Chapter 5: Continuous-time systems

July 30, 2015

Outline

Linear differential equations: definitions 1/2

Linear differential equations (LDE) are of the following form:

$$L[y(t)] = f(t),$$

where L is some linear operator.
The linear operator L is of the following form:

$$L_n(y) = \sum_{i=0}^{n} A_i(t) \frac{d^{n-i} y}{dt^{n-i}},$$

with given functions $A_{1:n}$.
The **order of a LDE** is the index of the highest derivative of y.

Linear differential equations: definitions 2/2

$$L_n(y) = \sum_{i=0}^{n} A_i(t) \frac{d^{n-i} y}{dt^{n-i}} = f(t).$$

- y is a scalar function $\to$ **ordinary differential equation** (ODE);
- y is a vector function $\to$ **partial differential equation** (PDE);
- $f = 0 \to$ **homogeneous equation**
  $\to$ solutions are called **complementary functions**;
- if $A_{0:n}(t)$ are constants (i.e. not functions of time), the LDE is said to have **constant coefficients**.

Example: radioactive decay 1/2

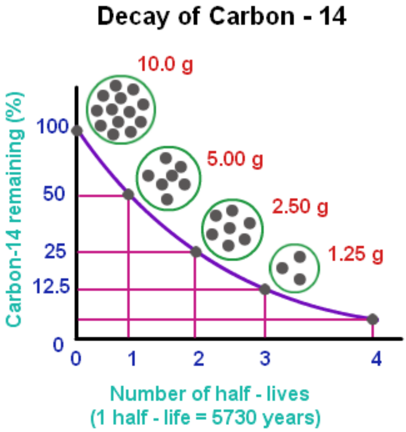
Let $N(t)$ be the number of radioactive atoms at time t, then:

$$\frac{dN(t)}{dt} = -kN(t),$$

for some constant $k > 0$.

This is a first order homogeneous LDE with constant coefficients.

Example: radioactive decay 2/2



Solving homogeneous LDEs with constant cofficients 1/3

Solutions of LDEs must be of the form e^{zt} with $z \in \mathbb{C}$.
We assume an LDE with constant coefficients:

$$\sum_{i=0}^{n} A_i y^{(n-i)} = 0.$$

Replacing $y = e^{zt}$ leads to:

$$\sum_{i=0}^{n} A_i z^{n-i} e^{zt} = 0$$

Dividing by e^{zt} yields the nth order **characteristic polynomial**:

$$F(z) = \sum_{i=0}^{n} A_i z^{n-i} = 0.$$

Solving homogeneous LDEs with constant cofficients 2/3

Characteristic equation:

$$F(z) = \sum_{i=0}^{n} A_i z^{n-i} = 0.$$

1. Solving the polynomial $F(z)$ yields n zeros z_1 to z_n;
2. Substituting a given zero z_i into e^{zt} gives a solution $e^{z_i t}$.

Homogeneous LDEs obey the superposition position:
$\rightarrow$ any linear combination of solutions $e^{z_1 t}, \ldots, e^{z_n t}$ is a solution

Solving homogeneous LDEs with constant cofficients 2/3

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Homogeneous LDEs obey the superposition position:
$\rightarrow$ any linear combination of solutions $e^{z_1 t}, \ldots, e^{z_n t}$ is a solution
$\rightarrow$ $e^{z_1 t}, \ldots, e^{z_n t}$ form a basis of the solution space of the LDE
The specific linear combination depends on initial conditions.

Solving homogeneous LDEs with constant cofficients 3/3

Example

$$y^{(4)}(t) - 2y^{(3)}(t) + 2y^{(2)}(t) - 2y^{(1)}(t) + y(t) = 0.$$

This is a 4th order homogeneous LDE with constant coefficients. The corresponding characteristic equation:

$$F(z) = z^4 - 2z^3 + 2z^2 - 2z + 1 = 0.$$

The zeros of $F(z)$ are ($j = \sqrt{-1}$):

$$z_1 = j, \quad z_2 = -j, \quad z_{3,4} = 1.$$

These zeros correspond to the following basis functions t:

$$e^{jt}, \quad e^{-jt}, \quad e^{t}, \quad te^{t}.$$

Outline

The Laplace transform

Definition

The Laplace transform of $f(t)$, for all real numbers $t \geq 0$:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt.$$

The parameter $s = \sigma + j\omega$ is the complex number frequency.

