Chapter 5: Continuous time systems

July 10, 2015

Outline

Linear differential equations (LDE) are of the following form:

$$L[y(t)] = f(t),$$

where L is some linear operator.

Linear differential equations: definitions 1/2

Linear differential equations (LDE) are of the following form:

$$L[y(t)] = f(t),$$

where L is some linear operator.

The linear operator L is of the following form:

$$L_n(y) = \sum_{i=0}^{n} A_i(t) \frac{d^{n-i} y}{dt^{n-i}},$$

with given functions $A_{1:n}$.

Linear differential equations: definitions 1/2

Linear differential equations (LDE) are of the following form:

$$L[y(t)] = f(t),$$

where L is some linear operator.

The linear operator L is of the following form:

$$L_n(y) = \sum_{i=0}^{n} A_i(t) \frac{d^{n-i}y}{dt^{n-i}},$$

with given functions $A_{1:n}$.

The **order of a LDE** is the index of the highest derivative of y.

Linear differential equations: definitions 2/2

$$L_n(y) = \sum_{i=0}^{n} A_i(t) \frac{d^{n-i}y}{dt^{n-i}} = f(t).$$

- y is a scalar function $\rightarrow$ **ordinary differential equation** (ODE)

Linear differential equations: definitions 2/2

$$L_n(y) = \sum_{i=0}^{n} A_i(t) \frac{d^{n-i} y}{dt^{n-i}} = f(t).$$

- y is a scalar function $\rightarrow$ **ordinary differential equation** (ODE)
- y is a vector function $\rightarrow$ **partial differential equation** (PDE)

Linear differential equations: definitions 2/2

$$L_n(y) = \sum_{i=0}^{n} A_i(t) \frac{d^{n-i}y}{dt^{n-i}} = f(t).$$

- y is a scalar function $\rightarrow$ **ordinary differential equation** (ODE)
- y is a vector function $\rightarrow$ **partial differential equation** (PDE)

- $f = 0 \rightarrow$ **homogeneous equation**
  $\rightarrow$ solutions are called **complementary functions**

Linear differential equations: definitions 2/2

$$L_n(y) = \sum_{i=0}^{n} A_i(t) \frac{d^{n-i} y}{dt^{n-i}} = f(t).$$

- y is a scalar function $\rightarrow$ **ordinary differential equation** (ODE)
- y is a vector function $\rightarrow$ **partial differential equation** (PDE)

- $f = 0 \rightarrow$ **homogeneous equation**
  $\rightarrow$ solutions are called **complementary functions**

- if $A_{0:n}(t)$ are constants (ie. not functions of time), the LDE is said to have **constant coefficients**

Example: radioactive decay 1/2

Let $N(t)$ be the number of radioactive atoms at time t, then:

$$\frac{dN(t)}{dt} = -kN(t),$$

for some constant $k > 0$.

Example: radioactive decay 1/2

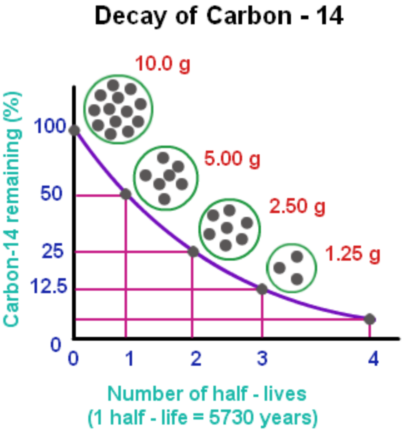
Let $N(t)$ be the number of radioactive atoms at time t, then:

$$\frac{dN(t)}{dt} = -kN(t),$$

for some constant $k > 0$.

This is a first order homogeneous LDE with constant coefficients.

Example: radioactive decay 2/2



Solving homogeneous LDEs with constant coefficients 1/3

Solutions of LDEs must be of the form e^{zt} with $z \in \mathbb{C}$.

Solutions of LDEs must be of the form e^{zt} with $z \in \mathbb{C}$.

We assume an LDE with constant coefficients:

$$\sum_{i=0}^{n} A_i y^{(n-i)} = 0.$$

Solving homogeneous LDEs with constant coefficients 1/3

Solutions of LDEs must be of the form e^{zt} with $z \in \mathbb{C}$.

We assume an LDE with constant coefficients:

$$\sum_{i=0}^{n} A_i y^{(n-i)} = 0.$$

Replacing $y = e^{zt}$ leads to:

$$\sum_{i=0}^{n} A_i z^{n-i} e^{zt} = 0$$

Solutions of LDEs must be of the form e^{zt} with $z \in \mathbb{C}$.

We assume an LDE with constant coefficients:

$$\sum_{i=0}^{n} A_i y^{(n-i)} = 0.$$

Replacing $y = e^{zt}$ leads to:

$$\sum_{i=0}^{n} A_i z^{n-i} e^{zt} = 0$$

Dividing by e^{zt} yields the nth order **characteristic polynomial**:

$$F(z) = \sum_{i=0}^{n} A_i z^{n-i} = 0.$$

Characteristic equation:

$$F(z) = \sum_{i=0}^{n} A_i z^{n-i} = 0.$$

Characteristic equation:

$$F(z) = \sum_{i=0}^{n} A_i z^{n-i} = 0.$$

1. Solving the polynomial $F(z)$ yields n zeros z_1 to z_n.
2. Substituting a given zero z_i into e^{zt} gives a solution $e^{z_i t}$.

Characteristic equation:

$$F(z) = \sum_{i=0}^{n} A_i z^{n-i} = 0.$$

1. Solving the polynomial $F(z)$ yields n zeros z_1 to z_n.
2. Substituting a given zero z_i into e^{zt} gives a solution $e^{z_i t}$.

Homogeneous LDEs obey the superposition position:
$\rightarrow$ any linear combination of solutions $e^{z_1 t}, \ldots, e^{z_n t}$ is a solution

Characteristic equation:

$$F(z) = \sum_{i=0}^{n} A_i z^{n-i} = 0.$$

1. Solving the polynomial $F(z)$ yields n zeros z_1 to z_n.
2. Substituting a given zero z_i into e^{zt} gives a solution $e^{z_i t}$.

Homogeneous LDEs obey the superposition position:
$\rightarrow$ any linear combination of solutions $e^{z_1 t}, \ldots, e^{z_n t}$ is a solution
$\rightarrow$ $e^{z_1 t}, \ldots, e^{z_n t}$ form a basis of the solution space of the LDE

Solving homogeneous LDEs with constant coefficients 2/3

Characteristic equation:

$$F(z) = \sum_{i=0}^{n} A_i z^{n-i} = 0.$$

1. Solving the polynomial $F(z)$ yields n zeros z_1 to z_n.
2. Substituting a given zero z_i into e^{zt} gives a solution $e^{z_i t}$.

Homogeneous LDEs obey the superposition position:
$\rightarrow$ any linear combination of solutions $e^{z_1 t}, \ldots, e^{z_n t}$ is a solution
$\rightarrow$ $e^{z_1 t}, \ldots, e^{z_n t}$ form a basis of the solution space of the LDE

The specific linear combination depends on initial conditions.

Example:

$$y^{(4)}(t) - 2y^{(3)}(t) + 2y^{(2)}(t) - 2y^{(1)}(t) + y(t) = 0.$$

Example:

$$y^{(4)}(t) - 2y^{(3)}(t) + 2y^{(2)}(t) - 2y^{(1)}(t) + y(t) = 0.$$

This is a 4th order homogeneous LDE with constant coefficients.

Example:

$$y^{(4)}(t) - 2y^{(3)}(t) + 2y^{(2)}(t) - 2y^{(1)}(t) + y(t) = 0.$$

This is a 4th order homogeneous LDE with constant coefficients.

