JOURNAL OF COMBINATORIAL THEORY (A) **20**, 80–88 (1976)

Strong Versions of Sperner's Theorem*

CURTIS GREENE AND DANIEL J. KLEITMAN

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 01239

Communicated by G.-C. Rota

Received March 27, 1975

A procedure for partitioning the collection of divisors of an integer into symmetric chains is described and analyzed in detail. As a consequence, several strengthenings of Sperner's theorem are obtained. The algorithm also leads to elementary combinatorial proofs of a number of results on lattice paths and plane partitions.

A well-known theorem of Sperner states that a collection of subsets of an n element set S, no two members of which are ordered by inclusion, can have no more than $\binom{n}{[n/2]}$ members. This result has been proved in many ways. Let us call such a collection an "antichain." One way consists of partitioning the entire collection (2^S) of subsets of S into $\binom{n}{[n/2]}$ chains or totally ordered elements. Since each chain can have only a single intersection with an antichain, the result immediately follows.

We shall see below that such a partition is easily found. It is the purpose of this note to point out that the existence and structure of such a partition has more powerful implications than the Sperner result. It implies, for example, that the same conclusion could be drawn in the partial order having the same members but with ordering only within the chains of the partition. Similarly, in the original order it implies that the ordering restriction can be significantly weakened without changing the conclusion of Sperner's theorem. This fact gives rise to some interesting exensions of Sperner's theorem.

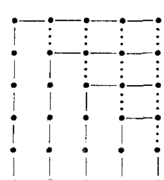
Below we describe a specific partition that has been used for the subset lattice and for the analogous problem in the lattice of divisors of an integer with "divides" as order relation. We then describe a number of weakenings of the hypothesis of Sperner's theorem which do not affect its conclusion, both for subsets and divisors.

* Research supported in part by ONR N00014-67-A-0204-0063.

The basic partition (which has been published in varying contexts by de Bruijn, Tengbergen, and Kruyswijk [1], Kleitman [5], Hansel [3], and undoubtedly others) can be obtained by induction on the number of elements of S or prime factors of an integer, as follows.

A *symmetric chain* of subsets is a totally ordered collection $A_{n/2-j} \subseteq A_{n/2-j+1} \subseteq \cdots \subseteq A_{n/2+j}$ such that $|A_i| = i$ for each i. (Here j is half-integral if n is odd.) If $|S| = n$, then a partition of 2^S into symmetric chains necessarily contains $\binom{n}{[n/2]}$ members. If such a partition has been constructed, and a is a new element, then 2^{S+a} can be partitioned into symmetric chains by the following procedure. To each symmetric chain in 2^S, associate two asymmetric chains in 2^{S+a}; the first is the original chain and the second is obtained by adding a to each member. Both of these chains become symmetric if the top member of the second is removed and added to the first. The resulting collection clearly partitions 2^{S+a}.

For divisors of an integer we can proceed in exactly the same manner as de Bruijn, Tengbergen, and Kruyswijk [1]. The only difference is that, upon adding a new prime-power factor p^k, there are $k + 1$ new asymmetric chains $C_0, ..., C_k$ (one for each of $1, p, p^2, ..., p^k$) instead of two as before. Also, the rank having half the total number of prime factors goes up by $k/2$ instead of $\frac{1}{2}$. To rearrange these into symmetric chains, remove the top element from each of $C_1, ..., C_k$ and add it to C_0. Then remove the next element from each of $C_2, ..., C_k$ and add it to C_1, and so forth, as illustrated in the diagram.



$C_0\ C_1\ C_2\ C_3\ C_4$

It is easy to see that the chains are all again symmetric. Thus, we have the desired partition.

Next we give an explicit description of the symmetric chains constructed above, first for families of subsets.

Suppose that the elements of S have been given a fixed ordering $x_1, x_2, ..., x_n$. Let A be a subset of S, and let $\lambda_k \in A$. We call x_k a *basic element of* A (with respect to the ordering) if, for some $j < k$, exactly half of the elements $x_j, x_{j+1}, ..., x_k$ are in A. If j is the largest index with this property, then $x_j \notin A$ and we say that x_j is *paired with* x_n. Thus every basic element $x_k \in A$ is paired with a unique element $x_j \notin A$.

