Schur's Theorem and Related Topics in Ramsey Theory

Summer Lynne Kisner
Advisor: Dr. Andrés E. Caicedo

Boise State University

March 4, 2013

Acknowledgments

- Advisor: Dr. Caicedo
- Committee: Dr. Scheepers and Dr. Teitler
- Mathematics department and Graduate program
- Classmates and friends
- Family and Dustin

"Ramsey theory ... but only the finite stuff!"

Overview

- Introduction to Ramsey Theory
- Definitions and Notation
- Ramsey's Theorem
- Schur's Theorem
- The Origin of Schur's Theorem
- Generalizations of Schur's Theorem

Ramsey Theory

Complete disorder is impossible
-Theodore S. Motzkin

Ramsey Theory: the study of preservation of properties under set partitions

Some basic notation

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
K_n	a complete graph on n vertices
$s(r)$	the Schur number of r
$R(k, l)$	the Ramsey number for k and l
$R(a_1, ..., a_r)$	the Ramsey number on r colors
$w(k, r)$	the Van der Waerden number for k and r

- Throughout the presentation, I will mostly be talking about the positive integers. I will denote the set $\{a, a+1, ..., b-1, b\}$, where $a < b$ are integers, as $[a, b]$.
- Sometimes, I will use numbers, such as $0, 1, 2, ...$ for various "colors". Simply because writing 0, 1, and 2 is a lot shorter to write than red, blue, and yellow.

Some basic notation

Definition

A *graph* $G = (V, E)$ is a set V of points, called *vertices*, and a set E of unordered pairs of vertices, called *edges*.

Definition

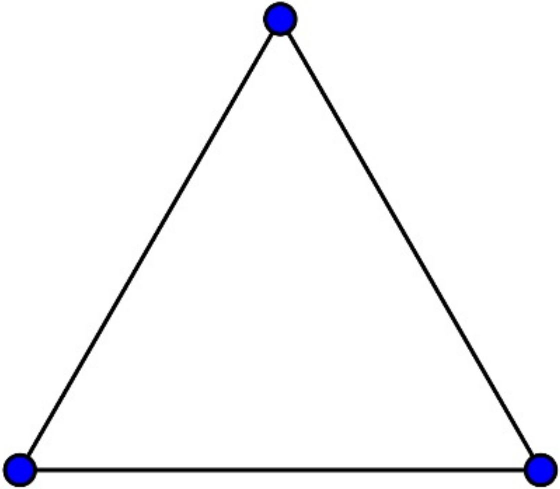
A *subgraph* $G' = (V', E')$ of a graph $G = (V, E)$ is a graph such that $V' \subseteq V$ and $E' \subseteq E$.

Definition

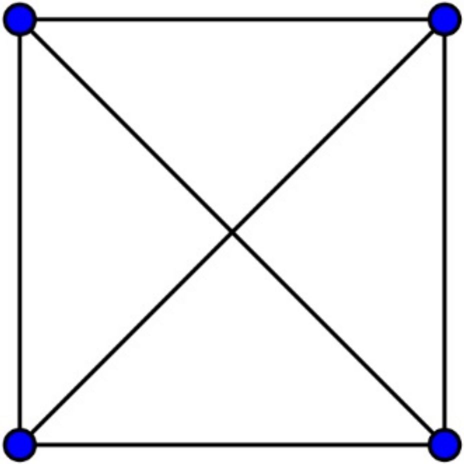
A *complete graph on* n *vertices*, denoted K_n, is a graph on n vertices, with the property that every pair of vertices is connected by an edge. If V is the set of vertices, we also write K_V for this graph.

