# Chapter 5: Continuous time systems

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July 10, 2015

# Outline

# Linear differential equations: definitions 1/2

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The **order of a LDE** is the index of the highest derivative of $y$.

# Linear differential equations: definitions 2/2

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- if $A_{0:n}(t)$ are constants (ie. not functions of time), the LDE is said to have **constant coefficients**

# Example: radioactive decay 1/2

Let $N(t)$ be the number of radioactive atoms at time $t$, then:

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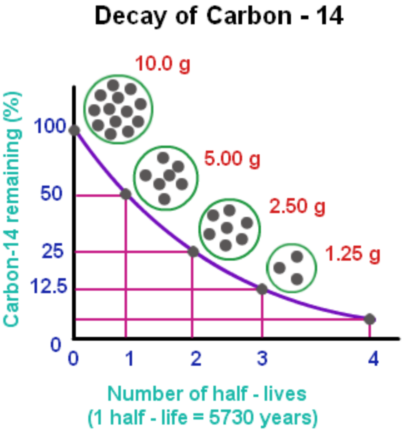

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Dividing by $e^{zt}$ yields the $n$th order **characteristic polynomial**:

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The specific linear combination depends on initial conditions.

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$$y^{(4)}(t) - 2y^{(3)}(t) + 2y^{(2)}(t) - 2y^{(1)}(t) + y(t) = 0.$$

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These zeros correspond to the following basis functions $t$:

$$e^{jt}, \quad e^{-jt}, \quad e^{t}, \quad te^{t}.$$

# Outline

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The final value theorem states $f(\infty) = \lim_{s \to 0} sF(s)$,
if all poles of $sF(s)$ are in the left half plane (ie. real part $< 0$).

# Important properties of the Laplace transform

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with $u(t) = \int_\infty^t \delta(t)dt$ (Heaviside) and $\delta(t)$ the Dirac delta.

# Inverse Laplace transform

The inverse Laplace transform converts $s$-domain to time domain:

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Practically, the inverse Laplace transform takes two steps:

1. write $F(s)$ in terms of partial fractions
2. transform each term in the partial fraction based on tables of $s/t$-domain pairs (course notes p 4.32-4.33)

# Outline

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This is done by using the following property (differentiation):

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Via induction, the Laplace transform of the $n$th order derivative:

$$\mathcal{L}\{f^{(n)}\} = s^nF(s) - \sum_{i=1}^{n} s^{n-i} f^{(n-i)}(0)$$

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Expanding Eq. (2) into (1) yields:

$$Y(s)\sum_{i=0}^{n} A_i s^i - \sum_{i=1}^{n}\sum_{j=1}^{i} A_i s^{i-j} y^{j-1}(0) = F(s)$$

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The solution in the time domain is obtained via the inverse Laplace transform: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

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This holds for linear, time-invariant systems with $n$ states if:

$$rank(\mathcal{O}) = n, \quad \mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}, \quad \mathcal{O} : \textbf{observability matrix}$$

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A linear, time-invariant system with $n$ states is controllable if:

$$rank(\mathcal{C}) = n, \quad \mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \ldots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix},$$

where $\mathcal{C}$ is called the **controllability matrix**.

# Outline

# Transfer function

The transfer function of input $i$ to output $j$ is defined as:

$$H_{i,j}(s) = \frac{Y_j(s)}{U_i(s)}, \quad \mathbf{U}(s) = \mathcal{L}\{u(t)\}, \quad \mathbf{Y}(s) = \mathcal{L}\{y(t)\}.$$

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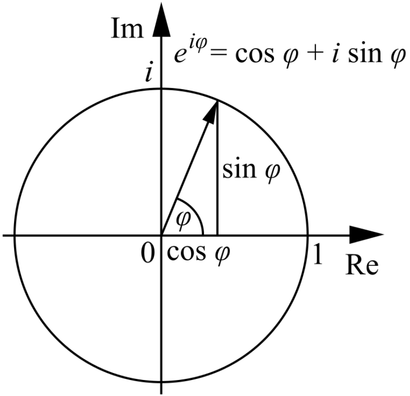
The complex Laplace variable can be rewritten: $s = \sigma + j\omega$.

The frequency response of a system can be analyzed via $\mathbf{H}(j\omega)$:

$$e^{\sigma + j\omega} = e^{\sigma}(\cos\omega + j\sin\omega).$$

# Illustration of Euler's formula



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Poles and zeros may cancel, ie. if $D(s) = N(s) = 0$ for some $s$.

# Steady state response

The output of a linear time-invariant system yields consists of:

- a steady-state output $y_{ss}(t)$, which similar periodicity to $u(t)$
  $\rightarrow$ $y_{ss}$ comprises the same frequencies as $u(t)$
- a transient output $y_{tr}(t)$
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The steady-state output $y_{ss}(t)$ of a linear time invariant system:

- consists of signals of same frequencies as the input signal $u(t)$
- which may have been magnified and/or phase changed

# Outline

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The speed of convergence depends on the position of the poles.