The initial value theorem states $f(0^+) = \lim_{s\to\infty} sF(s)$.
The final value theorem states $f(\infty) = \lim_{s\to 0} sF(s)$,
if all poles of $sF(s)$ are in the left half plane (i.e. real part < 0).

Important properties of the Laplace transform

property	**time domain**	**s-domain**
linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
differentiation	$f^{(1)}(t)$	$sF(s) - f(0)$
integration	$\int_0^t f(\tau)d\tau = (u * f)(t)$	$\frac{1}{s}F(s)$
convolution	$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$	$F(s) \cdot G(s)$
time scaling	$f(at)$	$\frac{1}{a}F(\frac{s}{a})$
time shifting	$f(t-a)u(t-a)$	$e^{-as}F(s)$

Inverse Laplace transform

Definition

The inverse Laplace transform converts s-domain to time domain:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{j2\pi}\int_{\gamma-jT}^{\gamma+jT} e^{st}F(s)ds.$$

Practically, the inverse Laplace transform takes two steps:

1. write $F(s)$ in terms of partial fractions
2. transform each term in the partial fraction based on tables of s/t-domain pairs (course notes p. 4.32-4.33)

Outline

Solving LDEs with the Laplace transform 1/3

The Laplace transform can be used to solve LDEs with given initial conditions (the previous approach gave us the basis functions). This is done by using the following property (differentiation):

$$\mathcal{L}\{f^{(1)}\} = sF(s) - f(0),$$
$$\mathcal{L}\{f^{(2)}\} = s^2F(s) - sf(0) - f^{(1)}(0).$$

Via induction, the Laplace transform of the nth order derivative:

$$\mathcal{L}\{f^{(n)}\} = s^nF(s) - \sum_{i=1}^{n} s^{n-i} f^{(n-i)}(0)$$

Solving LDEs with the Laplace transform 2/3

$$\mathcal{L}\{f^{(n)}\} = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(n-i)}(0)$$

We want to solve the following LDE:

$$\sum_{i=0}^{n} A_i y^{(n-i)}(t) = f(t),$$

$$y^{(i)}(0) = c_i \quad \forall i = 0 \dots n.$$

Via the linearity of the Laplace transform:

$$\sum_{i=0}^{n} A_i \mathcal{L}\{y^{(n-i)}(t)\} = \mathcal{L}\{f(t)\}$$

Solving LDEs with the Laplace transform 3/3

$$\sum_{i=0}^{n} A_i \mathcal{L}\{y^{(n-i)}(t)\} = \mathcal{L}\{f(t)\} \tag{1}$$

$$\mathcal{L}\{f^{(n)}\} = s^n F(s) - \sum_{i=1}^{n} s^{n-i} f^{(n-i)}(0) \tag{2}$$

Expanding Eq. (2) into (1) yields:

$$Y(s) \sum_{i=0}^{n} A_i s^i - \sum_{i=1}^{n} \sum_{j=1}^{i} A_i s^{i-j} y^{j-1}(0) = F(s)$$

The solution in the time domain is obtained via the inverse Laplace transform: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

Outline

Observability

Definition

A measure of how well a system's internal states $\mathbf{x}$ can be inferred by knowledge of its outputs $\mathbf{y}$.

Formally, a system is said to be observable if, for any possible sequence of state and control vectors, the current state can be determined in finite time using only the outputs.
This holds for linear, time-invariant systems with n states if:

$$rank(\mathcal{O}) = n, \quad \mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}, \quad \mathcal{O} : \textbf{observability matrix}$$

Controllability

Definition

A measure of the ability to move a system around in its entire configuration space using only certain admissible manipulations.