Complete Graph, K_2

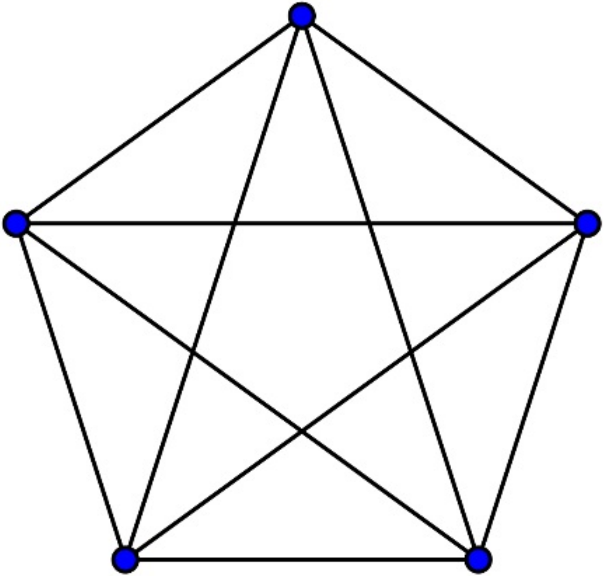
Complete Graph, K_3



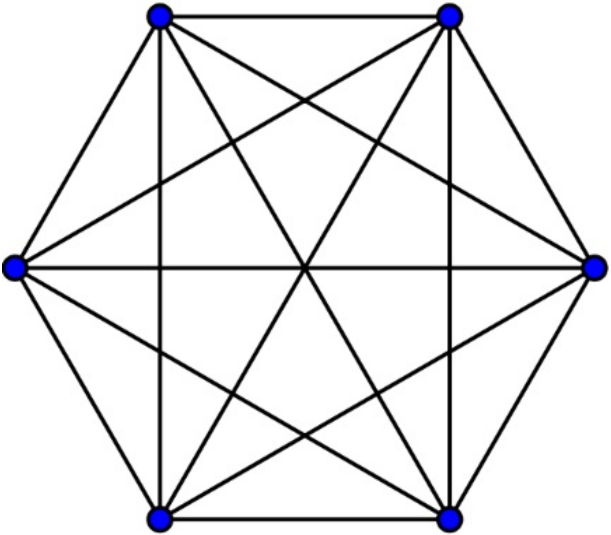
Complete Graph, K_4



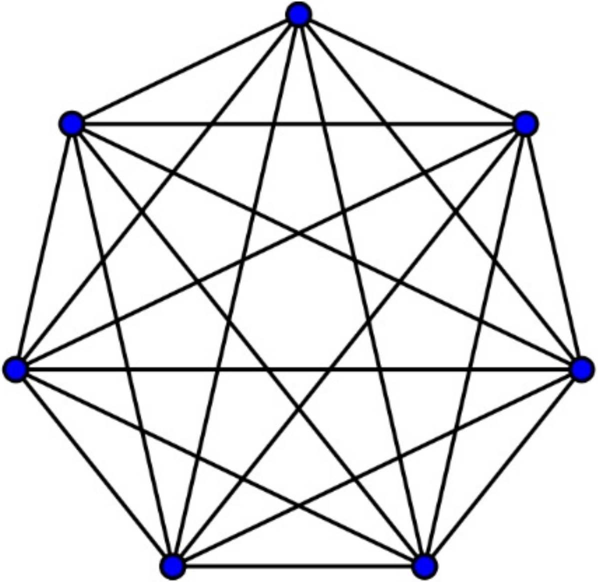
Complete Graph, K_5



Complete Graph, K_6



Complete Graph, K_7



Some more basic notation

Definition

An *edge-coloring* of a graph is an assignment of a color to each edge of the graph. A graph which has been edge-colored is called a *monochromatic graph* if all of its edges are the same color.

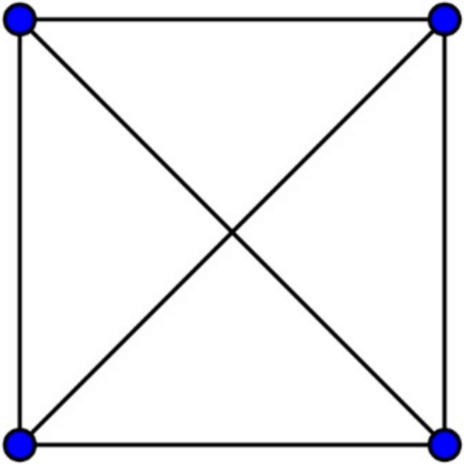
Definition

An *r-coloring* of a set S is a function $\chi : S \to C$, where $|C| = r$. We also refer to χ as a *C-coloring*.

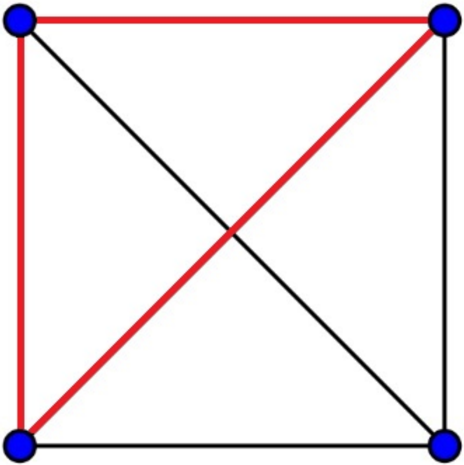
Definition

A coloring χ is *monochromatic* on a set S if χ is constant on S.

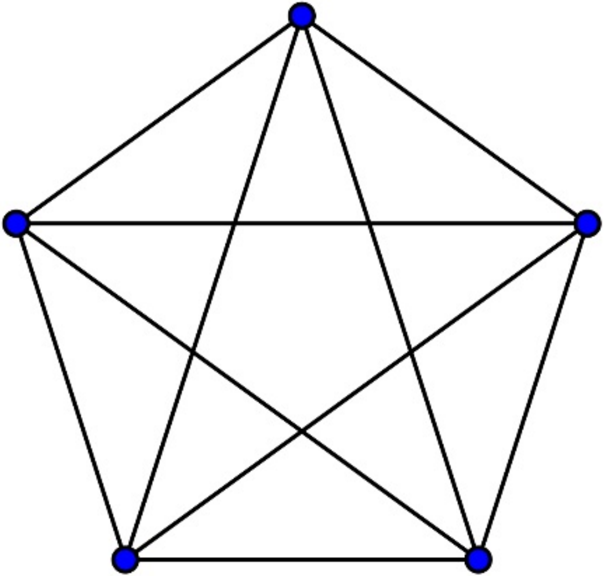
Let's color a monochromatic K_3 inside K_4



