Chapter 12: Lead and Lag Compensators

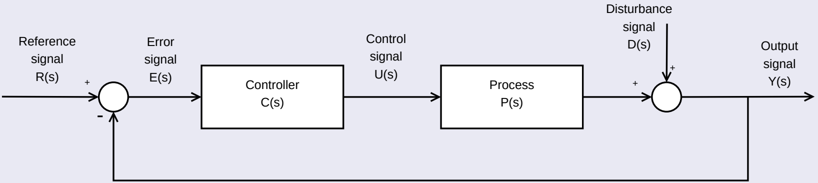
August 4, 2015

Outline

1. Definition of compensators
2. Lead compensators
3. Lag compensators

Lead Compensator vs Lag Compensator

Schematical representation



Transfer functions

Lead compensator : $C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with $0<\alpha<1$

Lag compensator : $C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}}$ with $\beta>1$

Lead Compensator vs Lag Compensator: zeros and poles

Transfer functions

Lead compensator : $C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with $0<\alpha<1$

Lag compensator : $C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}}$ with $\beta>1$

Zeros and poles

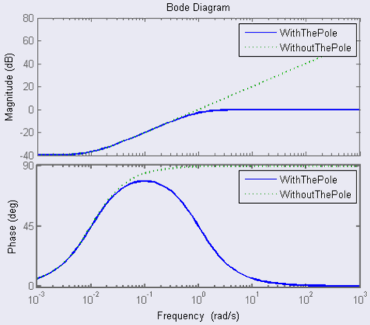
Zeros: $s = -\frac{1}{\tau}$
Poles: $s = -\frac{1}{\alpha\tau}$ or $s = -\frac{1}{\beta\tau}$
For lead compensators the pole lies more to the left in the complex plane than the zero and vice versa for lag compensators.

Outline

Transfer function and impact

Impact

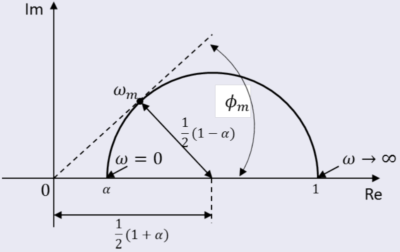
$C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with $0<\alpha<1$



Lead compensators

Determination of α

Use polar plot of $\frac{\alpha.(j\omega\tau+1)}{j\omega\alpha\tau+1}$



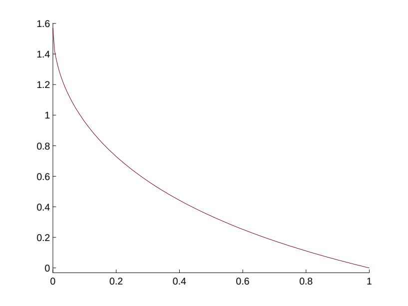
$\sin\phi_m = \frac{\frac{1}{2}.(1-\alpha)}{\frac{1}{2}.(1+\alpha)} = \frac{1-\alpha}{1+\alpha} \Rightarrow \alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$

This relation relates the maximum phase-lead angle and the value of α.

Lead compensators

Determination of α

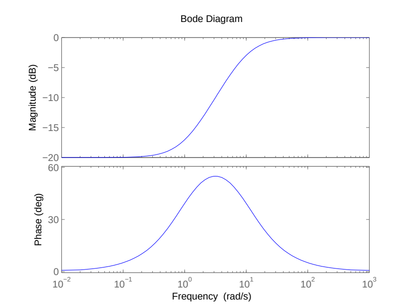
The figure shows the phase lead in function of α. Attention: the phase lead is in radians!



Usually, $\alpha \geqslant 0.05 \Rightarrow$ maximum phase lead about 65°.

Lead compensators

Determination of τ



The tangent point ω_m is the geometric mean of the two corner frequencies, so $\log \omega_m = \frac{1}{2}(\log \frac{1}{\tau} + \log \frac{1}{\alpha\tau})$ with $\tau = \frac{1}{\omega}$
$\Rightarrow \omega_m = \frac{1}{\sqrt{\alpha}\tau}$.
The lead compensator is a high-pass filter.

Lead compensators

Impact

- They push the pole of the closed loop system to the left.
  - Stabilisation of the system (see root locus)
  - Increase response speed (lead compensator will stimulate some larger frequencies)
- Increase of the phase margin: the phase of the lead compensator is positive for every frequency, and will hence only increase the phase.
- Thanks to the presence of a pole, the high frequencies (where most of the unwanted noise is located) are less amplified. Again, a lead compensator is a high-pass filter.

Lead compensators

Design with Bode plots

- Design process: tuning of the phase margin, with as a surplus (because we will have one extra degree of freedom) the tuning of the steady state error.
- Compensate for the excessive phase lag that is a result of the components of P(s).
- Increase in phase at gain crossover frequency (GCF) if GCF is around pole and zero of the lead compensator.
- Gain is impacted by the lead compensator: the GCF of P(s).C(s) is not equal to the GCF of P(s).

