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FACHBEREICH MATHEMATIK UND INFORMATIK

MASTERARBEIT

POLYHEDRAL STUDY ON BALDACCI'S MODEL FOR THE CAPACITATED VEHICLE ROUTING PROBLEM AND ITS EXTENSION TO THE RESOURCE CONSTRAINED SHORTEST PATH PROBLEM

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Chapter 1

Introduction

The Resource Constrained Shortest Path Problem (RCSPP) is a fundamental problem within the domain of discrete optimization, targeting the identification of a cost-optimized path subject to resource constraints. Within the transportation backdrop, a truck driver scheduling his delivery route out of a factory serves as an example. He will always unpack his vehicle at a market with precisely the items that are needed. The route's cost involves various factors, including fuel expenses, time penalties, and some potential costs. A typical example of resource constraints is the capacity of the truck's fuel tank.

Although the description looks very simple, RCSPP is classified as an $\mathcal{NP}$-hard problem by Hartmanis (1982), even under the assumption of non-negative costs. Nowadays, no polynomial-time algorithms have been found for solving RCSPP. This fact, RCSPP is a challenging problem makes it worth investigating into. Furthermore, RCSPP's connections with the real world, including transportation and management, highlight its importance.

A natural idea to construct linear programming for RCSPP is adding additional resource constraints to the linear programming model for the shortest path problem. Much work has been done on developing algorithms for solving this simple model. Aneja et al. (1983) proposed a network reduction algorithm, which deletes the nodes or arcs that can't appear in feasible solutions and remove them. This algorithm also records the current cost-least solution and keeps updating. Later, Beasley and Christofides (1989) utilized the same approach as Aneja to delete and remove nodes and arcs, but they additionally considered the reduced cost from the dual Lagrangian problem. A common technique for solving RCSPP is the Lagrangian relaxation. Mehlhorn and Ziegelmann (2000) solve the dual problem with the ellipsoid method. Through the ellipsoid method, the complexity of solving RCSPP with a single resource constraint could be determined. Besides these methods, the Path ranking based approach, dynamic programming, and Branch&Backtracking algorithm are considered to close the gap between the prime problem and the dual problem. The path ranking based approach is ranking feasible paths with respect to cost, which was first used by Handler and Zang (1980). They introduce a $k-$path algorithm, rank paths with respect to the reduced cost, derived from the dual Lagrangian problem. Santos et al. (2007) and Pugliese and Guerriero (2013) show the advantages of using convexity or Chebyshev-like norm in the path ranking approach. There are two primary techniques for dynamic programming strategies: the label establishing algorithm and the label correcting algorithm. Mehlhorn and Ziegelmann (2000) propose both these two algorithms. Dumitrescu and Boland (2003) offer an improved version of the label-setting algorithm. The earliest work that used the Branching and Backtracking algorithm was produced by Beasley and Christofides (1989). And a recent work with the Branching&Backtracking scheme was done by Muhandiramge and Boland (2009). The tableau 1.1 shows our literature research.

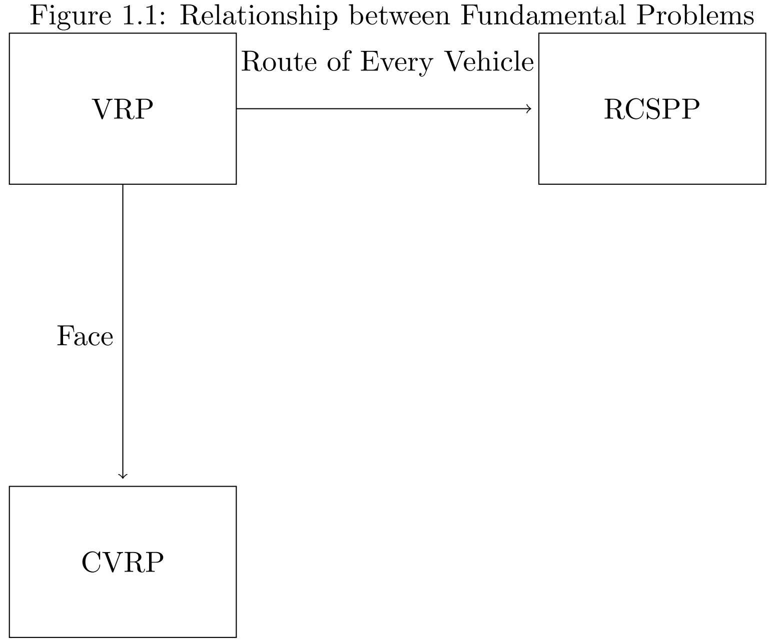
In discrete optimization, RCSPP is a sub-problem of the Vehicle Routing Problem (VRP). VRP seeks to establish a cost-efficient schedule for M vehicles tasked with delivering goods from a factory to various markets, ensuring that all market demands are met. Without lost of generality, assuming market demands are positive real numbers, demands are met implies every market should be visited by at least one vehicle. These conditions are called degree constraints. A notable variant of VRP is the Capacitated Vehicle Routing Problem (CVRP), which imposes each market must be visited exactly once. Consequently, the solution space of CVRP, which is usually a polytope, is a face of the polytope from VRP, since the degree constraints are tight.

To bridge RCSPP and VRP, one can introduce an artificial point to

	Network Reduction	Dual Problem	Path Ranking	Dynamic Programming	B&B scheme
Handler and Zang [1980]		×	×		
Aneja et al. [1983]	×			×	
Beasley and Christofides [1989]	×	×			×
Mehlhorn and Ziegelmann [2000]	×	×		×	
Dumitrescu and Boland [2003]	×	×		×	
Santros et al [2007]			×		
DPuglia Pugliese and Guerriero [2013]			×		
Muhandiramge and Boland [2009]	×	×			×

Table 1.1: Research on RCSPP Algorithms

Figure 1.1: Relationship between Fundamental Problems



transform a vehicle's route into a path. This is achieved by substituting the vehicle's return point with the artificial point, thereby converting the circuitous route into a path. Now every route in VRP is a valid path in RCSPP. Figure 1.1 illustrates the connection between these three problems.

Both VRP and CVRP are important in the field of discrete optimization, however, CVRP receiving more active research interest. The tight degree constraints attract more attention in real-world scenarios, because no drivers would like to back a visited market again. Over the years, researchers have developed various models for solving CVRP, reflecting its significance.

The VRP was firstly introduced by Dantzig and Ramser (1959) when they were doing a project. In five years, Balinski and Quandt (1964) introduced a set partitioning model for CVRP, an approach that treats all potential routes as variables and the degree constraints ensuring that each node is included precisely once in a route. Let R be the set of all possible routes, d_r be the cost of route r and $a_{i,r}$ be a binary variable indicate that if the consumer i is visited in route r, the mathematical formula is

$$
\begin{aligned}
&\min \sum_{r \in R} d_r y_r \\
&\text{subject to:} \\
&\sum_{\text{consumer } i} a_{i,r} = 1 \\
&\sum_{r \in R} y_r = m \\
&a_{i,r}, y_r \in \{0, 1\}
\end{aligned}
$$

Subsequently, the column generation algorithm and dynamic programming were introduced to solve Balinski and Quand's model. Notably, only Agarwal et al. (1989) managed to effectively solve instances with to 15-25 consumers using the column generation approach. A significant advancement came in 2008 when Baldacci et al. (2008) introduced additional inequalities and equalities to the classical set partitioning model. This enhancement expanded the model's solvability, allowing it to solve instances with 37 to 127 vertices.

Shifting perspective from treating potential routes as variables to considering edges as variables introduces an innovative approach to solving CVRP. Laporte and Nobert (1983) proposed a vehicle flow model to model CVRP as

$$
\begin{aligned}
&\sum_{\{i,j\}\in E} c_{ij}x_{ij} \\
&\text{subject to:} \\
&\sum_{j=1}^{n} x_{\{0,j\}} = 2M \\
&\sum_{i\neq j} x_{\{i,j\}} = 2 \quad \forall i \in [n] \\
&\sum_{i\in S}\sum_{j\in S^c} x_{\{i,j\}} \geq 2m(S) \quad \forall S \subseteq [n] \\
&x_{\{0,j\}} \in \{0,1,2\} \quad \forall j \in [n] \\
&x_{\{i,j\}} \in \{0,1\} \quad \forall i,j \in [n], i \neq j
\end{aligned}
$$

In this context, $m(S)$ is the minimum number of vehicles required to satisfy the demands of all consumers within the subset S. It is a typical integer programming model, for which the Branch and Cut algorithm is a common choice. However, it should be pointed out that computing $m(S)$ is expensive, indeed it is $\mathcal{NP}-$hard.

Gavish and Graves (1978) extended the idea in Garvin et al. (1957) for the oil delivery problem, and they proposed a commodity flow model for CVRP. In this flow based model, each encounter with a consumer reduces the flow value by the consumer's demand. Maintaining the non-negativity of flow values implies resource constraints. In a significant advancement of this methodology, Baldacci et al. (2004) introduced his two-commodity flow model.

Baldacci's model provides a significant forward for solving CVRP, particularly due to the number of variables or constraints are under control. Unlike the vehicle flow model and the set partition model, in which the number of variables or constraints increase exponentially with respect to the size of the problem, there are polynomial increases in Baldacci's framework. This distinction makes Baldacci's model more solvable in practice. As Baldacci represented, in various benchmark tests, his model has outperformed its predecessors. Moreover, a significant advance is that his model is able to solve an instance with 104 vertices, which is the largest instance among all earlier proposed models.

Our contribution includes a new mixed integer programming model for RCSPP, as well as the polyhedral study on Baldacci's model and this new model. For more detail, see Section 6.

Chapter 2

Notations and Preliminaries

The notations that Professor Ralf Borndörffer used in his slides are the same ones I am using. I also include a list of some uncommon notations here. Polytopes, integer programming, and linear programming novices may all readily understand this thesis. But first, I would like to review some fundamental polytope knowledge.

2.1 Notations

Let ij be an abbreviation of the tuple (i, j) if the context is clear, and for an undirected graph $G = (V, E)$, define the cut set $\gamma(u) := \{v : \{u, v\} \in E\}$, and for a directed graph $D = (V, A)$, define the cut sets $\gamma^+(u) := \{v : (u, v) \in A\}$ and $\gamma^-(u) := \{v : (v, u) \in A\}$. In simple graph, we use a sequence of vertices $(u_1, u_2, ..., u_n)$ to represent a path.

2.2 Preliminaries for Polytopes

We recall some definitions and theorems from Ziegler (2012).

Definition 2.2.1 (Convex Set)**.** *Consider a set C in $\mathbb{R}^d$. If for any $x, y \in C$, $\{\lambda x + (1-\lambda)y : \lambda \in [0, 1]\} := [x, y] \subseteq C$, then we say C is a convex set.*

Property 2.2.2. *Let $\{C_i\}_{i\in I}$ be a family of convex set, then $C = \bigcap\limits_{i\in I} C_i$ is convex.*

It's a similar property as closed sets, indeed, the convex hull could be defined as the closure.

Definition 2.2.3 (Convex Hull). *For any set $X \subseteq \mathbb{R}^d$, then the convex hull of X is the minimal convex set containing X, with respect to inclusion. This is exactly* $\operatorname{conv}(X) := \bigcap\limits_{\substack{X \subseteq C \\ C \text{ convex}}} C$. *In particular, if C is convex, then* $\operatorname{conv}(C) = C$.

Moreover, here is an equivalent description of convex hulls.

Theorem 2.2.4. *Let $X \subseteq \mathbb{R}^d$, then* $\operatorname{conv}(X) = \{\sum_{i=1}^n \lambda_i x_i : \sum_{i=1}^n \lambda_i = 1, n \in \mathbb{N},\ \forall i \in [n], \lambda_i \geq 0, x_i \in X\}$

Now that we have covered some of the fundamentals of convexity, we can discuss polytopes.

Definition 2.2.5 (V-polytope). *Let $\{x_i\}_{i=1}^n \subseteq \mathbb{R}^d$ be a set containing finite points. The $P = \operatorname{conv}(x_i : i \in [n])$ is a V-polytope, meanwhile* $\operatorname{conv}(x_i : i \in [n])$ *is a V-description of P.*

Definition 2.2.6 (H-polyhedron:). *Let $H^{\leq}(a_i, \alpha_i) := \{x : \mathbb{R}^d : \langle a_i, x \rangle \leq \alpha_i\}$ be a half space. If $P = \bigcap\limits_{i \in [n]} H^{\leq}(a_i, \alpha_i)$, then P is a H-polyhedra. Meanwhile $\bigcap\limits_{i \in [n]} H^{\leq}(a_i, \alpha_i)$ is an H-description of P and this description could be abbreviated as $\{x : Ax \leq b, A \in \mathbb{R}^{n \times d}\}$ such that the i^{th} row of A is a_i, and the coordinate i of b is α_i. Moreover, if P is bounded, then P is an H-polytope.*

Fortunately, these two definitions of polytopes are equivalent.

