# Geometric algebra

August 24, 2015

# Outline

# Vectors and spatential interpretation

## Properties of a vector

There are 3 properties of a vector $\overrightarrow{x}$:

- magnitude
- direction
- startpoint

**with respect to** a referention vector $\overrightarrow{0}$

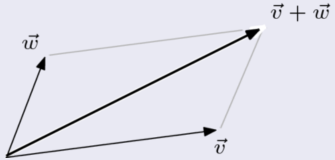
## Multiplication scalar and vector

r $\in \mathbb{R}$ (r $\in \mathbb{C}$ is possible, but hasn't a fysical representation)

- $|r| < 1$: shorten
- $|r| > 1$: increase
- $r < 0$: reverse the direction

## Addition of vectors

Parallellogramrule:



## vectorspace

### First condition

A vectorspace V over a body L (set of operators) is a set of vectors that satisfy:

1. A vectorsum is defined: $VxV \rightarrow V : (\vec{x}, \vec{y}) \rightarrow \vec{x} + \vec{y}$

   $\vec{x}, \vec{y}, \vec{z} \, \epsilon$ V

   a) $\vec{x} + \vec{y} \, \epsilon$ V
   b) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$
   c) $\exists! \vec{0} : \vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$
   d) $\forall \vec{x}, \exists(-\vec{x}) : \vec{x} + (-\vec{x}) = (-\vec{x}) + \vec{x} = 0$
   e) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

# vectorspace

## Second condition

2. A outside law is defined: $LxV \rightarrow V : (a, \overrightarrow{x}) \rightarrow a\overrightarrow{x}$

   $\overrightarrow{x}, \overrightarrow{y} \in$ V

   a,b $\in$ L

   a) $1\overrightarrow{x} = \overrightarrow{x}$
   b) $a(b\overrightarrow{x}) = (ab)\overrightarrow{x}$
   c) $(a+b)\overrightarrow{x} = a\overrightarrow{x} + b\overrightarrow{x}$
   d) $a(\overrightarrow{x} + \overrightarrow{y}) = a\overrightarrow{x} + a\overrightarrow{y}$

# Numberspaces of n-couples

## Defenition

This is the set of all n-couples like $\begin{bmatrix} \overrightarrow{x_1} \\ \overrightarrow{x_2} \\ . \\ . \\ \overrightarrow{x_n} \end{bmatrix}$ with $x_i \epsilon \mathbb{R}$ or $x_i \epsilon \mathbb{C}$.

This set together with the operator set $\mathbb{R}$ or $\mathbb{C}$ is a vectorspace.

# Subspaces

## Defenition

$V_1$ is a supspace of vectorspace V if:

1. $V_1 \subset$ V
2. With the same in- and outside law as V, is $V_1$ a vectorspace

## Properties

1. $\overrightarrow{0}$ $\epsilon$ every subspace
2. The intersection of two spaces is always a subspace
3. Given: p vectors $x_1, x_2, ..., x_p$ $\epsilon$ V.
   The set vectors $a_1x_1 + a_2x_2 + ... + a_nx_n$ with $a_i$ $\epsilon$ $\mathbb{R}$ is a subspace of V.

# Linear independance, basis, dimensions

## Defenition independance

Given: p vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p} \in$ V.
Construct the nullvector as a linear combination of those vectors (i.e. search the operators (numbers) $a_1, a_2, ..., a_p$ to form $a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + ... + a_n\overrightarrow{x_n} = \overrightarrow{0}$).
If the nullvector only can created by $a_1 = a_2 = ... = a_p = 0$, then are the vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ **linear independant**.

# Linear independance, basis, dimensions

## Properties

1. If the vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ are linear independent, then can't none of them be writed as a linear combination of the other p-1 vectors.
2. If the nullvector is one of the p vectors, then is the set $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ linear dependant (if $\overrightarrow{x_1} = 0$ then is $a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + ... + a_n\overrightarrow{x_n} = 0$ with $a_1 \neq 0$ and $a_2, a_3, ...a_p = 0$).
3. Basis and dimension: p linear independant vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ generate a vectrospace $V^p$. Every vector in $V^p$ can be writed **in only one way** as a linear combination of the p linear independant vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ using operators $a_1, a_2, ..., a_p$.

# Linear independance, basis, dimensions

## Basis, dimension

Given: $\overrightarrow{v} = a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + ... + a_p\overrightarrow{x_p}$.
The set operators $a_1, a_2, ..., a_p$ are called the **coordinates** of the vector $\overrightarrow{v}$ relative to the set vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$. This set vectors is a **basis** of vectorspace $V^p$, with **dimension** p.

# Linear independance, basis, dimensions

## Example

Given: $\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \overrightarrow{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The set $\{\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}\}$ is a linear independant combination. There doesn't exist numbers $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$ such that $a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + a_n\overrightarrow{x_3} = 0$. The set of all vectors $\overrightarrow{y} = a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + a_n\overrightarrow{x_3}$ is the three dimensional vectrospace $V^3$.

If $\overrightarrow{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Then is $3\overrightarrow{x_1} + 2\overrightarrow{x_2} + \overrightarrow{x_3}$ the only way to write $\overrightarrow{y}$ as a linear combination of $\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}$.

# Linear independance, basis, dimensions

## Example

The set of vectors $\overrightarrow{y} = a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2}$ is a **two dimensional** subspace $V^2$.

The vectors in this subspace are:

$$\overrightarrow{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}.$$

## Important difference

- All vectors of $V^2$ have **3** coordinates.
- The dimension of the subspace $V^2$ is **2**.

# Linear independance, basis, dimensions

## Convention of notation

Given: a n-dimensional vectorspace $V^n$.
The elements of this vectorspace are the elements: $\overrightarrow{x}, \overrightarrow{y}, \ldots$ If we choose $\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_n}$ as a basis of $V^n$. Then we can write every vector of $V^n$ as a linear combination of those basis vectors in only one way: $\overrightarrow{x} = x_1\overrightarrow{e_1} + x_2\overrightarrow{e_2} + \ldots + x_n\overrightarrow{e_n}$. The numbers $x_i$ are the coordinates of vector $\overrightarrow{x}$ relative to the basis $\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_n}$.

Between the vectorspace of dimension n and the number space of dimension n exists a isomorphism.

