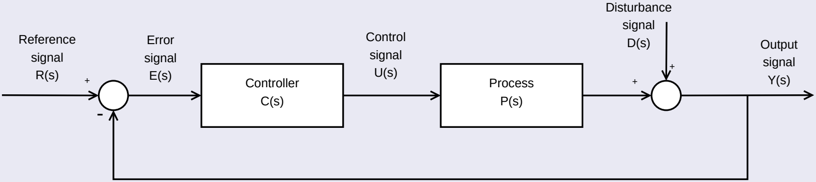
Chapter 12: Lead and Lag Compensators

August 7, 2015

Outline

Lead Compensator vs Lag Compensator

Schematical representation



Transfer functions

Lead compensator : $C(s) = K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with $0<\alpha<1$

Lag compensator : $C(s) = K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}}$ with $\beta>1$

Lead Compensator vs Lag Compensator: zeros and poles

Transfer functions

Lead compensator : $C(s) = K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with $0<\alpha<1$

Lag compensator : $C(s) = K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}}$ with $\beta>1$

Zeros and poles

Zeros: $s = -\frac{1}{\tau}$

Poles: $s = -\frac{1}{\alpha\tau}$ (for lead) or $s = -\frac{1}{\beta\tau}$ (for lag)

For lead compensators the pole lies more to the left in the complex plane than the zero and vice versa for lag compensators.

Important!

There are two ways of designing this kind of compensators:

- making use of frequency domain tools
- making use of root locus

In this chapter, we will only focus on **frequency domain tools**.

Outline

Lead compensators

Transfer function

$C(s) = K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with $0<\alpha<1$

Bode Diagram

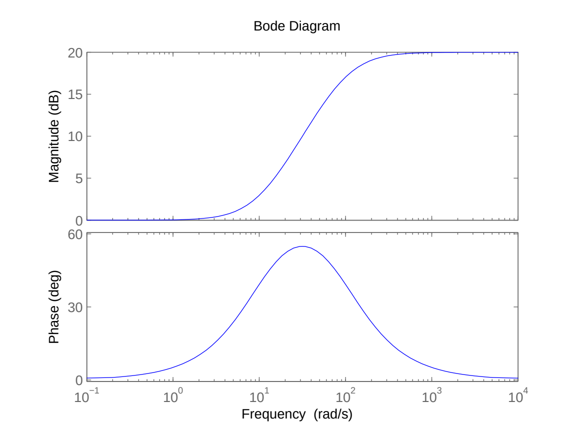
Example with K $= 10$ and $\alpha = 0.1$ (see next slide)
Magnitude of the lead compensator:

- becomes unity ($= 0$ dB) for small frequencies
- becomes 10 ($= 20$ dB) for high frequencies

$\Rightarrow$ Lead compensator is **high-pass filter**.

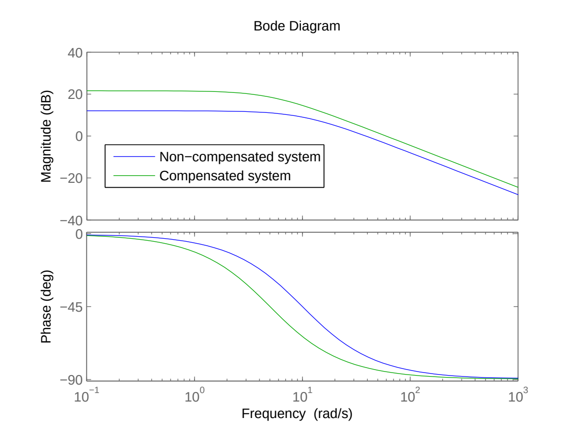
Lead compensators

Bode Diagram



Lead compensators

Impact of lead compensators: Bode Diagram



Lead compensators

Impact

- Push the poles of the closed loop system to the left.
  - Stabilization of the system (see root locus).
  - Increase response speed and bandwidth.
- Increase of the phase margin: the phase of the lead compensator is positive for every frequency, hence the phase will only increase.