Theorem 2.2.7 (Main Theorem of Polytopes). *Every V-polytope has an H-description, and every H-polytope has a V-description.*

Definition 2.2.8 (Affine subspace). *For any set $X \subseteq \mathbb{R}^d$, then the affine subspace* $\operatorname{aff}(X) := \{\sum_{i=1}^n \lambda_i x_i : \sum_{i=1}^n \lambda_i = 1, n \in \mathbb{N},\ \forall i \in [n], x_i \in X\}$ *generated by X*

Definition 2.2.9 (Dimension of polyhedra P). *The dimension of a polyhedra P is defined as the dimension of aff(P) (ie.* $\dim P = \dim \operatorname{aff}(P)$*)*

Definition 2.2.10 (Faces and Facets). *Consider an $H-$polyhedra $P = \{x : Ax \leq b\}$. And a half space $H^{\leq}(c, \alpha)$, if $P \subseteq H^{\leq}(c, \alpha)$, and $F := P \cap \{x : c^{\mathrm{T}}x = \alpha\} \neq \emptyset$, then F is a face of P. If $\dim F = \dim P - 1$, then F is a facet of P.*

Definition 2.2.11 (Valid Equations and Irredundant Equations). *Given a polyhedra P, if $c^{\mathrm{T}}x = \alpha$ for all $x \in P$, then we call $c^{\mathrm{T}}x = \alpha$ is a valid equation for P. Given a set of equations, an equation is called irredundant if this equation can't be represented as a linear combination of other equations.*

Theorem 2.2.12. *Let $\mathcal{A}$ be the maximal set of valid irredundant equations of $P \subseteq \mathbb{R}^n$, where the maximal respects to inclusion. Then,* $\dim P = n - |\mathcal{A}|$

Definition 2.2.13 (Linear Programming and Mixed Integer Programming). *The linear programming (LP) is solving the following problem*

$$\begin{aligned} &\max c^{\mathrm{T}}x \\ &s.t.\ \ Ax \leq b \end{aligned}$$

And the mixed integer programming (MIP) is solving the

$$\begin{aligned} &\max \begin{pmatrix} c \\ d \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} x \\ y \end{pmatrix} \\ &s.t.\ \ \begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \\ &x \in \mathbb{R}^n, y \in \mathbb{Z}^n \end{aligned}$$

Definition 2.2.14 (Polyhedron from Linear Programmings and Mixed Integer Programmings). *Given a Linear Programming of above form, the feasible solution $\{x : Ax \leq b\}$ is a polyhedra.*
Given a Mixed Integer Programming of above form, consider the polyhedra
$\mathrm{conv}\{\begin{pmatrix} x \\ y \end{pmatrix} : \begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, x \in \mathbb{R}^n, y \in \mathbb{Z}^n\}$

Chapter 3

Capacitated Vehicle Routing Problem

In this section, we introduce Baldacci's approach to the capacitated vehicle routing problem (CVRP) and our original study on his model. In subsection 3.1, we briefly describe CVRP. In subsection 3.2, we introduce the MIP suggested by Baldacci et al. (2004). In subsection 3.3, we discuss some polyhedral results on Bladacci's model. Finally, in subsection 3.4, we concentrate on a special case, where the demands are the same.

3.1 Problem Description

Definition 3.1.1 (Capacitated Vehicle Routing Problem (CVRP))**.** ***Given:*** *Undirected simple graph $G = (V, E)$, two special vertices, the sink $s \in V$ and the target $t \in V$, non-negative integer edge costs $c \in \mathbb{N}^E_{\geq 0}$, natural numbers $M, Q \in \mathbb{N}$ and node demands $q \in \mathbb{N}^N_{\geq 0}$, where $N = V \setminus \{s, t\}$. An $M-$route f consists of M simple st-paths $\{p_i\}_{i=1}^M$ in G, its cost and demand are $c(f)$ and $q(f)$, respectively, and it is feasible if $q(p_i) \leq Q$ holds for all i. An $M-$schedule is a feasible $M-$route that visits every nodes except s, t exactly once.* ***Sought:*** *A $M-$schedule of minimal total cost.*

3.2 Baldacci's Model

Given an undirected graph $G = (V, E)$, where $E = \{\{u, v\} : u, v \in N\} \cup \{\{s, u\} : u \in N\} \cup \{\{v, t\} : v \in N\}$. This graph could be transferred to a directed graph $D(G) = (V, A(E))$ such that $A(E) = \{uv : \{u, v\} \in E\} \cup \{vu : \{u, v\} \in E\}$. For each edge $\{u, v\} \in E$, assign an edge variable $\xi_{\{u,v\}}$, and for each arc $uv \in A(E)$, assign a flow variable x_{uv}. Then the Baldacci's model is the following

$$\min \sum_{\{u,v\}\in E} c_{\{u,v\}} \xi_{\{u,v\}}$$

subject to:

$$\sum_{v\in\gamma(u)} (x_{vu} - x_{uv}) = 2q_u \quad \forall u \in N \tag{3.1}$$

$$\sum_{v\in N} x_{sv} = q(N) \tag{3.2}$$

$$\sum_{v\in N} x_{vs} = MQ - q(N) \tag{3.3}$$

$$\sum_{v\in N} x_{tv} = MQ \tag{3.4}$$

$$x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \quad \forall \{u, v\} \in E \tag{3.5}$$

$$\sum_{v\in\gamma(u)} \xi_{\{u,v\}} = 2 \quad \forall u \in N \tag{3.6}$$

$$x_{uv} \geq 0,\ x_{vu} \geq 0,\ \xi_{\{u,v\}} \in \{0, 1\} \quad \forall\ \{u, v\} \in E$$

In Baldacci's model, (3.6) are for the degree constraints. (3.1) are for satisfying the demands of all consumers. In the solution, here is a flow and a residual flow represent the load and empty space of vehicles respectively. (3.5) implies the capacity of each vehicle and connects variables $\xi_{\{u,v\}}$, x_{uv} and x_{vu}. The remaining equation constraints are for the initial condition.

3.3 Polyhedral Study of Baldacci's Model

3.3.1 Complete the equation system

Let B_{IP} be the polytope behind Baldacci's model and B_{LP} be the polytope behind Baldacci's model after relaxing variable $\xi_{\{u,v\}} \in \{0,1\}$ to $\xi_{\{u,v\}} \in [0,1]$ for all $\{u,v\} \in E$. It's clear that $B_{IP} \subseteq B_{LP}$. Our first step is finding some implicit valid equations for B_{LP} to approach its dimension.

Observation 1. *$x_{ut} = 0$ for all $u \in N$*

Proof:

$$\begin{aligned}
0 &= \sum_{uv \in A(E)} (x_{uv} - x_{uv}) = \sum_{u \in V} \sum_{v \in \gamma(u)} (x_{uv} - x_{vu}) \\
&= \sum_{v \in N} (x_{sv} - x_{vs}) + \sum_{v \in N} (x_{tv} - x_{vt}) + \sum_{u \in N} \sum_{v \in \gamma(u)} (x_{uv} - x_{vu}) \\
&= \sum_{u \in N} q_u - (MQ - \sum_{u \in N} q_u) + MQ - \sum_{u \in N} x_{ut} + \sum_{u \in N} (-2q_u) \\
&= -\sum_{u \in N} x_{ut}
\end{aligned}$$

Since $x_{ut} \geq 0$ for all $u \in N$, therefore $x_{ut} = 0$ for all $u \in N$. Now we do some operations on the valid equations to draw equations that are independent to others. Plug (3.5) into (3.6), yields

$$\sum_{v \in \gamma(u)} x_{uv} + x_{vu} = 2Q$$

Combined with (3.1),

$$\begin{cases} \displaystyle\sum_{v \in \gamma(u)} x_{vu} - x_{uv} = 2q_u \\ \displaystyle\sum_{v \in \gamma(u)} x_{vu} + x_{uv} = 2Q \end{cases} \Rightarrow \begin{cases} \displaystyle\sum_{v \in \gamma(u)} x_{vu} = Q + q_i \\ \displaystyle\sum_{v \in \gamma(u)} x_{uv} = Q - q_i \end{cases}$$

Then here is an equivalent model to Baldacci's original model

$$\min \sum_{\{u,v\}\in E} c_{\{u,v\}}\xi_{\{u,v\}}$$

subject to:

$$\sum_{v\in\gamma(u)} x_{vu} = Q + q_u \quad \forall u \in N \tag{3.7}$$

$$\sum_{v\in\gamma(u)} x_{uv} = Q - q_u \quad \forall u \in N \tag{3.8}$$

$$\sum_{u\in N} x_{su} = \sum_{u\in N} q_u \tag{3.9}$$

$$\sum_{u\in N} x_{us} = MQ - \sum_{u\in N} q_u \tag{3.10}$$

$$\sum_{u\in N} x_{tu} = MQ \tag{3.11}$$

$$x_{ut} = 0 \quad \forall u \in N \tag{3.12}$$

$$x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \quad \forall \{u,v\} \in E \tag{3.13}$$

$$x_{uv} \geq 0,\ x_{vu} \geq 0,\ \xi_{\{u,v\}} \in \{0,1\} \quad \forall\ \{u,v\} \in E$$

In this model, the variables ξ_{ij} only appears exactly once in (3.13), which implies the equations (3.13) are independent to equations from (3.7)-(3.12).

Recall that

$$\begin{aligned} 0 &= \sum_{uv\in A(E)} (x_{uv} - x_{uv}) \\ &= \sum_{v\in N}(x_{sv} - x_{vs}) + \sum_{v\in N}(x_{tv} - x_{vt}) + \sum_{u\in N}\sum_{v\in\gamma(u)} (x_{uv} - x_{vu}) \end{aligned}$$

So we conclude the equation system from (3.7)-(3.12) is redundant. Our next goal is finding how many redundant equations are in (3.7)-(3.12).

3.3.2 A method for studying the irredundancy

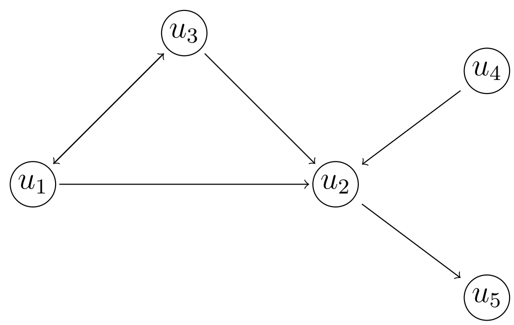
Given a directed connected graph $D_1(G) = (U, \bar{A})$. For every arc (u, v) in $\bar{A}$, assign a variable x_{uv}. Since $D_1(G)$ is connected, at least one of

$\gamma^+(u)$, $\gamma^-(u)$ is non-empty for an arbitrary $u \in U$. Then for every non-empty set $\gamma^+(u)$ and $\gamma^-(u)$, we construct the following equations with proper values $\beta_u, \bar{\beta}_u$ to obtain a equation system

$$\sum_{v \in \gamma^+(u)} x_{uv} = \beta_u \quad \text{or} \quad \sum_{v \in \gamma^-(u)} x_{vu} = \bar{\beta}_u \tag{3.14}$$

Here is an example shows how we construct the equation system.

Example 3.3.1. *Let the directed graph be*



From this graph we have

$$\begin{aligned} &\gamma^+(u_1) = \{u_2, u_3\} && \gamma^-(u_1) = \{u_3\} \\ &\gamma^+(u_2) = \{u_5\} && \gamma^-(u_2) = \{u_1, u_3, u_4\} \\ &\gamma^+(u_3) = \{u_1, u_2\} && \gamma^-(u_3) = \{u_1\} \\ &\gamma^+(u_4) = \{u_2\} && \gamma^-(u_4) = \emptyset \\ &\gamma^+(u_5) = \emptyset && \gamma^-(u_5) = \{u_2\} \end{aligned}$$

Therefore this graph will give the following equation system with proper choices of β and $\bar{\beta}$

$$\begin{aligned} &x_{u_1,u_2} + x_{u_1,u_3} = \beta_{u_1} \\ &x_{u_3,u_1} = \bar{\beta}_{u_1} \\ &x_{u_2,u_5} = \beta_{u_2} \\ &x_{u_1,u_2} + x_{u_3,u_2} + x_{u_4,u_2} = \bar{\beta}_{u_2} \\ &x_{u_3,u_1} + x_{u_3,u_2} = \beta_{u_3} \\ &x_{u_1,u_3} = \bar{\beta}_{u_3} \\ &x_{u_4,u_2} = \beta_{u_4} \\ &x_{u_2,u_5} = \bar{\beta}_{u_5} \end{aligned}$$

Now write the whole equation system as a matrix form $Cx = d$. Here is a bijection such that we can find the corresponding node and the arc direction for every row of C. Therefore, the rows in C can be labeled as $c_{\gamma^+(u)}$ and $c_{\gamma^-(u)}$ with respect to $\gamma^+(u)$ and $\gamma^-(u)$ for some u. Consider a vector a such that $\boldsymbol{a}^{\mathrm{T}}C = \mathbf{0}^{\mathrm{T}}$, the $\dim\{\boldsymbol{a} : \boldsymbol{a}^{\mathrm{T}}C = \mathbf{0}^{\mathrm{T}}\}$ will tell us $\mathrm{rank}(C)$. Let $\alpha^+(u)$, $\alpha^-(u)$ be entries of $\boldsymbol{a}$, such that

$$\boldsymbol{a}^{\mathrm{T}}C = \sum_{\substack{u\in V\\ \gamma^+(u)\neq\emptyset}} \alpha^+(u)c_{\gamma^+(u)} + \sum_{\substack{u\in V\\ \gamma^-(u)\neq\emptyset}} \alpha^-(u)c_{\gamma^-(u)}$$

At the beginning, here are two observations.

