Outline

Impulse response

Definition

The impulse response $h(t)$ of input i to output j is the output $y_j(t)$ of a system when an impulse $\delta(t)$ is applied at input $u_i(t)$. The impulse response is the inverse Laplace transform of the transfer function $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

For stable continuous time systems the impulse response always converges to 0:

$$\lim_{t\to\infty} h(t) = 0, \text{ because } \mathbf{D} = 0 \text{ and } \lim_{t\to\infty} \mathbf{x}(t) = 0.$$

The speed of convergence depends on the position of the poles.

Time constant

Definition

The transfer function of first order systems can be written as:

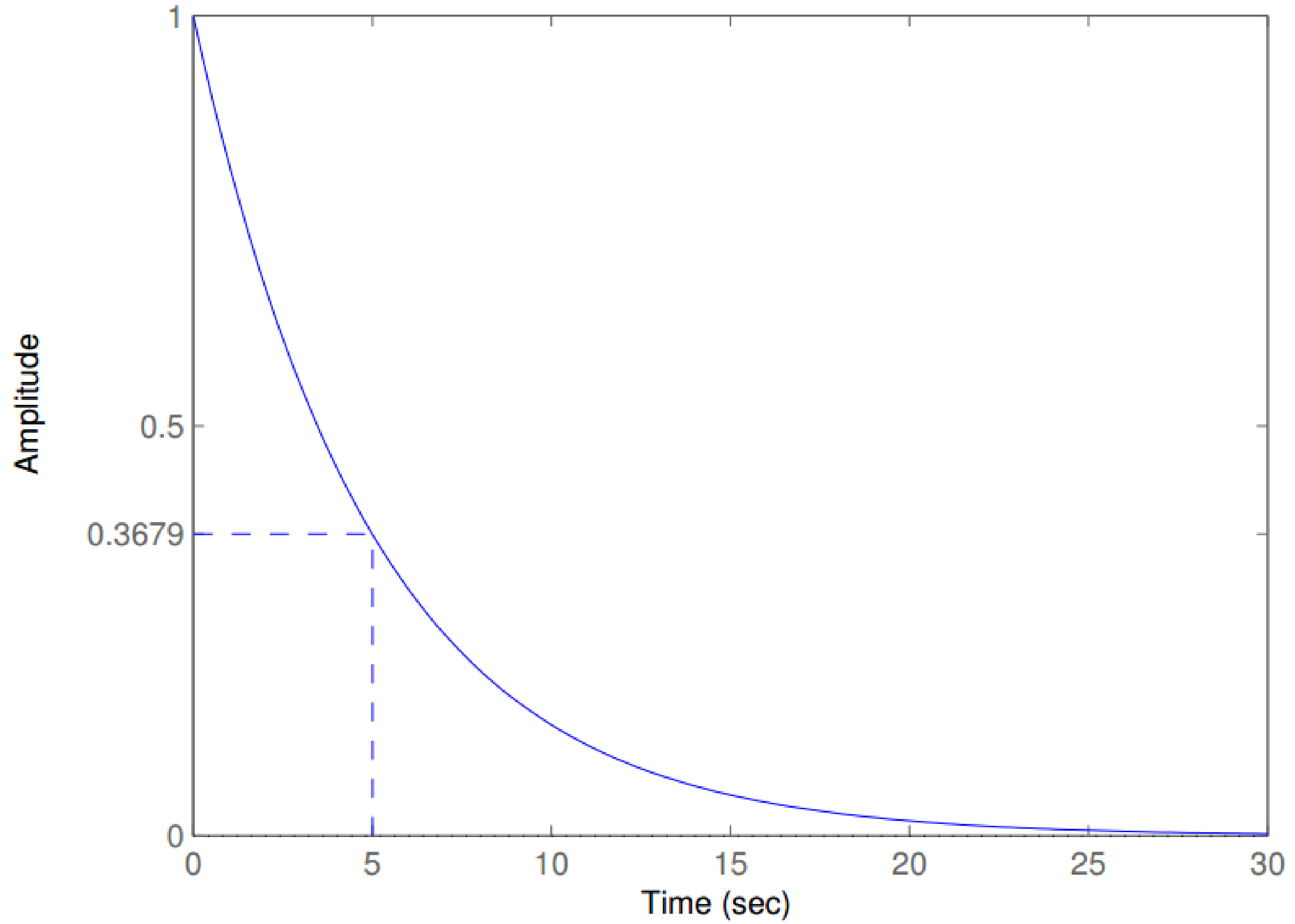
$$H(s) = \frac{K}{\tau s + 1} \quad \leftrightarrow \quad h(t) = \frac{K}{\tau} e^{-t/\tau},$$

where τ is called the system's **time constant**.

