The design of lead and lag compensator's by the root approach

September 4, 2015

Outline

Definitions

The main objective of this chapter: **design** and **compensation** of single-input-sigle-output, linear, time-invariant control systems.

- Compensation: the modification of the system dynamics to satisfy the given specifications.
- Specifications (transient response and steady-state requirements): given before the design.
- Design by root-locus method: making a new root locus by adding poles and zeros to the system's open-loop transfer function.
- Compensator: another system inserted in parallel or in cascade with the system for the purpose of satisfying the specifications of the original system (e.g. lead, lag, lag-lead compensator's or PID controllers).

Compensators

When a sinusoidal input is applied to the input of a network. We got a:

- lead network: if the steady-state output has a phase lead.
- lag network: if the steady-state output has a phase lag.
- lag-lead network: if we have phase lag and phase lead in the output but in different frequency regions (lag when the input has low frequency and lead in high frequency).

$\rightarrow$ The amount of lag/lead is a function of the frequency.

A compensator with characteristic of a lead network, lag network, or lag-lead network is called a lead compensator, lag compensator, or lag-lead compensator.

Remark: with trial and error we find the optimal compensator

Outline

1. Introduction
2. General design
3. Lead compensation
4. Lag compensation
5. Lag-lead compensator's
6. Parallel compensation

Controllers

Sketching the problem:

1. We got a plant that not fulfil all the specifications (and we cannot change the parameters of the original system).
2. We have to change other parameters such that the system will achieve all the specifications.
3. These other parameters can be changed by changing the root locus of the closed-loop system.
4. So we look for a compensator that changes the root locus of the overall system such that the overall system achieve the specifications.
5. We let the original system interact (cascade or parallel) with the compensator.

Remark: we only discuss continuous time systems.

Root locus approach

Looking at the transfer function $G(s)$, we can see that the closed-loop poles depend on the gain K. We can now plot the locus of all possible roots of the characteristic equation:

$$1 + KG(s) = 0$$

as K varies from 0 to ∞. This results in a graph which can help us in selecting the best value of K.

Furthermore, by studying the effects of additional poles and zeros, we can determine the consequences of additional dynamics in the loop. We can also extend this technique to examine the effect of other plant-parameter changes in order to achieve the best overall control design.

In **general**: we want to have the root loci of the system on the

Addition of poles/zeros

Addition of poles:

- Pulls the root locus to the right.
- Lowers the system's relative stability.
- Slows down the settling of the response.

Addition of zeros:

- Pulls the root locus to the left.
- Increases the system's relative stability.
- Speeds up the settling of the response.
- Speeds up the transient response.
- Increases the anticipation of the system.

Outline

Lead compensators

There are 3 ways to make a lead compensator:

1. Electronic networks using operational amplifiers.
2. Electrical RC networks.
3. Mechanical spring-dashpot systems.

When to use the root locus approach for lead compensator's:
$\rightarrow$ When the specifications are given in terms of time-domain quantities. Examples:

- damping ratio
- rise time
- settling time
- maximum overshoot
- undamped natural frequency

Lead compensators

Steps to make a lead compensator (in cascade with the original system):

1. Determine the locations of the desired dominant poles (from the specifications).
2. Draw the root locus of the uncompensated system. If we change the gain, then the poles will change. If we can achieve the locations of the desired poles on this way, then there is no need for a lead compensator. Else, if we cannot achieve the desired locations by adjusting the gain, then we have to make a lead compensator.
3. Assume the lead compensator with this transfer function: $C(s) = K_c\alpha\frac{Ts+1}{\alpha Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}$ with $0 < \alpha < 1$.

Lead compensators

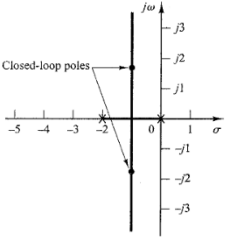
4. Determining α and T:
   α and T are determined by the angle ϕ. This is the angle $\angle(C(s))$ and must be equal to the difference between the angle of the desired pole and the angle of the original transfer function.
   Result: the closed-loop pole has the desired angle (angle of the desired pole). (And by determine T and α we determine the poles and zeros of the compensator.)
5. Determine K_c (=the open-loop gain) from the magnitude specifications.
   Result: the closed-loop pole has the desired magnitude.

Remark: if there is some freedom about the parameter α, then take α as high as possible.

Example 1: Lead compensators

Given:

- Feed-forward transfer function: $G(s) = \frac{4}{s(s+2)}$
- Root locus plot of the uncompensated open-loop system:



Specifications:

- Undamped frequency: $w_n = 4$ rad/s
- Don't change the damping ratio

Example 1: Lead compensators

Solution:

1. Determine the desired poles:
   - The closed-loop transfer function: $\frac{C(s)}{R(s)} = \frac{4}{s^2+2s+4}$ with poles $s = -1 \pm \sqrt{3}j$
   - Damping ratio: $\zeta = 0.5$
   - Undamped natural frequency: 2 rad/s
   - Static velocity constant: $K_v = 2s^{-1}$

$$\begin{cases} \zeta \text{ must be the same} \rightarrow \text{let the angle of the poles constant} \\ \text{Undamped frequency} = 4\text{rad/s} \rightarrow \text{magnitude of the poles} \end{cases}$$

The desired poles are: $s = -2 \pm j2\sqrt{3}$

Note: in some cases, the desired poles can be obtained by adjustment of the gain.