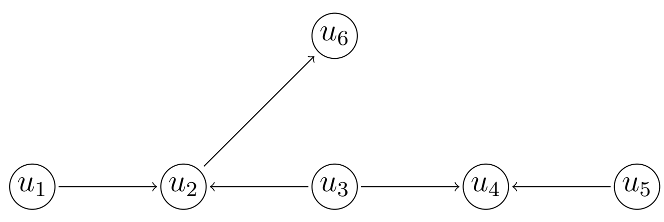
Observation 2. *Only in $c_{\gamma^+(u)}$ and $c_{\gamma^-(v)}$, the coefficient of variable x_{uv} in that row is non-zero.*

Observation 3. *An arc $(u, v) \in D_1(G)$ implies $\alpha^+(u) = -\alpha^-(v)$, because this is the only way to eliminate variable x_{uv}*

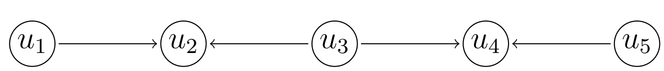
Because of the connectedness, we could extend our observations above to all vertices in $G(U, \bar{A})$. To do this, we need a special "path" to transfer this property from a vertex to another vertex.

Definition 3.3.1 (Alternating Path)**.** *Let $(u_1, \cdots, u_n)$ be a sequence of distinct vertices in $G(U, \bar{A})$. If here is a sequence of arcs $(a_1, \cdots, a_{n-1})$, such that for all $k \in [n-1]$, $a_{2k+1} = (u_{2k+1}, u_{2k+2})$, $a_{2k} = (u_{2k+1}, u_{2k})$ or $a_{2k+1} = (u_{2k+2}, u_{2k+1})$, $a_{2i} = (u_{2k}, u_{2k})$, then we call this sequence $(u_1, a_1, u_2, \cdots, a_{n-1}, u_n)$ an alternating path, and the number of arcs contained in an alternating path is the length of this alternating path.*

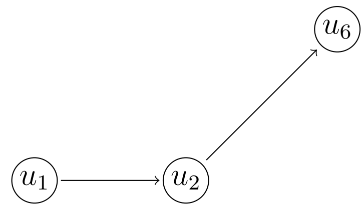
Example 3.3.2. *An example to illustrate the alternating path.*



The sequence $(u_1, e_{u_1,u_2}, u_2, e_{u_3,u_2}, u_3, e_{u_3,u_4}, u_4, e_{u_5,u_4}, u_5)$ is an alternating path.



And the sequence $(u_1, e_{u_1,u_2}, u_2, e_{u_2,u_6}, u_6)$ *is not an alternating path.*



Lemma 1. *If here is an alternating path* $(u_1, a_1, ..., a_{n-1}, u_n) = P$ *in the graph* $G(U, \bar{A})$*. Then for all* $k \in [n]$*, we have*

- *when* $a_1 = (u_1, u_2)$*:*

$$\alpha^+(u_1) = (-1)^{k-1}\alpha^{sign\left((-1)^{k-1}\right)}(u_k)$$

- *when* $a_1 = (u_2, u_1)$*:*

$$\alpha^-(u_1) = (-1)^{k-1}\alpha^{sign\left((-1)^{k-1}\right)}(u_k)$$

Proof: Induction on the length of alternating path.
For these alternating paths have length 1, it's just the observation 3.
Suppose the statement is true for all alternating paths with length $k-1$. Assume $a_1 = (u_1, u_2)$, then $a_k = (a_k, a_{k+1})$ if k is odd and $a_k = (a_{k+1}, a_k)$ if k is even. In the first case ($k \equiv 1 \pmod 2$), we have

$$\begin{aligned}\alpha^+(u_1) &= \alpha^+(u_k)\\ \alpha^+(u_k) &= -\alpha^-(u_{k+1})\end{aligned}$$

Therefore $\alpha^+(u_1) = \alpha^-(u_{k+1})$. In other cases, the proofs are similar.

Back in Baldacci's model, the directed graph is derived from an undirected graph by replacing edges with arcs in opposite directions. Therefore, here are two "parallel" alternating paths providing two series of equations. In this case, we have a stronger statement.

Theorem 3.3.2. *Let* $P = (u_1, a_1, \cdots, a_{2k}, u_{2k+1}) \subseteq G(U, \bar{A})$ *be an alternating path with length* $2k$*. If* $(-P) := (u_{2k+1}, -a_{2k}, u_{2k}, \cdots, -a_1, u_1)$ *exists in* $G(U, \bar{A})$*, and here is an arc* $(u_1, u_{2k+1}) \in \bar{A}$*. Then,*

$$\forall u \in V(P),\ \alpha^+(u) = -\alpha^-(u) \text{ and } \forall u, v \in V(P),\ |a^+(u)| = |a^+(v)|$$

Moreover, let N be the set of vertices are in P or adjacent to P. If $\boldsymbol{a}^{\mathrm{T}}C = \mathbf{0}^{\mathrm{T}}$, then $|\alpha^{\mathrm{sign}_1}(u)| = |\alpha^{\mathrm{sign}_2}(v)|$ for all $u, v \in N$, where $\mathrm{sign}_1, \mathrm{sign}_2 \in \{-, +\}$

Proof:
Since $k \geq 1$, then here exist a node u_l in path P, such that both $l-1$, $2k+1-l$ are odd. Therefore, from the Lemma 1, we have

$$\alpha^+(u_1) = -\alpha^-(u_l) \quad \alpha^-(u_1) = -\alpha^+(u_l)$$
$$\alpha^+(u_{2k+1}) = -\alpha^-(u_l) \quad \alpha^-(u_{2k+1}) = -\alpha^+(u_l)$$

Since here is an arc (u_1, u_{2k+1}), so $\alpha^+(u_1) = -\alpha^-(u_{2k+1})$. This means $\alpha^+(u_l) = -\alpha^-(u_l)$. Proof finished.

In this section, given a directed graph, for any vertices in this graph, we associate it with equations of the form (3.14) based on the structure of this graph. Then we can just read the graph and use the Theorem 3.3.2 to know if there exist redundant equations.

3.3.3 Dimension and Facets Results for B_{LP}

We return to Baldacci's model. In this model, every vertices has two equations of the form (3.14), and the graph has many edges, so we can always find a large even-length alternating path P with vertex ordered as $(u_1, \cdots, u_{2k+1})$, such that u_1, u_{2k+1} are adjacent, $-P$ exists in the induced graph $D(G) = (V, A(E))$ and all vertices in V are either contained in P or are adjacent to some vertices in P. So, exactly one equation in the equation system (3.7) – (3.12) is redundant. Assume $|N| = n \geq 3$, then

$$\dim B_{LP} \leq 3(\binom{n+2}{2} - 1) - (\binom{n+2}{2} - 1) - (3n+3) + 1 = n^2 - 2$$

This is an upper bound of dimension. The next part of this section is identifying some non-facet-defining inequalities by observing how dimension reduces when the inequality is active.

Observation 4. *$\xi_{\{u,v\}} \geq 0$, where $u, v \in N$, isn't a facet defining inequality when $|N| \geq 5$.*

proof:

$$\begin{cases} \xi_{\{u,v\}} = 0 \\ x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \\ x_{uv}, x_{vu} \geq 0 \end{cases} \Rightarrow \begin{cases} x_{uv} = 0 \\ x_{vu} = 0 \\ \xi_{\{u,v\}} = 0 \end{cases}$$

It's sufficient to check that $x_{uv} = 0$ isn't a linear combination of equations in (3.7)-(3.13) and $\xi_{\{u,v\}} = 0$. Applying then Theorem 3.3.2 for a subgraph induced by $V \setminus \{u, v\}$, and then check the whole system.

Observation 5. *If $M = 1$, then $\xi_{\{s,u\}} \leq 1$, $\xi_{\{u,t\}} \leq 1$ is not a facet defining inequality.*

Proof:
Note we have

$$\sum_{u \in N} x_{tu} = Q$$
$$x_{ut} = 0 \quad \forall u \in N$$
$$x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \quad \forall \{u,v\} \in E$$

Therefore, we have

$$\sum_{u \in N} Q\xi_{\{t,u\}} = \sum_{u \in N} (x_{tu} + x_{ut}) = \sum_{u \in N} x_{tu} = Q$$
$$\Rightarrow \sum_{u \in N} \xi_{\{t,u\}} = 1$$

If for one u_1 such that $\xi_{\{u_0,t\}} = 1$, then it implies that $\xi_{\{u,t\}} = 0$ for all $u \neq u_0$. Similarly, from

$$\sum_{u \in N} x_{us} = Q - \sum_{u \in N} q_u$$
$$\sum_{u \in N} x_{su} = \sum_{u \in N} q_u$$

We get

$$\sum_{u \in N} \xi_{\{s,u\}} = 1$$

This implies, for an arbitrary $u \in N$, if $\xi_{\{s,u\}} = 1$, then $\xi_{s\{s,v\}} = 0 \quad \forall v \in N, v \neq u$

3.4 Dimension and Facets Result for Unit Demand Case

If $q_u = 1$ for all $u \in V$, we call this case the "unit demand case". In this case, all vertices in N are "equivalent", and exchanging any two vertices preserves the feasibility. Recall Baldacci's model under this special case is

$$\min \sum_{\{u,v\}\in E} c_{\{u,v\}}\xi_{\{u,v\}}$$

subject to:

$$\sum_{v\in\gamma(u)} x_{vu} = Q + q_u \quad \forall u \in N \tag{3.15}$$

$$\sum_{v\in\gamma(u)} x_{uv} = Q - q_u \quad \forall u \in N \tag{3.16}$$

$$\sum_{u\in N} x_{su} = \sum_{u\in N} q_u \tag{3.17}$$

$$\sum_{u\in N} x_{us} = MQ - \sum_{u\in N} q_u \tag{3.18}$$

$$\sum_{u\in N} x_{tu} = MQ \tag{3.19}$$

$$x_{ut} = 0 \quad \forall j \in N \tag{3.20}$$

$$x_{uv} + x_{vu} = Q\xi_{\{u,v\}} \quad \forall \{u,v\} \in E \tag{3.21}$$

$$x_{uv} \geq 0,\ x_{vu} \geq 0,\ \xi_{\{u,v\}} \in \{0,1\} \quad \forall\ \{u,v\} \in E$$

In the previous section, we have obtained an upper bound of the dimension of B_{LP}. In this section, we will show in the unit demand case, we can determine the dimension of B_{LP} and B_{IP}. Precisely, that is the following theorem

Theorem 3.4.1. *Under the unit demand case, If $Q \geq 5$, $M \geq 2$ and $MQ > n = |N| > (M-1) + Q$, the polytopes from Baldacci's model are denoted by B_{IP}^{u+}, B_{LP}^{u+} respectively. A result is B_{IP}^{u+}, B_{LP}^{u+} are $(n^2-2)-$dimensional polytopes.*

It's sufficient to show $n^2 - 2 = \dim B_{IP}$. We will use a similar idea as Campos et al. (1991) by showing every valid equation for B_{IP} has to be a linear combination of equations in (3.15)-(3.21).

Let $\boldsymbol{c}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{\xi}\end{pmatrix}=\beta$ be an arbitrary valid equation for B_{IP}. Let $T\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{\xi}\end{pmatrix}=\boldsymbol{t}$ be the matrix of equations in (3.15)-(3.21). Since $\boldsymbol{c}^{\mathrm{T}}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{\xi}\end{pmatrix}=\beta$ is a valid equation, it's enough to show $\boldsymbol{c}^{\mathrm{T}}\in \mathrm{row}(T)$. Because every column of $\boldsymbol{c}^{\mathrm{T}}$ and T corresponding to one unique variable, we use $\boldsymbol{c}^{\mathrm{T}}\big|_{\{x_{uv}\}}$ and $T\big|_{\{x_{uv}\}}$ to denote the column corresponds to x_{uv}, same for other variables. We will take the following steps to prove Theorem 3.4.1.

