Classifications of systems

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Overview

Based on the number of inputs and outputs

1. **SISO**: Single Input Single Output
2. **SIMO**: Single Input Multiple Output
3. **MISO**: Multiple Input Single Output
4. **MIMO**: Multiple Input Multiple Output
5. **Autonomous**: No inputs and one or more outputs

Continuous vs. Discrete time

We will discuss both types simultaneously in order to emphasize the similarities (and differences).

Continuous system

1. It has continuous input and output signals
2. We denote continuous time by $t \in \Re$
3. We denote functions of continuous time with round brackets, e.g.: $x(t)$

Discrete system

1. It has discrete input and output signals
2. We denote discrete time by $k \in Z$
3. We denote functions of continuous time with square brackets, e.g.: $x[k]$

Continuous vs. Discrete time

Continuous
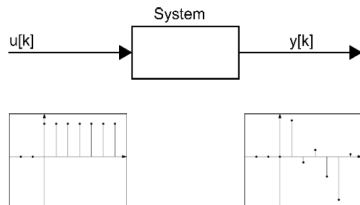
For every moments $t \in \Re$, the system has:

1. A vector of inputs $\mathbf{u}(t)$
2. A vector of outputs $\mathbf{y}(t)$
3. A vector of states $\mathbf{x}(t)$
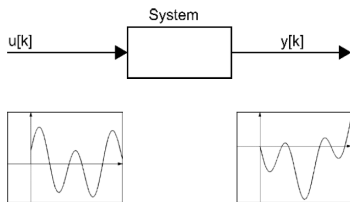
Discrete

For every moments $k \in Z$, the system has:

1. A vector of inputs $\mathbf{u}[k]$
2. A vector of outputs $\mathbf{y}[k]$
3. A vector of states $\mathbf{x}[k]$

Definition

A system is linear if $u_1(t) \rightarrow y_1(t)$ (*input $u_1(t)$ results in output $y_1(t)$*) and $u_2(t) \rightarrow y_2(t)$ imply that

$$\alpha u_1(t) + \beta u_2(t) \rightarrow \alpha y_1(t) + \beta y_2(t)$$

Properties of a linear system (contained in the definition):

- Superposition
- Homogeneity

Linear vs. Nonlinear: a linear system

Properties of a linear system (contained in the definition):

- Superposition

  $u_a(t) \rightarrow y_a(t),\ u_b(t) \rightarrow y_b(t) \Leftrightarrow u_a(t) + u_b(t) \rightarrow y_a(t) + y_b(t)$

  This means the output produced by simultaneous applications of two different inputs is the sum of the two individual outputs.

- Homogeneity

  $\alpha u(t) \rightarrow \alpha y(t)$

How to recognize a linear system:

- Linear in all of the variables
- No constant factors

Examples

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$$\begin{cases} \dot{x} = u \\ \dot{y} = x + 2u \end{cases}$$

Linearity of this system is easily verified, based on the linearity of the derivative:

$$\begin{cases} \alpha \dot{x}_a(t) + \beta \dot{x}_b(t) = \alpha u_a(t) + \beta u_b(t) \\ \alpha \dot{y}_a(t) + \beta \dot{y}_b(t) = \alpha x_a(t) + \beta x_b(t) + 2\alpha u_a(t) + 2\beta u_b(t) \end{cases}$$

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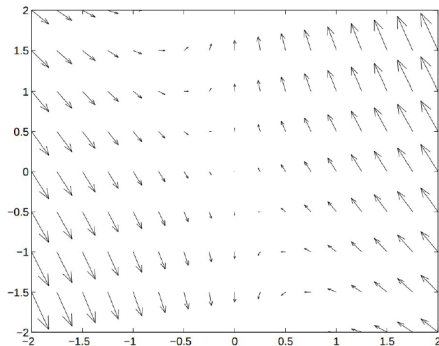$$\begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = \frac{3}{2}x_1 + u \\ \dot{y} = ax_1 - x_2 + 2u \end{cases}$$

