CS 217 – Algorithm Design and Analysis

Shanghai Jiaotong University, Fall 2019

NoCode

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1 Bit Complexity, Recursion, and Dynamic Programming

1.1 Bit Complexity of Euclid's Algorithm

We have proved that Euclid's algorithm for computing $\gcd(a, b)$ makes at most $O(\log a)$ iterations. What is the overall running time? Each iteration computes $u \mod v$ for some integers. This can be done by integer division. What is its running time? There are very sophisticated algorithms, but python probably does not come with them. Recall the "school method" for dividing integers. Have a look at the pdf slides on the webpage for an illustration of the school method. It is especially simple if we are dealing with binary numbers. If a and b have at most n bits, then the school method has complexity $O(n^2)$.

Exercise 1. Show the following, more precise bound of the school method for integer division: If a has n bits and b has k bits, then the school method can be implemented to run in $O(k(n - k))$ operations.

Solution Since b has k bits, each time we implement a subtraction costs us k operations. Also, for the school method, the divisor will move at least 1 bit right after a subtraction. During the whole process, the divisor will move at most $n - k$ times, since the divisor already has k bits. Thus, it will costs at most $O(k(n-k))$ operations to implement school method for integer division.

Exercise 2. Show that the bit complexity of Euclid's algorithm, using the school method to compute $a \mod b$, is $O(n^2)$. That is, if a and b have at most n bits, then $\gcd(a,b)$ makes $O(n^2)$ bit operations.

In order to do so, here is python code of the Euclidean algorithm:

```
def euclid(a,b):
  while (b > 0):
      r = a % b # so a = bu+r
      if (r == 0):
          return b
      s = b % r # so b = rv + s
      a = r
      b = s
  return a
```

Don't be afraid to introduce notation! I recommend to let n denote the number of bits of a. Take some other letters for the number of bits in b and so on.

Solution Let n denote the number of bits of a. We already know that Euclid's algorithm for computing $\gcd(a,b)$ makes at most $O(n)$ iterations. Now we first prove a lemma.

Lemma 1.1.2.1 For $a_1, a_2, \ldots, a_m$ and $\sum_{i=1}^m a_i = n$, $\sum_{i=1}^m a_i(n - \sum_{j=1}^i a_j)$ has maximum value only when $a_1 = a_2 = \ldots = a_m$

Proof. To maximize

$$\sum_{i=1}^{m} a_i(n - \sum_{j=1}^{i} a_j) = n^2 - \sum_{i=1}^{m}(a_i \sum_{j=1}^{i} a_j)$$
$$= n^2 - \sum_{i=1}^{m}\sum_{j=1}^{i} a_i a_j$$

we need to minimize

$$\sum_{i=1}^{m}\sum_{j=1}^{i} a_i a_j = \frac{a_1^2}{2} + \sum_{i=2}^{m}(\frac{2\sum_{j=1}^{i-1} a_j + a_i}{2}) + \sum_{i=1}^{m}\frac{a_i^2}{2}$$
$$= \frac{(n-1)^2}{2} + \frac{1}{2}\sum_{i=1}^{m} a_i^2$$

Based on the mean value inequality, when $a_1 = a_2 = \ldots = a_m$, $\sum_{i=1}^{m} a_i^2$ has the minimum value, which means $\sum_{i=1}^{m} a_i(n - \sum_{j=1}^{i} a_j)$ has maximum value.

Now we come back to the problem. Suppose we need to do Euclid's algorithm for $m(1 \leq m \leq n)$ iterations, and for every iterations we can shorten a by a_i bits. From the lemma, we can conclude that when $a_1 = a_2 = \ldots = a_m = \frac{n}{m}$ we would have the maximum operations which is the worst case.

