Polyhedral Study of Baldacci's Model for the Capacitated Vehicle Routing Problem and Its Extension to the Resource Constrained Shortest Path Problem

Haochi Jiang
Supervisor: Prof. Ralf Borndörfer

Freie Universität Berlin

February 11, 2024

Contents

Chapter 1

Introduction

The Resource Constrained Shortest Path Problem (RCSPP) is a fundamental problem in discrete optimization. RCSPP is aimed for finding a cost-optimal path, with additional resource constraints. For instance, in the transportation background, let us consider a truck departs from a factory for delivery. Assuming when this truck arrive a market, staffs of this visited market will take their demand. Clearly, the cost of the trip could include the fuel cost, time penalty and some other potential cost. The fuel tank of a truck is limited, so the resource constraint could be the capacity of fuel tank of this truck. This is a basic RCSPP in real world. The definition of RCSPP is very simple, howerver, indeed RCSPP is an $\mathcal{NP}-$hard problem [Hartmanis(1982)] even if the costs are all non-negative. The polynomial time solvable algorithms for RCSPP are not found yet. So, this is a challenging problem. However, it is worth to study how to solve RCSPP, because RCSPP is a sub-problem of many famous problems in transportation and has some applications in the transportation and management area.

The traditional linear programming model, which is directly derived the description of RCSPP. The idea behind the traditional model is very simple and natural. It just follows the linear programming model of shortest path problem and adds the resource constraints. Many researchers focus on developing algorithms to solve RCSPP based on this simple model. Aneja et al. proposed a network reduction algorithm [Aneja et al.(1983)Aneja, Aggarwal, and Nair]. The algorithm deletes and removes the nodes or arcs can't appear in feasible solutions, meanwhile record the current cost-least solution. After 6 years, Beasley and Christofides used the same rule as Aneja et al, but they also considered reduce the cost to solve a dual lagrangean problem [Beasley and Christofides(1989)]. In fact, using lagrangean relaxation is a most common technique in solving RCSPP. Mehlhorn and Ziegelmann use ellipsoid method to solve the dual problem [Mehlhorn and Ziegelmann(2000)]. They even computed the complexity of their algorithm for RCSPP with a single resource constraint. More tricks in solving RCSPP are Path ranking based approach, dynamic pro-

gramming and Branch&Backtracking algorithm. These method can close the gap of prime problem and the dual problem. The main idea of path ranking based approach is finding a feasible path with highest rank with respect to cost. The path ranking approach was first used by Handler and Zang [Handler and Zang(1980)]. They reduced the costs of arcs in the lagrangean problem to make their algorithm efficient. Apart of reducing the costs of arcs, using convexity or Chebyshev-like norm are common technique to accelerate the path ranking approach. The corresponding ideas can be found in [Santos et al.(2007)Santos, Coutinho-Rodrigues, and Current] and [Pugliese and Guerriero(2013)]. While trying use dynamic programming to solve RCSPP, every state in dynamic programming is a sub-path. Then the state-transformation function glues two sub-paths with the resource restriction. Dymanic programming are common used, for instances, Mehlhorn and Ziegelmann use the dynamic programming to solve the prime problem [Mehlhorn and Ziegelmann(2000)]. The last one, the Branching&Backtracking algorithm is based on the rules to search the feasible solutions. The oldest paper use this idea was written by Beasley and Christofides in 1989 [Beasley and Christofides(1989)]. And a recent work with Branching&Backtracking scheme was did by Muhandiramge and Boland [Muhandiramge and Boland(2009)].
The tableau shows our literature research.

	Network Reduction	Dual Problem	Path Ranking	Dynamic Programming	B&B scheme
Handler and Zang [1980]		×	×		
Aneja et al. [1983]	×			×	
Beasley and Christofides [1989]	×	×			×
Mehlhorn and Ziegelmann [2000]	×	×		×	
Santros et al [2007]			×		
DPuglia Pugliese and Guerriero [2013]			×		
Muhandiramge and Boland [2009]	×	×			×

Table 1.1: RCSPP Algorithms

In particular, RCSPP is a sub-problem of the Vehicle Routing Problem (VRP). VRP is aimed for finding a cost-optimal schedule for M many vehicles, such that all vehicles depart from a factory for delivery, and satisfy the demands of all markets. After delivery, all vehicle should return to the factory where they depart. A special case of VRP is named the Capacitated Vehicle Problem (CVRP) where all markets are required to be visited exactly once. Therefore, the solution set of CVRP, which is a polytope, is a face of solution set of VRP, because the degree constraints are tight. In order to connect with the RCSPP, through adding one artificial point, a trip, which means the destination is where the vehicles departure, can be transferred to a path by replacing the destination with this artificial point. After the transformation, every route of a vehicle in VRP is a resource constrained path. The following diagram shows the relationship

$$\begin{tikzcd}
\boxed{\text{VRP}} \arrow[r, "\text{route of every vehicle (path)}"] \arrow[d, "\text{Face}"'] & \boxed{\text{RCSPP}} \\
\boxed{\text{CVRP}} &
\end{tikzcd}$$

VRP and CVRP are important problems in discrete optimization. Researching CVRP is more active, because it has the degree constraints make sense in real world background. Drivers usually don't want to visit a same place again. Researchers have came up several different models to describe CVRP. In 1959, George Dantzig and John Ramser fristly asked the world about VRP [Dantzig and Ramser(1959)]. In 5 years, Balinski and Quand proposed a set partition model for VRP. They regarded all possible route as variables, and the degree constraints force that every node appears once in some routes. Let R be the set of all possible routes, d_r be the cost of route r and $a_{i,r}$ be a binary variable indicate that if the consumer i is visited in route r, the mathematical formula is

$$\begin{aligned}
&\min \sum_{r\in R} d_r y_r \\
&\text{subject to:} \\
&\sum_{\text{consumer } i} a_{i,r} = 1 \\
&\sum_{r\in R} y_r = m \\
&a_{i,r}, y_r \in \{0,1\}
\end{aligned}$$

Later researchers tried use column generation algorithm and dynamic programming to solve Balinski and Quand's model. But the results are not positive, only Agarwal, Mathur, and Salkin using column generation algorithm successfully solve instances with $15-25$ many consumers [Agarwal et al.(1989)Agarwal, Mathur, and Salkin]. In 2008, Baldacci, Christofides, and Mingozzi adding some inequalities and equalities constraints into this classical set partition model. Their improvement make the model is able to solve the instances with 37 to 127 many vertices [Baldacci et al.(2008)Baldacci, Christofides, and Mingozzi].

Apart from regarding the possible routes as variables, a natural idea is using edges as variables. This idea derives the vehicle flow model, which is first modeled by Laporte and Nobert [Laporte and Nobert(1983)]. They describe the problem as

$$
\begin{aligned}
&\sum_{\{i,j\}\in E} c_{ij}x_{ij} \\
&\text{subject to:} \\
&\sum_{j=1}^{n} x_{\{0,j\}} = 2M \\
&\sum_{i\neq j} x_{\{i,j\}} = 2 \quad \forall i \in [n] \\
&\sum_{i\in S}\sum_{j\in S^c} x_{\{i,j\}} \geq 2m(S) \quad \forall S \subseteq [n] \\
&x_{\{0,j\}} \in \{0,1,2\} \quad \forall j \in [n] \\
&x_{\{i,j\}} \in \{0,1\} \quad \forall i,j \in [n], i \neq j
\end{aligned}
$$

Here $m(S)$ represent the minimal number of vehicles such that the demand of all consumers in S can be met. Branch and Cut algorithm the most common trick in solving this vehicle flow model.

The disadvantage in the vehicle flow model is computing $m(S)$ is expensive. One idea to deal with the resource constraints is constructing a flow , every time, flow meet a consumer, the flow value will decrease by the demand of this consumer. Therefore, the resource constraints are equivalent to all flow values are non-negative. The commodity flow model follows this idea. Gavish and Graves [Gavish and Graves(1978)] extended Garvin's idea in an oil delivery problem [Garvin et al.(1957)Garvin, Crandall, John, and Spellman], gave their commodity flow model. For every consumer, they set variables indicate the flow from every edge in order to satisfies this consumer. In 2004, Baldacci raise his two commodity flow [Baldacci et al.(2004)Baldacci, Hadjiconstantinou, and Mingozzi], his computation result shows a progress.

Baldacci's model has advantages while compared to other models. The variables and constraints increase polynomially while other two typical models, the vehicle flow model and set partition model, increase exponentially, which make Baldacci's model is more solvable by current linear programming solver. For examples, Baldacci's model can solve a problem with 104 vertices size, which is not solved until Baldacci did it, and for some test benchmark, Baldacci's model has the best performance.

The main topic of my thesis, is studying the Baldacci's model and follow his idea to propose a new mixed integer programming model for RCSPP. We claim our new model is better than the classic RCSPP model in some sense.

In section 3, we delve into Baldacci's two-commodity flow model for the Vehicle Routing Problem. Deriving implicit equations from the original model and analyzing the redundancy within these equations lead to an estimation of the upper bound for the dimension of LP-relaxation polytopes. By observing how dimension reduces, we identify some non-facet-defining inequalities. In the case of unit demand, where all consumers share the same demand, we determine the dimension of MIP polytops and identify some facet-defining inequalities.

In section 4, we follow Baldacci's idea and propose a new model for the Resource-Constrained Shortest Path Problem. Using a simple example, we show the potential advantages of our model. And we find a graph transformation, such that the solution of MIP of original graph has the same objective value as the solution of LP of the transferred graph. Using the similar analysis as previous part, along with constructive proofs, we determine the dimensions of LP-relaxation polytopes. Similar methods lead to the dimensions of MIP polytopes and facet-defining inequalities in the unit demand case. Furthermore, we introduce some cuts for our new model, which can completely describe the MIP polytopes in a simple case. Finally, our model is compared to the traditional model using graphs derived from Delaunay triangulation.

Chapter 2

Notations and Preliminaries

The preliminaries are basic knowledgement about polytopes and optimization.
I am using the same notations as Prof. Ralf Borndörfer used. I emphasis some uncommon notation here. Let ij be abbreviation of the tuple (i, j) if the context is clear, and in a undirection graph $G = (V, E)$ $\gamma(u) := \{v : \{u, v\} \in E\}$, and in a directed graph $D = (V, A)$, $\gamma^+(u) := \{v : (u, v) \in A\}$ and $\gamma^-(u) := \{v : (v, u) \in A\}$. And in simple graph, we use a sequence of vertices $(u_1, u_2, ..., u_n)$ to represent a path.

Chapter 3

Capacitated Vehicle Routing Problem

This section introduces Baldacci's approach to the capacitated vehicle routing problem. Subsection 3.1 gives a formal statement of the problem. Subsection 3.2 states the integer program that was suggested by Baldacci [Baldacci et al.(2004)Baldacci, Hadjiconstantinou, and Mingozzi]. Subsection 3.3 shows some polyhedral results on Bladacci's model. Subsection 3.4 shows more polyhedral results under a special case.

3.1 Problem Description

Baldacci proposed an mixed integer programming model for the CVRP, that is based on the idea of flows and residual flows.

Definition 3.1.1 (Capacitated Vehicle Routing Problem (CVRP)) ***Given:*** *Undirected simple graph $G = (V, E)$, two special vertices, the sink $s \in V$ and the target $t \in V$, non-negative integer edge costs $c \in \mathbb{N}^E_{\geq 0}$, natural numbers $M, Q \in \mathbb{N}$ and node demands $q \in \mathbb{N}^N_{\geq 0}$, where $N = V \setminus \{s, t\}$. A $M-$route f consists of M simple st-path $\{p_i\}_{i=1}^M$ in G, its cost and demand are $c(f)$ and $q(f)$, respectively, and it is feasible if $q(p_i) \leq Q$ holds for all i. A $M-$schedule is a feasible $M-$route that visits every nodes except s, t exactly once.* ***Sought:*** *A $M-$schedule of minimal total cost.*

3.2 Baldacci's Model

Consider a undirected graph $G = (V, E)$, where $E = \{\{u, v\} : u, v \in N\} \cup \{\{s, u\} : u \in N\} \cup \{\{v, t\} : v \in N\}$, and a directed graph $D(G) = (V, A(E))$ such that $A(E) = \{uv : \{u, v\} \in E\} \cup \{vu : \{u, v\} \in E\}$. For all edges $\{u, v\} \in E$, assign an edge variable $\xi_{\{u,v\}}$, and for all arc $uv \in A(E)$, assign a flow variable x_{uv}. Then the Baldacci's model [Baldacci et al.(2004)Baldacci, Hadjiconstantinou, and Mingozzi] is the following

$$\min \sum_{\{u,v\}\in E} c_{\{u,v\}}\xi_{\{u,v\}}$$

subject to:

$$\sum_{v\in\gamma(u)} (x_{vu} - x_{uv}) = 2q_u \quad \forall u \in N \tag{3.1}$$

$$\sum_{v\in N} x_{sv} = q(N) \tag{3.2}$$

$$\sum_{v\in N} x_{vs} = MQ - q(N) \tag{3.3}$$

$$\sum_{v\in N} x_{tv} = MQ \tag{3.4}$$

$$x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \quad \forall \{u,v\} \in E \tag{3.5}$$

$$\sum_{v\in\gamma(u)} \xi_{\{u,v\}} = 2 \quad \forall u \in N \tag{3.6}$$

$$x_{uv} \geq 0,\ x_{vu} \geq 0,\ \xi_{\{u,v\}} \in \{0,1\} \quad \forall\ \{u,v\} \in E$$

In this model, (3.6) is for the degree constraint. (3.1) is for satisfying the demands of all consumers. In the solution, here is a flow and a residual flow represent the load and empty space of vehicle respectively. (3.5) implies the capacity of each vehicle and connects variables $\xi_{\{u,v\}}$, x_{uv} and x_{vu}. The remaining equation constraints are for setting the initial condition.

3.3 Polyhedral Study of Baldacci's Model

3.3.1 Complete the equation system

Let B_{IP} be the polytope of Baldacci's model and B_{LP} be the polytope of Baldacci's model after relaxing variable $\xi_{\{u,v\}} \in \{0,1\}$ to $\xi_{\{u,v\}} \in [0,1]$ for all $\{u,v\} \in E$. It's clear that $B_{IP} \subseteq B_{LP}$. Our first step is finding some implicit valid equations for B_{LP}, these inequalities can help us to approach the dimension of B_{LP}.

