Chapter 5: Continuous time systems

July 17, 2015

Outline

Linear differential equations: definitions 1/2

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The **order of a LDE** is the index of the highest derivative of y.

Linear differential equations: definitions 2/2

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- if $A_{0:n}(t)$ are constants (ie. not functions of time), the LDE is said to have **constant coefficients**

Example: radioactive decay 1/2

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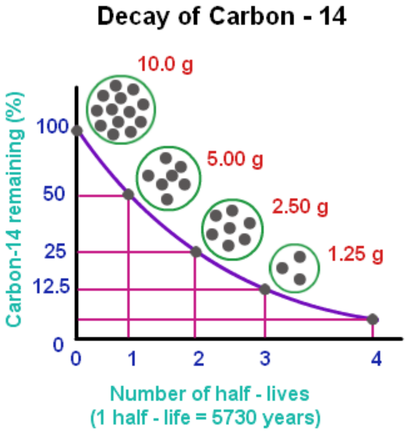

Solving homogeneous LDEs with constant coefficients 1/3

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Dividing by e^{zt} yields the nth order **characteristic polynomial**:

$$F(z) = \sum_{i=0}^{n} A_i z^{n-i} = 0.$$

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The specific linear combination depends on initial conditions.

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These zeros correspond to the following basis functions t:

$$e^{jt}, \quad e^{-jt}, \quad e^{t}, \quad te^{t}.$$

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The final value theorem states $f(\infty) = \lim_{s\to 0} sF(s)$,
if all poles of $sF(s)$ are in the left half plane (ie. real part < 0).

Important properties of the Laplace transform

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with $u(t) = \int^t \delta(t)dt$ (Heaviside) and $\delta(t)$ the Dirac delta

Inverse Laplace transform

The inverse Laplace transform converts s-domain to time domain:

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Practically, the inverse Laplace transform takes two steps:

1. write $F(s)$ in terms of partial fractions
2. transform each term in the partial fraction based on tables of s/t-domain pairs (course notes p 4.32-4.33)

Outline

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Via induction, the Laplace transform of the nth order derivative:

$$\mathcal{L}\{f^{(n)}\} = s^nF(s) - \sum_{i=1}^{n} s^{n-i} f^{(n-i)}(0)$$

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$$\sum_{i=0}^{n} A_i \mathcal{L}\{y^{(n-i)}(t)\} = \mathcal{L}\{f(t)\}$$

Solving LDEs with the Laplace transform 3/3

$$\sum_{i=0}^{n} A_i \mathcal{L}\{y^{(n-i)}(t)\} = \mathcal{L}\{f(t)\} \tag{1}$$

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Expanding Eq. (2) into (1) yields:

$$Y(s)\sum_{i=0}^{n} A_i s^i - \sum_{i=1}^{n}\sum_{j=1}^{i} A_i s^{i-j} y^{j-1}(0) = F(s)$$

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The solution in the time domain is obtained via the inverse Laplace transform: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

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This holds for linear, time-invariant systems with n states if:

$$rank(\mathcal{O}) = n, \quad \mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}, \quad \mathcal{O} : \textbf{observability matrix}$$

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A linear, time-invariant system with n states is controllable if:

$$rank(\mathcal{C}) = n, \quad \mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \ldots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix},$$

where $\mathcal{C}$ is called the **controllability matrix**.

Outline

Transfer function

The transfer function of input i to output j is defined as:

$$H_{i,j}(s) = \frac{Y_j(s)}{U_i(s)}, \quad \mathbf{U}(s) = \mathcal{L}\{u(t)\}, \quad \mathbf{Y}(s) = \mathcal{L}\{y(t)\}.$$

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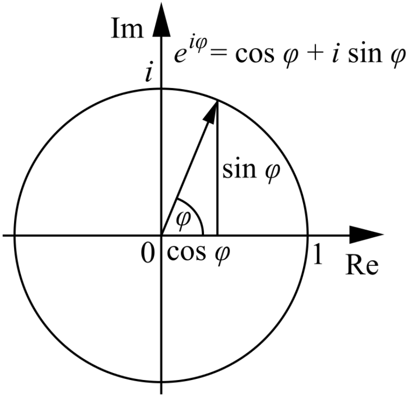
The frequency response of a system can be analyzed via $\mathbf{H}(j\omega)$:

$$e^{\sigma + j\omega} = e^{\sigma}(\cos\omega + j\sin\omega).$$

Illustration of Euler's formula



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Poles and zeros may cancel, ie. if $D(s) = N(s) = 0$ for some s.