Step 1:

Find a full-rank square submatrix M of T. Fix two vertices v_1, v_2 in V, let

$$\begin{aligned}
I &= \{x_{su}\}_{u\in N}\cup\{x_{us}\}_{u\in N}\cup\{x_{ut}\}_{u\in N}\cup\{x_{v_1v_2},x_{tv_1}\}\cup\{\xi_{\{u,v\}}\}_{uv\in E}\\
I_x &= \{x_{su}\}_{u\in N}\cup\{x_{us}\}_{u\in N}\cup\{x_{ut}\}_{u\in N}\cup\{x_{v_1v_2},x_{tv_1}\}
\end{aligned}$$

Then $M := T\big|_I$ is a full-rank square matrix.

Proof:

It's clear that every row vector of $T\big|_I$ from (3.21) is independent to all others, since the variable $\xi_{\{uv\}}$ only appears in (3.21). Therefore, it's enough to show that the matrix $T'\big|_I$, which consists of the rows from (3.15)-(3.20) in $T\big|_I$, is full rank.

Construct a directed graph $D_2(G)=(V,A')$ such that $A'=\{(u,v): x_{uv}\in I_x\}$. Then the equations system T'_I is of the form (3.14) in $D_2(G)$. The similar analysis in the previous section shows $T'\big|_I$ is row full rank matrix.

Step 2:

Since M is full rank, $\boldsymbol{\nu}^{\mathrm{T}}(T\big|_I)=(\boldsymbol{c}^{\mathrm{T}}\big|_I)$ has a unique solution. Let $\boldsymbol{r}^{\mathrm{T}}=\boldsymbol{c}^{\mathrm{T}}-\boldsymbol{\nu}^{\mathrm{T}}T$, it sufficient to show $\boldsymbol{r}^{\mathrm{T}}=\boldsymbol{0}^{\mathrm{T}}$. Since $\boldsymbol{c}^{\mathrm{T}}$ is a valid equality. So, let s_1, s_2 be two different schedules and $\boldsymbol{z}(s_1),\boldsymbol{z}(s_2)$ be the embedded points in vector space, we have

$$\langle \boldsymbol{r},(\boldsymbol{z}(s_1)-\boldsymbol{z}(s_2))\rangle = 0 \tag{3.22}$$

Step 3:

Firstly, we pick two arbitrary vertices u_1, u_2, then take two $M-$schedules

s_1, s_2, such that $p_1 \subseteq s_1, p_2 \subseteq s_2$ and $s_1 \setminus p_1 = s_2 \setminus p_2$

$$\begin{aligned} p_1 &= (s, u_{i_1}, u_1, u_2, u_{i_3}, \cdots, u_{i_m}, t) \\ p_2 &= (s, u_{i_1}, u_2, u_1, u_{i_3}, \cdots, u_{i_m}, t) \end{aligned}$$

Use the equation (3.22), we have the following equation. For convenience, in this proof, $\boldsymbol{r}^{\mathrm{T}}_{u_i u_j}$ are abbreviate as $\boldsymbol{r}^{\mathrm{T}}_{ij}$

$$\begin{aligned} &(m-1)(\boldsymbol{r}^{\mathrm{T}}_{i_1 1} - \boldsymbol{r}^{\mathrm{T}}_{i_1 2}) + (m-2)(\boldsymbol{r}^{\mathrm{T}}_{12} - \boldsymbol{r}^{\mathrm{T}}_{21}) + (m-3)(\boldsymbol{r}^{\mathrm{T}}_{2 i_3} - \boldsymbol{r}^{\mathrm{T}}_{1 i_3}) \\ &+ \big(Q-(m-1)\big)(\boldsymbol{r}^{\mathrm{T}}_{1 i_1} - \boldsymbol{r}^{\mathrm{T}}_{2 i_1}) + \big(Q-(m-2)\big)(\boldsymbol{r}^{\mathrm{T}}_{21} - \boldsymbol{r}^{\mathrm{T}}_{12}) \\ &+ \big(Q-(m-3)\big)(\boldsymbol{r}^{\mathrm{T}}_{i_3 2} - \boldsymbol{r}^{\mathrm{T}}_{i_3 1}) = 0 \end{aligned} \tag{3.23}$$

Step 4:
Similarly, take two $M-$schedules with different paths

$$\begin{aligned} p_1 &= (s, u_{i_m}, u_{i_1}, u_1, u_2, u_{i_3}, \cdots, u_{i_{m-1}}, t) \\ p_2 &= (s, u_{i_m}, u_{i_1}, u_2, u_1, u_{i_3}, \cdots, u_{i_{m-1}}, t) \end{aligned}$$

And drive an equation

$$\begin{aligned} &(m-2)(\boldsymbol{r}^{\mathrm{T}}_{i_1 1} - \boldsymbol{r}^{\mathrm{T}}_{i_1 2}) + (m-3)(\boldsymbol{r}^{\mathrm{T}}_{12} - \boldsymbol{r}^{\mathrm{T}}_{21}) + (m-4)(\boldsymbol{r}^{\mathrm{T}}_{2 i_3} - \boldsymbol{r}^{\mathrm{T}}_{1 i_3}) \\ &+ \big(Q-(m-2)\big)(\boldsymbol{r}^{\mathrm{T}}_{1 i_1} - \boldsymbol{r}^{\mathrm{T}}_{2 i_1}) + \big(Q-(m-3)\big)(\boldsymbol{r}^{\mathrm{T}}_{21} - \boldsymbol{r}^{\mathrm{T}}_{12}) \\ &+ \big(Q-(m-4)\big)(\boldsymbol{r}^{\mathrm{T}}_{i_3 2} - \boldsymbol{r}^{\mathrm{T}}_{i_3 1}) = 0 \end{aligned} \tag{3.24}$$

Subtract (3.24) from (3.23)

$$\begin{aligned} &(\boldsymbol{r}^{\mathrm{T}}_{i_1 1} - \boldsymbol{r}^{\mathrm{T}}_{i_1 2}) + (\boldsymbol{r}^{\mathrm{T}}_{12} - \boldsymbol{r}^{\mathrm{T}}_{21}) + (\boldsymbol{r}^{\mathrm{T}}_{2 i_3} - \boldsymbol{r}^{\mathrm{T}}_{1 i_3}) \\ &= (\boldsymbol{r}^{\mathrm{T}}_{1 i_1} - \boldsymbol{r}^{\mathrm{T}}_{2 i_1}) + (\boldsymbol{r}^{\mathrm{T}}_{21} - \boldsymbol{r}^{\mathrm{T}}_{12}) + (\boldsymbol{r}^{\mathrm{T}}_{i_3 2} - \boldsymbol{r}^{\mathrm{T}}_{i_3 1}) \end{aligned} \tag{3.25}$$

After that take

$$\begin{aligned} p_1 &= (s, u_1, u_2, u_{i_3}, \cdots, u_{i_m}, u_{i_1}, t) \\ p_2 &= (s, u_2, u_1, u_{i_3}, \cdots, u_{i_m}, u_{i_1}, t) \end{aligned}$$

yields

$$\begin{aligned} &(m-1)(\boldsymbol{r}^{\mathrm{T}}_{12} - \boldsymbol{r}^{\mathrm{T}}_{21}) + (m-2)(\boldsymbol{r}^{\mathrm{T}}_{2 i_3} - \boldsymbol{r}^{\mathrm{T}}_{1 i_3}) \\ &+ \big(Q-(m-1)\big)(\boldsymbol{r}^{\mathrm{T}}_{21} - \boldsymbol{r}^{\mathrm{T}}_{12}) + \big(Q-(m-2)\big)(\boldsymbol{r}^{\mathrm{T}}_{i_3 2} - \boldsymbol{r}^{\mathrm{T}}_{i_3 1}) = 0 \end{aligned} \tag{3.26}$$

From the equations above, get

$$m(\boldsymbol{r}^{\mathrm{T}}_{i_1 1} - \boldsymbol{r}^{\mathrm{T}}_{i_1 2}) = (m - Q)(\boldsymbol{r}^{\mathrm{T}}_{1 i_1} - \boldsymbol{r}^{\mathrm{T}}_{2 i_1}) \tag{3.27}$$

If here is a $st-$path with length $Q+1$ in the solution set, then $\boldsymbol{r}^{\mathrm{T}}_{i_1 1} = \boldsymbol{r}^{\mathrm{T}}_{i_1 2}$. Since the choice of u_{i_1}, u_1, u_2 are arbitrary, therefore, indeed we have $\boldsymbol{r}^{\mathrm{T}}_{ij} = \boldsymbol{r}^{\mathrm{T}}_{ik}$ for any $u_i, u_j, u_k \in N$. Similarly, If here is a $st-$path with length at most Q in the solution set, then $\boldsymbol{r}^{\mathrm{T}}_{ji} = \boldsymbol{r}^{\mathrm{T}}_{ki}$ for any $u_i, u_j, u_k \in N$. So, $\boldsymbol{r}^{\mathrm{T}}_{ij} = \boldsymbol{r}^{\mathrm{T}}_{kl}$ for any $u_i, u_j, u_k, u_l \in N$.
Step 5: Let

$$p_1 = (s, u_{i_1}, u_{i_2}, \cdots, u_{i_m}, u_1, u_2, t)$$
$$p_2 = (s, u_{i_1}, u_{i_2}, \cdots, u_{i_m}, u_2, u_1, t)$$

With the result in Step 4, we get

$$\boldsymbol{r}^{\mathrm{T}}_{ti} = \boldsymbol{r}^{\mathrm{T}}_{tj} \quad \forall u_i, u_j \in N$$

Because of the construction of $\boldsymbol{\nu}$, $\boldsymbol{r}^{\mathrm{T}}\big|_I = \mathbf{0}^{\mathrm{T}}$, therefore $\boldsymbol{r}^{\mathrm{T}} = \mathbf{0}^{\mathrm{T}}$.

The requirement on the length of a path needs the extra assumption about M, Q

Now, since the dimension of polytopes are determined, we could discuss the facets defining inequality of B^{u+}_{LP}, B^{u+}_{IP}

Observation 6. *$\xi_{\{u,t\}} \geq 0$ and $x_{tu} \geq 0$ defines facet of B^{u}_{LP} and B^{u}_{IP}*

Proof :
Similar to the dimension proof, by taking

$$I = \{\{x_{su}\}_{u \in N}, \{x_{us}\}_{u \in N}, \{x_{ut}\}_{u \in N}, x_{u_1,u_2}, x_{tu_j}, x_{tu_k}, \{\xi_{u,v}\}_{\{u,v\} \in E}\}$$

Observation 7. *The polytope B_{LP} is invariant under relabelling the vertices in N.*

This observation is based on the graph is symmetric, relabelling the vertices doesn't change the equation system of B_{LP}.

Theorem 3.4.2. *$x_{uv} \geq 0$ defines a facet of B^{u+}_{LP} for all $u, v \in N$, if* $|N| \geq 16$

Proof :
Let $n = |N|$, we know $\dim B_{LP}^{u+} = n^2 - 2$, so the number of facet defining inequalities is at least $n^2 - 1$. Since B_{LP}^{u+} is a special case of B_{LP} and $N \geq Q \geq 5$, we know that $\xi_{\{u,v\}} \geq 0$ isn't a facet. Here are totally $2(n^2 + 3n) - \binom{n}{2}$ many potential facet defining inequalities and $n^2 - n$ many inequalities of the form $x_{uv} \geq 0$ for $u, v \in N$. Therefore, if $n \geq 16$, then by pigeonhole principle, $x_{uv} \geq 0$ is a facet for some $u, v \in N$. Applying the observation 7, $x_{uv} \geq 0$ is a facet for all $u, v \in N$

Chapter 4

Resource Constrained Shortest Path Problem

In section 4.1, we describe the RSCPP and the traditional model for RCSPP. In section 4.2, we give a new MIP model for RCSPP, following Baldacci's idea. A simple example illustrates the new model is better than the traditional model. In section 4.3, we discuss the dimension of polytope behind the LP-relaxation. In section 4.4, we discuss the dimension of polytope for MIP, while all demands are units. In section 4.5, we give a series of cuts, these cuts are facets under the unit demand case. In particular, adding these cuts could give a complete description when $Q = 2$. In section 4.6, we will show some computation results.

