Mathematical Logic

homework 2

Problem 1 *Prove that:*

(a) If A and B are both groups, then so is $A \times B$.

(b) If A and B are both equivalence relations, then so is $A \times B$.

(c) For two fields A and B, their direct product $A \times B$ is not necessarily a field

Solution: .

(a) $a_0, a_1, a_2 \in A, b_0, b_1, b_3 \in B$，$\epsilon_a, \epsilon_b$ 分别为A, B中的单位元。

- **封闭性**: 因为 $a_0 \cdot a_1 \in A$, $b_0 \cdot b_1 \in B$, 所以 $(a_0, b_0) \cdot (a_1, b_1) = (a_0 \cdot a_1, b_0 \cdot b_1) \in A \times B$。
- **结合律**:

$$\begin{aligned}((a_0, b_0) \cdot (a_1, b_1)) \cdot (a_2, b_2) &= ((a_0 \cdot a_1) \cdot a_2, (b_0 \cdot b_1) \cdot b_2) \\ &= (a_0 \cdot (a_1 \cdot a_2), b_0 \cdot (b_1 \cdot b_2)) \\ &= (a_0, b_0) \cdot ((a_1, b_1) \cdot (a_2, b_2))\end{aligned}$$

- **单位元**：易验证(ϵ_a, ϵ_b)为单位元。
- **逆元元**：易验证(a_0^{-1}, b_0^{-1})为(a_0, b_0)单位元。

(b)

- **传递性**: $(a_0, b_0) \sim (a_1, b_1) \wedge (a_1, b_1) \sim (a_2, b_2) \Rightarrow a_0 \sim a_2 \wedge b_0 \sim b_2 \Rightarrow (a_0, b_0) \sim (a_2, b_2)$。
- **自反性**：$a_0 \sim a_0 \wedge b_0 \sim b_0 \Rightarrow (a_0, b_0) \sim (a_0, b_0)$

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- **对称性**：$(a_0, b_0) \sim (a_1, b_1) \Rightarrow a_0 \sim a_1 \wedge b_0 \sim b_1 \wedge a_1 \sim a_0 \wedge b_1 \sim b_0 \Rightarrow (a_0, b_0) \sim (a_1, b_1)$

(c) 例如A,B都为R, $(a, 0)$ 没有乘法逆元$a \neq 0$。

Problem 2 *Prove Example 1.12.*

Solution: 首先 $\forall v_0 \exists v_0 v_0 \circ v_1 \equiv e$, 又 $\exists v_2, v_1 \circ v_2 \equiv e$。

所以 $\forall v_0 \exists v_1$

$$\begin{aligned} v_1 \circ v_0 &\equiv (v_1 \circ v_0) \circ e \\ &\equiv (v_1 \circ v_0) \circ (v_1 \circ v_2) \\ &\equiv v_1 \circ (v_0 \circ v_1) \circ v_2 \\ &\equiv v_1 \circ e \circ v_2 \\ &\equiv v_1 \circ v_2 \\ &\equiv e \end{aligned}$$

有可以推出

$$e \circ v_0 \equiv (v_0 \circ v_1) \circ v_0 \equiv v_0 \circ (v_1 \circ v_0) \equiv v_0 \circ r \equiv v_0$$

Problem 3 *An S-formula is positive if it contains no logic symbols $\neg$, $\rightarrow$, and $\leftrightarrow$. Prove that every positive formula is satisfiable.*

Solution: *For a $\mathfrak{A}$, Let $A = \{\epsilon\}$，for a positive ϕ*。

- $\phi = t_1 \equiv t_2$. *we have*

$$\begin{aligned} \mathfrak{A} \models \epsilon \equiv \epsilon \Leftrightarrow & \mathfrak{A}(t_1) \equiv \mathfrak{A}(t_2) \\ \Leftrightarrow & t_1 \equiv t_2 \end{aligned}$$

- $\phi = Rt_1 \ldots t_n$, *Then we just need Let all relations can be satisfiable.*

- $\phi = \psi \vee \mathcal{X}$,

$$\mathfrak{A} \models \psi \wedge \mathfrak{A} \models \mathcal{X} \Rightarrow \mathfrak{A} \models \psi \wedge \mathcal{X}$$

- $\phi = \exists x \psi$, *x can just can be* ϵ, *so* $\mathfrak{A}\frac{\epsilon}{x} \models \phi$

Problem 4 *S-structures* $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$

(1) $\mathfrak{A} \approxeq \mathfrak{A}$

(2) $\mathfrak{A} \approxeq \mathfrak{B}$ *implies* $\mathfrak{B} \approxeq \mathfrak{A}$

(3) $\mathfrak{A} \approxeq \mathfrak{B}$ *and* $\mathfrak{B} \approxeq \mathfrak{C}$ *then* $\mathfrak{A} \approxeq \mathfrak{C}$

Solution: *(1) A mapping* $\pi : A \to A$。

Because $\forall x \in A, \pi(x) = x$, *so*

- π *is bijection.*
- *For any n-ary relation symbol* $R \in S$ *and* $a_0, a_1, \ldots a_{n-1} \in A$,

$$(a_0 \ldots a_{n-1}) \in R^{\mathfrak{A}} \Rightarrow (\pi(a_0) \ldots \pi(a_{n-1}) = (a_0 \ldots a_{n-1}) \in R^{\mathfrak{A}}$$

