Outline

1. Analysis of the sample and hold
2. Fourier transform
3. Spectrum of a sampled signal
   - Aliasing
   - Sampling theorem
   - Hidden oscillations
4. Data extrapolation (reconstruction)

Hidden oscillations

There is the possibility that a signal could contain some frequencies that the samples do not show at all.

Such signals, when they occur in digital control systems, are called **hidden oscillations**.

They can only occur at multiples of the Nyquist frequency (π/T).

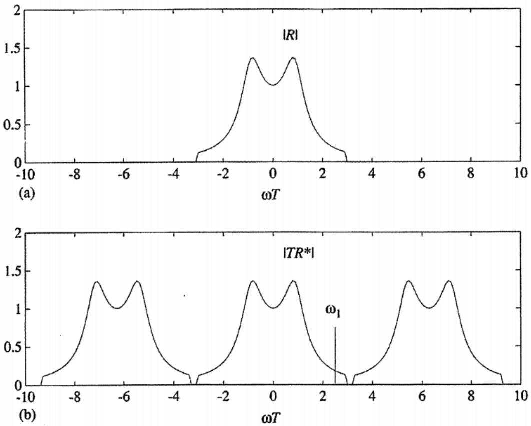
Outline

1. Analysis of the sample and hold
2. Fourier transform
3. Spectrum of a sampled signal
   - Aliasing
   - Sampling theorem
   - Hidden oscillations
4. Data extrapolation (reconstruction)

Reconstruction

<ins>Sampling theorem</ins>: *under the right conditions* it is possible to recover a signal from its samples.

The figure to the right shows the spectrum of $R(j\omega)$. It is contained in the low-frequency part of $R^*(j\omega)$. Therefore, to recover $R(j\omega)$ we need to process $R^*(j\omega)$ through a low-pass filter and multiply by T.



Reconstruction

If $R(j\omega)$ has zero energy for frequencies in the bands above the Nyquist frequency, in other words R is band-limited, then an ideal low-pass filter with gain T for $-\pi/T \leq \omega \leq \pi/T$ and zero elsewhere would recover $R(j\omega)$ from $R^*(j\omega)$ exactly.

If we define the ideal low-pass filter characteristic as $L(j\omega)$, we have:

$$R(j\omega) = L(j\omega)R^*(j\omega).$$

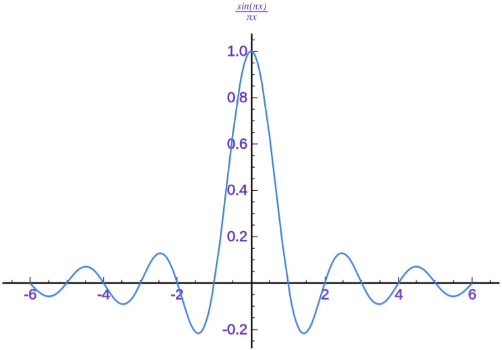
The signal $r(t)$ is the inverse Fourier transform of $R(j\omega)$. Because $R(j\omega)$ is the *product* of two Fourier transforms, $r(t)$ is the *convolution* of the time functions $\ell(t)$ and $r^*(t)$.

$$r(t) = \ell(t) * r^*(t)$$

Ideal low-pass filter

The impulse response of the filter can be computed using this definition

$$\begin{aligned}
\ell(t) &= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} Te^{j\omega t}\,d\omega \\
&= \frac{T}{2\pi}\frac{e^{j\omega t}}{jt}\Big|_{-\pi/T}^{\pi/T} \\
&= \frac{T}{2\pi jt}\left(e^{j(\pi t/T)} - e^{-j(\pi t/T)}\right) \\
&= \frac{sin(\pi t/T)}{\pi t/T} \\
&\triangleq sinc\frac{\pi t}{T}
\end{aligned}$$



The sinc functions are the interpolators that fill in the time gaps between samples with a signal that has no frequencies above π/T.

Reconstruction

Using the previous equations, we find:

$r(t) = \int_{-\infty}^{\infty} r(\tau) \sum_{k=-\infty}^{\infty} \delta(\tau - kT) sinc \frac{\pi(t-\tau)}{T} d\tau.$

Using the shifting property of the impulse, we have:

$$\boxed{r(t) = \sum_{k=-\infty}^{\infty} r(kT) sinc \frac{\pi(t - kT)}{T}}$$

This equation is a constructive statement of the sampling theorem. There is one disadvantage. Because $\ell(t)$ is nonzero for $t < 0$, this filter is noncausal. $\ell(t)$ starts at $t = -\infty$ while the impulse that triggers it does not occur until $t = 0$. The noncausality can be overcome by adding a phase lag, $e^{-j\omega\lambda}$, to $L(j\omega)$, which adds a delay to the filter and to the signals processed through it.

Zero-order hold

The transfer function of the zero-order hold was introduced as

$$ZOH(j\omega) = \frac{1-e^{-j\omega T}}{j\omega}.$$

We express this function in magnitude and phase form, to discover the frequency properties of $ZOH(j\omega)$.

We factor out $e^{-j\omega T/2}$ and multiply and divide by $2j$:

$$\begin{aligned} ZOH(j\omega) &= e^{-j\omega T/2}\left\{\frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j}\right\}\frac{2j}{j\omega} \\ &= Te^{-j\omega T/2}\frac{sin(\omega T/2)}{\omega T/2} \\ &= e^{-j\omega T/2}Tsinc(\omega T/2) \end{aligned}$$

Zero-order hold

The magnitude function is

$$|ZOH(j\omega)| = T\left|sinc\frac{\omega T}{2}\right|$$

and the phase is

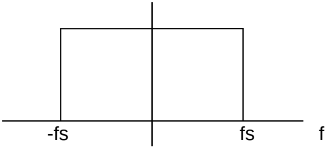
$$\angle ZOH(j\omega) = \frac{-\omega T}{2}$$

plus the 180° shifts where the sinc function changes sign.

Thus the effect of the zero-order hold is to introduce a phase shift of $\omega T/2$ (a time delay of $T/2$ seconds) and to multiply the gain by a function with the magnitude of $sinc(\omega T/2)$.

Zero-order hold

Spectrum of ideal filter



Spectrum of ZOH