4.1 Problem Description and Traditional Model

Definition 4.1.1 (Resource Constrained Shortest Path Problem (RCSPP))**.** ***Given:*** *directed simple graph* $G = (V, A)$*, two special vertices, the sink* $s \in V$ *and the target* $t \in V$*, non-negative integer edge costs* $c \in \mathbb{N}^E_{\geq 0}$*, natural numbers* $Q \in \mathbb{N}$ *and node demands* $q \in \mathbb{N}^N_{\geq 0}$*, where* $N = V \setminus \{s, t\}$*. A* $1-$*route is a simple st-path* p *in* G*, its cost and demand are* $c(p)$ *and* $q(p)$*, respectively, and it is feasible if* $q(p) \leq Q$ *holds.* ***Sought:*** *A feasible* 1*-route with minimal total cost.*

The traditional model for RCSPP is adding resource constraints to the shortest path problem. In this thesis, we consider the single resource constraint case.

$$\min \sum_{uv \in A} c_{uv} x_{uv}$$

subject to:

$$\sum_{u \in N} x_{su} = 1 \tag{4.1}$$

$$\sum_{u \in N} x_{ut} = 1 \tag{4.2}$$

$$x_{us} = x_{tu} = 0 \quad \forall u \in N \tag{4.3}$$

$$\sum_{v \in \gamma^-(u)} x_{vu} - \sum_{v \in \gamma^+(u)} x_{uv} = 0 \quad \forall u \in N \tag{4.4}$$

$$\sum_{u \in N} q_u \big(\sum_{v \in \gamma^+(u)} x_{uv} \big) \leq Q \tag{4.5}$$

$$x_{uv} \in \{0, 1\} \ \forall uv \in A \tag{4.6}$$

4.2 A New Model

Transfer a given undirected graph $G = (V, E)$ into a directed graph $D(G) = (V, A(E))$ with $c_{\{u,v\}} = c_{uv} = c_{vu}$. Assign an arc variable x_{uv}

and a flow variable y_{uv} for all $uv \in A(E)$.

$$\min \sum_{uv \in A(E)} c_{uv} x_{uv}$$

subject to:

$$\sum_{u \in N} x_{su} = 1 \tag{4.7}$$

$$x_{us} = x_{tu} = 0 \quad \forall u \in N \tag{4.8}$$

$$\sum_{u \in N} x_{ut} = 1 \tag{4.9}$$

$$y_{us} = 0 \quad \forall u \in N \tag{4.10}$$

$$\sum_{v \in \gamma(u)} x_{vu} - \sum_{v \in \gamma(u)} x_{uv} = 0 \quad \forall u \in N \tag{4.11}$$

$$y_{uv} + y_{vu} = Q(x_{uv} + x_{vu}) \quad \forall \{u, v\} \in E \tag{4.12}$$

$$\sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) = q_u \sum_{v \in \gamma(u)} (x_{vu} + x_{uv}) \quad \forall u \in N \tag{4.13}$$

$$x_{uv}, x_{vu} \in \{0, 1\},\ y_{uv} \geq 0, y_{vu} \geq 0\ \forall \{u, v\} \in E \tag{4.14}$$

This model is also based on the flow conservation formula, replace the resource constraints with (4.12) and (4.13), which follow the idea of Baldacci's model.

Observation 8. *The set of feasible solutions of the new model projected onto the arc variables space is a subset of the set of feasible solutions of the basic model. Moreover, it can be a proper subset.*

Proof : It's sufficient to check the feasible solutions of the new model satisfies (4.5):

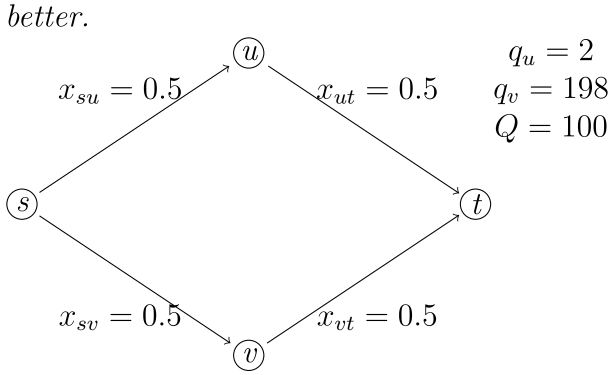
$$\begin{aligned}
\sum_{u\in N}\sum_{v\in\gamma(u)} q_u(x_{uv}+x_{vu}) &= \sum_{u\in N}\sum_{v\in\gamma(u)}(y_{vu}-y_{uv}) \\
&= \sum_{u\in N}\sum_{\substack{v\in\gamma(u)\\ v\in N}}(y_{vu}-y_{uv}) + \sum_{u\in N}(y_{su}-y_{us}+y_{tu}-y_{ut}) \\
&\leq \sum_{u\in N}(y_{su}+y_{us}+y_{tu}+y_{ut}) \\
&= \sum_{u\in N} Q(x_{su}+x_{us}) + \sum_{u\in N} Q(x_{tu}+x_{ut}) = 2Q
\end{aligned}$$

Then the flow conversation constraint says:

$$\sum_{u\in N} q_u \sum_{v\in\gamma(u)} x_{uv} = \sum_{u\in N} q_u \sum_{v\in\gamma(u)} x_{vu}$$

So, the solutions of new model satisfies (4.5).

Example 4.2.1. *Here is a simple example shows the new model is better.*



This example shows a solution of the basic model. However, this can't be a solution of the new model. Consider the equation on v, we have

$$\begin{aligned}
&y_{sv}+y_{tv}-y_{vs}-y_{vt} = q_v(x_{sv}+x_{tv}+x_{vs}+x_{vt}) = 198 \\
&y_{sv}+y_{vs} = Q(x_{sv}+x_{vs}) = 50 \\
&y_{vt}+y_{tv} = Q(x_{tv}+x_{vt}) = 50
\end{aligned}$$

thus

$$198 = y_{sv}+y_{tv}-y_{vs}-y_{vt} > y_{sv}+y_{tv}+y_{vs}+y_{vt} = 100$$

Since all variables are non-negative, this inequality above is impossible, which means this new model is better than the traditional model.

Moreover, we can compute the gap between the linear relaxations and the original mix-integer programming of the two models on this graph. For this example, it's clear that the only integer solution is (s, u, t) with cost $c_{su} + c_{ut}$. Assume that $c_{sv} + c_{vt} < c_{su} + c_{ut}$, then

- For the LP-relaxation of the new model, the only possible solution is (s, u, t)
- For the LP-relaxation of the traditional model, the optimal solution is $\frac{1}{2}(s, u, t) + \frac{1}{2}(s, v, t)$

Let

$$\text{gap} = \frac{\text{Optimal value of MIP} - \text{Optimial Value of LP}}{\text{Optimal Value of MIP}}$$

$$\begin{aligned} \text{gap}_n &= 0 \\ \text{gap}_c &= \frac{c_{su} + c_{ut} - c_{sv} - c_{vt}}{2(c_{su} + c_{ut})} \in (0, \frac{1}{2}) \end{aligned}$$

Where the $\text{gap}_n, \text{gap}_t$ are the gaps of the new model and traditional model respectively.

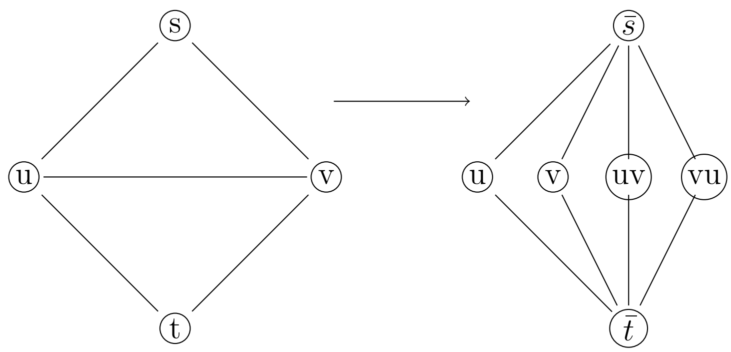
Actually, we have a stronger result when paths only share the sink and the target.

Theorem 4.2.1. *Let f be a flow with respect to the arcs such that $f = \sum_{i=1}^{m} \mu_i p_i$ for some paths p_i with $\sum_{i=1}^{m} \mu_i = 1$. If $\forall i, j \in [m]$, $V(p_i) \cap V(p_j) = \{s, t\}$. f is the projection of a solution of the new model onto arc variables space if and only if*

$$q(p_i) := \sum_{u \in V(p_i)} q_u \leq Q \quad \forall i \in [m]$$

Proof :
Take an arbitrary path $p_i = (s, u_{i_0}, \cdots, u_{i_m}, t)$ with weight λ_i. Then $y_{s,u_{i_0}} = \lambda_i Q$ proves the theorem.

Corollary 4.2.2. *Let $G = (V, E)$ be an undirected graph with the same notation we defined. Let $\mathcal{P}$ be the set of all $st-$paths. Consider a transformation T. Then $T(G) = (V_{\mathcal{P}}, E_{\mathcal{P}}$ such that $V_{\mathcal{P}}$ consists of two artificial nodes $\bar{s}, \bar{t}$ and nodes set $\{v_p\}_{p\in\mathcal{P}}$ with the demand $q(p)$ respectively. And $E_{\mathcal{P}} := \{\{\bar{s}, v_p\} : p \in \mathcal{P}\} \cup \{\{\bar{t}, v_p\} : p \in \mathcal{P}\}$ such that $c_{\bar{s},v_p} + c_{\bar{t},v_p} = c(p)$. Then the optimal solution of our LP model in the corresponding graph $A\big(T(G)\big)$ is the optimal solution of our MIP model on graph $A(G)$.*



Observation 9. *Let path p be $(s = u_0, u_1, \cdots, u_m, u_{m+1} = t)$, here are trivial values for flow variables that $y_{u_iu_{i+1}} = Q - \sum_{j=1}^{i} q_{u_j}$ and $y_{u_{i+1}u_i} = \sum_{j=1}^{i} q_{u_j}$ can associate to this path to get a solution for the new model. Moreover, if the vertices in p are distinct, then the value of flow variables are unique. (ie. the trivial values for flow variables are unique solution of the new model associated to p.)*

Proof:
This construction gives a solution f. Suppose here is another flow solution f' for p, then $f - f'$ is a solution for the homogeneous equation system with all arc variables equal to 0. Then, plug $x_{uv} = x_{vu} = 0$ for all $\{u, v\} \in E$ into the equation (4.12) and (4.13), we get

$$y_{uv} + y_{vu} = 0 \quad \forall \{u, v\} \in E$$
$$\sum_{v\in\gamma(u)} (y_{vu} - y_{uv}) = 0 \quad \forall u \in N$$

Therefore we conclude that $f - f' = \mathbf{0}$. That means the solution associates to an $st-$ path is unique.

Observation 10. *If Q is large enough, then the arc set of the optimal solutions is integral. Indeed, let p be the directed path with lowest cost in $D(G) = (V, A(E))$, then we only need $Q \geq c(p)$.*

If Q is larger than the demands on a directed path p with the lowest costs. Then for any solutions, we can replace the edge sets with the edge sets of p. And by Observation 9, the solution is unique.

4.3 Dimension of a Polytope from the New Model

Now consider $E = \{\{u, v\} : u, v \in N\} \cup \{\{s, u\} : u \in N\} \cup \{\{v, t\} : v \in N\}$, where $|N| = n$. Let P_{LP} be the polytope of the LP-relaxation of model derived from this graph and transferred graph. Use the similar idea in previous section, the model is equivalent to

$$\sum_{u \in N} x_{su} = 1 \tag{4.15}$$

$$\sum_{u \in N} x_{ut} = 1 \tag{4.16}$$

$$\sum_{v \in \gamma(u)} x_{uv} - \frac{1}{2} \sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) = 0 \quad \forall u \in N \tag{4.17}$$

$$\sum_{v \in \gamma(u)} x_{vu} - \frac{1}{2} \sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) = 0 \quad \forall u \in N \tag{4.18}$$

$$x_{us} = 0 \quad \forall u \in N \tag{4.19}$$

$$x_{tu} = 0 \quad \forall u \in N \tag{4.20}$$

$$y_{us} = 0 \quad \forall u \in N \tag{4.21}$$

$$y_{uv} + y_{vu} = Q(x_{uv} + x_{vu}) \quad \forall \{u, v\} \in E \tag{4.22}$$

Write the system as $\begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \boldsymbol{c} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$, where C represents the equations from (4.15)-(4.18), X represents equations from (4.19)-(4.20), Y represents (4.21) and G represents equations from (4.22). Actually, from these equations, we can conclude that $y_{us} = Qx_{us}$ $\forall u \in N$, but These equations are already in the row space of this matrix.

Lemma 2. *Exactly one equation in* $\begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \boldsymbol{c} \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}$ *is redundant.* *That means the rank of matrix* $\begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix}$ *is* $\frac{1}{2}n^2 + \frac{13}{2}n + 1$.