# Linear independance, basis, dimensions

## Vectorspace $V^p$

Given: a p-dimensional vectorspace $V^p$ where the vectors are n-couples (with $n \geq p$).

1. In $V^p$ you can choose a basis with p linear independant vectors.
2. Every vector $\overrightarrow{x} \epsilon V^p$ can be writed in only one way as a linear combination of the p basis vectors using coordinates.

## Example 1

Given: n=5, p=2, $\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \overrightarrow{x_2} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

# Linear independance, basis, dimensions

## Example 1

The vectors $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ are linear independant, so they span a two dimensional subspace: $\overrightarrow{y} = a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2}$ with $a_1, a_2 \in \mathbb{R}$.

The coordinates of the vector $y_1 = \begin{bmatrix} 5 \\ -6 \\ 1 \\ 2 \\ 5 \end{bmatrix}$, relative to the basis $\{\overrightarrow{x_1}, \overrightarrow{x_2}\}$, are $a_1 = 1$ and $a_2 = 2$.

# Linear independance, basis, dimensions

## Example 1

The vector $\overrightarrow{y_2} = \begin{bmatrix} 5 \\ -7 \\ 1 \\ 2 \\ 5 \end{bmatrix}$ can't be writen as a linear combination of the vectors $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$. So $y_2$ doesn't belong to the subspace spanned by $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$.
This implies that $\overrightarrow{y_2}$ is linear independant of $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$. Thus the subspace spanned by $\overrightarrow{y_2}$, $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ is a 3 dimensional subspace.

# Linear independance, basis, dimensions

## In general

When the set vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, \ldots, \overrightarrow{x_p}$ is linear independant, then lays $\overrightarrow{x_i}$ not totally in the subspace spanned by the vectors $\overrightarrow{x_1}, \ldots, \overrightarrow{x}_{i-1}, \overrightarrow{x}_{i+1}, \ldots, \overrightarrow{x_p}$.
The vector $\overrightarrow{x_i}$ can be writen as a sum of 2 components: $\overrightarrow{x}_{i\alpha}$ and $\overrightarrow{x}_{i\beta}$.

1. $\overrightarrow{x}_{i\alpha} \epsilon$ subspace spanned by $\overrightarrow{x_1}, \ldots, \overrightarrow{x}_{i-1}, \overrightarrow{x}_{i+1}, \ldots, \overrightarrow{x_p}$.
2. $\overrightarrow{x}_{i\beta} \perp$ subspace spanned by $\overrightarrow{x_1}, \ldots, \overrightarrow{x}_{i-1}, \overrightarrow{x}_{i+1}, \ldots, \overrightarrow{x_p}$.

# Inner product

## Defenition

The inproduct of two vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ $\epsilon$ $E^n$ (n-couples) is defined as the image: $E^n x E^n \to \mathbb{R} : \{\overrightarrow{x}, \overrightarrow{y}\} \to \overrightarrow{x}.\overrightarrow{y}$ $\epsilon$ $\mathbb{R}$. This image is:

1. Bilinear:
   $(\overrightarrow{x} + \overrightarrow{v}).\overrightarrow{y} = \overrightarrow{x}.\overrightarrow{y} + \overrightarrow{v}.\overrightarrow{y}$
   $\overrightarrow{x} + (\overrightarrow{v}).\overrightarrow{y}) = \overrightarrow{x}.\overrightarrow{v} + \overrightarrow{x}.\overrightarrow{y}$
   $(a\overrightarrow{x})\overrightarrow{y} = a(\overrightarrow{x}.\overrightarrow{y})$
   $\overrightarrow{x}(a\overrightarrow{y}) = a(\overrightarrow{x}.\overrightarrow{y})$
2. Symetric:
   $\overrightarrow{x}.\overrightarrow{y} = \overrightarrow{y}.\overrightarrow{x}$
3. Positive definite:
   $\forall \overrightarrow{x} \neq \overrightarrow{0}: \overrightarrow{x}.\overrightarrow{x} > 0$

# Inner product

## Matricial notation

The inproduct is a **scalar**. If $\overrightarrow{x}$, $\overrightarrow{y}$ and the basis $\epsilon$ $E^n$ then can the inproduct be noted matricial:

$$\overrightarrow{x}.\overrightarrow{y} = y^t Ax = x^t Ay = (x_1 ... x_n) A \begin{pmatrix} y_1 \\ ... \\ y_n \end{pmatrix}$$

with A positive definite and symetric $(A = A^t)$.

# Inner product

## Norm of a vector

$\|\overrightarrow{x}\|^2 = \overrightarrow{x}.\overrightarrow{x}$ and because $\overrightarrow{x}.\overrightarrow{x} > 0$ applies:
$\|\overrightarrow{x}\| = \sqrt{\overrightarrow{x}.\overrightarrow{x}}$ where $\|\overrightarrow{x}\|$ is called the norm of $\overrightarrow{x}$.
Normalizing is dividing a vector by its norm. The result is a vector with norm $= 1$.
$\|\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}\| = \sqrt{\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}} = \sqrt{\frac{\|\overrightarrow{x}\|^2}{\|\overrightarrow{x}\|\|\overrightarrow{x}\|}} = 1.$

# Inner product

## CauchySchwarz inequality

$|\overrightarrow{x}.\overrightarrow{y}| \leq \|\overrightarrow{x}\|\|\overrightarrow{y}\|$ or
$-\|\overrightarrow{x}\|\|\overrightarrow{y}\| \leq \overrightarrow{x}.\overrightarrow{y} \leq \|\overrightarrow{x}\|\|\overrightarrow{y}\|$ from wich follows:
$-1 \leq \frac{\overrightarrow{x}\,\overrightarrow{y}}{\|\overrightarrow{x}\|\|\overrightarrow{y}\|} \leq 1$
By defenition follows:
$\cos(\theta) = cos(\angle(\overrightarrow{x}, \overrightarrow{y})) = \frac{\overrightarrow{x}\,\overrightarrow{y}}{\|\overrightarrow{x}\|\|\overrightarrow{y}\|}$
Therefor: the angle between the vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ $=$
Bgcos(inproduct of $\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}$ and $\frac{\overrightarrow{y}}{\|\overrightarrow{y}\|}$).

