COMBINATORIAL RELATIONS AND CHROMATIC GRAPHS

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1. Introduction. The following elementary logical problem was a question in the William Lowell Putnam Mathematical Competition held in March 1953 **(1)**:

> Six points are in general position in space (no three in a line, no four in a plane). The fifteen line segments joining them in pairs are drawn, and then painted, some segments red, some blue. Prove that some triangle has all its sides the same color.

This problem may be generalized as follows:

> N vertex points are given, and r colors are available. Each of the $\frac{1}{2}N(N-1)$ segments joining two vertices is colored by one and one only of the r given colors. Find the minimum integer $n = n(k_1, k_2, \ldots, k_r)$ such that, if $N \geqslant n$, either there will exist a subset of k_1 vertices with all interconnecting segments of the first color, or there will exist a subset of k_2 vertices with all interconnecting segments of the second color, . . . , or there will exist a subset of k_r vertices with all interconnecting segments of the rth color.

The elementary problem is then the special case $r = 2$, $k_1 = 3$, $k_2 = 3$, and essentially asserts that $n(3, 3) \leqslant 6$. A slightly more general form would assert that $n(3, 3) = 6$.

THEOREM 1. $$n(3, 3) = 6.$$

Proof. To show $n(3, 3) > 5$, consider the five points to be the vertices of a regular pentagon. Let the two colors be red and blue. Color the edges red and the interior diagonals blue. In this assemblage of 5 vertices and 10 segments there is no triangle with all its sides the same color.

To show 6 is a sufficient number of vertices, select some one vertex point and call it a. Of the five line segments terminating at a, select a set of three all of which have the same color. Consider the three segments joining their farther ends in pairs. If no one of these segments is of the same color as the initial set, then all three segments are of the other color and do form a required triangle.

Note. The second part of the above proof is essentially the same as that given by one of the Putnam examination candidates.

2. General case for 2 colors. We define

$$n(1, m) = 1 \qquad (m = 1, 2, 3, \ldots),$$
$$n(k, 1) = 1 \qquad (k = 1, 2, 3, \ldots).$$

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It is easy to see that

$$n(2, m) = m \qquad (m = 2, 3, 4, \ldots),$$
$$n(k, 2) = k \qquad (k = 2, 3, 4, \ldots).$$

The relation $n(k, m) = n(m, k)$ is an obvious consequence of the symmetry or duality of the problem.

It is possible to establish two inequalities for $n(k, m)$ which are similar in appearance to a well-known identity for binomial coefficients.

For convenience, a set of vertices K will be said to be a red interconnected set if all segments joining pairs of vertices in K are colored red. A set of vertices M will be said to be a blue interconnected set if all segments joining pairs of vertices in M are colored blue. In the minimum vertex number $n(k, m)$, the first argument will be used for the red color, and the second argument for the blue color.

THEOREM 2. $$n(k, m) \leqslant n(k - 1, m) + n(k, m - 1).$$

Proof. Let T be any set of $n(k - 1, m) + n(k, m - 1)$ vertices. Form a chromatic graph by coloring each interconnecting segment either red or blue. Select one vertex and call it a. For vertex a consider the two associated numbers:

n_1 = number of vertices such that the segments joining them to a are red,
n_2 = number of vertices such that the segments joining them to a are blue.

Call these sets of vertices T_1 and T_2.

Since all vertices other than a belong to either T_1 or T_2, one obtains

$$n_1 + n_2 + 1 = n(k - 1, m) + n(k, m - 1).$$

If $n_1 < n(k - 1, m)$, then $n_2 \geqslant n(k, m - 1)$ and one selects the set T_2. In T_2 there is either a red interconnected set S_1 with k vertices or a blue interconnected set S_2 with $m - 1$ vertices. If the latter holds, the set $\{S_2 + \text{vertex } a\}$ is a blue interconnected set with m vertices. Hence the inequality of Theorem 2 is established for this case.

If $n_1 \geqslant n(k - 1, m)$, one selects the set T_1. In T_1 there is either a red interconnected set S_3 with $k - 1$ vertices or a blue interconnected set S_4 with m vertices. In the first case the set $\{S_3 + \text{vertex } a\}$ is seen to be a red interconnected set with k vertices. The proof of Theorem 2 is now complete.