A system is controllable if its state can be moved from any initial state $\mathbf{x}_0$ to any final state $\mathbf{x}_f$ via some finite sequence of inputs $\mathbf{u}_0 \ldots \mathbf{u}_f$.
A linear, time-invariant system with n states is controllable if:

$$rank(\mathcal{C}) = n, \quad \mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \ldots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix},$$

where $\mathcal{C}$ is called the **controllability matrix**.

Outline

Transfer function

Definition

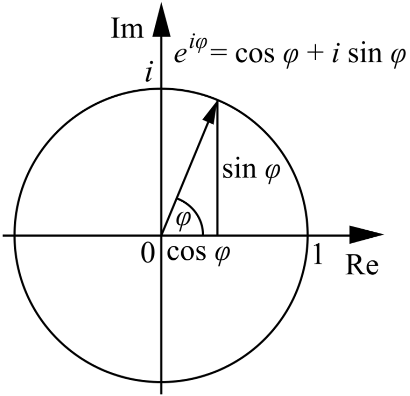
The transfer function of input i to output j is defined as:

$$H_{i,j}(s) = \frac{Y_j(s)}{U_i(s)}, \quad \mathbf{U}(s) = \mathcal{L}\{u(t)\}, \quad \mathbf{Y}(s) = \mathcal{L}\{y(t)\}.$$

MIMO systems with n inputs and m outputs have $n \times m$ transfer functions, one for each input-output pair.
The complex Laplace variable can be rewritten: $s = \sigma + j\omega$.
The frequency response of a system can be analyzed via $\mathbf{H}(j\omega)$:

$$e^{\sigma + j\omega} = e^{\sigma}(\cos\omega + j\sin\omega).$$

Illustration of Euler's formula



Poles and zeros

In general, the transfer function can be written as:

$$H(s) = \frac{N(s)}{D(s)}.$$

The poles of $H(s)$ are zeros of $D(s)$, i.e. $\{s \ : \ D(s) = 0\}$.

- $|H(s)| = \infty$ if s is a pole.

The zeros of $H(s)$ are zeros of $N(s)$, i.e. $\{s \ : \ N(s) = 0\}$.

- $H(s) = 0$ if s is a zero.

Poles and zeros may cancel, i.e. if $D(s) = N(s) = 0$ for some s.

Steady state response

The output of a linear time-invariant system consists of:

- a steady-state output $y_{ss}(t)$, with the same period as $u(t)$
  $\rightarrow$ y_{ss} comprises the same frequencies as $u(t)$;
- a transient output $y_{tr}(t)$
  $\rightarrow$ if the system is stable, then $\lim_{t\rightarrow\infty} y_{tr}(t) = 0$
  $\rightarrow$ $y_{tr}(t)$ depends on the initial state $\mathbf{x}_0(t)$ of the system.

If we apply an input $u(t) = cos(\alpha t + \theta)$, then:

$$y_{ss}(t) = |H(j\alpha)| cos(\alpha t + \theta + \angle H(j\alpha)).$$

The steady-state output $y_{ss}(t)$ of a linear time invariant system:

- consists of signals of same frequencies as the input signal $u(t)$;
- which may have been magnified and/or phase changed.

Outline

Impulse response

Definition

The impulse response $h(t)$ of input i to output j is the output $y_j(t)$ of a system when an impulse $\delta(t)$ is applied at input $u_i(t)$. The impulse response is the inverse Laplace transform of the transfer function $h(t) = \mathcal{L}^{-1}\{H(s)\}$.

For stable continuous time systems the impulse response always converges to 0:

$$\lim_{t\to\infty} h(t) = 0, \text{ because } \mathbf{D} = 0 \text{ and } \lim_{t\to\infty} \mathbf{x}(t) = 0.$$

The speed of convergence depends on the position of the poles.