Observation 1 $x_{ut} = 0$ *for all* $u \in N$

Proof:

$$\begin{aligned}
0 &= \sum_{uv\in A(E)} (x_{uv} - x_{uv}) = \sum_{u\in V}\sum_{v\in\gamma(u)} (x_{uv} - x_{vu}) \\
&= \sum_{v\in N}(x_{sv} - x_{vs}) + \sum_{v\in N}(x_{tv} - x_{vt}) + \sum_{u\in N}\sum_{v\in\gamma(u)} (x_{uv} - x_{vu}) \\
&= \sum_{u\in N} q_u - (MQ - \sum_{u\in N} q_u) + MQ - \sum_{u\in N} x_{ut} + \sum_{u\in N}(-2q_u) \\
&= -\sum_{u\in N} x_{ut}
\end{aligned}$$

Since $x_{ut} \geq 0$ for all $u \in N$, therefore $x_{ut} = 0$ for all $u \in N$, and these equations are valid for B_{LP}. Now we do some operations to draw equations that are independent to others. Plug (3.5) into (3.6), yields

$$\sum_{v\in\gamma(u)} x_{uv} + x_{vu} = 2Q$$

Combined with (3.1),

$$\begin{cases} \displaystyle\sum_{v\in\gamma(u)} x_{vu} - x_{uv} = 2q_u \\ \displaystyle\sum_{v\in\gamma(u)} x_{vu} + x_{uv} = 2Q \end{cases} \Rightarrow \begin{cases} \displaystyle\sum_{v\in\gamma(u)} x_{vu} = Q + q_i \\ \displaystyle\sum_{v\in\gamma(u)} x_{uv} = Q - q_i \end{cases}$$

The Baldacci's model is equivalent to

$$\min \sum_{\{u,v\}\in E} c_{\{u,v\}}\xi_{\{u,v\}}$$

subject to:

$$\sum_{v\in\gamma(u)} x_{vu} = Q + q_u \quad \forall u \in N \tag{3.7}$$

$$\sum_{v\in\gamma(u)} x_{uv} = Q - q_u \quad \forall u \in N \tag{3.8}$$

$$\sum_{u\in N} x_{su} = \sum_{u\in N} q_u \tag{3.9}$$

$$\sum_{u\in N} x_{us} = MQ - \sum_{u\in N} q_u \tag{3.10}$$

$$\sum_{u\in N} x_{tu} = MQ \tag{3.11}$$

$$x_{ut} = 0 \quad \forall u \in N \tag{3.12}$$

$$x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \quad \forall\{u,v\} \in E \tag{3.13}$$

$$x_{uv} \geq 0,\ x_{vu} \geq 0,\ \xi_{\{u,v\}} \in \{0,1\} \quad \forall\ \{u,v\} \in E$$

We eliminate some equation constraints involve the variables ξ_{ij}, such that the variable ξ_{ij} only appears exactly once in (3.13), so these equations from (3.13) are independent to equation system in (3.7)-(3.12). Therefore, after the operation, equations from (3.13) are irredundant.

Recall that

$$0 = \sum_{uv\in A(E)} (x_{uv} - x_{uv}) = \sum_{v\in N}(x_{sv} - x_{vs}) + \sum_{v\in N}(x_{tv} - x_{vt}) + \sum_{u\in N}\sum_{v\in\gamma(u)} (x_{uv} - x_{vu})$$

The equation above tells the equation system from (3.7)-(3.12) is redundant. Therefore, we need to determine the number of redundant equations in this model. In the next section, we will explain how to find it.

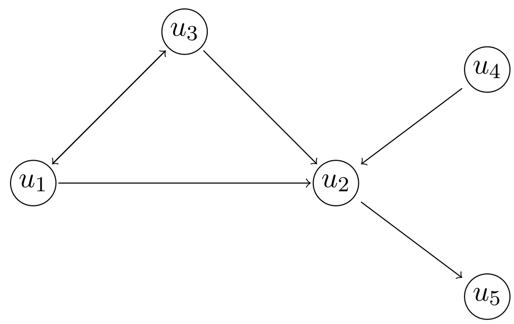
3.3.2 A method for reading the irredundancy

Consider a directed connected graph $D_1(G) = (U, \bar{A})$. For every arc (u, v) in $\bar{A}$, assign a variable x_{uv}. Since $D_1(G)$ is connected, at least one of $\gamma^+(u)$, $\gamma^-(u)$ is non-empty for an arbitrary $u \in U$. Then for every non-empty set $\gamma^+(u)$ and $\gamma^-(u)$, we assign the following equations with proper value $\beta_u, \bar{\beta}_u$

$$\sum_{v \in \gamma^+(u)} x_{uv} = \beta_u \quad \text{or} \quad \sum_{v \in \gamma^-(u)} x_{vu} = \bar{\beta}_u \tag{3.14}$$

Here is an example about how we construct the equation system.

Example 3.3.1 *Let the directed graph be*



From this graph we have

$$\begin{aligned}
\gamma^+(u_1) &= \{u_2, u_3\} & \qquad \gamma^-(u_1) &= \{u_3\} \\
\gamma^+(u_2) &= \{u_5\} & \qquad \gamma^-(u_2) &= \{u_1, u_3, u_4\} \\
\gamma^+(u_3) &= \{u_1, u_2\} & \qquad \gamma^-(u_3) &= \{u_1\} \\
\gamma^+(u_4) &= \{u_2\} & \qquad \gamma^-(u_4) &= \emptyset \\
\gamma^+(u_5) &= \emptyset & \qquad \gamma^-(u_5) &= \{u_2\}
\end{aligned}$$

Therefore this graph will give the following equation system with proper choice of β and $\bar{\beta}$

$$\begin{aligned}
& x_{u_1,u_2} + x_{u_1,u_3} = \beta_{u_1} \\
& x_{u_3,u_1} = \bar{\beta}_{u_1} \\
& x_{u_2,u_5} = \beta_{u_2} \\
& x_{u_1,u_2} + x_{u_3,u_2} + x_{u_4,u_2} = \bar{\beta}_{u_2} \\
& x_{u_3,u_1} + x_{u_3,u_2} = \beta_{u_3} \\
& x_{u_1,u_3} = \bar{\beta}_{u_3} \\
& x_{u_4,u_2} = \beta_{u_4} \\
& x_{u_2,u_5} = \bar{\beta}_{u_5}
\end{aligned}$$

Now write the whole equation system as a matrix form $Cx = d$. Here is a bijection such that we can find the corresponding node and the arc direction of every row of C. Therefore, the rows in C can be represent as $c_{\gamma^+(u)}$ and $c_{\gamma^-(u)}$ with respect to $\gamma^+(u)$ and $\gamma^-(u)$. Now, we are going to discuss about a vector a such that $\boldsymbol{a}^{\mathrm{T}}C = \mathbf{0}^{\mathrm{T}}$, since $\dim\{\boldsymbol{a} : \boldsymbol{a}^{\mathrm{T}}C = \mathbf{0}^{\mathrm{T}}\}$ will give us the rank(C). Let $\alpha^{+}(u)$, $\alpha^{-}(u)$ be entries of $\boldsymbol{a}$, such that

$$\boldsymbol{a}^{\mathrm{T}}C = \sum_{\substack{u\in V\\ \gamma^{+}(u)\neq\emptyset}} \alpha^{+}(u)c_{\gamma^{+}(u)} + \sum_{\substack{u\in V\\ \gamma^{-}(u)\neq\emptyset}} \alpha^{-}(u)c_{\gamma^{-}(u)}$$

At the beginning, here are two observations.

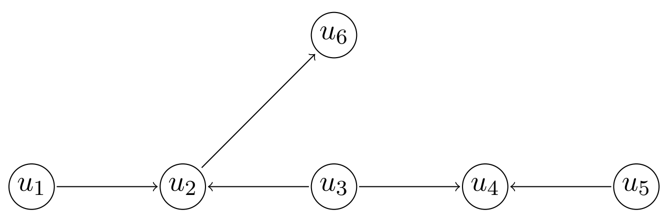
Observation 2 *Only in $c_{\gamma^{+}(u)}$ and $c_{\gamma^{-}(v)}$, the column respects to variable x_{uv} is non-zero.*

Observation 3 *An arc $(u, v) \in D_1(G)$ implies a relationship that $\alpha^{+}(u) = -\alpha^{-}(v)$ since this is the only way to eliminate variable x_{uv}*

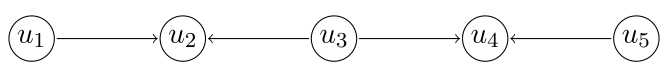
Since the graph $G(U, \bar{A})$ is connected, we could and we want to extend our observations such that we obtain an equation involve all vertices of $G(U, \bar{A})$. Therefore, we need a special "path" to transfer this property from a vertex to another vertex.

Definition 3.3.1 (Alternating Path) *Let $(u_1, \cdots, u_n)$ be a sequence of distinct vertices in $G(U, \bar{A})$. If here is a sequence of arcs $(a_1, \cdots, a_{n-1})$, such that for all $k \in [n-1]$, $a_{2k+1} = (u_{2k+1}, u_{2k+2})$, $a_{2k} = (u_{2k+1}, u_{2k})$ or $a_{2k+1} = (u_{2k+2}, u_{2k+1})$, $a_{2i} = (u_{2k}, u_{2k})$, then we call this sequence $(u_1, a_1, u_2, \cdots, a_{n-1}, u_n)$ alternating path, and the number of arcs contained in an alternating path is called the length of this alternating path.*

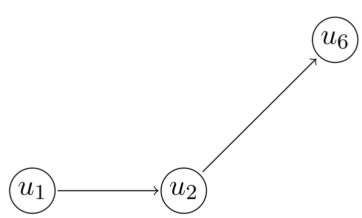
Example 3.3.2 *Here is an example for the alternating path I defined*



The sequence $(u_1, e_{u_1,u_2}, u_2, e_{u_3,u_2}, u_3, e_{u_3,u_4}, u_4, e_{u_5,u_4}, u_5)$ is an alternating path.



And the sequence $(u_1, e_{u_1,u_2}, u_2, e_{u_2,u_6}, u_6)$ is not an alternating path.



Lemma 1 *If here is an alternating path $(u_1, a_1, ..., a_{n-1}, u_n) = P$ in the graph $G(U, \bar{A})$. Then for all $k \in [n]$, we have*

- *when $a_1 = (u_1, u_2)$:*

$$\alpha^+(u_1) = (-1)^{k-1}\alpha^{sign\left((-1)^{k-1}\right)}(u_k)$$

- *when $a_1 = (u_2, u_1)$:*

$$\alpha^-(u_1) = (-1)^{k-1}\alpha^{sign\left((-1)^{k-1}\right)}(u_k)$$

Proof: Induction on the length of alternating path.
For these alternating paths have length 1, it's just the observation 3.
Suppose the statement is true for all alternating paths with length $k-1$.
Assume $a_1 = (u_1, u_2)$, then $a_k = (a_k, a_{k+1})$ if $k \equiv 1 \pmod 2$ and $a_k = (a_{k+1}, a_k)$ if $k \equiv 0 \pmod 2$. In the first case ($k \equiv 1 \pmod 2$), we have

$$\alpha^+(u_1) = \alpha^+(u_k)$$
$$\alpha^+(u_k) = -\alpha^-(u_{k+1})$$

Therefore $\alpha^+(u_1) = \alpha^-(u_{k+1})$. In all other cases, the proofs are similar.

When we transfer a undirected graph $G = (V, E)$ to a directed graph $D(G) = (V, A)$ by replacing a edge with two arcs which have opposite direction, we actually could obtain two "parallel" alternating path and therefore two series of equation. Now, we want to connect these two equations.

Theorem 3.3.2 *Let $P = (u_1, a_1, \cdots, a_{2k}, u_{2k+1}) \subseteq G(U, \bar{A})$ be an alternating path with length $2k$. If $-P := (u_{2k+1}, -a_{2k}, u_{2k}, \cdots, -a_1, u_1)$ exists in $G(U, \bar{A})$, and here is an arc $(u_1, u_{2k+1}) \in \bar{A}$. Then,*

$$\forall u \in V(P),\ \alpha^+(u) = -\alpha^-(u) \text{ and } \forall u, v \in V(P),\ |a^+(u)| = |a^+(v)|$$

Moreover, let N be the set of vertices are in P or adjacent to P. If $\boldsymbol{a}^{\mathrm{T}}C = \mathbf{0}^{\mathrm{T}}$, then $|\alpha^{\mathrm{sign}_1}(u)| = |\alpha^{\mathrm{sign}_2}(v)|$ for all $u, v \in N$, where $\mathrm{sign}_1, \mathrm{sign}_2 \in \{-, +\}$

Proof:
Since $k \geq 1$, then here exist a point u_l, such that both $l-1$, $2k+1-l$ are odd. Therefore, from the Lemma 1, we have

$$\alpha^+(u_1) = -\alpha^-(u_l) \quad \alpha^-(u_1) = -\alpha^+(u_l)$$
$$\alpha^+(u_{2k+1}) = -\alpha^-(u_l) \quad \alpha^-(u_{2k+1}) = -\alpha^+(u_l)$$

Since here is an arc (u_1, u_{2k+1}), so $\alpha^+(u_1) = -\alpha^-(u_{2k+1})$. This means $\alpha^+(u_l) = -\alpha^-(u_l)$. Then Lemma 1 can be extended, and have an equation involve all vertices.

In this section, given a directed graph, for any vertices in this graph, we associate it with equations of the form (3.14). Then we can just read the graph and use the Theorem 3.3.2 to know if here exists redundant equations.

3.3.3 Dimension and Facets Results for B_{LP}

Now, we return to (3.7) − (3.12) in Baldacci's model. In this model, every vertices has two equations of the form (3.14), and the graph has many edges, so we can always find a large even alternating path P with vertex ordered as $(u_1, \cdots, u_{2k+1})$, such that u_1, u_{2k+1} are adjacent, $-P$ exist in the induced graph $D(G) = (V, A(E))$ and all vertices in V are either contained in P or adjacent to some vertices in P. So, exactly one equation in the equation system is redundant. Assume $|N| = n \geq 3$, then

$$\dim B_{LP} \leq 3(\binom{n+2}{2} - 1) - (\binom{n+2}{2} - 1) - (3n+3) + 1 = n^2 - 2$$

We haven't got dimension yet, we only know the upper bound of dimension. But we can decide some inequalities aren't facets by observing how dimension changes when the inequality is active.

Observation 4 *$\xi_{\{u,v\}} \geq 0$, where $u, v \in N$, isn't a facet defining inequality when $|N| \geq 5$.*

proof:

$$\begin{cases} \xi_{\{u,v\}} = 0 \\ x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \\ x_{uv}, x_{vu} \geq 0 \end{cases} \Rightarrow x_{uv} = x_{vu} = \xi_{\{u,v\}} = 0$$

It's sufficient to check that $x_{uv} = 0$ isn't a linear combination of equations in (3.7)-(3.13) and $\xi_{\{u,v\}} = 0$. Applying then Theorem 3.3.2 for a subgraph induced by $V \setminus \{u, v\}$, and then check the whole system.