Design with Bode plots

Step 1

- Remember the steady state error for references of the shape: $\frac{at^n \epsilon(\tau)}{n!}$ with $\epsilon(t)$ the step function.
- Translate your steady state requirement in terms of the error constant:
    - $K_p = \lim_{s \to 0} P(s)C(s)$ (position)
    - $K_v = \lim_{s \to 0} sP(s)C(s)$ (velocity)
    - $K_a = \lim_{s \to 0} s^2 P(s)C(s)$ (acceleration)
- With this $K_p/K_v/K_a$ and $\lim_{s \to 0} P(s)$, we can determine $\lim_{s \to 0} C(s) = \lim_{s \to 0} K \frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}} = K\alpha$.
- Verify whether a proportional controller with gain $K\alpha$ would suffice.

Design with Bode plots

Step 2

Determine ϕ as the extra PM we have to add to $K\alpha P(s)$; if the PM is all right, you do not need a lead compensator; a proportional controller with gain $K\alpha$ suffices.

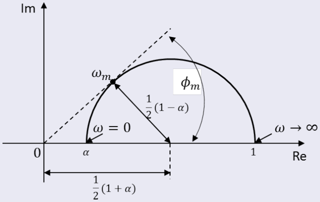
Step 3

Add 5° to get $\phi_m = \phi + 5^\circ$ (if $\phi_m > 60^\circ$, you will need more than one lead compensator). The addition of the lead compensator shifts the gain crossover frequency to the right and decreases the phase margin.

Design with Bode plots

Step 4

Determine α making use of the polar plot of $\frac{\alpha(j\omega\tau+1)}{j\omega\alpha\tau+1}$



$\sin\phi_m = \frac{\frac{1}{2}(1-\alpha)}{\frac{1}{2}(1+\alpha)} = \frac{1-\alpha}{1+\alpha} \Rightarrow \alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$. Usually, $\alpha \geqslant 0.05$.
We know α and we know $K\alpha$ (step 1), so we can calculate K.

Design with Bode plots

Step 5

Use the gain crossover frequency of P(s)C(s), denoted by ω_m:

$$| P(j\omega_m)C(j\omega_m) |= 1$$

$$| P(j\omega_m) | K \frac{\sqrt{\frac{1}{\alpha\tau^2}+\frac{1}{\tau^2}}}{\sqrt{\frac{1}{\alpha\tau^2}+\frac{1}{\alpha^2\tau^2}}} =| P(j\omega_m) | K\sqrt{\alpha} = 1$$

$$20\log | P\left(j\omega_m\right) |= -20\log\left(K\sqrt{\alpha}\right)$$

$$\Rightarrow GCF\left(P(s)K\sqrt{\alpha}\right) = \omega_m$$

The value of the tangent point ω_m can be determined from the Bode plot of $K\sqrt{\alpha}P(s)$, we know K and α from step 4.

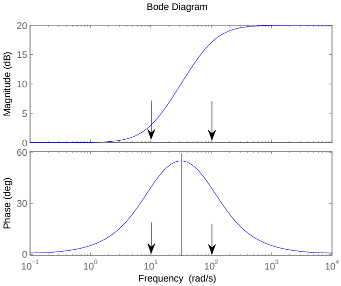
Design with Bode plots

Step 6

The tangent point ω_m is the geometric mean (frequency scale is exponential!) of the two corner frequencies (zeros and poles), so

$\log \omega_m = \frac{1}{2}(\log \frac{1}{\tau} + \log \frac{1}{\alpha\tau})$ with $\tau = \frac{1}{\omega}$

$\Rightarrow \omega_m = \frac{1}{\sqrt{\alpha}\tau} \Rightarrow \tau = \frac{1}{\sqrt{\alpha}\omega_m}$.



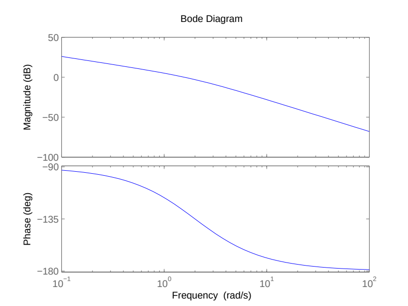
Design with Bode plots

Step 7

Verify that the system behaves as desired. Check the phase margin and steady state to make sure it is satisfactory. If not, repeat the design process by modifying the pole-zero location of the compensator until a satisfactory result is obtained.

