Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Design in the frequency domain
Nyquist stability criterion

July 27, 2015

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Outline

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Introduction

The root-locus method dealt with the s- and z-domain and the location of the poles and zeros formed the basis for that method.

For the **frequency domain**, we use the following substitution: $s = j\omega$, which means we will only regard perfect oscillations.

Sines, cosines and exponential signals are eigenfunctions of Linear Time Invariant systems (LTI).
Perfect oscillations form the natural decomposition of each signal when you are dealing with LTIs.

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Introduction

We need to translate our **design criteria** to something that fits the discussed method. For the root-locus method, we had to express the criteria in positions of poles and zeros.
For the frequency domain, typical design criteria are:

- Phase and gain margin
- Bandwidth
- Zero-frequency magnitude ($=$ DC gain)

We will discuss two different **graphical representations** to design compensators in the frequency domain:

- Nyquist stability diagram and Nyquist plots
- Bode plots (for the design of lead, lag and lead-lag compensators)

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Outline

1 **Introduction**

2 **Stability**

3 **Cauchy's argument principle**
   - Complex function
   - Cauchy's argument principle
   - Examples

4 **Nyquist stability criterion**
   - Nyquist stability criterion
   - Nyquist plot

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Stability of the closed loop system
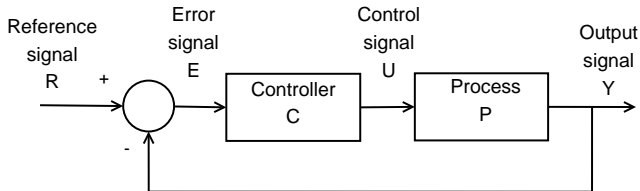
We write the output signal as a function of the input signal:

$$Y(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}R(s).$$

The closed loop system stability is determined by the poles of $\frac{P(s)C(s)}{1+P(s)C(s)}$, or the roots of $1 + P(s)C(s) = 0$.

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Stability of the closed loop system

In the root-locus method, we determined the positions of the poles
by plotting the roots of $1 + P(s)C(s) = 0$.

The system is stable if all the roots remain in the left half plane.

We are however not interested in the positions of the poles.
We only need to know whether there are poles in the right half
plane.

There is a cheaper alternative: **the Nyquist stability criterion**.

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Stability: Nyquist criterion

The Nyquist stability criterion avoids determining the roots of $1 + P(s)C(s)$ exactly (expensive). It only determines the number of roots in the right half plane.

It uses a theorem from complex calculus that finds the difference between the number of poles and the number of zeros within a contour (a closed curve).

We will apply this theorem to $1 + P(s)C(s)$ (which can be seen as a complex function) and the contour will encircle the entire right half plane ($=$ the Nyquist contour).

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Complex function
Cauchy's argument principle
Examples

Outline

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Complex function
Cauchy's argument principle
Examples

Outline

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Complex function
Cauchy's argument principle
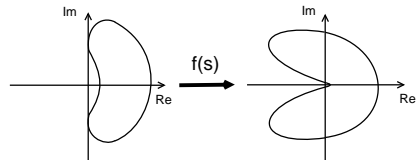Examples

Complex function

Before we get to the theorem, we discuss the concept of a complex function.

A complex function $f(s) = u(x, y) + jv(x, y)$ maps the complex number $s = x + jy$ onto the complex number $w = u + jv$.

For a complex number:
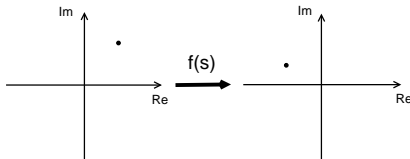
or for a contour:

The function $1 + P(s)C(s)$ can also be regarded as a complex function, so we can use it as a mapping.

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Complex function
Cauchy's argument principle
Examples

Outline

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Complex function
Cauchy's argument principle
Examples

Cauchy's argument principle

This is the engine behind the Nyquist stability criterion.

If a contour Γ in the s-plane circles Z zeros and P poles $f(s)$ in clockwise rotation, then the contour Γ′, which is the image of Γ as mapped by $f(s)$, circles the origin (in the ω-plane) Z-P times in the clockwise direction.

So the only thing we are looking at is the encirclements of the origin.
On the next slides, we will prove this.

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Complex function
Cauchy's argument principle
Examples

Cauchy's argument principle

Let's take a complex function $f(s) = \frac{(s-z_1)(s-z_2)(s-z_3)...}{(s-p_1)(s-p_2)(s-p_3)...}$.