Proof: We use the same idea as the previous section. Consider $(\boldsymbol{a}^{\mathrm{T}}, \boldsymbol{\mu}^{\mathrm{T}}, \lambda, \boldsymbol{b}^{\mathrm{T}})$ such that

$$(\boldsymbol{a}^{\mathrm{T}}, \boldsymbol{\mu}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}, \boldsymbol{b}^{\mathrm{T}}) \begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix} = \mathbf{0}^{\mathrm{T}}$$

Notice that each entry of $\boldsymbol{a}^{\mathrm{T}}$ corresponds to a equation, let

- $\alpha^+(u)$ be the entry in $\boldsymbol{a}^{\mathrm{T}}$ corresponding to

$$\sum_{v \in \gamma^+(u)} x_{uv} = \begin{cases} 1 & \text{if } u = s \\ \dfrac{1}{2} \displaystyle\sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) & \text{if } u \in N \end{cases}$$

- $\alpha^+(u)$ be the entry in $\boldsymbol{a}^{\mathrm{T}}$ corresponding to

$$\sum_{v \in \gamma^-(u)} x_{vu} = \begin{cases} 1 & \text{if } u = t \\ \dfrac{1}{2} \displaystyle\sum_{v \in \gamma(u)} (y_{vu} - y_{uv}) & \text{if } u \in N \end{cases}$$

- $\mu^+(u)$ be the entry in $\boldsymbol{\mu}^{\mathrm{T}}$ corresponding to

$$x_{us} = 0$$

- $\mu^-(u)$ be the entry in $\boldsymbol{\mu}^{\mathrm{T}}$ corresponding to

$$x_{tu} = 0$$

- $\omega(u)$ be the entry in $\boldsymbol{\omega}^{\mathrm{T}}$ corresponding to

$$y_{us} = 0$$

- $\beta_{\{u,v\}}$ be the entry in $\boldsymbol{b}^{\mathrm{T}}$ corresponding to

$$y_{uv} + y_{vu} - Q(x_{uv} + x_{vu}) = 0$$

Firstly, we arbitrarily choose $u, v \in N$, these variables only appears in (4.17),(4.18) and (4.22), in order to cancel the coefficients of variables x_{uv} and x_{vu}, we have

$$\alpha^{+}(u) + \alpha^{-}(v) + Q\beta_{\{u,v\}} = 0 \text{ and } \alpha^{-}(u) + \alpha^{+}(v) + Q\beta_{\{u,v\}} = 0$$

Then consider variables y_{uv} and y_{vu} which also only appears in (4.17),(4.18) and (4.22), yields

$$\begin{aligned} -2\beta_{\{u,v\}} + \alpha^{+}(u) + \alpha^{-}(u) - \alpha^{-}(v) - \alpha^{+}(v) = 0 \\ -2\beta_{\{u,v\}} - \alpha^{+}(u) - \alpha^{-}(u) + \alpha^{-}(v) + \alpha^{+}(v) = 0 \end{aligned}$$

Since u, v are arbitrary, for any $u, v \in N$, $\beta_{\{u,v\}} = 0$, $\alpha^{+}(u) = -\alpha^{-}(v)$. Then we start to cancel the coefficients of variables y_{ut}:

$$\alpha^{+}(u) + \alpha^{-}(u) - \beta_{\{u,t\}} = 0 \text{ and } \alpha^{+}(u) + \alpha^{-}(u) + \beta_{\{u,t\}} = 0$$

This implies $\beta_{\{u,t\}} = 0$ and $\alpha^{+}(u) = \alpha^{-}(u)$ for all $u \in N$. Now we cancel the coefficients of variables x_{ut}, x_{tu}

$$\mu^{-}(u) = -\alpha^{-}(u) \text{ and } \alpha^{-}(t) = -\alpha^{+}(u)$$

The next step, we cancel the coefficients of variables y_{us} and y_{su}

$$-2\beta_{\{u,s\}} + \alpha^{+}(u) + \alpha^{-}(u) + 2\omega_u = 0 \text{ and } -2\beta_{\{u,s\}} + \alpha^{+}(u) + \alpha^{-}(u) = 0$$

And we get $\beta_{\{u,s\}} = 0, \omega(u) = 0$ for all $u \in \omega$. Finally, we can cancel the x_{us} and x_{su}

$$\alpha^{+}(s) = -\alpha^{-}(u) \text{ and } \mu^{+}(u) = -\alpha^{+}(u)$$

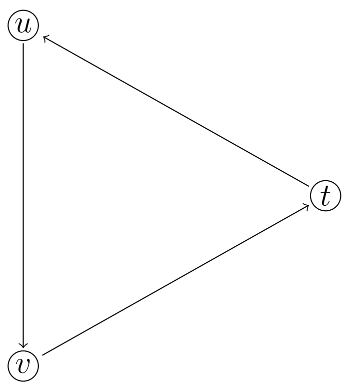
We have canceled all variables, and this process implies

$$\dim\{(\boldsymbol{a}^{\mathrm{T}}, \boldsymbol{\mu}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}, \boldsymbol{b}^{\mathrm{T}}) : (\boldsymbol{a}^{\mathrm{T}}, \boldsymbol{\mu}^{\mathrm{T}}, \boldsymbol{\omega}^{\mathrm{T}}, \boldsymbol{b}^{\mathrm{T}}) \begin{pmatrix} C \\ X \\ Y \\ G \end{pmatrix} = \mathbf{0}^{\mathrm{T}}\} = 1$$

The rank of the matrix follows immediately. Now, an upper bound of dimension has been obtained, then we obtain the lower bound by constructing affinely independent points.

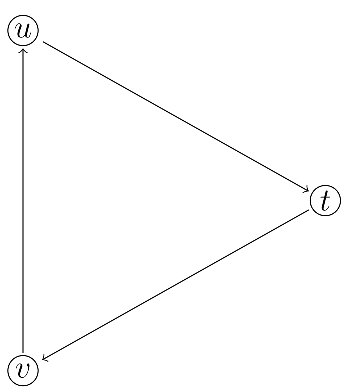
Theorem 4.3.1. *If* $3Q > 5\max_{u\in N} q_u$*, then* $\dim P_{LP} = 3\binom{n}{2} + n - 1$

Proof :
$\dim P_{LP} \le 3\binom{n}{2} + n - 1$ is immediately from Lemma 2. So, it's sufficient to find $3\binom{n}{2} + n$ many affinely independent points cotained in P_{LP}. It's clear that path $p_u := (s, u, t)$ with associated flow variables are affinely independent. Let $f = \frac{1}{n}\sum_{u\in N} p_u$. Then for every two $u, v \in N$, we construct three different solutions.
c^1_{uv}:



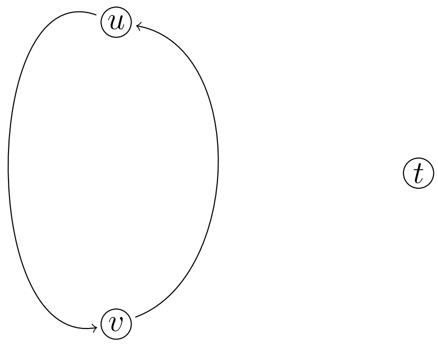
$$
\begin{aligned}
x_{vu} = x_{tu} = x_{tv} &= 0\\
x_{uv} &= 0.5\\
x_{ut} &= -0.5\\
x_{vt} &= 0.5\\
y_{uv} &= 0.5Q\\
y_{vu} &= 0\\
y_{ut} &= -0.25Q - 0.25q_u\\
y_{tu} &= 0.25Q + 0.25q_u\\
y_{vt} &= 0.25Q - 0.25q_v\\
y_{tv} &= -0.25Q + 0.25q_v
\end{aligned}
$$

c^2_{uv}:



$$
\begin{aligned}
x_{uv} = x_{tu} = x_{vt} &= 0\\
x_{vu} &= 0.5\\
x_{ut} &= 0.5\\
x_{vt} &= -0.5\\
y_{vu} &= 0.5Q\\
y_{uv} &= 0\\
y_{ut} &= 0.25Q - 0.25q_u\\
y_{tu} &= -0.25Q + 0.25q_u\\
y_{vt} &= -0.25Q - 0.25q_v\\
y_{tv} &= 0.25Q + 0.25q_v
\end{aligned}
$$

Take a small enough positive parameter ϵ, then c^3_{uv} is constructed as:



$$
\begin{gathered}
x_{ut} = x_{tu} = x_{vt} = x_{tv} = 0 \\
x_{vu} = 0.25 \\
x_{uv} = 0.25 \\
y_{uv} = 0.25Q + \epsilon \\
y_{vu} = 0.25Q - \epsilon \\
y_{ut} = -\epsilon - 0.25q_u \\
y_{tu} = \epsilon + 0.25q_u \\
y_{vt} = \epsilon - 0.25q_v \\
y_{tv} = -\epsilon + 0.25q_v
\end{gathered}
$$

It's easy to check that $f + \frac{1}{n}c^i_{uv}$ is a solution for arbitrary $u, v \in N$ and $i \in [3]$. Also, we can check $\{f + \frac{1}{n}c^i_{uv}\}_{\substack{u,v\in N \\ i\in[3]}} \cup \{p_u\}_{u\in N}$ is set of affinely independent points. Therefore,

$$\dim P_{LP} = 3\binom{n}{2} + n - 1$$

4.4 Integer Solution Polytope for Unit Demand Case

Now we consider the case that $q_u = 1$ for all $u \in N$, and in this case we can determine the dimension of the convex hull of integral solutions P^u_{IP}. Assume $|N| = n$

Observation 11. *Let $|N| \geq Q$. If $Q = 1$, then $\dim P^u_{IP} = n - 1$, and if $Q = 2$, then $\dim P^u_{IP} = 2\binom{n}{2} + n - 1$*

Proof :
Recall the constraints (4.13)

$$\sum_{v\in\gamma(u)} (y_{vu} - y_{uv}) = q_u \sum_{v\in\gamma(u)} (x_{vu} + x_{uv}) \quad \forall u \in N$$

Take the summation over all $u \in N$, and in the unit case

$$\begin{aligned}
\sum_{u\in N}\Big(\sum_{v\in\gamma(u)}(x_{uv}+x_{vu})\Big) &= \sum_{u\in N}\sum_{v\in\gamma(u)}(y_{vu}-y_{uv}) \\
&= \sum_{u\in N}(y_{su}+y_{tu}-y_{us}-y_{ut}) \\
&= Q+\sum_{u\in N}(y_{tu}-y_{ut}) \leq 2Q
\end{aligned}$$

since

$$\sum_{u\in N} x_{su} = \sum_{u\in N} x_{ut} = 1 \text{ and } \forall u \in N,\ x_{us} = x_{tu} = 0$$

If $Q = 1$, then $\sum\limits_{u,v\in N}(x_{uv}+x_{vu}) = 0$, so all integer solutions projected onto the arc variables are of the form (s,u,t)

If $Q = 2$, then $2\sum\limits_{v\in\gamma(u)\setminus\{s,t\}}\sum\limits_{u\in N} x_{uv} = \sum\limits_{u,v\in N}(x_{uv}+x_{vu}) \leq 2$, that means we can at most take one arc with head and tail in N. So, all integer solution w.r.t arcs are either of the form (s,u,t) or (s,u,v,t).

In Observation 9, we know that the way to associate flow is unique, and all solutions are affinely independent. So, we have the dimension.

The dimensions in two specific cases have been obtained. Now, we are going to prove the dimension for $Q \geq 3$.

Theorem 4.4.1. *Let $|N| \geq Q$. If $Q \geq 3$, then* $\dim P^u_{IP} = 3\binom{n}{2} + n - 1$

Proof:
Since $3Q \geq 9 > 5$, $\dim P^u_{IP} \leq \dim P_{LP} = 3\binom{n}{2} + n - 1$, then we will construct affinely independent points. Label the vertices in N such that $N = \{u_1, u_2, \cdots, u_n\}$. Take path (s, u_i, t) and (s, u_i, u_j, t) for all $i, j \in [n]$. And we take paths (s, u_i, u_j, u_k, t) such that $k > j$ and $i \equiv j - 1 \pmod n$. We have $\binom{3}{n}2 + n - 1$ many different paths, now we add one more paths $(s, u_{n-1}, u_1, u_n, t)$. Let $\mathcal{P}_3$ be the set of all paths with length 3, and $S := \{i, j) : (s, u_i, u_j, u_k, t) \in \mathcal{P}_3\}$. Indeed $S = \{(n, 1), (n-1, 1)\} \cup \{(1, 2), (2, 3), \cdots, (n-2, n-1)\}$

- Let $z(i)$ be the corresponding solution of the model with respect to the path (s, u_i, t)
- Let $z(i, j)$ be the corresponding solution of the model with respect to the path(s, u_i, u_j, t)
- Let $z(i, j, k)$ be the corresponding solution of the model with respect to the path (s, u_i, u_j, u_k, t)

Let $\alpha(i), \alpha(i, j), \alpha(i, j, k)$ be the coefficients of solutions $z(i), z(i, j)$ and $z(i, j, k)$ respectively. Consider

$$\sum_{i\in[n]} \alpha(i)z(i) + \sum_{\substack{i,j\in[n]\\ a\neq j}} \alpha(i, j)z(i, j) + \sum_{(s,u_i,u_j,u_k,t)\in\mathcal{P}_3} \alpha(i, j, k)z(i, j, k) = 0$$

$$\text{where} \sum_{i\in[n]} \alpha(i) + \sum_{i,j\in[n]} \alpha(i, j) + \sum_{(s,u_i,u_j,u_k,t)\in\mathcal{P}_3} \alpha(i, j, k)z(i, j, k) = 0$$

Firstly, note that for all $(i, j) \notin S$ with $i > j$, then only in the $z(i, j)$ the coordinate corresponds to this arc is non-zero. So $\alpha(i, j) = 0$ for all $i > j$ with $(i, j) \notin S$. Then we consider $(j, k) \notin S$ with $j \leq k$, then the coordinate corresponding to the arc is non-zero only in $z(j, k)$ and $z(i, j, k)$. Therefore, for these (j, k), $\alpha(j, k) = -\alpha(i, j, k)$. Then we check the flow value on (u_j, u_k), we get $\alpha(j, k) = \alpha(i, j, k) = 0$.