Observation 5 *If $M = 1$, then $\xi_{\{s,u\}} \leq 1$, $\xi_{\{u,t\}} \leq 1$ is not a facet defining inequality.*

Proof:
Note we have

$$\sum_{u \in N} x_{tu} = Q$$
$$x_{ut} = 0 \quad \forall u \in N$$
$$x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \quad \forall \{u, v\} \in E$$

Therefore, we have

$$\sum_{u \in N} Q\xi_{\{t,u\}} = \sum_{u \in N} (x_{tu} + x_{ut}) = \sum_{u \in N} x_{tu} = Q$$
$$\Rightarrow \sum_{u \in N} \xi_{\{t,u\}} = 1$$

If for one u_1 such that $\xi_{\{u_0,t\}} = 1$, then it implies that $\xi_{\{u,t\}} = 0$ for all $u \neq u_0$. Similarly, from

$$\sum_{u \in N} x_{us} = Q - \sum_{u \in N} q_u$$
$$\sum_{u \in N} x_{su} = \sum_{u \in N} q_u$$

We get

$$\sum_{u \in N} \xi_{\{s,u\}} = 1$$

This implies, for an arbitrary $u \in N$, if $\xi_{\{s,u\}} = 1$, then $\xi_{s\{s,v\}} = 0 \quad \forall v \in N, v \neq u$

3.4 Dimension and Facets Result for Unit Demand Case

If $q_u = 1$ for all $u \in V$, we call this case the unit demand case. Therefore, in this case all vertices in N are "equivalent", in the sense we construct get another feasible by exchange vertices. Recall Baldacci's model under this special case is

$$\min \sum_{\{u,v\}\in E} c_{\{u,v\}}\xi_{\{u,v\}}$$

subject to:

$$\sum_{v\in\gamma(u)} x_{vu} = Q + q_u \quad \forall u \in N \tag{3.15}$$

$$\sum_{v\in\gamma(u)} x_{uv} = Q - q_u \quad \forall u \in N \tag{3.16}$$

$$\sum_{u\in N} x_{su} = \sum_{u\in N} q_u \tag{3.17}$$

$$\sum_{u\in N} x_{us} = MQ - \sum_{u\in N} q_u \tag{3.18}$$

$$\sum_{u\in N} x_{tu} = MQ \tag{3.19}$$

$$x_{ut} = 0 \quad \forall j \in N \tag{3.20}$$

$$x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \quad \forall \{u,v\} \in E \tag{3.21}$$

$$x_{uv} \geq 0,\ x_{vu} \geq 0,\ \xi_{\{u,v\}} \in \{0,1\} \quad \forall\ \{u,v\} \in E$$

In previous section, we have obtained a upper bound of the dimension of B_{LP}. In this section, we will show in this special case, we can determine dimension of B_{LP} and B_{IP}. Precisely, that is the following theorem

Theorem 3.4.1 *Under the unit demand case, If $Q \geq 5$, $M \geq 2$ and $MQ > n = |N| > (M-1)+Q$, the polytopes from Baldacci's model are denoted by B^{u+}_{IP}, B^{u+}_{LP} respectively. A result is B^{u+}_{IP}, B^{u+}_{LP} are $(n^2-2)-$dimensional polytopes.*

It's sufficient to show $n^2-2 = \dim B_{IP}$. We will use a similar idea as [Campos et al.(1991)Campos, Corberan, and Mota] by showing every valid equation for B_{IP} has to be a linear combination of equations in (3.15)-(3.21).

Let $\boldsymbol{c}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{\xi}\end{pmatrix} = \beta$ be an arbitrary valid equation for B_{IP}. Let $T\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{\xi}\end{pmatrix} = \boldsymbol{t}$ be the matrix of equations in (3.15)-(3.21). Since $\boldsymbol{c}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{\xi}\end{pmatrix} = \beta$ is a valid equation, it's enough to show $\boldsymbol{c}^{\mathrm{T}} \in \mathrm{row}(T)$. We know the every column of $\boldsymbol{c}^{\mathrm{T}}$ and T corresponding to one unique variable. So, we use $\boldsymbol{c}^{\mathrm{T}}\big|_{\{x_{uv}\}}$ and $T\big|_{\{x_{uv}\}}$ to denote the column corresponding to x_{uv}, same for other variables. We will take the following steps to prove Theorem 3.4.1.

Step 1:

Finding a full-rank square submatrix M of T. Fix two vertices v_1, v_2 in V, let

$$\begin{aligned} I &= \{x_{su}\}_{u\in N} \cup \{x_{us}\}_{u\in N} \cup \{x_{ut}\}_{u\in N} \cup \{x_{v_1v_2}, x_{tv_1}\} \cup \{\xi_{\{u,v\}}\}_{uv\in E} \\ I_x &= \{x_{su}\}_{u\in N} \cup \{x_{us}\}_{u\in N} \cup \{x_{ut}\}_{u\in N} \cup \{x_{v_1v_2}, x_{tv_1}\} \end{aligned}$$

Then $M := T\big|_I$ is a full-rank square matrix.
Proof:
It's clear that every row of $T\big|_I$ from (3.21) is independent to all other rows, since the variable $\xi_{\{uv\}}$ only appears in (3.21). Therefore, it's enough to show that the matrix $T'\big|_I$ which consists of the rows from (3.15)-(3.20) in $T\big|_I$ is full rank.
Construct a directed graph $D_2(G) = (V, A')$ such that $A' = \{(u,v) : x_{uv} \in I_x\}$. Then the equations system T'_I represents is of the form (3.14) in $D_2(G)$. Therefore, using the similar analysis in previous section shows $T'\big|_I$ is row full rank matrix.
Step 2:
Since M is full rank, $\boldsymbol{\nu}^{\mathrm{T}}(T\big|_I) = (\boldsymbol{c}^{\mathrm{T}}\big|_I)$ has a unique solution. Let $\boldsymbol{r}^{\mathrm{T}} = \boldsymbol{c}^{\mathrm{T}} - \boldsymbol{\nu}^{\mathrm{T}}T$, it sufficient to show $\boldsymbol{r}^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}}$. Since $\boldsymbol{c}^{\mathrm{T}}$ is a valid equality. So, let s_1, s_2 be two different schedule and $\boldsymbol{z}(s_1), \boldsymbol{z}(s_2)$ be the embedded points in vector space, we have

$$\langle \boldsymbol{r}, (\boldsymbol{z}(s_1) - \boldsymbol{z}(s_2))\rangle = 0 \tag{3.22}$$

Step 3:
First we pick two arbitrary vertices u_1, u_2, then take two schedule s_1, s_2, such that $p_1 \subseteq s_1, p_2 \subseteq s_2$ and $s_1 \setminus p_1 = s_2 \setminus p_2$

$$\begin{aligned} p_1 &= (s, u_{i_1}, u_1, u_2, u_{i_3}, \cdots, u_{i_m}, t) \\ p_2 &= (s, u_{i_1}, u_2, u_1, u_{i_3}, \cdots, u_{i_m}, t) \end{aligned}$$

Use the equation (3.22), we have a equation. For convenience, in this proof, $\boldsymbol{r}^{\mathrm{T}}_{u_iu_j}$ are abbreviate as $\boldsymbol{r}^{\mathrm{T}}_{ij}$

$$\begin{aligned} &(m-1)(\boldsymbol{r}^{\mathrm{T}}_{i_11} - \boldsymbol{r}^{\mathrm{T}}_{i_12}) + (m-2)(\boldsymbol{r}^{\mathrm{T}}_{12} - \boldsymbol{r}^{\mathrm{T}}_{21}) + (m-3)(\boldsymbol{r}^{\mathrm{T}}_{2i_3} - \boldsymbol{r}^{\mathrm{T}}_{1i_3}) + \\ &\big(Q-(m-1)\big)(\boldsymbol{r}^{\mathrm{T}}_{1i_1} - \boldsymbol{r}^{\mathrm{T}}_{2i_1}) + \big(Q-(m-2)\big)(\boldsymbol{r}^{\mathrm{T}}_{21} - \boldsymbol{r}^{\mathrm{T}}_{12}) + \big(Q-(m-3)\big)(\boldsymbol{r}^{\mathrm{T}}_{i_32} - \boldsymbol{r}^{\mathrm{T}}_{i_31}) = 0 \end{aligned} \tag{3.23}$$

Step 4:
Similarly, take two schedule with different paths

$$\begin{aligned} p_1 &= (s, u_{i_m}, u_{i_1}, u_1, u_2, u_{i_3}, \cdots, u_{i_{m-1}}, t) \\ p_2 &= (s, u_{i_m}, u_{i_1}, u_2, u_1, u_{i_3}, \cdots, u_{i_{m-1}}, t) \end{aligned}$$

And drive an equation

$$\begin{aligned} &(m-2)(\boldsymbol{r}^{\mathrm{T}}_{i_11} - \boldsymbol{r}^{\mathrm{T}}_{i_12}) + (m-3)(\boldsymbol{r}^{\mathrm{T}}_{12} - \boldsymbol{r}^{\mathrm{T}}_{21}) + (m-4)(\boldsymbol{r}^{\mathrm{T}}_{2i_3} - \boldsymbol{r}^{\mathrm{T}}_{1i_3}) \\ &+ \big(Q-(m-2)\big)(\boldsymbol{r}^{\mathrm{T}}_{1i_1} - \boldsymbol{r}^{\mathrm{T}}_{2i_1}) + \big(Q-(m-3)\big)(\boldsymbol{r}^{\mathrm{T}}_{21} - \boldsymbol{r}^{\mathrm{T}}_{12}) + \big(Q-(m-4)\big)(\boldsymbol{r}^{\mathrm{T}}_{i_32} - \boldsymbol{r}^{\mathrm{T}}_{i_31}) = 0 \end{aligned} \tag{3.24}$$

Subtract (3.24) from (3.23)

$$(\boldsymbol{r}_{i_1 1}^{\mathrm{T}} - \boldsymbol{r}_{i_1 2}^{\mathrm{T}}) + (\boldsymbol{r}_{12}^{\mathrm{T}} - \boldsymbol{r}_{21}^{\mathrm{T}}) + (\boldsymbol{r}_{2 i_3}^{\mathrm{T}} - \boldsymbol{r}_{1 i_3}^{\mathrm{T}}) = (\boldsymbol{r}_{1 i_1}^{\mathrm{T}} - \boldsymbol{r}_{2 i_1}^{\mathrm{T}}) + (\boldsymbol{r}_{21}^{\mathrm{T}} - \boldsymbol{r}_{12}^{\mathrm{T}}) + (\boldsymbol{r}_{i_3 2}^{\mathrm{T}} - \boldsymbol{r}_{i_3 1}^{\mathrm{T}}) \tag{3.25}$$

After that take

$$p_1 = (s, u_1, u_2, u_{i_3}, \cdots, u_{i_m}, u_{i_1}, t)$$
$$p_2 = (s, u_2, u_1, u_{i_3}, \cdots, u_{i_m}, u_{i_1}, t)$$

drive

$$\begin{aligned} &(m-1)(\boldsymbol{r}_{12}^{\mathrm{T}} - \boldsymbol{r}_{21}^{\mathrm{T}}) + (m-2)(\boldsymbol{r}_{2 i_3}^{\mathrm{T}} - \boldsymbol{r}_{1 i_3}^{\mathrm{T}}) \\ &+ \big(Q-(m-1)\big)(\boldsymbol{r}_{21}^{\mathrm{T}} - \boldsymbol{r}_{12}^{\mathrm{T}}) + \big(Q-(m-2)\big)(\boldsymbol{r}_{i_3 2}^{\mathrm{T}} - \boldsymbol{r}_{i_3 1}^{\mathrm{T}}) = 0 \end{aligned} \tag{3.26}$$

From the equations above, we can drive

$$m(\boldsymbol{r}_{i_1 1}^{\mathrm{T}} - \boldsymbol{r}_{i_1 2}^{\mathrm{T}}) = (m-Q)(\boldsymbol{r}_{1 i_1}^{\mathrm{T}} - \boldsymbol{r}_{2 i_1}^{\mathrm{T}}) \tag{3.27}$$

If here is a $st-$path with length $Q+1$ in the solution set, then $\boldsymbol{r}_{i_1 1}^{\mathrm{T}} = \boldsymbol{r}_{i_1 2}^{\mathrm{T}}$. Since the choice of u_{i_1}, u_1, u_2 are arbitrary, therefore, indeed we have $\boldsymbol{r}_{ij}^{\mathrm{T}} = \boldsymbol{r}_{ik}^{\mathrm{T}}$ for any $u_i, u_j, u_k \in N$. Similarly, If here is a $st-$path with length at most Q in the solution set, then $\boldsymbol{r}_{ji}^{\mathrm{T}} = \boldsymbol{r}_{ki}^{\mathrm{T}}$ for any $u_i, u_j, u_k \in N$. So, $\boldsymbol{r}_{ij}^{\mathrm{T}} = \boldsymbol{r}_{kl}^{\mathrm{T}}$ for any $u_i, u_j, u_k, u_l \in N$.
Step 5: Let

$$p_1 = (s, u_{i_1}, u_{i_2}, \cdots, u_{i_m}, u_1, u_2, t)$$
$$p_2 = (s, u_{i_1}, u_{i_2}, \cdots, u_{i_m}, u_2, u_1, t)$$

With the result in Step 4, we get

$$\boldsymbol{r}_{ti}^{\mathrm{T}} = \boldsymbol{r}_{tj}^{\mathrm{T}} \quad \forall u_i, u_j \in N$$

Because of the construction of $\boldsymbol{\nu}$, $\boldsymbol{r}^{\mathrm{T}}\big|_I = \boldsymbol{0}^{\mathrm{T}}$, therefore $\boldsymbol{r}^{\mathrm{T}} = \boldsymbol{0}^{\mathrm{T}}$.
The requirement on the length of a path needs the extra assumption about M, Q
Now, since we know the dimension of polytope, we can go to decide the facets defining inequality of B_{LP}^{u+}, B_{IP}^{u+}

Observation 6 *$\xi_{\{u,t\}} \geq 0$ and $x_{tu} \geq 0$ defines facet of B_{LP}^{u} and B_{IP}^{u}*

Proof :
Similar to the dimension proof, by taking

$$I = \{\{x_{su}\}_{u\in N}, \{x_{us}\}_{u\in N}, \{x_{ut}\}_{u\in N}, x_{u_1,u_2}, x_{tu_j}, x_{tu_k}, \{\xi_{u,v}\}_{\{u,v\}\in E}\}$$

Observation 7 *The polytope B_{LP} is invariant under relabelling the vertices in N.*

This observation is based on the graph is very "complete", relabelling the vertices doesn't change the equation system of B_{LP}.

