Chapter 6: Frequency response of dynamical systems

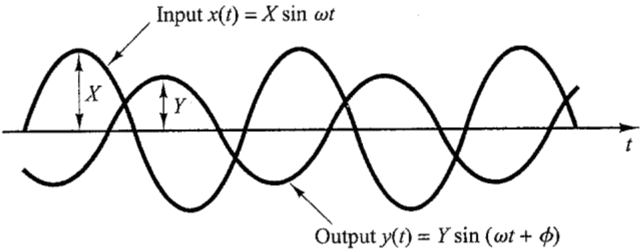
July 9, 2015

Outline

What is the frequency response of a system?

- Frequency response = steady state response of a system to a sinusoidal input.
- Assume an input $x(t) = X \sin(\omega t)$.
- The output in a linear system is then also sinusoidal, with a change in the magnitude and phase, i.e. $y(t) = Y \sin(\omega t + \phi)$
- It can be shown that: $Y = X \cdot |H(j\omega)|$ and $\phi = \angle H(j\omega)$
- $H(j\omega)$ is therefore also called the sinusoidal transfer function.

Relation of sinusoidal input/output in a linear system



Outline

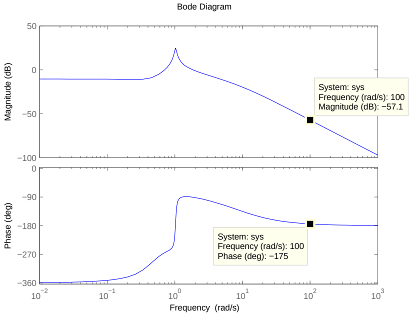
The bode plot

- A bode plot is a graphical representation of the sinusoidal transfer function $H(j\omega)$
- It consists of two seperate plots, a magnitude and a phase plot
- In this way, we can see the relation of a sinusoidal input with a given frequency ω to a linear system and its output
- After all, the relation between the amplitudes is given by $|H(j\omega)|$, and the phase shift by $\angle H(j\omega)$

Example bode plot

$$H(s) = \frac{14s^2+7s+3}{s^4+10s^3+10s^2+10s+10}$$

- Say we use an input $x(t) = 2\sin(100t)$ in this system.
- The steady state response would be



$$y(t) = |H(j100)| \cdot 2\sin(100t + \angle H(j100))$$

The magnitude plot

Convention:

- for the ordinate (y-axis) we use $20\log_{10}|H(j\omega)|$ with the unit dB
- for the abscissa (x-axis) we use a logarithmic plot of ω

This is thus a bi-log plot

The reason for using the logarithm of the modulus of $H(j\omega)$ will become clear later

The phase plot

Convention:

- for the ordinate (y-axis) we use $\angle H(j\omega)$ in degrees
- for the abscissa (x-axis) we use a logarithmic plot of ω

This is thus a semi-log plot

Discrete time systems

- The transfer function is a function of z, i.e. $H(z)$
- In contrast to continuous time systems, we do not use $H(j\omega)$. Instead, $H(e^{j\omega T_s})$ is now the sinusoidal transfer function.
- T_s is the sample time, i.e. the amount of time in between each sample.

Outline

A new representation of the transfer function

From before:

$$H(s) = \frac{\beta_0 s^r + \beta_1 s^{r-1} + \ldots + \beta_r}{s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n}$$

Factorization in zeros and poles

$$\Rightarrow \quad H(s) = \frac{\beta_0 (s - n_1)(s - n_2) \ldots (s - n_r)}{(s - p_1)(s - p_2) \ldots (s - p_n)}$$

This is the usual representation. Now however, we will look for factors $(1 + \frac{s}{s_i})$, with s_i a so-called breakpoint.

A new representation of the transfer function

We can do this by bringing all the zeros and poles not equal to zero outside the brackets, as follows:

$$H(s) = \beta_0 \frac{\prod(-n_i)}{\prod(-p_j)} \frac{(1+\frac{s}{-n_1})(1+\frac{s}{-n_2})\ldots(1+\frac{s}{-n_i})}{s^l(1+\frac{s}{-p_1})(1+\frac{s}{-p_2})\ldots(1+\frac{s}{-p_j})}$$

Replacing the constants by K, and setting

$$r_k = -n_k$$

$$s_k = -p_k$$

A new representation of the transfer function

We ultimately get:

$$H(s) = K\frac{(1+\frac{s}{r_1})(1+\frac{s}{r_2})\dots(1+\frac{s}{r_i})}{s^l(1+\frac{s}{s_1})(1+\frac{s}{s_2})\dots(1+\frac{s}{s_j})}$$

Now we are able to construct the bode plot of each different factor of $H(s)$. Afterwards we can just add up these plots using the calculation rules of complex numbers.

