Outline

Hidden oscillations

Definition

There is the possibility that a signal contains some frequencies that the samples do not show at all.

Such signals, when they occur in digital control systems, are called **hidden oscillations**.

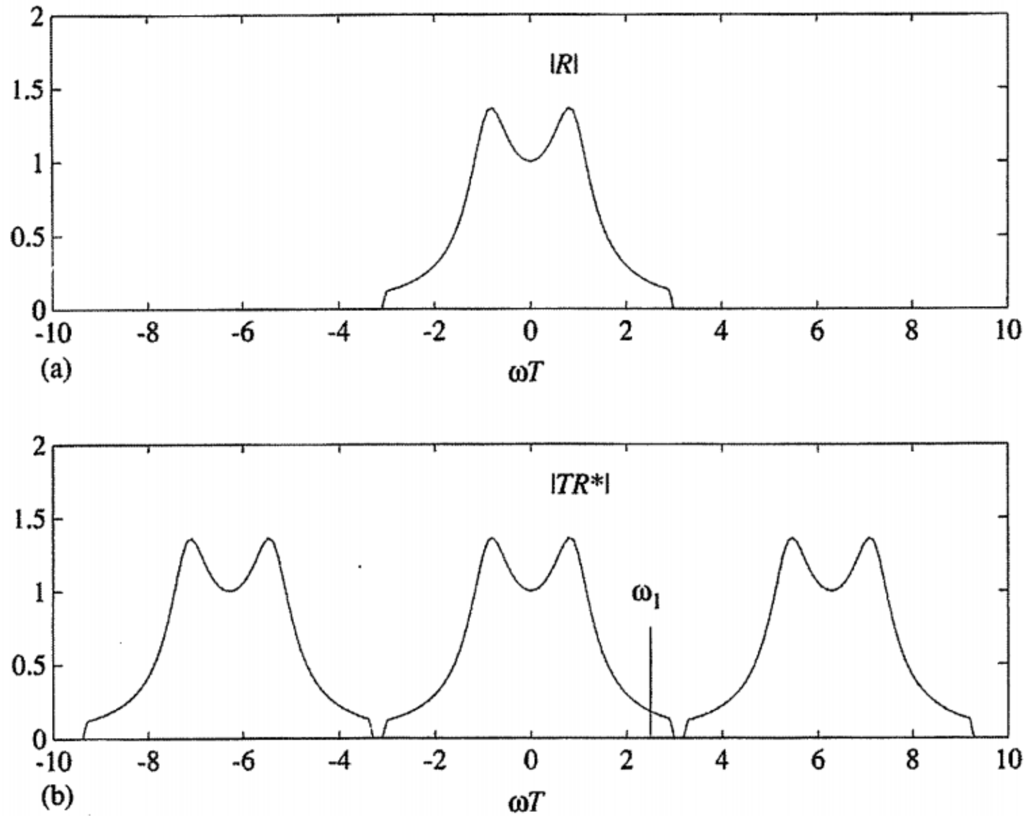
They can only occur at multiples of the Nyquist frequency (π/T_s).

Outline

Reconstruction

Sampling theorem: *under the right conditions* it is possible to recover a signal from its samples.

The figure to the right shows the spectrum of $R(j\omega)$. It is contained in the low-frequency part of $R^*(j\omega)$. Therefore, to recover $R(j\omega)$ we need to process $R^*(j\omega)$ through a low-pass filter and multiply by T_s.



Reconstruction

If $R(j\omega)$ has zero energy for frequencies in the bands above the Nyquist frequency, in other words R is band-limited, then an ideal low-pass filter with gain T_s for $-\pi/T_s \leq \omega \leq \pi/T_s$ and zero elsewhere would recover $R(j\omega)$ from $R^*(j\omega)$ exactly.

If we define the ideal low-pass filter characteristic as $L(j\omega)$, we have:

$$R(j\omega) = L(j\omega)R^*(j\omega).$$

The signal $r(t)$ is the inverse Fourier transform of $R(j\omega)$. Because $R(j\omega)$ is the *product* of two Fourier transforms, $r(t)$ is the *convolution* of the time functions $\ell(t)$ and $r^*(t)$.