Theorem 3.4.2 *$x_{uv} \geq 0$ defines a facet of B_{LP}^{u+} for all $u, v \in N$, if $|N| \geq 16$*

Proof : Let $n = |N|$, we know $\dim B_{LP}^{u+} = n^2-2$, so the number of facet defining inequalities is at least n^2-1. Since B_{LP}^{u+} is a special case of B_{LP} and $N \geq Q \geq 5$, we know that $\xi_{\{u,v\}} \geq 0$ isn't a facet. Here are totally $2(n^2+3n) - \binom{n}{2}$ many potential facet defining inequalities and $n^2 - n$ many inequalities of the form $x_{uv} \geq 0$ for $u, v \in N$. Therefore, if $n \geq 16$, then by pigeonhole principle, $x_{uv} \geq 0$ is a facet for some $u, v \in N$. However, because of Observation 7, $x_{uv} \geq 0$ is a facet for all $u, v \in N$

Chapter 4

Resource Constrained Shortest Path Problem

In section 4.1, we state the RSCPP and the basic traditional model. In section 4.2, we follow the idea of Baldacci and give a new model for RCSPP, and illustrate this model is better than the basic model. In section 4.3, we discuss the dimension of the LP-relaxation. In section 4.4, we discuss the dimension of the lattice polytope while all demand are units. In section 4.5, we give a series of cuts, these cuts are facets under the unit demand case and works well when capacity $Q = 2$. In section 4.6, we will show some computation results

4.1 Problem Description and Traditional Model

Definition 4.1.1 (Resource Constrained Shortest Path Problem (RCSPP)) ***Given:*** *directed simple graph $G = (V, A)$, two special vertices, the sink $s \in V$ and the target $t \in V$, non-negative integer edge costs $c \in \mathbb{N}^E_{\geq 0}$, natural numbers $Q \in \mathbb{N}$ and node demands $q \in \mathbb{N}^N_{\geq 0}$, where $N = V \setminus \{s, t\}$. A $1-$route is a simple st-path p in G, its cost and demand are $c(p)$ and $q(p)$, respectively, and it is feasible if $q(p) \leq Q$ holds.* ***Sought:*** *A feasible 1-route of minimal total cost.*

And a natural model for RCSPP is adding a resource constraint to the shortest path problem.

$$\min \sum_{uv \in A} c_{uv} x_{uv}$$

subject to:

$$\sum_{u \in N} x_{su} = 1 \tag{4.1}$$

$$\sum_{u \in N} x_{ut} = 1 \tag{4.2}$$

$$x_{us} = x_{tu} = 0 \quad \forall u \in N \tag{4.3}$$

$$\sum_{v \in \gamma^-(u)} x_{vu} - \sum_{v \in \gamma^+(u)} x_{uv} = 0 \quad \forall u \in N \tag{4.4}$$

$$\sum_{u \in N} q_u (\sum_{v \in \gamma^+(u)} x_{uv}) \leq Q \tag{4.5}$$

$$x_{uv} \in \{0, 1\} \; \forall uv \in A \tag{4.6}$$

4.2 A New Model

Transfer a given undirected graph $G = (V, E)$ into a directed graph $D(G) = (V, A(E))$ with $c_{\{u,v\}} = c_{uv} = c_{vu}$. Assign an arc variable x_{uv} and a flow variable y_{uv} for all $uv \in A(E)$.

$$\min \sum_{uv \in A(E)} c_{uv} x_{uv}$$

subject to:

$$\sum_{u \in N} x_{su} = 1 \tag{4.7}$$

$$x_{us} = x_{tu} = 0 \quad \forall u \in N \tag{4.8}$$

$$\sum_{u \in N} x_{ut} = 1 \tag{4.9}$$

$$y_{us} = 0 \quad \forall u \in N \tag{4.10}$$

$$\sum_{v \in \gamma(u)} x_{vu} - \sum_{v \in \gamma(u)} x_{uv} = 0 \quad \forall u \in N \tag{4.11}$$

$$y_{uv} + y_{vu} = Q(x_{uv} + x_{vu}) \quad \forall \{u, v\} \in E \tag{4.12}$$

$$\sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) = q_u \sum_{v \in \gamma(u)} (x_{vu} + x_{uv}) \quad \forall u \in N \tag{4.13}$$

$$x_{uv}, x_{vu} \in \{0, 1\}, \;\; y_{uv} \geq 0, y_{vu} \geq 0 \; \forall \{u, v\} \in E \tag{4.14}$$

This model is also based on the flow conservation formula, replace the resource constraints by (4.12) and (4.13), which follow the idea of Baldacci's model.

Since we have a new model, let's compare these two models.

Observation 8 *The set of feasible solutions of the new model project onto the arc variables space is a subset of the set of feasible solutions of the basic model. Moreover, it can be a proper subset.*

Proof : It's sufficient to check the feasible solutions of the new model satisfies (4.5):

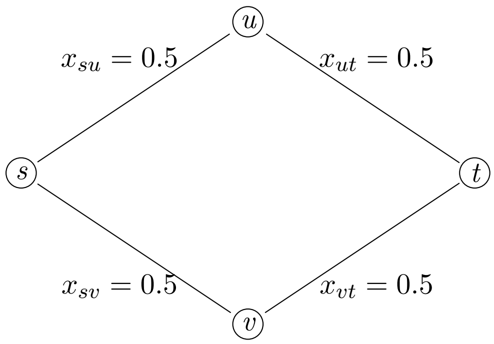
$$\begin{aligned}
\sum_{u \in N} \sum_{v \in \gamma(u)} q_u(x_{uv} + x_{vu}) &= \sum_{u \in N} \sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) \\
&= \sum_{u \in N} \sum_{\substack{v \in \gamma(u) \\ v \in N}} (y_{vu} - y_{uv}) + \sum_{u \in N} (y_{su} - y_{us} + y_{tu} - y_{ut}) \\
&\leq \sum_{u \in N} (y_{su} + y_{us} + y_{tu} + y_{ut}) \\
&= \sum_{u \in N} Q(x_{su} + x_{us}) + \sum_{u \in N} Q(x_{tu} + x_{ut}) = 2Q
\end{aligned}$$

Then the flow conversation constraint says:

$$\sum_{u\in N} q_u \sum_{v\in\gamma(u)} x_{uv} = \sum_{u\in N} q_u \sum_{v\in\gamma(u)} x_{vu}$$

So, the solutions of new model satisfies (4.5).

Example 4.2.1 *Here is simple example to show the new model is better.*



$$q_u = 2$$
$$q_v = 198$$
$$Q = 100$$

This exmple shows a solution of the basic model. However, this can't be a solution of the new model. Consider the equation on v, we have

$$\begin{aligned}
&y_{sv} + y_{tv} - y_{vs} - y_{vt} = q_v(x_{sv} + x_{tv} + x_{vs} + x_{vt}) = 198 \\
&y_{sv} + y_{vs} = Q(x_{sv} + x_{vs}) = 50 \\
&y_{vt} + y_{tv} = Q(x_{tv} + x_{vt}) = 50
\end{aligned}$$

thus

$$198 = y_{sv} + y_{tv} - y_{vs} - y_{vt} > y_{sv} + y_{tv} + y_{vs} + y_{vt} = 100$$

Since all variables are non-negative, the following inequality is impossible, which means this new model is better than the traditional model.

Moreover, we can compute the gap between linear relaxation and the original mix-integer programming of the two model on this graph. First, it's clear that the only interger solution is (s, u, t), and the optimal value of MIP $c_{su} + c_{ut}$. Then we aim to construct solution for LP-relaxation such that using the edge x_{sv} as much as possible for both models.
For the LP-relaxation of the new model, the only possible solution is (s, u, t).
For the LP-relaxation of the traditional model, the solution with largest value on x_{sv} is $\frac{1}{2}(s, u, t) + \frac{1}{2}(s, v, t)$.
Let

$$\text{gap} = \frac{\text{Optimal value of MIP} - \text{Optimial Value of LP}}{\text{Optimal Value of MIP}}$$

Assume that $c_{sv} + c_{vt} < c_{su} + c_{ut}$, then

$$\begin{aligned} \text{gap}_n &= 0 \\ \text{gap}_c &= \frac{c_{su} + c_{ut} - c_{sv} - c_{vt}}{2(c_{su} + c_{ut})} \end{aligned}$$

Where, $\text{gap}_n, \text{gap}_c$ are the gaps of the new model and classic model respectively.

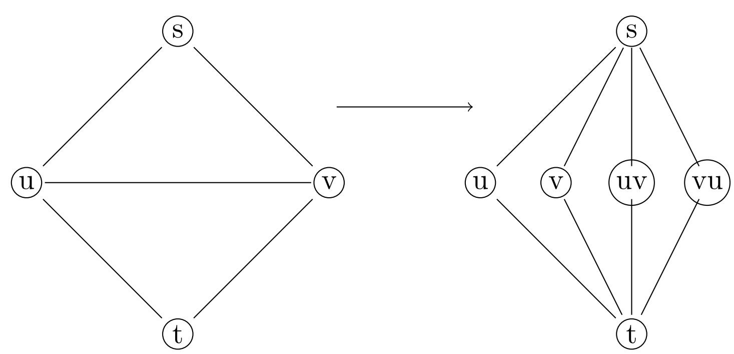
Actually, here is a stronger result.

Theorem 4.2.1 *Let f be a flow with respect to the arcs such that $f = \sum_{i=1}^{m} \mu_i p_i$ for some paths p_i with $\sum_{i=1}^{m} \mu_i = 1$. If $\forall i, j \in [m]$, $V(p_i) \cap V(p_j) = \{s, t\}$. f is the projection of a solution of the new model onto arc variables space if and only if*

$$q(p_i) := \sum_{u \in V(p_i)} q_u \leq Q \quad \forall i \in [m]$$

Proof :
Take an arbitrary path $p_i = (s, u_{i_0}, \cdots, u_{i_m}, t)$ with weight λ_i. Then $y_{s,u_{i_0}} = \lambda_i Q$ proves the theorem.

Corollary 4.2.2 *Let $G = (V, E)$ be a undirected graph with the same notation we defined s, t, N. Let $\mathcal{P}$ be the set of all $s,t-$paths. Consider a transformation T. Then $T(G) = (V_{\mathcal{P}}, E_{\mathcal{P}}$ such that $V_{\mathcal{P}}$ consists of two artificial nodes $\bar{s}, \bar{t}$ and nodes set $\{v_p\}_{p \in \mathcal{P}}$ with the demand $q(p)$ respectively. And $E_{\mathcal{P}} := \{\{\bar{s}, v_p\} : p \in \mathcal{P}\} \cup \{\{\bar{t}, v_p\} : p \in \mathcal{P}\}$ such that $c_{\bar{s},v_p} + c_{\bar{t},v_p} = c(p)$. Then the optimal solution of our LP model in the corresponding directed graph $A\big(T(G)\big)$ is the optimal solution of our MIP model on graph $A(G)$.*



Observation 9 *Let path p be $(s = u_0, u_1, \cdots, u_m, u_{m+1} = t)$, here are trivial value for flow variables that $y_{u_i u_{i+1}} = Q - \sum_{j=1}^{i} q_{u_j}$ and $y_{u_{i+1} u_i} = \sum_{j=1}^{i} q_{u_j}$ can associate to this path to get a solution for the new model. Moreover, if the vertices in p are distinct, then the value of flow variables are unique, ie the trivial value for flow variables are the only way to associate p to a solution of the new model.*

Proof:
This construction gives a solution f. Suppose here is another flow solution f' for p, then $f - f'$ is a solution for the homogeneous equation system with all arc variables equal to 0. Then, plug $x_{uv} = x_{vu} = 0$ for all $\{u, v\} \in E$ into the equation (4.12) and (4.13), we get

$$y_{uv} + y_{vu} = 0 \quad \forall \{u, v\} \in E$$
$$\sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) = 0 \quad \forall u \in N$$

Therefore we conclude that $f - f' = \mathbf{0}$. That means the solution associates to a $s - t$ path is unique.

Observation 10 *If Q is large enough, then the arc set of the optimal solutions of both models form the same directed path. Indeed, let p be the directed path with lowest cost in $D(G) = (V, A(E))$, then we only need $Q \geq c(p)$.*

If Q larger than the demands on a directed path p with lowest costs. Then for any solutions, we can replace the edge sets by the edge sets of p. And by the Observation 9, the solution is unique.

4.3 Dimension of a Polytope from the Model

Now consider $E = \{\{u, v\} : u, v \in N\} \cup \{\{s, u\} : u \in N\} \cup \{\{v, t\} : v \in N\}$, where $|N| = n$. In this case forall $u \in V$, $\gamma(u) = \gamma^+(u) = \gamma^-(u)$. Let P_{LP} be the polytope of the LP-relaxation of model in this graph. Use the similar idea in previous section, the equations can be transferred to

$$\sum_{u \in N} x_{su} = 1 \tag{4.15}$$
$$\sum_{u \in N} x_{ut} = 1 \tag{4.16}$$
$$\sum_{v \in \gamma(u)} x_{uv} - \frac{1}{2} \sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) = 0 \quad \forall u \in N \tag{4.17}$$
$$\sum_{v \in \gamma(u)} x_{vu} - \frac{1}{2} \sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) = 0 \quad \forall u \in N \tag{4.18}$$
$$x_{us} = 0 \quad \forall u \in N \tag{4.19}$$
$$x_{tu} = 0 \quad \forall u \in N \tag{4.20}$$
$$y_{us} = 0 \quad \forall u \in N \tag{4.21}$$
$$y_{uv} + y_{vu} = Q(x_{uv} + x_{vu}) \quad \forall \{u, v\} \in E \tag{4.22}$$

Write the model as $\begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \boldsymbol{c} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$, where C represents the equations from (4.15)-(4.18), X represents equations from (4.19)-(4.20), Y represents (4.21) and G represents equations from (4.22). Actually, from these equations, we can conclude that $y_{us} = Qx_{us}\ \forall u \in N$, but These equations are already in the row space of this matrix.

Lemma 2 *Only one equation in* $\begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \boldsymbol{c} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$ *is redundant. That means the rank of matrix* $\begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix}$ *is* $\frac{1}{2}n^2 + \frac{13}{2}n + 1$.