# Time constant

The transfer function of first order systems can be written as:

$$H(s) = \frac{K}{\tau s + 1} \quad \leftrightarrow \quad h(t) = \frac{K}{\tau} e^{-t/\tau},$$

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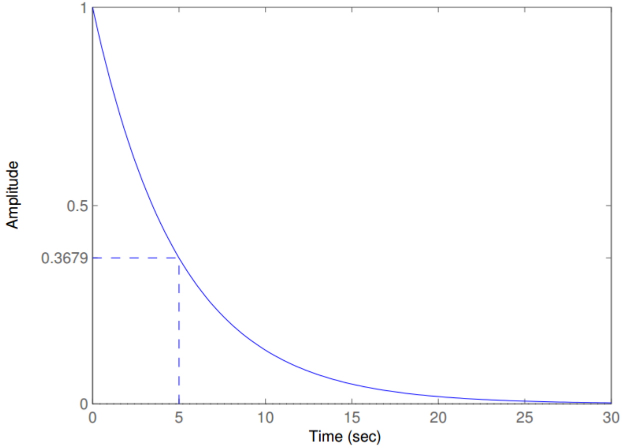
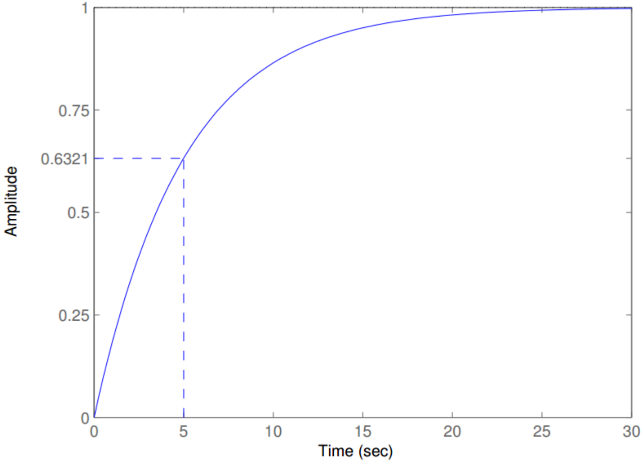
The time constant summarizes the speed of a system's dynamics:

- after $\tau$ seconds, the impulse response reaches $h(0)/e$.
- after $\tau$ seconds, the step response has reached $1 - e^{-1} \approx 63\%$ of its regime value.


1
0.5
0.3679
0
0
5
10
15
20
25
30
Amplitude
Time (sec)

# Step response $H(s) = 5/(5s+1) \leftrightarrow h(t) = exp(-t/5)$



# Outline

We start from the linear state-space representation:

time domain | Laplace domain

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \quad \leftrightarrow \quad \begin{cases} s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{cases}$$

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$$\begin{aligned} (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{B}\mathbf{U}(s) \\ \mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \\ \Rightarrow \mathbf{Y}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) \\ \Rightarrow \mathbf{H}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \end{aligned}$$

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Consider the following SISO system with 2 states:

$$\begin{bmatrix} sX_1(s) \\ sX_2(s) \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} + \begin{bmatrix} \beta \\ 2 \end{bmatrix} U(s)$$

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The transfer function $H(s) = \frac{\beta}{s-\alpha}$ has only one pole ($s_1 = \alpha$).
$\rightarrow$ not all eigenvalues of **A** are poles in transfer functions $\mathbf{H}(s)$.

# Outline

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# Transient Response

The time response of a control system may be written as:

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

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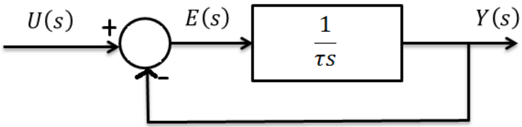
Transient response: when input of system changes, output does not change immediately but takes time to go to steady state

# First order systems

E.g.RC circuit, thermal system, ...