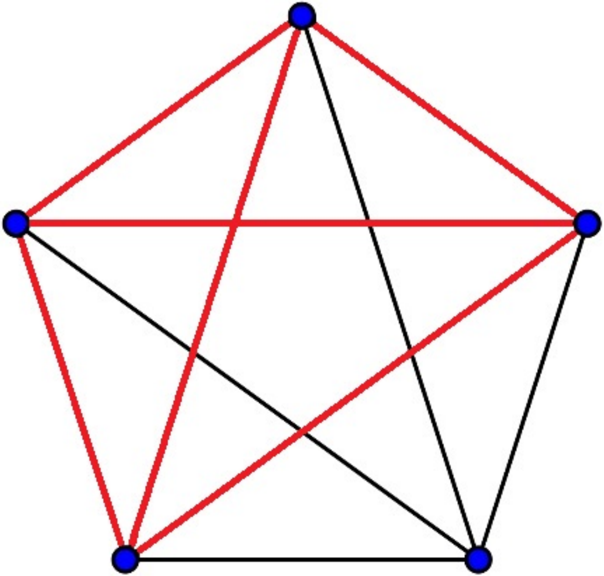
Monochromatic triangle in K_4



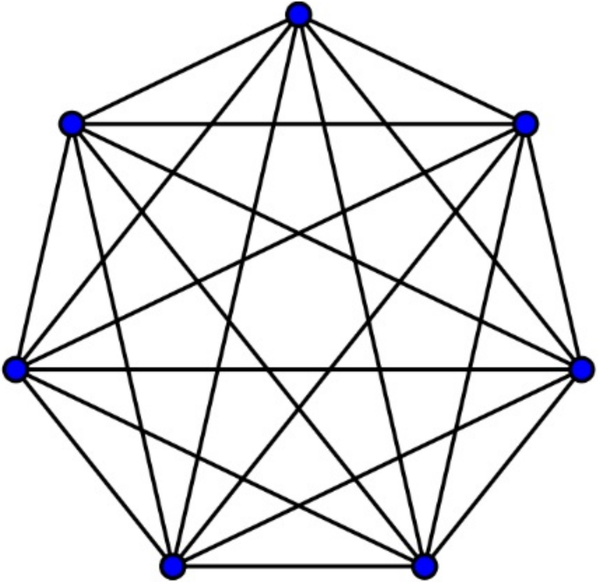
Let's color a monochromatic K_4 inside K_5



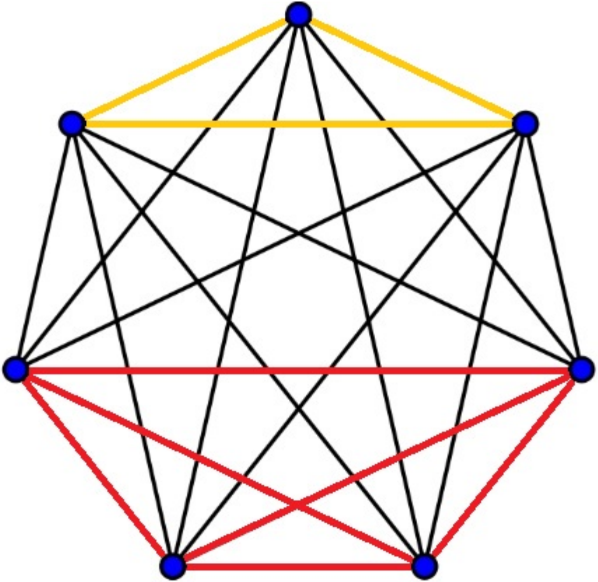
Monochromatic K_4 in K_5



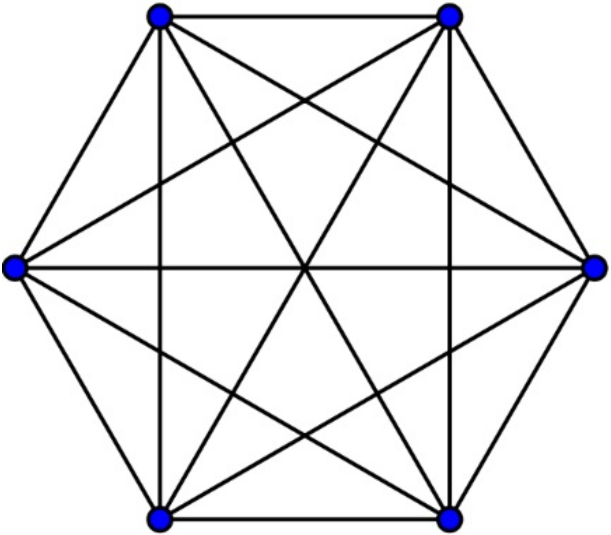
Let's color monochromatic K_4 and K_3 inside K_7



