Catalan number

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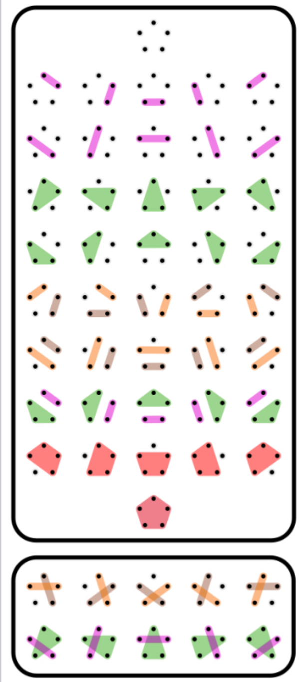
For names of numbers in Catalan, see List of numbers in various languages § Occitano-Romance.

In combinatorial mathematics, the **Catalan numbers** form a sequence of natural numbers that occur in various counting problems, often involving recursively-defined objects. They are named after the Belgian mathematician Eugène Charles Catalan (1814–1894).



The $C_5 = 42$ noncrossing partitions of a 5-element set (below, the other 10 of the 52 partitions)

The *n*th Catalan number is given directly in terms of binomial coefficients by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!} = \prod_{k=2}^{n} \frac{n+k}{k} \qquad \text{for } n \geq 0.$$

The first Catalan numbers for n = 0, 1, 2, 3, ... are

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452, ... (sequence A000108 in the OEIS).

Contents [hide]

Properties [edit]

An alternative expression for C_n is

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n} \qquad \text{for } n \geq 0,$$

which is equivalent to the expression given above because $\binom{2n}{n+1} = \frac{n}{n+1}\binom{2n}{n}$. This shows that C_n is an integer, which is not immediately obvious from the first formula given. This expression forms the basis for a proof of the correctness of the formula.

The Catalan numbers satisfy the recurrence relations[1]

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{i=0}^{n} C_i\, C_{n-i} \quad \text{for } n \geq 0,$$

$$\sum_{i_1+\cdots+i_m=n,\, i_1,\ldots,i_m\geq 0} C_{i_1}\cdots C_{i_m} = \begin{cases} \dfrac{m(n+1)(n+2)\cdots(n+m/2-1)}{2(n+m/2+2)(n+m/2+3)\cdots(n+m)} C_{n+m/2}, & m \text{ even} \\[2ex] \dfrac{m(n+1)(n+2)\cdots(n+(m-1)/2)}{(n+(m+3)/2)(n+(m+3)/2+1)\cdots(n+m)} C_{n+(m-1)/2}, & m \text{ odd}, \end{cases}$$

and

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \frac{2(2n+1)}{n+2} C_n.$$

Asymptotically, the Catalan numbers grow as

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$$

in the sense that the quotient of the nth Catalan number and the expression on the right tends towards 1 as n approaches infinity. This can be proved by using Stirling's approximation for n! or via generating functions.

The only Catalan numbers C_n that are odd are those for which $n = 2^k - 1$; all others are even. The only prime Catalan numbers are $C_2 = 2$ and $C_3 = 5$.[2]

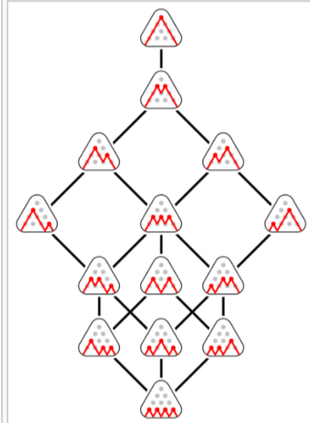
The Catalan numbers have an integral representation

$$C_n = \int_0^4 x^n \rho(x)\, dx,$$

where $\rho(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}$. This means that the Catalan numbers are a solution of a version of the Hausdorff moment problem.[3]

Applications in combinatorics [edit]

There are many counting problems in combinatorics whose solution is given by the Catalan numbers. The book *Enumerative Combinatorics: Volume 2* by combinatorialist Richard P. Stanley contains a set of exercises which describe 66 different interpretations of the Catalan numbers. Following are some examples, with illustrations of the cases $C_3 = 5$ and $C_4 = 14$.