Example

Given the system $P(s) = \frac{4}{s(s+2)}$ with Bode Diagram below. We want the phase margin to be at least 50° and a steady state error $K_v = \frac{20}{sec}$.

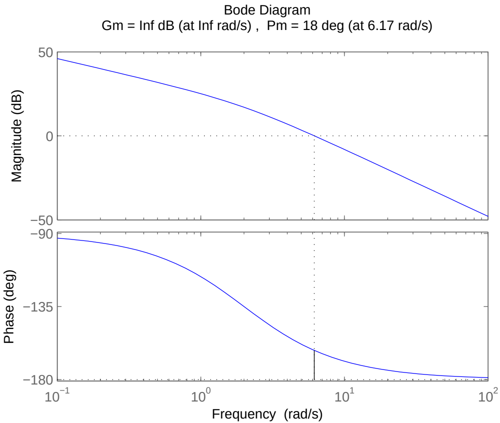


Step 1

- Steady state requirement: $K_v = \frac{20}{sec}$
  So, $\lim_{s\to 0} sP(s)C(s) = \lim_{s\to 0} s\frac{4}{s(s+2)}K\alpha = 2K\alpha = 20.$
  $\Rightarrow K\alpha = 10$
- Would a proportional controller with gain $K\alpha$ suffice? We have a look at the Bode Diagram of $K\alpha P(s)$ (see next slide) $\Rightarrow$ does not suffice! The phase margin is obviously smaller than $50\,^\circ$.

Step 2

- Phase margin of $K\alpha P(s) = 18\,^\circ$.
- We want a phase margin of at least $50\,^\circ \Rightarrow \phi = 32\,^\circ$.


Bode Diagram
Gm = Inf dB (at Inf rad/s) , Pm = 18 deg (at 6.17 rad/s)
Magnitude (dB)
50
0
−50
Phase (deg)
−90
−135
−180
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)

Step 2: alternative way

Calculation of phase margin without phase diagram:
We need the frequency where the magnitude is 0 dB.
So, $20 \log \mid K\alpha P(s) \mid = 0 \Rightarrow \mid K\alpha P(s) \mid = 1 \Rightarrow \mid P(s) \mid = 0.1$
When substituting $s = j\omega$ and calculating the modulus of the complex number at the left side, the equation becomes:

$$\left(\frac{-4\omega}{\omega(4+\omega^2)}\right)^2 + \left(\frac{-8}{\omega(4+\omega^2)}\right)^2 = 0.01.$$

This equation has just one real positive solution: $\omega = 6.168$.
Now, we have the right frequency. We can calculate $10P(6.168j)$ $=$ -0.951 - 0.308j. With the arctangent, we find that the phase of $10P(6.168j)$ is equal to $-162.05\,^\circ$. The phase margin is the difference between this value and $-180\,^\circ$, so the phase margin is equal to $17.95\,^\circ$.

Step 3

$\phi_m = \phi + 5^\circ = 37^\circ$

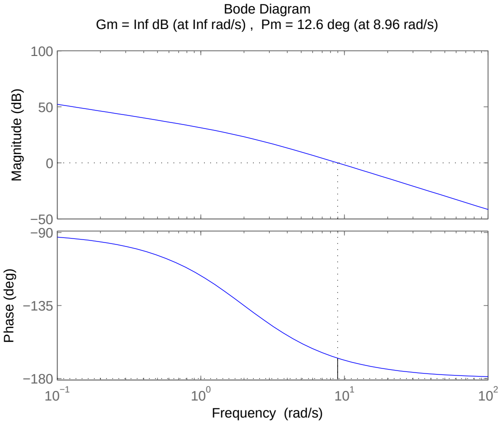
Step 4

$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m} = 0.24$

From step 1, we know that $K = \frac{10}{\alpha} = 42$

Step 5

Find ω_m, the frequency at which the gain is $-20\log(K\sqrt{\alpha})$ dB.
$GCF\,(P(s)K\sqrt{\alpha}) = \omega_m = 9\frac{rad}{s}$ (see Bode Diagram of $P(s)K\sqrt{\alpha}$ next slide)