Proof: We use the same idea as previous section. Consider $(\boldsymbol{a}^{\mathrm{T}}, \boldsymbol{\mu}^{\mathrm{T}}, \lambda, \boldsymbol{b}^{\mathrm{T}})$ such that

$$(\boldsymbol{a}^{\mathrm{T}}, \boldsymbol{\mu}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}, \boldsymbol{b}^{\mathrm{T}}) \begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix} = \mathbf{0}^{\mathrm{T}}$$

Let

- $\alpha^+(u)$ be the entry in $\boldsymbol{a}^{\mathrm{T}}$ corresponding to

$$\sum_{v \in \gamma^+(u)} x_{uv} = \begin{cases} 1 & \text{if } u = s \\ \dfrac{1}{2} \displaystyle\sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) & \text{if } u \in N \end{cases}$$

- $\alpha^+(u)$ be the entry in $\boldsymbol{a}^{\mathrm{T}}$ corresponding to

$$\sum_{v \in \gamma^-(u)} x_{vu} = \begin{cases} 1 & \text{if } u = t \\ \dfrac{1}{2} \displaystyle\sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) & \text{if } u \in N \end{cases}$$

- $\mu^+(u)$ be the entry in $\boldsymbol{\mu}^{\mathrm{T}}$ corresponding to

$$x_{us} = 0$$

- $\mu^-(u)$ be the entry in $\boldsymbol{\mu}^{\mathrm{T}}$ corresponding to

$$x_{tu} = 0$$

- $\omega(u)$ be the entry in $\boldsymbol{\omega}^{\mathrm{T}}$ corresponding to

$$y_{us} = 0$$

- $\beta_{\{u,v\}}$ be the entry in $\boldsymbol{b}^{\mathrm{T}}$ corresponding to

$$y_{uv} + y_{vu} - Q(x_{uv} + x_{vu}) = 0$$

Firstly, we arbitrarily choose $u, v \in N$, these variables only appears in (4.17),(4.18) and (4.22), in order to cancel the variables x_{uv} and x_{vu}, we have

$$\alpha^+(u) + \alpha^-(v) + Q\beta_{\{u,v\}} = 0 \text{ and } \alpha^-(u) + \alpha^+(v) + Q\beta_{\{u,v\}} = 0$$

Then consider variables y_{uv} and y_{vu} which also only appears in (4.17),(4.18) and (4.22), yields

$$\begin{aligned} -2\beta_{\{u,v\}} + \alpha^+(u) + \alpha^-(u) - \alpha^-(v) - \alpha^+(v) = 0 \\ -2\beta_{\{u,v\}} - \alpha^+(u) - \alpha^-(u) + \alpha^-(v) + \alpha^+(v) = 0 \end{aligned}$$

Since u, v are arbitrary, for any $u, v \in N$, $\beta_{\{u,v\}} = 0$, $\alpha^+(u) = -\alpha^-(v)$. Then we start to cancel the variable y_{ut}:

$$\alpha^+(u) + \alpha^-(u) - \beta_{\{u,t\}} = 0 \text{ and } \alpha^+(u) + \alpha^-(u) + \beta_{\{u,t\}} = 0$$

This implies $\beta_{\{u,t\}} = 0$ and $\alpha^+(u) = \alpha^-(u)$ for all $u \in N$. Now we cancel the variables x_{ut}, x_{tu}

$$\mu^-(u) = -\alpha^-(u) \text{ and } \alpha^-(t) = -\alpha^+(u)$$

After this setep, we cancel the variables y_{us} and y_{su}

$$-2\beta_{\{u,s\}} + \alpha^+(u) + \alpha^-(u) + 2\omega_u = 0 \text{ and } -2\beta_{\{u,s\}} + \alpha^+(u) + \alpha^-(u) = 0$$

And we get $\beta_{\{u,s\}} = 0, \omega(u) = 0$ forall $u \in \omega$. Finally, we can cancel the x_{us} and x_{su}

$$\alpha^+(s) = -\alpha^-(u) \text{ and } \mu^+(u) = -\alpha^+(u)$$

We have canceled all variables, and these relationships tell

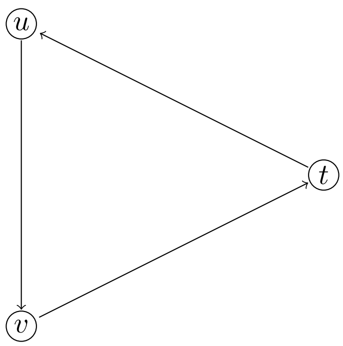
$$\dim\{(\boldsymbol{a}^{\mathrm{T}}, \boldsymbol{\mu}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}, \boldsymbol{b}^{\mathrm{T}}) : (\boldsymbol{a}^{\mathrm{T}}, \boldsymbol{\mu}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}, \boldsymbol{b}^{\mathrm{T}}) \begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix} = \mathbf{0}^{\mathrm{T}}\} = 1$$

The rank of matrix follows immediately. Now, this lemma provides a upper bound of dimension, the next task is constructing affinely independent points.

Theorem 4.3.1 *If* $3Q > 5\max_{u\in N} q_u$, *then* $\dim P_{LP} = 3\binom{n}{2} + n - 1$

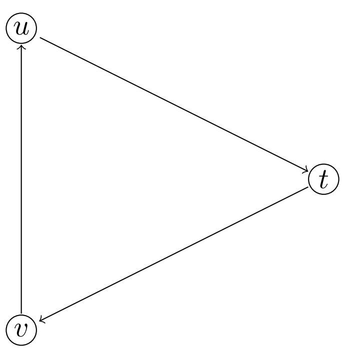
Proof :
$\dim P_{LP} \leq 3\binom{n}{2} + n - 1$ is immediately from Lemma 2. So, it's sufficient to find $3\binom{n}{2} + n$ many affinely independent points. It's clear that path $p_u := (s, u, t)$ with associated flow variables are affinely independent. Let $f = \frac{1}{n}\sum_{u\in N} p_u$. Then for every two $u, v \in N$, we construct three different solution.

c^1_{uv}:



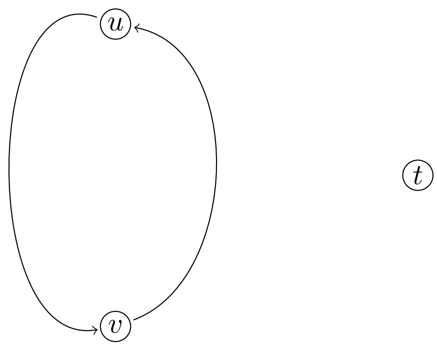
$$
\begin{aligned}
x_{vu} = x_{tu} = x_{tv} &= 0 \\
x_{uv} &= 0.5 \\
x_{ut} &= -0.5 \\
x_{vt} &= 0.5 \\
y_{uv} &= 0.5Q \\
y_{vu} &= 0 \\
y_{ut} &= -0.25Q - 0.25q_u \\
y_{tu} &= 0.25Q + 0.25q_u \\
y_{vt} &= 0.25Q - 0.25q_v \\
y_{tv} &= -0.25Q + 0.25q_v
\end{aligned}
$$

c^2_{uv}:



$$
\begin{aligned}
x_{uv} = x_{tu} = x_{vt} &= 0 \\
x_{vu} &= 0.5 \\
x_{ut} &= 0.5 \\
x_{vt} &= -0.5 \\
y_{vu} &= 0.5Q \\
y_{uv} &= 0 \\
y_{ut} &= 0.25Q - 0.25q_u \\
y_{tu} &= -0.25Q + 0.25q_u \\
y_{vt} &= -0.25Q - 0.25q_v \\
y_{tv} &= 0.25Q + 0.25q_v
\end{aligned}
$$

Take a small enough positive parameter ϵ, then c^3_{uv} is constructed as:



$$
\begin{aligned}
x_{ut} = x_{tu} = x_{vt} = x_{tv} &= 0 \\
x_{vu} &= 0.25 \\
x_{uv} &= 0.25 \\
y_{uv} &= 0.25Q + \epsilon \\
y_{vu} &= 0.25Q - \epsilon \\
y_{ut} &= -\epsilon - 0.25q_u \\
y_{tu} &= \epsilon + 0.25q_u \\
y_{vt} &= \epsilon - 0.25q_v \\
y_{tv} &= -\epsilon + 0.25q_v
\end{aligned}
$$

It's easy to check that $f + \frac{1}{n}c^i_{uv}$ is a solution for arbitrary $u, v \in N$ and $i \in [3]$. Also, we can check $\{f + \frac{1}{n}c^i_{uv}\}_{\substack{u,v\in N \\ i\in[3]}} \cup \{p_u\}_{u\in N}$ is set of affinely independent points. Therefore,

$$\dim P_{LP} = 3\binom{n}{2} + n - 1$$

4.4 Integer Solution Polytope for Unit Demand Case

Now we consider the case that $q_u = 1$ for all $u \in N$, and in this case we can compute the dimension of the convex hull of integer solution P^u_{IP}. Assume $|N| = n$

Observation 11 *Let* $|N| \geq Q$*. If Q=1, then* $\dim P^u_{IP} = n-1$*, and if Q=2, then* $\dim P^u_{IP} = 2\binom{n}{2} + n - 1$

Proof :
Recall the constraints (4.13)

$$\sum_{v\in\gamma(u)} (y_{vu} - y_{uv}) = q_u \sum_{v\in\gamma(u)} (x_{vu} + x_{uv}) \quad \forall u \in N$$

I sum them over all $u \in N$, and since the unit case

$$
\begin{aligned}
\sum_{u\in N} \Big(\sum_{v\in\gamma(u)} (x_{uv} + x_{vu}) \Big) &= \sum_{u\in N} \sum_{v\in\gamma(u)} (y_{vu} - y_{uv}) \\
&= \sum_{u\in N} (y_{su} + y_{tu} - y_{us} - y_{ut}) = Q + \sum_{u\in N} (y_{tu} - y_{ut}) \leq 2Q
\end{aligned}
$$

Since

$$\sum_{u\in N} x_{su} = \sum_{u\in N} x_{ut} = 1 \text{ and } \forall u \in N,\ x_{us} = x_{tu} = 0$$

If $Q = 1$, then $\sum\limits_{u,v \in N} (x_{uv} + x_{vu}) = 0$, so all integer solutions project onto the arc variables are of the form (s, u, t)
If $Q = 2$, then $2 \sum\limits_{v \in \gamma(u) \setminus \{s,t\}} \sum\limits_{u \in N} x_{uv} = \sum\limits_{u,v \in N} (x_{uv} + x_{vu}) \leq 2$, that means we can at most take one arc with head or tail in N. So, all integer solution w.r.t arcs are either of the form (s, u, t) or (s, u, v, t).
In Observation 9, we know that the way to associate flow is unique, and all solutions are affinely independent. So, we have the dimension.
We have observed the dimension in two simple and specific case, Now, we are going to prove the dimension for $Q \geq 3$.

Theorem 4.4.1 *Let $|N| \geq Q$. If $Q \geq 3$, then* $\dim P^u_{IP} = 3\binom{n}{2} + n - 1$

Proof:
Since $3Q \geq 9 > 5$, $\dim P^u_{IP} \leq \dim P_{LP} = 3\binom{n}{2} + n - 1$, then we will construct affinely independent points. Label the vertices in N such that $N = \{u_1, u_2, \cdots, u_n\}$. Take path (s, u_i, t) and (s, u_i, u_j, t) for all $i, j \in [n]$. And we take paths (s, u_i, u_j, u_k, t) such that $k > j$ and $i \equiv j - 1 \pmod{n}$. We have $\binom{3}{n}2 + n - 1$ many different paths, now we add one more paths $(s, u_{n-1}, u_1, u_n, t)$. Let $\mathcal{P}_3$ be the set of all paths with length 3, and $S := \{i, j) : (s, u_i, u_j, u_k, t) \in \mathcal{P}_3\}$. Indeed $S = \{(n, 1), (n-1, 1)\} \cup \{(1, 2), (2, 3), \cdots, (n-2, n-1)\}$

- Let $z(i)$ be the corresponding solution of the model w.r.t (s, u_i, t)
- Let $z(i, j)$ be the corresponding solution of the model w.r.t (s, u_i, u_j, t)
- Let $z(i, j, k)$ be the corresponding solution of the model w.r.t (s, u_i, u_j, u_k, t)

The following is show our construction gives $3\binom{n}{2} + n$ many affinely independent points. Let $\alpha(i), \alpha(i, j), \alpha(i, j, k)$ be the coefficient of solutions $z(i), z(i, j), z(i, j, k)$ respectively. Consider

$$\sum_{i \in [n]} \alpha(i) z(i) + \sum_{\substack{i,j \in [n] \\ a \neq j}} \alpha(i, j) z(i, j) + \sum_{(s, u_i, u_j, u_k, t) \in \mathcal{P}_3} \alpha(i, j, k) z(i, j, k) = 0$$
$$\text{where} \sum_{i \in [n]} \alpha(i) + \sum_{i,j \in [n]} \alpha(i, j) + \sum_{(s, u_i, u_j, u_k, t) \in \mathcal{P}_3} \alpha(i, j, k) z(i, j, k) = 0$$

Firstly, note that for all (u_i, u_j) sith $i > j$ and $(u_i, u_j) \notin S$, then only in the $z(i, j)$ this coordinate is non-zero. So $\alpha(i, j) = 0$ for all $i > j$ with $(i, j) \notin S$. Then we look at (u_j, u_k) such that $(j, k) \notin S$, then the coordinate corresponding to the arc is non-zero only in $z(j, k)$ and $z(i, j, k)$. Therefore, for these (j, k), $\alpha(j, k) = -\alpha(i, j, k)$. Then we check the flow value on (u_j, u_k), we get $\alpha(j, k) = \alpha(i, j, k) = 0$.
The coefficients may be non-zero yet are $\{\alpha(i)\}_{i \in [n]} \cup \{\alpha(i, j) : (i, j) \in S\} \cup \{\alpha(i, j, k) : (s, u_i, u_j, u_k, t) \in \mathrm{P}_3, j < k, (j, k) \in S\}$. Note the only way that $(i, j) = (n - 2.n - 1)$ in

(s, u_i, u_j, u_k, t) is $(s, u_{n-2}, u_{n-1}, u_n, t)$. Since $\alpha(n-2, n-1, n) = -\alpha(n-2, n-1) = 0$, therefore by the same discuss $\alpha(n-3, n-2, n-1) = -\alpha(n-3, n-2) = 0$, and then $\alpha(i, j, k)$ where $i = n-3, j = n-2$ are all zero. So $\alpha(n-4, n-3, n-2) = -\alpha(n-4, n-3) = 0$. Repeat this process, then this means we know that the coefficients $\{\alpha(i,j) : (i,j) \in S, 1 < i < j\} \cup \{\alpha(i,j,k) : (s, u_i, u_j, u_k, t) \in \mathrm{P}_3, 1 < j < k, (j,k) \in S\}$ are all zero. Also, we can remove $\{\alpha(i) : i \in [n-2], i \neq 1\}$. Therefore, only $\alpha(1), \alpha(n), \alpha(n-1), \alpha(n-1,1), \alpha(1,n-1), \alpha(1,n), \alpha(n,1), \alpha(n-1,1,n), \alpha(n,1,n-1)$ are not known yet. However, by checking the flow value, we know all them are 0. Therefore, we find $3\binom{n}{2} + n$ many affinely independent points.

