Geometric algebra

August 25, 2015

Outline

Vectors and spatential interpretation

Properties of a vector

There are 3 properties of a vector $\overrightarrow{x}$:

- magnitude
- direction
- startpoint

with respect to a referention vector $\overrightarrow{0}$

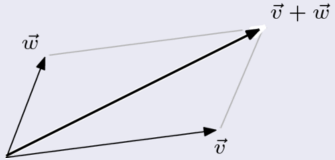
Multiplication scalar and vector

r $\in \mathbb{R}$ (r $\in \mathbb{C}$ is possible, but hasn't a fysical representation)

- $|r| < 1$: shorten
- $|r| > 1$: increase
- $r < 0$: reverse the direction

Addition of vectors

Parallellogramrule:



vectorspace

First condition

A vectorspace V over a body L (set of operators) is a set of vectors that satisfy:

1. A vectorsum is defined: $VxV \rightarrow V : (\vec{x}, \vec{y}) \rightarrow \vec{x} + \vec{y}$

   $\vec{x}, \vec{y}, \vec{z} \epsilon$ V

   a) $\vec{x} + \vec{y} \epsilon$ V

   b) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$

   c) $\exists! \vec{0} : \vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$

   d) $\forall \vec{x}, \exists(-\vec{x}) : \vec{x} + (-\vec{x}) = (-\vec{x}) + \vec{x} = 0$

   e) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

vectorspace

Second condition

2. A outside law is defined: $LxV \to V : (a, \overrightarrow{x}) \to a\overrightarrow{x}$

   $\overrightarrow{x}$, $\overrightarrow{y}$ ϵ V

   a,b ϵ L

   a) $1\overrightarrow{x} = \overrightarrow{x}$
   b) $a(b\overrightarrow{x}) = (ab)\overrightarrow{x}$
   c) $(a+b)\overrightarrow{x} = a\overrightarrow{x} + b\overrightarrow{x}$
   d) $a(\overrightarrow{x} + \overrightarrow{y}) = a\overrightarrow{x} + a\overrightarrow{y}$

Numberspaces of n-couples

Defenition

This is the set of all n-couples like $\begin{bmatrix} \overrightarrow{x_1} \\ \overrightarrow{x_2} \\ . \\ . \\ \overrightarrow{x_n} \end{bmatrix}$ with $x_i \epsilon \mathbb{R}$ or $x_i \epsilon \mathbb{C}$.

This set together with the operator set $\mathbb{R}$ or $\mathbb{C}$ is a vectorspace.

Subspaces

Defenition

V_1 is a supspace of vectorspace V if:

1. $V_1 \subset V$
2. With the same in- and outside law as V, is V_1 a vectorspace

Properties

1. $\overrightarrow{0} \ \epsilon$ every subspace
2. The intersection of two spaces is always a subspace
3. Given: p vectors $x_1, x_2, ..., x_p \ \epsilon$ V.
   The set vectors $a_1x_1 + a_2x_2 + ... + a_nx_n$ with $a_i \ \epsilon \ \mathbb{R}$ is a subspace of V.

Linear independance, basis, dimensions

Defenition independance

Given: p vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p} \epsilon$ V.
Construct the nullvector as a linear combination of those vectors (i.e. search the operators (numbers) $a_1, a_2, ..., a_p$ to form $a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + ... + a_n\overrightarrow{x_n} = \overrightarrow{0}$).
If the nullvector only can created by $a_1 = a_2 = ... = a_p = 0$, then are the vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ **linear independant**.

Linear independance, basis, dimensions

Properties

1. If the vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ are linear independent, then can't none of them be writed as a linear combination of the other p-1 vectors.
2. If the nullvector is one of the p vectors, then is the set $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ linear dependant (if $\overrightarrow{x_1} = 0$ then is $a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + ... + a_n\overrightarrow{x_n} = 0$ with $a_1 \neq 0$ and $a_2, a_3, ...a_p = 0$).
3. Basis and dimension: p linear independant vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ generate a vectrospace V^p. Every vector in V^p can be writed **in only one way** as a linear combination of the p linear independant vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$ using operators $a_1, a_2, ..., a_p$.

Linear independance, basis, dimensions

Basis, dimension

Given: $\overrightarrow{v} = a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + ... + a_p\overrightarrow{x_p}$.
The set operators $a_1, a_2, ..., a_p$ are called the **coordinates** of the vector $\overrightarrow{v}$ relative to the set vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$. This set vectors is a **basis** of vectorspace V^p, with **dimension** p.

Linear independance, basis, dimensions

Example

Given: $\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \overrightarrow{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The set $\{\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}\}$ is a linear independant combination. There doesn't exist numbers $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$ such that $a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + a_n\overrightarrow{x_3} = 0$. The set of all vectors $\overrightarrow{y} = a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2} + a_n\overrightarrow{x_3}$ is the three dimensional vectrospace V^3.

If $\overrightarrow{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Then is $3\overrightarrow{x_1} + 2\overrightarrow{x_2} + \overrightarrow{x_3}$ the only way to write $\overrightarrow{y}$ as a linear combination of $\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}$.

Linear independance, basis, dimensions

Example

The set of vectors $\overrightarrow{y} = a_1 \overrightarrow{x_1} + a_2 \overrightarrow{x_2}$ is a **two dimensional** subspace V^2.

The vectors in this subspace are:

$$\overrightarrow{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}.$$

Important difference

- All vectors of V^2 have **3** coordinates.
- The dimension of the subspace V^2 is **2**.

Linear independance, basis, dimensions

Convention of notation

Given: a n-dimensional vectorspace V^n.
The elements of this vectorspace are the elements: $\overrightarrow{x}, \overrightarrow{y}, \ldots$ If we choose $\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_n}$ as a basis of V^n. Then we can write every vector of V^n as a linear combination of those basis vectors in only one way: $\overrightarrow{x} = x_1\overrightarrow{e_1} + x_2\overrightarrow{e_2} + \ldots + x_n\overrightarrow{e_n}$. The numbers x_i are the coordinates of vector $\overrightarrow{x}$ relative to the basis $\overrightarrow{e_1}, \overrightarrow{e_2}, \ldots, \overrightarrow{e_n}$.

Between the vectorspace of dimension n and the number space of dimension n exists a isomorphism.

Linear independance, basis, dimensions

Vectorspace V^p

Given: a p-dimensional vectorspace V^p where the vectors are n-couples (with $n \geq p$).

1. In V^p you can choose a basis with p linear independant vectors.
2. Every vector $\overrightarrow{x} \in V^p$ can be writed in only one way as a linear combination of the p basis vectors using coordinates.

Example 1

Given: n=5, p=2, $\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \overrightarrow{x_2} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$

Linear independance, basis, dimensions

Example 1

The vectors $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ are linear independant, so they span a two dimensional subspace: $\overrightarrow{y} = a_1\overrightarrow{x_1} + a_2\overrightarrow{x_2}$ with $a_1, a_2 \epsilon \mathbb{R}$.

The coordinates of the vector $y_1 = \begin{bmatrix} 5 \\ -6 \\ 1 \\ 2 \\ 5 \end{bmatrix}$, relative to the basis $\left\{\overrightarrow{x_1}, \overrightarrow{x_2}\right\}$, are $a_1 = 1$ and $a_2 = 2$.

Linear independance, basis, dimensions

Example 1

The vector $\overrightarrow{y_2} = \begin{bmatrix} 5 \\ -7 \\ 1 \\ 2 \\ 5 \end{bmatrix}$ can't be writen as a linear combination of the vectors $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$. So y_2 doesn't belong to the subspace spanned by $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$.
This implies that $\overrightarrow{y_2}$ is linear independant of $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$. Thus the subspace spanned by $\overrightarrow{y_2}$, $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ is a 3 dimensional subspace.