Bode Diagram
Gm = Inf dB (at Inf rad/s) , Pm = 12.6 deg (at 8.96 rad/s)
Magnitude (dB)
100
50
0
−50
Phase (deg)
−90
−135
−180
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)

Step 6

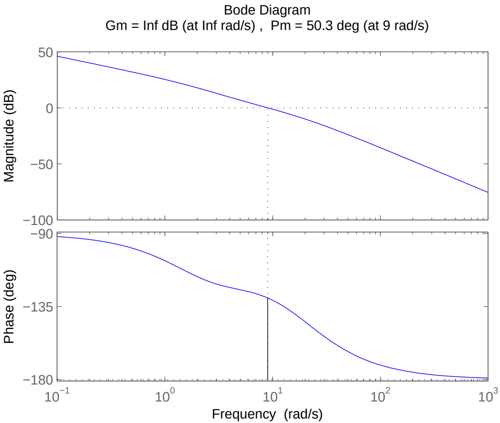
$\tau = \frac{1}{\omega_m\sqrt{\alpha}} = 0.23$

Step 7

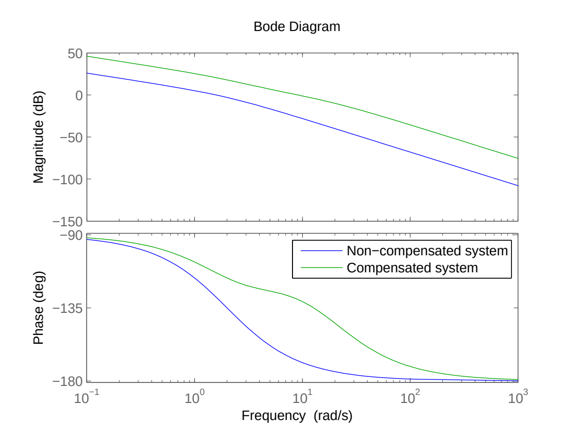
We verify whether or not our solution is correct. We plot the Bode Diagram of $\frac{4}{s(s+2)}K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with the α, τ and K we found. (see next slide)

We see that:

- the phase margin is indeed more than 50°
- the new tangent point is indeed about $9\frac{rad}{s}$
- the steady state error is all right


Bode Diagram
Gm = Inf dB (at Inf rad/s) , Pm = 50.3 deg (at 9 rad/s)
Magnitude (dB)
50
0
−50
−100
Phase (deg)
−90
−135
−180
10^−1
10^0
10^1
10^2
10^3
Frequency (rad/s)

Comparing compensated system vs non-compensated system



Outline

Lag compensators

Transfer function

$C(s) = K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}}$ with $\beta>1$

Bode Diagram

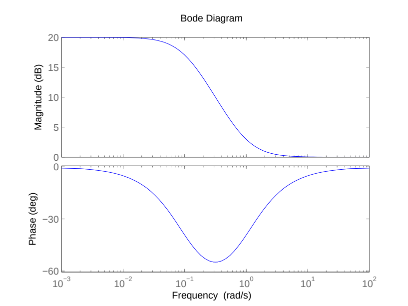
Example with $K = 1$ and $\beta = 10$ (see next slide)
Magnitude of the lag compensator:

- becomes 10 (= 20 dB) for small frequencies
- becomes unity (= 0 dB) for high frequencies

$\Rightarrow$ Lag compensator is **low-pass filter**.