Lead compensators

Design

- Required increase in phase gain: ϕ
- To compensate for increase GCF due to $C(s) \Rightarrow \phi_m = \phi + 5^\circ$. This will be needed to determinate α and τ
- K will be used to tune the steady state error.

Lead compensators

Determination of the tangent point

Use the gain crossover frequency of P(s)C(s) as ω_m:

$$| P(j\omega_m)C(j\omega_m) |= 1$$

$$| P(j\omega_m) | K\frac{\sqrt{\frac{1}{\alpha\tau^2}+\frac{1}{\tau^2}}}{\sqrt{\frac{1}{\alpha\tau^2}+\frac{1}{\alpha^2\tau^2}}} =| P(j\omega_m) | K\sqrt{\alpha} = 1$$

$$20\log | P(j\omega_m) |= -20\log(K\sqrt{\alpha})$$

The value of the tangent point ω_m can be determined from P(s)'s Bode plot, if you know K (the last freedom).

Lead compensators

Determination of K

- Remember the steady state error for references of the shape: $\frac{At^n\epsilon(\tau)}{n!}$ with $\epsilon(t)$ the step function.
- We found the error constants K_p, K_v and K_a as measures for the steady state error for a proportional (n=0), linear (n=1) and accelerating (n=2) reference.
- These error constants can be used to find proper values of K: $\lim_{s\to 0} K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}} s^n P(s) = K\alpha \lim_{s\to 0} s^n P(s)$

Lead Compensation Techniques Based on the Frequency-Response Approach

Step 1

Find $K\alpha$ from your steady-state requirement.

Step 2

Determine ϕ, the amount with which you want to increase the PM; if the PM is OK, you dont need a lead compensator; a proportional controller with gain $K\alpha$ suffices.

Step 3

Add $5\,^\circ$, to get $\phi_m = \phi + 5\,^\circ$ (if $\phi_m > 5\,^\circ$, you will need more than one lead compensator). The addition of the lead compensator shifts the gain crossover frequency to the right and decreases the phase margin.

Lead Compensation Techniques Based on the Frequency-Response Approach

Step 4

You will find α from this ϕ_m: $\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$ and hence also K (see step 1).

Step 5

Find the desired ω_m by looking at the Bode plot of P(s) and finding the frequency at which the gain equals $-20\log(K\sqrt{\alpha})$ dB.

Step 6

Find τ as $\frac{1}{\sqrt{\alpha}\omega_m}$.

Lead Compensation Techniques Based on the Frequency-Response Approach

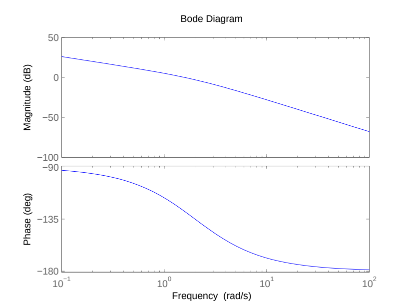
Step 7

Verify if the system works as asked. Check the gain margin to be sure it is satisfactory. If not, repeat the design process by modifying the pole-zero location of the compensator until a satisfactory result is obtained.

Example

Example

Given the system $P(s) = \frac{4}{s(s+2)}$. We want a phase margin of at least 50° and a steady state error for slope reference of maximal $\frac{A}{20}$.



Example

Step 1

Steady-state requirement: $K_v = \frac{20}{s}$
So, $\lim_{s \to 0} sP(s)C(s) = \lim_{s \to 0} s \frac{s}{s(s+2)} K\alpha = 2K\alpha = 20.$
$\Rightarrow K\alpha = 10$

Step 2

Phase margin of $K\alpha P(s) = 18^\circ$ (see phase diagram)
Calculation of phase margin without phase diagram:
We need the frequency where the magnitude is 0 dB.
So, $20 \log \mid K\alpha P(s) \mid = 0 \Rightarrow \mid K\alpha P(s) \mid = 1 \Rightarrow \mid P(s) \mid = 0.1$
When substituting $s = j\omega$ and calculating the modulus of the complex number at the left side, the equation becomes:
$\frac{-4\omega}{\omega(4+\omega^2)}^2 + \frac{-8}{\omega(4+\omega^2)}^2 = 0.01.$

Example

Step 2 continued

This equation has just one real positive solution in ω, $\omega = 6.168$.
Now, you have the right frequency. You find the phase margin by calculating the difference between -180° and the phase of $K\alpha P(6.168j)$.
We want a phase margin of at least $50^\circ \Rightarrow \phi = 32^\circ$

Step 3

$\phi_m = \phi + 5^\circ = 37^\circ$

Example

Step 4

$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m} = 0.24$
From step 1, we know that $K = \frac{\alpha}{10} = 42$