Lattice of the 14 Dyck words of length 8 – (and) interpreted as *up* and *down*

- C_n is the number of Dyck words[4] of length $2n$. A Dyck word is a string consisting of n X's and n Y's such that no initial segment of the string has more Y's than X's. For example, the following are the Dyck words of length 6:

  XXXYYY XYXXYY XYXYXY XXYYXY XXYXYY.

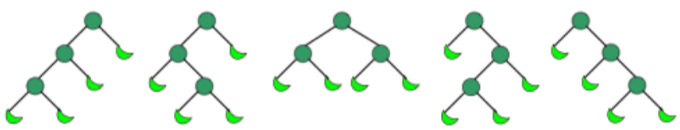
- Re-interpreting the symbol X as an open parenthesis and Y as a close parenthesis, C_n counts the number of expressions containing n pairs of parentheses which are correctly matched:

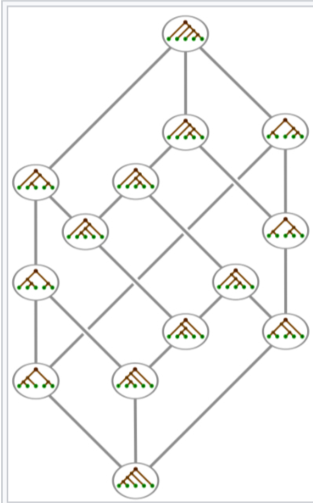
  ((())) ()(()) ()()() (())() (()())

- C_n is the number of different ways n + 1 factors can be completely parenthesized (or the number of ways of associating n applications of a binary operator). For n = 3, for example, we have the following five different parenthesizations of four factors:

  ((ab)c)d (a(bc))d (ab)(cd) a((bc)d) a(b(cd))

- Successive applications of a binary operator can be represented in terms of a full binary tree. (A rooted binary tree is *full* if every vertex has either two children or no children.) It follows that C_n is the number of full binary trees with n + 1 leaves:

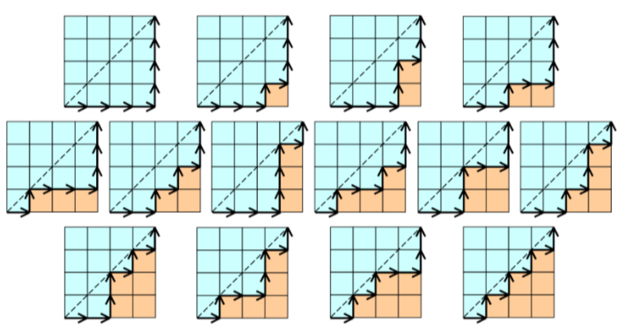




The associahedron of order 4 with the C_4=14 full binary trees with 5 leaves

- C_n is the number of non-isomorphic ordered trees with n + 1 vertices. (An ordered tree is a rooted tree in which the children of each vertex are given a fixed left-to-right order.)[5]
- C_n is the number of monotonic lattice paths along the edges of a grid with $n \times n$ square cells, which do not pass above the diagonal. A monotonic path is one which starts in the lower left corner, finishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards. Counting such paths is equivalent to counting Dyck words: X stands for "move right" and Y stands for "move up".

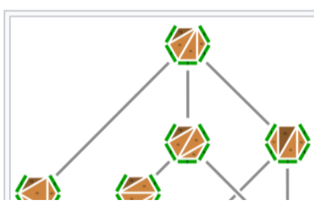
The following diagrams show the case n = 4:

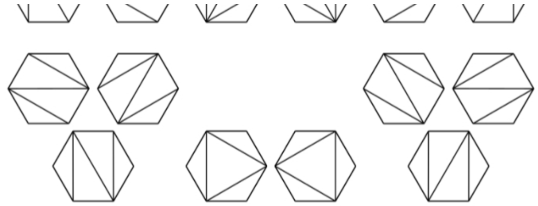


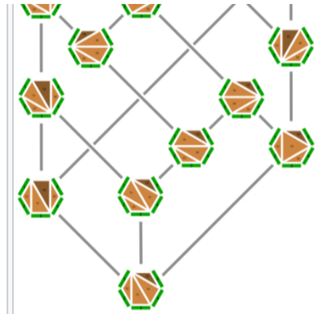
This can be succinctly represented by listing the Catalan elements by column height:[6]

