Advanced Algorithm Homework 1

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Problem 1.

proof: (1) Let Y_i be the r.v. if the ith position is the start of one streak of length $\log_2 n + 1$ and Y is the number of streaks $\log_2 n + 1$.

$$\mathrm{E}(Y) = \sum_{i=1}^{n-\log_2 n} \mathrm{E}(Y_i) = \sum_{i=1}^{n-\log_2 n} 2^{-\log_2 n} = 1 - o(1)$$

(2) $m = \lfloor \log_2 n - 2\log_2 \log_2 n \rfloor$. Break the sequence of flips up into disjoint blocks of m consecutive. So the number of blocks will less than $n/\log_2 n - 1$

Let Y be the event there is no streak of length at least m is less than $1/n$ and X be the event any blocks is not a streak. Note that $X \subset Y$, Hence

$$\Pr(Y) \leq \Pr(X)$$

then we bound it

$$\begin{aligned}
\Pr(X) &\leq \left(1 - 2^{1-m}\right)^{n/m-1} \\
&\leq \left(1 - 2^{2\log_2 \log_2 n - \log_2 n}\right)^{n/m-1} \\
&\leq \left(1 - \frac{\log_2^2 n}{n}\right)^{n/\log_2 n - 1} \\
&\leq e^{\frac{\log_2^2 n}{n} - \log_2 n} \\
&= n^{\log_2 e(\frac{\log_2 n}{n} - 1)}
\end{aligned}$$

So we just need proof $\log_2 e(\frac{\log_2 n}{n} - 1) \leq -1$ when n is large, it means

$$\log_2 e > 1 \geq \frac{n}{n - \log_2 n}$$

Obviously, the formula is satisfied when n is large enough.

$\square$

Problem 2.

proof: Let X be the r.v. of number of rolls until one sixes appears, Y be the r.v. of number of rolls until the first pair of consecutive sixes appears, it is easy to get $\mathrm{E}(X) = 6$. Hence,

$$\mathrm{E}(Y) = \mathrm{E}(X) + \mathrm{E}(\frac{1}{6} + \frac{5}{6}(1+Y)) \Rightarrow \mathrm{E}(Y) = 42$$

$\square$

Problem 3.

proof: A permutation can be represented as set of cycles, so consider insert 1 to n to a empty graph. We have two choice,

- As a self-loop, there will be new cycle.
- Become the successor of a node that already exists in graph.

Let X_i be the r.v., the number of cycle increased after insert the i, then $\mathrm{E}(X_i) = \frac{1}{i}$ X be the number of cycles in permutation, $\mathrm{E}(X) = \sum_{i=1}^{n} \mathrm{E}(X_i) = \sum_{i=1}^{n} \frac{1}{i} = H_n$. $\square$

Problem 4.

proof: Enumerate the number of HEADs to calculate the expectation of $|a-b|$,

$$\begin{aligned}
\frac{1}{2^n}\sum_{i=0}^{n}|n-2i|\binom{n}{i} &= \frac{1}{2^{n-1}}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}(n-2i)\binom{n}{i} \\
&= \frac{n}{2^{n-1}}(1+\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{i} - 2\binom{n-1}{i-1}) \\
&= \frac{n}{2^{n-1}}(1+\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor}\binom{n-1}{i} - \binom{n-1}{i-1}) \\
&= \frac{n}{2^{n-1}}\binom{n-1}{\lfloor \frac{n}{2} \rfloor}
\end{aligned}$$

It is easy to see $\frac{n}{2^{n-1}} = \Theta(n/2^n)$, and note that $\frac{1}{2}\binom{n}{\lfloor n/2 \rfloor} \le \binom{n-1}{\lfloor n/2 \rfloor} \le \binom{n}{\lfloor n/2 \rfloor}$, so we can just estimate the $\binom{n}{\lfloor n/2 \rfloor}$. When n is even,

$$\begin{aligned}
\frac{n}{2^{n-1}}\binom{n-1}{n/2} &= \Theta\left(\frac{n}{2^n}\right)\Theta\left(\binom{n}{n/2}\right) \\
&= \Theta\left(\frac{n}{2^n}\right)\Theta\left(n!/(\frac{n}{2}!)^2\right) \\
&= \Theta\left(\frac{n}{2^n}\right)\Theta\left(\frac{\sqrt{n}(n/e)^n}{n(n/(2e))^n}\right) \\
&= \Theta(\sqrt{n})
\end{aligned}$$

When n is odd,

$$\begin{aligned}
\frac{n}{2^{n-1}}\binom{n-1}{\frac{n-1}{2}} &= \Theta\left(\frac{n}{2^n}\right)\Theta\left((n-1)!/(\frac{n-1}{2}!)^2\right) \\
&= \Theta\left(\frac{n}{2^n}\right)\Theta\left(\frac{\sqrt{n-1}((n-1)/e)^{n-1}}{(n-1)((n-1)/(2e))^{n-1}}\right) \\
&= \Theta(\sqrt{n})
\end{aligned}$$

Anthoer way

Define the r.v. X_i, If the direction of the i-th coin occuers more, the $X_i = +1$, if less, the $X_i = -1$. Hence. If equal, the $X_i = 0$, $|a-b| = \sum X_i$

When $|a-b| \neq 0$ we reverse the i-th coin, the contribution will become to $-X_i$ if and only if the $|a-b| > 2$.

If n is odd, a coin will have contribution only when $|a-b| = 1$, so $\mathrm{E}(X_i) = 2\binom{n}{\frac{n-1}{2}}/2^n = \binom{n+1}{\frac{n+1}{2}}/2^n$. n is even is same, $\mathrm{E}(X_i) = \binom{n}{n/2}/2^n$.

Using the Stirling's formula, we can know that $\mathrm{E}(X_i) = \Theta(\frac{1}{\sqrt{n}})$, so $\mathrm{E}(|a-b|) = \Theta(\sqrt{n})$. $\square$