The undetermined coefficients are $\{\alpha(i)\}_{i\in[n]} \cup \{\alpha(i, j) : (i, j) \in S\} \cup \{\alpha(i, j, k) : (s, u_i, u_j, u_k, t) \in \mathrm{P}_3, j < k, (j, k) \in S\}$. Note that non-zero coefficient for arc $(n-2, n-1)$ only appears in $(s, u_{n-2}, u_{n-1}, u_n, t)$. So $\alpha(n-2, n-1, n) = -\alpha(n-2, n-1) = 0$, the similar discussion leads to $\alpha(n-3, n-2, n-1) = -\alpha(n-3, n-2) = 0$, and then $\alpha(i, j, k)$ where $i = n-3, j = n-2$ are all zero. So $\alpha(n-4, n-3, n-2) = -\alpha(n-4, n-3) = 0$. Repeat this process, then this means we know that the coefficients $\{\alpha(i, j) : (i, j) \in S, 1 < i < j\} \cup \{\alpha(i, j, k) : (s, u_i, u_j, u_k, t) \in \mathrm{P}_3, 1 < j < k, (j, k) \in S\}$ are all zero. Also, we can remove $\{\alpha(i) : i \in [n-2], i \neq 1\}$. Therefore, only $\alpha(1), \alpha(n), \alpha(n-1), \alpha(n-1, 1), \alpha(1, n-1), \alpha(1, n), \alpha(n, 1), \alpha(n-1, 1, n), \alpha(n, 1, n-1)$ are not known yet. However, by checking the flow value, we know all them are 0. Therefore, we find $3\binom{n}{2} + n$ many affinely independent points.

4.5 Cuts

In this section, we use the graphs as in the previous section. Let $p = (s, \cdots, u, t)$ be a path. If all vertices of p are distinct, then before the vehicle enters the last vertex u, it should carry at least q_u many loads. However, if we have a cycle when a vertex $u \in N$ is visited more than once. Here is a freedom to manage how to satisfy the demand of u. For example in the path $p = (s, \cdots, w, u, v, u, t)$, assume the node u is visited exactly twice. Then we only need to satisfy the following conditions

$$
\begin{aligned}
y_{wu} + y_{vu} - y_{uw} - y_{uv} &= 2q_u \\
y_{vu} + y_{tu} - y_{uv} - y_{ut} &= 0
\end{aligned}
$$

Here are many solutions for equation system above.

However, because the objective function only cares about arc variables, then we can choose a rule to construct the value of flow variables from the edge variables. We require when a node in N is visited, the flow should lose exact q_u units. Under this rule, we have cuts

$$y_{vu} \geq \min_{w \in N} q_w(x_{uv} + x_{vu}) \quad \forall u \in N, v \in \gamma(u) \tag{4.23}$$

These cuts work very well in unit demand case.

Theorem 4.5.1. *Let $|N| \geq Q$. If $q_u = 1$ for all $u \in N$ and $Q \geq 3$, then the inequality in (4.23) defines a facet for the polytope of LP-relaxation and convex hull of integer solution P^u_{LP}, P^u_{IP}*

Proof : We firstly show that $y_{vu} = \min_{w \in N} q_w(x_{uv} + x_{vu})$ is independent to the original equation system. From Section 4.3, we know that all equations from (4.22) are independent. And since $u, v \in N$, so here are only one equation in (4.22) is relevant to y_{uv}. That means $y_{vu} = \min_{w \in N} q_w(x_{uv} + x_{vu})$ is independent for any $u, v \in N$.

Then we claim that we can construct $3\binom{n}{2} + n - 1$ many affinely independent points with the cut is tight. Let $N = \{u_i\}_{i=1}^n$ We take paths in the three forms. Based on the symmetry of this polytope, W.L.O.G, let $u = u_l, v = u_r$ such that $l < r - 1$, $l, r \in \{2, 3, \cdots, n-1\}$

- (s, u_i, t) for all $i \in [n]$

- (s, u_i, u_j, t) for all $i, j \in [n]$, but $(i, j) \neq (l, r)$
- $(s, u_{i-1}, u_i, u_j, t)$ for all $i, j \in [n]$ with $i < j$ but $(i, j) \neq (l, r)$. Here the $i - 1$ is also similar to Section 4.4, that is respect to the remainder of modulo n.
- $(s, u_r, u_l, u_{l+1}.t)$. The plus is respect to the remainder of modulo n.

The proof is similar to want we did in Section 4.4, and these points are affinely independent. Therefore $y_{vu} \geq \min_{w \in N}(x_{uv} + x_{vu})$ is a facet defining inequality for any $u, v \in N$.

Note that this method can be generalized to the general case, but we need requirements on $\{q_u\}_{u \in N}$.

Theorem 4.5.2. *Let $|N| \geq Q$. If $Q = 2$, then adding these cuts will give a complete description of P^u_{IP}*

Proof :
Since $Q = 2$, By the cuts

$$y_{vu} \geq (x_{uv} + x_{vu}) \quad \forall u, v \in N$$
$$y_{uv} \geq (x_{uv} + x_{vu}) \quad \forall u, v \in N$$

and

$$y_{uv} + y_{vu} = 2(x_{uv} + x_{vu}) \tag{4.24}$$

Therefore, indeed, we get $y_{uv} = y_{vu}$ for all $u, v \in N$. And all possible paths (s, u, t) or (s, u, v, t) are satisfy (4.24).

Let f be a solution satisfies the model include the cuts (4.23). By flow decomposition theorem, $f = \sum_{i=1}^m \mu_i p_i + \sum_{i=1}^l \lambda_i c_i$ with $\sum_{i=1}^m \mu_i = 1$, where p_i are paths with distinct vertices, c_i are cycles. Because of observation 9, a path with distinct vertices has unique solution. So, it sufficient to show that $\lambda_i = 0$ for all i. Let $f' = \sum_{i=1}^m \mu_i p_i$, and $z(f), z(f')$ be the solutions of the model with flow f, f' respectively. Then $z(f) - z(f')$ is a solution of the homogeneous equations. In this model, cycles can't contain the sink s and the target t. Let $x_{uv}(f -$

$f'), y_{uv}(f-f')$ represent the variable x_{uv}, y_{uv} in $f-f'$. Then use (4.13) and note $y_{vu} = y_{uv}$, for an arbitrary $u \in N$

$$0 = \sum_{v \in \gamma(u)} (y_{vu} - y_{uv})(f - f') = \sum_{v \in \gamma(u)} (x_{uv} + x_{vu})(f - f')$$

Since in $f - f'$, $x_{uv} \geq 0$ for all $(u, v) \in A$

$$0 = 2(x_{uv} + x_{vu})(f - f') = y_{uv}(f - f') = y_{vu}(f - f')$$

So, here are no cycles in f. Every solution is a convex combination of integer solutions, in particular, all vertices are integral.

4.6 Computational Results

We generate various instances at random using the following settings in order to compare this model with the conventional model. And my Github contains all instances.

- number of nodes n: 100-1000
- position of nodes: random
- graph: generated by Delaunay triangulation
- demand of every node: random number in [0,1)
- cost: Euclidean distance
- solver: SCIP
- parameter Q: $0.25\sqrt{n}$
- x axis: instances
- y axis: the improved ratio is defined as

$$y = \begin{cases} 0 & \text{If } \mathrm{OptTrad}_{IP} = \mathrm{OptTrad}_{LP} \\ \dfrac{\mathrm{OptNew}_{LP} - \mathrm{OptTrad}_{LP}}{\mathrm{OptTrad}_{IP} - \mathrm{OptTrad}_{LP}} & \text{Otherwise} \end{cases}$$

  where $\mathrm{OptTrad}_{IP}$ represents the optimal solution from solving the traditional model, OptNew_{LP}, $\mathrm{OptTrad}_{LP}$ correspond to solving

LP-relaxation of new model, solving LP-relaxation of traditional model respectively.

Figure 4.1 shows some advantages of our model. Among these 100 instances, here are 15 instances $(62, 27, 81, 21, 15, 65, 34, 37, 59, 86, 61, 64, 17, 70, 92)$ have improved solutions compared to solving them in the traditional model. Indeed, we could expect among 100 random generalized instances, $5\% - 15\%$ instances have better solutions. In particular, instance 70 and instance 92 show that solving the new LP-relaxation model provides integral solutions. However, this new model does have disadvantages. Because the new model double the number of variables and has many more constraints, solving the LP-relaxation of the new model always costs more time. However, here is an interesting discovery after the pre-solver, the number of constraints is significantly reduced. Sometimes, a better solution is worth spending more time on it.

To visualize the solution of these two models, we test an instance with less vertices. This instance is uploaded to my Github. The four Figures 4.2, 4.3, 4.4, and 4.5 show the graph generated by Delaunay triangulation, solutions from the traditional model, and the new model for instance 24 with 32 vertices in the underlying graph. In this instance, solving the LP relaxation of the new model directly gives an integral solution $(0, 28, 25, 24, 31)$, while solving the LP relaxation of the traditional model still provides a factional solution.