The corresponding characteristic equation:

$$F(z) = z^4 - 2z^3 + 2z^2 - 2z + 1 = 0.$$

Example:

$$y^{(4)}(t) - 2y^{(3)}(t) + 2y^{(2)}(t) - 2y^{(1)}(t) + y(t) = 0.$$

This is a 4th order homogeneous LDE with constant coefficients.

The corresponding characteristic equation:

$$F(z) = z^4 - 2z^3 + 2z^2 - 2z + 1 = 0.$$

The zeros of $F(z)$ are ($j = \sqrt{-1}$):

$$z_1 = j, \quad z_2 = -j, \quad z_{3,4} = 1.$$

Example:

$$y^{(4)}(t) - 2y^{(3)}(t) + 2y^{(2)}(t) - 2y^{(1)}(t) + y(t) = 0.$$

This is a 4th order homogeneous LDE with constant coefficients.

The corresponding characteristic equation:

$$F(z) = z^4 - 2z^3 + 2z^2 - 2z + 1 = 0.$$

The zeros of $F(z)$ are ($j = \sqrt{-1}$):

$$z_1 = j, \quad z_2 = -j, \quad z_{3,4} = 1.$$

These zeros correspond to the following basis functions t:

$$e^{jt}, \quad e^{-jt}, \quad e^{t}, \quad te^{t}.$$

Outline

The Laplace transform

The Laplace transform of $f(t)$, for all real numbers $t \geq 0$:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt.$$

The Laplace transform

The Laplace transform of $f(t)$, for all real numbers $t \geq 0$:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

The parameter $s = \sigma + j\omega$ is the complex number frequency.

The Laplace transform

The Laplace transform of $f(t)$, for all real numbers $t \geq 0$:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)dt.$$

The parameter $s = \sigma + j\omega$ is the complex number frequency.

The initial value theorem states $f(0^+) = \lim_{s \to \infty} sF(s)$.

The Laplace transform of $f(t)$, for all real numbers $t \geq 0$:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt.$$

The parameter $s = \sigma + j\omega$ is the complex number frequency.

The initial value theorem states $f(0^+) = \lim_{s\to\infty} sF(s)$.

The final value theorem states $f(\infty) = \lim_{s\to 0} sF(s)$,
if all poles of $sF(s)$ are in the left half plane (ie. real part < 0).

Important properties of the Laplace transform

property	**time domain**	**s-domain**
linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$

property	**time domain**	**s-domain**
linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
differentiation	$f^{(1)}(t)$	$sF(s) - f(0)$

Important properties of the Laplace transform

property	**time domain**	***s*-domain**
linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
differentiation	$f^{(1)}(t)$	$sF(s) - f(0)$
integration	$\int_0^t f(\tau)d\tau = (u * f)(t)$	$\frac{1}{s}F(s)$

Important properties of the Laplace transform

property	time domain	s-domain
linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
differentiation	$f^{(1)}(t)$	$sF(s) - f(0)$
integration	$\int_0^t f(\tau)d\tau = (u * f)(t)$	$\frac{1}{s}F(s)$
convolution	$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$	$F(s) \cdot G(s)$

Important properties of the Laplace transform

property	time domain	s-domain
linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
differentiation	$f^{(1)}(t)$	$sF(s) - f(0)$
integration	$\int_0^t f(\tau)d\tau = (u * f)(t)$	$\frac{1}{s}F(s)$
convolution	$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$	$F(s) \cdot G(s)$
time scaling	$f(at)$	$\frac{1}{a}F(\frac{s}{a})$

Important properties of the Laplace transform

property	**time domain**	***s*-domain**
linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
differentiation	$f^{(1)}(t)$	$sF(s) - f(0)$
integration	$\int_0^t f(\tau)d\tau = (u * f)(t)$	$\frac{1}{s}F(s)$
convolution	$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$	$F(s) \cdot G(s)$
time scaling	$f(at)$	$\frac{1}{a}F(\frac{s}{a})$
time shifting	$f(t-a)u(t-a)$	$e^{-as}F(s)$

Important properties of the Laplace transform

property	**time domain**	**s-domain**
linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
differentiation	$f^{(1)}(t)$	$sF(s) - f(0)$
integration	$\int_0^t f(\tau)d\tau = (u * f)(t)$	$\frac{1}{s}F(s)$
convolution	$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$	$F(s) \cdot G(s)$
time scaling	$f(at)$	$\frac{1}{a}F(\frac{s}{a})$
time shifting	$f(t-a)u(t-a)$	$e^{-as}F(s)$

with $u(t) = \int_\infty^t \delta(t)dt$ (Heaviside) and $\delta(t)$ the Dirac delta.

Inverse Laplace transform

The inverse Laplace transform converts s-domain to time domain:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{j2\pi}\int_{\gamma-jT}^{\gamma+jT} e^{st}F(s)ds.$$

Inverse Laplace transform

The inverse Laplace transform converts s-domain to time domain:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{j2\pi}\int_{\gamma-jT}^{\gamma+jT} e^{st}F(s)ds.$$

Practically, the inverse Laplace transform takes two steps:

1. write $F(s)$ in terms of partial fractions
2. transform each term in the partial fraction based on tables of s/t-domain pairs (course notes p 4.32-4.33)

Outline

The Laplace transform can be used to solve LDEs with given initial conditions (the previous approach gave us the basis functions).

Solving LDEs with the Laplace transform 1/3

The Laplace transform can be used to solve LDEs with given initial conditions (the previous approach gave us the basis functions).

This is done by using the following property (differentiation):

$$\mathcal{L}\{f^{(1)}\} = sF(s) - f(0),$$
$$\mathcal{L}\{f^{(2)}\} = s^2F(s) - sf(0) - f^{(1)}(0).$$

Solving LDEs with the Laplace transform 1/3

The Laplace transform can be used to solve LDEs with given initial conditions (the previous approach gave us the basis functions).

This is done by using the following property (differentiation):

$$\mathcal{L}\{f^{(1)}\} = sF(s) - f(0),$$
$$\mathcal{L}\{f^{(2)}\} = s^2F(s) - sf(0) - f^{(1)}(0).$$

Via induction, the Laplace transform of the nth order derivative:

$$\mathcal{L}\{f^{(n)}\} = s^nF(s) - \sum_{i=1}^{n} s^{n-i}f^{(n-i)}(0)$$

$$\mathcal{L}\{f^{(n)}\} = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(n-i)}(0)$$

Solving LDEs with the Laplace transform 2/3

$$\mathcal{L}\{f^{(n)}\} = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(n-i)}(0)$$

We want to solve the following LDE:

$$\sum_{i=0}^{n} A_i y^{(n-i)}(t) = f(t),$$

$$y^{(i)}(0) = c_i \quad \forall i = 0 \ldots n.$$

$$\mathcal{L}\{f^{(n)}\} = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(n-i)}(0)$$

We want to solve the following LDE:

$$\sum_{i=0}^{n} A_i y^{(n-i)}(t) = f(t),$$
$$y^{(i)}(0) = c_i \quad \forall i = 0 \dots n.$$

Via the linearity of the Laplace transform:

$$\sum_{i=0}^{n} A_i \mathcal{L}\{y^{(n-i)}(t)\} = \mathcal{L}\{f(t)\}$$

Solving LDEs with the Laplace transform 3/3

$$\sum_{i=0}^{n} A_i \mathcal{L}\{y^{(n-i)}(t)\} = \mathcal{L}\{f(t)\} \tag{1}$$

$$\mathcal{L}\{f^{(n)}\} = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(n-i)}(0) \tag{2}$$

$$\sum_{i=0}^{n} A_i \mathcal{L}\{y^{(n-i)}(t)\} = \mathcal{L}\{f(t)\} \tag{1}$$

$$\mathcal{L}\{f^{(n)}\} = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(n-i)}(0) \tag{2}$$

Expanding Eq. (2) into (1) yields:

$$Y(s)\sum_{i=0}^{n} A_i s^i - \sum_{i=1}^{n}\sum_{j=1}^{i} A_i s^{i-j} y^{j-1}(0) = F(s)$$

$$\sum_{i=0}^{n} A_i \mathcal{L}\{y^{(n-i)}(t)\} = \mathcal{L}\{f(t)\} \tag{1}$$

$$\mathcal{L}\{f^{(n)}\} = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(n-i)}(0) \tag{2}$$

Expanding Eq. (2) into (1) yields:

$$Y(s)\sum_{i=0}^{n} A_i s^i - \sum_{i=1}^{n}\sum_{j=1}^{i} A_i s^{i-j} y^{j-1}(0) = F(s)$$

The solution in the time domain is obtained via the inverse Laplace transform: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Outline

Observability

A measure of how well a system's internal states $\mathbf{x}$ can be inferred by knowledge of its outputs $\mathbf{y}$.