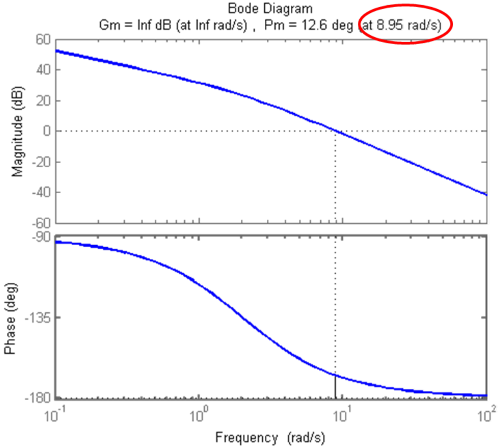
Step 5

Find ω_m, the frequency at which the gain is $-20\log(K\sqrt{\alpha})$ dB.
$GCF(P(s)K\sqrt{\alpha}) = GCF(P(s)C(s)) \Rightarrow \omega_m = 9\frac{rad}{s}$
(see Bode diagram next slide)

Step 6

$\tau = \frac{1}{\omega_m\sqrt{\alpha}} = 0.23$

Example


Bode Diagram
Gm = Inf dB (at Inf rad/s) , Pm = 12.6 deg (at 8.95 rad/s)
Magnitude (dB)
60
40
20
0
-20
-40
-60
Phase (deg)
-90
-135
-180
10^-1
10^0
10^1
10^2
Frequency (rad/s)

Example

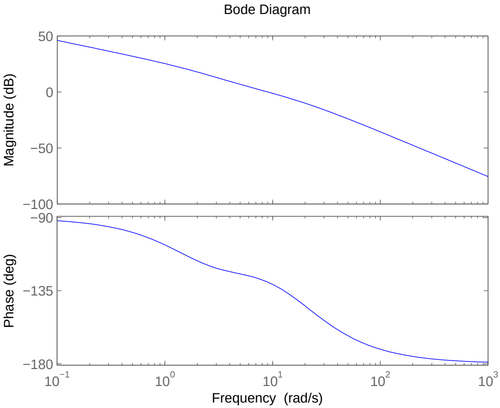
Step 7

We verify whether or not our solution is correct. We ask the Bode diagram of $K\frac{4}{s(s+2)}\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with α, τ and K the results of our calculations. (see next slide)

We see that:

- the phase margin is indeed more than $32\,^\circ$
- the new tangent point is indeed about $9\frac{rad}{s}$

Example

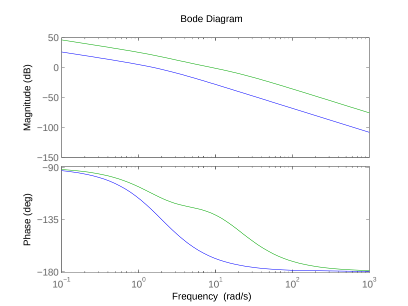

Bode Diagram
50
0
−50
−100
Magnitude (dB)
−90
−135
−180
Phase (deg)
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/s)

Example

Comparing compensated system vs non-compensated system

Blue: non-compensated system

Green: compensated system



Summary lead compensators

Evaluation of impact

- Pushing the poles to the left: this is not directly visible here, but is linked to the increased band width.
- The increase in bandwidth (this is linked to the response speed) and the increase in the phase margin were apparent in the Bode plot of P(s)C(s).
- A (small) decrease in the steady-state error occurs, since we designed it as such.
  Why small? The steady-state error decreases when the DC gain gets larger, but a lead compensators impact on the gain is not really built to increase the DC gain, the shape of a lag compensator is much more fit for this.

Summary lead compensators

Design with root locus

Design lead compensators with root locus for time-domain quantities - use dominant pole locations to fulfill overshoot, rise time, settling time, damping ratio, requirements.

Outline

1. Definition of compensators
2. Lead compensators
3. Lag compensators

Lag compensators

Transfer function

$C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}}$ with $\beta{>}1$

Bode Diagram

Example with K = 1 and $\beta = 10$ (see next slide)

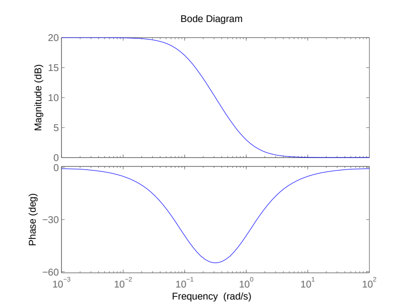
Magnitude of the lag compensator:

- becomes 10 (= 20 dB) for small frequencies
- becomes unity (= 0 dB) for high frequencies

$\Rightarrow$ Lag compensator is low-pass filter.