[0,0,0,0] [0,0,0,1] [0,0,0,2] [0,0,1,1]
[0,1,1,1] [0,0,1,2] [0,0,0,3] [0,1,1,2] [0,0,2,2] [0,0,1,3]
[0,0,2,3] [0,1,1,3] [0,1,2,2] [0,1,2,3]

- A convex polygon with n + 2 sides can be cut into triangles by connecting vertices with non-crossing line segments (a form of polygon triangulation). The number of triangles formed is n and the number of different ways that this can be achieved is C_n. The following hexagons illustrate the case n = 4:

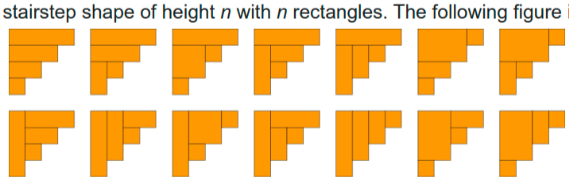






The triangles correspond to internal nodes of the binary trees.

- C_n is the number of stack-sortable permutations of {1, ..., *n*}. A permutation *w* is called stack-sortable if *S*(*w*) = (1, ..., *n*), where *S*(*w*) is defined recursively as follows: write *w* = *unv* where *n* is the largest element in *w* and *u* and *v* are shorter sequences, and set *S*(*w*) = *S*(*u*)*S*(*v*)*n*, with *S* being the identity for one-element sequences.
- C_n is the number of permutations of {1, ..., *n*} that avoid the permutation pattern 123 (or, alternatively, any of the other patterns of length 3); that is, the number of permutations with no three-term increasing subsequence. For *n* = 3, these permutations are 132, 213, 231, 312 and 321. For *n* = 4, they are 1432, 2143, 2413, 2431, 3142, 3214, 3241, 3412, 3421, 4132, 4213, 4231, 4312 and 4321.
- C_n is the number of noncrossing partitions of the set {1, ..., *n*}. *A fortiori*, C_n never exceeds the *n*th Bell number. C_n is also the number of noncrossing partitions of the set {1, ..., 2*n*} in which every block is of size 2. The conjunction of these two facts may be used in a proof by mathematical induction that all of the *free* cumulants of degree more than 2 of the Wigner semicircle law are zero. This law is important in free probability theory and the theory of random matrices.
- C_n is the number of ways to tile a stairstep shape of height *n* with *n* rectangles. The following figure illustrates the case *n* = 4:



- C_n is the number of ways to form a "mountain range" with *n* upstrokes and *n* downstrokes that all stay above a horizontal line. The mountain range interpretation is that the mountains will never go below the horizon.

$n=0$:	*	1 way
$n=1$:	/\	1 way
$n=2$:	/\ /\/\, / \	2 ways
$n=3$:	/\ /\ /\ /\/\ / \ /\/\/\, /\/ \, / \/\, / \, / \	5 ways

Mountain Ranges

- C_n is the number of standard Young tableaux whose diagram is a 2-by-*n* rectangle. In other words, it is the number of ways the numbers 1, 2, ..., 2*n* can be arranged in a 2-by-*n* rectangle so that each row and each column is increasing. As such, the formula can be derived as a special case of the hook-length formula.
- C_n is the number of ways that the vertices of a convex 2*n*-gon can be paired so that the line segments joining paired vertices do not intersect. This is precisely the condition that guarantees that the paired edges can be identified (sewn together) to form a closed surface of genus zero (a topological 2-sphere).
- C_n is the number of semiorders on *n* unlabeled items.[7]
- In chemical engineering C_{n-1} is the number of possible separation sequences which can separate a mixture of *n* components.[8]

Proof of the formula [edit]

There are several ways of explaining why the formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

solves the combinatorial problems listed above. The first proof below uses a generating function. The other proofs are examples of bijective proofs; they involve literally counting a collection of some kind of object to arrive at the correct formula.