Time constant

Definition

The transfer function of first order systems can be written as:

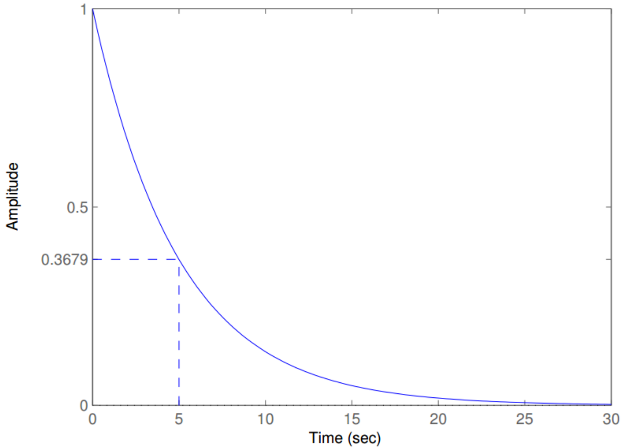
$$H(s) = \frac{K}{\tau s + 1} \quad \leftrightarrow \quad h(t) = \frac{K}{\tau} e^{-t/\tau},$$

where τ is called the system's **time constant**.

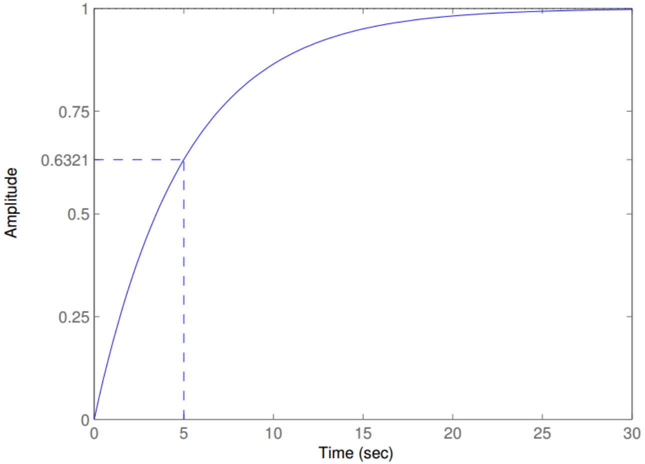
The time constant summarizes the speed of a system's dynamics:

- after τ seconds, the impulse response reaches $h(0)/e$.
- after τ seconds, the step response has reached $1 - e^{-1} \approx 63\%$ of its regime value.

Impulse response $H(s) = 5/(5s + 1) \leftrightarrow h(t) = exp(-t/5)$



Step response $H(s) = 5/(5s+1) \leftrightarrow h(t) = exp(-t/5)$



Outline

From state-space to transfer functions

We start from the linear state-space representation:

time domain | Laplace domain

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \quad \leftrightarrow \quad \begin{cases} s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{cases}$$

A transfer function $\mathbf{H}(s) = \frac{\mathbf{Y}(s)}{\mathbf{U}(s)}$ relates an input and an output in the Laplace-domain $\rightarrow$ to obtain it, we must eliminate $\mathbf{X}(s)$.

$$\begin{aligned} (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{B}\mathbf{U}(s) \\ \mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \\ \Rightarrow \mathbf{Y}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) \\ \Rightarrow \mathbf{H}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \end{aligned}$$

Relationship between poles and eigenvalues of **A** 1/2

Poles are zeros of the denominator of $\mathbf{H}(s)$, e.g. those values of s for which $\mathbf{H}(s)$ is singular.
The relationship between state-space representation (matrices **A**, **B**, **C** and **D**) and transfer functions is given by

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$H(s)$ cannot be computed when $(s\mathbf{I} - \mathbf{A})^{-1}$ does not exist, ie.

$$\det(s\mathbf{I} - \mathbf{A}) = 0$$

The determinant is zero if s is an eigenvalue of **A**.
$\rightarrow$ all poles of $\mathbf{H}(s)$ are eigenvalues of **A**.

