Geometric algebra

August 20, 2015

Outline

Vectors and spatential interpretation

Properties of a vector

There are 3 properties of a vector $\overrightarrow{x}$:

- magnitude
- direction
- startpoint

with respect to a referention vector $\overrightarrow{0}$

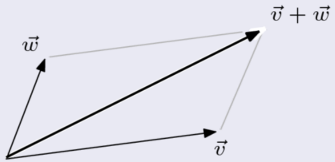
Multiplication scalar and vector

r ϵ $\mathbb{R}$ (r ϵ $\mathbb{C}$ is possible, but hasn't a fysical representation)

- $|r| < 1$: shorten
- $|r| > 1$: increase
- $r < 0$: reverse the direction

Addition of vectors

Parallellogramrule:



vectorspace

First condition

A vectorspace V over a body L (set of operators) is a set of vectors that satisfy:

1. A vectorsum is defined: $VxV \rightarrow V : (\vec{x}, \vec{y}) \rightarrow \vec{x} + \vec{y}$

$\vec{x}, \vec{y}, \vec{z} \in$ V

a) $\vec{x} + \vec{y} \in$ V

b) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$

c) $\exists! \vec{0} : \vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$

d) $\forall \vec{x}, \exists(-\vec{x}) : \vec{x} + (-\vec{x}) = (-\vec{x}) + \vec{x} = 0$

e) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

vectorspace

Second condition

2. A outside law is defined: $LxV \rightarrow V : (a, \overrightarrow{x}) \rightarrow a\overrightarrow{x}$

   $\overrightarrow{x}$, $\overrightarrow{y}$ ϵ V

   a,b ϵ L

   a) $1\overrightarrow{x} = \overrightarrow{x}$
   b) $a(b\overrightarrow{x}) = (ab)\overrightarrow{x}$
   c) $(a+b)\overrightarrow{x} = a\overrightarrow{x} + b\overrightarrow{x}$
   d) $a(\overrightarrow{x} + \overrightarrow{y}) = a\overrightarrow{x} + a\overrightarrow{y}$

Numberspaces of n-couples

Defenition

This is the set of all n-couples like $\begin{bmatrix} \overrightarrow{x_1} \\ \overrightarrow{x_2} \\ . \\ . \\ \overrightarrow{x_n} \end{bmatrix}$ with $x_i \ \epsilon \ \mathbb{R}$ or $x_i \ \epsilon \ \mathbb{C}$.

This set together with the operator set $\mathbb{R}$ or $\mathbb{C}$ is a vectorspace.

Subspaces

Defenition

V_1 is a supspace of vectorspace V if:

1. $V_1 \subset$ V
2. With the same in- and outside law as V, is V_1 a vectorspace

Properties

1. $\overrightarrow{0}$ ϵ every subspace
2. The intersection of two spaces is always a subspace
3. Given: p vectors $x_1, x_2, ..., x_p$ ϵ V.
   The set vectors $a_1 x_1 + a_2 x_2 + ... + a_n x_n$ with a_i ϵ $\mathbb{R}$ is a subspace of V.

Linear independance, basis, dimensions

Defenition independance

Given: p vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p} \epsilon$ V.
Construct the nullvector as a linear combination of those vectors (i.e. search the operators (numbers) $a_1, a_2, ..., a_p$ to form $a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + ... + a_n\overrightarrow{x_n} = \overrightarrow{0}$).
If the nullvector only can created by $a_1 = a_2 = ... = a_p = 0$, then are the vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ **linear independant**.

Linear independance, basis, dimensions

Properties

1. If the vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ are linear independent, then can't none of them be writed as a linear combination of the other p-1 vectors.
2. If the nullvector is one of the p vectors, then is the set $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ linear dependant (if $\overrightarrow{x_1} = 0$ then is $a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + ... + a_n\overrightarrow{x_n} = 0$ with $a_1 \neq 0$ and $a_2, a_3, ...a_p = 0$).
3. Basis and dimension: p linear independant vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ generate a vectrospace V^p. Every vector in V^p can be writed **in only one way** as a linear combination of the p linear independant vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ using operators $a_1, a_2, ..., a_p$.

Linear independance, basis, dimensions

Basis, dimension

Given: $\overrightarrow{v} = a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + ... + a_p\overrightarrow{x_p}$.
The set operators $a_1, a_2, ..., a_p$ are called the **coordinates** of the vector $\overrightarrow{v}$ relative to the set vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$. This set vectors is a **basis** of vectorspace V^p, with **dimension** p.