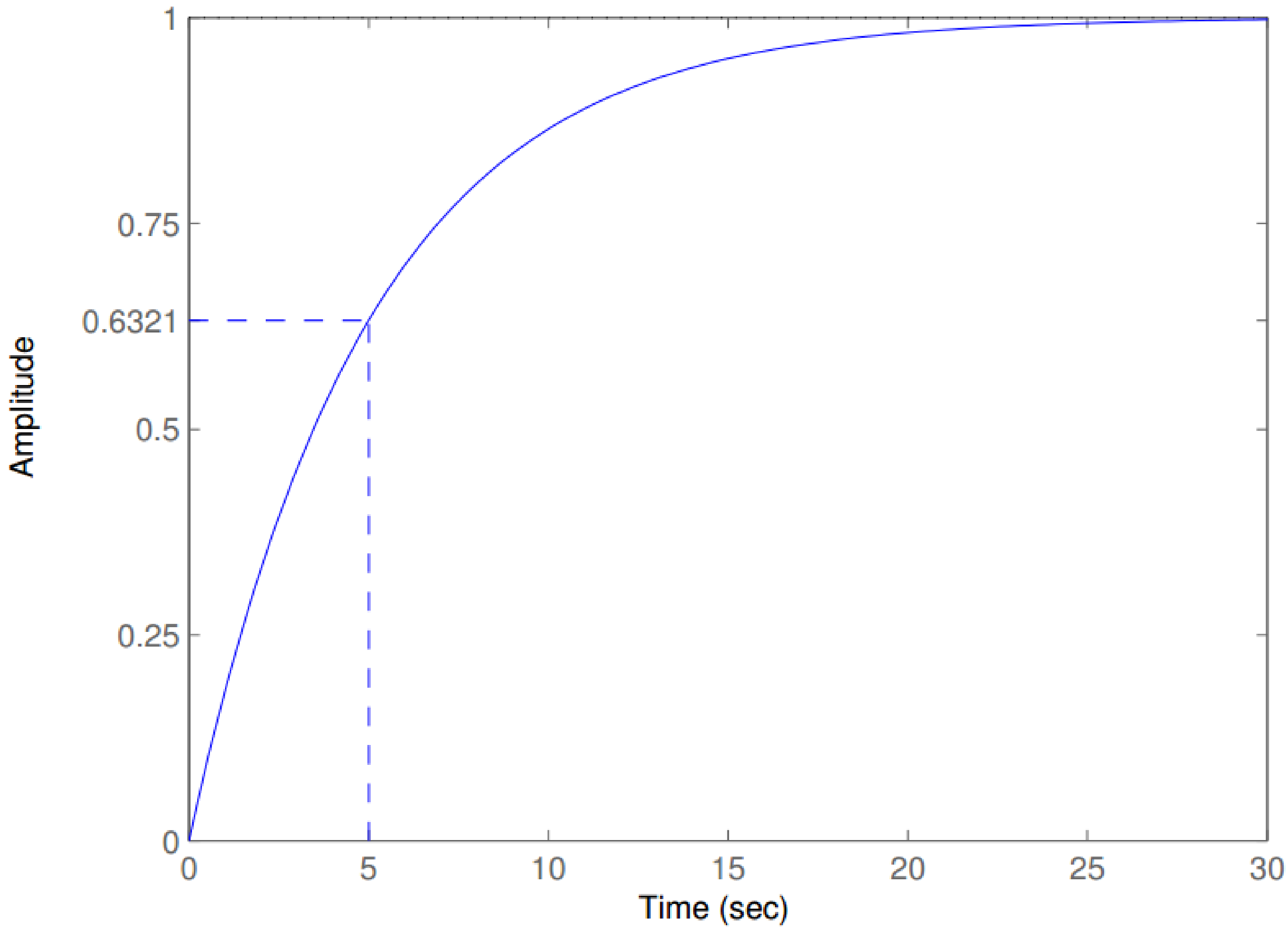
The time constant summarizes the speed of a system's dynamics:

- after τ seconds, the impulse response reaches $h(0)/e$.
- after τ seconds, the step response has reached $1 - e^{-1} \approx 63\%$ of its regime value.

Impulse response $H(s) = 5/(5s+1) \leftrightarrow h(t) = exp(-t/5)$



Step response $H(s) = 5/(5s+1) \leftrightarrow h(t) = exp(-t/5)$



Outline

From state-space to transfer functions

We start from the linear state-space representation:

time domain | Laplace domain

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \leftrightarrow \begin{cases} s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{cases}$$

A transfer function $\mathbf{H}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)}$ relates an input and an output in the Laplace-domain $\rightarrow$ to obtain it, we must eliminate $\mathbf{X}(s)$.

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\Rightarrow \mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s)$$

$$\Rightarrow \mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Relationship between poles and eigenvalues of **A** 1/2

Poles are zeros of the denominator of $\mathbf{H}(s)$, e.g. those values of s for which $\mathbf{H}(s)$ is singular.
The relationship between state-space representation (matrices **A**, **B**, **C** and **D**) and transfer functions is given by

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$H(s)$ cannot be computed when $(s\mathbf{I} - \mathbf{A})^{-1}$ does not exist, ie.

$$\det(s\mathbf{I} - \mathbf{A}) = 0$$

The determinant is zero if s is an eigenvalue of **A**.
$\rightarrow$ all poles of $\mathbf{H}(s)$ are eigenvalues of **A**.

Relationship between poles and eigenvalues of **A** 2/2

Transfer functions only capture what is relevant to describe an input-output relationship, but not all states necessarily contribute.
$\rightarrow$ *unobservable* modes of **A** are not poles in $\mathbf{H}(s)$.
Consider the following SISO system with 2 states:

$$\begin{bmatrix} sX_1(s) \\ sX_2(s) \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} + \begin{bmatrix} \beta \\ 2 \end{bmatrix} U(s)$$

$$Y(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$$

The transfer function $H(s) = \frac{\beta}{s-\alpha}$ has only one pole ($s_1 = \alpha$).
$\rightarrow$ not all eigenvalues of **A** are poles in transfer functions $\mathbf{H}(s)$.