Steady state response

The output of a linear time-invariant system yields consists of:

- a steady-state output $y_{ss}(t)$, which similar periodicity to $u(t)$
  $\rightarrow$ y_{ss} comprises the same frequencies as $u(t)$
- a transient output $y_{tr}(t)$
  $\rightarrow$ if the system is stable, then $\lim_{t\rightarrow\infty} y_{tr}(t) = 0$
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The steady-state output $y_{ss}(t)$ of a linear time invariant system:

- consists of signals of same frequencies as the input signal $u(t)$
- which may have been magnified and/or phase changed

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The speed of convergence depends on the position of the poles.

Time constant

The transfer function of first order systems can be written as:

$$H(s) = \frac{K}{\tau s + 1} \quad \leftrightarrow \quad h(t) = \frac{K}{\tau} e^{-t/\tau},$$

where τ is called the system's **time constant**.

Time constant

The transfer function of first order systems can be written as:

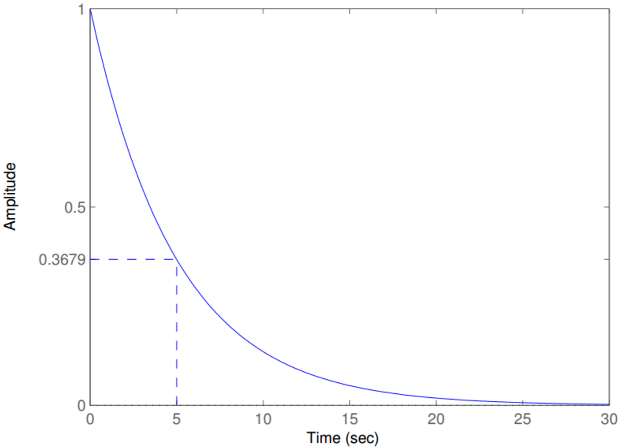
$$H(s) = \frac{K}{\tau s + 1} \quad \leftrightarrow \quad h(t) = \frac{K}{\tau} e^{-t/\tau},$$

where τ is called the system's **time constant**.

The time constant summarizes the speed of a system's dynamics:

- after τ seconds, the impulse response reaches $h(0)/e$.
- after τ seconds, the step response has reached $1 - e^{-1} \approx 63\%$ of its regime value.

Impulse response $H(s) = 5/(5s+1) \leftrightarrow h(t) = exp(-t/5)$



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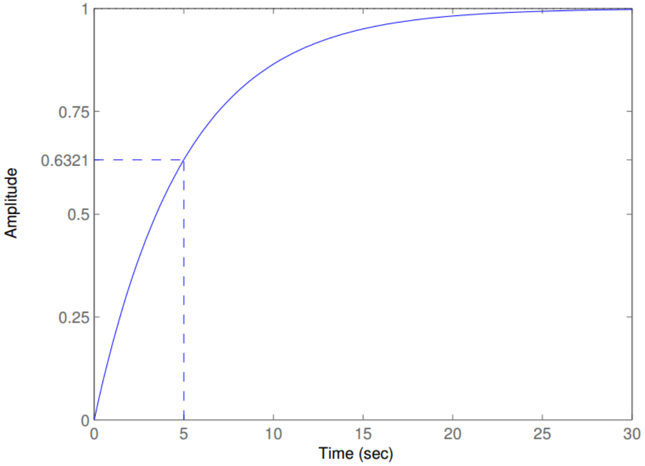

Outline

From state-space to transfer functions

We start from the linear state-space representation:

time domain Laplace domain

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \quad \leftrightarrow \quad \begin{cases} s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{cases}$$

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$$\begin{aligned} (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{B}\mathbf{U}(s) \\ \mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \\ \Rightarrow \mathbf{Y}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) \\ \Rightarrow \mathbf{H}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \end{aligned}$$

Relationship between poles and eigenvalues of $\mathbf{A}$ 1/2

Poles are zeros of the denominator of $\mathbf{H}(s)$, e.g. those values of s for which $\mathbf{H}(s)$ is singular.

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Consider the following SISO system with 2 states:

$$\begin{bmatrix} sX_1(s) \\ sX_2(s) \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} + \begin{bmatrix} \beta \\ 2 \end{bmatrix} U(s)$$

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The transfer function $H(s) = \frac{\beta}{s-\alpha}$ has only one pole ($s_1 = \alpha$).
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Outline

1. Linear differential equations
2. Laplace transform
3. Solving LDEs with the Laplace transform
4. Properties of state-space representation
5. Transfer functions
   - Impulse response and time constant
   - Relationship between state space and transfer functions
6. Transient response analysis of first order and second order systems
   - First order systems
   - Second order systems

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Transient Response

The time response of a control system may be written as:

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

Where $y_{tr}(t)$ is the transient response and $y_{ss}(t)$ is the steady state response. Most important characteristic of dynamic system is absolute stability.