Intermezzo complex numbers

- The magnitude of the product of complex numbers is equal to the product of the magnitudes of these numbers
- The phase of the product of complex numbers is equal to the sum of the phases of these numbers
- The logarithm of a product of numbers is equal to the sum of the logarithms of these numbers

This comes down to

$$20\log_{10}|H(j\omega)| = \sum 20\log_{10}|\text{factors}|$$

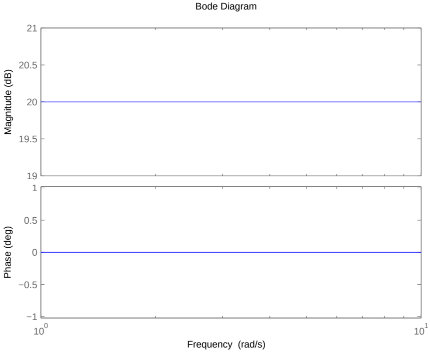
$$\angle H(j\omega) = \sum(\angle\text{factors})$$

Next we will quickly go over the simple bodeplots of the different factors of $H(s)$

The constant K

- $20\log_{10}|K| = \text{constant}$
- $\angle K = 0^\circ$ or $\pm 180^\circ$ (resp $K > 0$ and $K < 0$)

Example: $K = 10$



$(1 + \frac{j\omega}{r_i})$ in the numerator

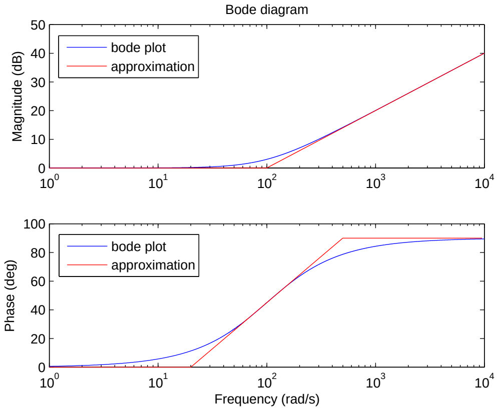
(Assume $r_i > 0$)

- What if $\omega \to 0$? $\quad (1 + \frac{j\omega}{r_i}) \to 1$
  - $20 \log_{10} |1| = 0$
  - $\angle 1 = 0°$
- What if $\omega \to \infty$? $\quad (1 + \frac{j\omega}{r_i}) \to j\infty$
  - $20 \log_{10} |j\infty| = \infty$
  - $\angle j\infty = 90°$
- The two terms balance each other out for $\omega = r_i$ (remember, this is called a breakpoint).
  - $20 \log_{10} |1 + j| = 20 \log_{10}(\sqrt{2}) \approx 3\text{dB}$
  - $\angle(1 + j) = 45°$

A breakpoint is therefore also called a 3dB point

$(1 + \frac{j\omega}{r_i})$ in the numerator

Example: $r_i = 100$



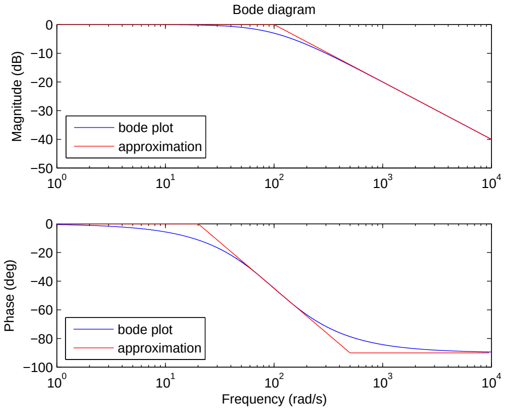
$(1+\frac{j\omega}{s_i})$ in the denominator

This factor is equivalent to the previous one. The only difference is the sign change in both plots:

- What if $\omega \to 0$? $\quad \frac{1}{1+\frac{j\omega}{s_i}} \to 1$
  - $20\log_{10}|1| = 0$
  - $\angle 1 = 0^\circ$
- What if $\omega \to \infty$? $\quad \frac{1}{1+\frac{j\omega}{s_i}} \to \frac{1}{j\infty} \to -j0$
  - $20\log_{10}|-j0| = -\infty$
  - $\angle -j0 = -90^\circ$
- The two terms balance each other out for $\omega = s_i$
  - $20\log_{10}\frac{1}{|1+j|} = 20\log_{10}(\frac{1}{\sqrt{2}}) \approx -3\text{dB}$
  - $\angle\frac{1}{(1+j)} = -45^\circ$

$(1 + \frac{j\omega}{s_i})$ in the denominator

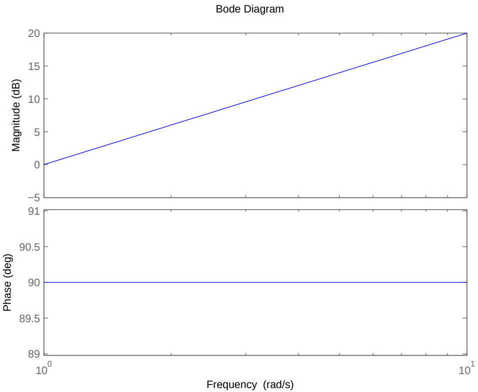
Example $s_i = 100$



$j\omega$ in the numerator

- This is simply a (ascending) straight line in the magnitude plot, with a slope of 20 dB/decade
- Constant phase of 90°
- What if $\omega \to 0$? $\quad j\omega \to j0$
  - $20\log_{10}|j0| = -\infty$
  - $\angle j0 = 90^\circ$
- What if $\omega \to \infty$? $\quad j\omega \to j\infty$
  - $20\log_{10}|j\infty| = \infty$
  - $\angle j\infty = 90^\circ$