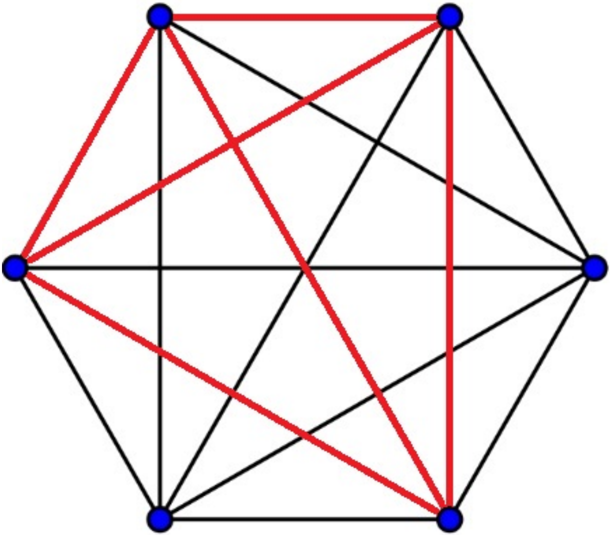
Monochromatic K_4 and K_3 in K_7



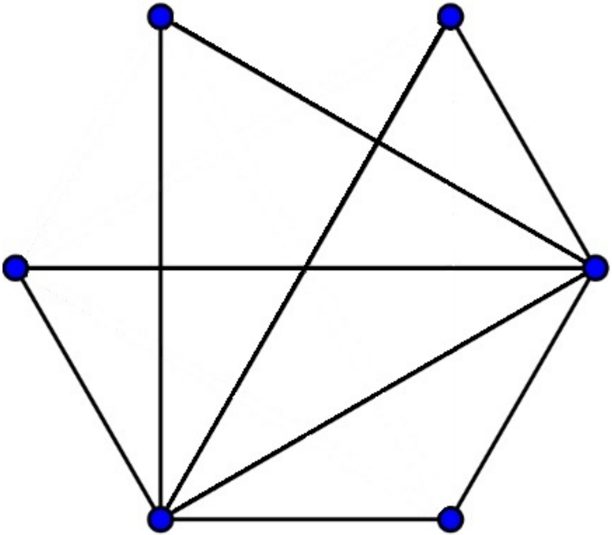
Sometimes, it's easier to color and not-color...



So instead of doing this,



We will do this.



The Pigeonhole Principle

Theorem

If a set of more than mn elements is partitioned into n sets, then some set contains more than m elements.

Ramsey's Theorem

Theorem (Ramsey, 1928)

Let $k, l \geq 2$. There exists a positive integer R such that every edge-coloring of K_R, with the colors red and blue, admits either a red K_k subgraph or a blue K_l subgraph.

We call the smallest number that satisfies this theorem the *Ramsey number of k and l* and denote it $R(k, l)$.

Ramsey's Theorem for more than 2 colors

We let $R_r(3)$ denote $R(3, 3, ..., 3)$, where we are using r colors.

Theorem

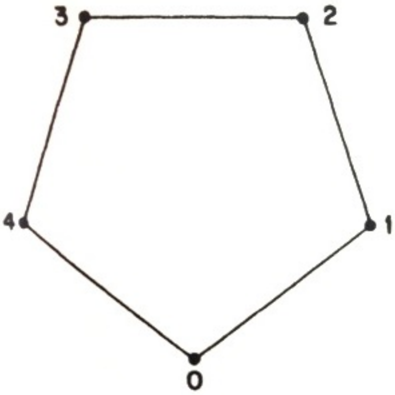
For $r \geq 1$, $R_r(3) \leq 3r!$.

Ramsey's Theorem

The typical proof of a Ramsey number is a counterexample proving a lower bound, and an argument to show an upper bound.

Let's take $R(3,3)$ for example.

Example of $R(3,3) > 5$



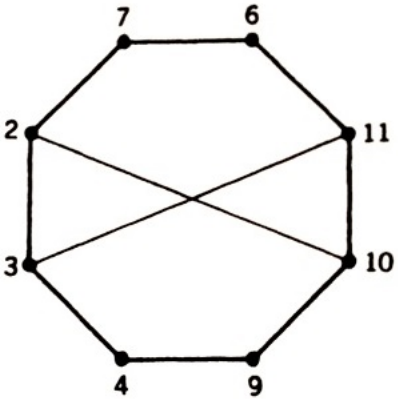
To show that $R(3,3) = 6$, we must now show that $R(3,3) \leq 6$.

Theorem

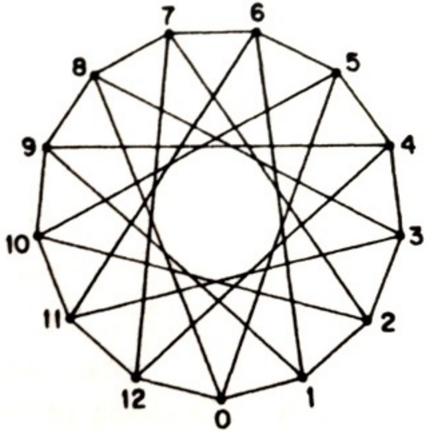
For $r \geq 1$, $R_r(3) \leq 3r!$.

From this, we see that $R(3,3) \leq 6$.

