Polyhedral Study of Baldacci's Model for the Capacitated Vehicle Routing Problem and Its Extension to the Resource Constrained Shortest Path Problem

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Contents

Chapter 1

Introduction

The Resource Constrained Shortest Path Problem (RCSPP) is a fundamental problem in discrete optimization. RCSPP is aimed for finding a cost-optimal path, with additional resource constraints. For instance, under the transportation background, consider a vehicle from a factory to some markets for delivery and satisfies the demand of visited markets. The cost could be the fuel cost, time penalty and the resource constraint could be the capacity of fuel tank of a vehicle. The constraint make sense, since the gas stations are separated in the whole region. RCSPP is a $\mathcal{NP}-$hard problem [Hartmanis(1982)] even if the costs are all non-negative. $\mathcal{NP}$ versus $\mathcal{P}$ is still an open question yet, that means when the problem has a large size, all algorithms researchers have yet are not polynomial. However, since RCSPP has some applications in the transportation and management area, indeed, this problem is a sub-problem of many famous problems in transportation. Therefore, researching the RCSPP is meaningful, particularly, for the real-world transportation companies.

The classic linear programming model directly from the description of RCSPP is quiet simple and natural, just follow the model of shortest path problem and add the resource constraints. Therefore, researcher are focus on developing algorithms to solve RCSPP. Aneja et al. proposed a network reduction algorithm [Aneja et al.(1983)Aneja, Aggarwal, and Nair]. The algorithm deletes and remove the nodes or arcs can't appear in feasible solutions, meanwhile record the current cost-least solution. After 6 years, Beasley and Christofides use the same rule as Aneja et al, but they also considered reduce the cost to solve a dual lagrangean problem [Beasley and Christofides(1989)]. Indeed, using lagrangean relaxation is a most common technique in solving RCSPP. Mehlhorn and Ziegelmann use ellipsoid method to solve the dual problem [Mehlhorn and Ziegelmann(2000)], and they computed the complexity of their alogrithm for RCSPP with a single resource constraint. It's also possible to use Path ranking based approach, dynamic programming and Branch&Backtracking algorithm to close the gap of prime problem and dual problem. The main idea of path ranking based approach is finding a feasible path with highest

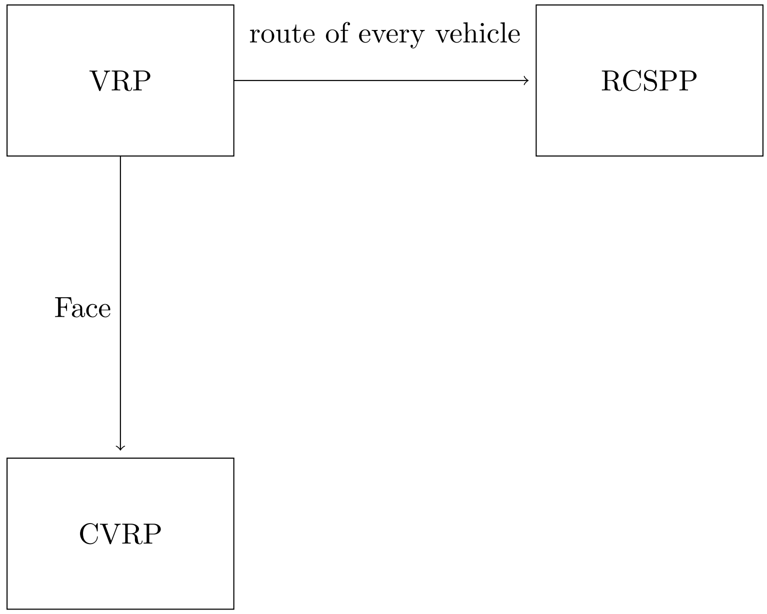
rank with respect to cost. The path ranking approach was first used by Handler and Zang [Handler and Zang(1980)], they reduced the cost of arcs in lagrangean problem to make the algorithm efficient. Apart of this trick, convexity and Chebyshev-like norm are common technique to make the path ranking based approach efficient, these ideas can be found in [Santos et al.(2007)Santos, Coutinho-Rodrigues, and Current] and [Pugliese and Guerriero(2013)]. While trying use dynamic programming to solve RCSPP, every state is a sub-path, then the algorithm glue two sub-paths with the resource restriction. Dynamic programming is a common trick, for instances, Mehlhorn and Ziegelmann use the dynamic programming while solving the prime problem [Mehlhorn and Ziegelmann(2000)]. The Branching&Backtracking algorithm is based on the rules to search the feasible solutions. The oldest paper use this idea was written by Beasley and Christofides in 1989 [Beasley and Christofides(1989)]. And a recent work with Branching&Backtracking scheme was did by Muhandiramge and Boland [Muhandiramge and Boland(2009)].
The tableau shows our literature research.

In particularly, RCSPP is usually used to help solving the Vehicle Routing Problem

	Network Reduction	Dual Problem	Path Ranking	Dynamic Programming	B&B scheme
Handler and Zang [1980]		×	×		
Aneja et al. [1983]	×			×	
Beasley and Christofides [1989]	×	×			×
Mehlhorn and Ziegelmann [2000]	×	×		×	
Santros et al [2007]			×		
DPuglia Pugliese and Guerriero [2013]			×		
Muhandiramge and Boland [2009]	×	×			×

Table 1.1: RCSPP Algorithms

(VRP). The VRP is aimed for finding a cost-optimal schedule for M many vehicles, such that all vehicles depart from a factory for delivery, and satisfy the demands of all markets. After delivery, all vehicle will return to the factory. A special case of VRP is the Capacitated Vehicle Problem (CVRP) where all markets are visited exactly once. And since in VRP, all markets should be satisfied and vehicles can only visit a market with integer times, so in VRP, all markets are visited at least once. Therefore, the solution set of CVRP is a face of solution set of VRP. After adding one artificial point, a trip, which means the destination is the place of departure, can be transferred to a path by replacing the destination with the artificial point. After the transformation, the route of a vehicle in VRP is a resource constrained path. The following diagram shows the relationship



VRP and CVRP are usually considered as fundamental problems in discrete optimization. Actually, CVRP is more popular, because it has the degree constraints, which make it make sence in reality background. Researchers came up several different models to describe CVRP. In 1959, George Dantzig and John Ramser asked the world about VRP [Dantzig and Ramser(1959)]. In 5 years, Balinski and Quand proposed a set partition model for VRP. They regarded all possible route as variables, and the degree constraints force that every node appears once in some routes. Let R be the set of all possible routes, d_r be the cost of route r and $a_{i,r}$ be a binary variable indicate that if the consumer i is visited in route r, the mathematical formula is

$$\begin{aligned}
&\min \sum_{r\in R} d_r y_r \\
&\text{subject to:} \\
&\sum_{\text{consumer } i} a_{i,r} = 1 \\
&\sum_{r\in R} y_r = m \\
&a_{i,r}, y_r \in \{0,1\}
\end{aligned}$$

Later researchers tried use column generation algorithm and dynamic programming to solve Balinski and Quand's model. But the results are not positive, only Agarwal, Mathur, and Salkin using column generation algorithm successful solve instances with $15-25$ many consumers [Agarwal et al.(1989)Agarwal, Mathur, and Salkin]. In 2008, Baldacci, Christofides,

and Mingozzi adding some inequalities and equalities constraints into this classical set partition model. Their method is able to solve the instances with 37 to 127 many vertices [Baldacci et al.(2008)Baldacci, Christofides, and Mingozzi].
Apart from regarding the possible routes as variables, it's natural to directly use edges as variables. This idea derives the vehicle flow model, which is first modeled by Laporte and Nobert [Laporte and Nobert(1983)]. They describe the problem as

$$\begin{aligned}
&\sum_{\{i,j\}\in E} c_{ij}x_{ij}\\
&\text{subject to:}\\
&\sum_{j=1}^{n} x_{\{0,j\}} = 2M\\
&\sum_{i\neq j} x_{\{i,j\}} = 2 \quad \forall i \in [n]\\
&\sum_{i\in S}\sum_{j\in S^c} x_{\{i,j\}} \geq 2m(S) \quad \forall S \subseteq [n]\\
&x_{\{0,j\}} \in \{0,1,2\} \quad \forall j \in [n]\\
&x_{\{i,j\}} \in \{0,1\} \quad \forall i,j \in [n], i \neq j
\end{aligned}$$

Here $m(S)$ represent the minimal number of vehicles such that the demand of all consumer in S can be met. Branch and Cut algorithm is common used in solving this model.
In the vehicle flow model, $m(S)$ cost a lot of computation. One idea to deal with the resource constraints is constructing a flow , every time, flow meet a consumer, the flow value will decrease by the demand of this consumer. Therefore, the resource constraints are equivalent to all flow values are non-negative. The commodity flow model follows this idea. Gavish and Graves [Gavish and Graves(1978)] extended Garvin's idea in an oil delivery problem [Garvin et al.(1957)Garvin, Crandall, John, and Spellman], gave their commodity flow model. For every consumer, they set variables indicate the flow from every edge in order to satisfies this consumer. In 2004, Baldacci raise his two commodity flow [Baldacci et al.(2004)Baldacci, Hadjiconstantinou, and Mingozzi], his computation result shows a progress.
Baldacci's model has advantages compared to other models. The variables and constraints are increase polynomially while other two models, the vehicle flow model and set partition model, increase exponentially. This make the model is solvable by current linear programming solver. For examples, Baldacci's model can solve a problem with 104 vertices size, which is not solved until Baldacci did it, and for some test benchmark, this model is optimal.
The main topic of this thesis, is studying the Baldacci's model and follow his idea to propose a new mixed integer programming model for RCSPP. We claim this new model is

better than the classic RCSPP model.
In section 3, we study the Baldacci's model. We first find some implicit equations, then we develop a method to read the redundancy of the equation system and find redundant equation. We also use this method to exclude some inequality from facet defining inequality. In with an extra condition that every consumer has the same demand, we prove the dimension of the polytopes from linear relaxation and Baldacci's model. Since the dimension is proved, we find some facet defining inequalities.
In section 4, we propose a new model for RCSPP followed Baldacci's idea. We use an example to show that this model is better, that is the polytope project onto the edge variable space is a proper subset of classic model. Then similar to the section 3, we first find the redundant equation and by constructing the affinely independent points, we claim the dimension of the polytope after linear relaxation. With the assumption that every consumer has the same demand, we can find the dimension of the polytope from the new model. Moreover, we find some simple cuts, but when the capacity is exactly twice of the demand, the cuts can provide a complete description of polytope from mixed integer programming.