# Inner product

## Orthogonality

$\overrightarrow{x}$ and $\overrightarrow{y}$ are orthogonal $\Leftrightarrow$
$\theta = \angle(\overrightarrow{x}, \overrightarrow{y}) = 90^\circ = \frac{\Pi}{2} rad \Leftrightarrow$
$cos(\theta) = 0 \Leftrightarrow$
$\overrightarrow{x} . \overrightarrow{y} = 0$

Hence, if $\overrightarrow{x}, \overrightarrow{y} \neq 0$:
$\overrightarrow{x} \perp \overrightarrow{y} \Leftrightarrow \overrightarrow{x} . \overrightarrow{y} = 0.$

## Parallellism

$\overrightarrow{x} \parallel \overrightarrow{y} \Leftrightarrow \theta = 0^\circ$ or $180^\circ \Leftrightarrow cos(\theta) = \pm 1 \Leftrightarrow |\overrightarrow{x}\,\overrightarrow{y}| = \|\overrightarrow{x}\|\|\overrightarrow{y}\|$

# Inner product

## Distance between two vectors

Distance $= \|\overrightarrow{x} - \overrightarrow{y}\| = \|\overrightarrow{z}\|$ with $\overrightarrow{z} = \overrightarrow{x} - \overrightarrow{y}$.
$\|\overrightarrow{x} - \overrightarrow{y}\|^2 = (\overrightarrow{x} - \overrightarrow{y})(\overrightarrow{x} - \overrightarrow{y})$
$= \overrightarrow{x}\,\overrightarrow{x} - \overrightarrow{x}\,\overrightarrow{y} - \overrightarrow{y}\,\overrightarrow{x} + \overrightarrow{y}\,\overrightarrow{y}$
$= \overrightarrow{x}\,\overrightarrow{x} + \overrightarrow{y}\,\overrightarrow{y} - 2\overrightarrow{x}\,\overrightarrow{y}$
$= \|\overrightarrow{x}\|^2 + \|\overrightarrow{y}\|^2 - 2\|\overrightarrow{x}\|\|\overrightarrow{y}\|cos(\theta)$ with $\theta$ the angle between $\overrightarrow{x}$ and $\overrightarrow{y}$.

## Pythagorean theorem

If $\overrightarrow{x} \perp \overrightarrow{y}$ then $cos(\theta) = 0$ and thus:
$\|\overrightarrow{x} - \overrightarrow{y}\|^2 = \|\overrightarrow{x}\|^2 + \|\overrightarrow{y}\|^2$.

# Inner product

## The 'simple' inproduct

If in the definition $\overrightarrow{x}.\overrightarrow{y} = y^t Ax = x^t Ay$ (with A positive definite and symetric) A=I, then the inproduct becomes the simple inproduct: $\overrightarrow{x}.\overrightarrow{y} = y^t Ix = x^t Iy = (x_1 ... x_n) \begin{pmatrix} y_1 \\ ... \\ y_n \end{pmatrix} = \sum_{i=1}^{n} x_i y_i$.

This simple inproduct can always be found by a basis transformation: x=Rx' and y=Ry', then $\overrightarrow{x}.\overrightarrow{y} = y'^t(R^t AR)x'$. Now, R must be taken such that $R^t AR = I$. This can be done by converting A to its normal form by a congruent transformation (e.g. the method of kwadratic forms).

In what follows we mean by 'inproduct' always 'simple inproduct'.

# Gram Schmidt orthogonalization

## Making two independent vectors orthogonal

Geometric derivation:

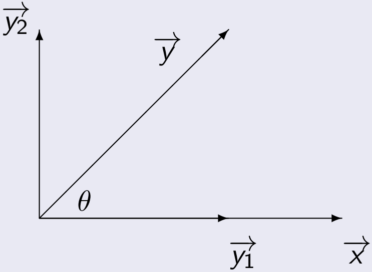



Figure 1: decomposition of vector $\overrightarrow{y}$ in a component parallel ($\overrightarrow{y_1}$) and a component orthogonal ($\overrightarrow{y_2}$) to $\overrightarrow{x}$.

# Gram Schmidt orthogonalization

## Making two indepentent vectors orthogonal

1. Project $\overrightarrow{y}$ orthogonal on $\overrightarrow{x}$, this generates the vector $\overrightarrow{y_1}$, the component parallel with $\overrightarrow{x}$.
2. Subtract $\overrightarrow{y}$ by $\overrightarrow{y_1}$, the result is $\overrightarrow{y_2}$ wich is orthogonal to $\overrightarrow{x}$.

$\overrightarrow{y_1}$ is a specific multiple of the normilised vector $\overrightarrow{x}$: $\overrightarrow{y_1} = \alpha \frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}$.
$\overrightarrow{y_1} \parallel \overrightarrow{x}$: $\overrightarrow{y_1}\overrightarrow{x} = \pm\|\overrightarrow{y_1}\|\|\overrightarrow{x}\|$ (+ if $\theta \leq 90°$ and - if $\theta > 90°$).
From fig. 1: $\|\overrightarrow{y_1}\| = cos(\theta)\|\overrightarrow{y}\|$.
From the inproduct: $cos(\theta) = \frac{\overrightarrow{x}\overrightarrow{y}}{\|\overrightarrow{x}\|\|\overrightarrow{y}\|}$. So
$\overrightarrow{y_1}\overrightarrow{x} = \alpha\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}\overrightarrow{x} = \alpha\|\overrightarrow{x}\| = \|\overrightarrow{y_1}\|\|\overrightarrow{x}\| = cos(\theta)\|\overrightarrow{y}\|\|\overrightarrow{x}\| = \overrightarrow{x}\overrightarrow{y}$.
So we get: $\alpha = \frac{\overrightarrow{x}\overrightarrow{y}}{\|\overrightarrow{x}\|}$.

# Gram Schmidt orthogonalization

## Conclusion

$\vec{y_1} = \alpha \frac{\vec{x}}{\|\vec{x}\|} = (\frac{\vec{x}\vec{y}}{\|\vec{x}\|^2})\vec{x}$ with $\frac{\vec{x}\vec{y}}{\|\vec{x}\|^2}$ a scalar.

And $\vec{y_2} = \vec{y} - \vec{y_1} = \vec{y} - (\frac{\vec{x}\vec{y}}{\|\vec{x}\|^2})\vec{x}$.