- *For any n-ary function symbol* $f \in S$ *and* $a_0, a_1, \ldots a_{n-1} \in A$,

$$\pi(f^{\mathfrak{A}}(a_0 \ldots a_{n-1})) = f^{\mathfrak{A}}(\pi(a_0) \ldots \pi(a_{n-1})) = f^{\mathfrak{A}}(a_0 \ldots a_{n-1})$$

(2) A mapping $\pi : A \to B$, *Because* π *is bijection, so* $\exists \pi^{-1} : B \to A$ *is bijection.*

- *For any n-ary relation symbol* $R \in S$ *and* $\pi(a_0) = b_0 \ldots \pi(a_{n-1}) = b_{n-1} \in B$.

$$(b_0 \ldots b_{n-1}) \in R^{\mathfrak{B}} \Rightarrow (\pi(a_0) \ldots \pi(a_{n-1}) \in R^{\mathfrak{B}}$$

Because π *is bijection, so* $(a_0 \ldots a_{n-1}) \in R^{\mathfrak{A}}$

- *For any n-ary function symaol* $f \in S$ *and* $\pi(a_0) = b_0 \ldots \pi(a_{n-1}) = b_{n-1} \in B$.

$$\pi(f^{\mathfrak{A}}(a_0 \ldots a_{n-1})) = f^{\mathfrak{B}}(\pi(a_0) \ldots \pi(a_{n-1}))$$

so

$$\pi^{-1}(f^{\mathfrak{B}}(b_0 \ldots b_{n-1})) = f^{\mathfrak{A}}(\pi^{-1}(b_0) \ldots \pi^{-1}(b_{n-1}))$$

(3) A mapping $\sigma : A \to B$, $\tau : B \to C$, $\pi = \tau\sigma$ *is bijection.*

- *For any n-ary relation symbol* $R \in S$,

$$\begin{aligned}(a_0 \ldots a_{n-1}) \in R^{\mathfrak{A}} &\Rightarrow (\sigma(a_0) \ldots \sigma(a_{n-1}) \in R^{\mathfrak{B}} \\ &\Rightarrow (\tau(\sigma(a_0)) \ldots \tau(\sigma(a_{n-1})) \in R^{\mathfrak{C}} \\ &\Rightarrow (\pi(a_0) \ldots \pi(a_{n-1}) \in R^{\mathfrak{C}}\end{aligned}$$

- *For any n-ary function symaol* $f \in S$,

$$\begin{aligned}\pi(f^{\mathfrak{A}}(a_0 \ldots a_{n-1})) &= \tau(f^{\mathfrak{B}}(\sigma(a_0) \ldots \sigma(a_{n-1}))) \\ &= f^{\mathfrak{C}}(\pi(a_0) \ldots \pi(a_{n-1}))\end{aligned}$$

Problem 5 *Let* ϕ, ψ, *and* $\mathcal{X}$ *be S-formulas. Prove that:*

(a) $(\phi \vee \psi) \models \mathcal{X}$ *if and only if* $\phi \models \mathcal{X}$ *and* $\psi \models \mathcal{X}$.

(b) $\models \phi \to \psi$ *if and only if* $\phi \models \psi$.

Solution: *(a) If* $\mathfrak{I} \models (\phi \vee \psi)$ *and* $\mathfrak{I} \models \mathcal{X}$, *then* $\mathfrak{I} \models \phi$ *and* $\mathfrak{I} \models \psi$, *so* $\phi \models \mathcal{X}$ *and* $\psi \models \mathcal{X}$。 *Similarly, if* $\mathfrak{I} \models \phi$, $\mathfrak{I} \models \psi$, $\mathfrak{I} \models \mathcal{X}$, $(\phi \vee \psi) \models \mathcal{X}$。

(b) If $\mathfrak{I} \models \phi \to \psi$, *then* $\mathfrak{I} \models \phi$ *implies* $\mathfrak{I} \models \psi$，*so* $\phi \models \psi$，*and vice versa*。

Problem 6 *Let S be finite, i.e., containing finitely many relation symbols, function symbols, and constants. Prove that two finite structures* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are isomorphic if and only if for any S-sentence* ϕ

$$\mathfrak{A} \models \phi \Leftrightarrow \mathfrak{B} \models \phi$$

Solution: *(1)* 如果$\mathfrak{A}$ 和 $\mathfrak{B}$ 同构，根据 *Lemma 1.5 (The Isomorphism Lemma)*这是对的。

(2) 若$\forall\phi$

$$\mathfrak{A} \models \phi \Leftrightarrow \mathfrak{B} \models \phi$$

假设$A = \{x_0, x_1, \ldots, x_{n-1}\}$,

我们可以找到这样一个命题ψ, 这个命题能够描述所有的命题，并且构造了一个双射π, 根据之前的构造，其他两个条件是显然的。所以$\mathfrak{A}$和$\mathfrak{B}$是同构的。