If we would apply this function to a complex number c, this comes down to multiplying factors $c - z_i = A_{z_i}e^{j\theta_{z_i}}$ and $\frac{1}{c-p_i} = \frac{1}{A_{p_i}}e^{-j\theta_{p_i}}$.
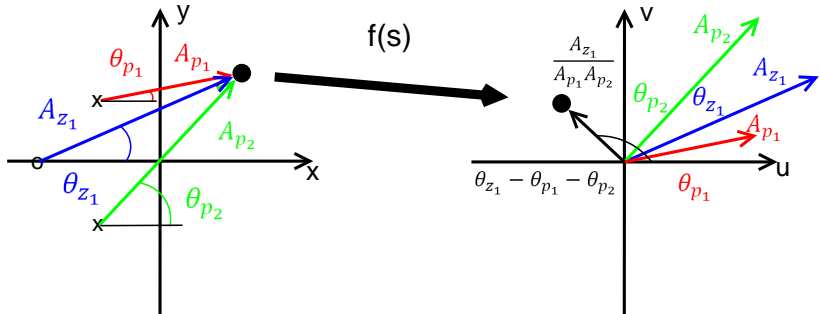
So the modulus of $f(c)$ can be easily found by evaluating $\frac{A_{z_1}A_{z_2}A_{z_3}...}{A_{p_1}A_{p_2}A_{p_3}...}$;

This might help if you want to map a point, but it is not important for us.

The evaluation of the argument of $f(c)$ is what will be interesting; $\angle f(c) = \theta_{z_1} + \theta_{z_2} + \theta_{z_3} + ... - \theta_{p_1} - \theta_{p_2} - \theta_{p_3} - ...$

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Complex function
Cauchy's argument principle
Examples

Cauchy's argument principle: graphically

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

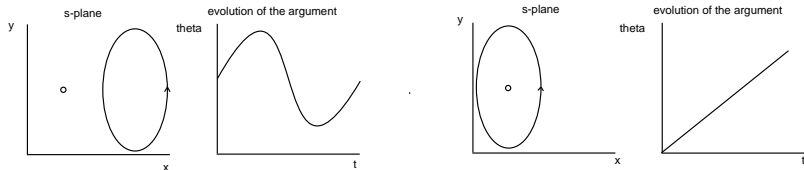Complex function
Cauchy's argument principle
Examples

Cauchy's argument principle

When does the image of a contour in the ω-plane encircle the origin?

That happens when the argument of the image of the contour at the beginning and the end differ by 2π.

A pole or zero outside the contour will never have that effect. One inside the contour has the following effect:

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Complex function
Cauchy's argument principle
Examples

Cauchy's argument principle

A pole results in -2π (counterclockwise rotation), if the contour is followed clockwise.

A zero results in $+2\pi$ (clockwise rotation).

This follows from the sign of their effect:
$\angle f(c) = \theta_{z_1} + \theta_{z_2} + \theta_{z_3} + ... - \theta_{p_1} - \theta_{p_2} - \theta_{p_3} - ...$
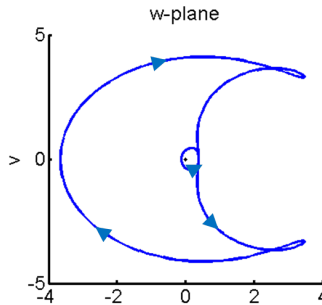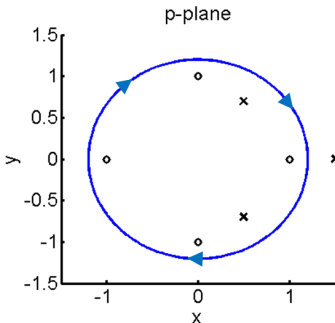
It is also possible that the origin is encircled when there are no poles or zeros in the contour (in the s-plane). But then the amount of clockwise encirclements equals the amount of counterclockwise encirclements hence no net encirclements.

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Complex function
Cauchy's argument principle
Examples

Outline

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Complex function
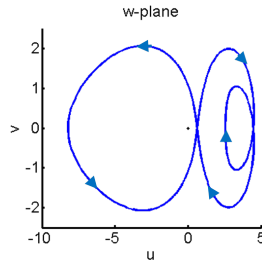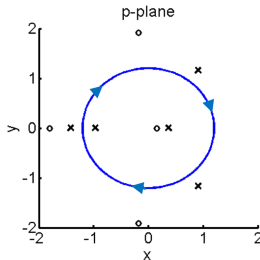Cauchy's argument principle
Examples

Example 1

- Two encircled poles (the x's): $P = 2$
- Four encircled zeros (the o's): $Z = 4$
- Hence: $N = Z - P = 2$
- Indeed, the image of the contour encircles the origin twice in the ω-plane (in clockwise direction).

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Complex function
Cauchy's argument principle
Examples

Example 2

- Two encircled poles: $P = 2$
- One encircled zero: $Z = 1$
- Hence: $N = Z - P = -1$
- Indeed, the image of the contour encircles the origin once in the ω-plane (in counterclockwise direction).

Introduction
Stability
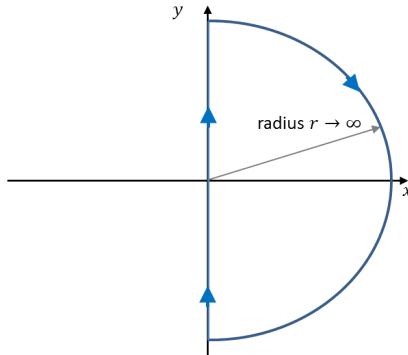Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Outline

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Outline

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist stability criterion

If we apply Cauchy's argument principle to the following contour (the Nyquist contour), the amount of clockwise encirclements around the origin of the image of this contour (called the Nyquist plot) in the ω-plane equals $Z - P$ of the right half plane (RHP).