Hence:

> The vector $\vec{y}$ gets orthogonalised on the vector $\vec{x}$ by subtract $\vec{y}$ by the component of $\vec{y}$ parallel with $\vec{x}$.

Control of $\vec{y_2} \perp \vec{x}$:

$\vec{y_2}\vec{x} = (\vec{y} - (\frac{\vec{x}\vec{y}}{\|\vec{x}\|^2})\vec{x})\vec{x} = \vec{y}\vec{x} - (\frac{\vec{x}\vec{y}}{\|\vec{x}\|^2})\|\vec{x}\|^2 = 0.$

# Gram Schmidt orthogonalization

## Generalization to multiple vectors

Given: 3 vectors: 2 orthogonal unit vectors $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ $(\|\overrightarrow{x_1}\| = 1 = \|\overrightarrow{x_2}\|, \overrightarrow{x_1}\overrightarrow{x_2} = 0)$ and a vector $\overrightarrow{y}$.
Asked: orthogonilise $\overrightarrow{y}$ on $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$.
Solution:
We first search the component of $\overrightarrow{y}$ parallel with $\overrightarrow{x_1}$ and subtract $\overrightarrow{y}$ by this component. This gives $\overrightarrow{y_1}$.
$\overrightarrow{y_1} = \overrightarrow{y} - (\frac{\overrightarrow{x_1}\overrightarrow{y}}{\|\overrightarrow{x_1}\|^2})\overrightarrow{x_1} = \overrightarrow{y} - (\overrightarrow{x_1}\overrightarrow{y})\overrightarrow{x_1}$ $(\|\overrightarrow{x_1}\|^2 = 1)$
$\overrightarrow{y_1} \perp \overrightarrow{x_1}$.

# Gram Schmidt orthogonalization

## Generalization to multiple vectors

Next, we subtract $\overrightarrow{y_1}$ by the component of $\overrightarrow{y_1}$ that is parallel with $\overrightarrow{x_2}$, to get $\overrightarrow{z}$ (wich is perpendicular to both $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$).
$\overrightarrow{z} = \overrightarrow{y_1} - \overrightarrow{x_2}(\overrightarrow{y_1}\overrightarrow{x_2})$. We can write $\overrightarrow{z}$ in another way:
$\overrightarrow{z} = \overrightarrow{y_1} - \overrightarrow{x_2}(\overrightarrow{y_1}\overrightarrow{x_2}) = \overrightarrow{y} - (\overrightarrow{x_1}\overrightarrow{y})\overrightarrow{x_1} - \overrightarrow{x_2}([\overrightarrow{y} - (\overrightarrow{x_1}\overrightarrow{y})\overrightarrow{x_1}]\overrightarrow{x_2})$
$\overrightarrow{z} = \overrightarrow{y} - \overrightarrow{x_1}(\overrightarrow{x_1\overrightarrow{y}}) - \overrightarrow{x_2}(\overrightarrow{x_2}\overrightarrow{y})$

## Conclusion

The vector $\overrightarrow{y}$ becomes orthogonilised on two orthogonal unit vectors $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ by subtracting $\overrightarrow{y}$ by the components of $\overrightarrow{y}$ parallel with $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$.

# Complementary subspace

## Defenition

Given: a n dimensional vector space $V^n$, with $p < n$ linear independent vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$. This vectors create a p dimensional subspace $V^p$ and can be orthogonilised via the Gram schidt method to a orthonormal basis $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_p}$ with $\overrightarrow{e_i}\,\overrightarrow{e_j} = \delta_{ij}$ (with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$).
This p vectors can be complemented by $n - p$ linear independent vectors $\overrightarrow{f_1}, \overrightarrow{f_2}, ..., \overrightarrow{f_{n-p}}$ that are linear independent with $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_p}$ and orthonormal.

These vectors $\overrightarrow{f_1}, \overrightarrow{f_2}, ..., \overrightarrow{f_{n-p}}$ generate the orthogonal complement of the subspace created by $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_p}$.
The orthogonal complement of the p-dimensional subspace $V^p$ of $V^n$ $(p < n)$, has dimension $n - p$.

# Complementary subspace

## Example

Given: $n = 5, p = 3, \overrightarrow{e_1} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{e_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \overrightarrow{e_3} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$

Asked: The orthogonal complement

Solution: The orthogonal complement has dimension $n - p = 5 - 3 = 2$ and consists of the set vectors that are perpendicular to the vectors $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_p}$.

$$\begin{cases} \overrightarrow{e_1}\overrightarrow{x} = 0 \\ \overrightarrow{e_2}\overrightarrow{x} = 0 \\ \overrightarrow{e_3}\overrightarrow{x} = 0 \end{cases}$$

# Complementary subspace

## Example

$$\begin{cases} \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \end{bmatrix} \overrightarrow{x} = 0 \\ \begin{bmatrix} 1 & 0 & 0 & 4 & 0 \end{bmatrix} \overrightarrow{x} = 0 \\ \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \end{bmatrix} \overrightarrow{x} = 0 \end{cases}$$

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 1 & 3 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{x_1} \\ \overrightarrow{x_2} \\ \overrightarrow{x_3} \\ \overrightarrow{x_4} \\ \overrightarrow{x_5} \end{bmatrix} = 0$$

This is a homogenous system of equetions. The solution of this system is the orthogonal complement.

# Outline

# Row- and column vectors

## Example

The rows of a mxn matrix A can be considered as m row vectors with n components.

$$A^{mxn} = \begin{bmatrix} \overrightarrow{r_1} \\ \overrightarrow{r_2} \\ \dots \\ \overrightarrow{r_m} \end{bmatrix}$$

The columns of a mxn matrix A can be considered as n column vectors with m components.

$$A^{mxn} = \begin{bmatrix} \overrightarrow{r_1} & \overrightarrow{r_2} & \dots & \overrightarrow{r_m} \end{bmatrix}$$

# Row- and column space, rank

## Column space

We consider the columns of $A^{mxn}$ as vectors with m components and difine the vectors $\overrightarrow{x}$ as every possible linear combination of the column vectors $\overrightarrow{k_i}$:
$\overrightarrow{x} = a_1\overrightarrow{k_1} + a_2\overrightarrow{k_2} + ... + a_n\overrightarrow{k_n}$ with $a_i \in \mathbb{R}$ and i means the $i^{th}$ column of A.
The set of all the vectors $\overrightarrow{x}$ is called the column space of A.
The column space = all possible linear combinations of columns of A .