First proof [edit]

We first observe that all of the combinatorial problems listed above satisfy Segner's[9] recurrence relation

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{i=0}^{n} C_i\, C_{n-i} \quad \text{for } n \geq 0.$$

For example, every Dyck word *w* of length ≥ 2 can be written in a unique way in the form

$w = Xw_1Yw_2$

with (possibly empty) Dyck words w_1 and w_2.

The generating function for the Catalan numbers is defined by

$$c(x) = \sum_{n=0}^{\infty} C_n x^n.$$

The recurrence relation given above can then be summarized in generating function form by the relation

$$c(x) = 1 + xc(x)^2;$$

in other words, this equation follows from the recurrence relation by expanding both sides into power series. On the one hand, the recurrence relation uniquely determines the Catalan numbers; on the other hand, the generating function relation can be algebraically solved to yield

$$c(x) = \frac{1 \mp \sqrt{1-4x}}{2x} = \frac{2}{1 \pm \sqrt{1-4x}}.$$

Choosing the minus sign (in the first expression), the fraction has a power series at 0 so its coefficients must therefore be the Catalan numbers. This solution satisfies

$$\lim_{x \to 0^+} c(x) = C_0 = 1$$

The other solution, with the plus sign, has a pole at 0 so it cannot be a valid solution for $c(x)$.

The square root term can be expanded as a power series using the identity

$$\sqrt{1+y} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} y^n = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{4^n(2n-1)} \binom{2n}{n} y^n = 1 + \frac{1}{2}y - \frac{1}{8}y^2 + \cdots.$$

This is a special case of Newton's generalized binomial theorem; as with the general theorem, it can be proved by computing derivatives to produce its Taylor series. Setting $y = -4x$ and substituting this power series into the expression for $c(x)$ and shifting the summation index n by 1, the expansion simplifies to

$$c(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{x^n}{n+1}.$$

The coefficients are now the desired formula for C_n.

Another way to get $c(x)$ is to solve for $xc(x)$ and observe that $\int_0^x t^n \, dt$ appears in each term of the power series.

Second proof [edit]

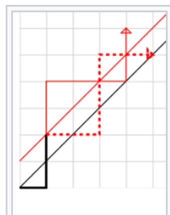


Figure 1. The invalid portion of the path (in solid red color) is flipped. Bad paths reach $(n-1, n+1)$ instead of (n,n).

This proof depends on a trick known as André's reflection method, which was originally used in connection with Bertrand's ballot theorem. (The reflection principle has been widely attributed to Désiré André, but his method did not actually use reflections; and the reflection method is a variation due to Aebly and Mirimanoff.[10]) We count the paths which start and end on the diagonal of the $n \times n$ grid. All such paths have n rightward and n upward steps. Since we can choose which of the $2n$ steps are upward (or, equivalently, rightward) ones, there are $\binom{2n}{n}$ total monotonic paths of this type. A *bad* path will cross the main diagonal and touch the next higher (*fatal*) diagonal (depicted red in the illustration). We flip the portion of the path occurring after that touch about that fatal diagonal, as illustrated; this geometric operation amounts to interchanging all the rightward and upward steps after that touch. In the section of the path that is not reflected, there is one more upward step than rightward steps, so the remaining section of the bad path has one more rightward than upward step (because it ends on the main diagonal). When this portion of the path is reflected, it will also have one more upward step than rightward steps. Since there are still $2n$ steps, there must now be $n + 1$ upward steps and $n - 1$ rightward steps. So, instead of reaching the target (n,n), all bad paths (after the portion of the path is reflected) will end in location $(n - 1, n + 1)$. As any monotonic path in the $(n - 1) \times (n + 1)$ grid must meet the fatal diagonal, this reflection process sets up a bijection between the bad paths of the original grid and the monotonic paths of this new grid because the reflection process is reversible. The number of bad paths is therefore,

$$\binom{n-1+n+1}{n-1} = \binom{2n}{n-1} = \binom{2n}{n+1}$$

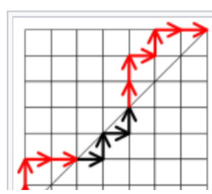
and the number of Catalan paths (i.e., good paths) is obtained by removing the number of bad paths from the total number of monotonic paths of the original grid,

$$C_n = \binom{2n}{n} - \binom{2n}{n+1}.$$

In terms of Dyck words, we start with a (non-Dyck) sequence of n X's and n Y's and interchange all X's and Y's after the first Y that violates the Dyck condition. At that first Y, there are $k + 1$ Y's and k X's for some k between 1 and $n - 1$.