Linear independance, basis, dimensions

Example

Given: $\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \overrightarrow{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The set $\{\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}\}$ is a linear independant combination. There doesn't exist numbers $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$ such that $a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + a_n\overrightarrow{x_3} = 0$. The set of all vectors $\overrightarrow{y} = a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + a_n\overrightarrow{x_3}$ is the three dimensional vectrospace V^3.

If $\overrightarrow{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Then is $3\overrightarrow{x_1} + 2\overrightarrow{x_2} + \overrightarrow{x_3}$ the only way to write $\overrightarrow{y}$ as a linear combination of $\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}$.

Linear independance, basis, dimensions

Example

The set of vectors $\overrightarrow{y} = a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2}$ is a **two dimensional** subspace V^2.
The vectors in this subspace are:

$$\overrightarrow{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}.$$

Important difference

- All vectors of V^2 have **3** coordinates.
- The dimension of the subspace V^2 is **2**.

Linear independance, basis, dimensions

Convention of notation

Given: a n-dimensional vectorspace V^n.
The elements of this vectorspace are the elements: $\overrightarrow{x}, \overrightarrow{y}, \ldots$ If we choose $\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_n}$ as a basis of V^n. Then we can write every vector of V^n as a linear combination of those basis vectors in only one way: $\overrightarrow{x} = x_1\overrightarrow{e_1} + x_2\overrightarrow{e_2} + \ldots + x_n\overrightarrow{e_n}$. The numbers x_i are the coordinates of vector $\overrightarrow{x}$ relative to the basis $\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_n}$.

Between the vectorspace of dimension n and the number space of dimension n exists a isomorphism.

Linear independance, basis, dimensions

Vectorspace V^p

Given: a p-dimensional vectorspace V^p where the vectors are n-couples (with $n \geq p$).

1. In V^p you can choose a basis with p linear independant vectors.
2. Every vector $\overrightarrow{x} \ \epsilon \ V^p$ can be writed in only one way as a linear combination of the p basis vectors using coordinates.

Example 1

Given: n=5, p=2, $\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \overrightarrow{x_2} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Linear independance, basis, dimensions

Example 1

The vectors $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ are linear independant, so they span a two dimensional subspace: $\overrightarrow{y} = a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2}$ with $a_1, a_2 \epsilon \mathbb{R}$.

The coordinates of the vector $y_1 = \begin{bmatrix} 5 \\ -6 \\ 1 \\ 2 \\ 5 \end{bmatrix}$, relative to the basis $\{\overrightarrow{x_1}, \overrightarrow{x_2}\}$, are $a_1 = 1$ and $a_2 = 2$.

Linear independance, basis, dimensions

Example 1

The vector $\overrightarrow{y_2} = \begin{bmatrix} 5 \\ -7 \\ 1 \\ 2 \\ 5 \end{bmatrix}$ can't be writen as a linear combination of the vectors $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$. So y_2 doesn't belong to the subspace spanned by $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$.
This implies that $\overrightarrow{y_2}$ is linear independant of $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$. Thus the subspace spanned by $\overrightarrow{y_2}$, $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ is a 3 dimensional subspace.

Linear independance, basis, dimensions

In general

When the set vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ is linear independant, then lays $\overrightarrow{x_i}$ not totally in the subspace spanned by the vectors $\overrightarrow{x_1}, ..., \overrightarrow{x}_{i-1}, \overrightarrow{x}_{i+1}, ..., \overrightarrow{x_p}$.
The vector $\overrightarrow{x_i}$ can be writen as a sum of 2 components: $\overrightarrow{x}_{i\alpha}$ and $\overrightarrow{x}_{i\beta}$.

1. $\overrightarrow{x}_{i\alpha} \in$ subspace spanned by $\overrightarrow{x_1}, ..., \overrightarrow{x}_{i-1}, \overrightarrow{x}_{i+1}, ..., \overrightarrow{x_p}$.
2. $\overrightarrow{x}_{i\beta} \perp$ subspace spanned by $\overrightarrow{x_1}, ..., \overrightarrow{x}_{i-1}, \overrightarrow{x}_{i+1}, ..., \overrightarrow{x_p}$.

