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Solutions to PFPL Problems

Chapter 3

3.1 Give an inductive definition of the *length* of a combinator as the number of occurrences of S and K within it.

$$\frac{}{\text{len}(k;1)}$$

$$\frac{}{\text{len}(s;1)}$$

$$\frac{a \text{ comb} \quad n \text{ nat} \quad \text{len}(a;n)}{\text{len}(\text{ap}(a;s);\mathbf{succ}(n))}$$

$$\frac{a \text{ comb} \quad n \text{ nat} \quad \text{len}(a;n)}{\text{len}(\text{ap}(s;a);\mathbf{succ}(n))}$$

$$\frac{a \text{ comb} \quad n \text{ nat} \quad \text{len}(a;n)}{\text{len}(\text{comb}(a;k);\mathbf{succ}(n))}$$

$$\frac{a \text{ comb} \quad n \text{ nat} \quad \text{len}(a;n)}{\text{len}(\text{comb}(k;a);\mathbf{succ}(n))}$$

3.2 Prove that if $x \mid x \text{ comb} \vdash_{\mathcal{C}} a_2 \text{ comb}$, and a_1 comb, then $[a_1/x]a_2$ comb.

Proof. Extend $\mathcal{C}$ with the axiom, a_1 comb. Substitute x for a_1, and by our first hypothesis, we conclude that $x \mid [a_2/x]x \text{ comb} \vdash_{\mathcal{C}} [a_2/]a_2$ comb, and proceed by induction. □

3.3 Show that $x \mid x \text{ comb} \vdash_{\mathcal{C}\cup\mathcal{E}} s\ k\ k\ x \equiv x$

Proof.

$$\frac{k \text{ comb} \quad k \text{ comb} \quad x \text{ comb}}{s\ k\ k\ x \equiv (k\ x)(k\ x)} \quad (3.16\text{f})$$

$$\frac{x \text{ comb} \quad (k\ x) \text{ comb}}{(k\ x)(k\ x) \equiv x} \quad (3.16\text{e})$$

$$\frac{s\ k\ k\ x \equiv (k\ x)(k\ x) \quad (k\ x)(k\ x) \equiv x}{s\ k\ k\ x \equiv x} \quad (3.16\text{c})$$

□

3.4 Show that if $x \mid x \text{ comb} \vdash_{\mathcal{C}} a \text{ comb}$, then there is a combinator a', written $[x]a$, such that $x \mid x \text{ comb} \vdash_{\mathcal{C}\cup\mathcal{E}} a'x \equiv a$.

Proof. We define $[x]$ with the following rules:

$$\frac{}{[x]x \equiv s\ k\ k}$$

$$\frac{}{[x]s \equiv k\ s}$$

$$\frac{}{[x]k \equiv k\ k}$$

$$\frac{[x]a_1 \equiv a_1' \quad [x]a_2 \equiv a_2'}{[x]a_1 a_2 \equiv s\ ([x]a_1)\ ([x]a_2)}$$

To finish the proof, use the first three axioms as base cases, then use the inductive hypothesis to write a comb as $a_1\ a_2$, and use the fourth rule. □

3.5 Prove that bracket abstraction is non-compositional by exhibiting a and b such that a comb and

$$x\ y \mid x \text{ comb } y \text{ comb} \vdash_{\mathcal{C}} b \text{ comb}$$

such that $[a/y]\,([x]\,b) \neq [x]([a/y]b)$.

Proof. Consider the case that b is y. If y is k, and a is s, for example, then we can show the contradiction:

$$[a/y]([x]\ y) = [a/k](k\ k) = a\ a = s\ s \neq k\ s = [x]\ s = [x]\ a = [x]([a/y]y)$$

We can simplify the definition of bracket abstraction, by defining it as

$$x \mid x \text{ comb} \vdash_{\mathcal{C}} [x]\ a \equiv k\ a, \text{ when } x \neq a, \text{ otherwise } [x]\ a \equiv s\ k\ k\ a$$

□

3.6 Give an inductive definition of the judgement b closed, which specifies that b has no free occurrences of the variables in $\mathcal{X}$.

First note that variables $x_1, \ldots x_n \notin \mathcal{X}$. Now we specify the rules:

$$\frac{x_1, \ldots, x_n \mid x_1 \text{ closed}, \ldots x_n \text{ closed} \vdash a_1 \text{ closed} \quad a_2 \text{ closed}}{x_1, \ldots, x_n \mid x_1 \text{ closed}, \ldots x_n \text{ closed} \vdash \text{ap}(a_1, a_2) \text{ closed}}$$

$$\frac{x_1, \ldots, x_n \mid x_1 \text{ closed}, \ldots x_n \text{ closed } y \text{ closed} \vdash b \text{ closed}}{x_1, \ldots, x_n \mid x_1 \text{ closed}, \ldots x_n \text{ closed } y \text{ closed} \vdash \lambda(y.b) \text{ closed}}$$

We see that $x_1, \ldots, x_n$ are closed, because we presuppose that they do not belong to the variable set $\mathcal{X}$. This would be implicitly true, if we did not first state that $x_1, \ldots, x_n \notin \mathcal{X}$.