Chapter 2

Notations and Preliminaries

The preliminaries are basic knowledgement about polytopes and optimization.
I am using the same notations as Prof. Ralf Borndörfer used. I emphasis some uncommon notation here. Let ij be abbreviation of the tuple (i, j) if the context is clear, and in a undirection graph $G = (V, E)$ $\gamma(u) := \{v : \{u, v\} \in E\}$, and in a directed graph $D = (V, A)$, $\gamma^+(u) := \{v : (u, v) \in A\}$ and $\gamma^-(u) := \{v : (v, u) \in A\}$. And in simple graph, we use a sequence of vertices $(u_1, u_2, ..., u_n)$ to represent a path.

Chapter 3

Capacitated Vehicle Routing Problem

This section introduces Baldacci's approach to the capacitated vehicle routing problem. Subsection 3.1 gives a formal statement of the problem. Subsection 3.2 states the integer program that was suggested by Baldacci [year]. Subsection 3.3 shows some polyhedral result on Bladacci's model. Subsection 3.4 shows more polyhedral result under a special case.

3.1 Problem Description

Baldacci proposed an integer programming model for the CVRP, that is based on the idea of a resource flow. Informal description of the idea.

Definition 3.1.1 (Capacitated Vehicle Routing Problem (CVRP)) ***Given:*** *Undirected simple graph $G = (V, E)$, two special vertices, the sink $s \in V$ and the target $t \in V$, non-negative integer edge costs $c \in \mathbb{N}^E_{\geq 0}$, natural numbers $M, Q \in \mathbb{N}$ and node demands $q \in \mathbb{N}^N_{\geq 0}$, where $N = V \setminus \{s, t\}$. A route f consists of M simple st-path in G, its cost and demand are $c(f)$ and $q(f)$, respectively, and it is feasible if $q(f) \leq Q$ holds. A schedule is a set of feasible routes that visits every nodes except s, t exactly once.* ***Sought:*** *A schedule of minimal total cost.*

3.2 Baldacci's Model

Consider a undirected graph $G = (V, E)$, where $E = \{\{u, v\} : u, v \in N\} \cup \{\{s, u\} : u \in N\} \cup \{\{v, t\} : v \in N\}$, and a directed graph $D(G) = (V, A(E))$ such that $A(E) = \{uv : \{u, v\} \in E\} \cup \{vu : \{u, v\} \in E\}$. For all edge $\{u, v\} \in E$, assign an edge variable $\xi_{\{u,v\}}$, and for all arc $uv \in A(E)$, assign a flow variable x_{uv}. Then the Baldacci's model [Baldacci et al.(2004)Baldacci, Hadjiconstantinou, and Mingozzi] is the following

$$\min \sum_{\{u,v\}\in E} c_{\{u,v\}}\xi_{\{u,v\}}$$

subject to:

$$\sum_{v\in\gamma(u)} (x_{vu} - x_{uv}) = 2q_u \quad \forall u \in N \tag{3.1}$$

$$\sum_{v\in N} x_{sv} = q(N) \tag{3.2}$$

$$\sum_{v\in N} x_{vs} = MQ - q(N) \tag{3.3}$$

$$\sum_{v\in N} x_{tv} = MQ \tag{3.4}$$

$$x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \quad \forall \{u,v\} \in E \tag{3.5}$$

$$\sum_{v\in\gamma(u)} \xi_{\{u,v\}} = 2 \quad \forall u \in N \tag{3.6}$$

$$x_{uv} \geq 0,\ x_{vu} \geq 0,\ \xi_{\{u,v\}} \in \{0,1\} \quad \forall\ \{u,v\} \in E$$

In this model, (3.6) is for the degree constraint. (3.1) is for satisfying the demands of all consumers. In the solution, here is a flow and a residual flow represent the load and empty space of vehicle respectively. (3.5) implies the capacity of each vehicle and connects variables $\xi_{\{u,v\}}$, x_{uv} and x_{vu}. The rest of equations are for the initial condition.

3.3 Polyhedral Study of Baldacci's Model

3.3.1 Complete the equation system

Let B_{IP} be the polytope of Baldacci's model and B_{LP} be the polytope of Baldacci's model after relaxing variable $\xi_{\{u,v\}} \in \{0,1\}$ to $\xi_{\{u,v\}} \in [0,1]$ for all $\{u,v\} \in E$. It's clear that $B_{IP} \subseteq B_{LP}$. Our first step is finding some implicit valid equations for B_{LP}, these inequalities can help us to approach the dimension.

Observation 1 $x_{ut} = 0$ *for all* $u \in N$

Proof:

$$\begin{aligned}
0 &= \sum_{uv \in A(E)} (x_{uv} - x_{uv}) = \sum_{u \in V} \sum_{v \in \gamma(u)} (x_{uv} - x_{vu}) \\
&= \sum_{v \in N} (x_{sv} - x_{vs}) + \sum_{v \in N} (x_{tv} - x_{vt}) + \sum_{u \in N} \sum_{v \in \gamma(u)} (x_{uv} - x_{vu}) \\
&= \sum_{u \in N} q_u - (MQ - \sum_{u \in N} q_u) + MQ - \sum_{u \in N} x_{ut} + \sum_{u \in N} (-2q_u) \\
&= -\sum_{u \in N} x_{ut}
\end{aligned}$$

Since $x_{ut} \geq 0$ for all $u \in N$, therefore $x_{ut} = 0$ for all $u \in N$, and these equations are valid for B_{LP}. Now we do some operations to draw equations that are independent to others. Plug (3.5) into (3.6), yields

$$\sum_{v \in \gamma(u)} x_{uv} + x_{vu} = 2Q$$

Combined with (3.1),

$$\begin{cases} \displaystyle\sum_{v \in \gamma(u)} x_{vu} - x_{uv} = 2q_u \\ \displaystyle\sum_{v \in \gamma(u)} x_{vu} + x_{uv} = 2Q \end{cases} \Rightarrow \begin{cases} \displaystyle\sum_{v \in \gamma(u)} x_{vu} = Q + q_i \\ \displaystyle\sum_{v \in \gamma(u)} x_{uv} = Q - q_i \end{cases}$$

The Baldacci's model is equivalent to

$$\min \sum_{\{u,v\}\in E} c_{\{u,v\}}\xi_{\{u,v\}}$$

subject to:

$$\sum_{v\in\gamma(u)} x_{vu} = Q + q_u \quad \forall u \in N \tag{3.7}$$

$$\sum_{v\in\gamma(u)} x_{uv} = Q - q_u \quad \forall u \in N \tag{3.8}$$

$$\sum_{u\in N} x_{su} = \sum_{u\in N} q_u \tag{3.9}$$

$$\sum_{u\in N} x_{us} = MQ - \sum_{u\in N} q_u \tag{3.10}$$

$$\sum_{u\in N} x_{tu} = MQ \tag{3.11}$$

$$x_{ut} = 0 \quad \forall u \in N \tag{3.12}$$

$$x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \quad \forall \{u,v\} \in E \tag{3.13}$$

$$x_{uv} \geq 0,\ x_{vu} \geq 0,\ \xi_{\{u,v\}} \in \{0,1\} \quad \forall\ \{u,v\} \in E$$

We eliminate some equation constraints involve the variables ξ_{ij}, such that the variable ξ_{ij} only appears exactly once in (3.13), so these equations from (3.13) are independent to equation system in (3.7)-(3.12). Therefore, after the operation, equations from (3.13) can't be redundant.

Recall that

$$0 = \sum_{uv\in A(E)} (x_{uv} - x_{uv}) = \sum_{v\in N}(x_{sv} - x_{vs}) + \sum_{v\in N}(x_{tv} - x_{vt}) + \sum_{u\in N}\sum_{v\in\gamma(u)} (x_{uv} - x_{vu})$$

The equation above tells the equation system from (3.7)-(3.12) is redundant. Therefore, we need to find the how many redundant equations in this model. In the next section, we will explain how to find it.

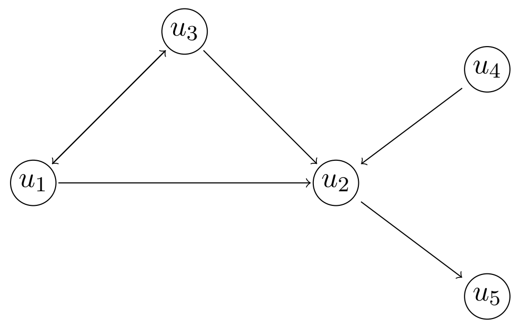
3.3.2 A method for reading the irredundancy

Consider a directed connected graph $D_1(G) = (U, \bar{A})$. For every arc (u,v) in $\bar{A}$, assign a variable x_{uv}. Since $D_1(G)$ is connected, at least one of $\gamma^+(u)$, $\gamma^-(u)$ is non-empty for an arbitrary $u \in U$. Then for every non-empty set $\gamma^+(u)$ and $\gamma^-(u)$, we assign the following equations with proper value $\beta_u, \bar{\beta}_u$

$$\sum_{v\in\gamma^+(u)} x_{uv} = \beta_u \quad \text{or} \quad \sum_{v\in\gamma^-(u)} x_{vu} = \bar{\beta}_u \tag{3.14}$$

Here is an example about how to this process works.