Linear independance, basis, dimensions

In general

When the set vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, \ldots, \overrightarrow{x_p}$ is linear independant, then lays $\overrightarrow{x_i}$ not totally in the subspace spanned by the vectors $\overrightarrow{x_1}, \ldots, \overrightarrow{x}_{i-1}, \overrightarrow{x}_{i+1}, \ldots, \overrightarrow{x_p}$.
The vector $\overrightarrow{x_i}$ can be writen as a sum of 2 components: $\overrightarrow{x}_{i\alpha}$ and $\overrightarrow{x}_{i\beta}$.

1. $\overrightarrow{x}_{i\alpha} \epsilon$ subspace spanned by $\overrightarrow{x_1}, \ldots, \overrightarrow{x}_{i-1}, \overrightarrow{x}_{i+1}, \ldots, \overrightarrow{x_p}$.
2. $\overrightarrow{x}_{i\beta} \perp$ subspace spanned by $\overrightarrow{x_1}, \ldots, \overrightarrow{x}_{i-1}, \overrightarrow{x}_{i+1}, \ldots, \overrightarrow{x_p}$.

Inner product

Defenition

The inproduct of two vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ ϵ E^n (n-couples) is defined as the image: $E^n x E^n \rightarrow \mathbb{R} : \{\overrightarrow{x}, \overrightarrow{y}\} \rightarrow \overrightarrow{x}.\overrightarrow{y}$ ϵ $\mathbb{R}$. This image is:

1. Bilinear:
   $(\overrightarrow{x} + \overrightarrow{v}).\overrightarrow{y} = \overrightarrow{x}.\overrightarrow{y} + \overrightarrow{v}.\overrightarrow{y}$
   $\overrightarrow{x} + (\overrightarrow{v}).\overrightarrow{y}) = \overrightarrow{x}.\overrightarrow{v} + \overrightarrow{x}.\overrightarrow{y}$
   $(a\overrightarrow{x})\overrightarrow{y} = a(\overrightarrow{x}.\overrightarrow{y})$
   $\overrightarrow{x}(a\overrightarrow{y}) = a(\overrightarrow{x}.\overrightarrow{y})$
2. Symetric:
   $\overrightarrow{x}.\overrightarrow{y} = \overrightarrow{y}.\overrightarrow{x}$
3. Positive definite:
   $\forall\overrightarrow{x} \neq \overrightarrow{0} : \overrightarrow{x}.\overrightarrow{x} > 0$

Inner product

Matricial notation

The inproduct is a **scalar**. If $\overrightarrow{x}$, $\overrightarrow{y}$ and the basis ϵ E^n then can the inproduct be noted matricial:

$$\overrightarrow{x}.\overrightarrow{y} = y^t Ax = x^t Ay = (x_1 ... x_n)A \begin{pmatrix} y_1 \\ ... \\ y_n \end{pmatrix}$$

with A positive definite and symetric $(A = A^t)$.

Inner product

Norm of a vector

$\|\vec{x}\|^2 = \vec{x}.\vec{x}$ and because $\vec{x}.\vec{x} > 0$ applies:
$\|\vec{x}\| = \sqrt{\vec{x}.\vec{x}}$ where $\|\vec{x}\|$ is called the norm of $\vec{x}$.
Normalizing is dividing a vector by its norm. The result is a vector with norm $= 1$.
$\|\frac{\vec{x}}{\|\vec{x}\|}\| = \sqrt{\frac{\vec{x}}{\|\vec{x}\|}\frac{\vec{x}}{\|\vec{x}\|}} = \sqrt{\frac{\|\vec{x}\|^2}{\|\vec{x}\|\|\vec{x}\|}} = 1.$

Inner product

CauchySchwarz inequality

$|\overrightarrow{x}.\overrightarrow{y}| \leq \|\overrightarrow{x}\|\|\overrightarrow{y}\|$ or
$-\|\overrightarrow{x}\|\|\overrightarrow{y}\| \leq \overrightarrow{x}.\overrightarrow{y} \leq \|\overrightarrow{x}\|\|\overrightarrow{y}\|$ from wich follows:
$-1 \leq \frac{\overrightarrow{x}\overrightarrow{y}}{\|\overrightarrow{x}\|\|\overrightarrow{y}\|} \leq 1$
By defenition follows:
$\cos(\theta) = cos(\angle(\overrightarrow{x}, \overrightarrow{y})) = \frac{\overrightarrow{x}\overrightarrow{y}}{\|\overrightarrow{x}\|\|\overrightarrow{y}\|}$
Therefor: the angle between the vectors $\overrightarrow{x}$ and $\overrightarrow{y}$ $=$
Bgcos(inproduct of $\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}$ and $\frac{\overrightarrow{y}}{\|\overrightarrow{y}\|}$).

Inner product

Orthogonality

$\overrightarrow{x}$ and $\overrightarrow{y}$ are orthogonal $\Leftrightarrow$
$\theta = \angle(\overrightarrow{x}, \overrightarrow{y}) = 90^\circ = \frac{\Pi}{2} rad \Leftrightarrow$
$cos(\theta) = 0 \Leftrightarrow$
$\overrightarrow{x} . \overrightarrow{y} = 0$

Hence, if $\overrightarrow{x}, \overrightarrow{y} \neq 0$:
$\overrightarrow{x} \perp \overrightarrow{y} \Leftrightarrow \overrightarrow{x} . \overrightarrow{y} = 0.$

Parallellism

$\overrightarrow{x} \parallel \overrightarrow{y} \Leftrightarrow \theta = 0^\circ$ or $180^\circ \Leftrightarrow cos(\theta) = \pm 1 \Leftrightarrow |\overrightarrow{x}\,\overrightarrow{y}| = \|\overrightarrow{x}\|\|\overrightarrow{y}\|$

Inner product

Distance between two vectors

Distance $= \|\overrightarrow{x} - \overrightarrow{y}\| = \|\overrightarrow{z}\|$ with $\overrightarrow{z} = \overrightarrow{x} - \overrightarrow{y}$.
$\|\overrightarrow{x} - \overrightarrow{y}\|^2 = (\overrightarrow{x} - \overrightarrow{y})(\overrightarrow{x} - \overrightarrow{y})$
$= \overrightarrow{x}\overrightarrow{x} - \overrightarrow{x}\overrightarrow{y} - \overrightarrow{y}\overrightarrow{x} + \overrightarrow{y}\overrightarrow{y}$
$= \overrightarrow{x}\overrightarrow{x} + \overrightarrow{y}\overrightarrow{y} - 2\overrightarrow{x}\overrightarrow{y}$
$= \|\overrightarrow{x}\|^2 + \|\overrightarrow{y}\|^2 - 2\|\overrightarrow{x}\|\|\overrightarrow{y}\|cos(\theta)$ with θ the angle between $\overrightarrow{x}$ and $\overrightarrow{y}$.

Pythagorean theorem

If $\overrightarrow{x} \perp \overrightarrow{y}$ then $cos(\theta) = 0$ and thus:
$\|\overrightarrow{x} - \overrightarrow{y}\|^2 = \|\overrightarrow{x}\|^2 + \|\overrightarrow{y}\|^2$.

Inner product

The 'simple' inproduct

If in the definition $\overrightarrow{x}.\overrightarrow{y} = y^t A x = x^t A y$ (with A positive definite and symetric) A=I, then the inproduct becomes the simple inproduct: $\overrightarrow{x}.\overrightarrow{y} = y^t I x = x^t I y = (x_1 ... x_n) \begin{pmatrix} y_1 \\ ... \\ y_n \end{pmatrix} = \sum_{i=1}^{n} x_i y_i$.

This simple inproduct can always be found by a basis transformation: x=Rx' and y=Ry', then $\overrightarrow{x}.\overrightarrow{y} = y'^t(R^t A R)x'$. Now, R must be taken such that $R^t A R = I$. This can be done by converting A to its normal form by a congruent transformation (e.g. the method of kwadratic forms).