Observability

A measure of how well a system's internal states $\mathbf{x}$ can be inferred by knowledge of its outputs $\mathbf{y}$.

Formally, a system is said to be observable if, for any possible sequence of state and control vectors, the current state can be determined in finite time using only the outputs.

Observability

A measure of how well a system's internal states $\mathbf{x}$ can be inferred by knowledge of its outputs $\mathbf{y}$.

Formally, a system is said to be observable if, for any possible sequence of state and control vectors, the current state can be determined in finite time using only the outputs.

This holds for linear, time-invariant systems with n states if:

$$rank(\mathcal{O}) = n, \quad \mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}, \quad \mathcal{O} : \textbf{observability matrix}$$

Controllability

A measure of the ability to move a system around in its entire configuration space using only certain admissible manipulations.

Controllability

A measure of the ability to move a system around in its entire configuration space using only certain admissible manipulations.

A system is controllable if its state can be moved from any initial state $\mathbf{x}_0$ to any final state $\mathbf{x}_f$ via some finite sequence of inputs $\mathbf{u}_0 \ldots \mathbf{u}_f$.

Controllability

A measure of the ability to move a system around in its entire configuration space using only certain admissible manipulations.

A system is controllable if its state can be moved from any initial state $\mathbf{x}_0$ to any final state $\mathbf{x}_f$ via some finite sequence of inputs $\mathbf{u}_0 \ldots \mathbf{u}_f$.

A linear, time-invariant system with n states is controllable if:

$$rank(\mathcal{C}) = n, \quad \mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \ldots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix},$$

where $\mathcal{C}$ is called the **controllability matrix**.

Outline

Transfer function

The transfer function of input i to output j is defined as:

$$H_{i,j}(s) = \frac{Y_j(s)}{U_i(s)}, \quad \mathbf{U}(s) = \mathcal{L}\{u(t)\}, \quad \mathbf{Y}(s) = \mathcal{L}\{y(t)\}.$$

MIMO systems with n inputs and m outputs have $n \times m$ transfer functions, one for each input-output pair.

The transfer function of input i to output j is defined as:

$$H_{i,j}(s) = \frac{Y_j(s)}{U_i(s)}, \quad \mathbf{U}(s) = \mathcal{L}\{u(t)\}, \quad \mathbf{Y}(s) = \mathcal{L}\{y(t)\}.$$

MIMO systems with n inputs and m outputs have $n \times m$ transfer functions, one for each input-output pair.

The complex Laplace variable can be rewritten: $s = \sigma + j\omega$.

Transfer function

The transfer function of input i to output j is defined as:

$$H_{i,j}(s) = \frac{Y_j(s)}{U_i(s)}, \quad \mathbf{U}(s) = \mathcal{L}\{u(t)\}, \quad \mathbf{Y}(s) = \mathcal{L}\{y(t)\}.$$

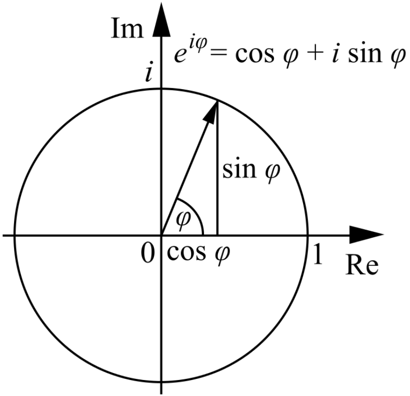
MIMO systems with n inputs and m outputs have $n \times m$ transfer functions, one for each input-output pair.

The complex Laplace variable can be rewritten: $s = \sigma + j\omega$.

The frequency response of a system can be analyzed via $\mathbf{H}(j\omega)$:

$$e^{\sigma + j\omega} = e^{\sigma}(\cos\omega + j\sin\omega).$$

Illustration of Euler's formula



Poles and zeros

In general, the transfer function can be written as:

$$H(s) = \frac{N(s)}{D(s)}.$$

Poles and zeros

In general, the transfer function can be written as:

$$H(s) = \frac{N(s)}{D(s)}.$$

The poles of $H(s)$ are zeros of $D(s)$, ie $\{s \ : \ D(s) = 0\}$.

- $|H(s)| = \infty$ if s is a pole.

Poles and zeros

In general, the transfer function can be written as:

$$H(s) = \frac{N(s)}{D(s)}.$$

The poles of $H(s)$ are zeros of $D(s)$, ie $\{s \ : \ D(s) = 0\}$.

- $|H(s)| = \infty$ if s is a pole.

The zeros of $H(s)$ are zeros of $N(s)$, ie $\{s \ : \ N(s) = 0\}$.

- $H(s) = 0$ if s is a zero.

Poles and zeros

In general, the transfer function can be written as:

$$H(s) = \frac{N(s)}{D(s)}.$$

The poles of $H(s)$ are zeros of $D(s)$, ie $\{s \ : \ D(s) = 0\}$.

- $|H(s)| = \infty$ if s is a pole.

The zeros of $H(s)$ are zeros of $N(s)$, ie $\{s \ : \ N(s) = 0\}$.

- $H(s) = 0$ if s is a zero.

Poles and zeros may cancel, ie. if $D(s) = N(s) = 0$ for some s.

Steady state response

The output of a linear time-invariant system yields consists of:

- a steady-state output $y_{ss}(t)$, which similar periodicity to $u(t)$
  $\rightarrow$ y_{ss} comprises the same frequencies as $u(t)$
- a transient output $y_{tr}(t)$
  $\rightarrow$ if the system is stable, then $\lim_{t\to\infty} y_{tr}(t) = 0$
  $\rightarrow$ $y_{tr}(t)$ depends on the initial state $\mathbf{x}_0(t)$ of the system

Steady state response

The output of a linear time-invariant system yields consists of:

- a steady-state output $y_{ss}(t)$, which similar periodicity to $u(t)$
  $\rightarrow$ y_{ss} comprises the same frequencies as $u(t)$
- a transient output $y_{tr}(t)$
  $\rightarrow$ if the system is stable, then $\lim_{t\rightarrow\infty} y_{tr}(t) = 0$
  $\rightarrow$ $y_{tr}(t)$ depends on the initial state $\mathbf{x}_0(t)$ of the system

If we apply an input $u(t) = cos(\alpha t + \theta)$, then:

$$y_{ss}(t) = |H(j\alpha)|cos(\alpha t + \theta + \angle H(j\alpha))$$

Steady state response

The output of a linear time-invariant system yields consists of:

- a steady-state output $y_{ss}(t)$, which similar periodicity to $u(t)$
  $\rightarrow$ y_{ss} comprises the same frequencies as $u(t)$
- a transient output $y_{tr}(t)$
  $\rightarrow$ if the system is stable, then $\lim_{t \to \infty} y_{tr}(t) = 0$
  $\rightarrow$ $y_{tr}(t)$ depends on the initial state $\mathbf{x}_0(t)$ of the system

If we apply an input $u(t) = cos(\alpha t + \theta)$, then:

$$y_{ss}(t) = |H(j\alpha)|cos(\alpha t + \theta + \angle H(j\alpha))$$

The steady-state output $y_{ss}(t)$ of a linear time invariant system:

- consists of signals of same frequencies as the input signal $u(t)$
- which may have been magnified and/or phase changed

Outline

Impulse response

The impulse response $h(t)$ of input i to output j is the output $y_j(t)$ of a system when an impulse $\delta(t)$ is applied at input $u_i(t)$.