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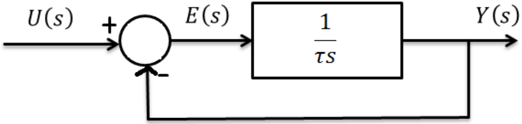


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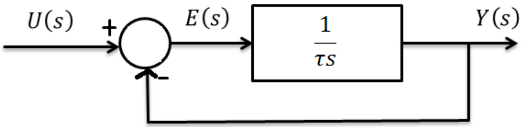


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Unit step response

- Laplace of unit-step is $\frac{1}{s}$ $\rightarrow$ substituting $U(s) = \frac{1}{s}$ into equation $Y(s) = \frac{1}{s}\frac{1}{\tau s+1}$
- Expanding into partial fractions gives $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$

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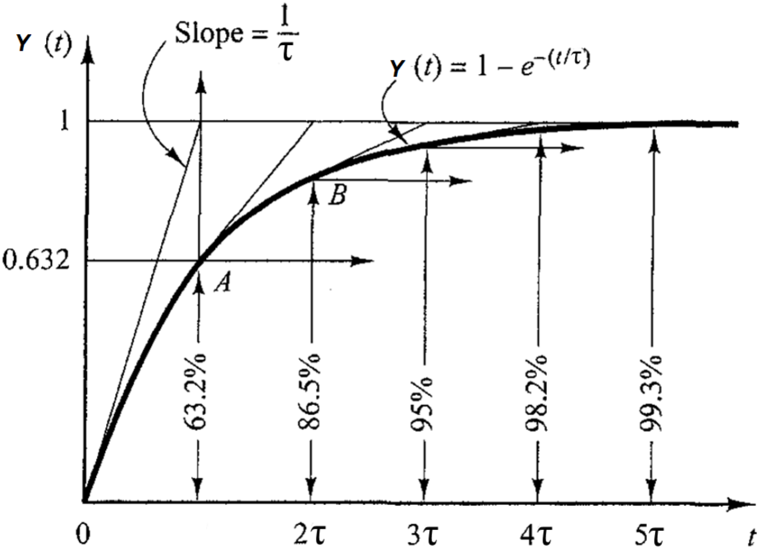
5. Slope at time $t = 0$ is $\frac{1}{\tau}$

   $\frac{dy}{dt}|_{t=0} = \frac{1}{\tau}e^{-\frac{t}{\tau}}|_{t=0} = \frac{1}{\tau}$

   Where $\tau$ is called the system time constant

# Unit step transient response



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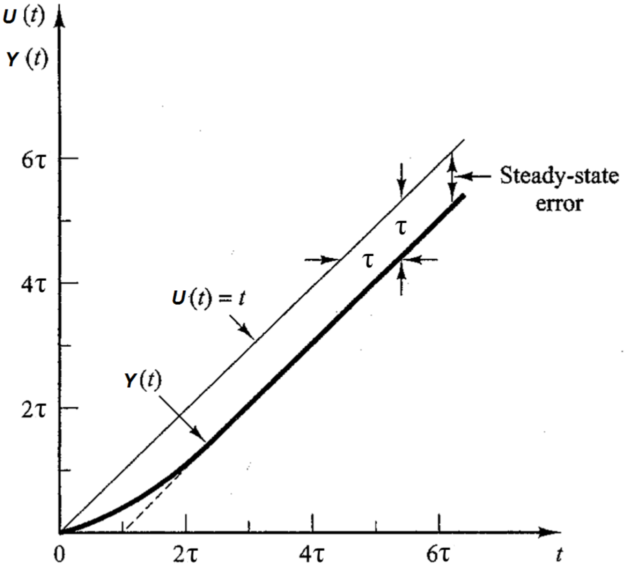
   $e(\infty) = \tau$

# Unit ramp transient response



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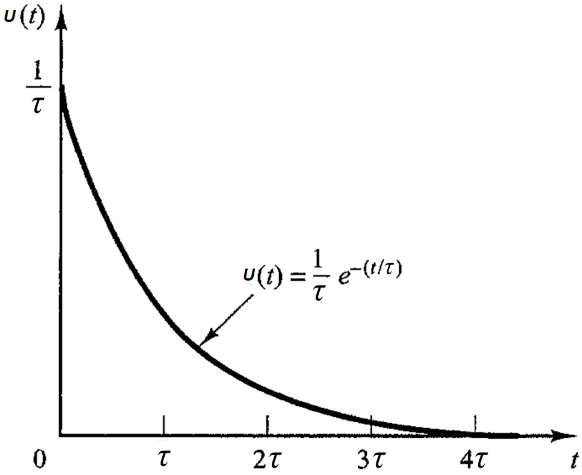

# Outline

# Second order systems

A second order system can generally be written as:

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If the transfer function has two real poles, the frequency response can be found by combining the effects of both poles

# Second order systems

Sometimes the transfer function has two complex conjugate poles. In that case we have to find a different solution for finding the frequency response.