# Row- and column space, rank

## Column space

if only r of the n vectors are linear independent, that means

- None of this r vectors can be written as a linear combination of the other $r-1$ vectors
- all others $n-r$ column vectors can be written as linear combinations of the r linear independant vectors

then r is called:

1. the rank of (column) matrix A
2. the dimension of the column space of matrix A

# Row- and column space, rank

## Row space

The concept row space and row rank can be derived in the same way as the column space is derived.

## Rank

It is a fundamental matrix property that:

$\boxed{\text{row rank A} = \text{column rank A}}$

That means: the number linear independant columns in a matrix is equal to the number linear independant rows. Thus:

rank A = row rank A = column rank A
= dimension row space A = dimension column space A

# Row- and column space, rank

## Example

$$A^{4x3} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \\ 2 & 4 & -2 \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

We determine the rank of A by converting it to his echlon form by elementary row operations (explaned in the appendix).

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \\ 2 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Canonical form: the first element of every row is 1. Above and below this ones is every number 0.

# Row- and column space, rank

## Conclusion

Rank = the number of rows that differs from zero. So the rank is 2. In other words: there are only 2 linear independent rows and columns in A.

Hence:

The column space of A is a 2 dimensional subspace of the 4 dimensional vectorspace.

The row space is a 2 dimensional subspace of the 3 dimensional space.

# Link between determinant and rank by square matrices

## Determinant-rank

If all the columns of a square matrix are linear dependent, then the determinant is 0. If $A^{mxm}$:
det A=0 $\Leftrightarrow$ rank A=0 $\Leftrightarrow$ columns depentent $\Leftrightarrow$ rows dependent.
If det A $\neq$ 0 then rank A = m and A is of **full rank**. A matrix can be inverted is his determinant when different from 0 (when its of full rank).

# Link between determinant and rank by square matrices

## Determinant-rank

If rank($B^{nxn}$)=r with r$\leq$n and rank($A^{nxn}$)=n (A is of full rank) then is:
rank(AB)=rank(BA)=rank(B).
Elementary row- and column operations are always of full rank (appendix). When there is a multiplication between a matrix and a elementary row- or column matrix, then has the product always the rank of the matrix.

# Column space test

## Belongs the vector to the column space?

Given: n vectors with m components relative to a basis in $V^m$:

$$\begin{bmatrix} x_{11} \\ x_{21} \\ \dots \\ x_{m1} \end{bmatrix}, \dots, \begin{bmatrix} x_{1n} \\ x_{2n} \\ \dots \\ x_{nn} \end{bmatrix}.$$

If r of this n vectors are linear independent then has the matrix

$$X^{mxn} = [\vec{x_1}, \dots, \vec{x_n}] = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \dots & & \dots \\ x_{m1} & \dots & x_{mn} \end{bmatrix}$$

rank r. The column space of X has dimension r.

# Column space test

## Belongs the vector to the column space?

How can you determine of a given vector $\overrightarrow{y}$ with m components belongs to the column space of X?
If $\overrightarrow{y}$ is a element of the column space of X, then can $\overrightarrow{y}$ be written as a linear combination of the vectors $\overrightarrow{x_1}, ..., \overrightarrow{x_n}$. By adding $\overrightarrow{y}$ to this vectors, the spanned space will be the same and the rank will still be r.
In other words, if $\overrightarrow{y}$ belongs to the column space of X, then will the rank of the expanded matrix $\left[\overrightarrow{x_1}, ..., \overrightarrow{x_n}, \overrightarrow{y}\right]$ be the same as the rank of the normal matrix X.

$$\boxed{\overrightarrow{y} \in \text{column space} \Leftrightarrow rank\left[X \quad \overrightarrow{y}\right] = rank\left[X\right] = r}$$

# Column space test

## Vector $\overrightarrow{y}$ not in column space

If $\overrightarrow{y}$ doesn't belong to the column space of X, then can't $\overrightarrow{y}$ be written as a linear combination of the vectors $\overrightarrow{x_1}, ..., \overrightarrow{x_n}$, in other words: $\overrightarrow{y}$ can be splitted in two vectors $\overrightarrow{y} = \overrightarrow{z_1} + \overrightarrow{z_2}$ with $\overrightarrow{z_1}$ in the column space of X and $\overrightarrow{z_2}$ not in the column space of X. By adding $\overrightarrow{y}$ to the vectors $\overrightarrow{x_1}, ..., \overrightarrow{x_n}$ increases the dimension of the spannend space.

$$\boxed{\overrightarrow{y} \notin \text{column space} \Leftrightarrow rank \begin{bmatrix} X & \overrightarrow{y} \end{bmatrix} = r + 1}$$

**Remark:** $\text{rank}\begin{bmatrix} X & \overrightarrow{z_1} \end{bmatrix} = r$ and $\text{rank}\begin{bmatrix} X & \overrightarrow{z_2} \end{bmatrix} = r + 1$.

# Column space test

## Resume

1. n vectors with m components $\overrightarrow{x_1}, \ldots, \overrightarrow{x_n}$
2. Belongs $\overrightarrow{y}$ to the space spanned by $\overrightarrow{x_1}, \ldots, \overrightarrow{x_n}$?
3. Solution:

a) Determine $r_1 = rank\left[X\right] = rank\left[\overrightarrow{x_1} \quad \ldots \quad \overrightarrow{x_n}\right]$

b) Determine $r_2 = rank\left[X \quad \overrightarrow{y}\right] = rank\left[\overrightarrow{x_1} \quad \ldots \quad \overrightarrow{x_n} \quad \overrightarrow{y}\right]$

c) Is $r_1 = r_2 \Rightarrow y \in$ column space or $r_1 + 1 = r_2 \Rightarrow y \notin$ column space

**Remark**: analogous for the row space test.

# Column space test

## Example

Given: $\overrightarrow{x_1} = \begin{bmatrix}1\\2\\3\\4\\5\end{bmatrix}, \overrightarrow{x_2} = \begin{bmatrix}-1\\2\\0\\5\\7\end{bmatrix}, \overrightarrow{x_3} = \begin{bmatrix}0\\4\\3\\9\\12\end{bmatrix}, \overrightarrow{y} = \begin{bmatrix}-1\\-1\\2\\1\\0\end{bmatrix}$

Asked: Belongs $\overrightarrow{y}$ to the space spanned by $\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}$?

