1(B_2)$. It is sufficient to verify this statement for "neighboring" sets, that is, when $|B_1 - B_2| = 1$. Let $B_1 = \{x_1, x_2, \dots, x_l\}, B_2 = \{x_2, x_3, \dots, x_l, x_{l+1}\}$. Consider the cyclic permutations $C = (x_2, x_3, \dots, x_l, x_1, x_{l+1}, x_{l+2} \dots, x_{n_2}), C' = (x_2, x_3, \dots, x_l, x_{l+1}, x_1, x_{l+2} \dots, x_{n_2})$. They define the chains (of length $n_2 + 1$) $\mathcal{L}_1$ and $\mathcal{L}_1'$, which differ only in one member. The function $\mathcal{F}_1$ associates families $\binom{X_1}{j}(\lfloor \frac{n_1-n_2}{2} \rfloor \le j \le \lceil \frac{n_1+n_2}{2} \rceil)$ with each member of these chains, where the js are different for one chain. If n_1 and n_2 have the same parities then there are $n_2 + 1$ choices for j therefore $\mathcal{F}(B_1) = \mathcal{F}(B_2)$. If their parities are different then $\mathcal{F}(B_1)$ and $\mathcal{F}(B_2)$ may be different: one of them is $\binom{X_1}{\lfloor \frac{n_1-n_2}{2} \rfloor}$ the other one is $\binom{X_1}{\lceil \frac{n_1+n_2}{2} \rceil}$. It is clear from the monotonicity (16) that this can happen only when $|B_1| = 1$ or $n_2 - 1$. This proves the statement $\mathcal{F}_1(B_1) = \mathcal{F}_1(B_2)$ for

$1 < |B_1| = |B_2| < n-1$. Moreover

$$\mathcal{F}_1(B) = \text{ either } \binom{X_1}{\lfloor \frac{n_1-n_2}{2} \rfloor} \text{ or } \binom{X_1}{\lceil \frac{n_1+n_2}{2} \rceil} \text{ if } |B| = 1, n-1.$$

Since $\mathcal{F}$ is a two-part Sperner family, $B \subset C$ implies $\mathcal{F}_1(B) \neq \mathcal{F}_1(C)$ (in fact, they must be disjoint). Suppose *e.g.* that $j(\{x\}) = \lfloor \frac{n_1-n_2}{2} \rfloor$ holds for some $x \in X_2$. Then $j(C)$ must be $\lceil \frac{n_1+n_2}{2} \rceil$ for all n_2-1-element C with the exception of $X_2 - x$. But these sets cover X_2 therefore $j(\{x\}) = \lfloor \frac{n_1-n_2}{2} \rfloor$ must hold for all $x \in X_2$, consequently $j(C) = \lceil \frac{n_1+n_2}{2} \rceil$ for all n_2-1-element $C \in X_2$. We have proved that $\mathcal{F}$ is homogeneous and full, the function f is defined by $f(i) = j(B)$ where $i = |B|$.

It is almost proved that $\mathcal{F}$ is well-paired, by (16). The only possible exception is that the right hand side of (2) does not hold for one or more of the pairs $(0,1), (0, n_2 - 1), (n_2, 1), (n_2, n_2-1)$. Suppose *e.g.* that the pair $(0,1)$ is such a one. Then

$$|\mathcal{F}| = \sum_{i=0}^{n_2} \binom{n_2}{i} \binom{n_1}{f(i)}$$

can be decreased by interchanging the values $f(0)$ and $f(1)$. (It decreases the sum only when $n_2 > 1$ but the case $n_2 = 1$ is trivial.) This contradiction shows that $\mathcal{F}$ is well-paired. □

We advice the interested reader to check [5] , where the optimal constructions for all four cases (depending on the parities of n_1 and n_2, resp.,) are illustrated with figures.

3.1 Extremal 2-part Sperner families and intervals

Here we give another proof for Theorem 1.1. We need two simple lemmas.

Lemma 3.1 *Suppose that $u \geq v \geq 1$ are integers, $a_1 \geq a_2 \geq \dots a_u \geq 0$, $b_1 \geq b_2 \geq \dots \geq b_v$ are reals and $f : [v] \to [u]$ an arbitrary injection (i.e., $f(i) \neq f(j)$ for $i \neq j$), then*

$$\sum_i a_{f(i)} b_i \leq \sum_{1 \leq i \leq v} a_i b_i,$$

and here equality holds if and only if $a_i = a_{f(i)}$ for all i. □

Lemma 3.2 *Let the $a_1, a_2, \dots, a_{n_1+1}$ be the sequence of binomial coefficients of rank n_1 in decreasing order, and let $b_1, \dots, b_{n_2+1}$ be the binomial coefficients of rank n_2 again in decreasing order. (We have $a_i = \binom{n_1}{\lfloor (n_1+i)/2 \rfloor}$ and $b_j = \binom{n_2}{\lfloor (n_2+j)/2 \rfloor}$.) Then $\sum_i a_i b_i = \binom{n}{\lfloor n/2 \rfloor}$.* □