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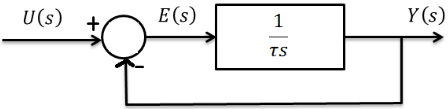
Transient response: when input of system changes, output does not change immediately but takes time to go to steady state

First order systems

E.g.RC circuit, thermal system, ...



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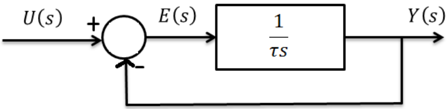

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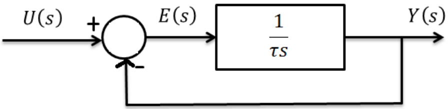

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Unit step response

- Laplace of unit-step is $\frac{1}{s}$ $\rightarrow$ substituting $U(s) = \frac{1}{s}$ into equation $Y(s) = \frac{1}{s}\frac{1}{\tau s+1}$
- Expanding into partial fractions gives $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$

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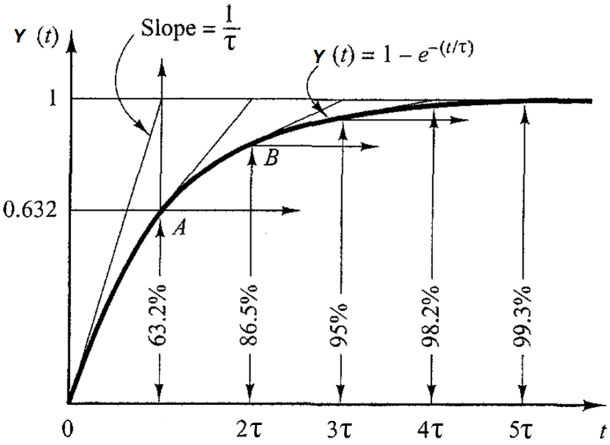
   $\frac{dy}{dt}|_{t=0} = \frac{1}{\tau}e^{-\frac{t}{\tau}}|_{t=0} = \frac{1}{\tau}$

   Where τ is called the system time constant

Unit step transient response



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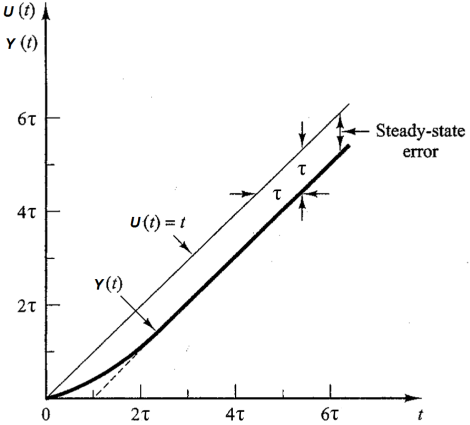
Unit ramp transient response



Unit-Impulse Response

For a unit-impulse input, $U(s) = 1$ and the output is

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For $t \to +\infty$, $y(t) \to 0$

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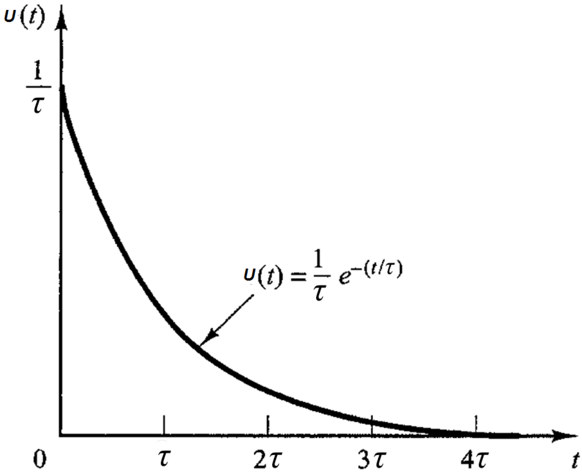

Outline

Second order systems

A second order system can generally be written as:

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If the transfer function has two real poles, the frequency response can be found by combining the effects of both poles

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Sometimes the transfer function has two complex conjugate poles. In that case we have to find a different solution for finding the frequency response.