COROLLARY 1.

$$n(k, m) \leqslant \binom{k + m - 2}{k - 1}.$$

By an obvious extension of this argument one may prove:

COROLLARY 2. *For the case of more than two colors,*

$$n(k_1, k_2, \ldots, k_r) \leqslant n(k_1 - 1, k_2, \ldots, k_r) + n(k_1, k_2 - 1, \ldots, k_r) + n(k_1, k_2, \ldots, k_r - 1).$$

COROLLARY 3. *For the multi-color case, an inequality on the size of the minimum number is afforded by the multinomial coefficient*

$$n(k_1 + 1, k_2 + 1, \ldots, k_r + 1) \leqslant \frac{(k_1 + k_2 + \ldots + k_r)!}{k_1!\, k_2! \ldots k_r!}.$$

THEOREM 3. *If* $n(k-1, m) = 2p$ *and* $n(k, m-1) = 2q$, *then*

$$n(k, m) < 2p + 2q = n(k-1, m) + n(k, m-1).$$

Proof. Take a set of $2p + 2q - 1$ vertices. Select one vertex and call it a. There are $2p + 2q - 2$ segments terminating at a. Three possibilities might arise:

(a) $2p$ or more segments terminating at a are red,
(b) $2q$ or more segments are blue, or
(c) $2p - 1$ segments are red and $2q - 1$ segments are blue.

For case (a) consider the set T_1 of the vertices at the farther ends of the $2p$ or more red segments. Since the number of vertices in T_1 is greater than or equal to $n(k-1, m)$, there is either a red interconnected set S_1 with $k - 1$ vertices or a blue interconnected set S_2 with m vertices. In the first situation, the set $\{S_1 + \text{vertex } a\}$ is a red interconnected k vertex set. Thus the theorem is true for case (a).

Likewise, a similar argument holds for case (b).

Case (c) cannot hold for each vertex of the chromatic graph. For if it did hold, then there would be $(2p + 2q - 1)(2p - 1)$ red ends. This calls for an odd number of red ends, but since each segment has two ends, the number of red ends is required to be even. Hence there must be at least one vertex where either case (a) or case (b) holds, and the theorem is true for both these cases.

3. Special values for the two color problem. It is known that $n(2, 4) = 4$ and $n(3, 3) = 6$. From Theorem 3 it follows that $n(3, 4) \leqslant 9$. This value shows that the strict inequality sign in Corollary 1 has to be used in some cases.

To assist in evaluating $n(3, 5)$ consider the 13 element field with the elements numbered from 0 to 12 inclusive and take each field element as a vertex in a graph. The cubic residues in this field are 1, 5, 8, and $12 \equiv -1 \pmod{13}$. If the difference of two vertices is a cubic residue, color the corresponding line segment red. If the difference is not a cubic residue, color the corresponding line segment blue. Since 1 and -1 are both cubic residues, the order of differencing is immaterial. In this chromatic graph there is no subset of three which is red interconnected and no subset of five which is blue interconnected. Hence $n(3, 5) > 13$.

Theorem 2 may now be used to remove the doubt as to the values $n(3, 4) \leqslant 9$ and $n(3, 5) \geqslant 14$. Since $n(2, 5) = 5$ and $n(3, 5) \leqslant n(2, 5) + n(3, 4)$, one readily obtains that $n(3, 4) = 9$ and $n(3, 5) = 14$.

It will now be shown that $n(4, 4) > 17$. Consider the field of 17 elements numbered from 0 to 16 inclusive. Let each element be a vertex. If two vertices have a numerical difference which is a quadratic residue of 17, the corresponding segment is to be colored red. Note that $-1 \equiv 16 \pmod{17}$, a square, and hence the order in which the subtraction is performed makes no difference. The other segments are to be colored blue.