Example 3.3.1 *Let the directed graph be*



Fro, this graph we have

$$\begin{aligned}
&\gamma^+(u_1) = \{u_2, u_3\} && \gamma^-(u_1) = \{u_3\} \\
&\gamma^+(u_2) = \{u_5\} && \gamma^-(u_2) = \{u_1, u_3, u_4\} \\
&\gamma^+(u_3) = \{u_1, u_2\} && \gamma^-(u_3) = \{u_1\} \\
&\gamma^+(u_4) = \{u_2\} && \gamma^-(u_4) = \emptyset \\
&\gamma^+(u_5) = \emptyset && \gamma^-(u_5) = \{u_2\}
\end{aligned}$$

Therefore this graph will give the following equation system with proper choice of β and $\bar{\beta}$

$$\begin{aligned}
&x_{u_1,u_2} + x_{u_1,u_3} = \beta_{u_1} \\
&x_{u_3,u_1} = \bar{\beta}_{u_1} \\
&x_{u_2,u_5} = \beta_{u_2} \\
&x_{u_1,u_2} + x_{u_3,u_2} + x_{u_4,u_2} = \bar{\beta}_{u_2} \\
&x_{u_3,u_1} + x_{u_3,u_2} = \beta_{u_3} \\
&x_{u_1,u_3} = \bar{\beta}_{u_3} \\
&x_{u_4,u_2} = \beta_{u_4} \\
&x_{u_2,u_5} = \bar{\beta}_{u_5}
\end{aligned}$$

Now write the whole equation system as a matrix form $Cx = d$. Here is a bijection such that we can find the corresponding node and the arc direction of every row of C. Therefore, the rows in C can be represent as $c_{\gamma^+(u)}$ and $c_{\gamma^-(u)}$ with respect to $\gamma^+(u)$ and $\gamma^-(u)$. Now, we are going to discuss about a vector a such that $\boldsymbol{a}^{\mathrm{T}}C = \mathbf{0}^{\mathrm{T}}$, since $\dim\{\boldsymbol{a} : \boldsymbol{a}^{\mathrm{T}}C = \mathbf{0}^{\mathrm{T}}\}$ will give us the rank(C). Let $\alpha^+(u)$, $\alpha^-(u)$ be entries of $\boldsymbol{a}$, such that

$$\boldsymbol{a}^{\mathrm{T}}C = \sum_{\substack{u\in V \\ \gamma^+(u)\neq\emptyset}} \alpha^+(u)c_{\gamma^+(u)} + \sum_{\substack{u\in V \\ \gamma^-(u)\neq\emptyset}} \alpha^-(u)c_{\gamma^-(u)}$$

At the beginning, here are two observations.

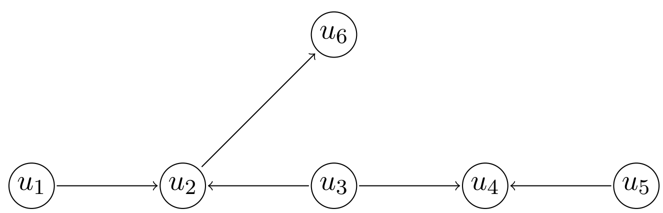
Observation 2 *Only in $c_{\gamma^+(u)}$ and $c_{\gamma^-(v)}$, the column respects to variable x_{uv} is non-zero.*

Observation 3 *An arc $(u, v) \in D_1(G)$ implies a relationship that $\alpha^+(u) = -\alpha^-(v)$ since this is the only way to eliminate variable x_{uv}*

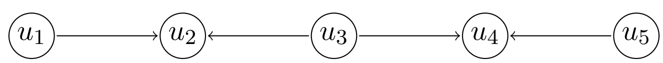
Since the graph $G(U, \bar{A})$ is connected, we could and we want to extend our observations to all vertices, that is finding a series of equations involving all the vertices of $G(U, \bar{A})$. Therefore, we need a special "path" to transfer this property.

Definition 3.3.1 (Alternating Path) *Let $(u_1, \cdots, u_n)$ be a sequence of distinct vertices in $G(U, \bar{A})$. If here is a sequence of arcs $(a_1, \cdots, a_{n-1})$, such that for all $k \in [n-1]$, $a_{2k+1} = (u_{2k+1}, u_{2k+2})$, $a_{2k} = (u_{2k+1}, u_{2k})$ or $a_{2k+1} = (u_{2k+2}, u_{2k+1})$, $a_{2i} = (u_{2k}, u_{2k})$, then we call this sequence $(u_1, a_1, u_2, \cdots, a_{n-1}, u_n)$ alternating path, and the number of arcs contained in an alternating path is called the length of this alternating path.*

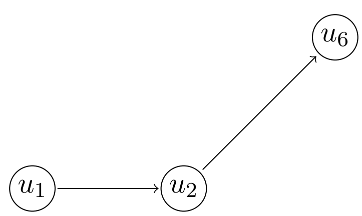
Example 3.3.2 *Here is an example for the alternating path I defined*



The sequence $(u_1, e_{u_1,u_2}, u_2, e_{u_3,u_2}, u_3, e_{u_3,u_4}, u_4, e_{u_5,u_4}, u_5)$ is an alternating path.



And the sequence $(u_1, e_{u_1,u_2}, u_2, e_{u_2,u_6}, u_6)$ is not an alternating path.



Lemma 1 *If here is an alternating path $(u_1, a_1, ..., a_{n-1}, u_n) = P$ in the graph $G(U, \bar{A})$. Then for all $k \in [n]$, we have*

- *when $a_1 = (u_1, u_2)$:*

$$\alpha^+(u_1) = (-1)^{k-1} \alpha^{sign\left((-1)^{k-1}\right)}(u_k)$$

- *when* $a_1 = (u_2, u_1)$*:*

$$\alpha^-(u_1) = (-1)^{k-1}\alpha^{sign\left((-1)^{k-1}\right)}(u_k)$$

Proof: Induction on the length of alternating path.
For these alternating paths have length 1, it's just the observation 3.
Suppose the statement is true for all alternating paths with length $k-1$.
Assume $a_1 = (u_1, u_2)$, then $a_k = (a_k, a_{k+1})$ if $k \equiv 1 \pmod 2$ and $a_k = (a_{k+1}, a_k)$ if $k \equiv 0 \pmod 2$. In the first case ($k \equiv 1 \pmod 2$), we have

$$\alpha^+(u_1) = \alpha^+(u_k)$$
$$\alpha^+(u_k) = -\alpha^-(u_{k+1})$$

Therefore $\alpha^+(u_1) = \alpha^-(u_{k+1})$. In all other cases, the proofs are similar.
One alternating path derives equations with respect to arcs, and in the graph $D(G)$, every arc has an arc with opposite direction associated with. So, we actually have two parallel alternating paths. And we want to connect the equations derived from the parallel alternating paths.

Theorem 3.3.2 *Let* $P = (u_1, a_1, \cdots, a_{2k}, u_{2k+1}) \subseteq G(U, \bar{A})$ *be an alternating path with length* $2k$*. If* $-P := (u_{2k+1}, -a_{2k}, u_{2k}, \cdots, -a_1, u_1)$ *exists in* $G(U, \bar{A})$*, and here is an arc* $(u_1, u_{2k+1}) \in \bar{A}$*. Then,*

$$\forall u \in V(P),\ \alpha^+(u) = -\alpha^-(u) \text{ and } \forall u, v \in V(P),\ |a^+(u)| = |a^+(v)|$$

Moreover, let N *be the set of vertices are in* P *or adjacent to* P*. If* $\boldsymbol{a}^{\mathrm{T}}C = \mathbf{0}^{\mathrm{T}}$*, then* $|\alpha^{\mathrm{sign}_1}(u)| = |\alpha^{\mathrm{sign}_2}(v)|$ *for all* $u, v \in N$*, where* $\mathrm{sign}_1, \mathrm{sign}_2 \in \{-, +\}$

Proof:
Since $k \geq 1$, then here exist a point u_l, such that both $l-1$, $2k+1-l$ are odd. Therefore, from the Lemma 1, we have

$$\alpha^+(u_1) = -\alpha^-(u_l) \quad \alpha^-(u_1) = -\alpha^+(u_l)$$
$$\alpha^+(u_{2k+1}) = -\alpha^-(u_l) \quad \alpha^-(u_{2k+1}) = -\alpha^+(u_l)$$

Since here is an arc (u_1, u_{2k+1}), so $\alpha^+(u_1) = -\alpha^-(u_{2k+1})$. This means $\alpha^+(u_l) = -\alpha^-(u_l)$.
Then Lemma 1 can extend the relation to all vertices.
In this section, given a directed graph, for any vertices in this graph, we associate it with equations of the form (3.14). Then we can just read the graph and use the Theorem 3.3.2 to know if here exists redundant equations.

3.3.3 Dimension and Facets Results for B_{LP}

Now, we return to (3.7) – (3.12) in Baldacci's model. In this model, every vertices has two equations of the form (3.14), and the graph has many edges, so we can always find a large even alternating path P with vertex ordered as $(u_1, \cdots, u_{2k+1})$, such that u_1, u_{2k+1} are adjacent, $-P$ exist in the induced graph $D(G) = (V, A(E))$ and all vertices in V are either contained in P or adjacent to some vertices in P. So, exactly one equation in the equation system is redundant. Assume $|N| = n \geq 3$, then

$$\dim B_{LP} \leq 3(\binom{n+2}{2} - 1) - (\binom{n+2}{2} - 1) - (3n+3) + 1 = n^2 - 2$$

Although we only know the upper bound of dimension yet. But we can decide some inequalities aren't facets by count how many inequalities will be tight

Observation 4 *$\xi_{\{u,v\}} \geq 0$, where $u, v \in N$, isn't a facet defining inequality when $|N| \geq 5$.*

proof:

$$\begin{cases} \xi_{\{u,v\}} = 0 \\ x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \Rightarrow x_{uv} = x_{vu} = \xi_{\{u,v\}} = 0 \\ x_{uv}, x_{vu} \geq 0 \end{cases}$$

It's sufficient to check that $x_{uv} = 0$ isn't a linear combination of equations in (3.7)-(3.13) and $\xi_{\{u,v\}} = 0$. At the beginning, apply then Theorem 3.3.2 for a subgraph induced by $V \setminus \{u, v\}$, and then check the whole system.

Observation 5 *If $M = 1$, then $\xi_{\{s,u\}} \leq 1$, $\xi_{\{u,t\}} \leq 1$ is not a facet defining inequality.*

Proof:
Note we have

$$\begin{aligned} &\sum_{u \in N} x_{tu} = Q \\ &x_{ut} = 0 \quad \forall u \in N \\ &x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \quad \forall \{u,v\} \in E \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\sum_{u \in N} Q\xi_{\{t,u\}} = \sum_{u \in N} (x_{tu} + x_{ut}) = \sum_{u \in N} x_{tu} = Q \\ &\Rightarrow \sum_{u \in N} \xi_{\{t,u\}} = 1 \end{aligned}$$

If for one u_1 such that $\xi_{\{u_0,t\}} = 1$, then it implies that $\xi_{\{u,t\}} = 0$ for all $u \neq u_0$. Similarly, from

$$\sum_{u \in N} x_{us} = Q - \sum_{u \in N} q_u$$
$$\sum_{u \in N} x_{su} = \sum_{u \in N} q_u$$

We get

$$\sum_{u \in N} \xi_{\{s,u\}} = 1$$

So, we have the observation.