The reader may find it useful to visualize basic elements in the following way. If $A \subseteq S$, make a list of the elements of S, replacing each $x \in A$ by a right parenthesis, and each $y \in S - A$ by a left parenthesis. Then a basic element of A corresponds to a right parenthesis which can be "closed" with a (unique closest) left parenthesis, following the usual rules. For example, if $S = \{0, 1, 2,..., 9\}$, and $A = \{0, 2, 3, 7, 8\}$, we associate with A the sequence

$$\begin{array}{cccccccccc}) & (&) &) & (& (& (&) &) & (\\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}$$

which can be parenthesized (uniquely) as shown. Thus the basic elements of A are 2, 7, and 8. Notice that the unpaired right parenthesis (which correspond to nonbasic elements of A) always occur to the left of the unpaired left parentheses (which correspond to unpaired elements of $S - A$).

With the above definitions in hand, we can now describe explicitly the partition of 2^S into symmetric chains which results from the construction referred to earlier. We assume that an ordering of the elements of S has been fixed, and that the partition has been built up by adding elements in this order.

THEOREM 1. *If $A \subseteq S$, let A^- denote the set of basic elements of A and let A^+ denote the result of adding to A the unpaired elements of $S - A$. Then the symmetric chain containing A stretches from A^- to A^+ and is constructed as follows. Start with A^- and add each of the nonbasic elements of A, in order, until A is reached. Then add the unpaired elements of $S - A$, in order, until A^+ is reached.*

In parenthesis notation, A^- is obtained by switching all unclosed right parentheses to left parentheses. If we are given a set A, we can describe the chain containing A by finding A^- in this manner, and then reversing the unpaired parentheses successively, from left to right. Two sets lie in the same chain if and only if they have the same "basic parenthesization."

For example, if S and A are defined as above, we obtain the symmetric chain $\{2, 7, 8\}, \{0, 2, 7, 8\}, \{0, 2, 3, 7, 8\}, \{0, 2, 3, 4, 7, 8\}, \{0, 2, 3, 4, 7, 8, 9\}$. (This is the result of adding 0, 3, 4, 9 successively to the set $\{2, 7, 8\}$.)

Since there are as many paired elements of $S - A$ as there are basic elements of A, it follows that $|A^-| = |S - A^+|$. Hence, the chains defined in Theorem 1 are always symmetric. Also, since every $A \subseteq S$ has a uniquely determined set of basic elements, the chains form a partition of 2^S. Moreover, an easy induction shows that this family of chains is precisely the one produced by the algorithm described earlier [1, 3, 5]. We leave the details of this last observation to the reader.[1]

Next, we obtain an analogous result for divisors of an integer. If $N = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$, associate to each divisor $p_1^{f_1} p_2^{f_2} \cdots p_n^{f_n}$ of N a sequence of parentheses, as follows. Put right parentheses in the first f_1 positions, followed by $k_1 - f_1$ left parentheses; then f_2 right parentheses, $k_2 - f_2$ left parentheses, and so forth. Thus, the divisors of N uniquely correspond to sequences of $\sum k_i$ parentheses, with the property that each successive block of size k_i consists of a string of right parentheses (possibly empty) followed by a string of left parentheses (possibly empty).

Each such sequence has a "basic parenthesization," just as before. If we switch all of the unpaired parentheses to left parentheses, the resulting sequence still corresponds to a divisor of N; in fact, if the unpaired left parentheses are now reversed, one by one starting from the left, each new sequence corresponds to a divisor of N (that is, the fixed blocks of size k_i begin with right parentheses and end with left parentheses).

It is not difficult to show that these operations produce a set of symmetric chains which partition the lattice of divisors of N. Moreover, it is not hard to see that these are the same chains obtained from the inductive algorithm of de Bruijn, Tengbergen, and Kruyswijk [1]. We state this result as our next theorem.

THEOREM 2. *If the divisors of* $N = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$ *are partitioned into symmetric chains by adding the factors* $p_i^{k_i}$ *in order (as described earlier), then two divisors are in the same chain if and only if, in the above notation, they have the same "basic parenthesization."*

For example, if $N = 2^3 \cdot 3^2 \cdot 5$, and $D = 2 \cdot 3$, we associate D with the sequence

$$\vdots\,)\ (\ \underbrace{(\, \vdots\,)}\ (\, \vdots\, (\, \vdots\,,$$

which is parenthesized as shown. Hence, the bottom of the chain containing D is the integer which corresponds to

$$\vdots\, (\ (\ \underbrace{(\, \vdots\,)}\ (\, \vdots\, (\, \vdots$$

(namely, $D^- = 3$). By changing unpaired left parentheses to right parentheses, we obtain the symmetric chain $3,\ 2 \cdot 3,\ 2^2 \cdot 3,\ 2^2 \cdot 3^2,\ 2^2 \cdot 3^2 \cdot 5$.