Inproduct

Defenition

The inproduct of two vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ ϵ E^n (n-couples) is defined as the image: $E^n x E^n \to \mathbb{R} : \{\overrightarrow{x}, \overrightarrow{y}\} \to \overrightarrow{x}.\overrightarrow{y} \; \epsilon \; \mathbb{R}$. This image is:

1. Bilinear:
   $(\overrightarrow{x} + \overrightarrow{v}).\overrightarrow{y} = \overrightarrow{x}.\overrightarrow{y} + \overrightarrow{v}.\overrightarrow{y}$
   $\overrightarrow{x} + (\overrightarrow{v}).\overrightarrow{y}) = \overrightarrow{x}.\overrightarrow{v} + \overrightarrow{x}.\overrightarrow{y}$
   $(a\overrightarrow{x})\overrightarrow{y} = a(\overrightarrow{x}.\overrightarrow{y})$
   $\overrightarrow{x}(a\overrightarrow{y}) = a(\overrightarrow{x}.\overrightarrow{y})$
2. Symetric:
   $\overrightarrow{x}.\overrightarrow{y} = \overrightarrow{y}.\overrightarrow{x}$
3. Positive definite:
   $\forall \overrightarrow{x} \neq \overrightarrow{0} : \; \overrightarrow{x}.\overrightarrow{x} > 0$

Inproduct

Matricial notation

The inproduct is a **scalar**. If $\overrightarrow{x}$, $\overrightarrow{y}$ and the basis ϵ E^n then can the inproduct be noted matricial:

$$\overrightarrow{x}.\overrightarrow{y} = y^t Ax = x^t Ay = (x_1 ... x_n) A \begin{pmatrix} y_1 \\ ... \\ y_n \end{pmatrix}$$

with A positive definite and symetric $(A = A^t)$.

Inproduct

Norm of a vector

$\|\vec{x}\|^2 = \vec{x}.\vec{x}$ and because $\vec{x}.\vec{x} > 0$ applies:
$\|\vec{x}\| = \sqrt{\vec{x}.\vec{x}}$ where $\|\vec{x}\|$ is called the norm of $\vec{x}$.
Normalizing is dividing a vector by its norm. The result is a vector with norm $= 1$.

$\|\frac{\vec{x}}{\|\vec{x}\|}\| = \sqrt{\frac{\vec{x}}{\|\vec{x}\|}\frac{\vec{x}}{\|\vec{x}\|}} = \sqrt{\frac{\|\vec{x}\|^2}{\|\vec{x}\|\|\vec{x}\|}} = 1.$

Inproduct

CauchySchwarz inequality

$|\overrightarrow{x}.\overrightarrow{y}| \leq \|\overrightarrow{x}\|\|\overrightarrow{y}\|$ or
$-\|\overrightarrow{x}\|\|\overrightarrow{y}\| \leq \overrightarrow{x}.\overrightarrow{y} \leq \|\overrightarrow{x}\|\|\overrightarrow{y}\|$ from wich follows:
$-1 \leq \frac{\overrightarrow{x}\overrightarrow{y}}{\|\overrightarrow{x}\|\|\overrightarrow{y}\|} \leq 1$
By defenition follows:
$\cos(\theta) = cos(\angle(\overrightarrow{x}, \overrightarrow{y})) = \frac{\overrightarrow{x}\overrightarrow{y}}{\|\overrightarrow{x}\|\|\overrightarrow{y}\|}$
Therefor: the angle between the vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ $=$
Bgcos(inproduct of $\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}$ and $\frac{\overrightarrow{y}}{\|\overrightarrow{y}\|}$).

Inproduct

Orthogonality

$\overrightarrow{x}$ and $\overrightarrow{y}$ are orthogonal $\Leftrightarrow$
$\theta = \angle(\overrightarrow{x}, \overrightarrow{y}) = 90^\circ = \frac{\Pi}{2} rad \Leftrightarrow$
$cos(\theta) = 0 \Leftrightarrow$
$\overrightarrow{x}.\overrightarrow{y} = 0$

Hence, if $\overrightarrow{x}, \overrightarrow{y} \neq 0$:
$\overrightarrow{x} \perp \overrightarrow{y} \Leftrightarrow \overrightarrow{x}.\overrightarrow{y} = 0$.