In what follows we mean by 'inproduct' always 'simple inproduct'.

Gram Schmidt orthogonalization

Making two independent vectors orthogonal

Geometric derivation:

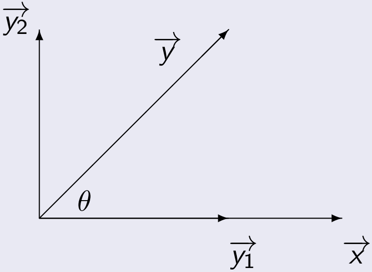


Figure 1: decomposition of vector $\overrightarrow{y}$ in a component parallel ($\overrightarrow{y_1}$) and a component orthogonal ($\overrightarrow{y_2}$) to $\overrightarrow{x}$.

Gram Schmidt orthogonalization

Making two indepentent vectors orthogonal

1. Project $\overrightarrow{y}$ orthogonal on $\overrightarrow{x}$, this generates the vector $\overrightarrow{y_1}$, the component parallel with $\overrightarrow{x}$.
2. Subtract $\overrightarrow{y}$ by $\overrightarrow{y_1}$, the result is $\overrightarrow{y_2}$ wich is orthogonal to $\overrightarrow{x}$.

$\overrightarrow{y_1}$ is a specific multiple of the normilised vector $\overrightarrow{x}$: $\overrightarrow{y_1} = \alpha\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}$.
$\overrightarrow{y_1} \parallel \overrightarrow{x}$: $\overrightarrow{y_1}\overrightarrow{x} = \pm\|\overrightarrow{y_1}\|\|\overrightarrow{x}\|$ ($+$ if $\theta \leq 90°$ and - if $\theta > 90°$).
From fig. 1: $\|\overrightarrow{y_1}\| = cos(\theta)\|\overrightarrow{y}\|$.
From the inproduct: $cos(\theta) = \frac{\overrightarrow{x}\overrightarrow{y}}{\|\overrightarrow{x}\|\|\overrightarrow{y}\|}$. So
$\overrightarrow{y_1}\overrightarrow{x} = \alpha\frac{\overrightarrow{x}}{\|\overrightarrow{x}\|}\overrightarrow{x} = \alpha\|\overrightarrow{x}\| = \|\overrightarrow{y_1}\|\|\overrightarrow{x}\| = cos(\theta)\|\overrightarrow{y}\|\|\overrightarrow{x}\| = \overrightarrow{x}\overrightarrow{y}$.
So we get: $\alpha = \frac{\overrightarrow{x}\overrightarrow{y}}{\|\overrightarrow{x}\|}$.

Gram Schmidt orthogonalization

Conclusion

$\overrightarrow{y_1} = \alpha \frac{\overrightarrow{x}}{\|\overrightarrow{x}\|} = (\frac{\overrightarrow{x}\,\overrightarrow{y}}{\|\overrightarrow{x}\|^2})\overrightarrow{x}$ with $\frac{\overrightarrow{x}\,\overrightarrow{y}}{\|\overrightarrow{x}\|^2}$ a scalar.

And $\overrightarrow{y_2} = \overrightarrow{y} - \overrightarrow{y_1} = \overrightarrow{y} - (\frac{\overrightarrow{x}\,\overrightarrow{y}}{\|\overrightarrow{x}\|^2})\overrightarrow{x}$.

Hence:

The vector $\overrightarrow{y}$ gets orthogonalised on the vector $\overrightarrow{x}$ by subtract $\overrightarrow{y}$ by the component of $\overrightarrow{y}$ parallel with $\overrightarrow{x}$.

Control of $\overrightarrow{y_2} \perp \overrightarrow{x}$:

$\overrightarrow{y_2}\,\overrightarrow{x} = (\overrightarrow{y} - (\frac{\overrightarrow{x}\,\overrightarrow{y}}{\|\overrightarrow{x}\|^2})\overrightarrow{x})\overrightarrow{x} = \overrightarrow{y}\,\overrightarrow{x} - (\frac{\overrightarrow{x}\,\overrightarrow{y}}{\|\overrightarrow{x}\|^2})\|\overrightarrow{x}\|^2 = 0.$

Gram Schmidt orthogonalization

Generalization to multiple vectors

Given: 3 vectors: 2 orthogonal unit vectors $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ $(\|\overrightarrow{x_1}\| = 1 = \|\overrightarrow{x_2}\|, \overrightarrow{x_1}\overrightarrow{x_2} = 0)$ and a vector $\overrightarrow{y}$.
Asked: orthogonilise $\overrightarrow{y}$ on $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$.
Solution:
We first search the component of $\overrightarrow{y}$ parallel with $\overrightarrow{x_1}$ and subtract $\overrightarrow{y}$ by this component. This gives $\overrightarrow{y_1}$.
$\overrightarrow{y_1} = \overrightarrow{y} - (\frac{\overrightarrow{x_1}\,\overrightarrow{y}}{\|\overrightarrow{x_1}\|^2})\overrightarrow{x_1} = \overrightarrow{y} - (\overrightarrow{x_1}\,\overrightarrow{y})\overrightarrow{x_1}\ (\|\overrightarrow{x_1}\|^2 = 1)$
$\overrightarrow{y_1} \perp \overrightarrow{x_1}$.

Gram Schmidt orthogonalization

Generalization to multiple vectors

Next, we subtract $\vec{y_1}$ by the component of $\vec{y_1}$ that is parallel with $\vec{x_2}$, to get $\vec{z}$ (wich is perpendicular to both $\vec{x_1}$ and $\vec{x_2}$).
$\vec{z} = \vec{y_1} - \vec{x_2}(\vec{y_1}\vec{x_2})$. We can write $\vec{z}$ in another way:
$\vec{z} = \vec{y_1} - \vec{x_2}(\vec{y_1}\vec{x_2}) = \vec{y} - (\vec{x_1}\vec{y})\vec{x_1} - \vec{x_2}([\vec{y} - (\vec{x_1}\vec{y})\vec{x_1}]\vec{x_2})$
$\vec{z} = \vec{y} - \vec{x_1}(\overline{x_1\vec{y}}) - \vec{x_2}(\vec{x_2}\vec{y})$

Conclusion

The vector $\vec{y}$ becomes orthogonilised on two orthogonal unit vectors $\vec{x_1}$ and $\vec{x_2}$ by subtracting $\vec{y}$ by the components of $\vec{y}$ parallel with $\vec{x_1}$ and $\vec{x_2}$.

Complementary subspace

Defenition

Given: a n dimensional vector space V^n, with $p < n$ linear independent vectors $\overrightarrow{x_1}, \overrightarrow{x_2}, ..., \overrightarrow{x_p}$. This vectors create a p dimensional subspace V^p and can be orthogonilised via the Gram schidt method to a orthonormal basis $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_p}$ with $\overrightarrow{e_i}\,\overrightarrow{e_j} = \delta_{ij}$ (with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$).
This p vectors can be complemented by $n - p$ linear independent vectors $\overrightarrow{f_1}, \overrightarrow{f_2}, ..., \overrightarrow{f_{n-p}}$ that are linear independent with $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_p}$ and orthonormal.

These vectors $\overrightarrow{f_1}, \overrightarrow{f_2}, ..., \overrightarrow{f_{n-p}}$ generate the orthogonal complement of the subspace created by $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_p}$.
The orthogonal complement of the p-dimensional subspace V^p of V^n $(p < n)$, has dimension $n - p$.

