Mathematical Logic

homework 5

Problem 1 *Let*

$$\Phi := \{v_0 \equiv t | t \in T^S\} \cup \{\exists v_0 \exists v_1 \neg v_0 \equiv v_1\}$$

Prove that Φ is consistent, but there is no consistent $\Phi \subseteq \Psi \subseteq L^S$ with which contains witnesses.

Solution: *We just construct $\mathfrak{I}$,*

$$\mathfrak{I} = \{\{0^{\mathfrak{I}}, 1^{\mathfrak{I}}\}, S^{\mathfrak{I}}, \beta\}$$

For any function f, relation R:

Let $f^{\mathfrak{I}}(v_0, v_1, \cdots v_{n-1}) = 0^{\mathfrak{I}}$, $R^{\mathfrak{I}} v_0 v_1 \cdots v_{n-1} = 0^{\mathfrak{I}}$。

Then $\Phi \dfrac{0, 1}{v_0, v_1} \models \Phi$。 So Φ is satisfiable, it is consistent.

If Ψ is contains witnesses, then for any variable x, and S-formula $\varphi \in \Psi$。
$\Psi \vdash \left(\exists x \varphi \to \varphi \dfrac{t}{x}\right)$

So $\Psi \vdash \Big(\exists v_0 \exists v_1 \neg v_0 \equiv v_1 \to \exists v_1 \neg t \equiv v_1\Big)$, then $\Psi \vdash \exists v_1 \neg t \equiv v_1$。

But $\Psi \vdash \{v_0 \equiv t | t \in T^S\}$。

So we can know Ψ is inconsistent。

Problem 2 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two S-structures. $\mathfrak{A}$ mapping $h : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ if the following properties hold.*

(1) For every n-ary relation symbol $R \in S$ and $a_1, \cdots, a_n \in A$ we have

$$(a_1, \cdots, a_n) \in R^{\mathfrak{A}} \text{ implies } (h(a_1), \cdots, h(a_n)) \in R^{\mathfrak{B}}.$$

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(2) For every n-ary function symbol $R \in S$ and $a_1, \cdots, a_n \in A$ we have

$$h(f^{\mathfrak{A}}(a_1, \cdots, a_n)) = f^{\mathfrak{B}}(h(a_1), \cdots, h(a_n)).$$

(3) For every constant $c \in S$

$$h(c^{\mathfrak{A}}) = c^{\mathfrak{B}}.$$

Now let $\Phi \subseteq L^S$ and $\mathfrak{A}$ be an S-structure with $\mathfrak{A} \models \Phi$. Prove that there is a homomorphism from the term model $\mathfrak{T}^{\Phi}$ to $\mathfrak{A}$.

Solution: *We define h*

$$\begin{aligned} h : T^{\Phi} &\mapsto A \\ \bar{t} &\mapsto \mathfrak{A}(t) \end{aligned}$$

(1) For every n-ary relation symbol $R \in S$ and $\overline{t_1}, \cdots, \overline{t_n} \in T^{\Phi}$:

$$\begin{aligned} (\overline{t_1}, \cdots, \overline{t_n}) \in R^{\mathfrak{T}^{\Phi}} &\Rightarrow \Phi \vdash Rt_1, \cdots, t_n \\ &\Rightarrow \Phi \models Rt_1, \cdots, t_n \\ &\Rightarrow \mathfrak{A} \models Rt_1, \cdots, t_n \\ &\Rightarrow (\mathfrak{A}(t_1), \cdots, \mathfrak{A}(t_n)) \in R^{\mathfrak{I}} \end{aligned}$$

(2) For every n-ary function symbol $f \in S$ and $\overline{t_1}, \cdots, \overline{t_n} \in T^{\Phi}$:

$$\begin{aligned} h(f^{\mathfrak{T}^{\Phi}}(\overline{t_1}, \cdots, \overline{t_n})) &= h(\overline{ft_1 \cdots t_n}) \\ &= \mathfrak{A}(ft_1 \cdots t_n) \\ &= f^{\mathfrak{A}}(\mathfrak{A}(t_1), \cdots, \mathfrak{A}(t_n)) \\ &= f^{\mathfrak{A}}(h(\overline{t_1}), \cdots, h(\overline{t_n}))) \end{aligned}$$

(3) For every constant $c \in S$

$$h(c^{\mathfrak{T}^{\Phi}}) = h(\bar{c}) = \mathfrak{A}(c) = c^{\mathfrak{A}}$$

So h is a homomorphism from the term model $\mathfrak{T}^{\Phi}$ to $\mathfrak{A}$.