Chapter 2: Classifications of systems

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Overview

Outline

Based on the number of inputs and outputs

1. **SISO**: Single Input Single Output
2. **SIMO**: Single Input Multiple Output
3. **MISO**: Multiple Input Single Output
4. **MIMO**: Multiple Input Multiple Output
5. **Autonomous**: No inputs and one or more outputs

Outline

Continuous vs. Discrete time systems

We will discuss both types simultaneously in order to emphasize the similarities (and differences).

Continuous system

1. It has continuous input and output signals
2. We denote continuous time by $t \in R$
3. We denote functions of continuous time with round brackets, e.g.: $x(t)$

Discrete system

1. It has discrete input and output signals
2. We denote discrete time by $k \in Z$
3. We denote functions of continuous time with square brackets, e.g.: $x[k]$

Continuous vs. Discrete time systems

Continuous

For every time instant $t \in R$, the system has:

1. A vector of inputs $\mathbf{u}(t)$
2. A vector of outputs $\mathbf{y}(t)$
3. A vector of states $\mathbf{x}(t)$

Discrete

For every time step $k \in Z$, the system has:

1. A vector of inputs $\mathbf{u}[k]$
2. A vector of outputs $\mathbf{y}[k]$
3. A vector of states $\mathbf{x}[k]$

Outline

Linear vs. Nonlinear: a linear system

Definition

A system is linear if

$u_1(t) \to y_1(t)$ (*input* $u_1(t)$ *results in output* $y_1(t)$)

$u_2(t) \to y_2(t)$

imply that

$$\alpha u_1(t) + \beta u_2(t) \to \alpha y_1(t) + \beta y_2(t)$$

Properties of a linear system (contained in the definition):

- Superposition
- Homogeneity

Linear vs. Nonlinear: a linear system

Properties of a linear system (contained in the definition):

- Superposition

$$u_a(t) \to y_a(t) \;\; and \;\; u_b(t) \to y_b(t)$$
$$\Updownarrow$$
$$u_a(t) + u_b(t) \to y_a(t) + y_b(t)$$

  This means the output produced by simultaneous applications of two different inputs is the sum of the two individual outputs.

- Homogeneity

  $\alpha u(t) \to \alpha y(t)$

How to recognize a linear system:

- Linear in all of the variables
- No constant factors

Linear vs. Nonlinear: a linear system

Example

$$\begin{cases} \dot{x} = u \\ \dot{y} = x + 2u \end{cases}$$

Linearity of this system is easily verified, based on the linearity of the derivative:

$$\begin{cases} \alpha\dot{x}_a(t) + \beta\dot{x}_b(t) = \alpha u_a(t) + \beta u_b(t) \\ \alpha\dot{y}_a(t) + \beta\dot{y}_b(t) = \alpha x_a(t) + \beta x_b(t) + 2\alpha u_a(t) + 2\beta u_b(t) \end{cases}$$

Linear vs. Nonlinear: autonomous linear systems

Continuous-time autonomous linear dynamical systems are described by:

$$\dot{x}(t) = Ax(t)$$

Example: $\dot{x}(t) = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} x(t)$

Linear vs. Nonlinear: violating homogeneity

All nonhomogeneous systems are strictly speaking nonlinear, e.g.:

$$\begin{cases} \dot{x}(t) = x(t) + u(t)^2 \\ \dot{y}(t) = x(t) \end{cases} \qquad \Rightarrow \text{nonhomogeneous}$$

This is nonlinear, because the term $u(t)^2$ violates homogeneity. It can be turned into a linear system with inputs $z(t) = u(t)^2$.

$$\begin{cases} \dot{x}(t) = x(t) + z(t) \\ \dot{y}(t) = x(t) \end{cases} \qquad \Rightarrow \text{linear}$$

$\rightarrow$ nonhomogeneous systems that are linear apart from some function of inputs are often treated as linear systems.

Linear vs. Nonlinear: nonlinear systems

Some examples of nonlinear systems:

$$\begin{cases} \dot{x}_1(t) = x_1(t) + u(t) \\ \dot{x}_2(t) = x_1(t)x_2(t) \\ y(t) = x_1(t) + x_2(t) \end{cases}$$

$$\begin{cases} \dot{x}(t) = sin(x(t)) + u(t) \\ y(t) = x(t) \end{cases}$$

$$\begin{cases} \dot{x}(t) = 2u(t) + 1 \\ y(t) = cos(x(t)) \end{cases}$$

Linear vs. Nonlinear systems

Predominantly linear

Simple electrical systems

- Circuits with ideal resistors, capacitors and inductors

Simple mechanical systems

- Systems with ideal springs

Inherently nonlinear

Chemical systems

Biological systems

Economical systems

More involved electrical or mechanical systems

...