Example of $R(3,4) > 8$



Example of $R(3,5) > 13$



Example of $R(4,4) > 17$

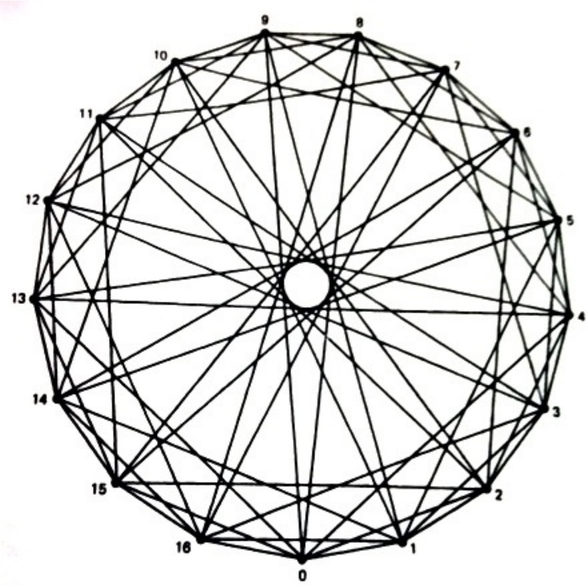


Table : Known Ramsey Numbers, $R(r, s)$

$r \backslash s$	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10
3	1	3	6	9	14	18	23	28	36	
4	1	4	9	18	25					
5	1	5	14	25						
6	1	6	18							
7	1	7	23							
8	1	8	28							
9	1	9	36							
10	1	10								

Suppose aliens invade the Earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

-Paul Erdős

Some Known Bounds for Ramsey Numbers, $R(r,s)$

$r \backslash s$	3	4	5	6	7	8	9	10
3								40–43
4				36–41	49–61	57–84	73–115	92–149
5			43–49	58–87	80–143	101–216	126–316	144–442
6		36–41	57–87	102–165	113–298	132–495	169–780	179–1171
7		49–61	80–143	113–298	205–540	217–1031	241–1713	289–2826
8		56–84	101–216	132–495	217–1031	282–1870	317–3583	331–6090
9		73–115	126–316	169–780	241–1713	317–3583	565–6588	581–12677
10	40–43	92–149	144–442	179–1171	289–2826	331–6090	581–12677	798–23556

Schur's Theorem

Theorem (Schur's Theorem)

For any $r \geq 1$, there exists a positive integer s such that, for any r-coloring of $[1, s]$, there exists a monochromatic solution to $x + y = z$.

We call the smallest number that satisfies this theorem the *Schur number* of r, and denote it $s(r)$.

We call the triple $\{x, y, z\}$ a *monochromatic Schur triple*.

Proof of Schur's Theorem

Proof

By Ramsey's theorem there exists an integer $n = R_r(3)$ *such that for any* r*-coloring* χ *of the edges of* K_n *there is a monochromatic triangle.*

Proof of Schur's Theorem

Proof

By Ramsey's theorem there exists an integer $n = R_r(3)$ such that for any r-coloring χ of the edges of K_n there is a monochromatic triangle.

We claim that $s \leq n - 1$. To see this, suppose an r-coloring of $[1, n-1]$ is given.

Proof of Schur's Theorem

Proof

By Ramsey's theorem there exists an integer $n = R_r(3)$ *such that for any* r*-coloring* χ *of the edges of* K_n *there is a monochromatic triangle.*

We claim that $s \leq n - 1$*. To see this, suppose an* r*-coloring of* $[1, n-1]$ *is given.*

We can construct a coloring of K_n *yielding a monochromatic solution to* $x + y = z$ *as follows. Label each of the vertices of the graph* K_n *with the numbers 1 through* n*. Then assign to the edge connecting a pair of vertices (the color of) the positive difference between them.*

Proof of Schur's Theorem

Proof

By Ramsey's theorem there exists an integer $n = R_r(3)$ such that for any r-coloring χ of the edges of K_n there is a monochromatic triangle.

We claim that $s \leq n-1$. To see this, suppose an r-coloring of $[1, n-1]$ is given.

We can construct a coloring of K_n yielding a monochromatic solution to $x + y = z$ as follows. Label each of the vertices of the graph K_n with the numbers 1 through n. Then assign to the edge connecting a pair of vertices (the color of) the positive difference between them.

Now, by Ramsey's theorem, we must have a triangle such that all the edges are assigned the same color.

Proof of Schur's Theorem

Proof

By Ramsey's theorem there exists an integer $n = R_r(3)$ such that for any r-coloring χ of the edges of K_n there is a monochromatic triangle.

We claim that $s \leq n-1$. To see this, suppose an r-coloring of $[1, n-1]$ is given.

We can construct a coloring of K_n yielding a monochromatic solution to $x + y = z$ as follows. Label each of the vertices of the graph K_n with the numbers 1 through n. Then assign to the edge connecting a pair of vertices (the color of) the positive difference between them.

Now, by Ramsey's theorem, we must have a triangle such that all the edges are assigned the same color.

Let the vertices of this triangle be named $a < b < c$, so we know that $b - a$, $c - b$, and $c - a$ are all the same color.

Proof of Schur's Theorem

Proof

Finally, let $x = b - a$, $y = c - b$ *and* $z = c - a$, *and note that* $x + y = (b - a) + (c - b) = c - a = z$.