3.4 Dimension and Facets Result for Unit Demand Case

If $q_u = 1$ for all $u \in V$, we can this is the unit demand case. Therefore, in this case all vertices in N are "equivalent", in the sense we can get another feasible by exchange vertices. Recall Baldacci's model under this special case is

$$\min \sum_{\{u,v\} \in E} c_{\{u,v\}} \xi_{\{u,v\}}$$

subject to:

$$\sum_{v \in \gamma(u)} x_{vu} = Q + q_u \quad \forall u \in N \tag{3.15}$$
$$\sum_{v \in \gamma(u)} x_{uv} = Q - q_u \quad \forall u \in N \tag{3.16}$$
$$\sum_{u \in N} x_{su} = \sum_{u \in N} q_u \tag{3.17}$$
$$\sum_{u \in N} x_{us} = MQ - \sum_{u \in N} q_u \tag{3.18}$$
$$\sum_{u \in N} x_{tu} = MQ \tag{3.19}$$
$$x_{ut} = 0 \quad \forall j \in N \tag{3.20}$$
$$x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \quad \forall \{u,v\} \in E \tag{3.21}$$
$$x_{uv} \geq 0,\ x_{vu} \geq 0,\ \xi_{\{u,v\}} \in \{0,1\} \quad \forall\ \{u,v\} \in E$$

In previous section, we have obtain a upper bound of the dimension of B_{LP}. In this section, we will show in this special case, we can find a lower bound of dimension. Precisely, that is the following theorem

Theorem 3.4.1 *Under the unit demand case, If $Q \geq 5$, $M \geq 2$ and $MQ > n = |N| > (M-1)+Q$, the polytopes from Baldacci's model are denoted by B^{u+}_{IP}, B^{u+}_{LP} respectively. A result is B^{u+}_{IP}, B^{u+}_{LP} are $(n^2-2)-$dimensional polytopes.*

It's sufficient to show $n^2-2 = \dim B_{IP}$. We will use a similar idea as [Campos et al.(1991)Campos, Corberan, and Mota] by showing every valid equation for B_{LP} has to be a linear combination of equations in (3.15)-(3.21).

Let $\boldsymbol{c}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{\xi}\end{pmatrix} = \beta$ be an arbitrary valid equation for B_{IP}. Let $T\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{\xi}\end{pmatrix} = \boldsymbol{t}$ be the matrix of equations in (3.15)-(3.21). Since $\boldsymbol{c}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{\xi}\end{pmatrix} = \beta$ is a valid equation, it's enough to show $\boldsymbol{c}^{\mathrm{T}} \in \mathrm{row}(T)$. We know the every column of $\boldsymbol{c}^{\mathrm{T}}$ and T corresponding to one unique variable. So, we use $\boldsymbol{c}^{\mathrm{T}}\big|_{\{x_{uv}\}}$ and $T\big|_{\{x_{uv}\}}$ to denote the column corresponding to x_{uv}, same for other variables. We will take the following steps to prove Theorem 3.4.1.

Step 1:

Finding a full-rank square submatrix M of T. Fix two vertices v_1, v_2 in V, let

$$I = \{x_{su}\}_{u\in N} \cup \{x_{us}\}_{u\in N} \cup \{x_{ut}\}_{u\in N} \cup \{x_{v_1v_2}, x_{tv_1}\} \cup \{\xi_{\{u,v\}}\}_{uv\in E}$$
$$I_x = \{x_{su}\}_{u\in N} \cup \{x_{us}\}_{u\in N} \cup \{x_{ut}\}_{u\in N} \cup \{x_{v_1v_2}, x_{tv_1}\}$$

Then $M := T\big|_I$ is a full-rank square matrix.

Proof:

It's clear that every row of $T\big|_I$ from (3.21) is independent to all other rows, since the variable $\xi_{\{uv\}}$ only appears in (3.21). Therefore, it's enough to show that the matrix $T'\big|_I$ which consists of the rows from (3.15)-(3.20) in $T\big|_I$ is full rank.

Construct a directed graph $D_2(G) = (V, A')$ such that $A' = \{(u,v) : x_{uv} \in I_x\}$. Then the equations system T'_I represents is of the form (3.14) in $D_2(G)$. Therefore, using the similar analysis in previous section shows $T'\big|_I$ is row full rank matrix.

Step 2:

Since M is full rank, $\boldsymbol{\nu}^{\mathrm{T}}(T\big|_I) = (\boldsymbol{c}^{\mathrm{T}}\big|_I)$ has a unique solution. Let $\boldsymbol{r}^{\mathrm{T}} = \boldsymbol{c}^{\mathrm{T}} - \boldsymbol{\nu}^{\mathrm{T}}T$, it sufficient to show $\boldsymbol{r}^{\mathrm{T}} = \boldsymbol{0}^{\mathrm{T}}$. Since $\boldsymbol{c}^{\mathrm{T}}$ is a valid equality. So, let s_1, s_2 be two different schedule and $\boldsymbol{z}(s_1), \boldsymbol{z}(s_2)$ be the embedded points in vector space, we have

$$\langle \boldsymbol{r}, (\boldsymbol{z}(s_1) - \boldsymbol{z}(s_2))\rangle = 0 \tag{3.22}$$

Step 3:

First we pick two arbitrary vertices u_1, u_2, then take two schedule s_1, s_2, such that $p_1 \subseteq s_1, p_2 \subseteq s_2$ and $s_1 \setminus p_1 = s_2 \setminus p_2$

$$p_1 = (s, u_{i_1}, u_1, u_2, u_{i_3}, \cdots, u_{i_m}, t)$$
$$p_2 = (s, u_{i_1}, u_2, u_1, u_{i_3}, \cdots, u_{i_m}, t)$$

Use the equation (3.22), we have a equation. For convenience, in this proof, $\boldsymbol{r}_{u_i u_j}^{\mathrm{T}}$ are abbreviate as $\boldsymbol{r}_{ij}^{\mathrm{T}}$

$$\begin{aligned}&(m-1)(\boldsymbol{r}_{i_1 1}^{\mathrm{T}}-\boldsymbol{r}_{i_1 2}^{\mathrm{T}})+(m-2)(\boldsymbol{r}_{12}^{\mathrm{T}}-\boldsymbol{r}_{21}^{\mathrm{T}})+(m-3)(\boldsymbol{r}_{2i_3}^{\mathrm{T}}-\boldsymbol{r}_{1i_3}^{\mathrm{T}})+\\&\big(Q-(m-1)\big)(\boldsymbol{r}_{1i_1}^{\mathrm{T}}-\boldsymbol{r}_{2i_1}^{\mathrm{T}})+\big(Q-(m-2)\big)(\boldsymbol{r}_{21}^{\mathrm{T}}-\boldsymbol{r}_{12}^{\mathrm{T}})+\big(Q-(m-3)\big)(\boldsymbol{r}_{i_3 2}^{\mathrm{T}}-\boldsymbol{r}_{i_3 1}^{\mathrm{T}})=0\end{aligned} \tag{3.23}$$

Step 4:
Similarly, take two schedule with different paths

$$\begin{aligned}p_1&=(s,u_{i_m},u_{i_1},u_1,u_2,u_{i_3},\cdots,u_{i_{m-1}},t)\\p_2&=(s,u_{i_m},u_{i_1},u_2,u_1,u_{i_3},\cdots,u_{i_{m-1}},t)\end{aligned}$$

And drive an equation

$$\begin{aligned}&(m-2)(\boldsymbol{r}_{i_1 1}^{\mathrm{T}}-\boldsymbol{r}_{i_1 2}^{\mathrm{T}})+(m-3)(\boldsymbol{r}_{12}^{\mathrm{T}}-\boldsymbol{r}_{21}^{\mathrm{T}})+(m-4)(\boldsymbol{r}_{2i_3}^{\mathrm{T}}-\boldsymbol{r}_{1i_3}^{\mathrm{T}})\\&+\big(Q-(m-2)\big)(\boldsymbol{r}_{1i_1}^{\mathrm{T}}-\boldsymbol{r}_{2i_1}^{\mathrm{T}})+\big(Q-(m-3)\big)(\boldsymbol{r}_{21}^{\mathrm{T}}-\boldsymbol{r}_{12}^{\mathrm{T}})+\big(Q-(m-4)\big)(\boldsymbol{r}_{i_3 2}^{\mathrm{T}}-\boldsymbol{r}_{i_3 1}^{\mathrm{T}})=0\end{aligned} \tag{3.24}$$

Subtract (3.24) from (3.23)

$$(\boldsymbol{r}_{i_1 1}^{\mathrm{T}}-\boldsymbol{r}_{i_1 2}^{\mathrm{T}})+(\boldsymbol{r}_{12}^{\mathrm{T}}-\boldsymbol{r}_{21}^{\mathrm{T}})+(\boldsymbol{r}_{2i_3}^{\mathrm{T}}-\boldsymbol{r}_{1i_3}^{\mathrm{T}})=(\boldsymbol{r}_{1i_1}^{\mathrm{T}}-\boldsymbol{r}_{2i_1}^{\mathrm{T}})+(\boldsymbol{r}_{21}^{\mathrm{T}}-\boldsymbol{r}_{12}^{\mathrm{T}})+(\boldsymbol{r}_{i_3 2}^{\mathrm{T}}-\boldsymbol{r}_{i_3 1}^{\mathrm{T}}) \tag{3.25}$$

After that take

$$\begin{aligned}p_1&=(s,u_1,u_2,u_{i_3},\cdots,u_{i_m},u_{i_1},t)\\p_2&=(s,u_2,u_1,u_{i_3},\cdots,u_{i_m},u_{i_1},t)\end{aligned}$$

drive

$$\begin{aligned}&(m-1)(\boldsymbol{r}_{12}^{\mathrm{T}}-\boldsymbol{r}_{21}^{\mathrm{T}})+(m-2)(\boldsymbol{r}_{2i_3}^{\mathrm{T}}-\boldsymbol{r}_{1i_3}^{\mathrm{T}})\\&+\big(Q-(m-1)\big)(\boldsymbol{r}_{21}^{\mathrm{T}}-\boldsymbol{r}_{12}^{\mathrm{T}})+\big(Q-(m-2)\big)(\boldsymbol{r}_{i_3 2}^{\mathrm{T}}-\boldsymbol{r}_{i_3 1}^{\mathrm{T}})=0\end{aligned} \tag{3.26}$$