Relationship between poles and eigenvalues of **A** 2/2

Transfer functions only capture what is relevant to describe an input-output relationship, but not all states necessarily contribute.
$\rightarrow$ *unobservable* modes of **A** are not poles in $\mathbf{H}(s)$.
Consider the following SISO system with 2 states:

$$\begin{bmatrix} sX_1(s) \\ sX_2(s) \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} + \begin{bmatrix} \beta \\ 2 \end{bmatrix} U(s)$$

$$Y(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$$

The transfer function $H(s) = \frac{\beta}{s-\alpha}$ has only one pole ($s_1 = \alpha$).
$\rightarrow$ not all eigenvalues of **A** are poles in transfer functions $\mathbf{H}(s)$.

Outline

Outline

Transient Response

The time response of a control system can be written as:

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

where $y_{tr}(t)$ is the transient response and $y_{ss}(t)$ is the steady state response.

Definition

The transient response of a system is the time-difference between the change of the inputs and the change of the outputs: when the input of a system changes, the output does not change immediately but takes time to go to steady state.

First order systems: stability

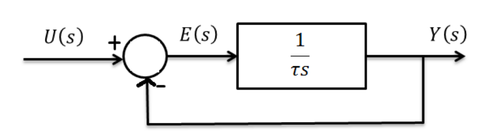
The most important characteristic of a dynamic system is absolute stability.

- A system is stable when it returns to equilibrium, if subject to initial condition
- A system is critically stable when oscillations of the output continue forever
- A system is unstable when the output diverges without bound from equilibrium, if subject to initial condition

First order systems

Example

Unit step response of RC circuit, thermal system, ...



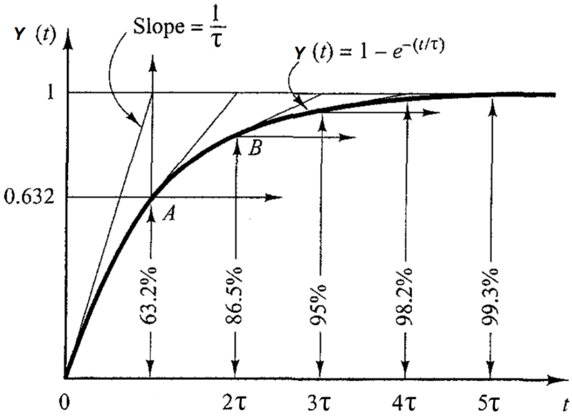
The transfer function is given by: $\frac{Y(s)}{U(s)} = \frac{1}{\tau s+1}$

- Laplace of unit-step is $\frac{1}{s}$ $\rightarrow$ substituting $U(s) = \frac{1}{s}$: $Y(s) = \frac{1}{s}\frac{1}{\tau s+1}$;
- Expanding into partial fractions gives $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$.

Unit step transient response

1. $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$;
2. Taking the inverse Laplace transform
   $y(t) = 1 - e^{-\frac{t}{\tau}}$, for $t \geq 0$;
3. At $t = 0$, the output $y(t) = 0$;
4. At $t = \tau$, the output $y(t) = 0.632$, or $y(t)$ has reached 63.2% of its total change $y(\tau) = 1 - e^{-1} = 0.632$;
5. Slope at time $t = 0$ is $\frac{1}{\tau}$
   $\frac{dy}{dt}|_{t=0} = \frac{1}{\tau}e^{-\frac{t}{\tau}}|_{t=0} = \frac{1}{\tau}$,
   where τ is called the system time constant.