Third proof [edit]

The following bijective proof, while being more involved than the previous one, provides a more natural explanation for the term $n + 1$ appearing in the denominator of the formula for C_n. A generalized version of this proof can be found in a paper of Rukavicka Josef (2011).[11]



Suppose we are given a monotonic path, which may happen to cross the diagonal. The **exceedance** of the path is defined to be the number of *vertical* edges which lie *above* the diagonal. For example, in Figure 2, the edges lying above the diagonal are marked in red, so the exceedance of the path is 5.

Now, if we are given a monotonic path whose exceedance is not zero, then we may apply the following algorithm to construct a new path whose exceedance is one less than the one we started with.

- Starting from the bottom left, follow the path until it first travels above the diagonal.

- Starting from the bottom left, follow the path until it first travels above the diagonal.
- Continue to follow the path until it *touches* the diagonal again. Denote by X the first such edge that is reached.
- Swap the portion of the path occurring before X with the portion occurring after X.

Figure 2. A path with exceedance 5.

The following example should make this clearer. In Figure 3, the black dot indicates the point where the path first crosses the diagonal. The black edge is X, and we swap the red portion with the green portion to make a new path, shown in the second diagram.

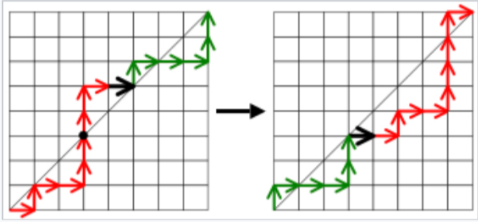


Figure 3. The green and red portions are being exchanged.

The exceedance has dropped from three to two. In fact, the algorithm will cause the exceedance to decrease by one, for any path that we feed it, because the first vertical step starting on the diagonal (at the point marked with a black dot) is the unique vertical edge that under the operation passes from above the diagonal to below it; all other vertical edges stay on the same side of the diagonal.

It is also not difficult to see that this process is *reversible*: given any path P whose exceedance is less than n, there is exactly one path which yields P when the algorithm is applied to it. Indeed, the (black) edge X, which originally was the first horizontal step ending on the diagonal, has become the *last* horizontal step *starting* on the diagonal.

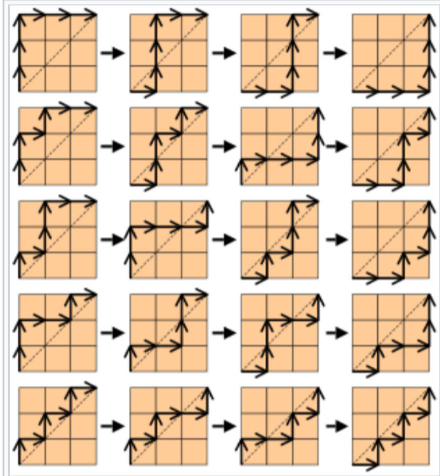


Figure 4. All monotonic paths in a 3×3 grid, illustrating the exceedance-decreasing algorithm.

This implies that the number of paths of exceedance n is equal to the number of paths of exceedance $n - 1$, which is equal to the number of paths of exceedance $n - 2$, and so on, down to zero. In other words, we have split up the set of *all* monotonic paths into $n + 1$ equally sized classes, corresponding to the possible exceedances between 0 and n. Since there are

$$\binom{2n}{n}$$

monotonic paths, we obtain the desired formula

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Figure 4 illustrates the situation for $n = 3$. Each of the 20 possible monotonic paths appears somewhere in the table. The first column shows all paths of exceedance three, which lie entirely above the diagonal. The columns to the right show the result of successive applications of the algorithm, with the exceedance decreasing one unit at a time. There are five rows, that is, $C_3 = 5$.