Impulse response

The impulse response $h(t)$ of input i to output j is the output $y_j(t)$ of a system when an impulse $\delta(t)$ is applied at input $u_i(t)$.

The impulse response is the inverse Laplace transform of the transfer function $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

The impulse response $h(t)$ of input i to output j is the output $y_j(t)$ of a system when an impulse $\delta(t)$ is applied at input $u_i(t)$.

The impulse response is the inverse Laplace transform of the transfer function $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

For stable continuous time systems the impulse response always converges to 0:

$$\lim_{t\to\infty} h(t) = 0, \text{ because } \mathbf{D} = 0 \text{ and } \lim_{t\to\infty} \mathbf{x}(t) = 0.$$

Impulse response

The impulse response $h(t)$ of input i to output j is the output $y_j(t)$ of a system when an impulse $\delta(t)$ is applied at input $u_i(t)$.

The impulse response is the inverse Laplace transform of the transfer function $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

For stable continuous time systems the impulse response always converges to 0:

$$\lim_{t\to\infty} h(t) = 0, \text{ because } \mathbf{D} = 0 \text{ and } \lim_{t\to\infty} \mathbf{x}(t) = 0.$$

The speed of convergence depends on the position of the poles.

Time constant

The transfer function of first order systems can be written as:

$$H(s) = \frac{K}{\tau s + 1} \quad \leftrightarrow \quad h(t) = \frac{K}{\tau} e^{-t/\tau},$$

where τ is called the system's **time constant**.

Time constant

The transfer function of first order systems can be written as:

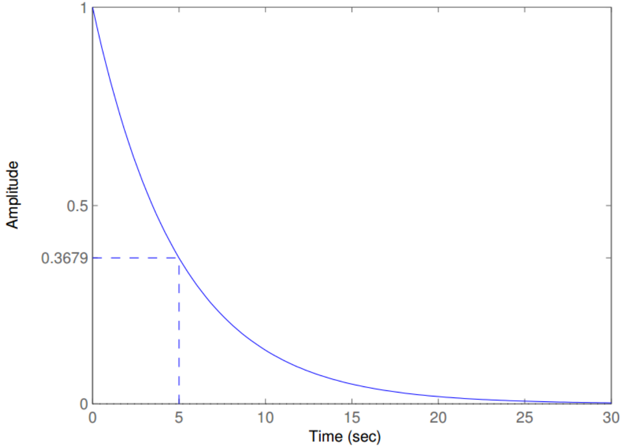
$$H(s) = \frac{K}{\tau s + 1} \quad \leftrightarrow \quad h(t) = \frac{K}{\tau} e^{-t/\tau},$$

where τ is called the system's **time constant**.

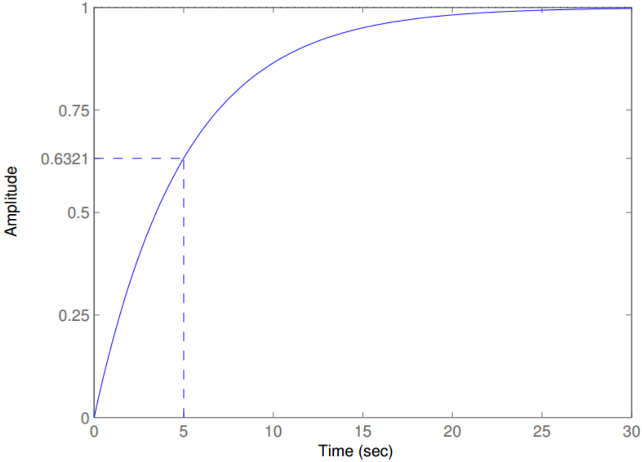
The time constant summarizes the speed of a system's dynamics:

- after τ seconds, the impulse response reaches $h(0)/e$.
- after τ seconds, the step response has reached $1 - e^{-1} \approx 63\%$ of its regime value.

Impulse response $H(s) = 5/(5s+1) \leftrightarrow h(t) = exp(-t/5)$



Step response $H(s) = 5/(5s+1) \leftrightarrow h(t) = exp(-t/5)$



Outline

1. Linear differential equations
2. Laplace transform
3. Solving LDEs with the Laplace transform
4. Properties of state-space representation
5. Transfer functions
   - Impulse response and time constant
   - Relationship between state space and transfer functions
6. Transient response analysis of first order and second order systems
   - First order systems
   - Second order systems

From state-space to transfer functions

We start from the linear state-space representation:

time domain   Laplace domain

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \quad \leftrightarrow \quad \begin{cases} s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{cases}$$

From state-space to transfer functions

We start from the linear state-space representation:

time domain Laplace domain

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \quad \leftrightarrow \quad \begin{cases} s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{cases}$$

A transfer function $\mathbf{H}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)}$ relates an input and an output in the Laplace-domain $\rightarrow$ to obtain it, we must eliminate $\mathbf{X}(s)$.

We start from the linear state-space representation:

time domain | Laplace domain

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \quad \leftrightarrow \quad \begin{cases} s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{cases}$$

A transfer function $\mathbf{H}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)}$ relates an input and an output in the Laplace-domain $\rightarrow$ to obtain it, we must eliminate $\mathbf{X}(s)$.

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{B}\mathbf{U}(s) \\ \mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \\ \Rightarrow \mathbf{Y}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) \\ \Rightarrow \mathbf{H}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \end{aligned}$$

Relationship between poles and eigenvalues of $\mathbf{A}$ 1/2

Poles are zeros of the denominator of $\mathbf{H}(s)$, e.g. those values of s for which $\mathbf{H}(s)$ is singular.

Relationship between poles and eigenvalues of $\mathbf{A}$ 1/2

Poles are zeros of the denominator of $\mathbf{H}(s)$, e.g. those values of s for which $\mathbf{H}(s)$ is singular.

The relationship between state-space representation (matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$) and transfer functions is given by

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Relationship between poles and eigenvalues of $\mathbf{A}$ 1/2

Poles are zeros of the denominator of $\mathbf{H}(s)$, e.g. those values of s for which $\mathbf{H}(s)$ is singular.

The relationship between state-space representation (matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$) and transfer functions is given by

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$H(s)$ cannot be computed when $(s\mathbf{I} - \mathbf{A})^{-1}$ does not exist, ie.

$$\det(s\mathbf{I} - \mathbf{A}) = 0$$

Relationship between poles and eigenvalues of $\mathbf{A}$ 1/2

Poles are zeros of the denominator of $\mathbf{H}(s)$, e.g. those values of s for which $\mathbf{H}(s)$ is singular.

The relationship between state-space representation (matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$) and transfer functions is given by

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$H(s)$ cannot be computed when $(s\mathbf{I} - \mathbf{A})^{-1}$ does not exist, ie.

$$\det(s\mathbf{I} - \mathbf{A}) = 0$$

The determinant is zero if s is an eigenvalue of $\mathbf{A}$.