Parallellism

$\overrightarrow{x} \parallel \overrightarrow{y} \Leftrightarrow \theta = 0^\circ$ or $180^\circ \Leftrightarrow cos(\theta) = \pm 1 \Leftrightarrow |\overrightarrow{x}\,\overrightarrow{y}| = \|\overrightarrow{x}\|\|\overrightarrow{y}\|$

Inproduct

Distance between two vectors

Distance $= \|\overrightarrow{x} - \overrightarrow{y}\| = \|\overrightarrow{z}\|$ with $\overrightarrow{z} = \overrightarrow{x} - \overrightarrow{y}$.
$\|\overrightarrow{x} - \overrightarrow{y}\|^2 = (\overrightarrow{x} - \overrightarrow{y})(\overrightarrow{x} - \overrightarrow{y})$
$= \overrightarrow{x}\overrightarrow{x} - \overrightarrow{x}\overrightarrow{y} - \overrightarrow{y}\overrightarrow{x} + \overrightarrow{y}\overrightarrow{y}$
$= \overrightarrow{x}\overrightarrow{x} + \overrightarrow{y}\overrightarrow{y} - 2\overrightarrow{x}\overrightarrow{y}$
$= \|\overrightarrow{x}\|^2 + \|\overrightarrow{y}\|^2 - 2\|\overrightarrow{x}\|\|\overrightarrow{y}\| cos(\theta)$ with θ the angle between $\overrightarrow{x}$ and $\overrightarrow{y}$.

Pythagorean theorem

If $\overrightarrow{x} \perp \overrightarrow{y}$ then $cos(\theta) = 0$ and thus:
$\|\overrightarrow{x} - \overrightarrow{y}\|^2 = \|\overrightarrow{x}\|^2 + \|\overrightarrow{y}\|^2$.

Inproduct

The 'simple' inproduct

If in the definition $\overrightarrow{x}.\overrightarrow{y} = y^t Ax = x^t Ay$ (with A positive definite and symetric) A=I, then the inproduct becomes the simple inproduct: $\overrightarrow{x}.\overrightarrow{y} = y^t Ix = x^t Iy = (x_1 ... x_n)\begin{pmatrix} y_1 \\ ... \\ y_n \end{pmatrix} = \sum_{i=1}^n x_i y_i$.

This simple inproduct can always be found by a basis transformation: x=Rx' and y=Ry', then $\overrightarrow{x}.\overrightarrow{y} = y'^t(R^t AR)x'$. Now, R must be taken such that $R^t AR = I$. This can be done by converting A to its normal form by a congruent transformation (e.g. the method of kwadratic forms).

In what follows we mean by 'inproduct' always 'simple inproduct'.

Gram Schmidt orthogonalization

Making two independent vectors orthogonal

Geometric derivation:

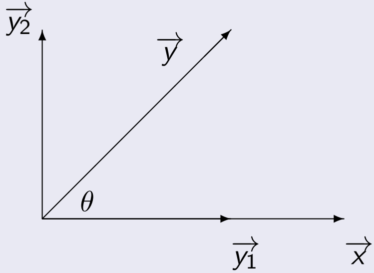


Figure 1: decomposition of vector $\overrightarrow{y}$ in a component parallel ($\overrightarrow{y_1}$) and a component orthogonal ($\overrightarrow{y_2}$) to $\overrightarrow{x}$.

Gram Schmidt orthogonalization

Making two indepentent vectors orthogonal

1. Project $\vec{y}$ orthogonal on $\vec{x}$, this generates the vector $\vec{y_1}$, the component parallel with $\vec{x}$.
2. Subtract $\vec{y}$ by $\vec{y_1}$, the result is $\vec{y_2}$ wich is orthogonal to $\vec{x}$.

$\vec{y_1}$ is a specific multiple of the normilised vector $\vec{x}$: $\vec{y_1} = \alpha\frac{\vec{x}}{\|\vec{x}\|}$.
$\vec{y_1} \parallel \vec{x}$: $\vec{y_1}\vec{x} = \pm\|\vec{y_1}\|\|\vec{x}\|$ (+ if $\theta \leq 90^\circ$ and - if $\theta > 90^\circ$).
From fig. 1: $\|\vec{y_1}\| = cos(\theta)\|\vec{y}\|$.
From the inproduct: $cos(\theta) = \frac{\vec{x}\vec{y}}{\|\vec{x}\|\|\vec{y}\|}$. So
$\vec{y_1}\vec{x} = \alpha\frac{\vec{x}}{\|\vec{x}\|}\vec{x} = \alpha\|\vec{x}\| = \|\vec{y_1}\|\|\vec{x}\| = cos(\theta)\|\vec{y}\|\|\vec{x}\| = \vec{x}\vec{y}$.
So we get: $\alpha = \frac{\vec{x}\vec{y}}{\|\vec{x}\|}$.