Fourth proof [edit]

This proof uses the triangulation definition of Catalan numbers to establish a relation between C_n and C_{n+1}. Given a polygon P with $n + 2$ sides, first mark one of its sides as the base. If P is then triangulated, we can further choose and orient one of its $2n + 1$ edges. There are $(4n + 2)C_n$ such decorated triangulations. Now given a polygon Q with $n + 3$ sides, again mark one of its sides as the base. If Q is triangulated, we can further mark one of the sides other than the base side. There are $(n + 2)C_{n+1}$ such decorated triangulations. Then there is a simple bijection between these two kinds of decorated triangulations: We can either collapse the triangle in Q whose side is marked, or in reverse expand the oriented edge in P to a triangle and mark its new side. Thus

$$(4n+2)C_n = (n+2)C_{n+1}.$$

The binomial formula for C_n follows immediately from this relation and the initial condition $C_1 = 1$.

Fifth proof [edit]

This proof is based on the Dyck words interpretation of the Catalan numbers, so C_n is the number of ways to correctly match n pairs of brackets. We denote a (possibly empty) *correct* string with c and its inverse (where "[" and "]" are exchanged) with c^+. Since any c can be uniquely decomposed into c = [c_1] c_2, summing over the possible spots to place the closing bracket immediately gives the recursive definition

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{i=0}^{n} C_i\, C_{n-i} \quad \text{for } n \geq 0.$$

Now let b stand for a *balanced* string of length $2n$—that is, containing an equal number of "[" and "]"—and $B_n = \binom{2n}{n} = d_n C_n$ with some factor $d_n \geq 1$. As above, any balanced string can be uniquely decomposed into either [c] b or] c^+ [b, so

$$B_{n+1} = 2\sum_{i=0}^{n} B_i C_{n-i}.$$

Also, any incorrect balanced string starts with c], so

Also, any incorrect balanced string starts with c_i], so

$$B_{n+1} - C_{n+1} = \sum_{i=0}^{n} \binom{2i+1}{i} C_{n-i} = \sum_{i=0}^{n} \frac{2i+1}{i+1} B_i C_{n-i}.$$

Subtracting the above equations and using $B_i = d_i\, C_i$ gives

$$C_{n+1} = 2\sum_{i=0}^{n} d_i C_i C_{n-i} - \sum_{i=0}^{n} \frac{2i+1}{i+1} d_i C_i C_{n-i} = \sum_{i=0}^{n} \frac{d_i}{i+1} C_i C_{n-i}.$$

Comparing coefficients with the original recursion formula for C_n gives $d_i = i + 1$, so

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Sixth proof [edit]

This simple proof[12] is also based on the Dyck words interpretation of the Catalan numbers but uses the beautiful Cycle Lemma of Dvoretzky and Motzkin.[13] Call a sequence of X's and Y's *dominating* if, reading from left to right, the *imbalance* is always positive, that is, the number of X's is always strictly greater than the number of Y's. The Cycle Lemma asserts that any sequence of m X's and n Y's, where $m > n$, has precisely $m - n$ dominating cyclic permutations. To see this, just arrange the given sequence of $m + n$ X's and Y's in a circle and repeatedly remove adjacent pairs XY until only $m - n$ X's remain. Each of these X's was the start of a dominating cyclic permutation before anything was removed. In particular, when $m = n + 1$, there is exactly one dominating cyclic permutation. Removing the leading X from it (a dominating sequence must begin with X) leaves a Dyck sequence. Since there are $C_n = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}$ distinct cycles of $n + 1$ X's and n Y's, each of which corresponds to exactly one Dyck sequence, C_n counts Dyck sequences.