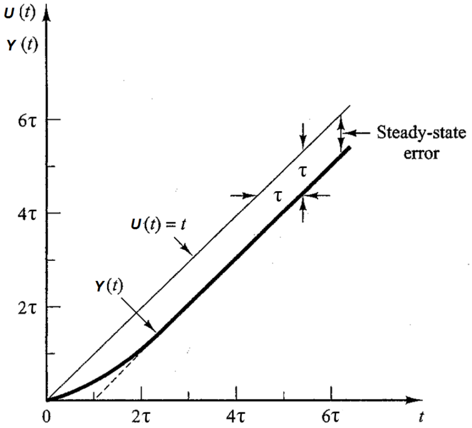
Unit step transient response



Unit ramp transient response

1. Laplace transform of unit ramp is $\frac{1}{s^2}$
   $Y(s) = \frac{1}{\tau s+1}\frac{1}{s^2}$;
2. Expanding into partial fractions gives
   $Y(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s+1}$;
3. Taking the inverse Laplace transform
   $y(t) = t - \tau + \tau e^{-\frac{t}{\tau}}$, for $t \geq 0$;
4. The error signal $e(t)$ is then
   $e(t) = u(t) - y(t) = \tau(1 - e^{-\frac{t}{\tau}})$;
5. For t approaching infinity, $e(t)$ approaches τ
   $e(\infty) = \tau$.

Unit ramp transient response



Unit-Impulse Response

For a unit-impulse input, $U(s) = 1$ and the output is:

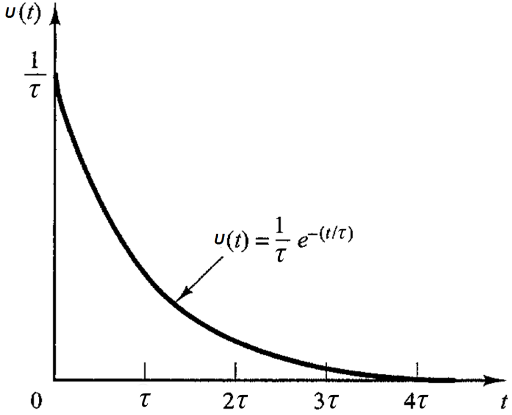
$$Y(s) = \frac{1}{\tau s + 1}.$$

The inverse Laplace transform gives:

$$y(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}, \text{ for } t \geq 0.$$

For $t \to +\infty$, $y(t) \to 0$.

Unit-Impulse Response



Outline

Second order systems

A second order system can generally be written as:

$$\frac{Y(s)}{U(s)} = H(s) = \frac{as^2 + bs + c}{ds^2 + es + f}$$

A system where the closed-loop transfer function possesses two poles is called a second-order system.

If the transfer function has two real poles, the frequency response can be found by combining the effects of both poles

Second order systems

Sometimes the transfer function has two complex conjugate poles. In that case we have to find a different solution for finding the frequency response.

In order to study the transient behaviour, let us first consider the following simplified example of a second order system:

$$H(s) = \frac{c}{ds^2 + es + c}.$$

Step response of a second order system

1. $H(s) = \frac{c}{ds^2+es+c}$;
2. The transfer function can be rewritten as:

$$H(s) = \frac{\frac{c}{d}}{s^2 + \frac{e}{d}s + \frac{c}{d}}$$
$$= \frac{\frac{c}{d}}{\left[s + \frac{e}{2d} + \sqrt{\left(\frac{e}{2d}\right)^2 - \frac{c}{d}}\right]\left[s + \frac{e}{2d} - \sqrt{\left(\frac{e}{2d}\right)^2 - \frac{c}{d}}\right]};$$

3. The poles are complex conjugates if $e^2 - 4dc < 0$;
4. The poles are real if $e^2 - 4dc \geq 0$.

Step response of a second order system

To simplify the transient analysis, it is convenient to write:

- $\frac{f}{d} = \omega_n^2$,
- $\frac{e}{d} = 2\zeta\omega_n = 2\sigma$

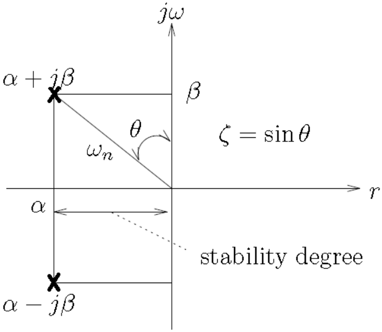
where σ is the attenuation, ω_n is the natural frequency and ζ is the damping ratio.