Relationship between poles and eigenvalues of $\mathbf{A}$ 1/2

Poles are zeros of the denominator of $\mathbf{H}(s)$, e.g. those values of s for which $\mathbf{H}(s)$ is singular.

The relationship between state-space representation (matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$) and transfer functions is given by

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$H(s)$ cannot be computed when $(s\mathbf{I} - \mathbf{A})^{-1}$ does not exist, ie.

$$\det(s\mathbf{I} - \mathbf{A}) = 0$$

The determinant is zero if s is an eigenvalue of $\mathbf{A}$.
$\rightarrow$ all poles of $\mathbf{H}(s)$ are eigenvalues of $\mathbf{A}$

Relationship between poles and eigenvalues of $\mathbf{A}$ 2/2

Transfer functions only capture what is relevant to describe an input-output relationship, but not all states necessarily contribute.

Relationship between poles and eigenvalues of $\mathbf{A}$ 2/2

Transfer functions only capture what is relevant to describe an input-output relationship, but not all states necessarily contribute.
$\rightarrow$ *unobservable* modes of $\mathbf{A}$ are not poles in $\mathbf{H}(s)$.

Relationship between poles and eigenvalues of $\mathbf{A}$ 2/2

Transfer functions only capture what is relevant to describe an input-output relationship, but not all states necessarily contribute.
$\rightarrow$ *unobservable* modes of $\mathbf{A}$ are not poles in $\mathbf{H}(s)$.

Consider the following SISO system with 2 states:

$$\begin{bmatrix} sX_1(s) \\ sX_2(s) \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} + \begin{bmatrix} \beta \\ 2 \end{bmatrix} U(s)$$

$$Y(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$$

Transfer functions only capture what is relevant to describe an input-output relationship, but not all states necessarily contribute.
→ *unobservable* modes of **A** are not poles in $\mathbf{H}(s)$.

Consider the following SISO system with 2 states:

$$\begin{bmatrix} sX_1(s) \\ sX_2(s) \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} + \begin{bmatrix} \beta \\ 2 \end{bmatrix} U(s)$$

$$Y(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$$

The transfer function $H(s) = \frac{\beta}{s-\alpha}$ has only one pole ($s_1 = \alpha$).
→ not all eigenvalues of **A** are poles in transfer functions $\mathbf{H}(s)$.

Outline

Outline

The time response of a control system may be written as:

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

Where $y_{tr}(t)$ is the transient response and $y_{ss}(t)$ is the steady state response. Most important characteristic of dynamic system is absolute stability.

Transient Response

The time response of a control system may be written as:

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

Where $y_{tr}(t)$ is the transient response and $y_{ss}(t)$ is the steady state response. Most important characteristic of dynamic system is absolute stability.

- System is stable when returns to equilibrium if subject to initial condition
- System is critically stable when oscillations of the output continue forever
- System is unstable when output diverges without bound from equilibrium if subject to initial condition

Transient Response

The time response of a control system may be written as:

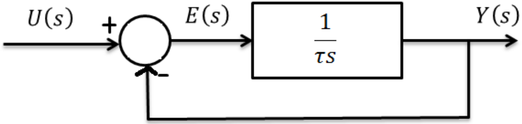
$$y(t) = y_{tr}(t) + y_{ss}(t)$$

Where $y_{tr}(t)$ is the transient response and $y_{ss}(t)$ is the steady state response. Most important characteristic of dynamic system is absolute stability.

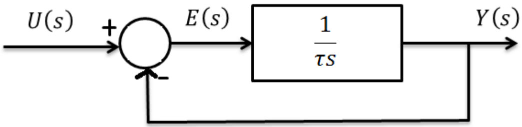
- System is stable when returns to equilibrium if subject to initial condition
- System is critically stable when oscillations of the output continue forever
- System is unstable when output diverges without bound from equilibrium if subject to initial condition

Transient response: when input of system changes, output does not change immediately but takes time to go to steady state

E.g.RC circuit, thermal system, ...

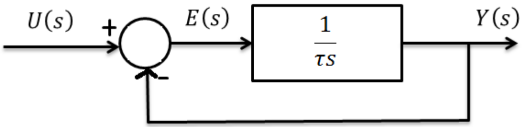


E.g.RC circuit, thermal system, ...



Transfer function is given by: $\frac{Y(s)}{U(s)} = \frac{1}{\tau s+1}$

E.g.RC circuit, thermal system, ...



Transfer function is given by: $\frac{Y(s)}{U(s)} = \frac{1}{\tau s+1}$

Unit step response

- Laplace of unit-step is $\frac{1}{s}$ $\rightarrow$ substituting $U(s) = \frac{1}{s}$ into equation $Y(s) = \frac{1}{s}\frac{1}{\tau s+1}$
- Expanding into partial fractions gives $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$

Unit step transient response

1. $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$

Unit step transient response

1. $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$

2. Taking the inverse Laplace transform

   $y(t) = 1 - e^{-\frac{t}{\tau}}$, for $t \geq 0$

Unit step transient response

1. $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$

2. Taking the inverse Laplace transform

   $y(t) = 1 - e^{-\frac{t}{\tau}}$, for $t \geq 0$

3. At $t = 0$, the output $y(t) = 0$

1. $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$
2. Taking the inverse Laplace transform

   $y(t) = 1 - e^{-\frac{t}{\tau}}$, for $t \geq 0$
3. At $t = 0$, the output $y(t) = 0$
4. At $t = \tau$, the output $y(t) = 0.632$, or $y(t)$ has reached 63.2%

   of its total change $y(\tau) = 1 - e^{-1} = 0.632$

Unit step transient response

1. $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$
2. Taking the inverse Laplace transform

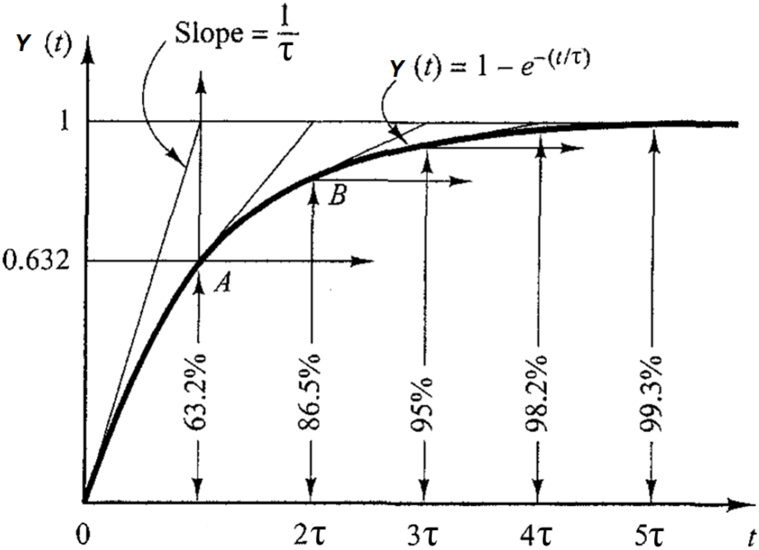
   $y(t) = 1 - e^{-\frac{t}{\tau}}$, for $t \geq 0$
3. At $t = 0$, the output $y(t) = 0$
4. At $t = \tau$, the output $y(t) = 0.632$, or $y(t)$ has reached 63.2%

   of its total change $y(\tau) = 1 - e^{-1} = 0.632$
5. Slope at time $t = 0$ is $\frac{1}{\tau}$

   $\frac{dy}{dt}|_{t=0} = \frac{1}{\tau}e^{-\frac{t}{\tau}}|_{t=0} = \frac{1}{\tau}$