Continuous-time autonomous linear dynamical systems are described by:

$$\dot{x}(t) = Ax(t)$$

Example: $\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} x(t)$

Linear vs. Nonlinear: violating homogeneity

All nonhomogeneous systems are strictly speaking nonlinear, e.g.:

$$\begin{cases} \dot{x}(t) = x(t) + u(t)^2 \\ \dot{y}(t) = x(t) \end{cases} \qquad \Rightarrow \text{ nonhomogeneous}$$

This is nonlinear, because the term $u(t)^2$ violates homogeneity.
It can be turned into a linear system with inputs $z(t) = u(t)^2$.

$$\begin{cases} \dot{x}(t) = x(t) + z(t) \\ \dot{y}(t) = x(t) \end{cases} \qquad \Rightarrow \text{ linear}$$

$\rightarrow$ nonhomogeneous systems that are linear apart from some function of inputs are often treated as linear systems.

Linear vs. Nonlinear: nonlinear systems

Some examples of nonlinear systems:

$$\begin{cases} \dot{x}_1(t) = x_1(t) + u(t) \\ \dot{x}_2(t) = x_1(t)x_2(t) \\ y(t) = x_1(t) + x_2(t) \end{cases}$$

$$\begin{cases} \dot{x}(t) = sin(x(t)) + u(t) \\ y(t) = x(t) \end{cases}$$

$$\begin{cases} \dot{x}(t) = 2u(t) + 1 \\ y(t) = cos(x(t)) \end{cases}$$

Predominantly linear

Simple electrical systems

- Circuits with ideal resistors, capacitors and inductors

Simple mechanical systems

- Systems with ideal springs

Inherently nonlinear

Chemical systems
Biological systems
Economical systems
More involved electrical or mechanical systems
...

Linear vs. Nonlinear

- Reality is nonlinear
- However, this course will only deal with linear systems
- Why we prefer linear systems:

  The previously mentioned properties will allow for a thorough study of the system
- Why we are allowed to use linear systems, even in a nonlinear setting:

  You can linearize around an equilibrium point (we will do this in the next lecture)

Causal vs. Non-causal

- A causal system only depends on the present and the past, not on the future
- A non-causal system (also) depends on the future
- (Almost) all physical systems are causal
  - A telephone:
    - It will not ring for future calls
  - Any human:
    - Is a system that will only react on inputs it has already received
    - If we react because we expect something to happen in the future, then that expectation arose from past or present inputs

source:
http://www.deekshith.in/2013/03/causal-and-non-causal-systems-better-explained.html

Causal vs. Non-causal

How do non-causal systems arise?

A possibility is by greatly reducing the complexity of a system, in which some causes of events are taken out of the equations.
Example:

- A model of the economical consumption (**output**)
- A lot of influencing factors, but the only **input** is the employment numbers
- Current and past employment numbers determine consumption, but when someone gets fired, they will continue to work for several weeks in most instances, but their consumption will drop immediately
  $\rightarrow$ A correcct model for this relation would have to be **non-causal**
- The non-causal model for this input-output relation is not useful if you want to determine the level of consumption
- You could use the relation to see a drop in employment, before it is visible in the employment numbers

Causal vs. Non-causal

Examples of non-causal systems: **expectations**

Modelling housing prices

- People are willing to offer more for houses if they expect rising prices
- It is hard to measure the expectations or housing prices
- Sometimes economists use their own predictions of housing prices to replace the expectations
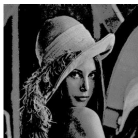
Causal vs. Non-causal

Examples of non-causal systems: **image processing**

- The input to our system (the image processor) is a two dimensional series of values ($u(k, l)$): the color values at the different pixels of the original image
- The output is a processed image ($y(k, l)$)
- There is now no reason to want causality; the input depends on position and not on time
- $y(k, l)$ can rely on 'later' values like $u(k + 1, l + 1)$, without that being a problem