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist stability criterion

That way we will find the difference between the number of poles and zeros of $1 + P(s)C(s)$ in the RHP: $N = Z - P$.

So if we know P (the number of poles in the RHP of $1 + P(s)C(s)$), we know how many zeros of $1 + P(s)C(s)$ there are in the RHP.
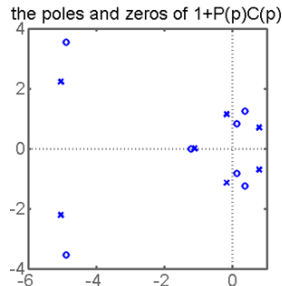
Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist stability criterion

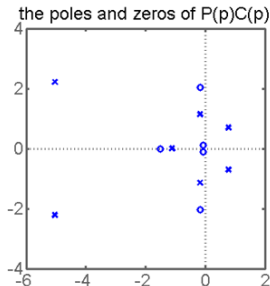Luckily this last aspect is simple, since the poles of $1 + P(s)C(s)$ equal those of $P(s)C(s)$. Hence the amount of RHP poles is equal (the connection between the zeros is not clear).

So if we assume the number of RHP poles of $P(s)C(s)$ is known (which we'll assume), we know whether the system with unity (negative) feedback is stable.

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist stability criterion

If we apply this to $1 + P(s)C(s)$, we need to count the number of encirclements of the origin.

However, the Nyquist stability criterion uses $P(s)C(s)$.

- The zeros of $1 + P(s)C(s)$ and the poles and zeros of $P(s)C(s)$ are hard to relate.

- This is in sharp contrast with how easily the Nyquist plots relate.

- The Nyquist plot of $P(s)C(s)$ equals the one of $(1 + P(s)C(s))$, after it has been moved to the right over a distance 1.

- To find $Z - P$, one has to count the number of clockwise encirclements of the image of $P(s)C(s)$ around the point $(-1, 0)$, since this equals the number of clockwise encirclements of the image of $1 + P(s)C(s)$ around the origin.

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist stability criterion

Nyquist stability criterion

If the open loop system $P(s)C(s)$ has ℓ poles in the open right half plane (of the s-plane),
then the system with unity feedback is stable if and only if the Nyquist plot encircles the point (-1,0) ℓ times in the counter clockwise direction.

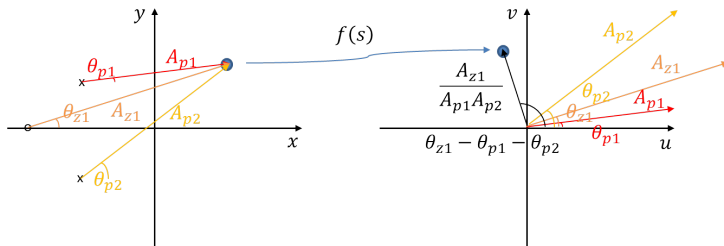Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Outline

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Cauchys argument principle

- Lets take a complex function $f(s) = \frac{(s-z_1)(s-z_2)(s-z_3)...}{(s-p_1)(s-p_2)(s-p_3)...}$
- So if we would apply this function to a complex number c then this comes down to multiplying factors $c - z_1 = A_{z_1} e^{j\theta_{z_1}}$ and $\frac{1}{c-p_1} = \frac{1}{A_{p_1}} e^{-j\theta_{p_1}}$
- So the modulus of $f(c)$ can be easily found by evaluating $\frac{A_{z_1} A_{z_2} A_{z_3}...}{A_{p_1} A_{p_2} A_{p_3}...}$;this might help if you want to map a point, but it is not important for us.
- The evaluation of the argument of $f(c)$ is what will be interesting; it is also very easy to find:
  $\angle f(c) = \theta_{z_1} + \theta_{z_2} + \theta_{z_3} + ... - \theta_{p_1} - \theta_{p_2} - \theta_{p_3} - ...$

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Cauchys argument principle

Lets first show this graphically:

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Cauchys argument principle

- When does the image of a contour in the w-plane encircle the origin?
- That happens when the image of the beginning and the end of the contour their argument differs by 2π
- A pole/zero outside the contour will never have that effect and one inside the contour does:

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot
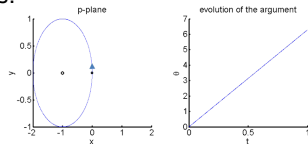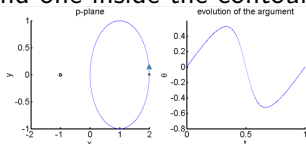
Cauchys argument principle