Now consider the worst case, that is, we have to do the school method division for m times, each time we can shorten the dividen by $t(= \frac{n}{m})$ bits. Use the conclusion from **Exercise 1.**, we can write the total operations needed as below:

$$\sum_{i=1}^{m}(n - it)t = n^2 - t^2\sum_{i=1}^{m} i$$
$$= n^2 - \frac{n^2}{m^2}\frac{m(m+1)}{2}$$
$$= \frac{n^2}{2} - \frac{n^2}{2m}$$

When $m = 1$, we have the worst case to compute $\gcd(a, b)$, which is $O(n^2)$. Thus, the bit complexity of Euclid's algorithm, using the school method to compute $a \mod b$, is $O(n^2)$.

1.2 Computing the Binomial Coefficient

Next, we will investigate the binomial coefficient $\binom{n}{k}$, which you might also know by the notation C_n^k. The number $\binom{n}{k}$ is defined as the number of subsets of $\{1, \ldots, n\}$ which have size exactly k. This immediately shows that $\binom{n}{k}$ is 0 if k is negative or larger than n. You might have seen the following recurrence:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \text{ if } n, k \geq 0 \ .$$

Exercise 3. [A Recursive Algorithm for the Binomial Coefficient] Using pseudocode, write a recursive algorithm computing $\binom{n}{k}$. Implement it in python! What is the running time of your algorithm, in terms of n and k? Would you say it is an efficient algorithm? Why or why not?

Solution

```
def Binomial(n, k):
    if (k > n or k < 0):
        return 0
    elif (n == k or k == 0):
        return 1
    else:
        return Binomial(n - 1, k - 1) + Binomial(n - 1, k)
```

The running time of this algorithm in terms of n and k is $\mathcal{O}(\binom{n}{k})$* the time of each adding, I think it isn't an efficient algorithm, because it computes too many useless and duplicated result.

Exercise 4. [A Dynamic Programming Algorithm for the Binomial Coefficient] Using pseudocode, write a dynamic programming algorithm computing $\binom{n}{k}$. Implement it in python! What is it running time in terms of n and k? Would you say your algorithm is efficient? Why or why not?

Solution

```
def Binomial(n, k):
    C = []
    if (k > n or k < 0):
        return 0
    C.append(1)
    for i in range (1, n+1):
        left_top = 1
        for j in range (1, k+1):
            if (j == i):
```

```
                C.append(1)
                break
            else:
                tmp = C[j]
                C[j] += left_top
                left_top = tmp
    return C[k]
```

The running time of this algorithm in terms of n and k is $\mathcal{O}((n-k)*k)$* the time of each adding, I think it is an efficient algorithm, because it computes Binomial Coefficient really fast, it computes each coefficient just once which is always useful for the final result.

Exercise 5. [Binomial Coefficient modulo 2] Suppose we are only interested in whether $\binom{n}{k}$ is even or odd, i.e., we want to compute $\binom{n}{k} \mod 2$. You could do this by computing $\binom{n}{k}$ using dynamic programming and then taking the result modulo 2. What is the running time? Would you say this algorithm is efficient? Why or why not?

Solution One possible algorithm is this, let f_n be the number of 2 factors of n g_n be the number of 2 factors of $n!$. We can compute f_n using dynamic programming.

$$f(x) = \begin{cases} 0 & \text{x is odd} \\ f(x/2)+1 & \text{x is even} \end{cases}$$

the process begin at $f(1) = 0$, and $g_n = \sum_{i=1}^{n} f_i$. Then whether $\binom{n}{k}$ is odd depend on $g_n = g_k + g_{n-k}$.

Compute single f_n we need do one division and one addition, but the division can be instead by bit right shift. Let m be the number of bit of n, so the bit complexity of the algorithm is $\Theta(m2^m)$.