Gram Schmidt orthogonalization

Conclusion

$\overrightarrow{y_1} = \alpha \frac{\overrightarrow{x}}{\|\overrightarrow{x}\|} = (\frac{\overrightarrow{x}\overrightarrow{y}}{\|\overrightarrow{x}\|^2})\overrightarrow{x}$ with $\frac{\overrightarrow{x}\overrightarrow{y}}{\|\overrightarrow{x}\|^2}$ a scalar.

And $\overrightarrow{y_2} = \overrightarrow{y} - \overrightarrow{y_1} = \overrightarrow{y} - (\frac{\overrightarrow{x}\overrightarrow{y}}{\|\overrightarrow{x}\|^2})\overrightarrow{x}$.

Hence:

The vector $\overrightarrow{y}$ gets orthogonalised on the vector $\overrightarrow{x}$ by subtract $\overrightarrow{y}$ by the component of $\overrightarrow{y}$ parallel with $\overrightarrow{x}$.

Control of $\overrightarrow{y_2} \perp \overrightarrow{x}$:

$\overrightarrow{y_2}\overrightarrow{x} = (\overrightarrow{y} - (\frac{\overrightarrow{x}\overrightarrow{y}}{\|\overrightarrow{x}\|^2})\overrightarrow{x})\overrightarrow{x} = \overrightarrow{y}\overrightarrow{x} - (\frac{\overrightarrow{x}\overrightarrow{y}}{\|\overrightarrow{x}\|^2})\|\overrightarrow{x}\|^2 = 0.$

Gram Schmidt orthogonalization

Generalization to multiple vectors

Given: 3 vectors: 2 orthogonal unit vectors $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ $(\|\overrightarrow{x_1}\| = 1 = \|\overrightarrow{x_2}\|, \overrightarrow{x_1}\overrightarrow{x_2} = 0)$ and a vector $\overrightarrow{y}$.
Asked: orthogonilise $\overrightarrow{y}$ on $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$.
Solution:
We first search the component of $\overrightarrow{y}$ parallel with $\overrightarrow{x_1}$ and subtract $\overrightarrow{y}$ by this component. This gives $\overrightarrow{y_1}$.
$\overrightarrow{y_1} = \overrightarrow{y} - (\frac{\overrightarrow{x_1}\overrightarrow{y}}{\|\overrightarrow{x_1}\|^2})\overrightarrow{x_1} = \overrightarrow{y} - (\overrightarrow{x_1}\overrightarrow{y})\overrightarrow{x_1}$ $(\|\overrightarrow{x_1}\|^2 = 1)$
$\overrightarrow{y_1} \perp \overrightarrow{x_1}$.

Gram Schmidt orthogonalization

Generalization to multiple vectors

Next, we subtract $\overrightarrow{y_1}$ by the component of $\overrightarrow{y_1}$ that is parallel with $\overrightarrow{x_2}$, to get $\overrightarrow{z}$ (wich is perpendicular to both $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$).
$\overrightarrow{z} = \overrightarrow{y_1} - \overrightarrow{x_2}(\overrightarrow{y_1}\overrightarrow{x_2})$. We can write $\overrightarrow{z}$ in another way:
$\overrightarrow{z} = \overrightarrow{y_1} - \overrightarrow{x_2}(\overrightarrow{y_1}\overrightarrow{x_2}) = \overrightarrow{y} - (\overrightarrow{x_1}\overrightarrow{y})\overrightarrow{x_1} - \overrightarrow{x_2}([\overrightarrow{y} - (\overrightarrow{x_1}\overrightarrow{y})\overrightarrow{x_1}]\overrightarrow{x_2})$
$\overrightarrow{z} = \overrightarrow{y} - \overrightarrow{x_1}(\overrightarrow{x_1\overrightarrow{y}}) - \overrightarrow{x_2}(\overrightarrow{x_2}\overrightarrow{y})$

Conclusion

The vector $\overrightarrow{y}$ becomes orthogonilised on two orthogonal unit vectors $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ by subtracting $\overrightarrow{y}$ by the components of $\overrightarrow{y}$ parallel with $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$.

Complementary subspace

Defenition

Given: a n dimensional vector space V^n, with $p < n$ linear independent vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$. This vectors create a p dimensional subspace V^p and can be orthogonilised via the Gram schidt method to a orthonormal basis $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_p}$ with $\overrightarrow{e_i}\,\overrightarrow{e_j} = \delta_{ij}$ (with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$).
This p vectors can be complemented by $n - p$ linear independent vectors $\overrightarrow{f_1}, \overrightarrow{f_2}, ..., \overrightarrow{f_{n-p}}$ that are linear independent with $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_p}$ and orthonormal.