- A pole results in -2π (CCW rotation), if the contour is followed CW, and a zero results in $+2\pi$ (CW rotation) follows from the sign of their effect:
  $$\angle f(c) = \theta_{z_1} + \theta_{z_2} + \theta_{z_3} + ... - \theta_{p_1} - \theta_{p_1} - \theta_{p_1} - ...$$
- Watch out: it is also possible that the origin is encircled when there are no poles or zeros in the contour (in the s-plane); but then the amount of CW encirclements equals the amount of CCW encirclements hence no net encirclements

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Cauchys argument principle: examples

- Two encircled poles (the x's): $P = 2$
- Four encircled zeros (the o's): $Z = 4$
- Hence: $N = Z - P = 2$
- Indeed, the image of the contour encircles the origin twice in the w-plane (in the CW direction)
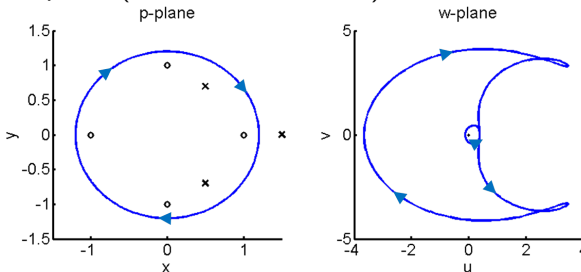
Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
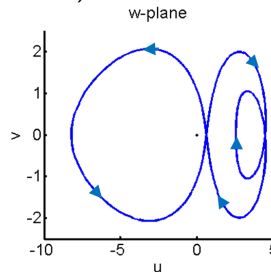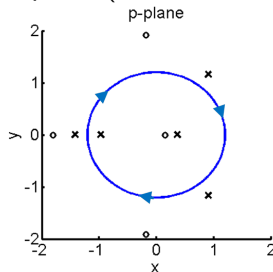Nyquist plot

Cauchys argument principle: examples

- Two encircled poles (the x' s): $P = 2$
- Four encircled zeros (the o' s): $Z = 1$
- Hence: $N = Z - P = -1$
- Indeed, the image of the contour encircles the origin once in the w-plane (in the CCW direction)

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist stability criterion

If we apply Cauchys argument principle to the following contour (the Nyquistcontour), the amount of CW encirclements around the origin of the image of this contour (called the Nyquist plot) in the w-plane equals $Z - P$ of the RHP
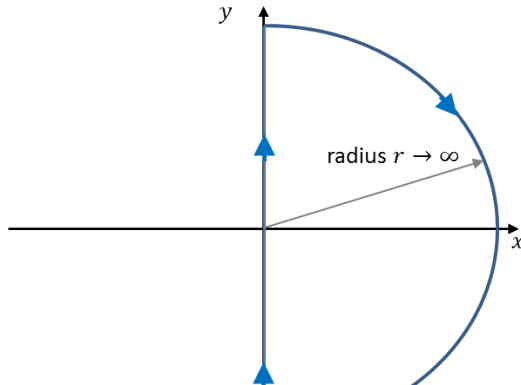
Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist stability criterion

- That way we will find the difference between the number of poles and zeros of $1 + P(s)C(s)$ in the RHP: $N = Z - P$
- So if we now know P (the number of poles in the RHP of $1 + P(s)C(s)$), we can know whether (and how many) zeros of $1 + P(s)C(s)$ are in the RHP
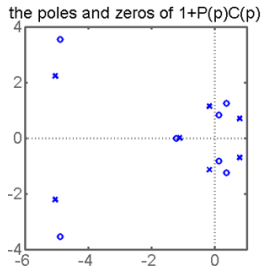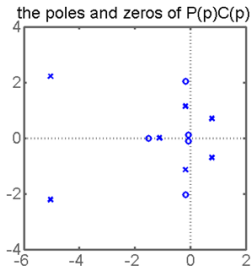
the poles and zeros of P(p)C(p)

the poles and zeros of 1+P(p)C(p)

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist stability criterion

- Luckily this last aspect is simple; since the poles of $1 + P(s)C(s)$ equal those of $P(s)C(s)$; hence the amount of RHP poles is equal
  (the connection between the zeros is not clear)
- So if we assume the number of RHP poles of $P(s)C(s)$ is known (which we'll assume), we can know whether the system with unity feedback is stable

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist stability criterion

So we could apply this to $1 + P(s)C(s)$, then the encirclements of the origin counts indeed

But the Nyquist stability criterion applies this to $P(s)C(s)$

- We saw that the zeros of $1 + P(s)C(s)$ and the poles and zeros of $P(s)C(s)$ are hard to relate.
- This is in sharp contrast with how easily the Nyquist plots relate
- The Nyquist plots of $P(s)C(s)$ equals the one of $1 + P(s)C(s)$, after he has been moved 1 to the right
- So to find $Z - P$ one has to count the number of CW encirclements of the image of $P(s)C(s)$ around $(-1,0)$ since this equals the number of CW encirclements of the image of $1 + P(s)C(s)$ around the origin.