Lag compensators

Bode Diagram of example

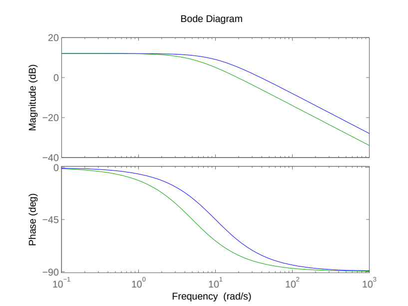


Lag compensators

Impact of lag compensators: Bode diagram

Blue: non-compensated system

Green: compensated system



Lag compensators

Impact of lag compensators: Bode diagram

- Lead compensators: increase the stability and tune the steady-state error by increasing the phase at the crossover frequency.
- Impact lag compensator = lead compensator, but different approach! By decreasing the gain, the gain crossover frequency comes down to a frequency at which the corresponding phase is higher.

Lag compensators

Impact of lag compensators

Large difference between lead and lag: their effect on the bandwidth of the system and hence on its speed of response.

- A lead compensator increases the bandwidth/speed of response
  - good if you want the system to react fast
- A lag compensator decreases the bandwidth/speed of response
  - good if your model if bad at high frequencies
  - good to reduce the impact of (mostly high-frequency) noise

Lag compensators

Design with Bode plots

We have three degrees of freedom:

- to have a sufficient drop in gain
- to push the drop in the phase to lower frequencies (that way we can use $\measuredangle P(s)$ as an approximation of $\measuredangle P(s)C(s)$ reliably to some extent
- to tune the steady state error

Lag compensators

Design with Bode plots

- Increase of phase margin $\Rightarrow$ decrease of the magnitude at some higher frequencies
- Decrease of the steady state error $\Rightarrow$ increase of the magnitude at DC

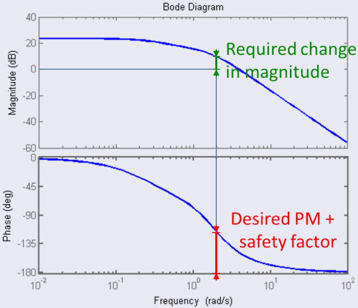
A lag compensator can realize both conditions.

- At DC value, the gain becomes: $\lim_{s\to 0} K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}} = K\beta$
- At high frequencies, the gain becomes: $\lim_{s\to\infty} K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}} = K$

Lag compensators

Design with Bode plots

K has to be such that the drop in magnitude is sufficient, the value of β has make the steady state error decrease enough and the value of τ has to be such that the transfer between from $K\beta$ to τ occurs at the right frequency.



Lag compensators

Determination of K

- Easily read from Bode plot (slide before).
- Find the frequency (ω) with desired phase margin ($+$ safety factor), then find the magnitude at that frequency; which is equal to the required change in magnitude ($=$ Q).
  $\Rightarrow K = \frac{1}{Q}$
  $\Rightarrow$ Safety factor about 10°
  - the drop in magnitude will not be complete (this is very marginal)
  - the lag compensator influences the phase plot

Lag compensators

Determination of β

$\Rightarrow$ in a similar way as we found $K\alpha$ for the lead compensator

- Translate steady state error requirement in a requirement on:
  - $K_p = \lim_{s\to 0}(P(s)C(s))$
  - $K_v = \lim_{s\to 0} s(P(s)C(s))$
  - $K_a = \lim_{s\to 0} s^2(P(s)C(s))$
  - or another error constant
- With this $K_p/K_v/K_a$ and $\lim_{s\to 0} P(s)$, we can determine $\lim_{s\to 0} C(s) = K\beta$

Lag compensators

Determination of τ

- Take τ large enough such that the magnitude is almost entirely dropped, and the phase drop has almost disappeared.
- Take the zero one decade smaller than the frequency (ω) at which P(s) had the desired phase (-180° + the desired phase margin + a safety factor of 10°). The addition of 10° compensates for the phase lag of the lag compensator.
- Verify the effect of a single zero at a frequency one decade smaller than ω.
    - The drop in magnitude is as good as complete.
    - The drop in phase cannot be more than -5.7°.

Lag Compensation Techniques Based on the Frequency-Response Approach

Step 1

Translate your steady-state requirement into a requirement on $\lim_{s\to 0} C(s) = K\beta$ and verify whether a proportional controller with gain $K\beta$ would suffice.

Step 2

Read ω, the frequency at which the phase margin equals $-180^\circ +$ "your desired phase margin" $+ 10^\circ$, off the Bode diagram. This allows us to compute $\tau = \frac{10}{\omega}$.

Lag Compensation Techniques Based on the Frequency-Response Approach

Step 3

Read Q, the magnitude at ω off the Bode plot and determine $K = \frac{1}{Q}$

Step 4

We just have calculated K (step 3) and we know $K\beta$ (step 1), so it's possible to determine β.

Step 5

Verify the behavior of the resulting system.