These vectors $\overrightarrow{f_1}, \overrightarrow{f_2}, ..., \overrightarrow{f_{n-p}}$ generate the orthogonal complement of the subspace created by $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_p}$.
The orthogonal complement of the p-dimensional subspace V^p of V^n $(p < n)$, has dimension $n - p$.

Complementary subspace

Example

Given: $n = 5, p = 3, \overrightarrow{e_1} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{e_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \overrightarrow{e_3} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$

Asked: The orthogonal complement

Solution: The orthogonal complement has dimension $n - p = 5 - 3 = 2$ and consists of the set vectors that are perpendicular to the vectors $\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_p}$.

$$\begin{cases} \overrightarrow{e_1}\overrightarrow{x} = 0 \\ \overrightarrow{e_2}\overrightarrow{x} = 0 \\ \overrightarrow{e_3}\overrightarrow{x} = 0 \end{cases}$$

Complementary subspace

Example

$$\begin{cases} \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \end{bmatrix} \overrightarrow{x} = 0 \\ \begin{bmatrix} 1 & 0 & 0 & 4 & 0 \end{bmatrix} \overrightarrow{x} = 0 \\ \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \end{bmatrix} \overrightarrow{x} = 0 \end{cases}$$

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 1 & 3 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{x_1} \\ \overrightarrow{x_2} \\ \overrightarrow{x_3} \\ \overrightarrow{x_4} \\ \overrightarrow{x_5} \end{bmatrix} = 0$$

This is a homogenous system of equetions. The solution of this system is the orthogonal complement.

Outline

Row- and column vectors

Example

The rows of a mxn matrix A can be considered as m row vectors with n components.

$$A^{mxn} = \begin{bmatrix} \overrightarrow{r_1} \\ \overrightarrow{r_2} \\ \dots \\ \overrightarrow{r_m} \end{bmatrix}$$

The columns of a mxn matrix A can be considered as n column vectors with m components.

$$A^{mxn} = \begin{bmatrix} \overrightarrow{r_1} & \overrightarrow{r_2} & \dots & \overrightarrow{r_m} \end{bmatrix}$$

Row- and column space, rank

Column space

We consider the columns of A^{mxn} as vectors with m components and difine the vectors $\overrightarrow{x}$ as every possible linear combination of the column vectors $\overrightarrow{k_i}$:
$\overrightarrow{x} = a_1\overrightarrow{k_1} + a_2\overrightarrow{k_2} + \dots + a_n\overrightarrow{k_n}$ with $a_i \in \mathbb{R}$ and i means the i^{th} column of A.
The set of all the vectors $\overrightarrow{x}$ is called the column space of A.
The column space = all possible linear combinations of columns of A .

Row- and column space, rank

Column space

if only r of the n vectors are linear independent, that means

- None of this r vectors can be written as a linear combination of the other $r-1$ vectors
- all others $n-r$ column vectors can be written as linear combinations of the r linear independant vectors

then r is called:

1. the rank of (column) matrix A
2. the dimension of the column space of matrix A

Row- and column space, rank

Row space

The concept row space and row rank can be derived in the same way as the column space is derived.

Rank

It is a fundamental matrix property that:

$\boxed{\text{row rank } A = \text{column rank } A}$

That means: the number linear independant columns in a matrix is equal to the number linear independant rows. Thus:

rank A = row rank A = column rank A
= dimension row space A = dimension column space A

Row- and column space, rank

Example

$$A^{4x3} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \\ 2 & 4 & -2 \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

We determine the rank of A by converting it to his echlon form by elementary row operations (explaned in the appendix).

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \\ 2 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Canonical form: the first element of every row is 1. Above and below this ones is every number 0.

Row- and column space, rank

Conclusion

Rank = the number of rows that differs from zero. So the rank is 2. In other words: there are only 2 linear independent rows and columns in A.
Hence:
The column space of A is a 2 dimensional subspace of the 4 dimensional vectorspace.
The row space is a 2 dimensional subspace of the 3 dimensional space.

Link between determinant and rank by square matrices

Determinant-rank

If all the columns of a square matrix are linear dependent, then the determinant is 0. If A^{mxm}:
det A=0 $\Leftrightarrow$ rank A=0 $\Leftrightarrow$ columns depentent $\Leftrightarrow$ rows dependent.
If det A $\neq$ 0 then rank A $=$ m and A is of **full rank**.