From the equations above, we can drive

$$m(\boldsymbol{r}_{i_1 1}^{\mathrm{T}}-\boldsymbol{r}_{i_1 2}^{\mathrm{T}})=(m-Q)(\boldsymbol{r}_{1i_1}^{\mathrm{T}}-\boldsymbol{r}_{2i_1}^{\mathrm{T}}) \tag{3.27}$$

If here is a $st-$path with length $Q+1$ in the solution set, then $\boldsymbol{r}_{i_1 1}^{\mathrm{T}}=\boldsymbol{r}_{i_1 2}^{\mathrm{T}}$. Since the choice of u_{i_1},u_1,u_2 are arbitrary, therefore, indeed we have $\boldsymbol{r}_{ij}^{\mathrm{T}}=\boldsymbol{r}_{ik}^{\mathrm{T}}$ for any $u_i,u_j,u_k\in N$. Similarly, If here is a $st-$path with length at most Q in the solution set, then $\boldsymbol{r}_{ji}^{\mathrm{T}}=\boldsymbol{r}_{ki}^{\mathrm{T}}$ for

any $u_i, u_j, u_k \in N$. So, $\boldsymbol{r}_{ij}^{\mathrm{T}} = \boldsymbol{r}_{kl}^{\mathrm{T}}$ for any $u_i, u_j, u_k, u_l \in N$.
Step 5: Let

$$p_1 = (s, u_{i_1}, u_{i_2}, \cdots, u_{i_m}, u_1, u_2, t)$$
$$p_2 = (s, u_{i_1}, u_{i_2}, \cdots, u_{i_m}, u_2, u_1, t)$$

With the result in Step 4, we get

$$\boldsymbol{r}_{ti}^{\mathrm{T}} = \boldsymbol{r}_{tj}^{\mathrm{T}} \quad \forall u_i, u_j \in N$$

Since by the construction of $\boldsymbol{\nu}$, $\boldsymbol{r}^{\mathrm{T}}\big|_I = \mathbf{0}^{\mathrm{T}}$, therefore $\boldsymbol{r}^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}}$.
The requirement on the length of a path needs the extra assumption about M, Q
Now, since we know the dimension of polytope, we can go to decide the facets defining inequality of B_{LP}^{u+}, B_{IP}^{u+}

Observation 6 *$\xi_{\{u,t\}} \geq 0$ and $x_{tu} \geq 0$ defines facet of B_{LP}^{u} and B_{IP}^{u}*

Proof :
Similar to the dimension proof and take

$$I = \{\{x_{su}\}_{u\in N}, \{x_{us}\}_{u\in N}, \{x_{ut}\}_{u\in N}, x_{u_1,u_2}, x_{tu_j}, x_{tu_k}, \{\xi_{u,v}\}_{\{u,v\}\in E}\}$$

Observation 7 *The polytope B_{LP} is invariant under relabel the vertices in N.*

This observation is based on the graph is very "complete", relabel the vertices doesn't change the equation system of B_{LP}, and therefore, the polytope B_{LP} is also symmetric.

Theorem 3.4.2 *$x_{uv} \geq 0$ defines a facet of B_{LP}^{u+} for all $u, v \in N$, if $|N| \geq 16$*

Proof : Let $n = |N|$, we know $\dim B_{LP}^{u+} = n^2 - 2$, so the number of facet defining inequalities is at least $n^2 - 1$. Since B_{LP}^{u+} is a special case of B_{LP} and $N \geq Q \geq 5$, we know that $\xi_{\{u,v\}} \geq 0$ isn't a facet. Here are totally $2(n^2 + 3n) - \binom{n}{2}$ many potential facet defining inequalities and $n^2 - n$ many inequalities of the form $x_{uv} \geq 0$ for $u, v \in N$. Therefore, if $n \geq 16$, then by pigeonhole principle, $x_{uv} \geq 0$ is a facet for some $u, v \in N$. However, because of Observation 7, $x_{uv} \geq 0$ is a facet for all $u, v \in N$

Chapter 4

Resource Constrained Shortest Path Problem

In section 4.1, we state the RSCPP and the basic classical model. In section 4.2, we follow the idea of Baldacci and give a new model for RCSPP, and show this model is better than the basic model. In section 4.3, we discuss the dimension of the LP-relaxation. In section 4.4, we discuss the dimension of the lattice polytope while all demand are units. In section 4.5, we give a series of cuts, these cuts are facets under the unit demand case and works well when capacity $Q = 2$.

4.1 Problem Description and Classical Model

Definition 4.1.1 (Resource Constrained Shortest Path Problem (RCSPP)) ***Given:*** *directed simple graph $G = (V, A)$, two special vertices, the sink $s \in V$ and the target $t \in V$, non-negative integer edge costs $c \in \mathbb{N}^E_{\geq 0}$, natural numbers $Q \in \mathbb{N}$ and node demands $q \in \mathbb{N}^N_{\geq 0}$, where $N = V \setminus \{s, t\}$. A route is simple st-path p in G, its cost and demand are $c(p)$ and $q(p)$, respectively, and it is feasible if $q(p) \leq Q$ holds.* ***Sought:*** *A route of minimal total cost.*

And a natural model for RCSPP is adding resource constraints to the shortest path problem.

$$\min \sum_{uv \in A} c_{uv} x_{uv}$$

subject to:

$$\sum_{u \in N} x_{su} = 1 \tag{4.1}$$

$$\sum_{u \in N} x_{ut} = 1 \tag{4.2}$$

$$x_{us} = x_{tu} = 0 \quad \forall u \in N \tag{4.3}$$

$$\sum_{v \in \gamma^-(u)} x_{vu} - \sum_{v \in \gamma^+(u)} x_{uv} = 0 \quad \forall u \in N \tag{4.4}$$

$$\sum_{u \in N} q_u (\sum_{v \in \gamma^+(u)} x_{uv}) \leq Q \tag{4.5}$$

$$x_{uv} \in \{0, 1\} \; \forall uv \in A \tag{4.6}$$

4.2 A New Model

Transfer a given undirected graph $G = (V, E)$ into a directed graph $D(G) = (V, A(E))$ with $c_{\{u,v\}} = c_{uv} = c_{vu}$. Assign an arc variable x_{uv} and a flow variable y_{uv} for all $uv \in A(E)$.

$$\min \sum_{uv \in A(E)} c_{uv} x_{uv}$$

subject to:

$$\sum_{u \in N} x_{su} = 1 \tag{4.7}$$

$$x_{us} = x_{tu} = 0 \quad \forall u \in N \tag{4.8}$$

$$\sum_{u \in N} x_{ut} = 1 \tag{4.9}$$

$$y_{us} = 0 \quad \forall u \in N \tag{4.10}$$

$$\sum_{v \in \gamma(u)} x_{vu} - \sum_{v \in \gamma(u)} x_{uv} = 0 \quad \forall u \in N \tag{4.11}$$

$$y_{uv} + y_{vu} = Q(x_{uv} + x_{vu}) \quad \forall \{u, v\} \in E \tag{4.12}$$

$$\sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) = q_u \sum_{v \in \gamma(u)} (x_{vu} + x_{uv}) \quad \forall u \in N \tag{4.13}$$

$$x_{uv}, x_{vu} \in \{0, 1\}, \; y_{uv} \geq 0, y_{vu} \geq 0 \; \forall \{u, v\} \in E \tag{4.14}$$

This model is also based on the flow conservation formula, replace the resource constraints by (4.12) and (4.13), which follow the idea of Baldacci's model.
Since we have a new model, the first thing is that we should compare this new model with the classical model.

Observation 8 *The set of feasible solutions of the new model project onto the arc variables space is a subset of the set of feasible solutions of the basic model. Moreover, it can be a proper subset.*

Proof : It's sufficient to check the feasible solutions of the new model satisfy (4.5):

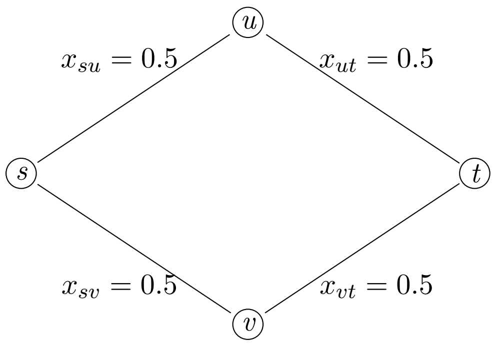
$$\begin{aligned}
\sum_{u \in N} \sum_{v \in \gamma(u)} q_u (x_{uv} + x_{vu}) &= \sum_{u \in N} \sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) \\
&= \sum_{u \in N} \sum_{\substack{v \in \gamma(u) \\ v \in N}} (y_{vu} - y_{uv}) + \sum_{u \in N} (y_{su} - y_{us} + y_{tu} - y_{ut}) \\
&\leq \sum_{u \in N} (y_{su} + y_{us} + y_{tu} + y_{ut}) \\
&= \sum_{u \in N} Q(x_{su} + x_{us}) + \sum_{u \in N} Q(x_{tu} + x_{ut}) = 2Q
\end{aligned}$$

Then the flow conversation constraint says:

$$\sum_{u\in N} q_u \sum_{v\in\gamma(u)} x_{uv} = \sum_{u\in N} q_u \sum_{v\in\gamma(u)} x_{vu}$$

So, the solutions of new model satisfy (4.5).

Example 4.2.1 *Here is simple example to show the new model is better.*



$$q_u = 2$$
$$q_v = 198$$
$$Q = 100$$

This is a solution of the basic model. However, this can't be a solution of the new model. Consider the equation on v

$$\begin{aligned} &y_{sv} + y_{tv} - y_{vs} - y_{vt} = q_v(x_{sv} + x_{tv} + x_{vs} + x_{vt}) = 198 \\ &y_{sv} + y_{vs} = Q(x_{sv} + x_{vs}) = 50 \\ &y_{vt} + y_{tv} = Q(x_{tv} + x_{vt}) = 50 \end{aligned}$$

Since all variables are non-negative, the following inequality is impossible, which means this new model is better than the classical model.

$$198 = y_{sv} + y_{tv} - y_{vs} - y_{vt} > y_{sv} + y_{tv} + y_{vs} + y_{vt} = 100$$

Actually, here is a stronger result. Let f be a flow with respect to the arcs such that $f = \sum_{i=1}^{m} \mu_i p_i$ for some paths p_i with $\sum_{i=1}^{m} \mu_i = 1$. If $\forall i, j \in [m]$, $V(p_i) \cap V(p_j) = \{s, t\}$, then a necessary condition such that f can be the projection of a solution of the new model is

$$q(p_i) := \sum_{u\in V(p_i)} q_u \le Q \quad \forall i \in [m]$$

The advantage in this case is that we don't need to consider the weights. Checking the demands of paths are enough to see if the solution is feasible.
Note the instance of graph should containing nodes, edges, weights and the parameter for capacity Q.