   Where τ is called the system time constant

Unit step transient response



Unit ramp transient response

1. Laplace transform of unit ramp is $\frac{1}{s^2}$

   $Y(s) = \frac{1}{\tau s+1}\frac{1}{s^2}$

Unit ramp transient response

1. Laplace transform of unit ramp is $\frac{1}{s^2}$

   $Y(s) = \frac{1}{\tau s+1}\frac{1}{s^2}$

2. Expanding into partial fractions gives

   $Y(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s+1}$

Unit ramp transient response

1. Laplace transform of unit ramp is $\frac{1}{s^2}$

   $Y(s) = \frac{1}{\tau s+1}\frac{1}{s^2}$

2. Expanding into partial fractions gives

   $Y(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s+1}$

3. Taking the inverse Laplace transform

   $y(t) = t - \tau + \tau e^{-\frac{t}{\tau}}$, for $t \geq 0$

Unit ramp transient response

1. Laplace transform of unit ramp is $\frac{1}{s^2}$

   $Y(s) = \frac{1}{\tau s+1}\frac{1}{s^2}$

2. Expanding into partial fractions gives

   $Y(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s+1}$

3. Taking the inverse Laplace transform

   $y(t) = t - \tau + \tau e^{-\frac{t}{\tau}}$, for $t \geq 0$

4. The error signal $e(t)$ is then

   $e(t) = u(t) - y(t) = \tau(1 - e^{-\frac{t}{\tau}})$

Unit ramp transient response

1. Laplace transform of unit ramp is $\frac{1}{s^2}$

   $Y(s) = \frac{1}{\tau s+1}\frac{1}{s^2}$

2. Expanding into partial fractions gives

   $Y(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s+1}$

3. Taking the inverse Laplace transform

   $y(t) = t - \tau + \tau e^{-\frac{t}{\tau}}$, for $t \geq 0$

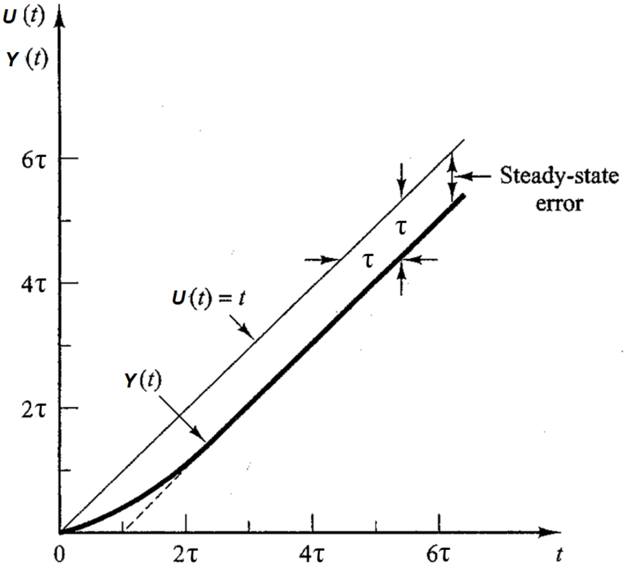
4. The error signal $e(t)$ is then

   $e(t) = u(t) - y(t) = \tau(1 - e^{-\frac{t}{\tau}})$

5. For t approaching infinity, $e(t)$ approaches τ

   $e(\infty) = \tau$

Unit ramp transient response



Unit-Impulse Response

For a unit-impulse input, $U(s) = 1$ and the output is

$$Y(s) = \frac{1}{\tau s + 1}$$

Unit-Impulse Response

For a unit-impulse input, $U(s) = 1$ and the output is

$$Y(s) = \frac{1}{\tau s + 1}$$

The inverse Laplace transform gives

$$y(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}, \text{ for } t \geq 0$$

Unit-Impulse Response

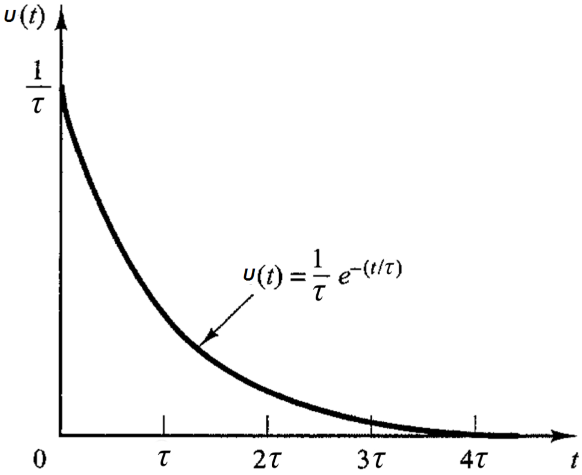
For a unit-impulse input, $U(s) = 1$ and the output is

$$Y(s) = \frac{1}{\tau s + 1}$$

The inverse Laplace transform gives

$$y(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}, \text{ for } t \geq 0$$

For $t \to +\infty$, $y(t) \to 0$


$u(t)$
$\frac{1}{\tau}$
$u(t) = \frac{1}{\tau} e^{-(t/\tau)}$
0
τ
2τ
3τ
4τ
t

Outline

Second order systems

A second order system can generally be written as:

$$\frac{Y(s)}{U(s)} = H(s) = \frac{as^2 + bs + c}{ds^2 + es + f}$$

A second order system can generally be written as:

$$\frac{Y(s)}{U(s)} = H(s) = \frac{as^2 + bs + c}{ds^2 + es + f}$$

A system where the closed-loop transfer function possesses two poles is called a second-order system

Second order systems

A second order system can generally be written as:

$$\frac{Y(s)}{U(s)} = H(s) = \frac{as^2 + bs + c}{ds^2 + es + f}$$

A system where the closed-loop transfer function possesses two poles is called a second-order system

If the transfer function has two real poles, the frequency response can be found by combining the effects of both poles

Sometimes the transfer function has two complex conjugate poles. In that case we have to find a different solution for finding the frequency response.

Second order systems

Sometimes the transfer function has two complex conjugate poles. In that case we have to find a different solution for finding the frequency response.

In order to study the transient behaviour, let us first consider the following simplified example of a second order system

$$H(s) = \frac{c}{ds^2 + es + c}$$

Step response second order system

1. $H(s) = \frac{c}{ds^2+es+c}$

1. $H(s) = \frac{c}{ds^2+es+c}$
2. The transfer function can be rewritten as:

$$H(s) = \frac{\frac{c}{d}}{s^2 + \frac{e}{d}s + \frac{c}{d}}$$
$$= \frac{\frac{c}{d}}{[s + \frac{e}{2d} + \sqrt{(\frac{e}{2d})^2 - \frac{c}{d}}][s + \frac{e}{2d} - \sqrt{(\frac{e}{2d})^2 - \frac{c}{d}}]}$$

1. $H(s) = \frac{c}{ds^2+es+c}$
2. The transfer function can be rewritten as:

$$H(s) = \frac{\frac{c}{d}}{s^2 + \frac{e}{d}s + \frac{c}{d}}$$
$$= \frac{\frac{c}{d}}{[s + \frac{e}{2d} + \sqrt{(\frac{e}{2d})^2 - \frac{c}{d}}][s + \frac{e}{2d} - \sqrt{(\frac{e}{2d})^2 - \frac{c}{d}}]}$$

3. The poles are complex conjugates if

$$e^2 - 4dc < 0$$

Step response second order system

1. $H(s) = \frac{c}{ds^2+es+c}$

2. The transfer function can be rewritten as:

$$H(s) = \frac{\frac{c}{d}}{s^2 + \frac{e}{d}s + \frac{c}{d}}$$
$$= \frac{\frac{c}{d}}{[s + \frac{e}{2d} + \sqrt{(\frac{e}{2d})^2 - \frac{c}{d}}][s + \frac{e}{2d} - \sqrt{(\frac{e}{2d})^2 - \frac{c}{d}}]}$$

3. The poles are complex conjugates if

$$e^2 - 4dc < 0$$

4. The poles are real if

$$e^2 - dc \geq 0$$

Step response second order system

To simplify the transient analysis, it is convenient to write

$$\frac{f}{d} = \omega_n^2,\ \frac{e}{d} = 2\zeta\omega_n = 2\sigma$$

Where
σ is the attenuation
ω_n is the natural frequency
ζ is the damping ratio

Step response second order system

To simplify the transient analysis, it is convenient to write

$$\frac{f}{d} = \omega_n^2,\ \frac{e}{d} = 2\zeta\omega_n = 2\sigma$$

Where
σ is the attenuation
ω_n is the natural frequency
ζ is the damping ratio

The transfer function can now be rewritten as

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega s + \omega_n^2}$$

Which is called the standard form of the second-order system.