$$r(t) = \ell(t) * r^*(t).$$

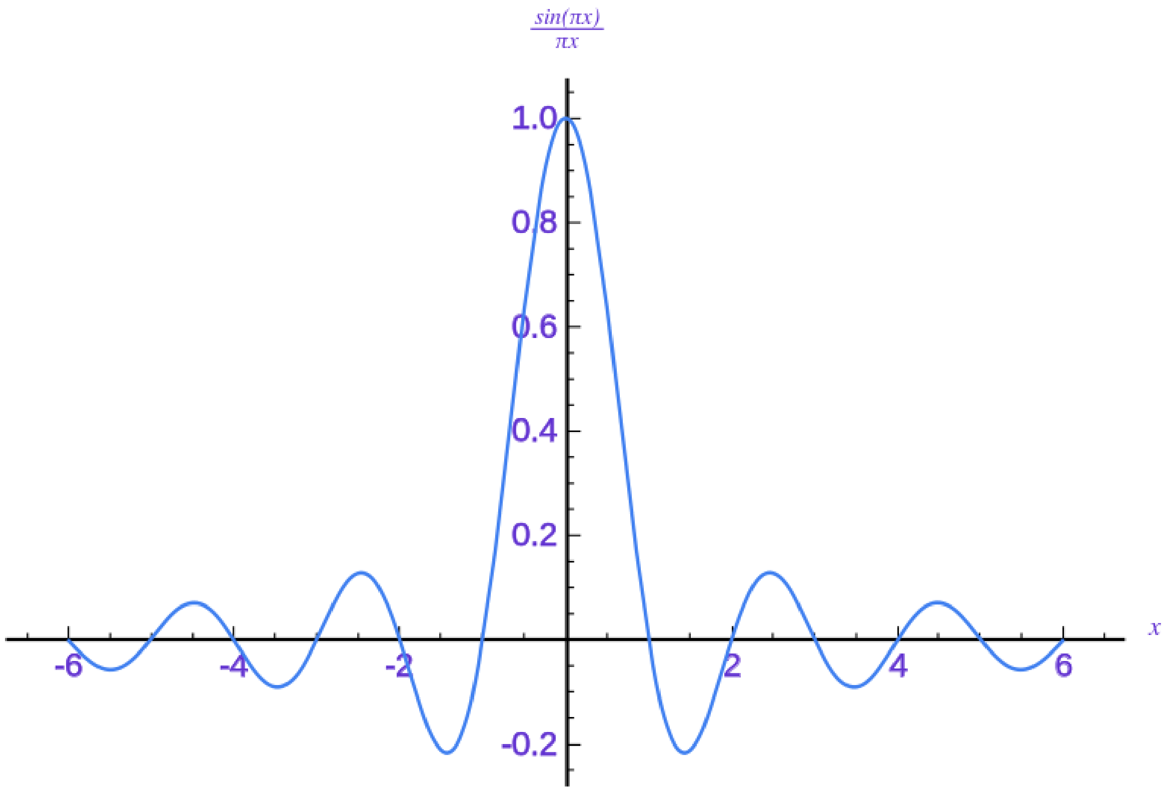
Ideal low-pass filter

The impulse response of the filter can be computed using this definition:

$$
\begin{aligned}
\ell(t) &= \frac{1}{2\pi}\int_{-\pi/T_s}^{\pi/T_s} Te^{j\omega t} d\omega \\
&= \frac{T_s}{2\pi}\frac{e^{j\omega t}}{jt}\bigg|_{-\pi/T_s}^{\pi/T_s} \\
&= \frac{T_s}{2\pi jt}(e^{j(\pi t/T_s)} - e^{-j(\pi t/T_s)}) \\
&= \frac{sin(\pi t/T_s)}{\pi t/T_s} \\
&\triangleq sinc\frac{\pi t}{T_s}
\end{aligned}
$$

Ideal low-pass filter

The sinc functions are the interpolators that fill in the time gaps between samples with a signal that has no frequencies above π/T_s.



Reconstruction

Using the previous equations, we find:

$$r(t) = \int_{-\infty}^{\infty} r(\tau) \sum_{k=-\infty}^{\infty} \delta(\tau - kT_s) sinc \frac{\pi(t-\tau)}{T_s} d\tau.$$

Using the shifting property of the impulse, we obtain:

$$r(t) = \sum_{k=-\infty}^{\infty} r(kT_s) sinc \frac{\pi(t-kT_s)}{T_s}.$$

This filter is non-causal because $\ell(t)$ is nonzero for $t < 0$. $\ell(t)$ starts at $t = -\infty$ while the impulse that triggers it does not occur until $t = 0$. The non-causality can be overcome by adding a phase lag, $e^{-j\omega\lambda}$, to $L(j\omega)$, which adds a delay to the filter and to the signals processed through it.

Zero-order hold

The transfer function of the zero-order hold was introduced as

$$ZOH(j\omega) = \frac{1-e^{-j\omega T_s}}{j\omega}.$$

We express this function in magnitude and phase form, to discover the frequency properties of $ZOH(j\omega)$.