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In order to study the transient behaviour, let us first consider the following simplified example of a second order system

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# Step response second order system

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4. The poles are real if

$$e^2 - dc \geq 0$$

To simplify the transient analysis, it is convenient to write

$$\frac{f}{d} = \omega_n^2,\ \frac{e}{d} = 2\zeta\omega_n = 2\sigma$$

Where
$\sigma$ is the attenuation
$\omega_n$ is the natural frequency
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The dynamic behavior of the second-order system can then be described in terms of only two parameters $\zeta$ and $\omega_n$

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- The system is then called **underdamped**
- The **transient response is oscillatory**

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If $\zeta > 1$, the system is called **overdamped**

We will now look at the unit step response for each of these cases

# Underdamped system

For the underdamped case ($0 < \zeta < 1$), the transfer function can be written as:

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

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For a unit-step input we can write

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}$$

# Underdamped system

Which can be rewritten as partial fractions

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} - \frac{\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

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\end{aligned}
$$

It can be shown that

$$
\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] &= e^{-\zeta\omega_n t} cos(\omega_d t) \\
\mathcal{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] &= e^{-\zeta\omega_n t} sin(\omega_d t)
\end{aligned}
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Therefore:

$$\mathcal{L}^{-1}\Big[Y(s)\Big] = y(t)$$

# Underdamped system

Therefore:

$$\mathcal{L}^{-1}\Big[Y(s)\Big] = y(t)$$

$$= 1 - e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t))$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} sin(\omega_d t + tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta}))$$

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$$= 1 - e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t))$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} sin(\omega_d t + tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta}))$$

It can be seen that the frequency of the transient oscillation is the damped natural frequency $\omega_d$ and thus varies with the damping ratio $\zeta$

# Underdamped system

The error signal is the difference between input and output

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Where

$$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n \text{ and } s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$$

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- Once the faster decaying exponential term has disappeared, the response is similar to that of a first-order system
- In that case, $H(s)$ can be approximated by

$$H(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} = \frac{s_2}{s + s_2}$$

# Overdamped system

With the approximate transfer function, the unit-step response becomes

$$Y(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

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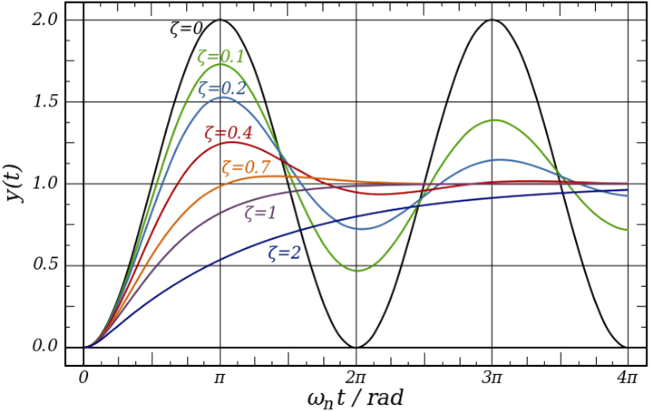
The time response for the approximate transfer function is then given as

$$y(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}, \text{ for } t \leq 0$$

Response on a step function



# Second order systems - characteristics

Overshoot: Highest amplitude above steady state.

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

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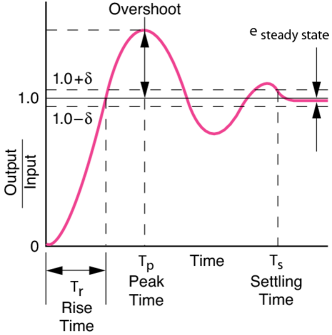
Peak Time: Time to reach overshoot.

$t_p = \frac{\pi}{\omega_d}$



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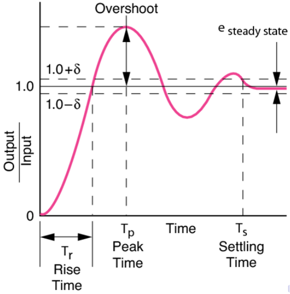
We find:

$$e^{-\zeta\omega_n T_s} < 0.02$$

$$T_s = \frac{4}{\omega_n \zeta}$$



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A different view on the Tacoma bridge disaster:
`https://www.youtube.com/watch?v=6ai2QFxStxo`

In fact the collapse was a result of a number of effects like Aerodynamic flutter and vortices. Read the full article here:
`http://www.ketchum.org/billah/Billah-Scanlan.pdf`

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# Second order systems - resonance

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Another phenomenon with bridges and resonance is that many people marching with the same rhythm can cause a bridge to start resonating like the Angers bridge in 1850. A more recent example is the Millennium bridge in London who started resonating.

# Second order systems - damping

When we want a system with no resonance, we choose one with damping $< 0.707$. This means a pole between $135^\circ$ and $225^\circ$:

$$arctan(\frac{\sqrt{1-\zeta^2}}{\zeta}) = +135^\circ$$

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We mostly want a short settling time ($< 4$s). This results in another restriction on the poles of the system:

$$\tau_n = \frac{4}{\omega\zeta} < 4\text{s}$$
$$\omega_n\zeta > 1$$

# Second order systems - damping