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist stability criterion

Nyquist stability criterion:

> If the open loop system $P(s)C(s)$ has l-poles in the open right half plane (of the s-plane),
> then the system with unity feedback is stable if and only if the Nyquist plot encircles the point
> $(-1,0)$ l times in the counter clockwise direction

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot
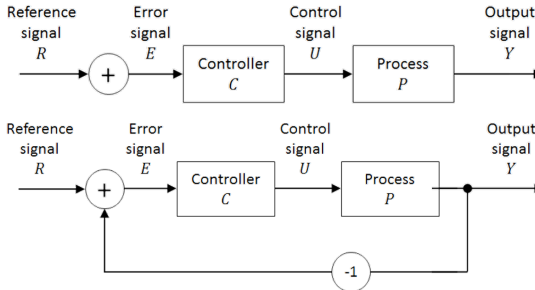
Nyquist plot

We now found how to deduce stability of the closed loop system
from the Nyquist plot
Now we will discuss how these plots can be found
But first some simplifications:

- For physically realizable systems (i.e. relevant systems to
  engineers) the circle bow will be mapped on a point
- The image of the positive imaginary axis is the mirror image
  of the image of the negative imaginary axis

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot
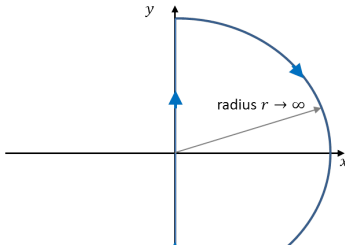
Nyquist plot: physically realizable systems

- Where does the very convenient property of physically realizable systems come from?
- The reason is:
    1. Every physically realizable system is causal
    2. Every causal system has a transfer function with an order of the denominator that is larger than or equal to the order of the numerator
    3. From this shape of the transfer function we can easily show that the circle bow maps onto a point

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot
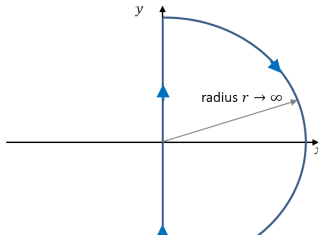
Nyquist plot: physically realizable systems

1. Every physically realizable system is causal
   This is logical: you cannot build a system that knows the future

2. Every causal system has a transfer function with an order of the denominator that is larger than or equal to the order of the numerator
   This can be easily understood in the discrete time case:
   $$a_0 y_n + a_1 y_{n-1} + a_2 y_{n-2} + ... = b_o u_n + b_1 u_{n-1} + b_2 u_{n-2} + ...$$
   Apply the z-transformation:
   $$\frac{Y(z)}{U(z)} = H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + ...}{a_0 + a_1 z^{-1} + a_2 z^{-2} + ...}$$

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist plot: physically realizable systems

1. From this shape of the transfer function we can easily show that the circle bow maps onto a point, in the case of physically realizable systems

    - If the order of the denominator is strictly higher than the order of the numerator:
      If $s \to \infty$, then $P(s)C(s) \to 0$
    - If the order of the denominator is equal to the order of the numerator:
      If $s \to \infty$, then $P(s)C(s) \to c$, a real number

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist plot: symmetry

- The symmetry follows directly from the way $f(c)$ can be evaluated (see previously)
- Since the position of $f(c)$ only depends on the location of the poles and zeros, and those poles and zeros only occur symmetrically round the real axis; the Nyquist plot will be symmetrical round the real axis

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist plot

So we will only have to study the positive (or the negative) imaginary axis!

- The circular part maps onto one point, which is the same as where $j\omega$ maps onto
- The other half of the imaginary axis will give the mirror image of the studied half

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot
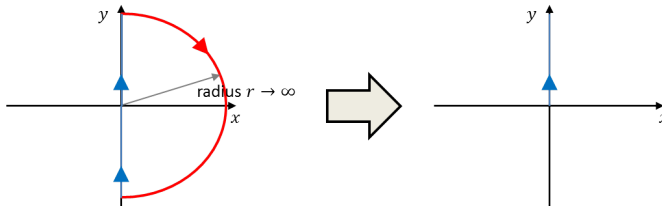
Nyquist plot

- Extracting the number of times $(-1,0)$ is circled now isn't that difficult anymore:
  - You search for the image of $j0^+$
  - You search for the image $j\infty$
  - then you serach for the positive real y for which f(jy)'s imaginary part changes sign
- This information allows you to determine if you encircle $(-1,0)$
- Lets show this with a simple example, but of course you can use software to do this! (e.g. nyquist in Matlab)