We factor out $e^{-j\omega T_s/2}$ and multiply and divide by $2j$:

$$\begin{aligned} ZOH(j\omega) &= e^{-j\omega T_s/2}\left\{\frac{e^{j\omega T_s/2} - e^{-j\omega T_s/2}}{2j}\right\}\frac{2j}{j\omega} \\ &= T_s e^{-j\omega T_s/2}\frac{sin(\omega T_s/2)}{\omega T_s/2} \\ &= e^{-j\omega T_s/2} T_s sinc(\omega T_s/2) \end{aligned}$$

Zero-order hold

The magnitude function is

$$|ZOH(j\omega)| = T_s \left| sinc \frac{\omega T_s}{2} \right|$$

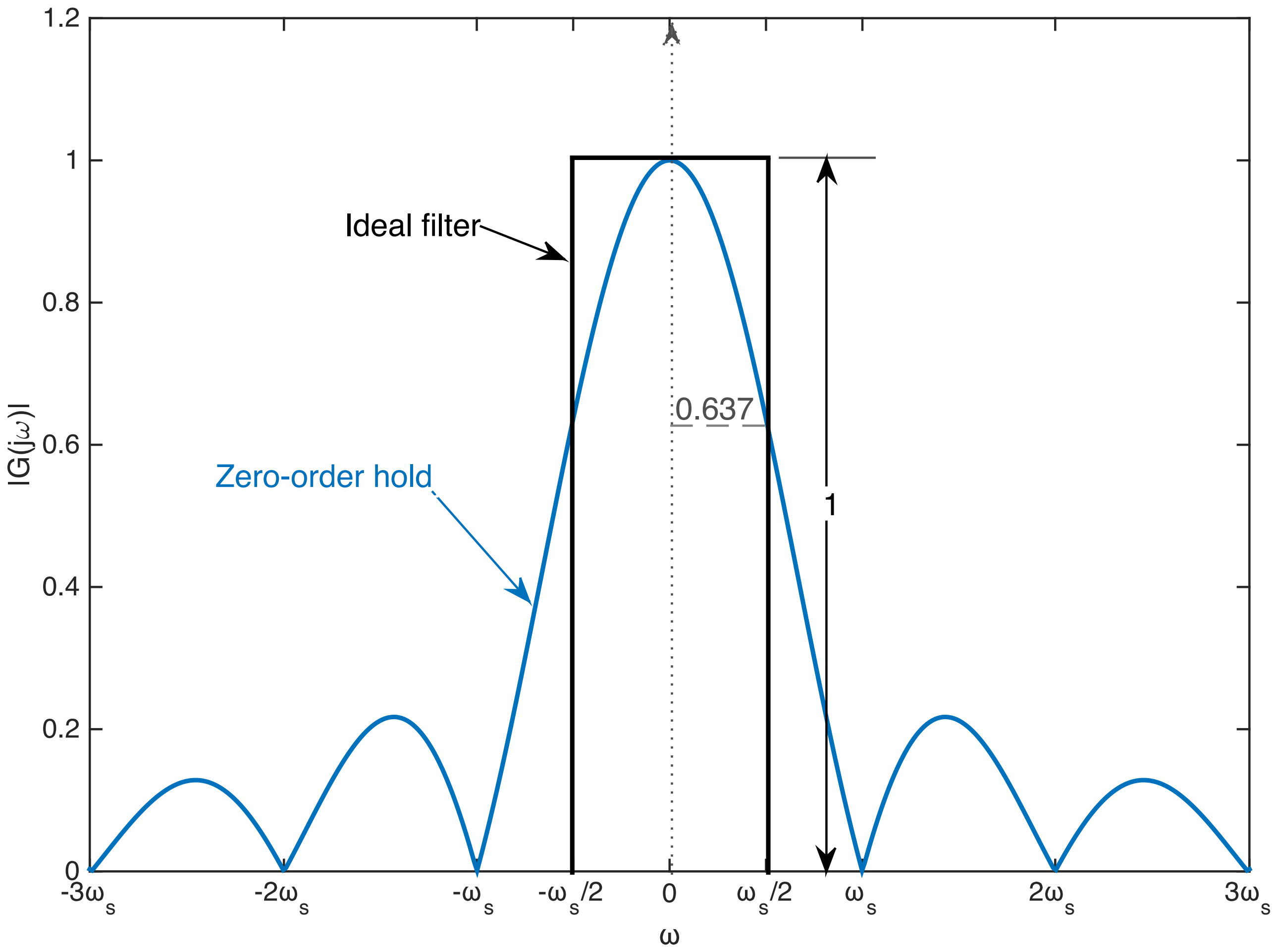
and the phase is

$$\angle ZOH(j\omega) = \frac{-\omega T_s}{2}$$

plus the 180° shifts where the sinc function changes sign.

Thus the effect of the zero-order hold is to introduce a phase shift of $\omega T_s/2$ (a time delay of $T_s/2$ seconds) and to multiply the gain by a function with the magnitude of $sinc(\omega T_s/2)$.

Zero-order hold filter vs ideal filter



Note: for the sake of comparison, the magnitudes $|G(j\omega)|$ are normalized

First-order hold

If the extrapolation is done by a first order polynomial, then the extrapolator is called a first-order hold and its transfer function is denoted $FOH(s)$.

$$FOH(s) = (1 - e^{-sT_s})\frac{sT_s+1}{sT_s^2}$$