Original image

Removed details

Highlight borders

Causal vs. Non-causal

Some mathematical examples:

Causal systems

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$$\begin{cases} x[k+1] = u[k] \\ y[k] = 3x[k+1] + u[k] \end{cases}$$

- $y(t) = 2u(t - \tau)$

Non-causal systems

-
$$\begin{cases} x[k+1] = u[k]) \\ y[k] = x[k+2] + \frac{1}{2}u[k] \end{cases}$$

- $y(t) = u(t + \tau)$

Time-invariant vs. Time-varying

Dynamics and properties of the system don't change through time.
If the same input is applied to a system that is in the same state, then the output of a time-invariant system will be the same.

Mathematically this looks as follows:

- In a time-invariant system:
  if $u(t) \xrightarrow{x(t)} y(t)$ then $u(t + \tau) \xrightarrow{x(t+\tau)} y(t + \tau)$

- In time-varying systems, the parameters of the system are functions of time, e.g.:

$$\begin{cases} \dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) \\ y(t) = \mathbf{C}x(t) + \mathbf{D}u(t) \end{cases} \quad \Rightarrow \quad \begin{cases} \dot{x}(t) = \mathbf{A}(t)x(t) + \mathbf{B}(t)u(t) \\ y(t) = \mathbf{C}(t)x(t) + \mathbf{D}(t)u(t) \end{cases}$$

Time-invariant vs. Time-varying

Time-invariant differential equation

The dependent variable and its derivatives appear as linear combinations.
The coefficients of all terms are constant.
Example: $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 10x = 0$

Time-varying differential equation

The dependent variable and its derivatives appear as linear combinations,
but coefficients of terms may involve the independent variable.
Example: $\frac{d^2x}{dt^2} + (1 - cos(2t)x = 0$

Time-invariant vs. Time-varying

- Examples of time-varying systems:
  - The properties of an electrical circuit slowly change over time
  - The human body also has many changing properties
  - Systems affected by night and day (heating of buildings), when those aspects were ignored in the model

- Examples of (de facto) time-invariant systems:
  - A system that describes a physical law, for instance a system with two masses as its input and their attractive force as an output
  - In practice we approximate all systems whose properties change much slower than the variables as time-invariant

Lumped vs. Distributed

Take the following example:

We have a reactor in which we send a mixture of X and Y, we take $u(t)$ to be the concentration of X in the input (the concentration of Y is implicitly determined by the concentration of X, since they sum to 1).

The output is a mixture of X, Y and a product P. We take $y_1(t)$ to be the concentration of X in the output and $y_2(t)$ to be the concentration of P. The concentration of Y is implicitly determined by the other concentrations).

u(t) ⟶ [Reactor] ⟶ y(t)
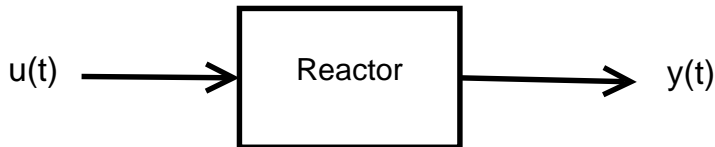
Lumped vs. Distributed

We want to determine the amount of states.

The states should describe the reactor in such a way that we will be able to determine the reaction rate throughout the reactor.

The rate of the reaction is determined by the local concentrations of the three compounds, which means we need two states at every point in the reactor (the reactor is not uniform).

This means we need an **infinite amount of states**, we say that this is a **distributed system**.

The relations in a distributed system are naturally expressed by **PDE's**.

Lumped vs. Distributed

Do we really need an infinitely large system for this problem?

Not if we discretize the system, in which we take a finite amount of points at which we keep track of the concentration.

Physically this would mean we assume that the concentrations are locally uniform.

For some types of reactors you can go as far as to assume the reactor is perfectly stirred; with a homogeneous content.

If we have a **finite amount of states** (for instance by discretizing), we call the system a **lumped system**.

It can be expressed in a finite amout of **ODE's**.