Complementary subspace

Example

Given: $n = 5, p = 3, \overrightarrow{e_1} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \overrightarrow{e_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \overrightarrow{e_3} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$

Asked: The orthogonal complement

Solution: The orthogonal complement has dimension $n - p = 5 - 3 = 2$ and consists of the set vectors that are perpendicular to the vectors $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_p}$.

$$\begin{cases} \overrightarrow{e_1}\overrightarrow{x} = 0 \\ \overrightarrow{e_2}\overrightarrow{x} = 0 \\ \overrightarrow{e_3}\overrightarrow{x} = 0 \end{cases}$$

Complementary subspace

Example

$$\begin{cases} \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \end{bmatrix} \overrightarrow{x} = 0 \\ \begin{bmatrix} 1 & 0 & 0 & 4 & 0 \end{bmatrix} \overrightarrow{x} = 0 \\ \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \end{bmatrix} \overrightarrow{x} = 0 \end{cases}$$

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 1 & 3 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \overrightarrow{x_1} \\ \overrightarrow{x_2} \\ \overrightarrow{x_3} \\ \overrightarrow{x_4} \\ \overrightarrow{x_5} \end{bmatrix} = 0$$

This is a homogenous system of equetions. The solution of this system is the orthogonal complement.

Outline

Row- and column vectors

Example

The rows of a mxn matrix A can be considered as m row vectors with n components.

$$A^{mxn} = \begin{bmatrix} \overrightarrow{r_1} \\ \overrightarrow{r_2} \\ \dots \\ \overrightarrow{r_m} \end{bmatrix}$$

The columns of a mxn matrix A can be considered as n column vectors with m components.

$$A^{mxn} = \begin{bmatrix} \overrightarrow{r_1} & \overrightarrow{r_2} & \dots & \overrightarrow{r_m} \end{bmatrix}$$

Row- and column space, rank

Column space

We consider the columns of A^{mxn} as vectors with m components and difine the vectors $\overrightarrow{x}$ as every possible linear combination of the column vectors $\overrightarrow{k_i}$:
$\overrightarrow{x} = a_1\overrightarrow{k_1} + a_2\overrightarrow{k_2} + ... + a_n\overrightarrow{k_n}$ with $a_i \in \mathbb{R}$ and i means the i^{th} column of A.
The set of all the vectors $\overrightarrow{x}$ is called the column space of A.
The column space = all possible linear combinations of columns of A .

Row- and column space, rank

Column space

if only r of the n vectors are linear independent, that means

- None of this r vectors can be written as a linear combination of the other $r - 1$ vectors
- all others $n - r$ column vectors can be written as linear combinations of the r linear independant vectors

then r is called:

1. the rank of (column) matrix A
2. the dimension of the column space of matrix A

Row- and column space, rank

Row space

The concept row space and row rank can be derived in the same way as the column space is derived.

Rank

It is a fundamental matrix property that:

$\boxed{\text{row rank A} = \text{column rank A}}$

That means: the number linear independant columns in a matrix is equal to the number linear independant rows. Thus:

rank A = row rank A = column rank A

= dimension row space A = dimension column space A

Row- and column space, rank

Example

$$A^{4x3} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \\ 2 & 4 & -2 \end{bmatrix} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

We determine the rank of A by converting it to his echlon form by elementary row operations (explaned in the appendix).

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \\ 2 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Canonical form: the first element of every row is 1. Above and below this ones is every number 0.

Row- and column space, rank

Conclusion

Rank = the number of rows that differs from zero. So the rank is 2. In other words: there are only 2 linear independent rows and columns in A.

Hence:

The column space of A is a 2 dimensional subspace of the 4 dimensional vectorspace.

The row space is a 2 dimensional subspace of the 3 dimensional space.

Link between determinant and rank by square matrices

Determinant-rank

If all the columns of a square matrix are linear dependent, then the determinant is 0. If A^{mxm}:
det A=0 $\Leftrightarrow$ rank A=0 $\Leftrightarrow$ columns depentent $\Leftrightarrow$ rows dependent.
If det A $\neq$ 0 then rank A $=$ m and A is of **full rank**. A matrix can be inverted is his determinant when different from 0 (when its of full rank).

Link between determinant and rank by square matrices

Determinant-rank

If rank(B^{nxn})=r with r$\leq$n and rank(A^{nxn})=n (A is of full rank) then is:
rank(AB)=rank(BA)=rank(B).
Elementary row- and column operations are always of full rank (appendix). When there is a multiplication between a matrix and a elementary row- or column matrix, then has the product always the rank of the matrix.

Column space test

Belongs the vector to the column space?

Given: n vectors with m components relative to a basis in V^m:

$$\begin{bmatrix} x_{11} \\ x_{21} \\ \dots \\ x_{m1} \end{bmatrix}, \dots, \begin{bmatrix} x_{1n} \\ x_{2n} \\ \dots \\ x_{nn} \end{bmatrix}.$$

If r of this n vectors are linear independent then has the matrix

$$X^{mxn} = [\vec{x_1}, \dots, \vec{x_n}] = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \dots & & \dots \\ x_{m1} & \dots & x_{mn} \end{bmatrix}$$

rank r. The column space of X has dimension r.

Column space test

Belongs the vector to the column space?

How can you determine of a given vector $\overrightarrow{y}$ with m components belongs to the column space of X?
If $\overrightarrow{y}$ is a element of the column space of X, then can $\overrightarrow{y}$ be written as a linear combination of the vectors $\overrightarrow{x_1}, ..., \overrightarrow{x_n}$. By adding $\overrightarrow{y}$ to this vectors, the spanned space will be the same and the rank will still be r.
In other words, if $\overrightarrow{y}$ belongs to the column space of X, then will the rank of the expanded matrix $\left[\overrightarrow{x_1}, ..., \overrightarrow{x_n}, \overrightarrow{y}\right]$ be the same as the rank of the normal matrix X.

$$\boxed{\overrightarrow{y} \in \text{column space} \Leftrightarrow rank\left[X \quad \overrightarrow{y}\right] = rank\left[X\right] = r}$$

Column space test

Vector $\overrightarrow{y}$ not in column space

If $\overrightarrow{y}$ doesn't belong to the column space of X, then can't $\overrightarrow{y}$ be written as a linear combination of the vectors $\overrightarrow{x_1}, \ldots, \overrightarrow{x_n}$, in other words: $\overrightarrow{y}$ can be splitted in two vectors $\overrightarrow{y} = \overrightarrow{z_1} + \overrightarrow{z_2}$ with $\overrightarrow{z_1}$ in the column space of X and $\overrightarrow{z_2}$ not in the column space of X. By adding $\overrightarrow{y}$ to the vectors $\overrightarrow{x_1}, \ldots, \overrightarrow{x_n}$ increases the dimension of the spannend space.

$$\boxed{\overrightarrow{y} \notin \text{column space} \Leftrightarrow rank \begin{bmatrix} X & \overrightarrow{y} \end{bmatrix} = r + 1}$$

Remark: rank$\begin{bmatrix} X & \overrightarrow{z_1} \end{bmatrix} = r$ and rank$\begin{bmatrix} X & \overrightarrow{z_2} \end{bmatrix} = r + 1$.

Column space test

Resume

1. n vectors with m components $\overrightarrow{x_1}, \ldots, \overrightarrow{x_n}$
2. Belongs $\overrightarrow{y}$ to the space spanned by $\overrightarrow{x_1}, \ldots, \overrightarrow{x_n}$?
3. Solution:

   a) Determine $r_1 = rank\left[X\right] = rank\left[\overrightarrow{x_1} \quad \ldots \quad \overrightarrow{x_n}\right]$

   b) Determine
   $r_2 = rank\left[X \quad \overrightarrow{y}\right] = rank\left[\overrightarrow{x_1} \quad \ldots \quad \overrightarrow{x_n} \quad \overrightarrow{y}\right]$

   c) Is $r_1 = r_2 \Rightarrow y \in$ column space or
   $r_1 + 1 = r_2 \Rightarrow y \notin$ column space

Remark: analogous for the row space test.

Column space test

Example

Given: $\vec{x_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \vec{x_2} = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 5 \\ 7 \end{bmatrix}, \vec{x_3} = \begin{bmatrix} 0 \\ 4 \\ 3 \\ 9 \\ 12 \end{bmatrix}, \vec{y} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$

Asked: Belongs $\vec{y}$ to the space spanned by $\vec{x_1}, \vec{x_2}, \vec{x_3}$?