Step response second order system

To simplify the transient analysis, it is convenient to write

$$\frac{f}{d} = \omega_n^2,\ \frac{e}{d} = 2\zeta\omega_n = 2\sigma$$

Where
σ is the attenuation
ω_n is the natural frequency
ζ is the damping ratio

The transfer function can now be rewritten as

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega s + \omega_n^2}$$

Which is called the standard form of the second-order system.

The dynamic behavior of the second-order system can then be described in terms of only two parameters ζ and ω_n

Step response second order system

If $0 < \zeta < 1$, the poles are complex conjugates and lie in the left-half s-plane

- The system is then called **underdamped**
- The **transient response is oscillatory**

Step response second order system

If $0 < \zeta < 1$, the poles are complex conjugates and lie in the left-half s-plane

- The system is then called **underdamped**
- The **transient response is oscillatory**

If $\zeta = 0$, the **transient response doesn't die out**

Step response second order system

If $0 < \zeta < 1$, the poles are complex conjugates and lie in the left-half s-plane

- The system is then called **underdamped**
- The **transient response is oscillatory**

If $\zeta = 0$, the **transient response doesn't die out**

If $\zeta = 1$, the system is called **critically damped**

Step response second order system

If $0 < \zeta < 1$, the poles are complex conjugates and lie in the left-half s-plane

- The system is then called **underdamped**
- The **transient response is oscillatory**

If $\zeta = 0$, the **transient response doesn't die out**

If $\zeta = 1$, the system is called **critically damped**

If $\zeta > 1$, the system is called **overdamped**

We will now look at the unit step response for each of these cases

Underdamped system

For the underdamped case ($0 < \zeta < 1$), the transfer function can be written as:

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

Underdamped system

For the underdamped case ($0 < \zeta < 1$), the transfer function can be written as:

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

Where ω_d is called the damped natural frequency

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

Underdamped system

For the underdamped case ($0 < \zeta < 1$), the transfer function can be written as:

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

Where ω_d is called the damped natural frequency

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

For a unit-step input we can write

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}$$

Which can be rewritten as partial fractions

$$
\begin{aligned}
Y(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
&= \frac{1}{s} - \frac{s + \zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} - \frac{\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\end{aligned}
$$

Underdamped system

Which can be rewritten as partial fractions

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} - \frac{\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

It can be shown that

$$\mathcal{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t} cos(\omega_d t)$$

$$\mathcal{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t} sin(\omega_d t)$$

Underdamped system

Therefore:

$$\mathcal{L}^{-1}\left[Y(s)\right] = y(t)$$

Underdamped system

Therefore:

$$\mathcal{L}^{-1}\left[Y(s)\right] = y(t)$$

$$= 1 - e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t))$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} sin(\omega_d t + tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta}))$$

Underdamped system

Therefore:

$$\mathcal{L}^{-1}\left[Y(s)\right] = y(t)$$

$$= 1 - e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t))$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} sin(\omega_d t + tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta}))$$

It can be seen that the frequency of the transient oscillation is the damped natural frequency ω_d and thus varies with the damping ratio ζ

The error signal is the difference between input and output

$$
\begin{aligned}
e(t) &= y(t) - u(t) \\
&= e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t))
\end{aligned}
$$

Underdamped system

The error signal is the difference between input and output

$$
\begin{aligned}
e(t) &= y(t) - u(t) \\
&= e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t))
\end{aligned}
$$

The error signal exhibits a damped sinusoidal oscillation

Underdamped system

The error signal is the difference between input and output

$$e(t) = y(t) - u(t)$$
$$= e^{-\zeta \omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t))$$

The error signal exhibits a damped sinusoidal oscillation

At steady state, or at $t = \infty$, the error goes to zero

Underdamped system

If damping $\zeta = 0$, the response becomes undamped

Underdamped system

If damping $\zeta = 0$, the response becomes undamped

- Oscillations continue indefinitely

Underdamped system

If damping $\zeta = 0$, the response becomes undamped

- Oscillations continue indefinitely
- Filling in $\zeta = 0$ into the equation for y(t) gives us $y(t) = 1 - cos(_n t)$, for $t \geq 0$

Underdamped system

If damping $\zeta = 0$, the response becomes undamped

- Oscillations continue indefinitely

- Filling in $\zeta = 0$ into the equation for y(t) gives us $y(t) = 1 - cos(_n t)$, for $t \geq 0$

- We see that the system now oscillates at the natural frequency ω_n

Underdamped system

If damping $\zeta = 0$, the response becomes undamped

- Oscillations continue indefinitely
- Filling in $\zeta = 0$ into the equation for y(t) gives us $y(t) = 1 - cos(_n t)$, for $t \geq 0$
- We see that the system now oscillates at the natural frequency ω_n
- If a linear system has any amount of damping, the undamped natural frequency cannot be observed experimentally, only ω_d can be observed

Underdamped system

If damping $\zeta = 0$, the response becomes undamped

- Oscillations continue indefinitely
- Filling in $\zeta = 0$ into the equation for y(t) gives us $y(t) = 1 - cos(_n t)$, for $t \geq 0$
- We see that the system now oscillates at the natural frequency ω_n
- If a linear system has any amount of damping, the undamped natural frequency cannot be observed experimentally, only ω_d can be observed
- ω_d is always lower than ω_n

Critically damped system

If the two poles of the system are equal, the system is critically damped and $\zeta = 1$

Critically damped system

If the two poles of the system are equal, the system is critically damped and $\zeta = 1$

For a unit-step, $R(s) = \frac{1}{s}$ and we can write

$$Y(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s}$$

Critically damped system

If the two poles of the system are equal, the system is critically damped and $\zeta = 1$

For a unit-step, $R(s) = \frac{1}{s}$ and we can write

$$Y(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s}$$

The inverse Laplace transform gives us

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) \text{ for } t \geq 0$$

Overdamped system

A system is overdamped ($\zeta > 1$) when the two poles are negative, real and unequal

Overdamped system

A system is overdamped ($\zeta > 1$) when the two poles are negative, real and unequal

For a unit-step $R(s) = \frac{1}{s}$, $Y(s)$ can be written as

$$Y(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n^2\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n^2\sqrt{\zeta^2 - 1})}$$

Overdamped system

A system is overdamped ($\zeta > 1$) when the two poles are negative, real and unequal

For a unit-step $R(s) = \frac{1}{s}$, $Y(s)$ can be written as

$$Y(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n^2\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n^2\sqrt{\zeta^2 - 1})}$$

The inverse Laplace transform is

$$y(t) = 1 + \frac{w_n}{2\sqrt{\zeta^2 - 1}}(\frac{e^{-s_1 t}}{s1} - \frac{e^{-s_2 t}}{s2}), \text{ for } t \geq 0$$

Overdamped system

A system is overdamped ($\zeta > 1$) when the two poles are negative, real and unequal

For a unit-step $R(s) = \frac{1}{s}$, $Y(s)$ can be written as

$$Y(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n^2\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n^2\sqrt{\zeta^2 - 1})}$$

