Chapter 10: Root Locus Analysis

August 5, 2015

Outline

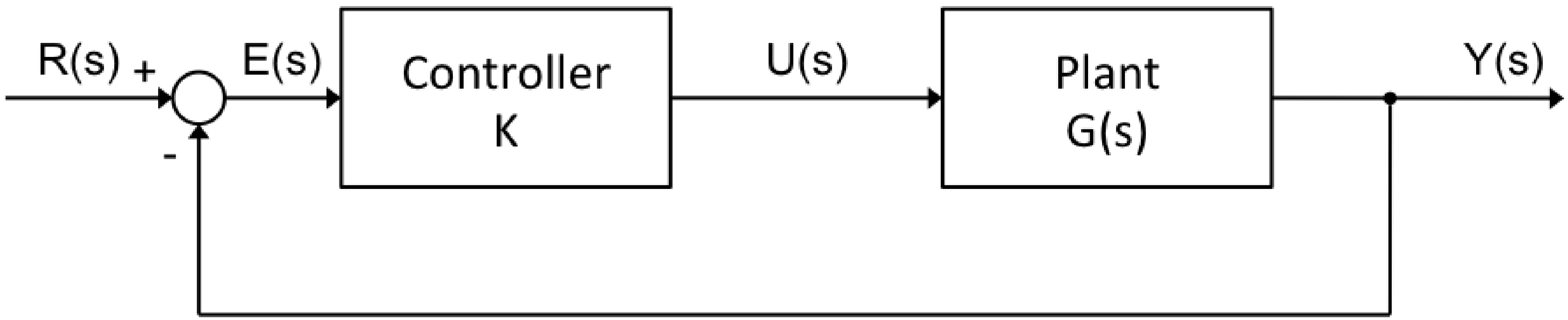
Concept of the technique

In this chapter the Root Locus Method is presented. This technique shows how changes in the system's feedback characteristics and other parameters influence the pole locations. The method permits us to plot the locus of the closed-loop poles in the s-plane as a function of a varying parameter.

It is very important to understand the background of root loci: how they take their shape, why they are useful, ... Therefore, we start this chapter by explaining the concept of the Root Locus Technique. Next we explain how to sketch a root locus. Finally we will give some examples in MATLAB.

Concept of the technique

We begin with the basic feedback system, shown in the figure below:



The closed-loop transfer function is:

$$H(s) = \frac{Y(s)}{R(s)} = \frac{KG(s)}{1+KG(s)}.$$

Concept of the technique

Looking at this transfer function, we can see that the closed-loop poles depend on the gain K. We can now plot the locus of all possible roots of the characteristic equation:

$$1 + KG(s) = 0 \tag{1}$$

as K varies from 0 to ∞. This results in a graph which can help us in selecting the best value of K.

Furthermore, by studying the effects of additional poles and zeros, we can determine the consequences of additional dynamics in the loop. We can also extend this technique to examine the effect of other plant-parameter changes in order to achieve the best overall control design.

Root Locus

Definition I

The root locus is the set of values of s for which $1 + KG(s) = 0$ is statisfied as the real parameter K varies from 0 to $+\infty$. Often $G(s)$ is the open-loop transfer function of a system; in this case, roots on the locus are closed-loop poles of that system.

Use of root locus

The root locus method provides a tool not only for selecting the gain but for designing lead and lag compensators, PID controllers, etc.

Concept of the technique

For the further derivation, we assume that the system's open-loop transfer function $G(s)$ is a rational function whose numerator is $b(s)$ and whose denominator is $a(s)$. $b(s)$ is a monic polynomial of degree m and $a(s)$ is a monic polynomial of degree n.

All positive values of K

As $K \to 0$, the closed-loop poles of system are the roots of $a(s) = 0$ or the open-loop poles. As $K \to \infty$, the closed-loop poles of the system are the roots of $b(s) = 0$ or the open-loop zeros.

Concept of the technique

If $H(s)$ has more poles than zeros ($m < n$) we say that $H(s)$ has zeros at infinity. In this case, the limit of $H(s)$ as $s \to \infty$ is zero. The number of zeros at infinity is $n - m$, the number of poles minus the number of zeros, and is the number of branches of the root locus that go to infinity (asymptotes).

Open-loop vs Closed-loop

The root-locus method can be thought of as a method for inferring properties of the closed-loop system given the open-loop transfer function $KG(s)$.

Concept of the technique

Root-Locus form

We can express Eq. (1) in different equivalent ways:

- $1 + KG(s) = 0;$
- $1 + K\frac{b(s)}{a(s)} = 0;$
- $a(s) + Kb(s) = 0;$
- $G(s) = -\frac{1}{K}.$

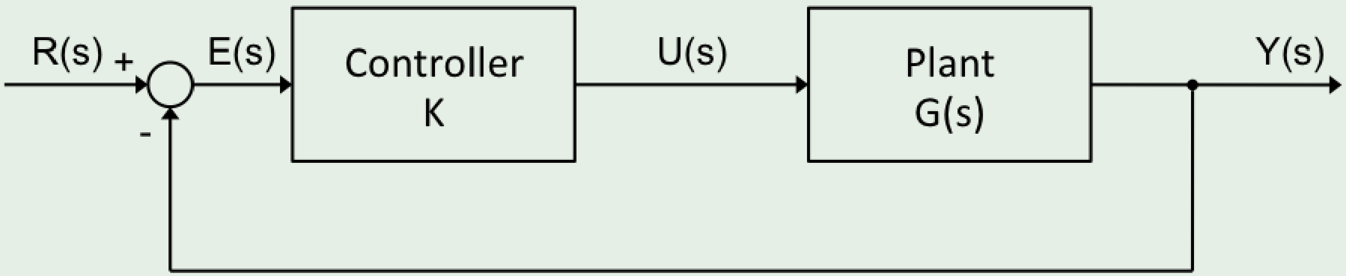
These different presentations are called the root-locus forms.

