System Modeling, part 2

Marc Claesen

July 23, 2015

1. Nonlinear systems & linearization

2. System identification (cont)
   - Grey box identification
   - Black box identification

Outline

Nonlinear systems

In this course we focus on the linear state-space representation:

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- *clipping*: e.g. $\alpha \leq x(t) \leq \beta$.

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Linearizing is done via first order Taylor expansions:

$$\begin{cases} \frac{dx}{dt} = \frac{d\Delta x}{dt} = f(x,u) = f(x_e + \Delta x, u_e + \Delta u), \\ y_e + \Delta y = g(x,u) = g(x_e + \Delta x, u_e + \Delta u). \end{cases}$$

Example: decalcification plant

Used to reduce concentration of calcium hydroxide in water:

- chemical reaction: $Ca(OH)_2 + CO_2 \rightarrow CaCO_3 + H_2O$
- reaction speed: $r = c[Ca(OH)_2][CO_2]$
- rate of change of concentration:

$$\frac{d[Ca(OH)_2]}{dt} = \frac{k}{V} - \frac{r}{V},$$
$$\frac{d[CO_2]}{dt} = \frac{u}{V} - \frac{r}{V},$$

  with inflow rates k and u in mol/s and tank volume V in L.
- input u: inflow of CO_2, output: $[Ca(OH)_2]$

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Linearization of the decalcification plant

For small deviations near the equilibrium:

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\begin{aligned}
\frac{d\Delta x_1}{dt} &= -\frac{c}{V}[CO_2]_{eq}\Delta x_1 - \frac{c}{V}[Ca(OH)_2]_{eq}\Delta x_2, \\
\frac{d\Delta x_2}{dt} &= -\frac{c}{V}[CO_2]_{eq}\Delta x_1 - \frac{c}{V}[Ca(OH)_2]_{eq}\Delta x_2 + \frac{\Delta u}{V}, \\
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$$y(t) = [Ca(OH)_2]$$

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White box modeling: based on first principles.

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Most popular approaches are forms of black box identification.

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"*All models are wrong, but some are useful.*" - George E. P. Box

Linear regression

Consider input matrix $\mathbf{X}$, output vector $\mathbf{y}$ and residuals ϵ:

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The OLS estimate minimizes the sum-of-squares of errors, i.e.:

$$\hat{\theta}_{OLS} = \arg\min_{\theta} \sum_{i=1}^{N} \Big(y(i) - \sum_{j=1}^{d} X(i,j)\theta(j)\Big)^2$$

Linear regression with ordinary least squares

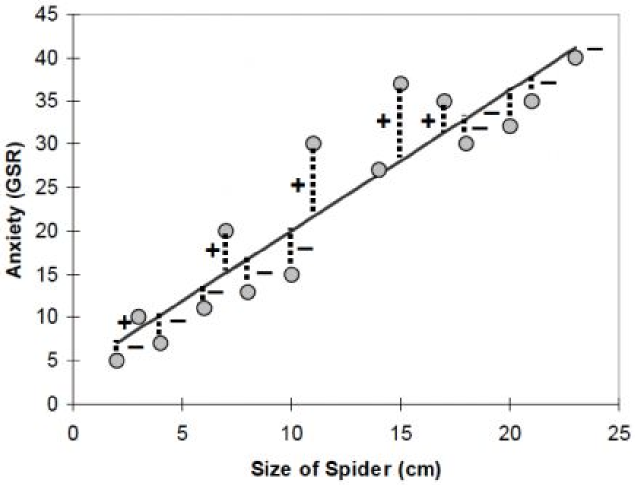

Image taken from `http://freakonometrics.hypotheses.org/2348`.

Maximum likelihood estimation

The maximum likelihood estimate $\hat{\theta}_{ML}$ is the parameter vector that maximizes the likelihood $\mathcal{L}(\cdot)$ of observing the (known) outputs $\mathbf{y}$, given the (known) inputs $\mathbf{X}$:

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Example: least squares estimators are the maximum likelihood estimators if the associated residuals ϵ are normally distributed.

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Additionally accounts for measurement errors in inputs.
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Typically described via *latent variables*:

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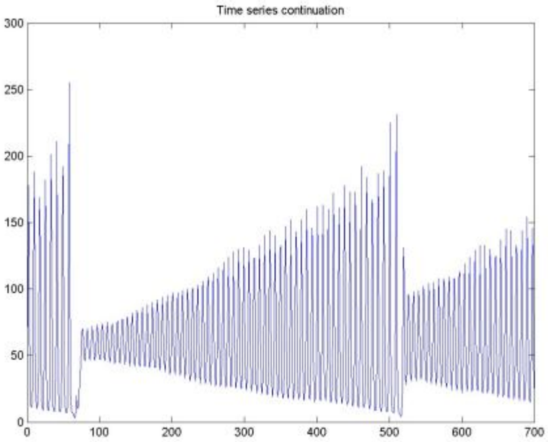
Time series: Santa Fe laser



Modelling the Santa Fe laser

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$\rightarrow$ for given N, we can estimate θ via grey box methods.