Solution: 1) find the rang of X:

$$X = \begin{bmatrix}1 & -1 & 0\\2 & 2 & 4\\3 & 0 & 3\\4 & 5 & 9\\5 & 7 & 12\end{bmatrix} \sim \begin{bmatrix}1 & -1 & 0\\2 & 2 & 0\\3 & 0 & 0\\4 & 5 & 0\\5 & 7 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\\frac{3}{2} & \frac{3}{4} & 0\\-\frac{1}{2} & \frac{9}{4} & 0\\-1 & \frac{12}{4} & 0\end{bmatrix}$$

Hence, rank(X)$= 2$.

# Column space test

## Example

2) Determine the rank $\begin{bmatrix} X & \vec{y} \end{bmatrix}$

$$\begin{bmatrix} X & \vec{y} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 2 & 2 & 4 & -1 \\ 3 & 0 & 3 & 2 \\ 4 & 5 & 9 & 1 \\ 5 & 7 & 12 & 0 \end{bmatrix} \qquad \text{rank} \begin{bmatrix} X & \vec{y} \end{bmatrix} = 3.$$

Hence, rank $\begin{bmatrix} X & \vec{y} \end{bmatrix} = rank \begin{bmatrix} X \end{bmatrix} + 1$.
That means: $\vec{y}$ can **not** be written as a linear combination of the vectors $\vec{x_1}, \vec{x_2}, \vec{x_3}$ and doesn't belong to the space spanned by this vectors. Because rank(X)$= 2$ is the dimension of the space 2. Hence $\vec{x_3}$ can be written as a linear combination of $\vec{x_1}, \vec{x_2}$.
Rank(X$\vec{y}$)=3, so the space spanned by $\vec{x_1}, \vec{x_2}, \vec{x_3}$ and $\vec{y}$ has dimension 3.

# Orthonormal matrices

## Definition

The set vectors $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_n}$ is orthonormal if:

- $\|\overrightarrow{e_i}\| = 1$ with $i = 1, ..., n$
- $\overrightarrow{e_i} \perp \overrightarrow{e_j}$ or $\overrightarrow{e_i} . \overrightarrow{e_j} = 0$ with $i \neq j$

This is noted as:
$\overrightarrow{e_i} . \overrightarrow{e_j} = \delta_{ij}$ with $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$ ($\delta_{ij}$ is called 'kronecker delta').

If $\overrightarrow{e_{1i}}$ has the components $\begin{bmatrix} \overrightarrow{e_{1i}} \\ ... \\ \overrightarrow{e_{ni}} \end{bmatrix}$ relative to some basis, then applies for the inproduct:

$$\overrightarrow{e_{1i}}.\overrightarrow{e_{1j}} = \begin{bmatrix} e_{1i} & ... & e_{ni} \end{bmatrix} \begin{bmatrix} e_{1i} \\ ... \\ e_{ni} \end{bmatrix} = e_{1i}e_{1j} + ... + e_{ni}e_{nj} = \delta_{ij}$$

# Orthonormal matrices

## Defenition

If we create from $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_n}$ the matrix

$$E = [\overrightarrow{e_1}, ..., \overrightarrow{e_n}] = \begin{bmatrix} e_{11} & ... & e_{1n} \\ ... & & ... \\ e_{n1} & ... & e_{nn} \end{bmatrix}$$

then applies

$$E^t E = \begin{bmatrix} e_{11} & ... & e_{n1} \\ ... & & ... \\ e_{1n} & ... & e_{nn} \end{bmatrix} \begin{bmatrix} e_{11} & ... & e_{1n} \\ ... & & ... \\ e_{n1} & ... & e_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix} =$$

$I^{nxn}$

Hence, if the columns and rows of a matrix form a orthonomal system then is the matrix onthonormal: $E^t E = I = EE^t$.

# Orthonormal matrices

## Properties

1. The inverse of a orthonormal matrix is his transform: $E^tE = I = EE^t \Rightarrow E^{-1}EE^t = E^{-1}I \Rightarrow IE^t = E^{-1} \Rightarrow E^t = E^{-1}$
2. Maintaining the norm of a vector: $\overrightarrow{x}$ is a vector with norm $\|\overrightarrow{x}\|$ and E an orthonormal matrix then applies $\|E\overrightarrow{x}\| = \|\overrightarrow{x}\|$.
   When there is a 'fault' $\Delta x$ on x, then is $\|E(\overrightarrow{x} + \Delta\overrightarrow{x})\| = \|\overrightarrow{x} + \Delta\overrightarrow{x}\|$.
   Hence, an orthogonal matrix doesn't change the magnitude of a fault. This is important in numerical applications.
3. The determinant of an orthonormal matrix is $\pm 1$, when it is 1 the matrix represent a rotation matrix.

# Change of basis and matrix representation

## Change of basis

In a n dimension vectorspace $V^n$ are two different bases given:
'old basis': $\overrightarrow{e_1}, ...., \overrightarrow{e_n}$
'new basis': $\overrightarrow{f_1}, ..., \overrightarrow{f_n}$

A vector x has coordinates $\begin{bmatrix} a_1 \\ ... \\ a_n \end{bmatrix}$ relative to the old basis and coordinates $\begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix}$ relative to the new basis.

# Change of basis and matrix representation

## Change of basis

Every basisvector $\overrightarrow{f_i}$ can be expressed as a linear combination of the old basis vectors: $\overrightarrow{f_i} = f_{1i}\overrightarrow{e_1} + ... + f_{ni}\overrightarrow{e_n}$. So $\overrightarrow{f_i}$ has coordinates $\begin{bmatrix} f_{1i} \\ ... \\ f_{ni} \end{bmatrix}$ relative to the old basis.