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist plot: a simple example

- Lets take the following open loop system:
  $P(s)C(s) = \frac{1}{s^2-2s+2}$

- Fill in
  $s = j\omega \rightarrow \frac{1}{-\omega^2-2j\omega+2} = \frac{1}{-\omega^2+2-2j\omega}\frac{-\omega^2+2+2j\omega}{-\omega^2+2+2j\omega} = \frac{-\omega^2+2+2j\omega}{(-\omega^2+2)^2+4\omega^2}$
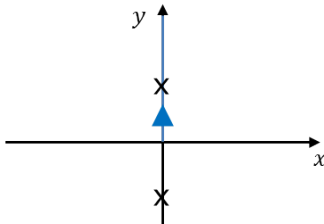
  - $f(j0^+) = \frac{1}{2}$
  - $f(j\infty) = 0$
  - Imaginary part: $\frac{2\omega}{(-\omega^2+2)^2+4\omega^2} = 0 \Rightarrow \omega = 0$

- The real axis get's crossed 2 times; first at $\frac{1}{2}$ ($\omega = 0^+$) an then at 0 ($\omega = \infty$)$\rightarrow$ (−1,0) is not encircled

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist plot: a simple example

- Remember our open loop system: $P(s)C(s) = \frac{1}{s^2 - 2s + 2}$
- Z and P are the number of zeros and poles of $1 + P(s)C(s)$
- The poles of $P(s)C(s)$ and $1 + P(s)C(s)$ are the same ($P = 2$ in our example)
- Since $(-1,0)$ is not encircled: $Z - P = 0$; hence there are 2 zeros in the right half plane
- Remember that the zeros of $1 + P(s)C(s)$ are the poles of the closed loop system and thus, the unity feedback controller is unstable

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist plot: poles on the imaginary axis

- There is one more thing to know about Nyquist plots: how to deal with poles on the imaginary axis
- Why are they a problem?
  - Take for instance the case with one pair of imaginary poles at jc and $-jc$
  - When coming close to jc the argument will remain 0 and the gain will increase to infinity
  - At jc itself the gain will be infinite, but the argument is undetermined, hence we cannot map this point

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Nyquist plot: poles on the imaginary axis

- How do you solve this?
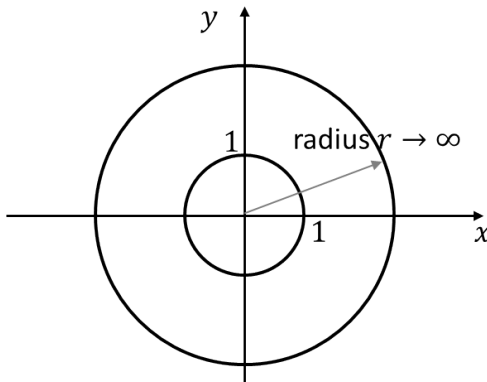   - Instead of going through the poles, we will evade them by an infinitesimally small amount (see figure)
    - That way we do not have the problem of an undetermined mapping at the pole
    - Since we avoid them by an infinitesimally small amount we also know we will not wrongly avoid a pole that lies in the right half plane
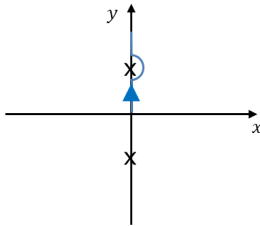
- Now the Nyquist plot will go to infinity as x is approached,

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

The Nyquist criterion in the discrete time case

- The Cauchy argument principle still applies, since $P(z)C(z)$ also has the shape of a rational polynomial
- The contour now will have to encircle the entire complex plane except for the unity circle

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

The Nyquist criterion in the discrete time case

This is not a contour however, but this can be solved with the following trick:
The two horizontal pieces are both infinitely close to the real axis, that way they are identical but with opposite signs. They will cancel each other.

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot
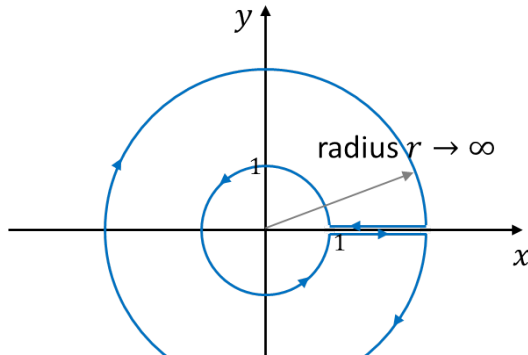
Margins

- The Nyquist plot allows us to determine how stable the system will be
- We know that the stability changes when $1 + P(s)C(s)$ has an imaginary zero (then the system is marginally stable)
- Can we see such a zero in the Nyquist plot of $P(s)C(s)$?
- Of course: the Nyquist plot is the image of the imaginary axis, so if there is a zero on the imaginary axis, the Nyquist plot of $1 + P(s)C(s)$ would go through 0 and the Nyquist plot of $P(s)C(s)$ will go through -1
- Hence the system is marginally stable if the Nyquist plot goes through -1

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Margins

- This explains why we can use the 'distance to -1' as a measure of stability
- We will see two different stability margins, which can be easily read off the Nyquist diagram:
  - The gain margin: the amount of extra gain you can allow before instability occurs (in dB)
  - The phase margin: the amount of phase you can add before instability occurs

