Chapter 12: Lead and Lag Compensators

August 3, 2015

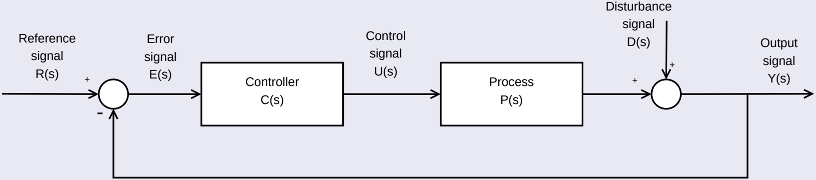
Outline

1. Definition of compensators

2. Lead compensators

Lead Compensator vs Lag Compensator

Schematical representation



Transfer functions

Lead compensator : $C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with $0<\alpha<1$

Lag compensator : $C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}}$ with $\beta>1$

Lead Compensator vs Lag Compensator: zeros and poles

Transfer functions

Lead compensator : $C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with $0<\alpha<1$

Lag compensator : $C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}}$ with $\beta>1$

Zeros and poles

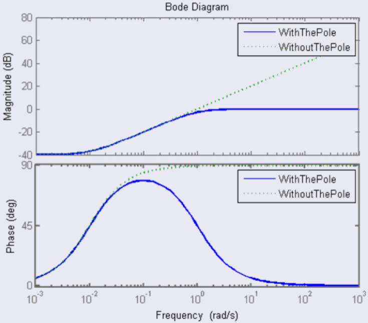
Zeros: $s = -\frac{1}{\tau}$
Poles: $s = -\frac{1}{\alpha\tau}$ or $s = -\frac{1}{\beta\tau}$
For lead compensators the pole lies more to the left in the complex plane than the zero and vice versa for lag compensators.

Outline

Lead compensators

Impact

$$C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}} \text{ with } 0<\alpha<1$$



Lead compensators

Impact

- They push the pole of the closed loop system to the left.
  - Stabilisation of the system (see root locus)
  - Increase response speed (lead compensator will stimulate some larger frequencies)
- Increase of the phase margin: the phase of the lead compensator is positive for every frequency, and will hence only increase the phase.
- Thanks to the presence of a pole, the high frequencies (where most of the unwanted noise is located) are less amplified.

Lead compensators

Design with Bode plots

- Design process: tuning of the phase margin, with as a surplus (because we will have one extra degree of freedom) the tuning of the steady state error.
- Compensate for the excessive phase lag that is a result of the components of P(s).
- Increase in phase at gain crossover frequency (GCF) if GCF is around pole and zero of the lead compensator.
- Gain is impacted by the lead compensator: the GCF of P(s).C(s) is not equal to the GCF of P(s).

Lead compensators

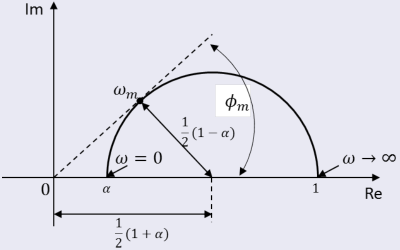
Design

- Required increase in phase gain: ϕ
- To compensate for increase GCF due to $C(s) \Rightarrow \phi_m = \phi + 5\,^\circ$. This will be needed to determinate α and τ
- K will be used to tune the steady state error.

Lead compensators

Determination of α

Use polar plot of $\frac{\alpha.(j\omega\tau+1)}{j\omega\alpha\tau+1}$



$\sin\phi_m = \frac{\frac{1}{2}.(1-\alpha)}{\frac{1}{2}.(1+\alpha)} = \frac{1-\alpha}{1+\alpha} \Rightarrow \alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$

This relation relates the maximum phase-lead angle and the value of α.

Lead compensators

Determination of τ

Tekening bodeplot zoals boek p 622. The tangent point ω_m is the geometric mean of the two corner frequencies, so
$\log \omega_m = \frac{1}{2}(\log \frac{1}{\tau} + \log \frac{1}{\alpha\tau})$ with $\tau = \frac{1}{\omega}$
$\Rightarrow \omega_m = \frac{1}{\sqrt{\alpha\tau}}$.
The lead compensator is a high pass filter.

Lead compensators

Determination of the tangent point

Use the gain crossover frequency of P(s)C(s) as ω_m:

$$| P(j\omega_m)C(j\omega_m) |= 1$$

$$| P(j\omega_m) | K \frac{\sqrt{\frac{1}{\alpha\tau^2}+\frac{1}{\tau^2}}}{\sqrt{\frac{1}{\alpha\tau^2}+\frac{1}{\alpha^2\tau^2}}} =| P(j\omega_m) | K\sqrt{\alpha} = 1$$

$$20\log | P(j\omega_m) |= -20\log(K\sqrt{\alpha})$$

The value of the tangent point ω_m can be determined from P(s)'s Bode plot, if you know K (the last freedom).