4.5 Cuts

We consider the undirected graph $G = (V, E)$ and its induced graph $D(G) = (V, A(E))$ as section 4.3. Let $p = (s, \cdots, u, t)$ be a path. If all vertices of p are distinct, then before the vehicle entering the last vertex u, it should carry at least q_u many flow. However, if we have a cycle, when a vertex $u \in N$ are visited more then once. Here is a freedom to manage how to satisfies the demand of u. For example in the path $p = (s, \cdots, w, u, v, u, t)$, assume u is visited exactly twice. Then we only need to satisfy the following conditions

$$y_{wu} + y_{vu} - y_{uw} - y_{uv} = 2q_u$$
$$y_{vu} + y_{tu} - y_{uv} - y_{ut} = 0$$

The solution of equation system above is not unique.

However, since the objective function only cares about arc variable, then we can choose a unique way to construct the value of flow variables from the edge variables. We require why a node in N is visted, them the flow should lost exact q_u unit. Under this rule, we have a cut

$$y_{vu} \geq \min_{w \in N} q_w(x_{uv} + x_{vu}) \quad \forall u \in N, v \in \gamma(u) \tag{4.23}$$

These cuts work very well in unit demand case.

Theorem 4.5.1 *Let $|N| \geq Q$. If $q_u = 1$ for all $u \in N$ and $Q \geq 3$, then the inequality in (4.23) define a facet for the polytope of LP-relaxation and convex hull of integer solution P^u_{LP}, P^u_{IP}*

Proof : We firstly show that $y_{vu} = \min_{w \in N} q_w(x_{uv} + x_{vu})$ is independent to the original equation system. From Section 4.3, we know that all equations from (4.22) are independent. And since $u, v \in N$, so here are only one equation in (4.22) is relevant to y_{uv}. That means $y_{vu} = \min_{w \in N} q_w(x_{uv} + x_{vu})$ is independent for any $u, v \in N$.
Then we claim that we can construct $3\binom{n}{2} + n - 1$ many affinely independent points with

the cut is tight. Let $N = \{u_i\}_{i=1}^n$ We take paths in the three forms. Based on the symmetry of this polytope, W.L.O.G, let $u = u_l, v = u_r$ such that $l < r - 1$, $l, r \in \{2, 3, \cdots, n-1\}$

- (s, u_i, t) for all $i \in [n]$
- (s, u_i, u_j, t) for all $i, j \in [n]$, but $(i, j) \neq (l, r)$
- $(s, u_{i-1}, u_i, u_j, t)$ for all $i, j \in [n]$ with $i < j$ but $(i, j) \neq (l, r)$. Here the $i - 1$ is also similar to Section 4.4, that is respect to the remainder of modulo n.
- $(s, u_r, u_l, u_{l+1}.t)$. The plus is respect to the remainder of modulo n.

The proof is similar to want we did in Section 4.4, they are affinely independent points. Therefore $y_{vu} \geq \min_{w \in N}(x_{uv} + x_{vu})$ is a facet defining inequality for any $u, v \in N$.
Note that this method can be generalized to the non-unit demand case, but the affinely independence dependent on $\{q_u\}_{u \in N}$.

Theorem 4.5.2 *Let $|N| \geq Q$. If $Q = 2$, then adding these cuts will give a complete description of P_{IP}^u*

Proof :
Since $Q = 2$, By the cuts

$$\begin{aligned} y_{vu} &\geq (x_{uv} + x_{vu}) \quad \forall u, v \in N \\ y_{uv} &\geq (x_{uv} + x_{vu}) \quad \forall u, v \in N \end{aligned}$$

and

$$y_{uv} + y_{vu} = 2(x_{uv} + x_{vu}) \tag{4.24}$$

Therefore, indeed, we get $y_{uv} = y_{vu}$ for all $u, v \in N$. And all possible paths (s, u, t) or (s, u, v, t) are satisfy (4.24).
Let f be a solution satisfies the model include the cuts (4.23). We first look at the arc variables. By flow decomposition theorem, $f = \sum_{i=1}^m \mu_i p_i + \sum_{i=1}^l \lambda_i c_i$, where p_i are paths with distinct vertices, c_i are cycles and $\sum_{i=1}^m \mu_i = 1$. Since the flow variable associated with a path with distinct vertices have unique value. So, it sufficient to show that $\lambda_i = 0$ for all i. Let $f' = \sum_{i=1}^m \mu_i p_i$, and $z(f), z(f')$ be the solutions of the model with flow f, f' respectively. Then $z(f) - z(f')$ is a solution of the homogeneous equations. Our cycle can't include vertices s, t, so $x_{su}, x_{us}, x_{tu}, x_{ut} = 0$, therefore $y_{su}, y_{us}, y_{tu}, y_{ut} = 0$. The use (4.13),

$$0 = \sum_{u,v \in N} (y_{vu} - y_{uv}) = (x_{uv} + x_{vu}) \Rightarrow x_{uv} + x_{vu} = 0$$

Therefore $\lambda_i = 0$, that is here is no cycles in f. So every solution is a convex combination of integer solutions, this implies that all vertices are integral.

4.6 Computational Results

4.6.1 Results on Delaunay Triangulation

To compare this model with the traditional model, we randomly generate different instances in the following ways.

- number of nodes n: 100-1000
- position of nodes: random
- graph: Delaunay Triangulation
- demand of every node: random number in [0,1)
- cost: Euclidean Distance
- solver: SCIP
- parameter Q: $0.25\sqrt{n}$
- x axis: instances
- y axis:

$$y = f(x) = \begin{cases} 0 & \text{If } \mathrm{OptTrad}_{IP} = \mathrm{OptTrad}_{LP} \\ \dfrac{\mathrm{OptNew}_{LP} - \mathrm{OptTrad}_{LP}}{\mathrm{OptTrad}_{IP} - \mathrm{OptTrad}_{LP}} & \text{Otherwise} \end{cases}$$

The result is

The curve shows we can always expect our model has advantages. In order to understand how this model works better, we generate the graph with much less vertices, for which we could draw the optimal paths. We choose an instance such that our model have good performance.


