Advanced Algorithm Homework 2

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Problem 1.

proof:

The permutation will be a derangement without fix point, we know that

$$D_n = n! \sum_{i=2}^{n} \frac{(-1)^i}{i!}$$

So we can calculate the variance, let X be the r.v. of number of fix points.

$$\begin{aligned}
\mathrm{E}(X^2) &= \frac{1}{n!} \sum_{i=0}^{n} i^2 \binom{n}{i} D_{n-i} \\
&= \sum_{i=1}^{n} \sum_{j=2}^{n-i} \frac{i}{(i-1)!} \frac{(-1)^j}{j!}
\end{aligned}$$

Let X_i be r.v. whether i-th node is a fix point, then $E(X_i) = \frac{1}{n}$, so

$$\mathrm{E}(X) = \sum \mathrm{E}(X_i) = 1 = \sum_{i=1}^{n} \sum_{j=2}^{n-i} \frac{1}{(i-1)!} \frac{(-1)^j}{j!}$$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}(X^2) - \mathrm{E}(X)^2 \\
&= \sum_{i=1}^{n} \sum_{j=2}^{n-i} \frac{1}{(i-1)!} \frac{(-1)^j}{j!} (i - \mathrm{E}(X)) \\
&= \sum_{i=2}^{n} \sum_{j=2}^{n-i} \frac{1}{(i-2)!} \frac{(-1)^j}{j!} \\
&= \sum_{i=1}^{n-1} \sum_{j=2}^{n-i-1} \frac{1}{(i-1)!} \frac{(-1)^j}{j!} \\
&= 1
\end{aligned}$$

At last step, note that the form of

$$\sum_{i=1}^{n-1}\sum_{j=2}^{n-i-1}\frac{1}{(i-1)!}\frac{(-1)^j}{j!}$$

is same as $\mathrm{E}(X)$, so the formula equals 1. In special, we know $\mathrm{Var}(X)=0$ when $n=1$. $\square$

Problem 2.

proof: For the first problem, we can calculate the probability

$$\begin{aligned}\Pr(\text{no two balls in same bin}) &= \prod_{i=1}^{c_1\sqrt{n}-1}\left(1-\frac{i}{n}\right)\\ &\le \exp(-\sum_{i=1}^{c_1\sqrt{n}-1}\frac{i}{n})\\ &\le e^{\frac{c_1\sqrt{n}-c_1^2 n}{2n}}\end{aligned}$$

So we need $\frac{c_1\sqrt{n}-c_1^2n}{2n}\le -1$ for all n, it means $\frac{c_1}{c_1^2-2}\le\sqrt{n}$ and $c_1>\sqrt{n}$. Hence, $c_1>2$ For the second constant,

$$\begin{aligned}\Pr(\text{no two balls in same bin}) &= \prod_{i=1}^{c_2\sqrt{n}-1}\left(1-\frac{i}{n}\right)\\ &\ge \exp(-\sum_{i=1}^{c_2\sqrt{n}-1}\frac{i}{n}+\frac{i^2}{n^2})\\ &\ge \exp(-\frac{c_2\sqrt{n}(c_2\sqrt{n}-1)}{2n}-\frac{(c_2\sqrt{n}-1)c_2\sqrt{n}(2c_2\sqrt{n}-1)}{6n^2})\\ &\ge \exp(-\frac{c_2^2}{2})\end{aligned}$$

So we need $\frac{c_2^2}{2}<\frac{1}{\log_2 e}$ for all n, so we get $0\le c_2\le\sqrt{2\ln 2}$ $\square$

Problem 3.

proof: (1) Note $X_i\sim\mathrm{Binom}(n,\frac{1}{n})$, so we let $Y_i\sim\mathrm{Poisson}(1)$. $\Pr(\bigcap Y_i=1)=\prod\Pr(Y_i=0)=\frac{1}{e^n}$. By Corollary 5.9 we get the upper bound is $\frac{\sqrt{n}}{e^{n-1}}$.

(2) Note that n balls in n different bins is a permutation, so $\Pr(\bigcap X_i=1)=\frac{n!}{n^n}$.

(3) Note that the probability that a Poisson random variable with parameter n takes on the value n is $\frac{e^{-n}n^n}{n!}$, we verify it is two probabilities differ by the multiplicative factor.

From Theorem 5.6, we have

$$\Pr(\bigcap X_i = 1) = \Pr(\bigcap Y_i = 1 | \sum Y_i = n) = \Pr(\bigcap Y_i = 1)/\Pr(\sum Y_i = n)$$

and the $\sum Y_i \sim \text{Poisson}(n)$, so we get such result. □

Problem 4.

proof: Without general, we assume $0 \le x_n \le x_{n-1} \le \cdots \le x_1$, and define r.v $X_S = \sum_{i \in S} \epsilon_i x_i$.

We split x into two sets,

- If $n = 1$,

$$\Pr(x_i \le 1) = 1$$

- If $x_1 \le \frac{1}{2}$, let $A = [1, k]$ and $B = [k+1, n]$, it holds

$$\sum_{i=1}^{k-1} x_i^2 < \frac{1}{4} \quad \text{and} \quad \sum_{i=1}^{k} x_i^2 \ge \frac{1}{4}$$

  Note that $\text{Var}(X_S) = \sum_{i \in S} x_i^2$, so $\text{Var}(X_A) \in [\frac{1}{4}, \frac{1}{2}]$ and $\text{Var}(X_B) \in [\frac{1}{2}, \frac{3}{4}]$, then we using Chebyshev inequality ,

$$\Pr(|X_A| \ge 1) \le \text{Var}(X_A) \le \frac{1}{2} \quad \text{and} \quad \Pr(|X_B| \ge 1) \le \text{Var}(X_B) \le \frac{3}{4}$$

  Let $p = \Pr(\text{the signs of } X_A \text{ and } X_B \text{ is different})$, we know that $p = \frac{1}{2}$.

  So

$$\Pr(|X_{[n]}| \le 1) \ge p\Pr(|X_A| < 1)\Pr(|X_B| < 1) \ge \frac{1}{16}$$

- If $x_1 > \frac{1}{2}$, let $A = [1], B = [2, n]$. Thereforce,

$$\Pr(|X_A| \le 1) = 1 \quad \text{and} \quad \Pr(|X_B| \ge 1) \le \text{Var}(X_B) \le \frac{3}{4}$$

  And

$$\Pr(|X_{[n]}| \le 1) \ge p\Pr(|X_A| < 1)\Pr(|X_B| < 1) \ge \frac{1}{8}$$

Hence, we can choose $c = \frac{1}{16}$. □