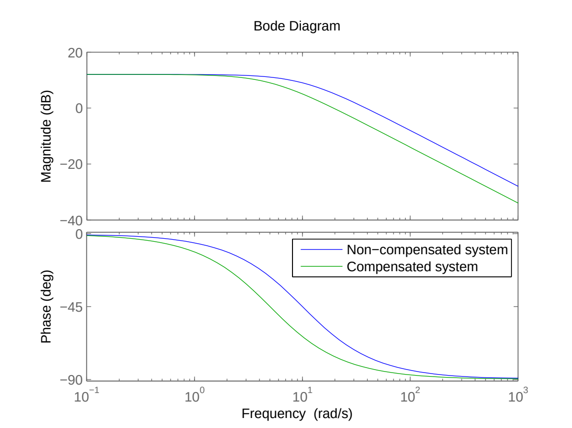
Lag compensators

Bode Diagram



Lag compensators

Impact of lag compensators: Bode Diagram



Lag compensators

Impact of lag compensators: Bode Diagram

- The primary function of a lag compensator is to provide attenuation in the high-frequency range to give a system sufficient phase margin. The phase-lag characteristic is of no consequence in lag compensation.
- A lag compensator decreases the bandwidth/speed of response:
  - good if your model is bad at high frequencies
  - good to reduce the impact of high-frequency noise
  - bad if you want the system to react fast $\Rightarrow$ use lead compensator

Lag compensators

Design with Bode plots

- Increase of phase margin implies decrease of the magnitude at high frequencies
- Decrease of the steady state error implies increase of the DC-level ($\omega = 0$)

A lag compensator can realize both conditions.

- At high frequencies, the gain becomes: $\lim_{s\to\infty} K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}} = K$
- At $\omega = 0$, the gain becomes: $\lim_{s\to 0} K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}} = K\beta$

Design with Bode plots

Step 1

- Determine $K\beta$ in a similar way as we found $K\alpha$ for the lead compensator.
- Translate your steady state requirement in terms of the error constant:
  - $K_p = \lim_{s\to 0} P(s)C(s)$
  - $K_v = \lim_{s\to 0} sP(s)C(s)$
  - $K_a = \lim_{s\to 0} s^2 P(s)C(s)$
- With this $K_p/K_v/K_a$ and $\lim_{s\to 0} P(s)$, we can determine $\lim_{s\to 0} C(s) = K\beta$.
- Verify whether a proportional controller with gain $K\beta$ would suffice.

Design with Bode plots

Step 2

- Take the frequency (ω) at which $P(s)$ has the desired phase (-180° + the desired phase margin + a safety factor of 10°). The addition of 10° compensates for the phase lag of the lag compensator. Choose this frequency as the new gain crossover frequency ω_{GCF}.

Design with Bode plots

Step 3

- Choose the corner frequency ω one decade below ω_{GCF} to avoid bad effects of phase lag due to the lag compensator. The pole and zero must be located at a lower frequency than ω_{GCF} so that it will not affect the phase margin.
- Compute $\tau = \frac{1}{\omega}$.

Design with Bode plots

Step 4

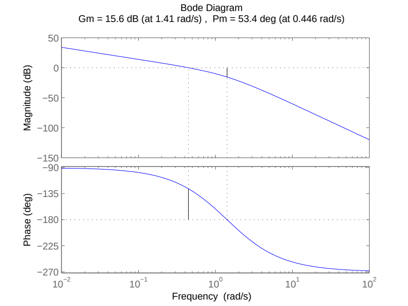
- Determine the attenuation Q (in dB!) necessary to bring the magnitude curve down to 0 dB at ω_{GCF}.
- Compute $\beta = 10^{\left(\frac{-Q}{20}\right)}$.

Step 5

- We just have calculated β (step 4) and we know $K\beta$ (step 1), so it's possible to determine K.
- Verify the behavior of the resulting system.

Example

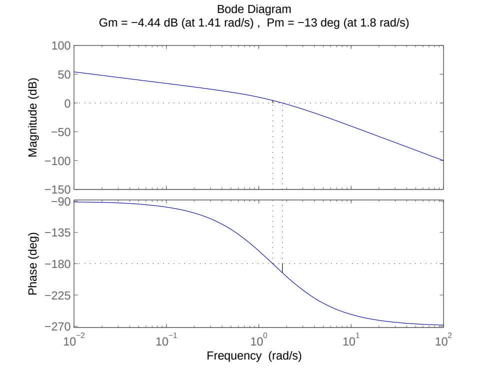
Given the system $P(s) = \frac{1}{s(s+1)(s+2)}$ with Bode Diagram below. We want a steady state error $K_v = \frac{5}{sec}$ and a PM $\geq 40^\circ$.