If we group the old basis vectors in a matrix: $E = \begin{bmatrix} \overrightarrow{e_1} & ... & \overrightarrow{e_n} \end{bmatrix}$ then you can write: $\overrightarrow{x} = \begin{bmatrix} \overrightarrow{e_1} & ... & \overrightarrow{e_n} \end{bmatrix} \begin{bmatrix} a_1 \\ ... \\ a_n \end{bmatrix} = E\overrightarrow{x}_{old}$ and

$$\overrightarrow{f_i} = \begin{bmatrix} \overrightarrow{e_1} & ... & \overrightarrow{e_n} \end{bmatrix} \begin{bmatrix} f_{1i} \\ ... \\ f_{ni} \end{bmatrix} = E \begin{bmatrix} f_{1i} \\ ... \\ f_{ni} \end{bmatrix}.$$

# Change of basis and matrix representation

## Change of basis

If we now consider x as a linear combination of the 'new' basis vectors $\overrightarrow{x} = b_1 \overrightarrow{f_1} + ... + b_n \overrightarrow{f_n} = \begin{bmatrix} \overrightarrow{f_1} & ... & \overrightarrow{f_n} \end{bmatrix} \begin{bmatrix} b_1 \\ ... \\ b_n \end{bmatrix} =$

$$\begin{bmatrix} \overrightarrow{f_1} & ... & \overrightarrow{f_n} \end{bmatrix} \overrightarrow{x}_{new} = \begin{bmatrix} E \begin{bmatrix} f_{11} \\ ... \\ f_{n1} \end{bmatrix} & ... & E \begin{bmatrix} f_{1n} \\ ... \\ f_{nn} \end{bmatrix} \end{bmatrix} \overrightarrow{x}_{new} = EF\overrightarrow{x}_{new}$$

with $F = \begin{bmatrix} f_{11} & ... & f_{1n} \\ ... & & ... \\ f_{n1} & ... & f_{nn} \end{bmatrix}$. The columns of F contains the coordinates of the new basis vectors relative to the old basis vectors $\overrightarrow{e_1}, ...., \overrightarrow{e_n}$.

# Change of basis and matrix representation

## Change of basis

So we get: $\vec{x} = E\vec{x}_{old} = EF\vec{x}_{new}$.

From wich follows:

$$\boxed{\vec{x}_{old} = \begin{bmatrix} a_1 \\ \dots \\ a_n \end{bmatrix} = F\vec{x}_{new} = F\begin{bmatrix} b_1 \\ \dots \\ b_n \end{bmatrix}}$$

with the columns of F the coordinates of the new basis vectors relative to the old basis.

So, $\vec{x}_{old} = F\vec{x}_{new}$

from wich:

$$\boxed{\vec{x}_{new} = F^{-1}\vec{x}_{old}}$$

# Change of basis and matrix representation

## Example

Given: the 3 dimensional vector space $V^3$ with basis vectors:

$$\vec{e_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{e_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The 'new' basis vectors are defined as:

$$\vec{f_1} = \vec{e_1} - \vec{e_2} + \vec{e_3}$$
$$\vec{f_2} = 2\vec{e_1} + 3\vec{e_2} + 0\vec{e_3}$$
$$\vec{f_3} = -\vec{e_1} + \vec{e_2} + 2\vec{e_3}.$$

And there is a vector $\vec{x}$:

$$\vec{x} = 2\vec{e_1} + \vec{e_2} - \vec{e_3}.$$

Asked: The coordinates of $\vec{x}$ relative to the basis $\vec{f_1}, \vec{f_2}, \vec{f_3}$.

# Change of basis and matrix representation

## Example

Solution: The components of $\overrightarrow{f_1}$ relative to the other basis are: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

The components of $\overrightarrow{f_2}$ relative to the other basis are: $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$.

The components of $\overrightarrow{f_3}$ relative to the other basis are: $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

The components of $\overrightarrow{x}$ relative to the other basis are: $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

# Change of basis and matrix representation

## Example

So, $\overrightarrow{x}_old = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. We know that $\overrightarrow{x}_{old} = F\overrightarrow{x}_{new}$ with

$F = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

Now is $\overrightarrow{x}_{new} = F^{-1}\overrightarrow{x}_{old} = \frac{1}{15}\begin{bmatrix} 6 & -4 & 5 \\ 3 & 3 & 0 \\ -3 & 2 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

So $\overrightarrow{x}_{new} = \frac{1}{15}\begin{bmatrix} 3 \\ 9 \\ -9 \end{bmatrix} = \frac{1}{5}\overrightarrow{f_1} + \frac{3}{5}\overrightarrow{f_2} - \frac{3}{5}\overrightarrow{f_3}$.

# Change of basis and matrix representation

## Remarks

1. If the old basis is orthonormal and the new basis too, then is the transition matrix orthonormal as well ($F^{-1} = F^T$).
2. If the old basis is from dimension n, then the new basis has the same dimension n too, because they span the same space. So F has to be of full rank: rank(F)=n.

# Outline

# Systems of linear equations

## Defenitions

In general, a system of linear equations is described in a matrix-vector identity: Ax=y. If y≠0 then the system is called not homogeneous, if y=0 then the system is called homogeneous. It will turn out that in all the possible cases, the rank r of the matrix A is important for the existance of a solution and the number of solutions.

A good <u>geometric</u> view in the properties and solutions of homogeneous and not homogeneous systems is essential for an understanding of linear images, eigenvalues, etc.

# Not homogeneous equations

## Existance of a solution

In general, a system of m equations in n variables $x_1, x_2, ..., x_n$ has the form: $A^{mxn} x^{nx1} = y^{mx1}$. The problem is: search the variables $x_1, x_2, ..., x_n$ wich statisfy this relationship (A and y are known).

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \dots \\ y_m \end{bmatrix} \text{ or}$$

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = y_1 \\ \dots \\ a_{m1}x_1 + \dots + a_{mn}x_n = y_m \end{cases}$$

# Not homogeneous equations

## Existance of a solution

If we think of A as column vectors with m components $A = \begin{bmatrix} \overrightarrow{a_1} & \dots & \overrightarrow{a_n} \end{bmatrix}$, then we can write the system in antother way:

$$x_1 \begin{bmatrix} a_{11} \\ \dots \\ a_{m1} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ \dots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} y_1 \\ \dots \\ y_m \end{bmatrix} \text{ or } x_1 \overrightarrow{a_1} + \dots + x_n \overrightarrow{a_n} = \overrightarrow{y}.$$

The geometric interpretation:

Given a vector $\overrightarrow{y}$ of m components and n vectors $\overrightarrow{a_i}$ of m components, search **all** numbers $x_i$, i=1,n , so that vector $\overrightarrow{y}$ can be written as a linear combination of the vectors $\overrightarrow{a_i}$.