Proof of Schur's Theorem

Proof.

Finally, let $x = b - a$, $y = c - b$ and $z = c - a$, and note that
$x + y = (b - a) + (c - b) = c - a = z$.

Thus, since x, y, and z are the same color, we have found a monochromatic solution to $x + y = z$. $\square$

From the proof of Schur's theorem, using Ramsey's theorem, we see:

Theorem

For $r \geq 1$, $s(r) \leq R_r(3) - 1$.

From the proof of Schur's theorem, using Ramsey's theorem, we see:

Theorem

For $r \geq 1$, $s(r) \leq R_r(3) - 1$.

We also have a lower bound:

Theorem

For $r \geq 1$, $s(r) \geq \frac{3^r - 1}{2}$.

Table : Known Schur Numbers

r	$s(r)$
1	2
2	5
3	14
4	45

s(2)

Theorem

$s(2) = 5$.

s(2)

Theorem

$s(2) = 5$.

Proof

First, we show that $s(2) \geq 5$.

s(2)

Theorem

$s(2) = 5$.

Proof

First, we show that $s(2) \geq 5$. *Suppose that* 1 *is colored red.*

s(2)

Theorem

$s(2) = 5$.

Proof

First, we show that $s(2) \geq 5$. *Suppose that* 1 *is colored red.*

If 2 *is colored red, then we have the triple* $\{1, 1, 2\}$, *thus let* 2 *be colored blue.*

s(2)

Theorem

$s(2) = 5$.

Proof

First, we show that $s(2) \geq 5$. *Suppose that* 1 *is colored red.*

If 2 *is colored red, then we have the triple* $\{1, 1, 2\}$, *thus let* 2 *be colored blue.*

If 4 *is colored blue, then we have the triple* $\{2, 2, 4\}$, *so let* 4 *be colored red.*

s(2)

Theorem

$s(2) = 5$.

Proof

First, we show that $s(2) \geq 5$. *Suppose that* 1 *is colored red.*

If 2 *is colored red, then we have the triple* $\{1, 1, 2\}$, *thus let* 2 *be colored blue.*

If 4 *is colored blue, then we have the triple* $\{2, 2, 4\}$, *so let* 4 *be colored red.*

Now, if 3 *is colored red, then we have the triple* $\{1, 3, 4\}$, *thus let* 3 *be colored blue.*

s(2)

Theorem

$s(2) = 5$.

Proof

First, we show that $s(2) \geq 5$. *Suppose that* 1 *is colored red.*

If 2 *is colored red, then we have the triple* $\{1, 1, 2\}$, *thus let* 2 *be colored blue.*

If 4 *is colored blue, then we have the triple* $\{2, 2, 4\}$, *so let* 4 *be colored red.*

Now, if 3 *is colored red, then we have the triple* $\{1, 3, 4\}$, *thus let* 3 *be colored blue.*

We have obtained a 2*-coloring* f *of* $[1, 4]$ *without monochromatic Schur triples. This means that* $s(2) \geq 5$.

s(2)

Theorem

$s(2) = 5$.

Proof

Now, we show that $s(2) \leq 5$.

s(2)

Theorem

$s(2) = 5$.

Proof

Now, we show that $s(2) \leq 5$.

Consider any 2-coloring χ *of* $[1, 5]$.

s(2)

Theorem

$s(2) = 5$.

Proof

Now, we show that $s(2) \leq 5$.

Consider any 2-coloring χ *of* $[1, 5]$.

If $\chi \upharpoonright [1, 4] \neq f$*, we are done because there must be a monochromatic triangle.*

s(2)

Theorem

$s(2) = 5$.

Proof.

Now, we show that $s(2) \leq 5$.

Consider any 2-coloring χ of $[1, 5]$.

If $\chi \restriction [1, 4] \neq f$, we are done because there must be a monochromatic triangle.

If $\chi \restriction [1, 4] = f$, we cannot assign 5 a color without creating a monochromatic Schur triple, so $s(2) \leq 5$. □

Example of $s(3) > 13$

Red	1	4	10	13	
Blue	2	3	11	12	
Yellow	5	6	7	8	9

Here is an example of a coloring that demonstrates $s(4) > 44$.

Red	1	3	5	15	17	19	26	28	40	42	44
Blue	2	7	8	18	21	24	27	33	37	38	43
Yellow	4	6	13	20	22	23	25	30	32	39	41
Green	9	10	11	12	14	16	29	31	34	35	36

Origin of Schur's Theorem

Schur's original proof, which preceded Ramsey's theorem, was published in 1917. It was introduced as a lemma in a paper meant to improve some results of Dickson. Dickson had written a paper about localized versions of Fermat's Last Theorem.

Origin of Schur's Theorem

Schur's original proof, which preceded Ramsey's theorem, was published in 1917. It was introduced as a lemma in a paper meant to improve some results of Dickson. Dickson had written a paper about localized versions of Fermat's Last Theorem. In the original proof, a bound can be extracted.

Theorem

For $r \geq 1$, $s(r) \leq \lceil r!e \rceil$.