improve_rate, %
0.0
0.2
0.4
0.6
0.8
1.0
name

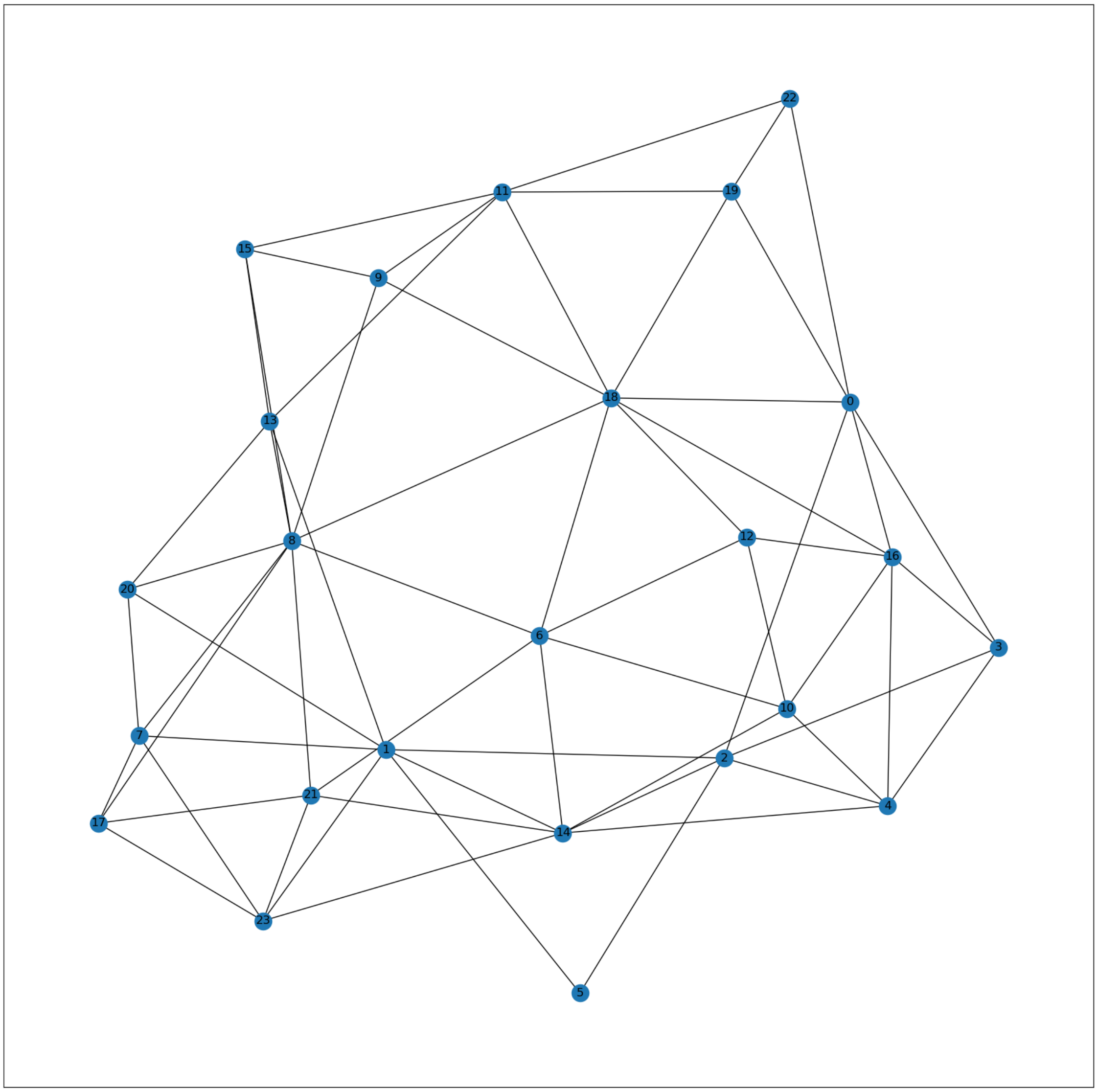


Figure 4.2: Delaunay Triangluation for small vertices l6v24

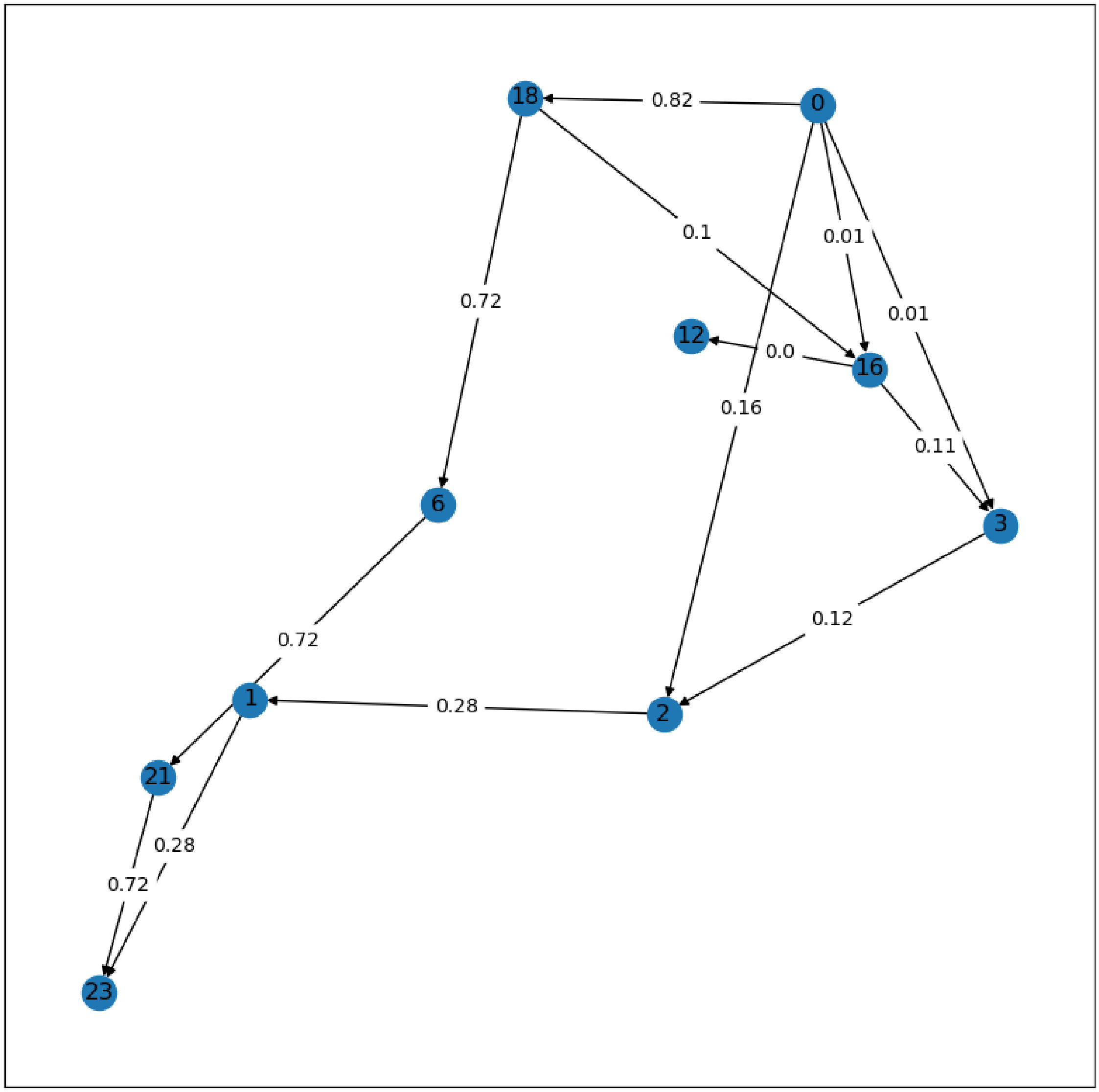


Figure 4.3: Path solution for Delaunay Triangluation l6v24

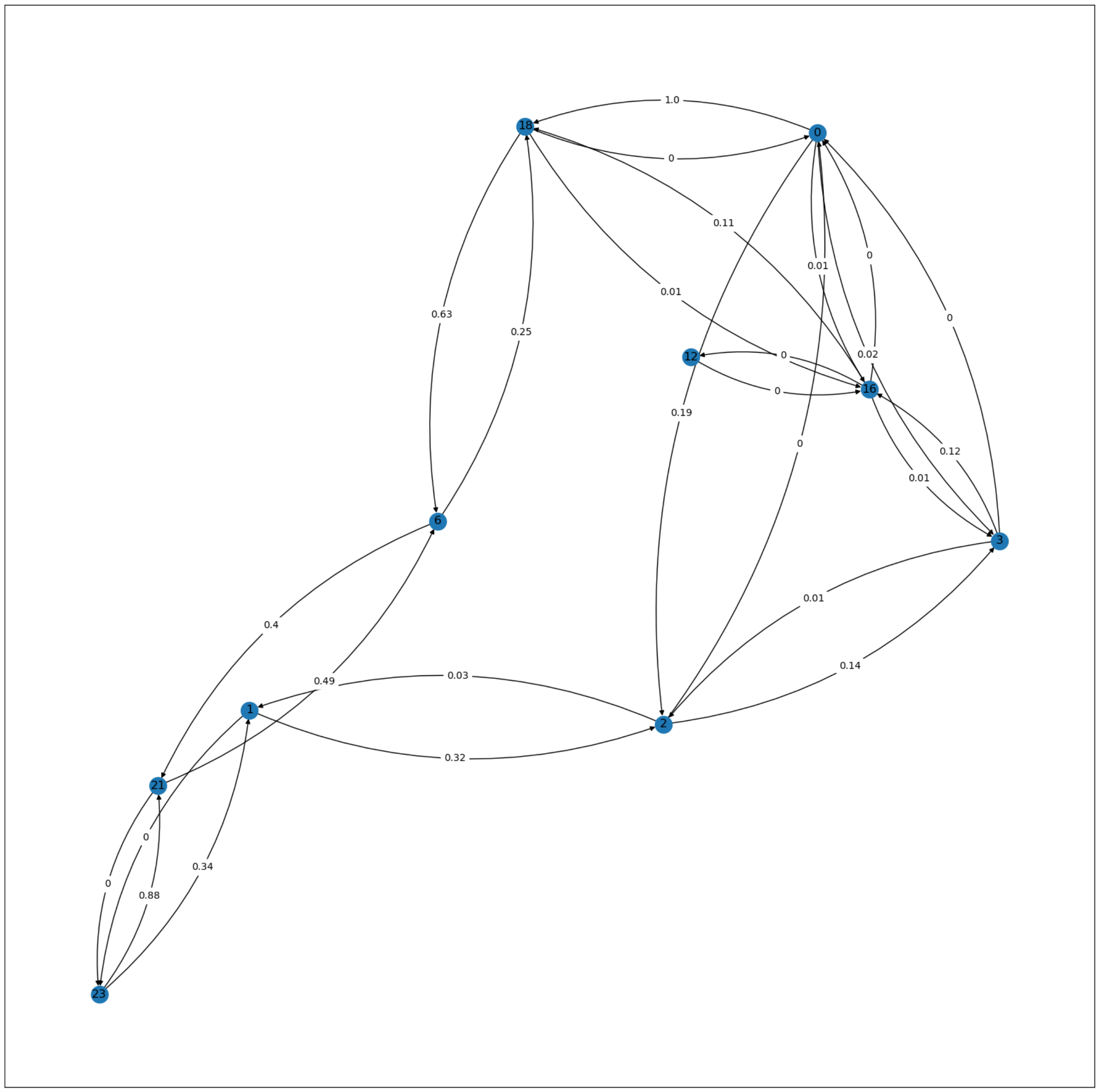


Figure 4.4: Flow solution for Delaunay Triangluation l6v24

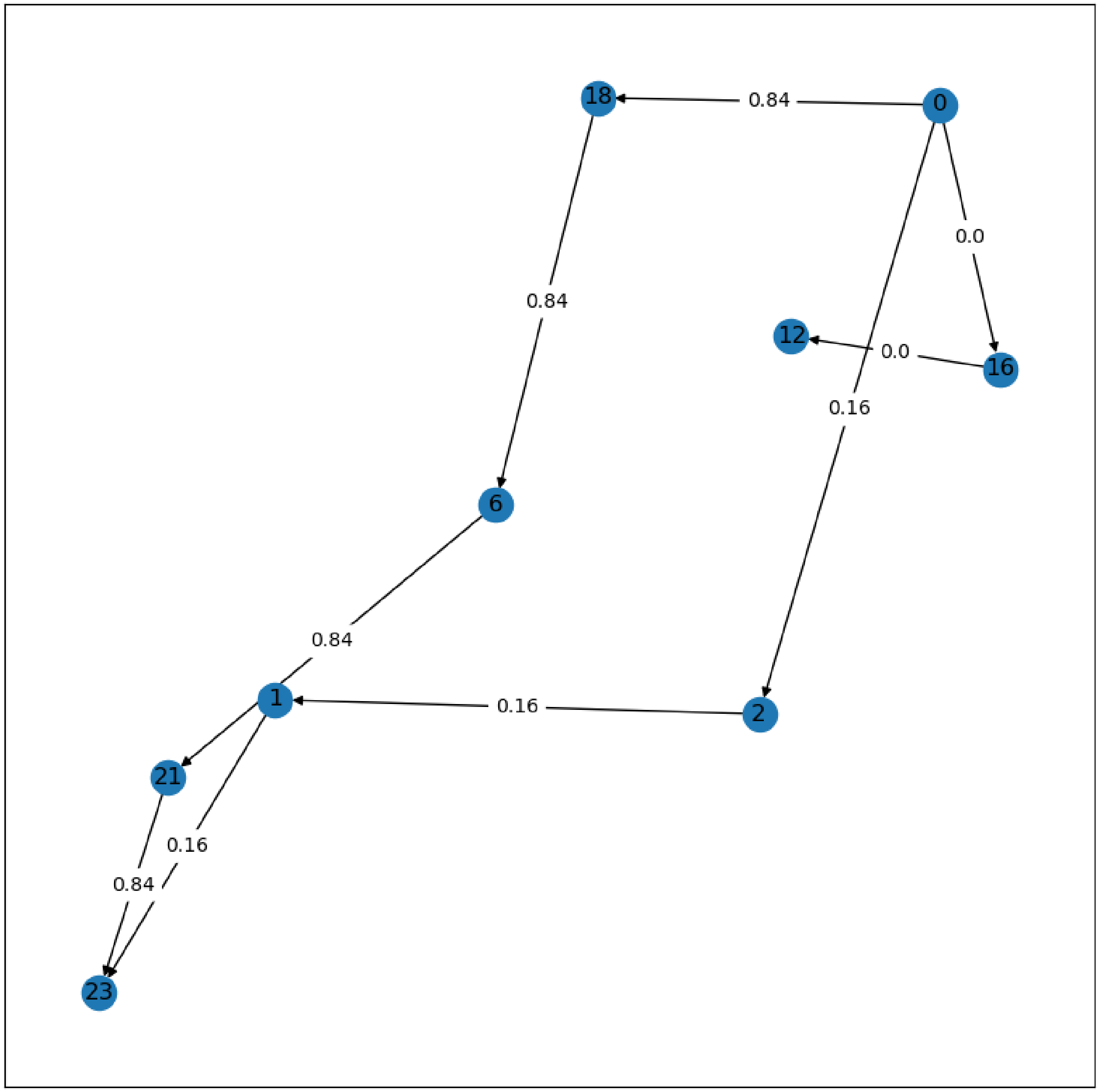


Figure 4.5: Standard model for Delaunay Triangluation l6v24

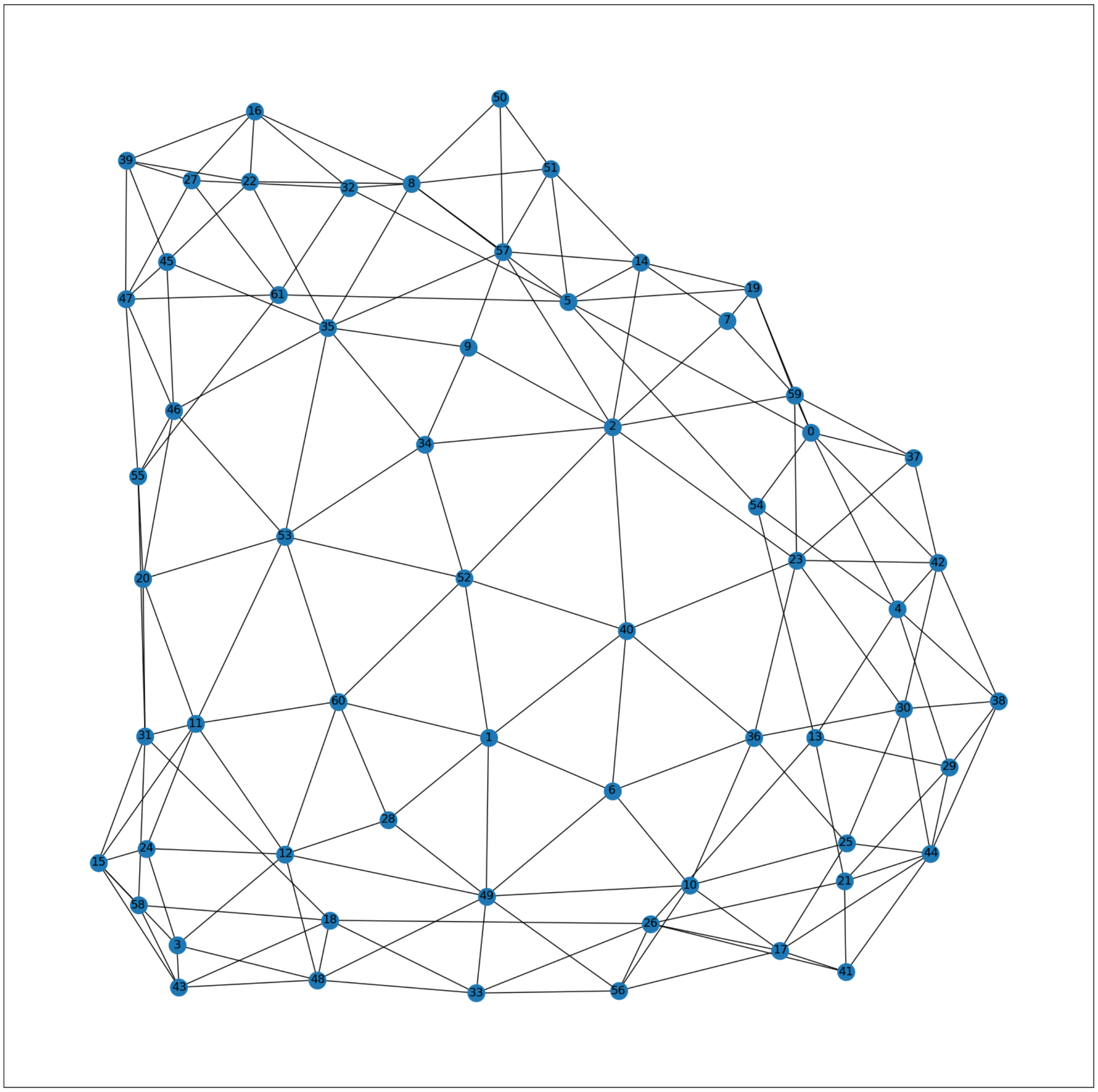


Figure 4.6: Delaunay Triangluation for small vertices l27v62

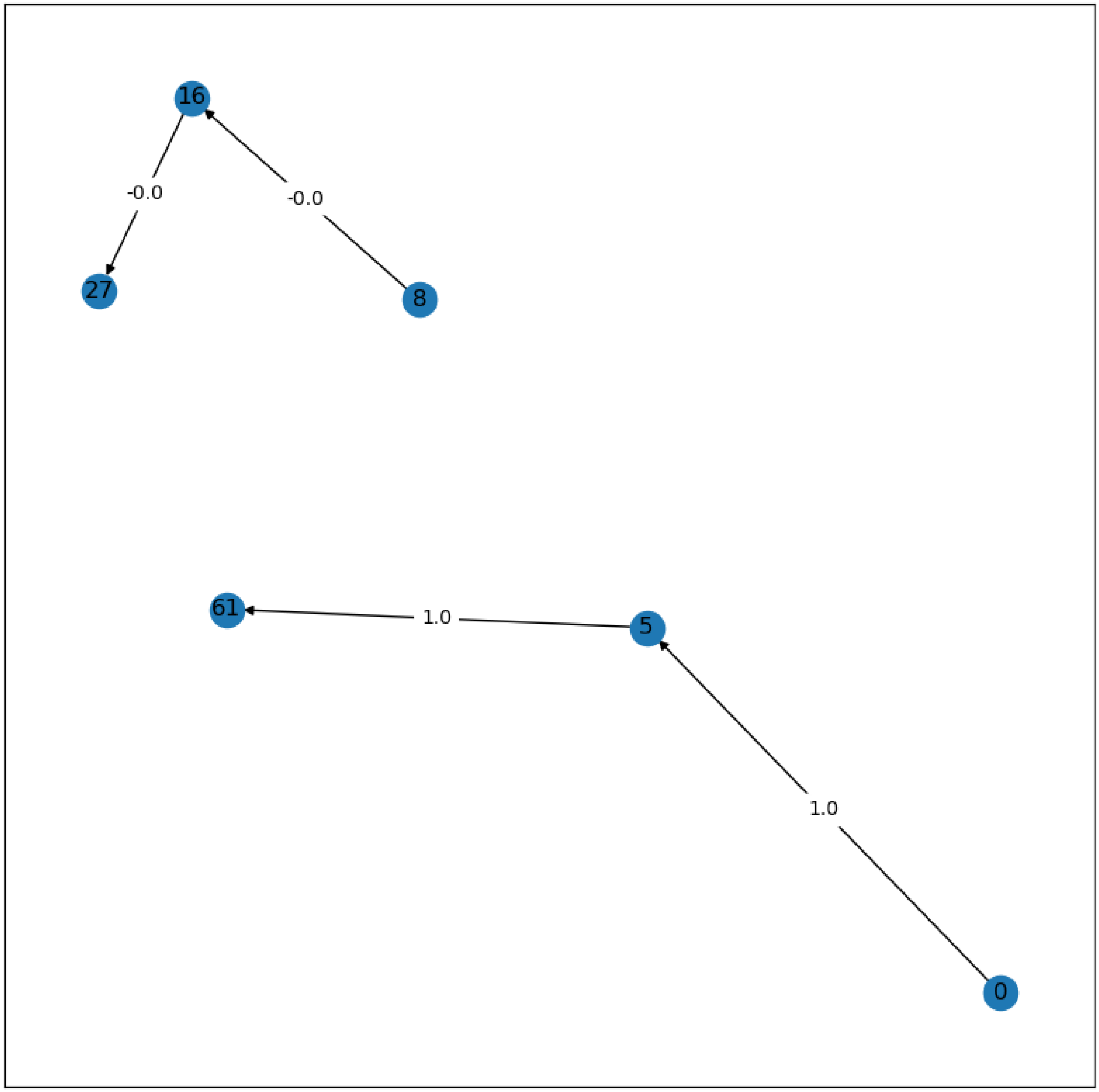


Figure 4.7: Path solution for Delaunay Triangluation l27v62

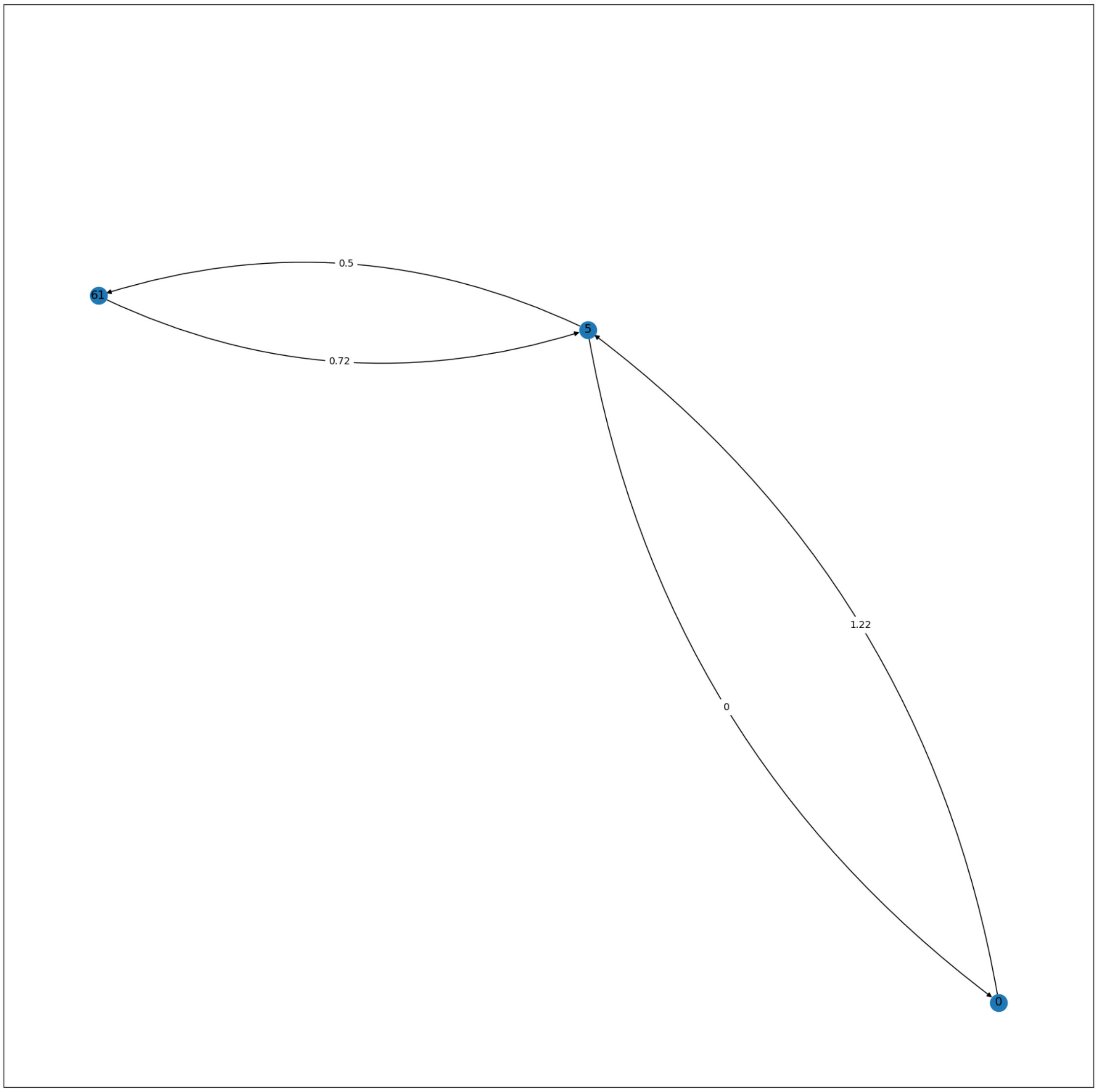


Figure 4.8: Flow solution for Delaunay Triangluation ll27v62

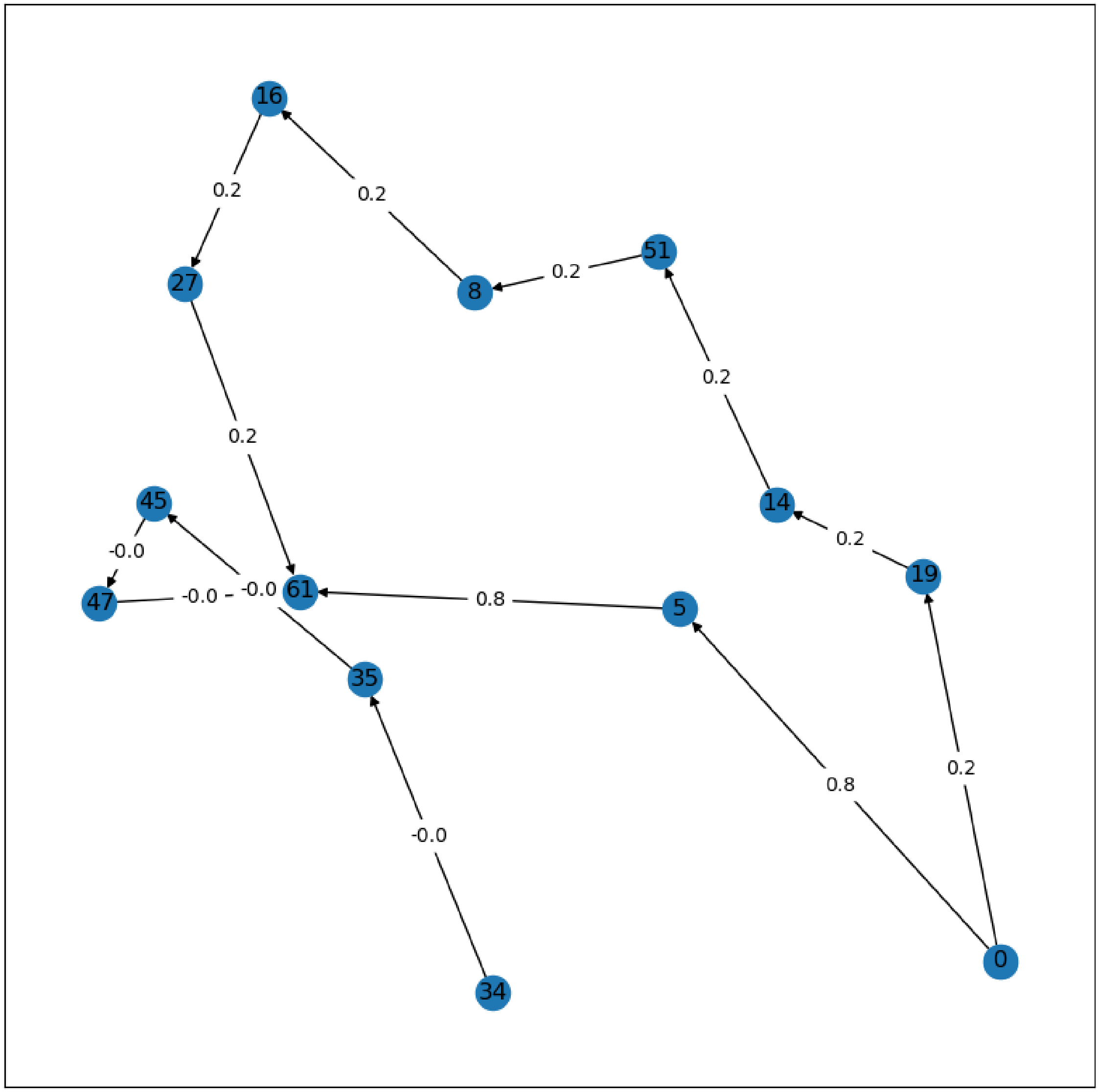


Figure 4.9: Standard model for Delaunay Triangluation l27v62

4.6.2 Results on Complete Graph

In the previous proof, we obtain a complete description of P_{IP} in a simple case. We also tried the experiment on this graph. In this section, we only focus on the running time, because the optimal solutions are too simple.

Chapter 5

Summary

Summary

In this thesis, we delve into Baldacci's two-commodity flow model for the Vehicle Routing Problem. Deriving implicit equations from the original model and analyzing the redundancy within these equations, leads to an estimation of the upper bound for the dimension of LP-relaxation polytopes. By observing how dimension reduces, we identify some non-facet-defining inequalities. In a uniform demand case, where all consumers share the same demand, we determine the dimension of MIP polytops and identify some facet-defining inequalities.

Following Baldacci's idea, a model for the Resource-Constrained Shortest Path Problem can be proposed. Through a simple example, we show the potential advantages of our model. Using the similar analysis as previous part, along with constructive proofs, we determine the dimensions of LP-relaxation polytopes. Similar methods lead to the dimensions of MIP polytopes and facet-defining inequality in the uniform demand case.

Furthermore, we introduce some cuts for our new model, which can completely describe the MIP polytopes in a simple case. Finally, we compare our model with the traditional model on graphs derived from Delaunay triangulation.

Zusammenfassung

In dieser Arbeit beschäftigen wir uns mit Baldaccis Zwei-Güter-Fluss-Modell für das Vehicle Routing Problem. Durch die Ableitung impliziter Gleichungen aus dem ursprünglichen Modell und die Analyse der Redundanz in diesen Gleichungen gelangen wir zu einer Schätzung der oberen Grenze für die Dimension von LP-Entspannungspolytopen. Durch die Beobachtung, wie sich die Dimension verringert, identifizieren wir einige nicht-facettendefinierende Ungleichungen. Im Fall einer einheitlichen Nachfrage, bei der alle Verbraucher dieselbe Nachfrage haben, bestimmen wir die Dimension von MIP-Polytopen und identifizieren einige facettendefinierende Ungleichungen.

In Anlehnung an Baldaccis Idee kann ein Modell für das Resource-Constrained Shortest Path Problem vorgeschlagen werden. Anhand eines einfachen Beispiels zeigen wir die potenziellen Vorteile unseres Modells. Unter Verwendung einer ähnlichen Analyse wie im vorherigen Teil, zusammen mit konstruktiven Beweisen, bestimmen wir die Dimensionen von LP-Entspannungspolytopen. Ähnliche Methoden führen zu den Dimensionen von MIP-Polytopen und facettendefinierenden Ungleichungen im Fall einer einheitlichen Nachfrage.