[1] *Note added in proof.* The concept of basic elements and parenthesizations has been considered independently by K. Leeb (unpublished).

We now describe some weakenings of the Sperner hypothesis which, by virtue of the previous results, do not affect its conclusion.

If the elements of a set are colored in two colors, we say that the coloring is *balanced* if the number of elements of one color differs by at most 1 from the number of elements of the other color.

THEOREM 3. *Let F be a family of subsets of S with the following property. There exists a balanced coloring of S such that no comparable pair $A \subseteq B$ in F has a difference $B - A$ which is balanced. Then $|F| \leqslant \binom{n}{[n/2]}$. If no chain of $k + 1$ members of F has all differences balanced, then $|F|$ does not exceed the sum of the k largest binomial coefficients.*

THEOREM 4. *Let F be a family of subsets of S for which there exists an ordering $a_1, \ldots, a_n$ (of S) such that no comparable pair $A \subseteq B$ has the property that the subscripts of the elements of $B - A$ alternate even–odd (when read in increasing order). Then $|F| \leqslant \binom{n}{[n/2]}$. (An analogous statement holds for families containing no chains of length $k + 1$.)*

The next theorem is the strongest possible strengthening which this approach allows.

THEOREM 5. *Let F be a family of subsets of S for which there exists an ordering of S such that no two members of F have the same basic elements. Then $|F| \leqslant \binom{n}{[n/2]}$.*

It is obvious that Theorem 4 implies Theorem 3, and that Theorem 5 follows immediately from our characterization of symmetric chains (Theorem 1). The connection between Theorems 5 and 4 is given by the following.

LEMMA. *If A and B are members of the same symmetric chain (or, equivalently, if A and B have the same basic elements) and $A \subseteq B$, then the subscripts of the elements of $B - A$ alternate even–odd.*

This can be proved easily by induction, or directly by the following argument. An element is basic if and only if it can be paired with a smaller "nonelement" adjacent to it or separated from it by other such pairs. Since these pairs are necessarily even–odd, the remaining elements must alternate even–odd. Two sets with the same basic elements differ by a consecutive segment of the remaining elements, which therefore alternate as asserted.

We have the obvious analog of Theorem 5 for divisors of an integer:

THEOREM 5′. *Given a collection F of divisors of an integer N, and an ordering of prime factors with the property that no $k+1$ divisors have the same "parenthesization," then the number of members of F is no greater than the number of divisors of N having the k middle weights. (The weight of $p_1^{f_1} \cdots p_n^{f_n}$ is defined to be $\sum_1^n f_i$).*

There are a number of further generalizations of Sperner's theorem which can be deduced from the existence of a symmetric chain partition. Many of these can be extended by weakening the hypothesis in the manner described above. For example, Katona [4] and also Kleitman [5] also showed that, if S is a two-colored set, and F is a family of subsets such that no two members whose difference is monochromatic are comparable, then F can have at most $\binom{n}{[n/2]}$ members.

We can extend this result to

THEOREM 6. *Suppose that S has been given two different 2-colorings where the first is arbitrary, and the second induces a balanced coloring on each block of the first. Let F be a family of subsets of S which contains no comparable pair of sets $A \subseteq B$ such that $B - A$ is monocolored with respect to the first coloring and balanced with respect to the second coloring. Then $|F| \leqslant \binom{n}{[n/2]}$.*

The proof is as follows. Order each block of the first coloring (separately) so that the second coloring is represented by "evenness and oddness." Partition the subsets of each block into symmetric chains (separately), and form all possible products of chains, one from the first block and one from the second. This partitions the family of subsets of S into symmetric "rectangles." Under the given conditions, the number of members of F in each rectangle cannot exceed its maximal dimension, which is the number of members of middle weight.

One can extend the result of Theorem 6 along the lines of Theorems 4 and 5 (in fact, the proof remains the same). Analogous results are easily proved for divisors of an integer, but we omit the details here.

Although they appear at first glance to be complicated beyond the point of usefulness, theorems of this type are especially well suited for applications to geometric problems, for example, to extensions of the Littlewood–Offord theorem on the distribution of sums of vectors. See [6] for details and further applications.

We conclude with some remarks on the connection between these results and some well-known problems in the theory of one-dimensional random walks (cf. [2]).

By Theorem 1, a subset $A \subseteq S = \{a_1, \ldots, a_n\}$ is the bottom of a chain if and only if each of its elements is basic. This is the same as saying that, for each $i = 1, \ldots, n$, the sequence $a_1, \ldots, a_i$ contains at least as many elements of $S - A$ as it does elements of A. If we replace elements of A by -1 and elements of $S - A$ by $+1$, and form partial sums, we obtain a nonnegative *lattice path* from $(0, 0)$ to (n, p), where $p = |S - A| - |A|$. Since the total number of symmetric chains is $\binom{n}{[n/2]}$, the following is immediate.