All the linear combinations of the vectors $\overrightarrow{a_i}$ can be written as: $x_1 \overrightarrow{a_1} + \dots + x_n \overrightarrow{a_n} = \overrightarrow{y}$ and is called the column space of A.

# Not homogeneous equations

## Existance of a solution

If rank(A)=r, then has the colums space dimension r. In other words: there are r linear independent columns in A and the other n-r columns in A can be written as a linear combination of the r linear independent columns.

Since $\overrightarrow{y}$ has to be a linear combination of the columns of A, $\overrightarrow{y}$ has to be in the columns space of A.
If $\overrightarrow{y}$ doesn't exist of the column space of A, then $\overrightarrow{y}$ can't be written as a linear combination of the columns of A, then the system **doesn't have** a solution.

# Not homogeneous equations

## Existance of a solution

The condition for the existence of a solution is that $\overrightarrow{y}$ belongs to the column space of A. Now, the column space test beccomes the solution test.

> The system Ax=y with rank(A)=r, has a solution only if rank(A)=rank(Ay)=r with Ay a mx(n+1) matrix.

# Not homogeneous equations

## Example 1

$$\begin{cases} -x_1 + 2x_2 = 0 \\ x_1 + 3x_2 = 1 \\ 0x_1 + x_2 = 2 \end{cases} \quad \text{or} \quad \underbrace{\begin{bmatrix} -1 & 2 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}}_{\text{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{x}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{\text{y}}$$

Rank(A)=2, rank(Ay)=3

Rank(A)≠rank(Ay), so there **isn't** a solution.

# Not homogeneous equations

## Example 2

$$\begin{cases} x_1 + x_2 + x_3 = 2 \\ -x_1 + 2x_2 + 3x_3 = -1 \end{cases} \quad \text{or} \quad \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{y}$$

Rank(A)=2=rank(Ay)=2, so this system can be solved.

In the next we suppose that there is always a solution, thus: rank(A)=rank(Ay).

# Number of solutions and solution methods

## Defenitions

$$A^{mxn}x^{nx1} = y^{mx1}$$

There are **three** cases:

1. m>n: overdetermined system (more equations than variables)
2. m=n: square system
3. m<n: underdetermined system

# Number of solutions and solution methods

## m>n: overdetermined system

There is a solution if rang(A)=rang(Ay)=r. If this condition is statisfied, we can write: $x_1\overrightarrow{a_1} + \ldots + x_n\overrightarrow{a_n} = \overrightarrow{y}$ with $a_i$ the i$^{th}$ column of A. As rank(A)=r, there are r linear independent columns $\overrightarrow{a_1}, \ldots, \overrightarrow{a_r}$. Now we can write:

$x_1\overrightarrow{a_1} + \ldots + x_r\overrightarrow{a_r} = \overrightarrow{y} - x_{r+1}\overrightarrow{a}_{r+1} - \ldots - x_n\overrightarrow{a_n}$

If: $\overrightarrow{y'} = \overrightarrow{y} - x_{r+1}\overrightarrow{a}_{r+1} - \ldots - x_n\overrightarrow{a_n}$ then $\begin{bmatrix}\overrightarrow{a_1} & \ldots & \overrightarrow{a_r}\end{bmatrix}\begin{bmatrix}x_1\\ \ldots \\ x_r\end{bmatrix} = \overrightarrow{y'}$.

From the columntest follows that $\overrightarrow{y}$ belongs to the columnspace of A. The vectors $\overrightarrow{a}_{r+1}, \ldots, \overrightarrow{a_n}$ belong to the same space. The vectors $\overrightarrow{a}_1, \ldots, \overrightarrow{a_r}$ create the **basis** of this space.

# Number of solutions and solution methods

## m>n: overdetermined system

For every choice of the vector $\overrightarrow{y'}$, there exist a **unique** set (only one solution) of numbers $x_1, \ldots, x_r$: this are the coordinates of the vector $\overrightarrow{y'}$ in the basis $\overrightarrow{a_1}, \ldots, \overrightarrow{a_r}$. There is only one solution because every vector in only one way can be descibed as a linear combination of basis vectors.
Now is: $\overrightarrow{y'} = \overrightarrow{y} - x_{r+1}\overrightarrow{a}_{r+1} - \ldots - x_n\overrightarrow{a_n}$.
For each set $[x_{r+1}, \ldots, x_n]$ there is a vector $\overrightarrow{y'}$, and for every vector $\overrightarrow{y'}$ exist just one solution $[x_1, \ldots, x_r]$. For the choise of the set $[x_{r+1}, \ldots, x_n]$, there are $\infty^{n-r}$ possibilities.

# Number of solutions and solution methods

## m>n: overdetermined system

If m>n and rank(Ay)=rank(A)=r, then has the system Ax=y $\infty^{n-r}$ solutions.

The variables $x_1, x_2, ..., x_r$ are the main variables and $x_{r+1}, x_{r+2}, ..., x_n$ are the help variables. For each set help variables exist only one set main variables.

If rank(A)=rank(Ay)=r and if n=r, then has the system Ax=y exact **one** solution ($\infty^0 = 1$).

# Number of solutions and solution methods

## m>n: overdetermined system

How do you determine the solution for a certain set help variables $[x_{r+1}, x_{r+2}, ..., x_n]$.

If $\begin{bmatrix} \overrightarrow{a_1} & \ldots & \overrightarrow{a_r} \end{bmatrix} \begin{bmatrix} x_1 \\ \ldots \\ x_r \end{bmatrix} = \overrightarrow{y} - x_{r+1}\overrightarrow{a}_{r+1} - \ldots - x_n\overrightarrow{a_n} \Rightarrow A'x' = y'$

with $A' = \begin{bmatrix} \overrightarrow{a_1} & \ldots & \overrightarrow{a_r} \end{bmatrix}$ and

$\overrightarrow{y'} = \overrightarrow{y} - x_{r+1}\overrightarrow{a}_{r+1} - \ldots - x_n\overrightarrow{a_n} = \begin{bmatrix} y'_1 \\ \ldots \\ y'_m \end{bmatrix}.$

We know that rang(A')=r. This means that r rows of A' are linear independent. The other rows of A' can be created by linear combinations of the linear independent rows.