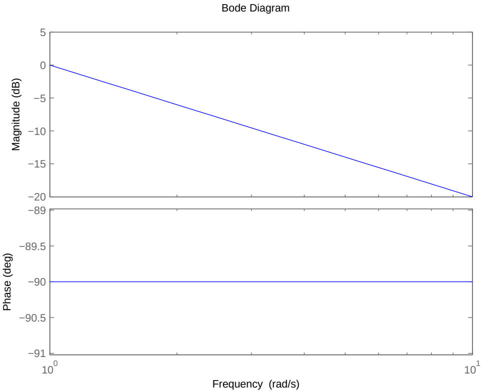
$j\omega$ in the numerator



$j\omega$ in the denominator

- This is simply a (descending) straight line in the magnitude plot, with a slope of -20 dB/decade
- Constant phase of -90°
- What if $\omega \to 0$? $\quad \frac{1}{j\omega} \to \frac{1}{j0} \to -j\infty$
  - $20\log_{10}|-j\infty| = \infty$
  - $\angle - j\infty = -90^\circ$
- What if $\omega \to \infty$? $\quad \frac{1}{j\omega} \to \frac{1}{j\infty} \to -j0$
  - $20\log_{10}|-j0| = -\infty$
  - $\angle - j0 = -90^\circ$

$j\omega$ in the denominator



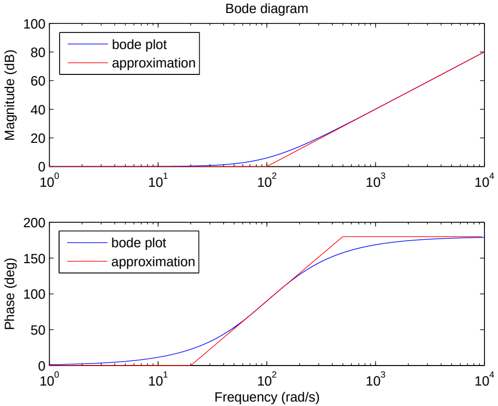
What if the multiplicity is higher than 1?

Take for example multiplicity 2, a second order factor.
A second order factor has twice the effect of a first order factor.
Consider for example the effect of a double zero:

- Slope of 40dB/decade instead of 20dB/decade after the breakpoint
- Phase shift of 180° instead of 90°

Second order factor

$$(1 + \frac{j\omega}{100})^2$$



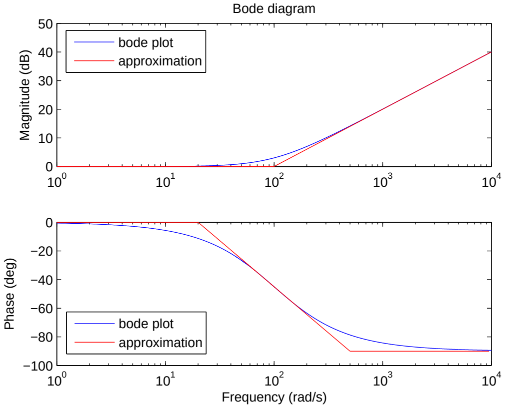
Exception

Up until now we always considered r_i and $s_i > 0$, but what if we have a factor $(1 - \frac{j\omega}{r_i})$ for example?

- The magnitude plot remains unchanged, as $|1 + \frac{j\omega}{r_i}| = |1 - \frac{j\omega}{r_i}|$
- The phase plot is reversed, as $\angle(1 + \frac{j\omega}{r_i}) = -\angle(1 - \frac{j\omega}{r_i})$
- If we have such a factor in the denominator, the system will be unstable!

Exception

Example $(1 - \frac{j\omega}{100})$ in the numerator



Quadratic factors and resonance

- Also possible is a quadratic factor $[1 + 2\zeta(j\frac{\omega}{\omega_n}) + (j\frac{\omega}{\omega_n})^2]^{\pm 1}$
- ζ is called the damping factor, ω_n is the natural frequency
- If $\zeta > 1$, this quadratic factor can be expressed as two first order factors with real zeros/poles
- But if $0 < \zeta < 1$, this quadratic factor is the product of two complex-conjugate factors
- For low ζ, the asymptotic approximations are not accurate. Instead, a peak occurs in the magnitude plot around ω_n
- This peak is a phenomenon known as resonance.

Quadratic factors and resonance