Generalizations of Schur's Theorem: Rado's Theorem

Schur's Theorem naturally leads us to wonder which equations admit monochromatic solutions under finite colorings of the nonzero integers. Schur's theorem corresponds to the equation

$$x + y + z = 0.$$

Richard Rado, a student of Schur's, turned his attention to the general linear setting, considering arbitrary coefficients and an arbitrary number of variables. He determined exactly when equations of the form

$$\sum_{i=1}^{k} c_i x_i = 0$$

are guaranteed to have monochromatic solutions under any finite coloring of the positive integers.

Generalizations of Schur's Theorem: Rado's Theorem

Let $\mathcal{S}$ be a linear homogeneous equation and let $r \geq 1$. We say that $\mathcal{S}$ is r-regular if for every r-coloring of $\mathbb{Z}$ there is a monochromatic solution to $\mathcal{S}$. If $\mathcal{S}$ is r-regular for all $r \geq 1$, we say that $\mathcal{S}$ is *regular*.

Theorem (Rado's Single Equation Theorem)

Let $k \geq 2$, and let $c_i \in \mathbb{Z}, 1 \leq i \leq k$, be constants. Then

$$\sum_{i=1}^{k} c_i x_i = 0$$

is regular if and only if there exists a nonempty $D \subseteq \{c_i : 1 \leq i \leq k\}$ such that

$$\sum_{d \in D} d = 0.$$

Example

Consider the equation

$$x + 2y - 4z = 0.$$

Note that no subset of the coefficients sums to 0.

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 2 \\ c_3 &= -4 \\ c_1 + c_3 &= -3 \\ c_2 + c_3 &= -2 \\ c_1 + c_2 &= 3 \\ c_1 + c_2 + c_3 &= -1 \end{aligned}$$

We will now suppose we have a monochromatic solution to the equation $x + 2y = 4z$, and arrive at a contradiction.

Example

We define a coloring $\chi : \mathbb{Z} \to [1,4]$ given by $\chi(5^k \cdot j) = j \pmod 5$.
Since we are assuming we have a monochromatic solution to the equation $x + 2y = 4z$, $\chi(x) = \chi(y) = \chi(z) = j$.

Example

We define a coloring $\chi : \mathbb{Z} \to [1, 4]$ given by $\chi(5^k \cdot j) = j \pmod 5$.
Since we are assuming we have a monochromatic solution to the equation $x + 2y = 4z$, $\chi(x) = \chi(y) = \chi(z) = j$.
Then, we have the following equation:

$$5^{k_1}(5 \cdot x_1 + j) + 2 \cdot 5^{k_2}(5 \cdot y_1 + j) - 4 \cdot 5^{k_3}(5 \cdot z_1 + j) = 0$$

for some x_1, y_1, z_1 and some k_1, k_2, k_3.

Example

We define a coloring $\chi : \mathbb{Z} \to [1,4]$ given by $\chi(5^k \cdot j) = j \pmod 5$.
Since we are assuming we have a monochromatic solution to the equation $x + 2y = 4z$, $\chi(x) = \chi(y) = \chi(z) = j$.
Then, we have the following equation:

$$5^{k_1}(5 \cdot x_1 + j) + 2 \cdot 5^{k_2}(5 \cdot y_1 + j) - 4 \cdot 5^{k_3}(5 \cdot z_1 + j) = 0$$

for some x_1, y_1, z_1 and some k_1, k_2, k_3.
We divide by the largest possible power of 5 to get:

$$5^{m_1}(5 \cdot x_1 + j) + 2 \cdot 5^{m_2}(5 \cdot y_1 + j) - 4 \cdot 5^{m_3}(5 \cdot z_1 + j) = 0$$

Example

We define a coloring $\chi : \mathbb{Z} \to [1, 4]$ given by $\chi(5^k \cdot j) = j \pmod 5$.
Since we are assuming we have a monochromatic solution to the equation $x + 2y = 4z$, $\chi(x) = \chi(y) = \chi(z) = j$.
Then, we have the following equation:

$$5^{k_1}(5 \cdot x_1 + j) + 2 \cdot 5^{k_2}(5 \cdot y_1 + j) - 4 \cdot 5^{k_3}(5 \cdot z_1 + j) = 0$$

for some x_1, y_1, z_1 and some k_1, k_2, k_3.
We divide by the largest possible power of 5 to get:

$$5^{m_1}(5 \cdot x_1 + j) + 2 \cdot 5^{m_2}(5 \cdot y_1 + j) - 4 \cdot 5^{m_3}(5 \cdot z_1 + j) = 0$$

We take this equation modulo 5 to get:

$$5^{m_1}(j) + 2 \cdot 5^{m_2}(j) - 4 \cdot 5^{m_3}(j) = 0 \pmod 5$$

Example

We define a coloring $\chi : \mathbb{Z} \to [1, 4]$ given by $\chi(5^k \cdot j) = j \pmod 5$.
Since we are assuming we have a monochromatic solution to the equation $x + 2y = 4z$, $\chi(x) = \chi(y) = \chi(z) = j$.
Then, we have the following equation:

$$5^{k_1}(5 \cdot x_1 + j) + 2 \cdot 5^{k_2}(5 \cdot y_1 + j) - 4 \cdot 5^{k_3}(5 \cdot z_1 + j) = 0$$

for some x_1, y_1, z_1 and some k_1, k_2, k_3.
We divide by the largest possible power of 5 to get:

$$5^{m_1}(5 \cdot x_1 + j) + 2 \cdot 5^{m_2}(5 \cdot y_1 + j) - 4 \cdot 5^{m_3}(5 \cdot z_1 + j) = 0$$