Suppose that some four vertices are connected by segments of one color. Without loss of generality, one of these vertices may be considered to be the field element marked 0. Call the other three a, b, c. Then the six numbers a, b, c, $a - b$, $a - c$, and $b - c$ are either all residues or all non-residues. Since a is not the zero element in the field, multiplication by a^{-1} is permissible. If one sets $B = ba^{-1}$ and $C = ca^{-1}$, one can consider the new set of six numbers $1, B, C, 1 - B, 1 - C, B - C$. All of these must be quadratic residues of 17, further none of them can be zero. But the residues of 17 are 1, 2, 4, 8, 9, 13, 15 and 16 and it is impossible for all members of the set above to be residues. Theorem 3 may now be used to establish that $n(4, 4) = 18$.

On the basis of these values, an array may be set up giving the known values $n(k, m)$ as entries in the body of Table 1.

TABLE I

$m =$	1	2	3	4	5	6	7
$k = 1$	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7
3	1	3	6	9	14	19 or less	
4	1	4	9	18	31 or less		
5	1	5	14	31 or less			
6	1	6	19 or less				
7	1	7					

The authors desire to point out that deep combinatorial questions are encountered in attempting evaluations of the minimum numbers. In this paper proof or disproof of the possibility of a chromatic graph construction leans heavily upon finite field theory.

4. Three or more colors. For three colors, red, green and blue, the evaluation of $n(3, 3, 3)$ may be of interest.

THEOREM 4. $$n(3, 3, 3) = 17.$$

Proof. It will be convenient to give the proof in two parts. First, one may show that $n(3, 3, 3) > 16$ by use of a $2^4 = 16$ element field. In such a field the elements may be identified by the symbols $0, x, x^2, \ldots, x^{15} = 1$. To assist in identifying cubic residues, one may factor

$$x^{15} - 1 \equiv (x^4 - x - 1)(x^{11} + x^8 + x^7 + x^5 + x^3 + x^2 + x + 1) \pmod 2,$$

and take x as a root of $x^4 - x - 1 \equiv 0 \pmod 2$. Then the cubic residues of the field elements are x^3, $x^3 + x^2$, $x^3 + x$, $x^3 + x^2 + x + 1$, and 1. The field elements are taken as vertices and a chromatic graph is constructed as follows. If the difference of two vertex elements is a cubic residue, the corresponding segment is colored red. If the difference of two elements belongs to the first coset of the cubic residues in the multiplicative group of the non-zero field elements, the segment is colored green. If the difference belongs to the second coset, the segment is colored blue. These specifications define completely the chromatic graph. Again one notes that $-1 \equiv 1 \pmod 2$, and the order of differencing is immaterial.

Suppose now that a set of three vertices is completely interconnected by segments of the same color. Without any loss of generality, the color may be considered as red and the vertices may be considered as having been numbered 0, 1 and A, where A and $(1 - A)$ are both cubic residues. But a comparison with the list of cubic residues above shows that this situation cannot arise. Hence there is no set of three vertices interconnected by segments of the same color, and $n(3, 3, 3) > 16$.

For the second part, it must be proved that 17 is a sufficient number. In any chromatic graph with 3 colors and 17 vertices, select any one vertex and call this vertex a. At least six segments terminating at a must all have the same color. If in the subgraph consisting of a and the six farther ends there is no triangle of the original color, then all interconnections among the six farther ends must be of the other two colors. But this is now a case of $n(3, 3) = 6$, and there is an interconnected set of three here.

Remark. This example serves also to prove that $n(6, 3) > 16$. One imagines the graph to be examined by a man who is red-green color blind and hence sees only two colors, namely blue and red-green. Supposing that there is no set of three vertices interconnected in blue, then the existence of a set of six interconnected in red-green would furnish a red triangle or a green triangle, since $n(3, 3) = 6$.

THEOREM 5. $$41 < n(3, 3, 3, 3) \leqslant 66.$$

Proof. The upper bound given in the theorem follows at once in the same way as used in the proof of Theorem 4, i.e., reducing to the next lower case.