Darüber hinaus führen wir einige Schnitte für unser neues Modell ein, die die MIP-Polytope in einem einfachen Fall vollständig beschreiben können. Abschließend vergleichen wir unser Modell mit dem traditionellen Modell auf Graphen, die aus Delaunay-Triangulationen abgeleitet wurden.

Bibliography

[Hartmanis(1982)] Juris Hartmanis. Computers and intractability: a guide to the theory of np-completeness (michael r. garey and david s. johnson). *Siam Review*, 24(1):90, 1982.

[Aneja et al.(1983)Aneja, Aggarwal, and Nair] Yash P Aneja, Vijay Aggarwal, and Kunhiraman PK Nair. Shortest chain subject to side constraints. *Networks*, 13(2):295–302, 1983.

[Beasley and Christofides(1989)] John E Beasley and Nicos Christofides. An algorithm for the resource constrained shortest path problem. *Networks*, 19(4):379–394, 1989.

[Mehlhorn and Ziegelmann(2000)] Kurt Mehlhorn and Mark Ziegelmann. Resource constrained shortest paths. In *European Symposium on Algorithms*, pages 326–337. Springer, 2000.

[Handler and Zang(1980)] Gabriel Y Handler and Israel Zang. A dual algorithm for the constrained shortest path problem. *Networks*, 10(4):293–309, 1980.

[Santos et al.(2007)Santos, Coutinho-Rodrigues, and Current] Luis Santos, Joao Coutinho-Rodrigues, and John R Current. An improved solution algorithm for the constrained shortest path problem. *Transportation research part B: methodological*, 41(7):756–771, 2007.

[Pugliese and Guerriero(2013)] Luigi Di Puglia Pugliese and Francesca Guerriero. A reference point approach for the resource constrained shortest path problems. *Transportation Science*, 47(2):247–265, 2013.

[Muhandiramge and Boland(2009)] Ranga Muhandiramge and Natashia Boland. Simultaneous solution of lagrangean dual problems interleaved with preprocessing for the weight constrained shortest path problem. *Networks: An International Journal*, 53 (4):358–381, 2009.

[Dantzig and Ramser(1959)] George B Dantzig and John H Ramser. The truck dispatching problem. *Management science*, 6(1):80–91, 1959.

[Agarwal et al.(1989)Agarwal, Mathur, and Salkin] Yogesh Agarwal, Kamlesh Mathur, and Harvey M Salkin. A set-partitioning-based exact algorithm for the vehicle routing problem. *Networks*, 19(7):731–749, 1989.

[Baldacci et al.(2008)Baldacci, Christofides, and Mingozzi] Roberto Baldacci, Nicos Christofides, and Aristide Mingozzi. An exact algorithm for the vehicle routing problem based on the set partitioning formulation with additional cuts. *Mathematical Programming*, 115:351–385, 2008.

[Laporte and Nobert(1983)] Gilbert Laporte and Yves Nobert. A branch and bound algorithm for the capacitated vehicle routing problem. *Operations-Research-Spektrum*, 5: 77–85, 1983.

[Gavish and Graves(1978)] Bezalel Gavish and Stephen C Graves. The travelling salesman problem and related problems. 1978.

[Garvin et al.(1957)Garvin, Crandall, John, and Spellman] WW Garvin, HW Crandall, JB John, and RA Spellman. Applications of vehicle routing in the oil industry. *Management Science*, 3(1):407–430, 1957.

[Baldacci et al.(2004)Baldacci, Hadjiconstantinou, and Mingozzi] Roberto Baldacci, Eleni Hadjiconstantinou, and Aristide Mingozzi. An exact algorithm for the capacitated vehicle routing problem based on a two-commodity network flow formulation. *Operations research*, 52(5):723–738, 2004.

[Campos et al.(1991)Campos, Corberan, and Mota] Vincente Campos, Angel Corberan, and Enrique Mota. Polyhedral results for a vehicle routing problem. *European Journal of Operational Research*, 52(1):75–85, 1991.