CS 217 – Algorithm Design and Analysis

no code

Handed out on Friday, 2020-06-05
First submission and questions due on Thursday, 2020-06-12
You will receive feedback from the TA.
Final submission due on Thursday, 2020-06-19

7 Farkas Lemma and LP Duality

7.1 Different Versions of Farkas Lemma

In the following, let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$, and let $\mathbf{x} = (x_1, \ldots, x_n)^T$ be a column vector of n variables and $\mathbf{y} = (y_1, \ldots, y_m)$ be a row vector of m variables.

Exercise 1. Show that the three versions of Farkas Lemma presented in class are all equivalent (I actually did not present the third version in class):

$$(\neg \exists \mathbf{x} : A\mathbf{x} \leq \mathbf{b}) \iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T A = \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) . \tag{1}$$

$$(\neg \exists \mathbf{x} \geq \mathbf{0} : A\mathbf{x} \leq \mathbf{b}) \iff (\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T A \geq \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) . \tag{2}$$

$$(\neg \exists \mathbf{x} \geq \mathbf{0} : A\mathbf{x} = \mathbf{b}) \iff (\exists \mathbf{y} \quad : \mathbf{y}^T A \geq \mathbf{0}, \mathbf{y}^T \mathbf{b} < 0) . \tag{3}$$

Note that the direction "$\Longleftarrow$" is easy in each case. We will show the "$\Longrightarrow$" of (1) in class using a technique called *Fourier-Motzkin Elimination.* This exercise is actually not that hard. The hardest part is keeping track of what you want to prove and what you can assume.

Solution First, we prove each $\Longleftarrow$ of the statements is true. For(1), we just need to show that $\exists \mathbf{y} \geq \mathbf{0} : \mathbf{y}^T A = \mathbf{0}$, $\mathbf{y}^T \mathbf{b} < 0$ and $\exists \mathbf{x} : A\mathbf{x} \leq \mathbf{b}$ can't hold simultaneously. Because of $\mathbf{y}^T A\mathbf{x} = (\mathbf{y}^T A)\mathbf{x} = 0$ and $\mathbf{y}^T A\mathbf{x} = \mathbf{y}^T(A\mathbf{x}) \leq$

$\mathbf{y}^T b < 0$ contradicts, so we prove (1). Similarly, $\mathbf{y}^T A\mathbf{x} = \mathbf{y}^T(A\mathbf{x}) \le \mathbf{y}^T b < 0$ and $\mathbf{y}^T A\mathbf{x} = (\mathbf{y}^T A)\mathbf{x} \ge 0$ contradicts,so we prove (2). And $\mathbf{y}^T A\mathbf{x} = \mathbf{y}^T(A\mathbf{x}) = \mathbf{y}^T b < 0$ and $\mathbf{y}^T A\mathbf{x} = (\mathbf{y}^T A)\mathbf{x} \ge 0$ contradicts, so we prove (3).
Next, we need to show the $\Longrightarrow$ of each statements are equivalent.
(1)$\Longrightarrow$(2): if $\neg\exists \mathbf{x} \ge \mathbf{0} : A\mathbf{x} \le \mathbf{b}$ then, we have $\neg\exists \mathbf{x} : \begin{bmatrix} A \\ -I \end{bmatrix} \mathbf{x} \le \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$, then we let $A' = \begin{bmatrix} A \\ -I \end{bmatrix}, b' = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$, according to version (1), then there exist $\mathbf{y}' = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \ge \mathbf{0}$ satisfies $\mathbf{y}'^T A' = \mathbf{0}, \mathbf{y}'^T b' < 0$, then we can conclude $\mathbf{y}_1 \ge 0, \mathbf{y}_1^T A \ge 0, \mathbf{y}_1^T b < 0$, then because we find the $\mathbf{y}_1$, so (2) holds.
(2)$\Longrightarrow$(3): if $\neg\exists \mathbf{x} \ge \mathbf{0} : A\mathbf{x} = \mathbf{b}$, then we have $\neg\exists \mathbf{x} : \begin{bmatrix} A \\ -A \end{bmatrix} \mathbf{x} \le \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}$, let $A' = \begin{bmatrix} A \\ -A \end{bmatrix}$, $b' = \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \end{bmatrix}$, then there exist $\mathbf{y}' = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \ge \mathbf{0}$ satisfies $\mathbf{y}'^T A' \ge \mathbf{0}, \mathbf{y}'^T b' < 0$, then we can conclude $\mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2, \mathbf{y}_1^T A \ge 0, \mathbf{y}_1^T b < 0$, then because we find the $\mathbf{y} = \mathbf{y}_1 - \mathbf{y}_2$, so (3) holds.
(3)$\Longrightarrow$(1): we will prove the contraposition of (1) from the contraposition of (3). $\mathbf{y} \ge \mathbf{0}, \mathbf{y}^T A = \mathbf{0} \iff y^T[A, -A, I] \ge \mathbf{0}$, let $A' = [A, -A, I]$ and according to (3), there exits $\mathbf{x}' = [\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3]^T$ satisfies $A'\mathbf{x}' = \mathbf{b}$ with $\mathbf{x}' \ge 0$. Let $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$, we have $A\mathbf{x} \le \mathbf{b}$, then (1) holds.