To establish a lower bound, one could consider chromatic graphs whose cosets of fourth power residues were employed in connection with fields in which the multiplicative group is of order $4k$. This suggests the primes 41, 53, 61 and the extension field of 49 elements. Unfortunately, -1 is not a quartic residue for the cases 53 and 61, and the order of subtraction of vertex elements would affect the question of what coset of residues the difference is in, and thus lead to a non-unique coloring scheme. For the 41 element field, the quartic residues are 1, 4, 10, 16, 18, 23, 25, 31, 37 and 40. There are no consecutives in this list, and thus a triangle whose vertices are 0, 1, and A cannot be interconnected by segments of the same color. Thus $n(3, 3, 3, 3) > 41$. The authors have not further narrowed the range on $n(3, 3, 3, 3)$.

5. Upper and Lower Bounds for $n(3^r)$. Let $t_r = n(3^r) = n(3, 3, \ldots, 3)$. The upper bounds used in Theorems 1, 4 and 5 can all be obtained by the use of

THEOREM 6. $$t_{r+1} \leqslant (r+1)(t_r - 1) + 2.$$

This theorem is easily proved by induction; and then it is trivial to establish, also by induction, that

$$t_{r+1} \leqslant 3(r+1)!.$$

A somewhat sharper inequality may be obtained, however, without any added difficulties. It has already been established that

$$t_r \leqslant [(r!)\, e] + 1, \qquad r = 2, 3, 4,$$

where $[M]$ means the greatest integer contained in M. Such a bound holds for all integers $r \geqslant 2$, for if it did not there would be a least integer, say $s+1$, for which the relation failed to hold. By Theorem 6,

$$t_{s+1} \leqslant (s+1)[(s!)\, e] + 2, \qquad s \geqslant 2.$$

But $[(s+1)!\, e] = (s+1)[(s!)\, e] + 1$, and hence

$$t_{s+1} \leqslant [(s+1)!\, e] + 1$$

and the stated upper bound follows.

For t_5, an upper bound of 327 is thus obtained. In order to determine a possible lower bound, one may generalize to some extent the previous arguments for lower bounds.

Assume a chromatic graph with p vertices (where the p vertices are to be thought of as field elements) and 5 colors. To ensure uniqueness of coloring, one would require that -1 be a 5th power residue. If p is odd, this means that 5 divides $\frac{1}{2}(p-1)$. If p is even, then $-1 \equiv 1 \pmod 2$ and the condition that -1 be a 5th power residue is trivially satisfied.

For $r = 5$, and restricting p to primes less than 327, one sees that 317, 313, 307, 293, 283 do not satisfy the divisibility condition and hence 311 and 281 (which do satisfy the divisibility condition) are the most likely cases. For the field with 311 elements, 168 and 169 are both quintic residues. Thus the argument that the triangle with vertices at 0, 1 and A cannot be interconnected by segments of the same color breaks down because of this unit difference in the quintic residues. The authors have not made a complete investigation of the case $p = 281$.

6. Relationship to Fermat's theorem and a certain trinomial congruence. Returning to the general case for a lower bound to $n(3^r)$, one requires that -1 be an rth power residue in a field of p elements. This requires a divisibility condition (already stated for 5), namely that r divides $\frac{1}{2}(p-1)$ when p is odd. One also requires that there be no solution of the trinomial congruence

$$1^r + x^r + y^r = 0$$

in the field. This congruence has been extensively studied by people interested in Fermat's theorem **(2; 3)**. Under the conditions assumed above a chromatic graph on p vertices with r colors can be so constructed that no monochromatic triangle appears.

Since an upper bound for $n(3^r)$ has been established by Theorem 6, it follows that when $p \geqslant n(3^r)$ and a chromatic graph has been constructed with r colors on the p field elements, a monochromatic triangle must appear. For such cases, then, the trinomial congruence is solvable.

The previous restriction that r divides $\frac{1}{2}(p-1)$ may be stated as r divides $p-1$ when r is odd. This may be interpreted as requiring the group of rth residues to have r cosets (including the original group of residues as one of its cosets). If r does not divide $p-1$, then in the field being an rth power is the same thing as being an mth power, where m is the greatest common divisor of r and $p-1$.

In view of the existence of the upper bound one obtains the result that there are only a finite number of finite fields for which the trinomial congruence is not solvable for given r **(3)**.

References

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