Lead compensators

Determination of K

- Remember the steady state error for references of the shape: $\frac{At^n\epsilon(\tau)}{n!}$ with $\epsilon(t)$ the step function.
- We found the error constants K_p, K_v and K_a as measures for the steady state error for a proportional (n=0), linear (n=1) and accelerating (n=2) reference.
- So these error constants can be used to find proper values of K: $\lim_{s\to 0} K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}s^nP(s) = K\alpha\lim_{s\to 0} s^nP(s)$

Lead Compensation Techniques Based on the Frequency-Response Approach

Step 1

Find $K\alpha$ from your steady-state requirement.

Step 2

Determine ϕ, the amount with which you want to increase the PM; if the PM is OK, you dont need a lead compensator; a proportional controller with gain $K\alpha$ suffices.

Step 3

Add 5°, to get $\phi_m = \phi + 5^\circ$ (if $\phi_m > 5^\circ$, you will need more than one lead compensator). The addition of the lead compensator shifts the gain crossover frequency to the right and decreases the phase margin.

Lead Compensation Techniques Based on the Frequency-Response Approach

Step 4

Youll find α from this ϕ_m: $\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$ and hence also K (see step 1).

Step 5

Find the desired ω_m by looking at the Bode plot of P(s) and finding the frequency at which the gain equals $-20\log(K\sqrt{\alpha})$ dB.

Step 6

Find τ as $\frac{1}{\sqrt{\alpha}\omega_m}$.

Lead Compensation Techniques Based on the Frequency-Response Approach

Step 7

Verify if the system works as asked. Check the gain margin to be sure it is satisfactory. If not, repeat the design process by modifying the pole-zero location of the compensator until a satisfactory result is obtained.

Example

Given the system $P(s) = \frac{4}{s(s+2)}$. We want a phase margin of at least 50° and a steady state error for slope reference of maximal $\frac{A}{20}$.

Example: Bode plot and phase diagram

Maken in MATLAB.

Example

Step 1

Steady-state requirement: $K_v = \frac{20}{s}$
So, $\lim_{s\to 0} sP(s)C(s) = \lim_{s\to 0} s\frac{s}{s(s+2)}K\alpha = 2K\alpha = 20.$
$\Rightarrow K\alpha = 10$

Step 2

Phase margin of $K(s) = 18^\circ$ (see phase diagram)
$\Rightarrow \phi = 32^\circ$

Step 3

$\phi_m = \phi + 5^\circ = 37^\circ$

Example

Step 4

$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m} = 0.24$
From step 1, we know that $K = \frac{\alpha}{10} = 42$

Step 5

Find ω_m, the frequency at which the gain is $-20\log(K\sqrt{\alpha})$ dB.
$GCF(P(s)K\sqrt{\alpha}) = GCF(P(s)C(s)) \Rightarrow \omega_m = 9\frac{rad}{s}$
(see Bode diagram next slide)

Step 6

$\tau = \frac{1}{\omega_m\sqrt{\alpha}} = 0.23$

Example

Step 7

Verify! MATLAB!!!!

Summary lead compensators

Evaluation of impact

- Pushing the poles to the left: this is not directly visible here, but is linked to the increased band width.
- The increase in bandwidth (this is linked to the response speed) and the increase in the phase margin were apparent in the Bode plot of P(s)C(s).
- A (small) decrease in the steady-state error occurs, since we designed it as such.
  Why small? The steady-state error decreases when the DC gain gets larger, but a lead compensators impact on the gain is not really built to increase the DC gain, the shape of a lag compensator is much more fit for this.

Summary lead compensators

Design with root locus

Design lead compensators with root locus for time-domain quantities - use dominant pole locations to fulfill overshoot, rise time, settling time, damping ratio, requirements.

Outline

Lag compensators

Impact of lag compensators: Bode diagram

MATLAB

Lag compensators

Transfer function

$$C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}} \text{ with } \beta>1$$

Impact of lag compensators: Bode diagram

- Lead compensators: increase the stability and tune the steady-state error by increasing the phase at the crossover frequency.
- Impact lag compensator = lead compensator, but different approach! By decreasing the gain, the gain crossover frequency comes down to a frequency at which the corresponding phase is higher.