Observation 9 *Let path p be $(s = u_0, u_1, \cdots, u_m, u_{m+1} = t)$, here are trivial value for flow variables that $y_{u_iu_{i+1}} = Q - \sum_{j=1}^{i} q_{u_j}$ and $y_{u_{i+1}u_i} = \sum_{j=1}^{i} q_{u_j}$ can associate to this path to get a solution for the new model. Moreover, if the vertices in p are distinct, then the value of flow variables are unique, ie the trivial value for flow variables are the only way to associate p to a solution of the new model.*

Proof:
This construction gives a solution f. Suppose here is another flow solution f' for p, then $f - f'$ is a solution for the homogeneous equation system with all arc variables equal to 0. Then, plug $x_{uv} = x_{vu} = 0$ for all $\{u, v\} \in E$ into the equation (4.12) and (4.13), we get

$$y_{uv} + y_{vu} = 0 \quad \forall \{u, v\} \in E$$
$$\sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) = 0 \quad \forall u \in N$$

Therefore we conclude that $f - f' = \mathbf{0}$. That means the solution associates to a $s - t$ path is unique.

Observation 10 *If Q is large enough, then the arc set of the optimal solutions of both models form the same directed path. Indeed, let p be the directed path with lowest cost in $D(G) = (V, A(E))$, then we only need $Q \geq c(p)$.*

If Q larger than the demands on a directed path p with lowest costs. Then for any solutions, we can replace the edge sets by the edge sets of p. And by the Observation 9, the solution is unique.

4.3 Dimension of a Polytope from the Model

Now consider $E = \{\{u, v\} : u, v \in N\} \cup \{\{s, u\} : u \in N\} \cup \{\{v, t\} : v \in N\}$, where $|N| = n$. In this case forall $u \in V$, $\gamma(u) = \gamma^+(u) = \gamma^-(u)$. Let P_{LP} be the polytope of the LP-relaxation of model in this graph. Use the similar idea in previous section, the equations can be transferred to

$$\sum_{u\in N} x_{su} = 1 \tag{4.15}$$

$$\sum_{u\in N} x_{ut} = 1 \tag{4.16}$$

$$\sum_{v\in\gamma(u)} x_{uv} - \frac{1}{2}\sum_{v\in\gamma(u)} (y_{vu} - y_{uv}) = 0 \quad \forall u \in N \tag{4.17}$$

$$\sum_{v\in\gamma(u)} x_{vu} - \frac{1}{2}\sum_{v\in\gamma(u)} (y_{vu} - y_{uv}) = 0 \quad \forall u \in N \tag{4.18}$$

$$x_{us} = 0 \quad \forall u \in N \tag{4.19}$$

$$x_{tu} = 0 \quad \forall u \in N \tag{4.20}$$

$$y_{us} = 0 \quad \forall u \in N \tag{4.21}$$

$$y_{uv} + y_{vu} = Q(x_{uv} + x_{vu}) \quad \forall \{u, v\} \in E \tag{4.22}$$

Write the model as $\begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \boldsymbol{c} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$, where C represents the equations from (4.15)-(4.18), X represents equations from (4.19)-(4.20), Y represents (4.21) and G represents equations from (4.22). Actually, from these equations, we can conclude that $y_{us} = Qx_{us}\ \forall u \in N$, but These equations are already in the row space of this matrix.

Lemma 2 *Only one equation in* $\begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \boldsymbol{c} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$ *is redundant. That means the rank of matrix* $\begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix}$ *is* $\frac{1}{2}n^2 + \frac{13}{2}n + 1$.

Proof: We use the same idea as previous section. Consider $(\boldsymbol{a}^{\mathrm{T}}, \boldsymbol{\mu}^{\mathrm{T}}, \lambda, \boldsymbol{b}^{\mathrm{T}})$ such that

$$(\boldsymbol{a}^{\mathrm{T}}, \boldsymbol{\mu}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}, \boldsymbol{b}^{\mathrm{T}}) \begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix} = \mathbf{0}^{\mathrm{T}}$$

Let

- $\alpha^+(u)$ be the entry in $\boldsymbol{a}^\mathrm{T}$ corresponding to

$$\sum_{v\in\gamma^+(u)} x_{uv} = \begin{cases} 1 & \text{if } u = s \\ \dfrac{1}{2}\displaystyle\sum_{v\in\gamma(u)}(y_{vu} - y_{uv}) & \text{if } u \in N \end{cases}$$

- $\alpha^+(u)$ be the entry in $\boldsymbol{a}^\mathrm{T}$ corresponding to

$$\sum_{v\in\gamma^-(u)} x_{vu} = \begin{cases} 1 & \text{if } u = t \\ \dfrac{1}{2}\displaystyle\sum_{v\in\gamma(u)}(y_{vu} - y_{uv}) & \text{if } u \in N \end{cases}$$

- $\mu^+(u)$ be the entry in $\boldsymbol{\mu}^\mathrm{T}$ corresponding to

$$x_{us} = 0$$

- $\mu^-(u)$ be the entry in $\boldsymbol{\mu}^\mathrm{T}$ corresponding to

$$x_{tu} = 0$$

- $\omega(u)$ be the entry in $\boldsymbol{\omega}^\mathrm{T}$ corresponding to

$$y_{us} = 0$$

- $\beta_{\{u,v\}}$ be the entry in $\boldsymbol{b}^\mathrm{T}$ corresponding to

$$y_{uv} + y_{vu} - Q(x_{uv} + x_{vu}) = 0$$

Firstly, we arbitrary choose $u, v \in N$, these variables only appears in (4.17),(4.18) and (4.22), in order to cancel the variables x_{uv} and x_{vu}, we have

$$\alpha^+(u) + \alpha^-(v) + Q\beta_{\{u,v\}} = 0 \text{ and } \alpha^-(u) + \alpha^+(v) + Q\beta_{\{u,v\}} = 0$$

Then consider variables y_{uv} and y_{vu} which also only appears in (4.17),(4.18) and (4.22), yields

$$\begin{aligned} -2\beta_{\{u,v\}} + \alpha^+(u) + \alpha^-(u) - \alpha^-(v) - \alpha^+(v) = 0 \\ -2\beta_{\{u,v\}} - \alpha^+(u) - \alpha^-(u) + \alpha^-(v) + \alpha^+(v) = 0 \end{aligned}$$

Since u, v are arbitrary, for any $u, v \in N$, $\beta_{\{u,v\}} = 0$, $\alpha^+(u) = -\alpha^-(v)$. Then we start to cancel the variable y_{ut}:

$$\alpha^+(u) + \alpha^-(u) - \beta_{\{u,t\}} = 0 \text{ and } \alpha^+(u) + \alpha^-(u) + \beta_{\{u,t\}} = 0$$

This implies $\beta_{\{u,t\}} = 0$ and $\alpha^+(u) = \alpha^-(u)$ for all $u \in N$. Now we cancel the variables x_{ut}, x_{tu}

$$\mu^-(u) = -\alpha^-(u) \text{ and } \alpha^-(t) = -\alpha^+(u)$$

After this, we cancel the variables y_{us} and y_{su}

$$-2\beta_{\{u,s\}} + \alpha^+(u) + \alpha^-(u) + 2\omega_u = 0 \text{ and } -2\beta_{\{u,s\}} + \alpha^+(u) + \alpha^-(u) = 0$$

And we get $\beta_{\{u,s\}} = 0, \omega(u) = 0$ forall $u \in \omega$. Finally, we can cancel the x_{us} and x_{su}

$$\alpha^+(s) = -\alpha^-(u) \text{ and } \mu^+(u) = -\alpha^+(u)$$

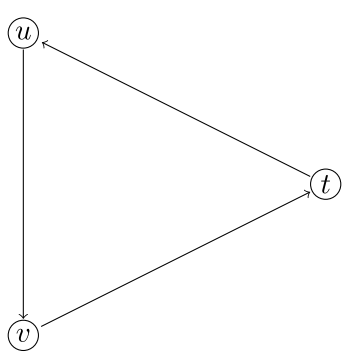
We have canceled all variables, and these relationships tell

$$\dim\{(\boldsymbol{a}^{\mathrm{T}}, \boldsymbol{\mu}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}, \boldsymbol{b}^{\mathrm{T}}) : (\boldsymbol{a}^{\mathrm{T}}, \boldsymbol{\mu}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}, \boldsymbol{b}^{\mathrm{T}}) \begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix} = \mathbf{0}^{\mathrm{T}}\} = 1$$

The rank of matrix follows immediately. Now, this lemma provides a upper bound of dimension, the next task is constructing affinely independent points.