Solution: 1) find the rang of X:

$$X = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 4 \\ 3 & 0 & 3 \\ 4 & 5 & 9 \\ 5 & 7 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 0 \\ 3 & 0 & 0 \\ 4 & 5 & 0 \\ 5 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{3}{2} & \frac{3}{4} & 0 \\ -\frac{1}{2} & \frac{9}{4} & 0 \\ -1 & \frac{12}{4} & 0 \end{bmatrix}$$

Hence, rank(X)$= 2$.

Column space test

Example

2) Determine the rank $\begin{bmatrix} X & \vec{y} \end{bmatrix}$

$$\begin{bmatrix} X & \vec{y} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 2 & 2 & 4 & -1 \\ 3 & 0 & 3 & 2 \\ 4 & 5 & 9 & 1 \\ 5 & 7 & 12 & 0 \end{bmatrix} \qquad \text{rank}\begin{bmatrix} X & \vec{y} \end{bmatrix} = 3.$$

Hence, rank $\begin{bmatrix} X & \vec{y} \end{bmatrix} = rank \begin{bmatrix} X \end{bmatrix} + 1$.

That means: $\vec{y}$ can **not** be written as a linear combination of the vectors $\vec{x_1}, \vec{x_2}, \vec{x_3}$ and doesn't belong to the space spanned by this vectors. Because rank(X)$=2$ is the dimension of the space 2. Hence $\vec{x_3}$ can be written as a linear combination of $\vec{x_1}, \vec{x_2}$. Rank(X$\vec{y}$)=3, so the space spanned by $\vec{x_1}, \vec{x_2}, \vec{x_3}$ and $\vec{y}$ has dimension 3.

Orthonormal matrices

Definition

The set vectors $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_n}$ is orthonormal if:

- $\|\overrightarrow{e_i}\| = 1$ with $i = 1, ..., n$
- $\overrightarrow{e_i} \perp \overrightarrow{e_j}$ or $\overrightarrow{e_i} . \overrightarrow{e_j} = 0$ with $i \neq j$

This is noted as:
$\overrightarrow{e_i} . \overrightarrow{e_j} = \delta_{ij}$ with $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$ (δ_{ij} is called 'kronecker delta').

If $\overrightarrow{e_{1i}}$ has the components $\begin{bmatrix} \overrightarrow{e_{1i}} \\ ... \\ \overrightarrow{e_{ni}} \end{bmatrix}$ relative to some basis, then applies for the inproduct:

$$\overrightarrow{e_{1i}}.\overrightarrow{e_{1j}} = \begin{bmatrix} e_{1i} & ... & e_{ni} \end{bmatrix} \begin{bmatrix} e_{1i} \\ ... \\ e_{ni} \end{bmatrix} = e_{1i}e_{1j} + ... + e_{ni}e_{nj} = \delta_{ij}$$

Orthonormal matrices

Defenition

If we create from $\overrightarrow{e_1}, \overrightarrow{e_2}, ..., \overrightarrow{e_n}$ the matrix

$$E = [\overrightarrow{e_1}, ..., \overrightarrow{e_n}] = \begin{bmatrix} e_{11} & ... & e_{1n} \\ ... & & ... \\ e_{n1} & ... & e_{nn} \end{bmatrix}$$

then applies

$$E^t E = \begin{bmatrix} e_{11} & ... & e_{n1} \\ ... & & ... \\ e_{1n} & ... & e_{nn} \end{bmatrix} \begin{bmatrix} e_{11} & ... & e_{1n} \\ ... & & ... \\ e_{n1} & ... & e_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & ... & 0 \\ 0 & 1 & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & 1 \end{bmatrix} =$$

$$I^{nxn}$$

Hence, if the columns and rows of a matrix form a orthonomal system then is the matrix onthonormal: $E^t E = I = EE^t$.

Orthonormal matrices

Properties

1. The inverse of a orthonormal matrix is his transform: $E^tE = I = EE^t \Rightarrow E^{-1}EE^t = E^{-1}I \Rightarrow IE^t = E^{-1} \Rightarrow E^t = E^{-1}$
2. Maintaining the norm of a vector: $\overrightarrow{x}$ is a vector with norm $\|\overrightarrow{x}\|$ and E an orthonormal matrix then applies $\|E\overrightarrow{x}\| = \|\overrightarrow{x}\|$.
   When there is a 'fault' Δx on x, then is $\|E(\overrightarrow{x} + \Delta\overrightarrow{x})\| = \|\overrightarrow{x} + \Delta\overrightarrow{x}\|$.
   Hence, an orthogonal matrix doesn't change the magnitude of a fault. This is important in numerical applications.
3. The determinant of an orthonormal matrix is ± 1, when it is 1 the matrix represent a rotation matrix.

Change of basis and matrix representation

Change of basis

In a n dimension vectorspace V^n are two different bases given:
'old basis': $\overrightarrow{e_1}, \ldots., \overrightarrow{e_n}$
'new basis': $\overrightarrow{f_1}, \ldots, \overrightarrow{f_n}$

A vector x has coordinates $\begin{bmatrix} a_1 \\ \ldots \\ a_n \end{bmatrix}$ relative to the old basis and coordinates $\begin{bmatrix} b_1 \\ \ldots \\ b_n \end{bmatrix}$ relative to the new basis.

Change of basis and matrix representation

Change of basis

Every basisvector $\overrightarrow{f_i}$ can be expressed as a linear combination of the old basis vectors: $\overrightarrow{f_i} = f_{1i}\overrightarrow{e_1} + \ldots + f_{ni}\overrightarrow{e_n}$. So $\overrightarrow{f_i}$ has coordinates $\begin{bmatrix} f_{1i} \\ \ldots \\ f_{ni} \end{bmatrix}$ relative to the old basis.

If we group the old basis vectors in a matrix: $E = \begin{bmatrix} \overrightarrow{e_1} & \ldots & \overrightarrow{e_n} \end{bmatrix}$

then you can write: $\overrightarrow{x} = \begin{bmatrix} \overrightarrow{e_1} & \ldots & \overrightarrow{e_n} \end{bmatrix} \begin{bmatrix} a_1 \\ \ldots \\ a_n \end{bmatrix} = E\overrightarrow{x}_{old}$ and

$$\overrightarrow{f_i} = \begin{bmatrix} \overrightarrow{e_1} & \ldots & \overrightarrow{e_n} \end{bmatrix} \begin{bmatrix} f_{1i} \\ \ldots \\ f_{ni} \end{bmatrix} = E \begin{bmatrix} f_{1i} \\ \ldots \\ f_{ni} \end{bmatrix}.$$

Change of basis and matrix representation

Change of basis

If we now consider x as a linear combination of the 'new' basis vectors $\overrightarrow{x} = b_1\overrightarrow{f_1} + \ldots + b_n\overrightarrow{f_n} = \begin{bmatrix} \overrightarrow{f_1} & \ldots & \overrightarrow{f_n} \end{bmatrix} \begin{bmatrix} b_1 \\ \ldots \\ b_n \end{bmatrix} =$

$\begin{bmatrix} \overrightarrow{f_1} & \ldots & \overrightarrow{f_n} \end{bmatrix} \overrightarrow{x}_{new} = \begin{bmatrix} E \begin{bmatrix} f_{11} \\ \ldots \\ f_{n1} \end{bmatrix} & \ldots & E \begin{bmatrix} f_{1n} \\ \ldots \\ f_{nn} \end{bmatrix} \end{bmatrix} \overrightarrow{x}_{new} = EF\overrightarrow{x}_{new}$

with $F = \begin{bmatrix} f_{11} & \ldots & f_{1n} \\ \ldots & & \ldots \\ f_{n1} & \ldots & f_{nn} \end{bmatrix}$. The columns of F contains the coordinates of the new basis vectors relative to the old basis vectors $\overrightarrow{e_1}, \ldots, \overrightarrow{e_n}$.