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This laser can be treated as an autonomous discrete time system:

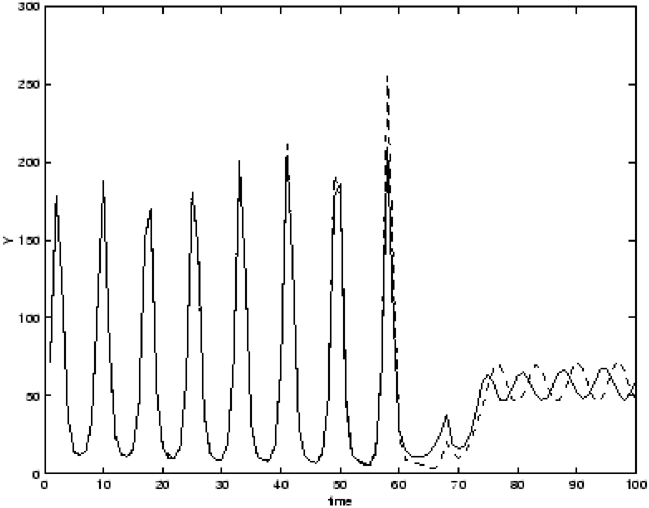
$$\begin{cases} x[k+1] = f\big(x[k-N+1], \dots, x[k]\big), \\ y[k] = x[k]. \end{cases}$$

The output depends on the past N states & no inputs.
$\rightarrow$ how large is N? $\rightarrow$ **unknown structure**

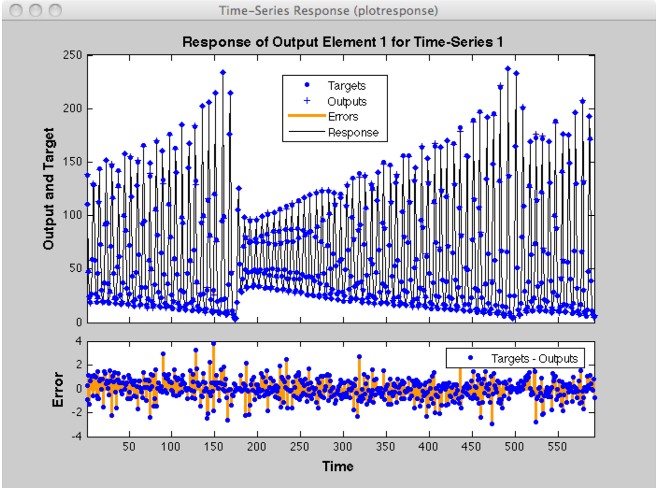
Treat it as a regression problem with N inputs: $y = f\big(X_1, \dots, X_N\big)$.
$\rightarrow$ lets say linear, i.e. $y = \mathbf{X}\theta$ $\rightarrow$ **unknown parameters** $\theta \in \mathbb{R}^N$.
$\rightarrow$ for given N, we can estimate θ via grey box methods.

Nonlinear models can be obtained via machine learning methods.
$\rightarrow$ neural networks, support vector machine, random forest, ...

Predictions of a least-squares support vector machine



Predictions of an artificial neural network



Neural network: biological

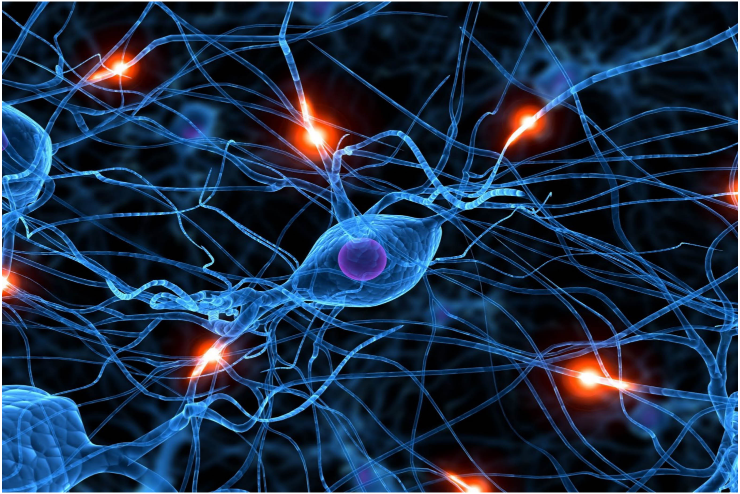


Image taken from http://www.extremetech.com/wp-content/uploads/2013/09/340.jpg.

Structure of a single neuron

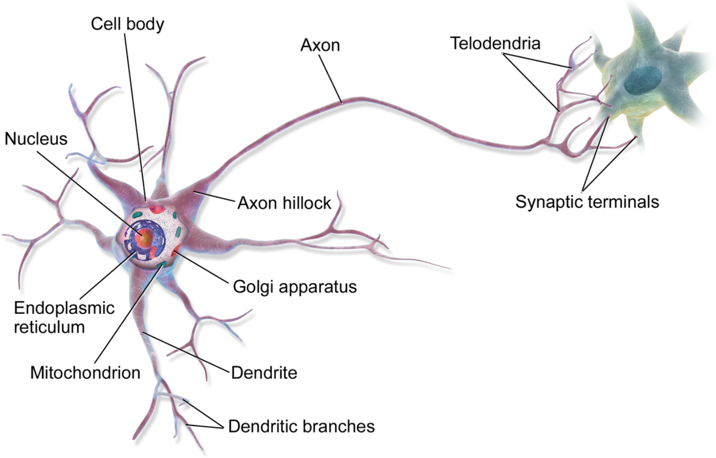


Image taken from `http://en.wikipedia.org/wiki/File:Blausen_0657_MultipolarNeuron.png`.

Neural network: artificial