Step 1

- Steady state requirement: $K_v = \frac{5}{sec}$
  $\Rightarrow \lim_{s\to 0} sP(s)C(s) = \frac{1}{2}\lim_{s\to 0} C(s) = \frac{5}{sec} \Rightarrow K\beta = 10$
- Would a proportional controller with gain $K\beta$ suffice? To answer this, we must have a look at the Bode plot of $K\beta P(s)$ (see next slide).
  $\Rightarrow$ Does not suffice! Adding a gain of $10 = 20dB$ to get the right steady state error, the phase margin would become negative. In other words, at the frequency where the magnitude of $K\beta P(s)$ equals 0 dB, the phase is less than -180° which means the system would become unstable.

Step 1 continued



Step 2

- Determine the desired phase
  $phase = -180^\circ + 40^\circ + 10^\circ = -130^\circ$
- $P(s)$ has the desired phase at a frequency of about $\omega_{GCF} = 0.5\frac{rad}{s}$. (see Bode Diagram $P(s)$)

Step 3

- We take ω one decade below ω_{GCF}, so $\omega = 0.05\frac{rad}{s}$.
- We compute $\tau = \frac{1}{\omega} = 20$.

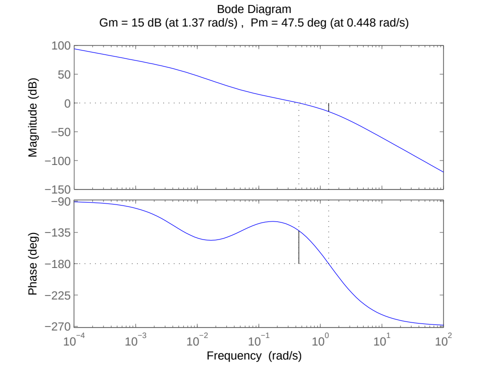
Step 4

- To have an amplitude of $0dB$ at ω_{GCF}, we need an attenuation $Q = -20dB$.
- $\beta = 10^{\frac{-Q}{20}} = 10$

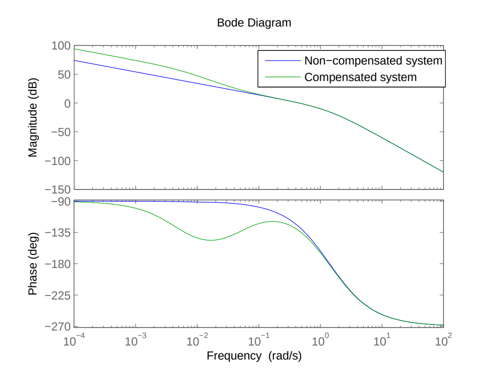
Step 5

- From the results of step 1 and step 4, we find $K = 1$.
- We find $C(s) = \frac{s+0.05}{s+0.005}$.
  We verify the behavior of $P(s)C(s) = \frac{1}{s(s+1)(s+2)}\frac{s+0.05}{s+0.005}$ on its Bode Diagram (see next slide).
  The phase margin is indeed greater than 40° and the steady state error is all right.

Step 5 continued



Comparing compensated system vs non-compensated system



Outline

Lag-lead compensators

Introduction

In some cases you would want to combine the effects of a lag and a lead compensator:
In most cases a lead compensator is more fit to increase the phase margin and a lag compensator is better at decreasing the steady state error.

Lag-lead compensators

Transfer function

$C(s) = K\frac{s+\frac{1}{\tau_1}}{s+\frac{1}{\alpha\tau_1}}\frac{s+\frac{1}{\tau_2}}{s+\frac{1}{\beta\tau_2}}$ with $\beta>1$, $0<\alpha<1$, $\tau_1<\tau_2$

Usually, we take $\beta = \frac{1}{\alpha}$, but that's not necessary.