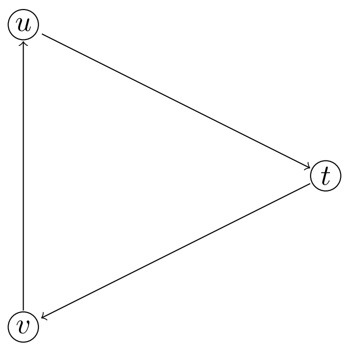
Theorem 4.3.1 *If* $3Q > 5\max_{u\in N} q_u$, *then* $\dim P_{LP} = 3\binom{n}{2} + n - 1$

Proof :
$\dim P_{LP} \leq 3\binom{n}{2} + n - 1$ is immediately from Lemma 2. So, it's sufficient to find $3\binom{n}{2} + n$ many affinely independent points. It's clear that path $p_u := (s, u, t)$ with associated flow variables are affinely independent. Let $f = \frac{1}{n}\sum_{u\in N} p_u$. Then for every two $u, v \in N$, we construct three different solution.
c^1_{uv}:



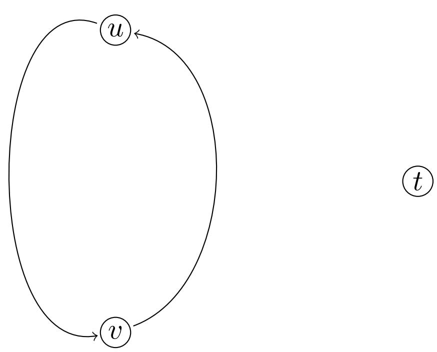
$$\begin{gathered} x_{vu} = x_{tu} = x_{tv} = 0 \\ x_{uv} = 0.5 \\ x_{ut} = -0.5 \\ x_{vt} = 0.5 \\ y_{uv} = 0.5Q \\ y_{vu} = 0 \\ y_{ut} = -0.25Q - 0.25q_u \\ y_{tu} = 0.25Q + 0.25q_u \\ y_{vt} = 0.25Q - 0.25q_v \\ y_{tv} = -0.25Q + 0.25q_v \end{gathered}$$

c^2_{uv}:



$$
\begin{gathered}
x_{uv} = x_{tu} = x_{vt} = 0 \\
x_{vu} = 0.5 \\
x_{ut} = 0.5 \\
x_{vt} = -0.5 \\
y_{vu} = 0.5Q \\
y_{uv} = 0 \\
y_{ut} = 0.25Q - 0.25q_u \\
y_{tu} = -0.25Q + 0.25q_u \\
y_{vt} = -0.25Q - 0.25q_v \\
y_{tv} = 0.25Q + 0.25q_v
\end{gathered}
$$

Take a small enough positive parameter ϵ, then c^3_{uv} is constructed as:



$$
\begin{gathered}
x_{ut} = x_{tu} = x_{vt} = x_{tv} = 0 \\
x_{vu} = 0.25 \\
x_{uv} = 0.25 \\
y_{uv} = 0.25Q + \epsilon \\
y_{vu} = 0.25Q - \epsilon \\
y_{ut} = -\epsilon - 0.25q_u \\
y_{tu} = \epsilon + 0.25q_u \\
y_{vt} = \epsilon - 0.25q_v \\
y_{tv} = -\epsilon + 0.25q_v
\end{gathered}
$$

It's easy to check that $f + \frac{1}{n}c^i_{uv}$ is a solution for arbitrary $u, v \in N$ and $i \in [3]$. Also, we can check $\{f + \frac{1}{n}c^i_{uv}\}_{\substack{u,v\in N \\ i\in[3]}} \cup \{p_u\}_{u\in N}$ are affinely independent. Therefore,

$$\dim P_{LP} = 3\binom{n}{2} + n - 1$$

4.4 Integer Solution Polytope for Unit Demand Case

Now we consider the case that $q_u = 1$ for all $u \in N$, and in this case we can compute the dimension of the convex hull of integer solution P^u_{IP}. Assume $|N| = n$

Observation 11 *Let $|N| \geq Q$. If Q=1, then $\dim P^u_{IP} = n-1$, and if Q=2, then $\dim P^u_{IP} = 2\binom{n}{2} + n - 1$*

Proof :
Recall the constraints (4.13)

$$\sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) = q_u \sum_{v \in \gamma(u)} (x_{vu} + x_{uv}) \quad \forall u \in N$$

I sum them over all $u \in N$, and since the unit case

$$\begin{aligned}\sum_{u \in N} \Big(\sum_{v \in \gamma(u)} (x_{uv} + x_{vu}) \Big) &= \sum_{u \in N} \sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) \\ &= \sum_{u \in N} (y_{su} + y_{tu} - y_{us} - y_{ut}) = Q + \sum_{u \in N} (y_{tu} - y_{ut}) \leq 2Q\end{aligned}$$

Since

$$\sum_{u \in N} x_{su} = \sum_{u \in N} x_{ut} = 1 \text{ and } \forall u \in N,\ x_{us} = x_{tu} = 0$$

If $Q = 1$, then $\sum\limits_{u,v \in N} (x_{uv} + x_{vu}) = 0$, so all integer solutions project onto the arc variables are of the form (s, u, t)
If $Q = 2$, then $2 \sum\limits_{v \in \gamma(u) \setminus \{s,t\}} \sum\limits_{u \in N} x_{uv} = \sum\limits_{u,v \in N} (x_{uv} + x_{vu}) \leq 2$, that means we can at most take one arc with head or tail in N. So, all integer solution w.r.t arcs are either of the form (s, u, t) or (s, u, v, t).
In Observation 9, we know that the way to associate flow is unique, and all solutions are affinely independent. So, we have the dimension.
We have observed the dimension in two simple and specific case, Now, we are going to prove the dimension for $Q \geq 3$.

Theorem 4.4.1 *Let $|N| \geq Q$. If $Q \geq 3$, then* $\dim P^u_{IP} = 3\binom{n}{2} + n - 1$

Proof:
Since $3Q \geq 9 > 5$, $\dim P^u_{IP} \leq \dim P_{LP} = 3\binom{n}{2} + n - 1$, then we will construct affinely independent points. Label the vertices in N such that $N = \{u_1, u_2, \cdots, u_n\}$. Take path (s, u_i, t) and (s, u_i, u_j, t) for all $i, j \in [n]$. And we take paths (s, u_i, u_j, u_k, t) such that $k > j$ and $i \equiv j - 1 \pmod{n}$. We have $\binom{3}{n}2 + n - 1$ many different paths, now we add one more paths $(s, u_{n-1}, u_1, u_n, t)$. Let $\mathcal{P}_3$ be the set of all paths with length 3, and $S := \{i, j) : (s, u_i, u_j, u_k, t) \in \mathcal{P}_3\}$. Indeed $S = \{(n, 1), (n-1, 1)\} \cup \{(1, 2), (2, 3), \cdots, (n-2, n-1)\}$

- Let $z(i)$ be the corresponding solution of the model w.r.t (s, u_i, t)
- Let $z(i, j)$ be the corresponding solution of the model w.r.t (s, u_i, u_j, t)
- Let $z(i, j, k)$ be the corresponding solution of the model w.r.t (s, u_i, u_j, u_k, t)

The following is show our construction gives $3\binom{n}{2}+n$ many affinely independent points. Let $\alpha(i), \alpha(i,j), \alpha(i,j,k)$ be the coefficient of solutions $z(i), z(i,j), z(i,j,k)$ respectively. Consider

$$\sum_{i\in[n]} \alpha(i)z(i) + \sum_{\substack{i,j\in[n]\\ a\neq j}} \alpha(i,j)z(i,j) + \sum_{(s,u_i,u_j,u_k,t)\in\mathcal{P}_3} \alpha(i,j,k)z(i,j,k) = 0$$

$$\text{where} \sum_{i\in[n]} \alpha(i) + \sum_{i,j\in[n]} \alpha(i,j) + \sum_{(s,u_i,u_j,u_k,t)\in\mathcal{P}_3} \alpha(i,j,k)z(i,j,k) = 0$$

Firstly, note that for all (u_i, u_j) sith $i > j$ and $(u_i, u_j) \notin S$, then only in the $z(i,j)$ this coordinate is non-zero. So $\alpha(i,j) = 0$ for all $i > j$ with $(i,j) \notin S$. Then we look at (u_j, u_k) such that $(j,k) \notin S$, then the coordinate corresponding to the arc is non-zero only in $z(j,k)$ and $z(i,j,k)$. Therefore, for these (j,k), $\alpha(j,k) = -\alpha(i,j,k)$. Then we check the flow value on (u_j, u_k), we get $\alpha(j,k) = \alpha(i,j,k) = 0$.
The coefficients may be non-zero yet are $\{\alpha(i)\}_{i\in[n]} \cup \{\alpha(i,j) : (i,j) \in S\} \cup \{\alpha(i,j,k) : (s,u_i,u_j,u_k,t) \in \mathrm{P}_3, j < k, (j,k) \in S\}$. Note the only way that $(i,j) = (n-2.n-1)$ in (s,u_i,u_j,u_k,t) is $(s,u_{n-2},u_{n-1},u_n,t)$. Since $\alpha(n-2,n-1,n) = -\alpha(n-2,n-1) = 0$, therefore by the same discuss $\alpha(n-3,n-2,n-1) = -\alpha(n-3,n-2) = 0$, and then $\alpha(i,j,k)$ where $i = n-3, j = n-2$ are all zero. So $\alpha(n-4,n-3,n-2) = -\alpha(n-4,n-3) = 0$. Repeat this process, then this means we know that the coefficients $\{\alpha(i,j) : (i,j) \in S, 1 < i < j\} \cup \{\alpha(i,j,k) : (s,u_i,u_j,u_k,t) \in \mathrm{P}_3, 1 < j < k, (j,k) \in S\}$ are all zero. Also, we can remove $\{\alpha(i) : i \in [n-2], i \neq 1\}$. Therefore, only $\alpha(1), \alpha(n), \alpha(n-1), \alpha(n-1,1), \alpha(1,n-1), \alpha(1,n), \alpha(n,1), \alpha(n-1,1,n), \alpha(n,1,n-1)$ are not known yet. However, by checking the flow value, we know all them are 0. Therefore, we find $3\binom{n}{2}+n$ many affinely independent points.

4.5 Cuts

We consider the undirected graph $G = (V,E)$ and its induced graph $G = (V, A(E))$ as section 4.3. Let $p = (s, \cdots, u, t)$ be a path. If all vertices of p are distinct, then before the vehicle entering the last vertex u, it should carry at least q_u many flow. However, if we have a cycle, when a vertex $u \in N$ are visited more then once. Here is a freedom to manage how to satisfies the demand of u. For example in the path $p = (s, \cdots, w, u, v, u, t)$, assume u is visited exactly twice. Then I can take

$$\begin{aligned} y_{wu} + y_{vu} - y_{uw} - y_{uv} &= 2q_u \\ y_{vu} + y_{tu} - y_{uv} - y_{ut} &= 0 \end{aligned}$$

However, since the objective function only cares about arc variable, then we can cut these solutions. Note, in this model we have two flows, one is a $st-$ flow and another is the residual $ts-$flow. So, what we discussed can be directly applied to the residual flow. Here is a cut

$$y_{vu} \geq \min_{w \in N} q_w(x_{uv} + x_{vu}) \quad \forall u \in N, v \in \gamma(u) \tag{4.23}$$

These cuts work very well in unit demand case.