Change of basis and matrix representation

Change of basis

So we get: $\overrightarrow{x} = E\overrightarrow{x}_{old} = EF\overrightarrow{x}_{new}$.

From wich follows:

$$\boxed{\overrightarrow{x}_{old} = \begin{bmatrix} a_1 \\ \dots \\ a_n \end{bmatrix} = F\overrightarrow{x}_{new} = F\begin{bmatrix} b_1 \\ \dots \\ b_n \end{bmatrix} \text{ with the columns of F the coordinates of the new basis vectors relative to the old basis.}}$$

So, $\overrightarrow{x}_{old} = F\overrightarrow{x}_{new}$

from wich:

$$\boxed{\overrightarrow{x}_{new} = F^{-1}\overrightarrow{x}_{old}}$$

Change of basis and matrix representation

Example

Given: the 3 dimensional vector space V^3 with basis vectors:

$$\vec{e_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{e_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The 'new' basis vectors are defined as:

$$\vec{f_1} = \vec{e_1} - \vec{e_2} + \vec{e_3}$$
$$\vec{f_2} = 2\vec{e_1} + 3\vec{e_2} + 0\vec{e_3}$$
$$\vec{f_3} = -\vec{e_1} + \vec{e_2} + 2\vec{e_3}.$$

And there is a vector $\vec{x}$:

$$\vec{x} = 2\vec{e_1} + \vec{e_2} - \vec{e_3}.$$

Asked: The coordinates of $\vec{x}$ relative to the basis $\vec{f_1}, \vec{f_2}, \vec{f_3}$.

Change of basis and matrix representation

Example

Solution: The components of $\overrightarrow{f_1}$ relative to the other basis are: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

The components of $\overrightarrow{f_2}$ relative to the other basis are: $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$.

The components of $\overrightarrow{f_3}$ relative to the other basis are: $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

The components of $\overrightarrow{x}$ relative to the other basis are: $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

Change of basis and matrix representation

Example

So, $\overrightarrow{x}_old = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. We know that $\overrightarrow{x}_{old} = F\overrightarrow{x}_{new}$ with

$F = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 1 \\ -1 & 1 & 2 \end{bmatrix}$.

Now is $\overrightarrow{x}_{new} = F^{-1}\overrightarrow{x}_{old} = \frac{1}{15}\begin{bmatrix} 6 & -4 & 5 \\ 3 & 3 & 0 \\ -3 & 2 & 5 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

So $\overrightarrow{x}_{new} = \frac{1}{15}\begin{bmatrix} 3 \\ 9 \\ -9 \end{bmatrix} = \frac{1}{5}\overrightarrow{f_1} + \frac{3}{5}\overrightarrow{f_2} - \frac{3}{5}\overrightarrow{f_3}$.

Change of basis and matrix representation

Remarks

1. If the old basis is orthonormal and the new basis too, then is the transition matrix orthonormal as well ($F^{-1} = F^T$).
2. If the old basis is from dimension n, then the new basis has the same dimension n too, because they span the same space. So F has to be of full rank: rank(F)=n.

Outline

Systems of linear equations

Defenitions

In general, a system of linear equations is described in a matrix-vector identity: Ax=y. If y≠0 then the system is called not homogeneous, if y=0 then the system is called homogeneous. It will turn out that in all the possible cases, the rank r of the matrix A is important for the existance of a solution and the number of solutions.
A good geometric view in the properties and solutions of homogeneous and not homogeneous systems is essential for an understanding of linear images, eigenvalues, etc.

Not homogeneous equations

Existance of a solution

In general, a system of m equations in n variables $x_1, x_2, ..., x_n$ has the form: $A^{mxn}x^{nx1} = y^{mx1}$. The problem is: search the variables $x_1, x_2, ..., x_n$ wich statisfy this relationship (A and y are known).

$$\begin{bmatrix} a_{11} & ... & a_{1n} \\ ... & & ... \\ a_{m1} & ... & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ ... \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ ... \\ y_m \end{bmatrix} \text{ or}$$

$$\begin{cases} a_{11}x_1 + ... + a_{1n}x_n = y_1 \\ ... \\ a_{m1}x_1 + ... + a_{mn}x_n = y_m \end{cases}$$

Not homogeneous equations

Existance of a solution

If we think of A as column vectors with m components $A = \begin{bmatrix} \overrightarrow{a_1} & \dots & \overrightarrow{a_n} \end{bmatrix}$, then we can write the system in antother way:

$$x_1 \begin{bmatrix} a_{11} \\ \dots \\ a_{m1} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ \dots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} y_1 \\ \dots \\ y_m \end{bmatrix} \text{ or } x_1 \overrightarrow{a_1} + \dots + x_n \overrightarrow{a_n} = \overrightarrow{y}.$$

The geometric interpretation:

Given a vector $\overrightarrow{y}$ of m components and n vectors $\overrightarrow{a_i}$ of m components, search **all** numbers x_i, i=1,n , so that vector $\overrightarrow{y}$ can be written as a linear combination of the vectors $\overrightarrow{a_i}$.

All the linear combinations of the vectors $\overrightarrow{a_i}$ can be written as: $x_1 \overrightarrow{a_1} + \dots + x_n \overrightarrow{a_n} = \overrightarrow{y}$ and is called the column space of A.

Not homogeneous equations

Existance of a solution

If rank(A)=r, then has the colums space dimension r. In other words: there are r linear independent columns in A and the other n-r columns in A can be written as a linear combination of the r linear independent columns.

Since $\overrightarrow{y}$ has to be a linear combination of the columns of A, $\overrightarrow{y}$ has to be in the columns space of A.
If $\overrightarrow{y}$ doesn't exist of the column space of A, then $\overrightarrow{y}$ can't be written as a linear combination of the columns of A, then the system **doesn't have** a solution.

Not homogeneous equations

Existance of a solution

The condition for the existence of a solution is that $\overrightarrow{y}$ belongs to the column space of A. Now, the column space test beccomes the solution test.

The system Ax=y with rank(A)=r, has a solution only if rank(A)=rank(Ay)=r with Ay a mx(n+1) matrix.

Not homogeneous equations

Example 1

$$\begin{cases} -x_1 + 2x_2 = 0 \\ x_1 + 3x_2 = 1 \\ 0x_1 + x_2 = 2 \end{cases} \quad \text{or} \quad \underbrace{\begin{bmatrix} -1 & 2 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_{y}$$

Rank(A)=2, rank(Ay)=3

Rank(A)≠rank(Ay), so there **isn't** a solution.

Not homogeneous equations

Example 2

$$\begin{cases} x_1 + x_2 + x_3 = 2 \\ -x_1 + 2x_2 + 3x_3 = -1 \end{cases} \quad \text{or} \quad \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix}}_{\text{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\text{x}} = \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_{\text{y}}$$

Rank(A)=2=rank(Ay)=2, so this system can be solved.

In the next we suppose that there is always a solution, thus: rank(A)=rank(Ay).

Number of solutions and solution methods

Defenitions

$$A^{mxn}x^{nx1} = y^{mx1}$$

There are **three** cases:

1. m>n: overdetermined system (more equations than variables)
2. m=n: square system
3. m<n: underdetermined system

Number of solutions and solution methods

m>n: overdetermined system

There is a solution if rang(A)=rang(Ay)=r. If this condition is statisfied, we can write: $x_1\overrightarrow{a_1} + ... + x_n\overrightarrow{a_n} = \overrightarrow{y}$ with a_i the ith column of A. As rank(A)=r, there are r linear independent columns $\overrightarrow{a_1}, ..., \overrightarrow{a_r}$. Now we can write:

$$x_1\overrightarrow{a_1} + ... + x_r\overrightarrow{a_r} = \overrightarrow{y} - x_{r+1}\overrightarrow{a}_{r+1} - ... - x_n\overrightarrow{a_n}$$

If: $\overrightarrow{y'} = \overrightarrow{y} - x_{r+1}\overrightarrow{a}_{r+1} - ... - x_n\overrightarrow{a_n}$ then $\begin{bmatrix}\overrightarrow{a_1} & ... & \overrightarrow{a_r}\end{bmatrix}\begin{bmatrix}x_1\\ ...\\ x_r\end{bmatrix} = \overrightarrow{y'}$.