The inverse Laplace transform is

$$y(t) = 1 + \frac{w_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{-s_1 t}}{s1} - \frac{e^{-s_2 t}}{s2}\right), \text{ for } t \geq 0$$

Where

$$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n \text{ and } s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$$

Overdamped system

$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$

Thus $y(t)$ includes two decaying exponential terms

Overdamped system

$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$

Thus $y(t)$ includes two decaying exponential terms

Overdamped system

$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$

Thus $y(t)$ includes two decaying exponential terms

- When $\zeta >> 1$, one of the two decreases much faster than the other, and then the faster decaying exponential may be neglected

Overdamped system

$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$

Thus $y(t)$ includes two decaying exponential terms

- When $\zeta >> 1$, one of the two decreases much faster than the other, and then the faster decaying exponential may be neglected
- If $-s_2$ is located much closer to the $j\omega$ axis than $-s_1$ ($|s_2| >> |s_1|$), then $-s_1$ may be neglected

Overdamped system

$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$

Thus $y(t)$ includes two decaying exponential terms

- When $\zeta >> 1$, one of the two decreases much faster than the other, and then the faster decaying exponential may be neglected
- If $-s_2$ is located much closer to the $j\omega$ axis than $-s_1$ ($|s_2| >> |s_1|$), then $-s_1$ may be neglected
- Once the faster decaying exponential term has disappeared, the response is similar to that of a first-order system

Overdamped system

$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$

Thus $y(t)$ includes two decaying exponential terms

- When $\zeta >> 1$, one of the two decreases much faster than the other, and then the faster decaying exponential may be neglected
- If $-s_2$ is located much closer to the $j\omega$ axis than $-s_1$ ($|s_2| >> |s_1|$), then $-s_1$ may be neglected
- Once the faster decaying exponential term has disappeared, the response is similar to that of a first-order system
- In that case, $H(s)$ can be approximated by

$$H(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} = \frac{s_2}{s + s_2}$$

Overdamped system

With the approximate transfer function, the unit-step response becomes

$$Y(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

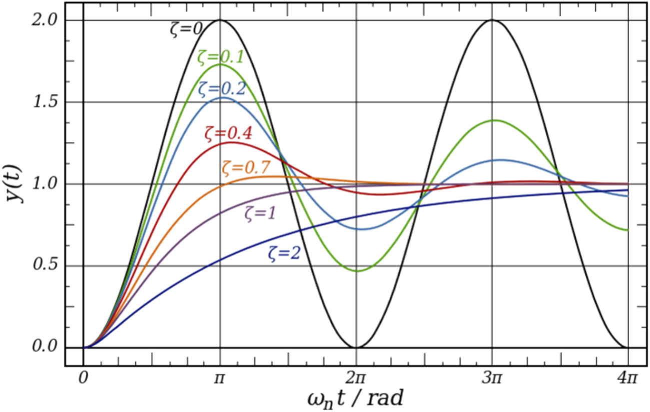
With the approximate transfer function, the unit-step response becomes

$$Y(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

The time response for the approximate transfer function is then given as

$$y(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}, \text{ for } t \leq 0$$

Response on a step function



Overshoot: Highest amplitude above steady state.

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Overshoot: Highest amplitude above steady state.

$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$

Rise Time: Time needed to reach the steady state for the first time. $t_r = \frac{1.8}{\omega_n}$

Second order systems - characteristics

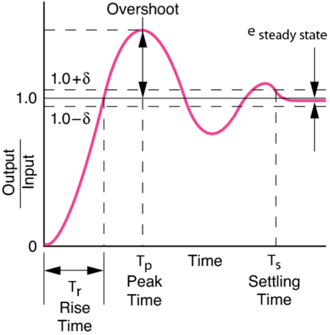
Overshoot: Highest amplitude above steady state.

$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$

Rise Time: Time needed to reach the steady state for the first time. $t_r = \frac{1.8}{\omega_n}$

Peak Time: Time to reach overshoot.

$t_p = \frac{\pi}{\omega_d}$



Second order systems - characteristics

Settling Time: Time needed to approximate the steady state.
$t_s = \frac{4.6}{\zeta\omega_n}$

Settling Time: Time needed to approximate the steady state.
$t_s = \frac{4.6}{\zeta\omega_n}$
Important: this formulas are only useable for $0 < \zeta < 1$!

Second order systems - characteristics

Settling Time: Time needed to approximate the steady state.
$t_s = \frac{4.6}{\zeta\omega_n}$
Important: this formulas are only useable for $0 < \zeta < 1$!

For: $\delta = \frac{0.02}{\sqrt{1-\zeta^2}}$

Second order systems - characteristics

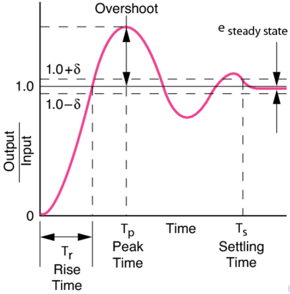
Settling Time: Time needed to approximate the steady state.
$t_s = \frac{4.6}{\zeta\omega_n}$
Important: this formulas are only useable for $0 < \zeta < 1$!

For: $\delta = \frac{0.02}{\sqrt{1-\zeta^2}}$

We find:

$$e^{-\zeta\omega_n T_s} < 0.02$$

$$T_s = \frac{4}{\omega_n\zeta}$$



Second order systems - resonace

The resonance frequency is the frequency at which the systems output has a larger amplitude than at other frequencies. This happens when underdamped functions oscillate at a greater magnitude than the input.

Second order systems - resonace

The resonance frequency is the frequency at which the systems output has a larger amplitude than at other frequencies. This happens when underdamped functions oscillate at a greater magnitude than the input.

An input with this frequency can sometime have catastrophic effects.

Second order systems - resonace

The resonance frequency is the frequency at which the systems output has a larger amplitude than at other frequencies. This happens when underdamped functions oscillate at a greater magnitude than the input.

An input with this frequency can sometime have catastrophic effects.

A different view on the Tacoma bridge disaster:
`https://www.youtube.com/watch?v=6ai2QFxStxo`

In fact the collapse was a result of a number of effects like Aerodynamic flutter and vortices. Read the full article here:
`http://www.ketchum.org/billah/Billah-Scanlan.pdf`

Second order systems - resonance

The resonance frequency is: $\omega_r = \omega_n\sqrt{1-\zeta^2}$

Second order systems - resonance

The resonance frequency is: $\omega_r = \omega_n \sqrt{1 - \zeta^2}$

Systems with a damping > 0.707 do not resonate

Second order systems - resonance

The resonance frequency is: $\omega_r = \omega_n\sqrt{1-\zeta^2}$

Systems with a damping > 0.707 do not resonate

The resonance frequency and the natural frequency are equal when a system has no damping.

The resonance frequency is: $\omega_r = \omega_n\sqrt{1-\zeta^2}$

Systems with a damping > 0.707 do not resonate

The resonance frequency and the natural frequency are equal when a system has no damping.

Another phenomenon with bridges and resonance is that many people marching with the same rhythm can cause a bridge to start resonating like the Angers bridge in 1850. A more recent example is the Millennium bridge in London who started resonating.

Second order systems - damping

When we want a system with no resonance, we choose one with damping < 0.707. This means a pole between 135° and 225°:

$$\arctan\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = +135^\circ$$

Second order systems - damping

When we want a system with no resonance, we choose one with damping < 0.707. This means a pole between 135° and 225°:

$$\arctan\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = +135^\circ$$

We mostly want a short settling time (< 4s). This results in another restriction on the poles of the system:

$$\tau_n = \frac{4}{\omega\zeta} < 4\text{s}$$

$$\omega_n\zeta > 1$$

Second order systems - damping