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot
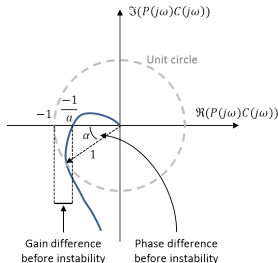
Margins: gain margin

- The gain margin is the amount of extra gain allowed before the system becomes unstable

- Or: how much larger the gain has to be, before the system becomes unstable

- The gain margin is **multiplicative**, so it is the factor with which you have to multiply the gain so the Nyquist plot goes through -1 in the
  w-plane

- It will be expressed in dB; take K a certain factor, then this can be expressed in dB: $20.log_{10}K$ dB

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Margins: gain margin

- Lets look at this the following way:
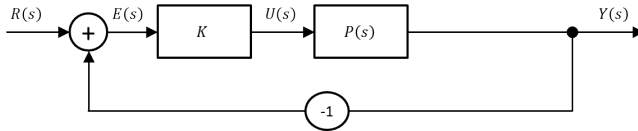
- The stability margin of $P(s)$ with unity feedback is the K for which the system above is marginally stable

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Margins: gain margin

- So $KP(s)$ should equal -1 for an imaginary $s = j\omega$
- This requires $\angle(KP(j\omega)) = \angle P(j\omega)$ to equal $-180°$; this ω_π is called the gain crossover frequency (GCF)
- K then has to be set such that $|KP(j\omega_\pi)| = 1$
- So now you should be able to understand that a large gain can lead to instability; and that this risk only exists when there exists a ω_π for which $\angle P(j\omega) = -180$
- We will illustrate this with a few examples

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Margins: gain margin examples

- Take $P(s) = \frac{1}{s(s+2)}$
- $\angle P(j\omega) = -tan^{-1}(\frac{\omega}{0}) - tan^{-1}(\frac{\omega}{2}) = -90° - tan^{-1}(\frac{\omega}{2})$
- So when is this equal to $-180°$?
- This would require $\omega \to \infty$
  which makes $P(j\omega) = 0$
- So the gain margin is infinite in this case

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Margins: gain margin examples

- We will show this again in the Nyquist plot:
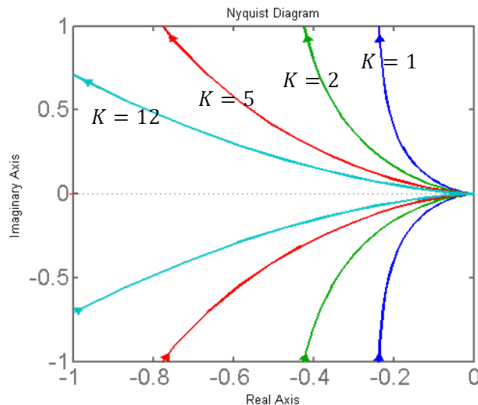
- The only crossover with the real axis is at $\omega = 0$ and that
  doesn't change with increasing K

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
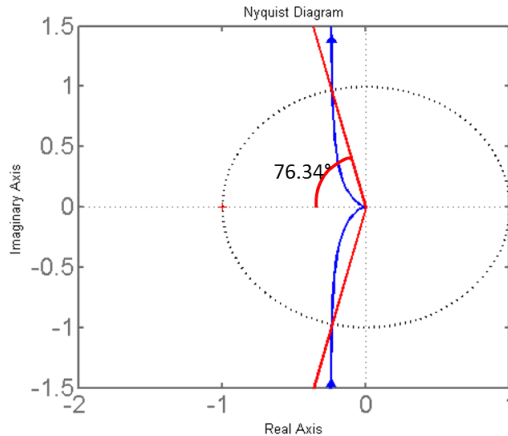Nyquist plot

Margins: phase margin

- The phase margin is how much you can rotate the Nyquist diagram before -1 is crossed
- This can be interpreted as multiplying $P(s)$ with $e^{j\theta}$ until -1 is crossed
- So $P(s)e^{j\theta} = -1$ for an imaginary $s = j\omega$
- This requires $|P(j\omega)e^{j\theta}| = |P(j\omega)|$ to equal 1; this ω_0 is called the Phase Crossover Frequency (PCF)
- $|\theta|$ then has to be set such that $\angle(P(s)e^{j\theta}) = \angle P(s) + \theta = -180°$
- The gain margin is defined as positive, but that doesnt matter, because of the symmetry with respect to the real axis: If a rotation of θ degrees results in a crossing of -1, then a rotation of $-\theta$ does the same

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Margins: phase margin

- Lets take $P(s) = \frac{1}{s(s+2)}$ again
- When is $|P(j\omega)| = 1$?
  $\frac{1}{\omega} \cdot \frac{1}{\sqrt{\omega^2+4}} = 1$, hence $\sqrt{\omega^4 + 4\omega^2} = 1$, or
  $\omega = \sqrt{\sqrt{5} - 2} = 0.486$
- Find θ so $\angle(P(j\omega)e^{j\theta}) = -180°$
- $\theta = -180° + tan^{-1}(\frac{\omega}{0}) + tan^{-1}(\frac{\omega}{2}) = -76.34°$
  The phase margin is $76.34°$