From the columntest follows that $\overrightarrow{y}$ belongs to the columnspace of A. The vectors $\overrightarrow{a}_{r+1}, ..., \overrightarrow{a_n}$ belong to the same space. The vectors $\overrightarrow{a}_1, ..., \overrightarrow{a_r}$ create the **basis** of this space.

Number of solutions and solution methods

m>n: overdetermined system

For every choice of the vector $\overrightarrow{y'}$, there exist a **unique** set (only one solution) of numbers $x_1, ..., x_r$: this are the coordinates of the vector $\overrightarrow{y'}$ in the basis $\overrightarrow{a_1}, ..., \overrightarrow{a_r}$. There is only one solution because every vector in only one way can be descibed as a linear combination of basis vectors.
Now is: $\overrightarrow{y'} = \overrightarrow{y} - x_{r+1}\overrightarrow{a}_{r+1} - ... - x_n\overrightarrow{a_n}$.
For each set $[x_{r+1}, ..., x_n]$ there is a vector $\overrightarrow{y'}$, and for every vector $\overrightarrow{y'}$ exist just one solution $[x_1, ..., x_r]$. For the choise of the set $[x_{r+1}, ..., x_n]$, there are ∞^{n-r} possibilities.

Number of solutions and solution methods

m>n: overdetermined system

If m>n and rank(Ay)=rank(A)=r, then has the system Ax=y ∞^{n-r} solutions.

The variables $x_1, x_2, \ldots, x_r$ are the main variables and $x_{r+1}, x_{r+2}, \ldots, x_n$ are the help variables. For each set help variables exist only one set main variables.

If rank(A)=rank(Ay)=r and if n=r, then has the system Ax=y exact **one** solution ($\infty^0 = 1$).

Number of solutions and solution methods

m>n: overdetermined system

How do you determine the solution for a certain set help variables $[x_{r+1}, x_{r+2}, \ldots, x_n]$.

If $\begin{bmatrix} \overrightarrow{a_1} & \ldots & \overrightarrow{a_r} \end{bmatrix} \begin{bmatrix} x_1 \\ \ldots \\ x_r \end{bmatrix} = \overrightarrow{y} - x_{r+1}\overrightarrow{a}_{r+1} - \ldots - x_n\overrightarrow{a_n} \Rightarrow A'x' = y'$

with $A' = \begin{bmatrix} \overrightarrow{a_1} & \ldots & \overrightarrow{a_r} \end{bmatrix}$ and

$\overrightarrow{y'} = \overrightarrow{y} - x_{r+1}\overrightarrow{a}_{r+1} - \ldots - x_n\overrightarrow{a_n} = \begin{bmatrix} y'_1 \\ \ldots \\ y'_m \end{bmatrix}.$

We know that rang(A')=r. This means that r rows of A' are linear independent. The other rows of A' can be created by linear combinations of the linear independent rows.

Number of solutions and solution methods

m>n: overdetermined system

The same is true for the matrix Ay, since rank(Ay)=r has matrix Ay r linear independent rows. The other rows can be created by linear combinations of the linear independent rows. This means that there are only r rows necessery to solve the system. The other m-r equations are linear combinations of the r independent equations.

Thus: select from A' r linear independent equations with the components of y' according to this equations:

$A'' = \begin{bmatrix} \overrightarrow{a_1}' \\ \dots \\ \overrightarrow{a_r}' \end{bmatrix}$. And we get: $A''x' = \begin{bmatrix} \overrightarrow{y_1}' \\ \dots \\ \overrightarrow{y_r}' \end{bmatrix} = y''$.

Number of solutions and solution methods

m>n: overdetermined system

A" is a square matrix and has full rank, so it is an invertible matrix.
$A''x' = y'' \Rightarrow (A'')^{-1}A''x' = (A'')^{-1}y''$ and we get:

$$\boxed{\underbrace{x'}_{\text{rx1}} = \underbrace{(A'')^{-1}}_{\text{rxr}} \underbrace{y'}_{\text{rx1}}}$$

This is the general solution of the overdetermined system. The

matrix A" is the main matrix of the system. A" is created by the choice of help variables and independent rows. There are m-r help variables, and for each set of them there is a vector y". So there exist ∞^{n-r} solutions in x.

Number of solutions and solution methods

m>n: overdetermined system

Example:

$$\begin{cases} x_1 - 2x_2 - x_3 = 3 \\ 2x_1 + 0x_2 + 2x_3 = -2 \\ 3x_1 + 3x_2 + 6x_3 = -9 \\ 4x_1 + 6x_2 + 10x_3 = -16 \\ 5x_2 + 7x_2 + 12x_3 = -19 \end{cases}$$

m=5, n=3

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 2 \\ 3 & 3 & 6 \\ 4 & 6 & 10 \\ 5 & 7 & 12 \end{bmatrix}, y = \begin{bmatrix} 3 \\ -2 \\ -9 \\ -16 \\ -19 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Number of solutions and solution methods

m>n: overdetermined system

1. Solution test:
   rank(A)=2=rank(Ay)=2 $\Rightarrow$ solvable
2. Solution:
   We choose the first two columns of A as linear independent columns. $\rightarrow$ So we fix the help and main variables.

$$\begin{bmatrix} 1 & -2 \\ 2 & 0 \\ 3 & 3 \\ 4 & 6 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -9 \\ -16 \\ -19 \end{bmatrix} - x_3 \begin{bmatrix} -1 \\ 2 \\ 6 \\ 10 \\ 12 \end{bmatrix}$$

Number of solutions and solution methods

m>n: overdetermined system

Rank(Ay)=2, so there are 2 linear independent rows. The other rows are linear combinations of the independent rows. We choose the first two rows to be linear independent:

$\begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - x_3 \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ this is $A''x' = y''$.

With $A''^{-1} = \frac{1}{4} \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix}$, we get:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \frac{x_3}{4} \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 - x_3 \\ -2 - x_3 \end{bmatrix}.$$

This is the general solution of the system (verify by doing substitution). For each choice of x_3 you get a different solution. In this case there are $\infty^{n-r} = \infty 3 - 2 = \infty$ solutions.

Number of solutions and solution methods

m=n: square system

We suppose there is a solution: rank(Ay)=rank(A)=r. The same rules can be apllied here: if $\overrightarrow{a_1}, \ldots, \overrightarrow{a_r}$ are the linear independent columns of A, we can write:

$\begin{bmatrix} \overrightarrow{a_1} & \ldots & \overrightarrow{a_r} \end{bmatrix} \begin{bmatrix} x_1 \\ \ldots \\ x_r \end{bmatrix} = y - r_{r+1} \overrightarrow{a}_{r+1} - \ldots - x_n \overrightarrow{a}_n$ and again there are ∞^{n-r} solutions (one solution for each set $[x_{r+1}, \ldots, x_n]$).

Remark: if m=n=r, there is exactly one solution, the system is called **Cramers system**.

Conclusion: A square system can be solved on the same way as a overdetermined system.

Number of solutions and solution methods

m<n: underdetermined system

Again there has to be statistified to: rank(A)=rank(Ay)=r. If this condition is true, then we can find a solution on the same way as described above.