The transfer function can now be rewritten as

$$H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega s+\omega_n^2} \text{ (= standard form).}$$

The dynamic behavior of the second-order system can then be described in terms of only two parameters ζ and ω_n.

Poles of the system



$$\alpha = -\zeta\omega_n, \beta = \omega_n\sqrt{1-\zeta^2}$$

Step response of a second order system

If $0 < \zeta < 1$, the poles are complex conjugates and lie in the left-half s-plane

- The system is then called **underdamped**
- The **transient response is oscillatory**

If $\zeta = 0$, the **transient response doesn't die out**. If $\zeta = 1$, the system is called **critically damped**. If $\zeta > 1$, the system is called **overdamped**. We will now look at the unit step response for each of these cases.

Underdamped system

For the underdamped case ($0 < \zeta < 1$), the transfer function can be written as:

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

Where ω_d is called the damped natural frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

For a unit-step input we can write

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}.$$

Underdamped system

Which can be rewritten as partial fractions

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} - \frac{\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \end{aligned}$$

It can be shown that

$$\mathcal{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t} cos(\omega_d t)$$

$$\mathcal{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t} sin(\omega_d t).$$

Underdamped system

Therefore:

$$\begin{aligned}\mathcal{L}^{-1}\{Y(s)\} &= y(t)\\ &= 1 - e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t))\\ &= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} sin(\omega_d t + tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta})).\end{aligned}$$

It can be seen that the frequency of the transient oscillation is the damped natural frequency ω_d and thus varies with the damping ratio ζ.

Underdamped system

The error signal is the difference between input and output

$$\begin{aligned} e(t) &= y(t) - u(t) \\ &= e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t)) \end{aligned}$$

The error signal exhibits a damped sinusoidal oscillation. At steady state, or at $t = \infty$, the error goes to zero.

Underdamped system

If damping $\zeta = 0$, the response becomes **undamped**

- Oscillations continue indefinitely;
- Filling in $\zeta = 0$ into the equation for y(t) gives us: $y(t) = 1 - cos(\omega_n t)$, for $t \geq 0$;
- We see that the system now oscillates at the natural frequency ω_n;
- If a linear system has any amount of damping, the undamped natural frequency cannot be observed experimentally, only ω_d can be observed;
- ω_d is always lower than ω_n.

Critically damped system

If the two poles of the system are equal, the system is critically damped and $\zeta = 1$. For a unit-step, $R(s) = \frac{1}{s}$ and we can write:

$$Y(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s}.$$

The inverse Laplace transform gives us:

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) \text{ for } t \geq 0.$$

Overdamped system

A system is overdamped ($\zeta > 1$) when the two poles are negative, real and unequal. For a unit-step $R(s) = \frac{1}{s}$, $Y(s)$ can be written as

$$Y(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n^2\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n^2\sqrt{\zeta^2 - 1})}.$$

The inverse Laplace transform is

$$y(t) = 1 + \frac{w_n}{2\sqrt{\zeta^2 - 1}}(\frac{e^{-s_1 t}}{s1} - \frac{e^{-s_2 t}}{s2}), \text{ for } t \geq 0.$$

Where

$$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n \text{ and } s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n.$$

Overdamped system

$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$
Thus $y(t)$ includes two decaying exponential terms

- When $\zeta >> 1$, one of the two decreases much faster than the other, the faster decaying exponential may be neglected;
- If $-s_2$ is located much closer to the $j\omega$ axis than $-s_1$ ($|s_2| >> |s_1|$), then $-s_1$ may be neglected;
- Once the faster decaying exponential term has disappeared, the response is similar to that of a first-order system:

$$H(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} = \frac{s_2}{s + s_2}.$$