THEOREM 7. *The total number of nonnegative lattice paths of length n, starting from $(0, 0)$ and ending at (n, p) for some $p \geqslant 0$, is $\binom{n}{[n/2]}$.*

Since the number of chains whose bottoms have size k is $\binom{n}{k} - \binom{n}{k-1}$, we obtain the following refinement (known as Bertrand's ballot theorem).

THEOREM 8. *For fixed $p \geqslant 0$, the number of nonnegative lattice paths from $(0, 0)$ to (n, p) is $\binom{n}{k} - \binom{n}{k-1} = [(n - 2k + 1)/(n - k + 1)]\binom{n}{k}$, where $k = (n - p)/2$ (assuming that k is integral).*

By applying the same arguments to the case of divisors of an integer, we obtain a more general result, which can be expressed in terms of lattice paths which are required to be unimodal in successive blocks of fixed size. On the other hand, such paths are equivalent to *column-strict plane partitions* having two rows (see [8] for a survey and references). We express our final result in the form:

Let $\bar{k} = (k_1, k_2, \ldots, k_n)$ be a sequence of nonnegative integers. A (two-rowed) *column-strict plane partition of type* $\bar{k}$ is an array of k_1 1's, k_2 2's, etc., in two rows of sizes r_1 and r_2 such that $r_1 \geqslant r_2$, with each row nondecreasing, and each column strictly increasing. If $r = (r_1, r_2)$, then $\bar{r}$ is called the *shape* of the partition.

Suppose that $N = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$, and $D = p_1^{f_1} p_2^{f_2} \cdots p_n^{f_n}$ is a divisor of N which is the bottom of a symmetric chain (assuming that the divisors of N have been partitioned as described earlier). We associate with D a column-strict plane partition, as follows. Place f_1 1's in the second row, $k_1 - f_1$ 1's in the first row, f_2 2's in the second row, $k_2 - f_2$ 2's in the first row, and so forth, always adding each new number to the second row first. (As a matter of fact, f_1 will always be zero, so no 1's will appear in the second row.) The reader can check that, by Theorem 2, the result will be a column-strict plane partition; conversely, every column-strict plane partition gives rise to a divisor which is the bottom of a chain.

Let $w_i(N)$ be the number of divisors of N having weight i. Then the number of symmetric chains whose bottom has weight i is $w_i(N) - w_{i-1}(N)$, and we have

THEOREM 9. *Let $\bar{k} = (k_1, k_2, \ldots, k_n)$ be a sequence of nonnegative* *integers whose sum is Σ. For $i \leqslant \Sigma/2$, let $\bar{r} = (\Sigma - i, i)$. Then the number of column-strict plane partitions of type $\bar{k}$ and shape $\bar{r}$ is equal to $w_i(N) - w_{i-1}(N)$, where $N = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$ and $p_1, p_2, \ldots, p_n$ is an arbitrary sequence of distinct primes. The total number of column-strict plane partitions of type $\bar{k}$ (with two rows) is $w_{\Sigma/2}(N)$.*

For example, if $N = 2^2 \cdot 3^3 \cdot 5$, the bottoms of symmetric chains are $1, 3, 3^2, 5, 3 \cdot 5, 3^2 \cdot 5$. These in turn correspond to plane partitions as indicated below.

$$1 \leftrightarrow (1\ 1\ 2\ 2\ 2\ 3),$$

$$3 \leftrightarrow \begin{pmatrix} 1 & 1 & 2 & 2 & 3 \\ 2 & & & & \end{pmatrix},$$

$$3^2 \leftrightarrow \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & & \end{pmatrix},$$

$$5 \leftrightarrow \begin{pmatrix} 1 & 1 & 2 & 2 & 2 \\ 3 & & & & \end{pmatrix},$$

$$3 \cdot 5 \leftrightarrow \begin{pmatrix} 1 & 1 & 2 & 2 \\ 2 & 3 & & \end{pmatrix},$$

$$3^2 \cdot 5 \leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}.$$

The generating function for the w_i's is trivially

$$\begin{aligned} \sum w_i(N)\, x^i &= (1 + x + x^2)(1 + x + x^2 + x^3)(1 + x) \\ &= 1 + 3x + 5x^2 + 6x^3 + 5x^4 + 3x^5 + x^6, \end{aligned}$$

and the above statements can be verified immediately.

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