Figure 4.1: result of 100 instances

Figure 4.2: Delaunay Trianglation for l24v32

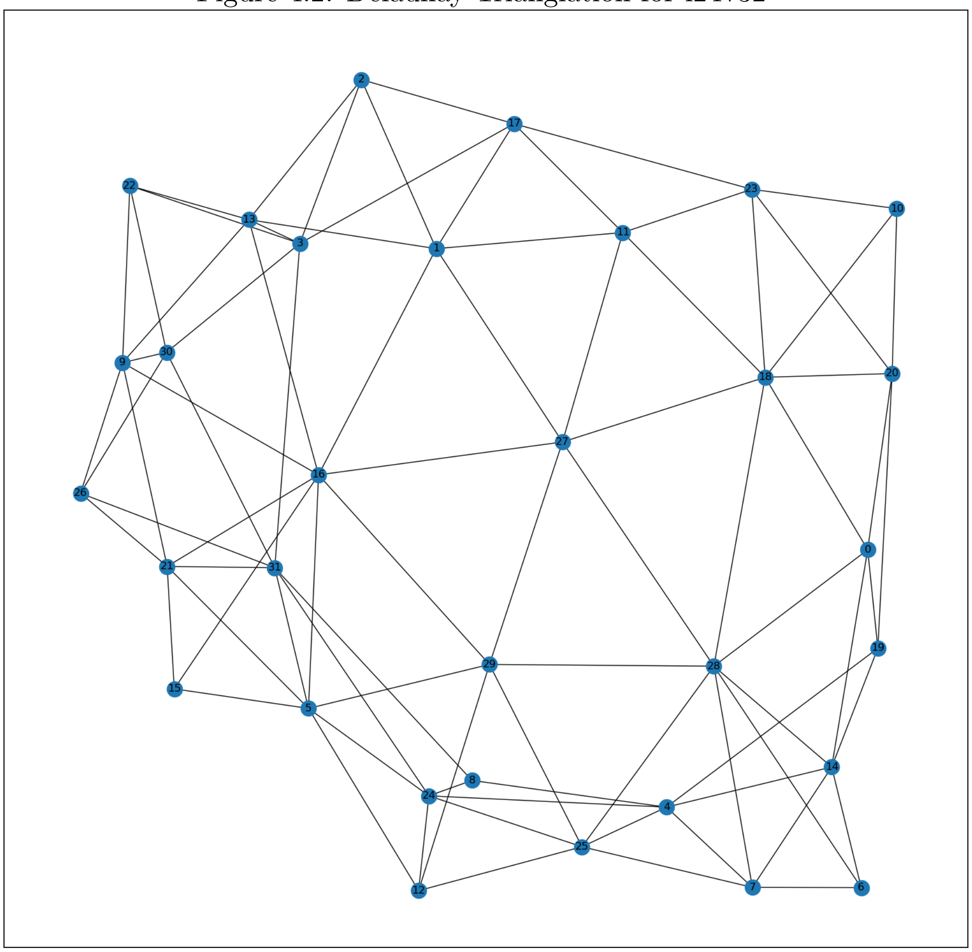


Figure 4.3: Path Solution of Traditional Modal (LP) for l24v32

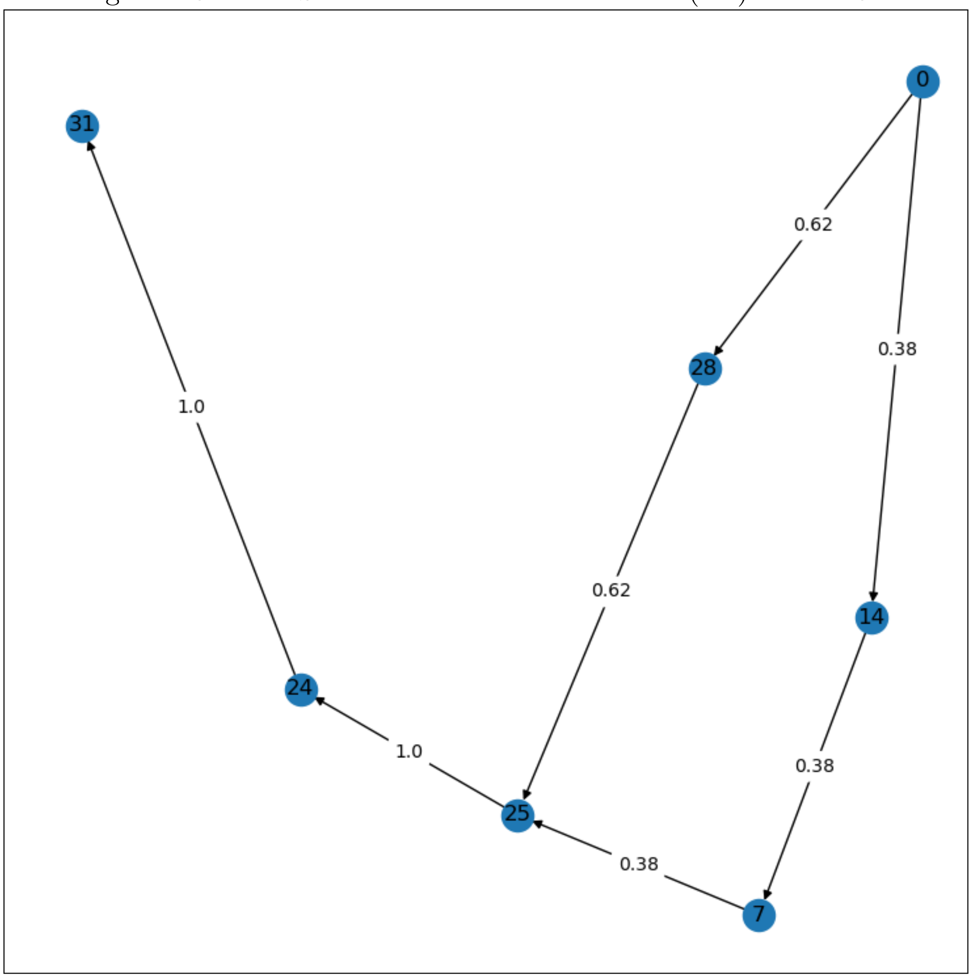


Figure 4.4: Flow Solution of New Modal (LP) for l24v32

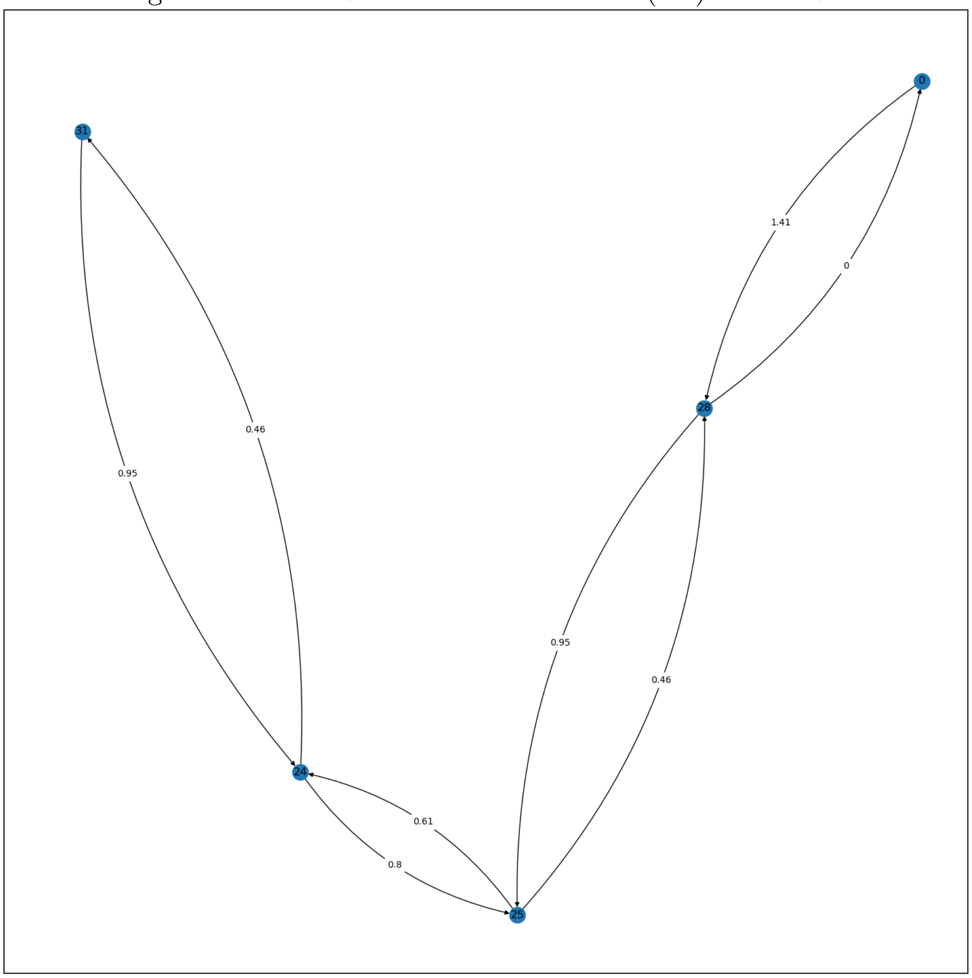
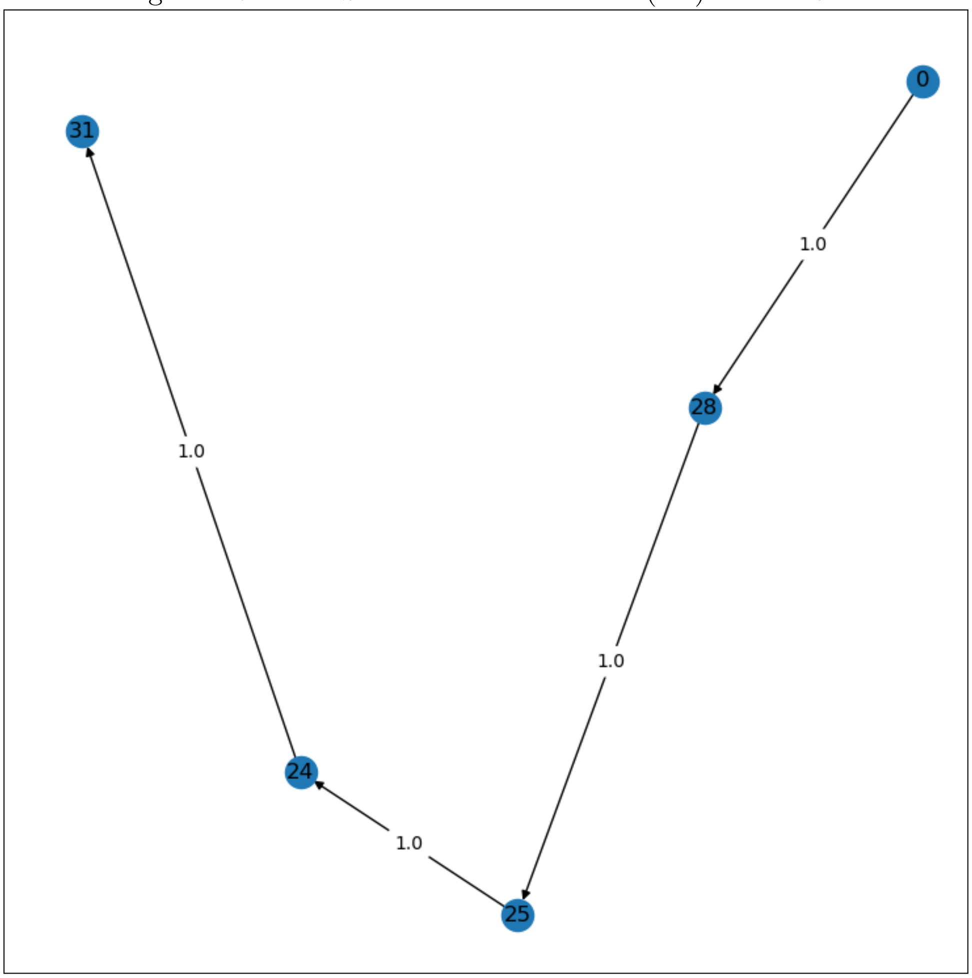


Figure 4.5: Path Solution of New Modal (LP) for l24v32



Chapter 5

Open Questions

During our research, there are some found open questions. They might be worth to look into.

Question 1

Observation 10 states that the capacity parameter Q shouldn't be excessively high. Indeed, over all feasible paths from sink to target, Q should lie between the highest total demand and the lowest total demand. The expectation of demand on every node is 0.5, therefore, Q could be a linear function with respect to the average of length of all possible paths from sink to target in the graph. In our experiment, we choose $Q = 0.25\sqrt{n}$ where n is the number of vertices in instances, and this parameter works well. So, maybe we can have the following conjecture:
Conjecture:
The expectation of diameter of a Delaunay Triangulation generated by n random-position points on a plane is asymptotic to $\mathcal{O}(\sqrt{n})$

Question 2

The optimal solution for the LP-relaxation of traditional model with positive objective function is the convex combination of two paths,where one path exceeds capacity but has a low cost and the other obeys the capacity constraint with a higher cost. We are also interested in characterizing the optimal solution for the LP-relaxation of this new model.

Some tested instances show the optimal solution can be a combination of 3 paths, and cycles (backward arcs) could appear in optimal solution. What we could show is if the cycles in optimal solution are backward arcs, then backward arcs can only appear at flow split point or flow merge point. However, characterizing cycles is not finished yet.

Conjecture:

With positive objective function, cycles in the optimal solution have length 2.

Chapter 6

Summary

Summary

In this thesis, we delve into Baldacci's two-commodity flow model for the Vehicle Routing Problem. Deriving implicit equations from the original model and analyzing the redundancy within these equations, leads to an estimation of the upper bound for the dimension of LP-relaxation polytopes. By observing how dimension reduces, we identify some non-facet-defining inequalities. In a uniform demand case, where all consumers share the same demand, we determine the dimension of MIP polytops and identify some facet-defining inequalities.

Following Baldacci's idea, a model for the Resource-Constrained Shortest Path Problem can be proposed. Through a basic example, we show the potential advantages of our model. Using the similar analysis as previous part, along with constructive proofs, we determine the dimensions of LP-relaxation polytopes. Similar methods lead to the dimensions of MIP polytopes and facet-defining inequality in the uniform demand case.

Furthermore, we introduce some cuts for our new model, which can completely describe the MIP polytopes in a simple case. Finally, we compare our model with the traditional model on some graphs generated by Delaunay triangulation.

Zusammenfassung

In dieser Dissertation tauchen wir in Baldaccis Zwei-Güter-Fluss-Modell für das Vehicle Routing Problem ein. Durch das Ableiten impliziter Gleichungen aus dem ursprünglichen Modell und die Analyse der Redundanz innerhalb dieser Gleichungen kommen wir zu einer Schätzung der Obergrenze für die Dimension von LP-Entspannungspolytopen. Durch die Beobachtung, wie sich die Dimension verringert, identifizieren wir einige nicht-facettendefinierende Ungleichungen. Im Fall einer einheitlichen Nachfrage, bei der alle Verbraucher dieselbe Nachfrage haben, bestimmen wir die Dimension von MIP-Polytopen und identifizieren einige facettendefinierende Ungleichungen.

In Anlehnung an Baldaccis Idee kann ein Modell für das ressourcenbeschränkte Kürzeste-Wege-Problem vorgeschlagen werden. Anhand eines grundlegenden Beispiels zeigen wir die potenziellen Vorteile unseres Modells. Unter Verwendung einer ähnlichen Analyse wie im vorherigen Teil, zusammen mit konstruktiven Beweisen, bestimmen wir die Dimensionen von LP-Entspannungspolytopen. Ähnliche Methoden führen zu den Dimensionen von MIP-Polytopen und facettendefinierenden Ungleichungen im Fall einer einheitlichen Nachfrage.

Darüber hinaus führen wir einige Schnitte für unser neues Modell ein, die die MIP-Polytope in einem einfachen Fall vollständig beschreiben können. Schließlich vergleichen wir unser Modell mit dem traditionellen Modell anhand einiger durch Delaunay-Triangulation generierter Graphen.

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