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Margins: phase margin

Well show this graphically with the Nyquist plot:

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
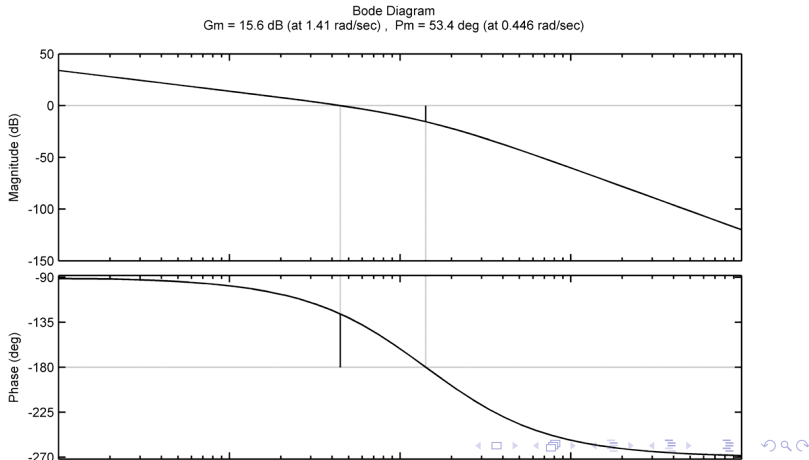Nyquist plot

Margins: what should the margins be?

Phase margin:

- This is more subtle than the gain margin
- If it is too small instability might occur due to practicalities
- If it is too small we get large overshoots and large oscillations that fade away very slowly(which is not something we associate with a good form of stability)
- Sometimes a good value is $60°$, but it is highly case dependent

A good margin does not offer certainty about the stability; whereas a bad phase margin (very large or very small) does give certainty about instability

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Margins using Bode plots

We can also easily derive the gain and phase margins from the Bode plots of $P(s)C(s)$:

Bode Diagram
Gm = 15.6 dB (at 1.41 rad/sec) , Pm = 53.4 deg (at 0.446 rad/sec)

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Margins: design example

- Well design a simple proportional controller for the system $P(s) = \frac{1-s}{(1+s)^2}$
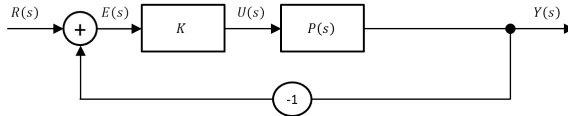
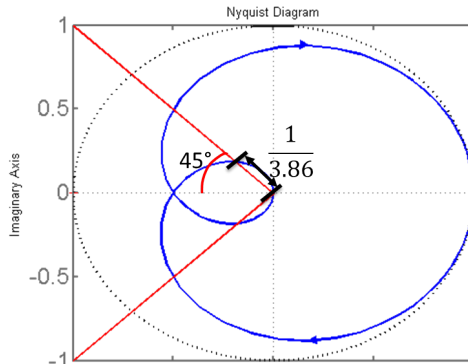- What will K have to be in order to have a phase margin of $45°$?

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Margins: design example

- We find ω by setting $\angle(P(j\omega)e^{\frac{j\pi}{4}}) = -180°$
  - $45° + tan^{-1}(-\omega) - tan^{-1}(\omega) - tan^{-1}(\omega) = -180°$
  - $-3tan^{-1}(\omega) = -225°$
  - $\omega = tan(75°) = 3.73$

- Now we can find K (which doesn't influence the argument) by setting $|KP(j\omega)| = 1$:
  $K\frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+1}^2} = \frac{K}{\sqrt{\omega^2+1}} = \frac{K}{3.86} = 1 \rightarrow K = 3.86 = 5.87\text{dB}$

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Margins: design example

- We can do this graphically with the Nyquist plot
- First determine the point that corresponds to $\theta = 45°$
- Then determine the modulus;
  K is then equal to the inverse of that modulus

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Let's recapitulate

- The Nyquist stability criterion came to existence as a cheap alternative to determine stability of a closed loop system with unity feedback

- It also allows to see the phase margin and the gain margin; which are used to measure how stable the system actually is

- Its relevance today is as a design tool, as shown in the final example

- In the next section we will discuss new kinds of classical control components

- Bode plots make up a similar tool, yet the Nyquist stability criterion is (slightly) more generally applicable

Introduction
Stability
Cauchy's argument principle
Nyquist stability criterion

Nyquist stability criterion
Nyquist plot

Interesting links

- About the Nyquist argument principle:
  - https://www.youtube.com/watch?v=sof3meN96MA&index=3&list=PLEJyHgf7y0ie3nwSzMDVzAdxf4dK9m89a
  - https://www.youtube.com/watch?v=tsgOstfoNhk
- About margins:
  - https://www.youtube.com/watch?v=Hw_hrxsX4_M
  - https://www.youtube.com/watch?v=BTNZ8SRs7Y8
  - https://www.youtube.com/watch?v=V1mfy7VhJNQ