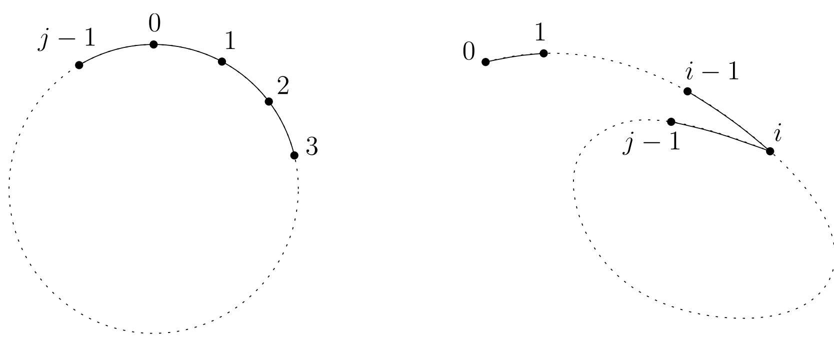
It is not a efficient algorithm, because we just need to check $\binom{n}{k}$, we can have better algorithm. It is simply check if n & k equals to k (& is bit and). If n & $k = k$, then $\binom{n}{k}$ is odd. The bit complexity of the algorithm is $\Theta(m)$.

The correctness of the algorithm can be proved by induction, but the more convenient way is to use Lucas theorem, if p is a prime.

$$\begin{aligned}\binom{n}{m} &= \prod_{i=0}^{k} \binom{m_i}{n_i} \pmod p \\ m =& m_k p^k + m_{k-1} p^{k-1} + \cdots + m_1 p + m_0 \\ n =& n_k p^k + n_{k-1} p^{k-1} + \cdots + n_1 p + n_0\end{aligned}$$

Let $p = 2$, we can know if $m_i = 0$, then n_i must be 0, or $\binom{n}{m}$ is even. This is equivalent to $\binom{n}{k}$ is odd $\Rightarrow n \ \& \ k = k$.

Exercise 6. Remember the "period" algorithm for computing $F'_n := (F_n \mod k)$ discussed in class: (1) find some i, j between 0 and k^2 for which $F'_i = F'_j$ and $F'_{i+1} = F'_{j+1} k$. Then for $d := j - i$ the sequence F'_n will repeat every d steps, as there will be a cycle. This cycle can either be a "true cycle" or a "lasso":



a cycle a lasso

Show that a lasso cannot happen. That is, show that the smallest i for which this happens is 0, i.e, for some j we have $F'_0 = F'_j$ and $F'_1 = F'_{j+1}$ and thus $F'_n = F'_{n \bmod j}$.

Solution If we already know that lasso cannot happen, we can just check F'_i from 1 to k^2 until $F'_i = 0$ and $F'_{i+1} = 1$.

If we do not know that, we can use Floyd Cycle Detection Algorithm. Initialy, we set two pointer $t = (F'_0, F'_1), s = (F'_0, F'_1)$. Every step t become from (F'_i, F'_{i+1}) to (F'_{i+1}, F'_{i+2}), s become from (F'_i, F'_{i+1}) to (F'_{i+2}, F'_{i+3}), after every step, check the if t equals s, if equals then stop, we already find the cycle.

$$\mathbf{F} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

To prove the lasso cannot happen. we just need to prove the $\exists p > 0, \mathbf{F}^p = I$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = I$$

So F is invertable, so F is in $GL_2(Z_k)$. Because $GL_2(Z_k)$ is finite, so must $\exists a, b(a > b), F^a = F^b$, then $F^{a-b} = I$, so lasso will not happen.

Python. Please write your code in python. It is a very simple programming language. If you do not know any python, I can put same example code online and you can learn by example. Also, for this homework you definitely need a Big Integer class since numbers with ten thousand digits do not fit into any `long long int` or similar. Python automatically supports Big Integer, so there is no problem here.

```
def find_fib_cycle(k):
  """ The lasso cannot happen """
  j = 0
  f0, f1 = 0, 1

  while True:
      f0, f1 = f1, (f0 + f1) % k
      if f0 == 0 and f1 == 1:
          return 0, j
      j += 1

def find_fib_lasso(k):
  """ Floyd Cycle Detection Algorithm """
  i = 0
  j = 0
  t = (0, 1)
  s = (0, 1)
  while True:
      t = (t[1], (t[0] + t[1]) % k)
      s = (s[1], (s[0] + s[1]) % k)
      s = (s[1], (s[0] + s[1]) % k)
```

```
i += 1
j += 2
if t == s:
    return i, j
```