7.2 A Linear Program for, well, for what?

Let $G = (V, E)$ be a directed graph, $s, t \in V$, and $c : E \to \mathbf{R}^+$ be a cost function. We want to find an $s-t$-flow f of value 1. Every edge e generates cost $f(e) \cdot c(e)$, and we want to minimize the overall cost. There are no capacity constraints. We can easily write this as a linear program MCF (Minimum Cost Flow):

$$\text{MCF}(G, s, t, c): \quad \begin{array}{ll} \text{minimize} & \sum_{e \in E} c(e) f(e) \\ & \\ \text{subject to} & \sum_{v \in V} f(v, t) = 1 \\ & \sum_{u \in V} f(u, v) - \sum_{w \in V} f(v, w) = 0 \quad \forall\, v \in V \setminus \{s, t\} \\ & \\ & f(e) \ge 0 \;\forall\, e \in E \end{array}$$

Note that we have m variables, one variable $f(e)$ for each edge e. The first constraint says that the value of the flow should be 1. The other constraints say that the inflow at v should equal the outflow.

Exercise 2. Let d be the shortest path distance from s to t in the directed graph G, where distance means sum of the $c(e)$ along the path. Show that $\text{opt}(MCF) = d$. **Hint.** Make sure you show both $\leq$ and $\geq$.

Solution

Proof of $opt(MCF) \leq d$ We can take out the shortest path, let $f(e) = 1$ if the edge is on the shortest path, and $f(e) = 0$ otherwise. Apparently it's a feasible solution.

Proof of $opt(MCF) \geq d$ Consider the residual network. We can arbitrarily choose a path at first, satisfying the restrictions of the MCF problem. To minimize the target value, we can try to eliminate all the circle C with $\Sigma_{e \in C} c(e) f(e) < 0$. By adding the circle to the old path, we've got a new path with smaller value.

Now suppose that we choose a shortest path first, and we get a residual network. There is no circle with $\Sigma_{e \in C} c(e) f(e) < 0$ because there is no path shorter than the shortest path. So the final answer won't be less than d.

In conclusion, $opt(MCF) = d$.

Exercise 3. Write down the dual of MCF. This will be a maximization problem. Don't use any matrix notation.

Solution

$$
\begin{array}{ll}
\text{maximize} & z_t \quad v \in V \\
\text{subject to} & z_v - z_u \leq c(u,v) \quad (u,v) \in E \\
 & z_v \in \mathbb{R} \quad \forall v \in V
\end{array}
$$

Exercise 4. Interpret the dual. Show that it is the LP formulation of a "natural" maximization problem on G.

Solution This is a system of difference constraints. We can build a graph $G^{'}$ with the same structure as G and have $dis(u,v) = c(u,v)$ as the length of an edge.

So to maximize z_t, we need to solve the shortest path distance from s to t in $G^{'}$. It's the same to solve the shortest path distance from s to t in G where distance means sum of the $c(e)$ along the path.

Exercise 5. Describe an optimal solution of the dual program.

Solution The optimal solution of the dual program is the shortest path distance from s to t in the graph G where distance means sum of the $c(e)$ along the path.