Hankel matrix [edit]

The $n \times n$ Hankel matrix whose (i, j) entry is the Catalan number C_{i+j-2} has determinant 1, regardless of the value of n. For example, for $n = 4$ we have

$$\det\begin{bmatrix} 1 & 1 & 2 & 5 \\ 1 & 2 & 5 & 14 \\ 2 & 5 & 14 & 42 \\ 5 & 14 & 42 & 132 \end{bmatrix} = 1.$$

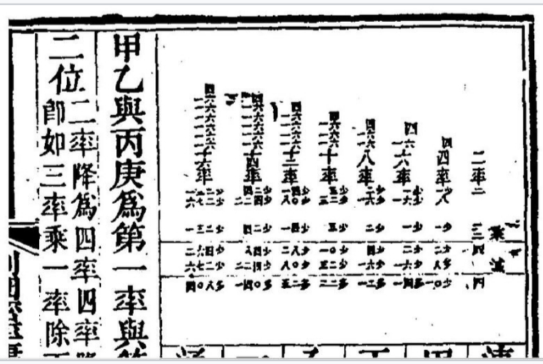
Moreover, if the indexing is "shifted" so that the (i, j) entry is filled with the Catalan number C_{i+j-1} then the determinant is still 1, regardless of the value of n. For example, for $n = 4$ we have

$$\det\begin{bmatrix} 1 & 2 & 5 & 14 \\ 2 & 5 & 14 & 42 \\ 5 & 14 & 42 & 132 \\ 14 & 42 & 132 & 429 \end{bmatrix} = 1.$$

Taken together, these two conditions uniquely define the Catalan numbers.

History [edit]

The Catalan sequence was described in the 18th century by Leonhard Euler, who was interested in the number of different ways of dividing a polygon into triangles. The sequence is named after Eugène Charles Catalan, who discovered the connection to parenthesized expressions during his exploration of the Towers of Hanoi puzzle. The counting trick for Dyck words was found by Désiré André in 1887.



Catalan numbers in Mingantu's book *The Quick Method for Obtaining the Precise Ratio of Division of a Circle* volume III

In 1988, it came to light that the Catalan number sequence had been used in China by the Mongolian mathematician Minggantu by 1730.[14][15] That is when he started to write his book *Ge Yuan Mi Lu Jie Fa [The Quick Method for Obtaining the Precise Ratio of Division of a Circle]*, which was completed by his student Chen Jixin in 1774 but published sixty years later. Peter J. Larcombe (1999) sketched some of the features of the work of Mingantu, including the stimulus of Pierre Jartoux, who brought three infinite series to China early in the 1700s.

For instance, Ming used the Catalan sequence to express series expansions of sin(2α) and sin(4α) in terms of sin(α).

Generalizations [edit]

The two-parameter sequence of non-negative integers $\frac{(2m)!(2n)!}{(m+n)!m!n!}$ is a generalization of the Catalan numbers. These are named **super-Catalan numbers**, by Ira Gessel. These number should not confused with the Schröder–Hipparchus numbers, which sometimes are also called super-Catalan numbers.

For $m = 1$, this is just two times the ordinary Catalan numbers, and for $m = n$, the numbers have an easy combinatorial description. However, other combinatorial descriptions are only known[16] for $m = 2$ and $m = 3$, and it is an open problem to find a general combinatorial interpretation.

Sergey Fomin and Nathan Reading have given a generalized Catalan number associated to any finite crystallographic Coxeter group, namely the number of fully commutative elements of the group; in terms of the associated root system, it is the number of anti-chains (or order ideals) in the poset of positive roots. The classical Catalan number C_n corresponds to the root system of type A_n. The classical recurrence relation generalizes: the Catalan number of a Coxeter diagram is equal to the sum of the Catalan numbers of all its maximal proper sub-diagrams.[17]

See also [edit]

Mathematics portal

- Associahedron
- Bertrand's ballot theorem
- Binomial transform
- Catalan's triangle
- Catalan–Mersenne number
- Fuss–Catalan number
- List of factorial and binomial topics
- Lobb numbers
- Narayana number
- Schröder–Hipparchus number
- Tamari lattice
- Wedderburn–Etherington number

Notes [edit]

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- Davis, Tom: Catalan numbers. Still more examples.
- "Equivalence of Three Catalan Number Interpretations" from The Wolfram Demonstrations Project [1]
- Learning materials related to Partition related number triangles at Wikiversity