MA150 Algebra

homework 7

Problem 1 *Page 88-1*

Solution:

- $\phi(x) = |x|\ \forall x, y \in R^*$, $\phi(xy) = |xy| = |x| \cdot |y| = \phi(x)\phi(y)$。所以$\phi$是同态映射。 $Ker\phi = \{1, -1\}$，$\phi(G) = R^+$。
- $\phi(x) = ax$, $\forall x, y \in R^*$, $\phi(xy) = axy$。若$a = 1$，则ϕ是同态映射，$Ker\phi = \{1\}$,$\phi(G) = G$。否则不是。
- $\phi(x) = x^2$, $\forall x, y \in R^*$, $\phi(xy) = (xy)^2 = x^2y^2 = \phi(x)\phi(y)$，则$\phi$是同态映射。 $Ker\phi = \{1, -1\}, \phi(G) = R^+$
- $\phi(x) = -\frac{1}{x}$, $\forall x, y \in R^*$, $\phi(xy) = -\frac{1}{xy} \neq \phi(x)\phi(y)$，则$\phi$不是同态映射。

Problem 2 *Page 88-6*

$$\phi(a + bi) = (a + bi)^6$$

则$\forall x, y \in \mathbb{C}^*$, $x = a + bi$, $y = c + di$。

$$\begin{aligned}\phi(xy) &= (xy)^6 \\ &= x^6y^6 \\ &= \phi(x)\phi(y)\end{aligned}$$

$Ker(\phi) = \{x \in C^* | x^6 = 1\} = \bigcup_{x=0}^{5}\{\cos(\frac{x}{3}\pi) + i\sin(\frac{x}{3}\pi)\}$

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Problem 3 *Page 88-7*

Solution: 设ϕ为$\mathbb{Z} \to \mathbb{Z}_m$的同态映射，令$\phi(x) = \overline{x'}$。

则$\phi(x+y) = \phi(x) + \phi(y) = \overline{x'} + \overline{y'} = \overline{(x' + y')} \Rightarrow \forall a \in \mathbb{Z}$,$\phi(ax) = a\overline{x'}$

所以$\phi(xy) = x\overline{y'} = y\overline{x'} \Rightarrow xy' \equiv yx' (\mod m)$。

由于x, y的任意性，即有可能$\gcd(x, y) = 1$,所以必有$x' = xa$,$y' = ya$。

推出$\phi(x) = x\overline{a}(a = 0, 1, \ldots, m-1)$共$m$个映射满足要求。

Problem 4 *Page 89-16*

Solution: **充分性**：若$aKer\phi = bKer\phi$, 则$\phi(aKer\phi) = \{x \in G' | x = \phi(a)\} = \phi(bKer\phi) = \{x \in G' | x = \phi(b)\}$。所以$\phi(a) = \phi(b)$。

必要性：若$\phi(a) = \phi(b)$, $\forall x \in aKer\phi$, $x = at(t \in Ker\phi)$,

又$\phi(b^{-1}al) = \phi(b^{-1})\phi(a) = \phi(b^{-1})\phi(b) = \phi(e) = e'$。

所以$b^{-1}al \in Ker\phi \Rightarrow x \in bKer\phi$。

即$\forall x \in aKer\phi, x \in bKer\phi$，$aKer\phi = bKer\phi$。

Problem 5 *Page 89-18*

Solution:

- $\forall a \in HK, \exists h \in H, k \in K, a = hk$。

  $\phi(a) = \phi(h)\phi(k) = \phi(h) \in \phi(H)$, 则$a \in \phi^{-1}(\phi(H))$。

- $\forall a \in \phi^{-1}(\phi(H))$, 则$\phi(a) \in \phi(H)$,

  于是$\exists h \in H$, $\phi(h) = \phi(a) \Rightarrow \phi(h^{-1})\phi(a) = \phi(h^{-1})\phi(h) = e^{'}$, 所以$h^{-1}a \in K$。所以$h \cdot h^{-1}a \in HK$。

  综上，$\phi^{-1}(\phi(H)) = HK$。

Problem 6 *Page 89-19*

Solution: 根据题意, $G_1 = \langle a \rangle$，$G_2 = \langle b \rangle$。

$\Leftarrow$: (1) 令$\phi(a^x) = b^x$, 显然这是一个从 $G_1 \rightarrow G_2$ 的映射，

(2) 又$\phi(a^{xy}) = b^x = b^x \cdot b^y = \phi(a^x)\phi(a^y)$，所以这是一个同态映射。

(3) $n_2 | n_1 \Rightarrow n_1 \leq n_2$, 所以$\forall b^x \in G_2, \exists a^x \in G_1$, 使得$\phi(a^x) = b^x$，所以$\phi$是一个满同态。

$\Rightarrow$: 因为$G_1 \sim G_2$, 所以$G_1/Ker\phi \cong G_2$, 所以$\dfrac{n_1}{n_2} = |Ker\phi| \Rightarrow n_2 | n_1$。