Let $\mathcal{F}$ be a 2-part Sperner family on the n-element underlying set $X = [n]$, with parts X_1, X_2, $|X_i| = n_i$, $n_1 \geq n_2 > 0$. Suppose that $|\mathcal{F}|$ is maximal, we have $|\mathcal{F}| \geq \binom{n}{\lfloor n/2 \rfloor}$ Let $\pi_i \in S_{[n_i]}$ a permutation of X_i, $i = 1, 2$. Define the $(n_1+1) \times (n_2+1)$

matrix $M = M(\pi_1, \pi_2)$ as follows. Label the rows by $0, 1, \ldots, n_1$, the columns by $0, 1, \ldots, n_2$ and the i, j entry, $M_{i,j}$ equals to 1 if the unions of the two initial segments $\{\pi_1(1), \pi_1(2), \ldots, \pi_1(i)\} \cup \{\pi_2(1), \ldots, \pi_2(j)\}$ belongs to $\mathcal{F}$. $M_{i,j} = 0$ for the other entries. Such an M contains at most one nonzero entry in each row and column.

Suppose that M is an arbitrary $(n_1+1) \times (n_2+1)$ matrix, labeled as above, and suppose that each entry is 0 or 1 and each row and column contains at most one 1. Define a 2-part Sperner family $\mathcal{H}(M)$ by taking all sets $F \subset X$ with $M_{|F \cap X_1|, |F \cap X_2|} = 1$. Then $|\mathcal{H}(M)| = \sum_{M_{i,j}=1} \binom{n_1}{i}\binom{n_2}{j}$. By Lemma 3.1 and 3.2 we have

$$|\mathcal{H}(M)| \le \sum_{i,j} a_i b_j = \binom{n}{\lfloor n/2 \rfloor}.$$

We obtain

$$\begin{aligned}
|\mathcal{F}| n_1! n_2! &\ge \binom{n}{\lfloor n/2 \rfloor} n_1! n_2! \ge \sum_{(\pi_1,\pi_2)} |\mathcal{H}(M(\pi_1,\pi_2))| \\
&= \sum_{F \in \mathcal{F}} \sum_{\substack{\pi_1,\pi_2 \\ F \cap X_i \text{ is initial in } \pi_i}} \binom{n_1}{|F \cap X_1|}\binom{n_2}{|F \cap X_2|} \\
= \sum_{F \in \mathcal{F}} |F \cap X_1|!&(n_1 - |F \cap X_1|)! |F \cap X_2|! (n_2 - |F \cap X_2|)! \binom{n_1}{|F \cap X_1|}\binom{n_2}{|F \cap X_2|} \\
&= |\mathcal{F}| n_1! n_2!.
\end{aligned}$$

Thus here equality holds, i.e., $|\mathcal{F}| = \binom{n}{\lfloor n/2 \rfloor}$ so does each $|\mathcal{H}(M(\pi_1, \pi_2))|$. It also follows, that for each (π_1, π_2) the family $\mathcal{H}$ is full and well-paired (cf. (2)). We have to show that $\mathcal{F}$ is homogeneous, too.

Check what happens if instead of (π_1, π_2) one considers the pair (π_1, π_2') where π_2' is obtained from π_2 by exchanging the elements v and $v+1$ in X_2 $(1 \le v < n_2)$. In the new matrix $M' = M(\pi_1, \pi_2')$ all columns, except eventually the uth, are unchanged. Because M' (and $M(\pi_1, \pi_2)$) are full, there is an entry $M'_{u',v} = 1$ and $M_{u,v} = 1$. We claim that $u = u'$, the two matrices are identical. Indeed, calculating the cardinalities $|\mathcal{H}(M(\pi_1, \pi_2))|$ and $|\mathcal{H}(M(\pi_1, \pi_2'))|$ both have maximal values. So Lemma 3.1 gives that the factors corresponding to $\binom{n_2}{v}$ are the same, $\binom{n_1}{u} = \binom{n_1}{u'}$. Hence either $u = u'$ (and we are done) or $u + u' = n_1$. In the later case consider again the sums

$$\sum_{M_{i,j}=1} \binom{n_1}{i}\binom{n_2}{j} = \sum_{M'_{i,j}=1} \binom{n_1}{i}\binom{n_2}{j}.$$

In the second sum there is no $\binom{n_1}{u}$, and in the first there is no $\binom{n_1}{n_1-u}$. Since both contains the largest n_2+1 values of binomial coefficients of rank n_1, this implies that both are having all of $\binom{n_1}{i}$ with i between u and $n_1 - u$ and exactly one of $\{\binom{n_1}{u}, \binom{n_1}{n_1-u}\}$. These are $(n_1 - 2u - 1) + 1$ coefficients, so $u = (n_1 - n_2 - 1)/2$ and then $v = 0$ or n_2, a contradiction (since $1 \le v < n_2$). Finally, a similar proof gives that M is unchanged if we exchange two neighboring elements u and $u+1$ in X_1, i.e., $\mathcal{F}$ is homogeneous. $\square$

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