- The term $\frac{s+\frac{1}{\tau_1}}{s+\frac{1}{\alpha\tau_1}}$ produces the effect of the lead network.
- The term $\frac{s+\frac{1}{\tau_2}}{s+\frac{1}{\beta\tau_2}}$ produces the effect of the lag network.
- Zeros: $\frac{1}{\tau_1}$ and $\frac{1}{\tau_2}$
- Poles: $\frac{1}{\alpha\tau_1}$ and $\frac{1}{\beta\tau_2}$

Lag-lead compensators

Bode Diagram

Example with K $= 10$, $\alpha = 0.1$, $\beta = 10$, $\tau_1 = 0.01$, $\tau_2 = 10$ (see next slide)

Magnitude of lag-lead compensator:

- becomes 10 ($= 20$ dB) at low frequencies
- becomes unity ($= 0$ dB) at frequencies of about $1\frac{rad}{s}$ to $10\frac{rad}{s}$
- becomes 10 ($= 20$ dB) at high frequencies

$\Rightarrow$ Lag-lead compensator is a **band stop filter**.

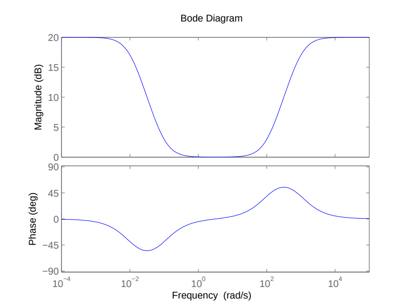
Explanation: $\beta\tau_2 > \tau_2 > \tau_1 > \alpha\tau_1$

$\Rightarrow \frac{1}{\beta\tau_2} < \frac{1}{\tau_2} < \frac{1}{\tau_1} < \frac{1}{\alpha\tau_1}$

$\Rightarrow pole_1 < zero_1 < zero_2 < pole_2$

Lag-lead compensators

Bode Diagram



Lag-lead compensators

Bode Diagram

Both the good properties of a lag and a lead compensator are used (see Bode Diagram last slide):

- At low frequencies, the lag compensator is meant to decrease the steady state error.
- At high frequencies, the lead compensator is meant to increase the phase margin.

Lag-lead compensators

Design with Bode plots

The design of a lag-lead compensator by the frequency-response approach is based on the combination of the design techniques discussed under lead compensation and lag compensation.

Disadvantage

The disadvantage of a lag-lead compensator over a lag compensator or a lead compensator is its increased complexity, and hence cost (the same way a lag or lead compensator is more complex/costly than a proportional controller).

Outline

Comparison lead, lag and lag-lead compensators

Method

- Lead compensation achieves the desired result through the merits of its phase lead contribution.
- Lag compensation accomplishes the result through the merits of its attenuation property at high frequencies.

Comparison lead, lag and lag-lead compensators

Bandwidth

Lead compensators are commonly used for improving stability margins and yields a higher gain crossover frequency than it is possible with lag compensation. The higher gain crossover frequency means a larger bandwidth.

- Advantage: reduction in the settling time $\Rightarrow$ fast response
- Disadvantage: it makes the system more susceptible to noise because of an increase in the high-frequency gain.

Comparison lead, lag and lag-lead compensators

Gain

Lead compensation requires an additional increase in gain to offset the attenuation inherent in the lead network.
$\Rightarrow$ Lead compensation will require a larger gain than that required by lag compensation.

Large control actions

The lead compensation may generate large control actions in the system, which might lead to saturation in the actuators.

Comparison lead, lag and lag-lead compensators

High and low frequencies

Lag compensation reduces the system gain at higher frequencies without reducing the system gain at lower frequencies. Thanks to this, the total system gain can be increased, as well as the low-frequency gain and the steady state accuracy can be improved. Also, any high-frequency noise involved in the system is attenuated.

Transient response

Lag compensation will introduce a pole-zero combination near the origin that will generate a long tail with small amplitude in the transient response.

Comparison lead, lag and lag-lead compensators

Lag-lead compensators

If both fast responses and good static accuracy are desired, a lag-lead compensator may be employed. By use of the lag-lead compensator, the low-frequency gain can be increased (which means an improvement in steady state accuracy), while at the same time the system bandwidth and stability margins can be increased.

Most important of all

In daily life, you will need a combination of all those techniques!