Example 1:

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 1 & -4 & -6 \\ 2 & 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -9 \\ 2 \end{bmatrix}$$

$$\text{Rank(A)} \Rightarrow \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 0 & -7 & -7 \\ 0 & 0 & -7 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow rank(A) = 2$$

Number of solutions and solution methods

m<n: underdetermined system

Rank(Ay)

$$\Rightarrow \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 1 & 1 & -4 & -6 & -9 \\ 2 & 2 & -1 & -5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & 0 & -7 & -7 & -15 \\ 0 & 0 & -7 & -7 & -14 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & 0 & 1 & 1 & \frac{15}{7} \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -2 & 6-\frac{45}{7} \\ 0 & 0 & 1 & 1 & \frac{15}{7} \\ 0 & 0 & 0 & 0 & \frac{-1}{7} \end{bmatrix} \Rightarrow rank(Ay) = 3$$

Rank(A)≠rank(Ay) so the system is **unsolvable**.

Number of solutions and solution methods

m<n: underdetermined system

Example 2:

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 1 & -4 & -6 \\ 2 & 2 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \end{bmatrix}$$

Rank(A)=rank(Ay)=r, so there is a solution. Column one and three are linear independent. Now we can write:

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \end{bmatrix} \text{ or}$$

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix} == \begin{bmatrix} 6 \\ -8 \\ -2 \end{bmatrix} - x_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - x_4 \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix}.$$

Number of solutions and solution methods

m<n: underdetermined system

We choose the first two rows:

$$\begin{bmatrix}1 & 3\\1 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_3\end{bmatrix} = \begin{bmatrix}6\\-8\end{bmatrix} - x_2\begin{bmatrix}1\\1\end{bmatrix} - x_4\begin{bmatrix}1\\-6\end{bmatrix}$$

$A''^{-1} = \frac{1}{7}\begin{bmatrix}4 & 3\\1 & -1\end{bmatrix}$ so we get:

$$\begin{bmatrix}x_1\\x_3\end{bmatrix} = \frac{1}{7}\begin{bmatrix}4 & 3\\1 & -1\end{bmatrix}\begin{bmatrix}6\\-8\end{bmatrix} - \frac{-x_2}{7}\begin{bmatrix}4 & 3\\1 & -1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} - \frac{x_4}{7}\begin{bmatrix}4 & 3\\1 & -1\end{bmatrix}\begin{bmatrix}1\\-6\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_3\end{bmatrix} = \frac{1}{7}\begin{bmatrix}-7x_2 + 14x_3\\14 + 0x_2 - 7x1\end{bmatrix} = \begin{bmatrix}-x_2 + 2x_4\\2 - x_4\end{bmatrix}$$

There are $\infty^{nr} = \infty^2$ solutions. Verify via substitution:

$$\begin{bmatrix}1 & 1 & 3 & 1\\1 & 1 & -4 & -6\\2 & 2 & -2 & -5\end{bmatrix}\begin{bmatrix}-x_2 + 2x_4\\x_2\\2 - x_4\\x_4\end{bmatrix} = \begin{bmatrix}6\\-8\\-2\end{bmatrix}.$$

Not homogeneous systems

Resume

$$\underbrace{A}_{\text{mxn}} \underbrace{x}_{\text{nx1}} = \underbrace{y}_{\text{mx1}}$$

1. Determine rank(A):
   rank(A)=r
2. Determine rank(Ay):
   rank(Ay)=r+1 $\rightarrow$ unsolvable
   rank(Ay)=r $\rightarrow$ solvable
3. If solvable: choose r independent columns of A and the components $[x_1, ..., x_r]$ according to it.

$$\begin{bmatrix} \overrightarrow{a_1} & \dots & \overrightarrow{a_r} \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_r \end{bmatrix} = \overrightarrow{y} - x_{r+1}\overrightarrow{a}_{r+1} - \dots - x_r \overrightarrow{a} = \overrightarrow{y'}$$

Not homogeneous systems

Resume

For every choice of $[x_{r+1}, ..., x_n]$ exist one solution. Hence, there are ∞^{n-r} solutions.
Because rank(Ay)=r, there are r rows of Ay linear independent and the other m-r rows are linear cominations of it. Choose r independent rows of Ay. Then is

$$\underbrace{A''}_{\text{rxr}} \underbrace{x'}_{\text{rx1}} = \underbrace{y''}_{\text{mx1}} \Rightarrow x' = A''^{-1} y''$$

the general solution of the system. Verify by substitution.

Not homogeneous systems

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Not homogeneous systems

Remark

The method above can be described in a **matricial** way:

$$A_{mxn} = \begin{bmatrix} A_{rxr} & A_{rx(n-r)} \\ A_{(m-r)xr} & A_{(m-r)x(n-r)} \end{bmatrix}$$

We get: $\begin{bmatrix} A_{rxr} & A_{rx(n-r)} \\ A_{(m-r)xr} & A_{(m-r)x(n-r)} \end{bmatrix} \begin{bmatrix} x_r \\ x_{n-r} \end{bmatrix} = \begin{bmatrix} y_r \\ y_{m-r} \end{bmatrix}$ with x_r the main variables and x_{n-r} the help variables.

Now we can write:

$\begin{cases} A_{rxr}x_r + A_{rx(n-r)}x_{n-r} = y_r \\ A_{(m-r)xr}x_r + A_{(m-r)x(n-r)}x_{n-r} = y_{m-r} \end{cases}$. Thus:

$\begin{cases} x_r = A_{rxr}^{-1}y_r - A_{rxr}^{-1}A_{rx(n-r)}x_{n-r} \\ x_{n-r} = x_{n-r} \end{cases}$ is the general ∞^{n-r} fold solution.

Homogeneous systems

General

$$\underbrace{A}_{\text{mxn}}\underbrace{x}_{\text{nx1}} = 0 \text{ and rank(A)=r}$$

x=0 is always a solution.

Existance of a solution

Now we consider the **row** vectors of A.

$$\begin{bmatrix} \overrightarrow{a_1} \\ \dots \\ \overrightarrow{a_m} \end{bmatrix} \overrightarrow{x} = 0 \text{ or } \begin{bmatrix} \overrightarrow{a_1}\,\overrightarrow{x} \\ \dots \\ \overrightarrow{a_m}\,\overrightarrow{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \dots \\ 0 \end{bmatrix}.$$

The homogeneous system is nothing more than m simple inner products that has to be 0.

Homogeneous systems

Existance of a solution

The inner product of two vectors is zero if the two vectors are orthogonal. In other words: $\overrightarrow{x} \perp$ on every row of A.
Hence, the solution of the homogeneous system consists of all the vectors $\overrightarrow{x}$ wich are perpendicular to the rows of A.

Because rank(A)=r is the dimension of the row space r. In A are r linear independent rows, the other rows are a linear combination of the independent rows.

Homogeneous systems

Existance of a solution

The rows of A are vectors with n components. The row space is a r dimensional subspace of the n dimensional vector space. This global n dimensional space consists of two subspaces.

1. The subspace spanned by the rows of A: subspace with dimension r.
2. The complementary subspace: the vectors in this space are linear independently from the rows of A. The complementar subspace has dimension n-r.

Homogeneous systems

Resume

The solution of $\underbrace{A}_{\text{mxn}}\underbrace{x}_{\text{nx1}}=0$ with rank(A)=r is a n-r dimension space: the complementary space of the row space of A.

When n=r, the **only** solution of the homogeneous system is 0.
There isn't a vector wich is perpendicular to the the rows of A.
Since n=r, the rows of A span the total n dimensional space and the complementary space has dimension 0.

$\underbrace{A}_{\text{mxn}}\underbrace{x}_{\text{nx1}}=0$ with rank(A)=r has a solution different from 0 if rank(A)<n

Homogeneous systems

Number of solutions

$$\underbrace{A}_{\text{mxn}}\underbrace{x}_{\text{nx1}} = 0 \Rightarrow \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = 0$$

Covert the matrix A by elementary row operations to this form:

$$A \sim \begin{bmatrix} r\{ \overbrace{\ 1\ }^{r} & 1 \\ 1 & 1 \end{bmatrix}$$