Mathematical Logic

homework 6

Problem 1 *Let $\Phi \subseteq L^S$ be finite, and let $\in L^S$ with $\Phi \vdash \varphi$. Note that a proof might use formulas built on any symbol in S.*

Define $S_0 \in S$ to be the set of symbols that occur in Φ and φ. Then there is a proof for $\Phi \vdash \varphi$ such that every formula occurs in the proof is an S_0-formula.

Solution: *By theorem 1.2, we have a S-interpretation $\mathfrak{I}^{\Phi}$.*

$$\mathfrak{I}^{\Phi} \models \varphi \Longleftrightarrow \Phi \vdash \varphi$$

same as S_0-interpretation $\mathfrak{I}_0^{\Phi}$, let $\mathfrak{I}_0^{\Phi}$ has same interpretation on function and relation with $\mathfrak{I}^{\Phi}$. By Coincidenc lemma

$$\mathfrak{I}^{\Phi} \models \varphi \Longrightarrow \mathfrak{I}_0^{\Phi} \models \varphi \Longrightarrow \Phi \vdash \varphi$$

Problem 2 *Assume that for every set A there is a well order $\leqslant \subseteq A \times A$. Prove Zorn's Lemma.*

Solution: *First we proof in any partially ordered set $(S, \leqslant)$ there is a maximal chain(a chain C for which no $C \cup \{s\}$ is a chain for any s in S/C:*

For one chain C_0, if for any s in S/C_0 that $C_0 \cup \{s\}$ is not a chain, then C_0 is maximal chain. Otherwise, let $C_1 = C_0 \cup \{s\}$, $C_0 \cup \{s\}$ is a chain. Similarly we can define $C_2, C_3, \cdots, C_n$. and define $M = \{C_0, C_1, C_2, \cdots, C_n\}$, Because $C_0 \subseteq C_1 \subseteq C_2 \cdots \subseteq C_n$, so M is well-ordering. and M is transfinite induction object by transfinite induction.

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So C_n *is the upperbound of* M *and the maximal chain of* S*, and* C_n *is also well-ordering. Let* a *is the upperbound of* C_n*, if* $\exists b, a \leq b$*, then* $C_n \cup \{b\} \subseteq C_n$*. hence* $a = b$*. So* a *is the maximal element of* S*.*