Mathematical Logic

homework 4

Problem 1 *Can you derive the rule of contradiction from the modified contradiction?*

Solution:

1.Γ	$\neg\varphi$	ψ	*(premise)*
2.Γ	$\neg\varphi$	$\neg\psi$	*(premise)*
3.Γ	$\neg\varphi$	φ	*(modified contradiction by 1 and 2)*
4.Γ	φ	φ	*(assumption)*
5.	Γ	φ	*(modified contradiction by 3 and 4)*

Problem 2 *Prove:* (a) $\dfrac{\Gamma \quad \varphi}{\Gamma \quad \neg\neg\varphi}$ (b) $\dfrac{\Gamma \quad \neg\neg\varphi}{\Gamma \quad \varphi}$

Solution: (a)

1.Γ		φ	*(premise)*
2.Γ	$\neg\varphi$	φ	*(assumption)*
3.Γ	$\neg\varphi$	$\neg\varphi$	*(assumption)*
4.Γ	$\neg\varphi$	$\neg\neg\varphi$	*(modified contradiction by 2 and 3)*
5.Γ	$\neg\neg\varphi$	$\neg\neg\varphi$	*(assumption)*
6.Γ		$\neg\neg\varphi$	*(case analysis by 4 and 5)*

Author(s): 于峥

(b)

1.Γ		$\neg\neg\varphi$	*(premise)*
2.Γ	$\neg\varphi$	$\neg\neg\varphi$	*(assumption)*
3.Γ	$\neg\varphi$	$\neg\varphi$	*(assumption)*
4.Γ	$\neg\varphi$	φ	*(modified contradiction by 2 and 3)*
5.Γ	φ	φ	*(assumption)*
6.Γ		φ	*(case analysis by 4 and 5)*

Problem 3 *Is the following derivable?*

$$\frac{}{\Gamma\ \exists x\varphi\ \forall x\varphi}$$

Solution: *We let*

$$\mathfrak{I} = \{\{0^{\mathfrak{I}}, 1^{\mathfrak{I}}\}, R^{\mathfrak{I}}, \beta\}$$

and $0^{\mathfrak{I}} \in R^{\mathfrak{I}}$,$1^{\mathfrak{I}} \notin R^{\mathfrak{I}}$. *if* $\varphi := Rx$.

Then $\mathfrak{I}\frac{0}{v_0}\frac{1}{v_1} \models \Gamma \wedge \exists x\varphi$. $\mathfrak{I}\frac{0}{v_0}\frac{1}{v_1} \not\models \forall x\varphi$.

So $\{\Gamma\ \exists x\varphi\} \not\models \forall x\varphi$, *it is not correct, so it is not derivable.*

Problem 4 *Let* $S = \{R\}$ *with unary relation symbol* R.

$$\Phi = \{\exists x Rx\} \cup \{\neg Ry | \textit{for every variable } y\}$$

Prove that:

– Φ *is consistent.*

– *There is no term* $t \in T^S$, *with* $\Phi \vdash Rt$.

Solution: *(1) We construct*

$$\mathfrak{I} = \{\{0^{\mathfrak{I}}, 1^{\mathfrak{I}}\}, R^{\mathfrak{I}}, \beta\}$$

where $\beta(v_i) = 1^{\mathfrak{I}}$, *and* $0^{\mathfrak{I}} \in R^{\mathfrak{I}}$, $1^{\mathfrak{I}} \notin R^{\mathfrak{I}}$。

(i) because $0^{\mathfrak{I}} \in R^{\mathfrak{I}}$, *so* $\exists x Rx$ *is satisfiable.*

(ii) and $\beta(v_i) = 1 \notin R^{\mathfrak{I}}$, *so* $\{\neg Ry | \text{for every variable } y\}$ *is satisfiable.*

Φ *is satisfiable* $\Rightarrow \Phi$ *is consistent (**Lemma 2.5**).*

(2) We construct

$$\mathfrak{I} = \{\{1^{\mathfrak{I}}\}, R^{\mathfrak{I}}, \beta\}$$

Where $\beta(v_i) = 1^{\mathfrak{I}}$, *and* $1^{\mathfrak{I}} \notin R^{\mathfrak{I}}$. *Because* $\{1\}$ *is universe.*

So no term $t \in T^S$, *with* $\Phi \vdash Rt$. *It is satisfiable.*

Problem 5 *Let* $S = \{R\}$ *with unary relation symbol* R. *and*

$$\Phi = \{Rx \vee Ry\}$$

Prove that:

– Φ *is consistent.*

– $\Phi \not\vdash Rx$, $\Phi \not\vdash Ry$.

\- $\mathfrak{I}^{\Phi} \not\models \Phi$

Solution: *(1) Because* $\Phi \not\vdash Rx \wedge Ry$, *so* Φ *is not inconsistent.*

(2) Construct

$$\mathfrak{I} = \{\{0^{\mathfrak{I}}, 1^{\mathfrak{I}}\}, R^{\mathfrak{I}}, \beta\}\}$$

and $0^{\mathfrak{I}} \in R^{\mathfrak{I}}$, $1^{\mathfrak{I}} \notin R^{\mathfrak{I}}$.

Then $\mathfrak{I}\frac{0}{x}\frac{1}{y} \models \Phi$, *and* $\mathfrak{I}\frac{0}{x}\frac{1}{y} \not\models Ry$.

$\Phi \not\models Ry \Rightarrow \Phi \vdash Ry$ ***is not correct***, *so* $\Phi \not\vdash Ry$.

same as $\Phi \not\vdash Rx$.

(3) Because $\Phi \not\vdash Rx$, *so does not exist* $\bar{t} \in R^{\mathfrak{I}^{\Phi}}$. *then* $\mathfrak{I}^{\Phi} \not\models Rx$ *and* $\mathfrak{I}^{\Phi} \not\models Ry \Rightarrow \mathfrak{I}^{\Phi} \not\models Rx \vee Ry$