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Solutions to PFPL Problems

Chapter 2 Exercises

2.1 Prove that for every **nat** m, **nat** n, there exists a unique **nat** p satisfying the judgement $\max(m; n; p)$.

Proof. We define the function $\max : \textbf{nat} \times \textbf{nat} \rightarrow \textbf{nat}$ with the following rules:

$$\frac{\textbf{nat}\ b}{\max(\textbf{zero}; b)} \qquad\qquad \frac{\max(\textbf{nat}\ n; \textbf{nat}\ m; \textbf{nat}\ p)}{\max(\textbf{succ}(n); \textbf{succ}(m); \textbf{succ}(p))} \tag{1}$$

Let $\mathcal{P}(m)$ be the proposition that if **nat** n then $\exists p$ s.t. $\max(m; n; p)$.

To prove existence, we note that $\mathcal{P}(\textbf{zero})$ for all **nat** n, given rule 1.1, which settles our base case. Now if we suppose that **nat** m, **nat** n, $\mathcal{P}(m)$, we conclude by rule 1.2 that $\mathcal{P}(\textbf{succ}(m))$.

To prove uniqueness, we first note that if n is **zero** then for all **nat** m we have $\mathcal{P}(m)$ with p is m. Now suppose $\max(m; n; p_1)$ and $\max(m; n; p_2)$. Letting $n = \textbf{succ}(n'), m = \textbf{succ}(m'), p_1 = \textbf{succ}(p_1'), p_2 = \textbf{succ}(p_2')$, we conclude by the inductive hypothesis that p_1' is p_2' and so p_1 is p_2. □

2.2 Prove that the judgement **hgt** defines a function from trees to natural numbers.

Proof. To show that **hgt** defines a function, we need to show that for all $t \in$ trees there exists some $n \in \mathbb{N}$, such that $\textbf{hgt}(t; n)$. We use induction:

For our base case suppose t is **empty**. Then by our first rule we have that $\textbf{hgt}(t; \textbf{zero})$.

Now consider t **tree**, where $\text{node}(t_1; t_2)$ **tree**. By the inductive hypothesis, we have that $\textbf{hgt}(t_1; n_1)$ and $\textbf{hgt}(t_2; n_2)$ for some **nat** n_1 and **nat** n_2. By exercise 2.1, we know there exists some **nat** n, such that $\max(n_1; n_2; n)$. Therefore by the second rule in this problem, we conclude that

$$\cfrac{\cfrac{\textbf{hgt}(t_1; n_1) \quad \textbf{hgt}(t_2; n_2) \quad \max(n_1; n_2; n)}{\textbf{hgt}(\textbf{node}(t_1; t_2); n)}}{\textbf{hgt}(t; n)}$$

□

2.3 Give a simultaneous inductive definition of a ordered variadic trees.

$$\frac{}{\textbf{empty forest}}$$

$$\frac{f\ \textbf{forest}}{\textbf{node}(\text{f})\ \textbf{tree}}$$

$$\frac{f\ \textbf{forest} \quad t\ \textbf{tree}}{\text{cons(t;f)}\ \textbf{forest}}$$

2.4 Give an inductive definition of the height of a variadic tree.

$$\frac{}{\textbf{hgtForest}(\textbf{empty};\ \textbf{zero})}$$

$$\frac{\textbf{hgtForest}(f;h)}{\textbf{hgtTree}(\textbf{node}(f);\ \textbf{succ}(n))}$$

$$\frac{\textbf{hgtForest}(f;h_1) \quad \textbf{hgtTree}(t;h_2) \quad \max(h_1;h_2;h)}{\textbf{hgtForest}(\text{cons}(t;f);h)}$$

2.5 Give an inductive definition of the binary natural numbers.

$$\frac{}{\textbf{zero bnn}} \qquad \frac{b\ \textbf{bnn}}{\text{twice}(b)\ \textbf{bnn}} \qquad \frac{b\ \textbf{bnn}}{\text{twiceplusone}(b)\ \textbf{bnn}}$$

2.6 Give an induction definition for the sum of binary natural numbers.

First the auxiliary definition of $\textbf{succ}(()p;q)$.

$$\frac{}{\textbf{succ}(\textbf{zero};\text{twiceplusone}(\textbf{zero}))} \qquad \frac{}{\textbf{succ}(\text{twice}(p);\text{twiceplusone}(p))} \qquad \frac{\textbf{succ}(p;q)}{\textbf{succ}(\text{twiceplusone}(p);\text{twice}(q))}$$

Then we define each of the cases of input for sum:

$$\frac{}{\text{sum}(\textbf{zero};\textbf{zero};\textbf{zero})}$$

$$\frac{\text{sum}(m;n;k)}{\text{sum}(\text{twice}(m);\ \text{twice}(n);\ \text{twice}(k))}$$

$$\frac{\text{sum}(m;n;k)}{\text{sum}(\text{twice}(m);\ \text{twiceplusone}(n);\ \text{twiceplusone}(k))}$$

$$\frac{\text{sum}(m;n;k)}{\text{sum}(\text{twiceplusone}(m);\ \text{twiceplusone}(n);\ \text{twice}(\textbf{succ}(k))}$$