Advanced Algorithm Homework 5

于峥 518030910437

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Problem 1.

proof: Construct three r.v. a, b, c are uniformly random in $[0, 3]$, and $a + b + c = 6$, then event $X_v = [v \leq 2]$. We can know that

$$\Pr[X_a] = \Pr[X_b] = \Pr[X_c] = 2/3$$

$$\Pr[X_a \wedge X_b] = \Pr[X_a \wedge X_c] = \Pr[X_b \wedge X_c] = 4/9$$

$$\Pr[X_a \wedge X_b \wedge X_c] = \Pr[\overline{X_a} \wedge \overline{X_b} \wedge \overline{X_c}] = 0 \quad (*)$$

So X_a, X_b, X_c are pairwise independent but not mutually independent, it means the dependency graph is three isolated point. We set $x(X_a) = x(X_b) = x(X_c) = \frac{5}{6}$, it is easily to verify the function satisfy asymmetric Lovász local lemma condition, then we get

$$\Pr[\overline{X_a} \wedge \overline{X_b} \wedge \overline{X_c}] \geq \frac{1}{6^3}$$

which contradict to $(*)$. Hence, the requirement that A and $A \backslash \{N^+(A)\}$ are mutually independent cannot be weaken to pairwise independent. $\square$

Problem 2.

proof: Let $C_u = \{C \text{ is a clause } | u \in C\}$, then we can induce on size of C_u.

Base Step $C_u = \emptyset$. σ be a random assignment, $\varphi|_\sigma = \texttt{true}$ means σ is a satisfying assignment of φ. We easily know that

$$\Pr_{\mathcal{D}}[u = \texttt{true}] = \Pr[u = \texttt{true} | \varphi|_\sigma = \texttt{true}] = \frac{1}{2}$$

Induction Step $|C_u| \geq 1$. Let $\varphi' = \varphi \backslash C (C \in C_u)$, then

$$\begin{aligned}
\Pr_{\mathcal{D}}[u = \texttt{true}] &= \Pr[u = \texttt{true} | \varphi|_\sigma = \texttt{true}] \\
&= \frac{\Pr[u = \texttt{true} \wedge \varphi|_\sigma = \texttt{true}]}{\Pr[\varphi|_\sigma = \texttt{true}]} \\
&\leq \frac{\Pr[u = \texttt{true} \wedge \varphi'|_\sigma = \texttt{true}]}{\Pr[\varphi|_\sigma = \texttt{true} | \varphi'|_\sigma = \texttt{true}]\Pr[\varphi'|_\sigma = \texttt{true}]} \\
&\leq \frac{\Pr[u = \texttt{true} \wedge \varphi'|_\sigma = \texttt{true}]}{(1 - x(C))\Pr[\varphi'|_\sigma = \texttt{true}]} \qquad (*) \\
&\leq \frac{\Pr[u = \texttt{true} | \varphi'|_\sigma = \texttt{true}]}{1 - x(C)} \\
&\leq \frac{1}{2} \prod_{u \in C} (1 - x(C))^{-1}
\end{aligned}$$

Step $(*)$ use the lemma proofed in class. If x satisfies the Lovasz Local Lemma condition, then

$$\forall i \notin S, \Pr[A_i | \bigcap_{i \in S} \overline{A_i}] \leq x(A_i)$$

It means

$$\Pr[\varphi|_\sigma = \texttt{true} | \varphi'|_\sigma = \texttt{true}] = \Pr[C|_\sigma = \texttt{true} | \varphi'|_\sigma = \texttt{true}] \geq 1 - x(C)$$

Hence, we finished the proof. $\square$

Problem 3.

proof: (1) The counterexample is $P_3(x - y - z)$. There are 5 possible kinds independent set $\emptyset, \{x\}, \{y\}, \{z\}, \{x, z\}$. We consider the probability of the result is $\{x, z\}$. Let X be the event of result is$y$. We can simply calculate it.

$$\Pr[X] = \frac{1}{8} + \frac{1}{8} \cdot \frac{1}{3} \neq \frac{1}{5}$$

So the algorithm is not uniform. $\square$