Overdamped system

With the approximate transfer function, the unit-step response becomes:

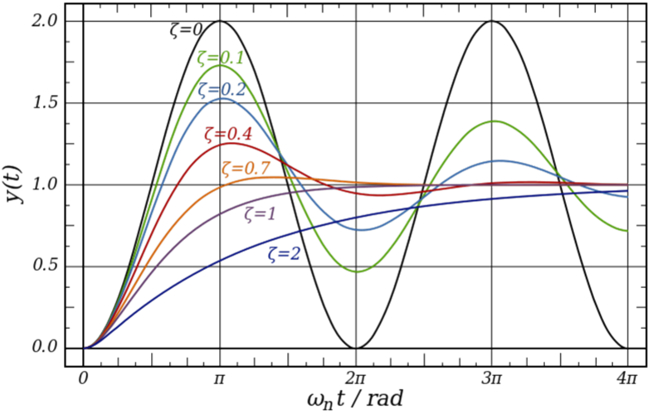
$$Y(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

The time response for the approximate transfer function is then given as:

$$y(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}, \text{ for } t \leq 0$$

Second order systems unit step response curves

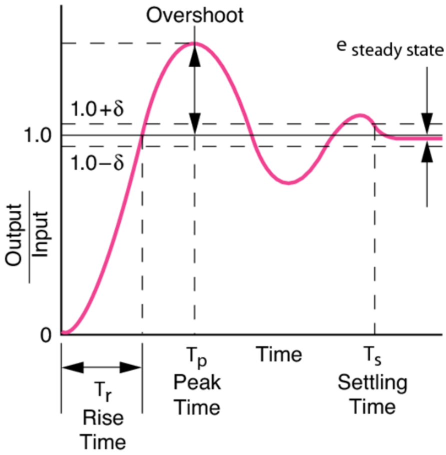
Response on a step function



Second order systems - characteristics

- Overshoot: Highest amplitude above steady state: $M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$;
- Rise Time: Time needed to reach the steady state for the first time. $t_r = \frac{1.8}{\omega_n}$;
- Peak Time: Time to reach overshoot. $t_p = \frac{\pi}{\omega_d}$;
- Settling Time: Time needed to approximate the steady state: $t_s = \frac{4.6}{\zeta\omega_n}$. Important note: this formulas can only be used when $0 < \zeta < 1$!

Second order systems - characteristics



Example

Given: $\delta = \frac{0.02}{\sqrt{1-\zeta^2}}$

We find a settling time of:

$$e^{-\zeta \omega_n t_s} < 0.02$$

$$t_s = \frac{4}{\omega_n \zeta}$$

Second order systems - resonance

The resonance frequency is the frequency at which the systems output has a larger amplitude than at other frequencies. This happens when underdamped functions oscillate at a greater magnitude than the input. An input with this frequency can sometime have catastrophic effects.

A different view on the Tacoma bridge disaster:
`https://www.youtube.com/watch?v=6ai2QFxStxo`

In fact the collapse was a result of a number of effects like Aerodynamic flutter and vortices. Read the full article here:
`http://www.ketchum.org/billah/Billah-Scanlan.pdf`

Second order systems - resonance

The resonance frequency is: $\omega_r = \omega_n\sqrt{1-\zeta^2}$.

Systems with a damping > 0.707 do not resonate. The resonance frequency and the natural frequency are equal when a system has no damping.

Another phenomenon with bridges and resonance is that many people marching with the same rhythm can cause a bridge to start resonating like the Angers bridge in 1850. A more recent example is the Millennium bridge in London which started resonating.

Second order systems - damping

When we want a system with no resonance, we choose one with damping < 0.707. This means a pole between 135° and 225°:

$$\arctan\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = +135^\circ$$

We mostly want a short settling time (< 4s). This results in another restriction on the poles of the system:

$$\tau_n = \frac{4}{\omega\zeta} < 4\text{s}$$

$$\omega_n\zeta > 1$$

Second order systems - damping