Example

Problem: a normalized transfer function of a DC motor is:

$$G(s) = \frac{1}{s(s+1)}. \tag{2}$$

Find the locus of roots with respect to the proportional gain K of the closed-loop system created by feeding back the output as shown in the figure:



Solve by using direct calculations of the root locations.

Solution: in terms of our notation

$$m = 0, \qquad b(s) = 1,$$
$$n = 2, \qquad a(s) = s(s+1).$$

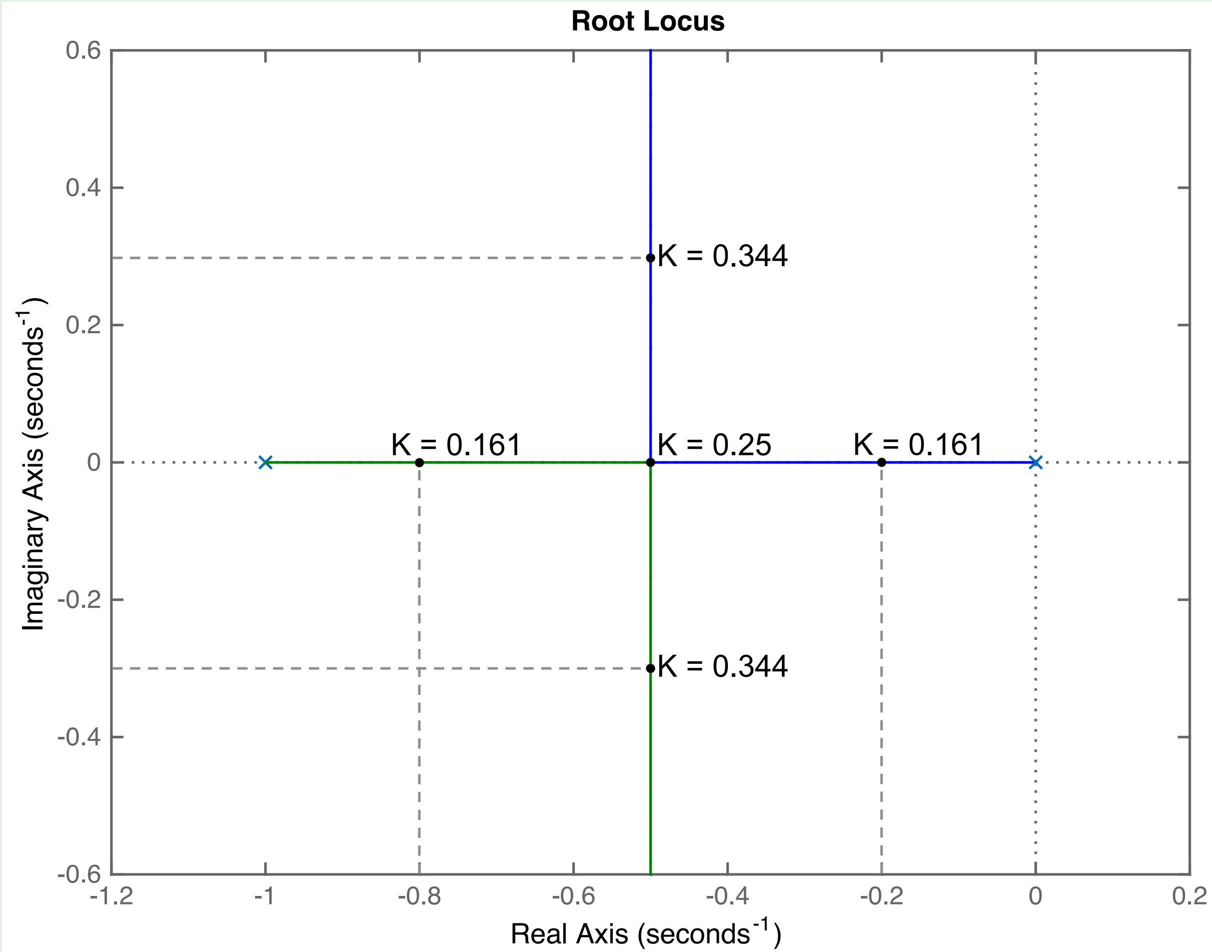
We can use the root-locus form $a(s) + Kb(s) = 0$ to obtain a quadratic equation of which the roots will produce a graph.
In this case, the quadratic equation is:

$$s^2 + s + K = 0 \tag{3}$$

which has the following roots:

$$r_1, r_2 = -\frac{1}{2} \pm \frac{\sqrt{1-4K}}{2}. \tag{4}$$

The root locus is shown below



- For $0 \leq K \leq \frac{1}{4}$, the roots are real between -1 and 0;
- At $K = \frac{1}{4}$ there are two roots at $-\frac{1}{2}$;
- For $K > \frac{1}{4}$ the roots become complex, with a real part of $-\frac{1}{2}$ and an imaginary part that increases essentially in proportion to the square root of K.