Linear vs. Nonlinear systems

- Reality is nonlinear
- However, this course will only deal with linear systems
- Why do we prefer linear systems?

  The previously mentioned properties will allow for a thorough study of the system
- Why are we allowed to use linear systems, even in a nonlinear setting?

  You can linearize around an equilibrium point

Outline

Causal vs. Non-causal systems

- A causal system only depends on the present and the past, not on the future
- A non-causal system (also) depends on the future
- All physical systems are causal
  - A telephone:
    - It will not ring for future calls
  - Any human:
    - Is a system that will only react on inputs it has already received
    - If we react because we expect something to happen in the future, then that expectation arose from past or present inputs

source: http://www.deekshith.in/2013/03/causal-and-non-causal-systems-better-explained.html

Causal vs. Non-causal systems

How do non-causal systems arise?

A possibility is by greatly reducing the complexity of a system, in which some causes of events are taken out of the equations.

Example:

- A model of the economical consumption (**output**)
- A lot of influencing factors, but the only **input** is the employment numbers
- Current and past employment numbers determine consumption, but when someone gets fired, they will continue to work for several weeks in most instances, but their consumption will drop immediately $\rightarrow$ A correct model would have to be **non-causal**
- The non-causal model for this input-output relation is not useful if you want to determine the level of consumption
- You could use the relation to see a drop in employment, before it is visible in the employment numbers

Causal vs. Non-causal systems

Example of non-causal systems: **image processing**

- The input to our system (the image processor) is a two dimensional series of values $(u(k, l))$: the color values at the different pixels of the original image
- The output is a processed image $(y(k, l))$
- There is now no reason to want causality; the input depends on position and not on time
- $y(k, l)$ can rely on 'later' values like $u(k + 1, l + 1)$, without that being a problem

Original image

Highlight borders

Removed details

Causal vs. Non-causal systems

Some mathematical examples:

Causal systems

- $$\begin{cases} x[k+1] = u[k] \\ y[k] = 3x[k+1] + u[k] \end{cases}$$
- $y(t) = 2u(t-\tau)$

Non-causal systems

- $$\begin{cases} x[k+1] = u[k] \\ y[k] = x[k+2] + \frac{1}{2}u[k] \end{cases}$$
- $y(t) = u(t+\tau)$

Outline

Time-invariant vs. Time-varying systems

Dynamics and properties of the system don't change through time. If the same input is applied to a system that is in the same state, then the output of a time-invariant system will be the same.

Mathematically this looks as follows:

- In a time-invariant system:
  if $u(t) \xrightarrow{x(t)} y(t)$ then $u(t+\tau) \xrightarrow{x(t+\tau)} y(t+\tau)$
- In time-varying systems, the parameters of the system are functions of time, e.g.:

$$\begin{cases} \dot{x}(t) = \mathbf{A}x(t) + \mathbf{B}u(t) \\ y(t) = \mathbf{C}x(t) + \mathbf{D}u(t) \end{cases} \Rightarrow \begin{cases} \dot{x}(t) = \mathbf{A}(t)x(t) + \mathbf{B}(t)u(t) \\ y(t) = \mathbf{C}(t)x(t) + \mathbf{D}(t)u(t) \end{cases}$$

Time-invariant vs. Time-varying systems

Time-invariant differential equation

The dependent variable and its derivatives appear as linear combinations. The coefficients of all terms are constant.
Example: $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 10x = 0$

Time-varying differential equation

The dependent variable and its derivatives appear as linear combinations, but coefficients of terms may involve the independent variable.
Example: $\frac{d^2x}{dt^2} + (1 - cos(2t))x = 0$

Time-invariant vs. Time-varying systems

- Examples of time-varying systems:
  - The properties of an electrical circuit slowly change over time
  - The human body also has many changing properties
  - Systems affected by night and day (heating of buildings), when those aspects were ignored in the model
- Examples of time-invariant systems:
  - A system that describes a physical law, for instance a system with two masses as its input and their attractive force as an output
  - In practice we approximate all systems whose properties change much slower than the variables as time-invariant

Outline

Lumped vs. Distributed parameter systems

Many physical phenomena are described mathematically by partial differential equations (PDEs). Such systems are called distributed parameter systems.
Lumped parameter systems are systems which can be described by ordinary differential equations (ODEs).
Example:

- Diffusion equation (discretize in space)
- Heat equation

This means we need an **infinite amount of states**, we say that this is a **distributed parameter system**.
If we have a **finite amount of states** (for instance by discretizing), we call the system a **lumped parameter system**.