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In order to study the transient behaviour, let us first consider the following simplified example of a second order system

$$H(s) = \frac{c}{ds^2 + es + c}$$

Step response second order system

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$$= \frac{\frac{c}{d}}{[s + \frac{e}{2d} + \sqrt{(\frac{e}{2d})^2 - \frac{c}{d}}][s + \frac{e}{2d} - \sqrt{(\frac{e}{2d})^2 - \frac{c}{d}}]}$$

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4. The poles are real if
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To simplify the transient analysis, it is convenient to write

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Where

σ is the attenuation

ω_n is the natural frequency

ζ is the damping ratio

Step response second order system

The transfer function can now be rewritten as

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega s + \omega_n^2}$$

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The dynamic behavior of the second-order system can then be described in terms of only two parameters ζ and ω_n

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If $0 < \zeta < 1$, the poles are complex conjugates and lie in the left-half s-plane

- The system is then called **underdamped**
- The **transient response is oscillatory**

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If $\zeta > 1$, the system is called **overdamped**

We will now look at the unit step response for each of these cases

Underdamped system

For the underdamped case ($0 < \zeta < 1$), the transfer function can be written as

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

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Where ω_d is called the damped natural frequency

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

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For a unit-step input we can write

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}$$

Underdamped system

Which can be rewritten as partial fractions

$$
\begin{aligned}
Y(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
&= \frac{1}{s} - \frac{s + \zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} - \frac{\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}
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\end{aligned}
$$

It can be shown that

$$
\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] &= e^{-\zeta\omega_n t} cos(\omega_d t) \\
\mathcal{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] &= e^{-\zeta\omega_n t} sin(\omega_d t)
\end{aligned}
$$

Underdamped system

Therefore:

$$\mathcal{L}^{-1}\Big[Y(s)\Big] = y(t)$$

Underdamped system

Therefore:

$$\mathcal{L}^{-1}\left[Y(s)\right] = y(t)$$

$$= 1 - e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t))$$

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Underdamped system

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It can be seen that the frequency of the transient oscillation is the damped natural frequency ω_d and thus varies with the damping ratio ζ

Underdamped system

The error signal is the difference between input and output

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e(t) &= y(t) - u(t) \\
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At steady state, or at $t = \infty$, the error goes to zero

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- Once the faster decaying exponential term has disappeared, the response is similar to that of a first-order system
- In that case, $H(s)$ can be approximated by

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With the approximate transfer function, the unit-step response becomes

$$Y(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

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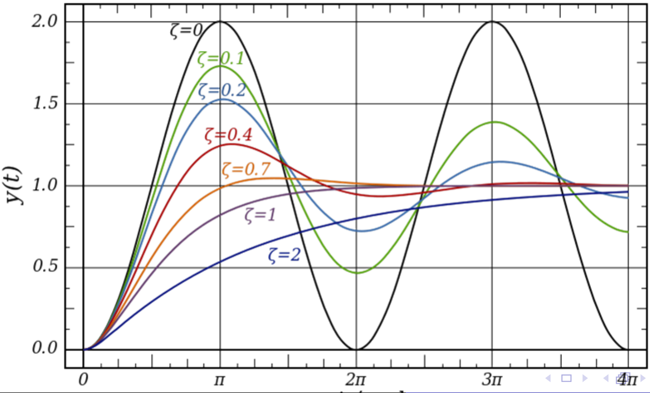
The time response for the approximate transfer function is then given as

$$y(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}, \text{ for } t \leq 0$$

Second order systems unit step response curves

Response on a step function



Second order systems - characteristics

Overshoot: Highest amplitude above steady state.

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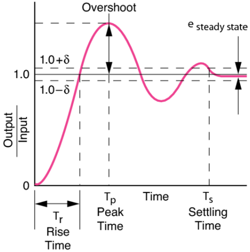
Peak Time: Time to reach overshoot.

$t_p = \frac{\pi}{\omega_d}$



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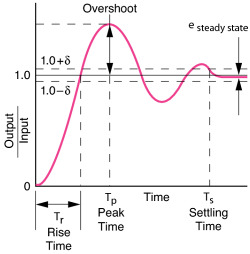
We find:

$$e^{-\zeta\omega_n T_s} < 0.02$$

$$T_s = \frac{4}{\omega_n \zeta}$$



Second order systems - resonace

The resonance frequency is the frequency at which the systems output has a larger amplitude than at other frequencies. This happens when underdamped functions oscillate at a greater magnitude than the input.

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A different view on the Tacoma bridge disaster:
`https://www.youtube.com/watch?v=6ai2QFxStxo`

In fact the collapse was a result of a number of effects like Aerodynamic flutter and vortices. Read the full article here:
`http://www.ketchum.org/billah/Billah-Scanlan.pdf`

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Another phenomenon with bridges and resonance is that many people marching with the same rhythm can cause a bridge to start resonating like the Angers bridge in 1850. A more recent example is the Millennium bridge in London who started resonating.

Second order systems - damping

When we want a system with no resonance, we choose one with damping < 0.707. This means a pole between 135° and 225°:

$$\arctan\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = +135^\circ$$

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We mostly want a short settling time (< 4s). This results in another restriction on the poles of the system:

$$\tau_n = \frac{4}{\omega\zeta} < 4\text{s}$$

$$\omega_n\zeta > 1$$

Second order systems - damping