Theorem 4.5.1 *Let $|N| \geq Q$. If $q_u = 1$ for all $u \in N$ and $Q \geq 3$, then the inequality in (4.23) define a facet for the polytope of LP-relaxation and convex hull of integer solution P^u_{LP}, P^u_{IP}*

Proof : We firstly show that $y_{vu} = \min_{w \in N} q_w(x_{uv} + x_{vu})$ is independent to the original equation system. From Section 4.3, we know that all equations from (4.22) are independent. And since $u, v \in N$, so here are only one equation in (4.22) is relevant to y_{uv}. That means $y_{vu} = \min_{w \in N} q_w(x_{uv} + x_{vu})$ is independent for any $u, v \in N$.
Then we claim that we can construct $3\binom{n}{2} + n - 1$ many affinely independent points with the cut is tight. Let $N = \{u_i\}_{i=1}^n$ We take paths in the three forms. Based on the symmetry of this polytope, W.L.O.G, let $u = u_l, v = u_r$ such that $l < r - 1$, $l, r \in \{2, 3, \cdots, n-1\}$

- (s, u_i, t) for all $i \in [n]$
- (s, u_i, u_j, t) for all $i, j \in [n]$, but $(i, j) \neq (l, r)$
- $(s, u_{i-1}, u_i, u_j, t)$ for all $i, j \in [n]$ with $i < j$ but $(i, j) \neq (l, r)$. Here the $i - 1$ is also similar to Section 4.4, that is respect to the remainder of modulo n.
- $(s, u_r, u_l, u_{l+1}.t)$. The plus is respect to the remainder of modulo n.

The proof is similar to want we did in Section 4.4, they are affinely independent points. Therefore $y_{vu} \geq \min_{w \in N}(x_{uv} + x_{vu})$ is a facet defining inequality for any $u, v \in N$.
Note that this method can be generalized to the non-unit demand case, but the affinely independence dependent on $\{q_u\}_{u \in N}$.

Theorem 4.5.2 *Let $|N| \geq Q$. If $Q = 2$, then adding these cuts will give a complete description of P^u_{IP}*

Proof :
Since $Q = 2$, By the cuts

$$\begin{aligned} y_{vu} &\geq (x_{uv} + x_{vu}) \quad \forall u, v \in N \\ y_{uv} &\geq (x_{uv} + x_{vu}) \quad \forall u, v \in N \end{aligned}$$

and

$$y_{uv} + y_{vu} = 2(x_{uv} + x_{vu}) \tag{4.24}$$

Therefore, indeed, we get $y_{uv} = y_{vu}$ for all $u, v \in N$. And all possible paths (s, u, t) or (s, u, v, t) are satisfy (4.24).
Let f be a solution satisfies the model include the cuts (4.23). We first look at the arc variables. By flow decomposition theorem, $f = \sum_{i=1}^{m} \mu_i p_i + \sum_{i=1}^{l} \lambda_i c_i$, where p_i are paths with distinct vertices, c_i are cycles and $\sum_{i=1}^{m} \mu_i = 1$. Since the flow variable associated with a path with distinct vertices have unique value. So, it sufficient to show that $\lambda_i = 0$ for all i. Let $f' = \sum_{i=1}^{m} \mu_i p_i$, and $z(f), z(f')$ be the solutions of the model with flow f, f' respectively. Then $z(f) - z(f')$ is a solution of the homogeneous equations. Our cycle can't include vertices s, t, so $x_{su}, x_{us}, x_{tu}, x_{ut} = 0$, therefore $y_{su}, y_{us}, y_{tu}, y_{ut} = 0$. The use (4.13),

$$0 = \sum_{u,v \in N} (y_{vu} - y_{uv}) = (x_{uv} + x_{vu}) \Rightarrow x_{uv} + x_{vu} = 0$$

Therefore $\lambda_i = 0$, that is here is no cycles in f. So every solution is a convex combination of integer solutions, this implies that all vertices are integral.

4.6 Computational Results

All instance are generated by the Delaunay Triangulation on random points on a plane, and they can be found in my github. We will use SCIP to solve these instances.
Because of the observation 10, for all instances, we need to choose the proper capacity Q. otherwise, RCSPP will reduced to the shortest path problem. Therefore, we provide a parameter Q in our instance files, which the sum of all demands divided by 2. Here the parameter coefficient means we set the capacity as $0.27Q$. We begin with simple instances containing totally 10 nodes. And a tabular for the MIP case Therefore, in this experiment,

Table 4.1: The performance of two LP-relaxation model with 10 nodes

	The new model				The classic model			
parameter coefficient=0.27	10.1	10.2	10.3	10.4	10.1	10.2	10.3	10.4
run time (s)	0.0043	0.0044	0.0041	0.0040	0.0048	0.0021	0.0026	0.0029
optimal sol	41.4	127.8	49.2	95.1	41.4	127.8	49.2	95.1

Table 4.2: The performance of two model with 10 nodes

	The new model				The classic model			
parameter coefficient=0.27	10.1	10.2	10.3	10.4	10.1	10.2	10.3	10.4
run time (s)	0.0018	0.0073	0.0049	0.0076	0.0046	0.0014	0.0011	0.0011
optimal sol	41.4	127.8	49.2	114.8	41.4	127.8	49.2	114.8

we can see our model usually return the same optimal solution as the classic model. It is understandable, because the graphs are small and were generated randomly. We can also see since our model has more variables, so it usually takes more time to obtain a solution. Now, we will try lager graphs with nodes between 1000 and 20000.

Table 4.3: The performance of two LP-relaxation model with 1000 nodes with euclidean distance

	The new model				The classic model			
parameter coefficient=0.018	1000.1	1000.2	1000.3	1000.4	1000.1	1000.2	1000.3	1000.4
run time (s)	0.754	0.637		3.52	0.119	0.117		0.207
optimal sol	43.8	37.627		104.71	43.8	37.612		104.44

Table 4.4: The performance of two model with 1000 nodes with euclidean distance

	The new model				The classic model			
parameter coefficient=0.018	1000.1	1000.2	1000.3	1000.4	1000.1	1000.2	1000.3	1000.4
run time (s)	2.67	1.24		8.63	0.68	0.39		0.67
optimal sol	45.5	37.7		112.69	45.5	37.7		112.69

The instance 3 can't be solved because the parameter 0.018 is too small.
We can also change the cost to the square of euclidean distance.

Table 4.5: The performance of two LP-relaxation model with 1000 nodes

	The new model				The classic model			
parameter coefficient=0.018	1000.1	1000.2	1000.3	1000.4	1000.1	1000.2	1000.3	1000.4
run time (s)	1.64	0.96			0.14	0.13		
optimal sol	192.1	151.6			192.1	151.6		

Table 4.6: The performance of two model with 1000 nodes

	The new model				The classic model			
parameter coefficient=0.018	1000.1	1000.2	1000.3	1000.4	1000.1	1000.2	1000.3	1000.4
run time (s)	15.80	8.39			0.57	0.67		
optimal sol	203.5	166.1			203.5	166.1		

Table 4.7: Instance 1000.3

	The new LP-model		The classic LP-model	
	run time (s)	optimal sol	run time (s)	optimal sol
parameter coefficient=0.024	3.54	145.9	0.22	143.6
parameter coefficient=0.023	3.90	157.1	0.22	155.0
parameter coefficient=0.022	3.64	167.4	0.25	166.4
parameter coefficient=0.021	4.24	188.8	0.25	188.8
parameter coefficient=0.020	4.13	222.0	0.25	222.0

Table 4.8: Instance 1000.3

	The new model		The classic model	
	run time (s)	optimal sol	run time (s)	optimal sol
parameter coefficient=0.024	26.94	171.2	1.25	171.2
parameter coefficient=0.023	11.05	171.2	0.71	171.2
parameter coefficient=0.022	9.60	171.3	0.59	171.3
parameter coefficient=0.021	8.64	200.4	0.63	200.4
parameter coefficient=0.020	6.41	236.1	0.61	236.1

Chapter 5

Summary

Summary

In this thesis, we delve into Baldacci's two-commodity flow model for the Vehicle Routing Problem. Deriving implicit equations from the original model and analyzing the redundancy within these equations, leads to an estimation of the upper bound for the dimension of LP-relaxation polytopes. By observing how dimension reduces, we identify some non-facet-defining inequalities. In a uniform demand case, where all consumers share the same demand, we determine the dimension of MIP polytops and identify some facet-defining inequalities.

Following Baldacci's idea, a model for the Resource-Constrained Shortest Path Problem can be proposed. Through a simple example, we show the potential advantages of our model. Using the similar analysis as previous part, along with constructive proofs, we determine the dimensions of LP-relaxation polytopes. Similar methods lead to the dimensions of MIP polytopes and facet-defining inequality in the uniform demand case.

Furthermore, we introduce some cuts for our new model, which can completely describe the MIP polytopes in a simple case. Finally, we compare our model with the traditional model on graphs derived from Delaunay triangulation.

Zusammenfassung

In dieser Arbeit beschäftigen wir uns mit Baldaccis Zwei-Güter-Fluss-Modell für das Vehicle Routing Problem. Durch die Ableitung impliziter Gleichungen aus dem ursprünglichen Modell und die Analyse der Redundanz in diesen Gleichungen gelangen wir zu einer Schätzung der oberen Grenze für die Dimension von LP-Entspannungspolytopen. Durch die Beobachtung, wie sich die Dimension verringert, identifizieren wir einige nicht-facettendefinierende Ungleichungen. Im Fall einer einheitlichen Nachfrage, bei der alle Verbraucher dieselbe Nachfrage haben, bestimmen wir die Dimension von MIP-Polytopen und identifizieren einige facettendefinierende Ungleichungen.

In Anlehnung an Baldaccis Idee kann ein Modell für das Resource-Constrained Shortest Path Problem vorgeschlagen werden. Anhand eines einfachen Beispiels zeigen wir die potenziellen Vorteile unseres Modells. Unter Verwendung einer ähnlichen Analyse wie im vorherigen Teil, zusammen mit konstruktiven Beweisen, bestimmen wir die Dimensionen von LP-Entspannungspolytopen. Ähnliche Methoden führen zu den Dimensionen von MIP-Polytopen und facettendefinierenden Ungleichungen im Fall einer einheitlichen Nachfrage.

Darüber hinaus führen wir einige Schnitte für unser neues Modell ein, die die MIP-Polytope in einem einfachen Fall vollständig beschreiben können. Abschließend vergleichen wir unser Modell mit dem traditionellen Modell auf Graphen, die aus Delaunay-Triangulationen abgeleitet wurden.

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