We take this equation modulo 5 to get:

$$5^{m_1}(j) + 2 \cdot 5^{m_2}(j) - 4 \cdot 5^{m_3}(j) = 0 \pmod 5$$

And we divide by j:

$$5^{m_1} + 2 \cdot 5^{m_2} - 4 \cdot 5^{m_3} = 0 \pmod 5$$

Example

We define a coloring $\chi : \mathbb{Z} \to [1, 4]$ given by $\chi(5^k \cdot j) = j \pmod 5$.
Since we are assuming we have a monochromatic solution to the equation $x + 2y = 4z$, $\chi(x) = \chi(y) = \chi(z) = j$.
Then, we have the following equation:

$$5^{k_1}(5 \cdot x_1 + j) + 2 \cdot 5^{k_2}(5 \cdot y_1 + j) - 4 \cdot 5^{k_3}(5 \cdot z_1 + j) = 0$$

for some x_1, y_1, z_1 and some k_1, k_2, k_3.
We divide by the largest possible power of 5 to get:

$$5^{m_1}(5 \cdot x_1 + j) + 2 \cdot 5^{m_2}(5 \cdot y_1 + j) - 4 \cdot 5^{m_3}(5 \cdot z_1 + j) = 0$$

We take this equation modulo 5 to get:

$$5^{m_1}(j) + 2 \cdot 5^{m_2}(j) - 4 \cdot 5^{m_3}(j) = 0 \pmod 5$$

And we divide by j:

$$5^{m_1} + 2 \cdot 5^{m_2} - 4 \cdot 5^{m_3} = 0 \pmod 5$$

But we know that at least one of m_1, m_2, m_3 is 0. Now it just takes an analysis by cases to see that this is impossible.

Another Example

E.G. Straus showed that 3-coloring $[1, 54]$ would admit a monochromatic solution to the equation $x + y = 3z$, and in fact that coloring $[1, 53]$ would not suffice.

Here is the example that he found. Let $[1, 53]$ be colored as follows:

$$R = \{x : x \equiv 1 \pmod 3\} \cup \{x : x \equiv 3 \pmod 9\}$$
$$B = \{x : x \equiv 2 \pmod 3\} \cup \{9, 27, 36\}$$
$$Y = \{6, 15, 18, 24, 33, 42, 45, 51\}$$

This is a 3-coloring of $[1, 53]$ that does not admit any monochromatic solution to the equation $x + y = 3z$. This means that $x + y = 3z$ is not 3-regular in $[1, 53]$ but it is 3-regular in $\mathbb{Z}^+$. It can be shown that this equation is not 4-regular by the argument used in the proof of Rado's Theorem.

Van der Waerden's Theorem

Theorem (Van der Waerden)

There exists a least positive integer $w(k, r)$ such that any r-coloring of $[1, w(k, r)]$ admits a monochromatic arithmetic progression of length k.

Generalizations of Schur's Theorem

Coloring elements of the symmetric group S_n and searching for quadruples of the form $\{x, y, xy, yx\}$ (S_n is non-abelian, so xy and yx could be different).

Theorem (McCutcheon)

Let $r \in \mathbb{N}$. There exists $n = n(r)$ such that for any r-coloring of the alternating group A_n, there is a monochromatic Schur quadruple.

Generalizations of Schur's Theorem

In 1979, Neil Hindman asked the following:

Question

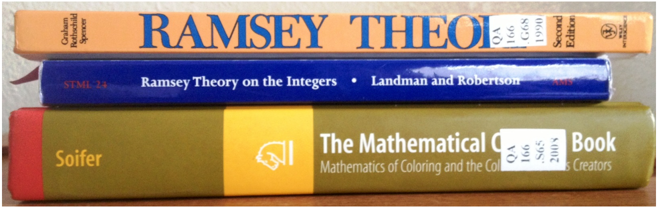
Suppose that the positive integers are partitioned into finitely many pieces,

$$\mathbb{N} = A_1 \cup A_2 \cup \ldots \cup A_n.$$

Must there be integers x, y such that $x, y, x + y$ are xy belong to the same A_i?

This question is still open.

References



Bruce M. Landman and Aaron Robertson. **Ramsey Theory on the Integers** Student Mathematical Library, 24. American Mathematical Society, Providence, RI, 2004. MR2020361 (2004h:05125)

Ronald L. Graham, Bruce L. Rothschild, and Joel H. Spencer. **Ramsey Theory**, Second edition. Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 1990. MR1044995 (90m:05003)

To see the thesis, and even links to most of the references, please see my thesis website

https://sites.google.com/site/slkthesis/

Thank you!