Example 1: Lead compensators

Solution:

2. Find the sum of the angles at the desired location of one of the dominant closed-loop poles with the open-loop poles and zeros of the original system, and determine the necessary angle ϕ to be added so that the total sum of the angles is equal to $\pm 180^\circ(2k+1)$. The lead compensator must contribute this angle ϕ.
   Calculation of ϕ:

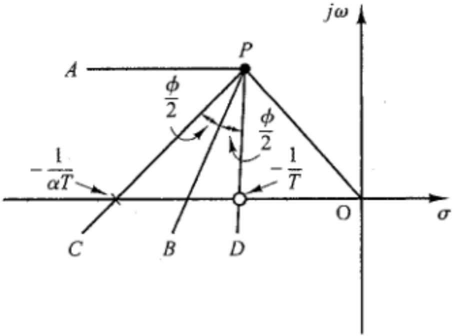
$$\angle\left(\frac{4}{s(s+2)}\right)\Big|_{s=-2+j2\sqrt{3}} = -210^\circ = -180^\circ - 30^\circ$$

   And so we find $\phi = 30^\circ$.

Example 1: Lead compensators

Solution:

4. Determine the pole and zero of the compensator (there are many possibilities): we use an algorithm to find the the pole and zero and to get an α as big as possible:



Example 1: Lead compensators

Solution: Algorithm (figure previous slide):

1. Draw a horizontal line trough the desired dominant closed-loop pole P $\rightarrow$ PA
2. Connect the origin with P $\rightarrow$ PO
3. Bisect the two previous lines $\rightarrow$ PB
4. Draw two lines that make an angle $\phi/2$ with the line PB $\rightarrow$ PC, PD
5. The intersections of the negative real axis and the lines PC and PD are the zero and pole of the compensator.

Example 1: Lead compensators

Solution

4. Transfer function of the lead compensator:

$$C(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} \text{ with } 0 < \alpha < 1$$

(there are many solutions for T and α).
From the algorithm we get:
zero: -2.9 and pole: -5.4.

$$T = \frac{1}{2.9} = 0.345, \ \alpha T = \frac{1}{5.4} = 0.185$$

So, we find α: $\alpha = \frac{1}{5.4T} = 0.537$.

Example 1: Lead compensators

Solution:

5. Determine K_c from the magnitude condition: the open-loop transfer function of the compensated system becomes: $C(s)G(s) = K_c\frac{s+2.9}{s+5.4}\frac{4}{s(s+2)} = \frac{K(s+2.9)}{s(s+2)(s+5.4)}$ with K=$4K_c$.

   The magnitude condition:

$$\left|\frac{K(s+2.9)}{s(s+2)(s+5.4)}\right|_{s=-2+j2\sqrt{3}} = 1 \Rightarrow K = 18.7 \Rightarrow K_c = \frac{K}{4} = 4.68$$

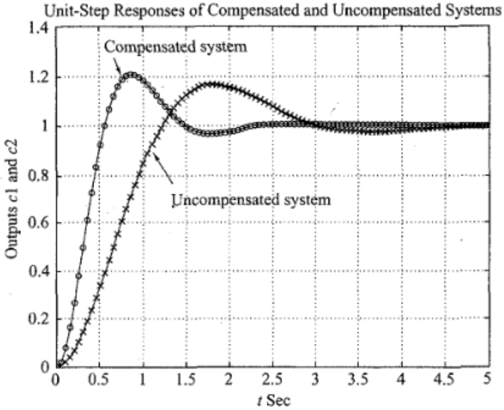
   And we get the transfer function of the lead compensator:

$$C(s) = 4.68\frac{s+2.9}{s+5.4}$$

.

Example 1: Lead compensators

Solution:



Outline

Lag compensators

Problem: A system that satisfies the desired transient response but don't satisfy the steady-state.
$\rightarrow$ We have to compensate the system in such a way that it also satisfy the steady-state specifications (we have to make a lag compensator in cascade with the original system).

Concrete:

- Changing the steady-state by increasing the open-loop gain.
- Untouch the transient response by untouch the root locus in the neighbourhood of dominant closed-loop poles:
  a) poles and zeros close to each other;
  b) poles and zeros close to the origin;
  c) angle of the compensator must be small, $\angle(C(s)) < 5^\circ$.

Lag compensators

$C(s) = K_c \beta \frac{Ts+1}{\beta Ts+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$ with $\beta > 1$

- Choose poles and zeros close together
- Choose K_c close to 1 (if K_c is exact 1, the transient response won't change)
- Take β as large as possible
- Take T as large as possible

Remark: by the choice of β and T, we have to take account of the reality (the physical realisation), so there is a limitation on the values.

The **downside** of the lag compensator: the settling time will increase because the pole and zero of the closed-loop system are close to the origin.

Steps to make a lag compensator

1. Draw the root locus of the uncompensated open loop system and search the dominant closed loop poles on the root-locus (from the specifications).
2. The lag compensator is $C(s) = K_c\beta\frac{Ts+1}{\beta Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$.
3. Calculate the static error constant K_c.
4. Calculate the new static error constant K_c' if we achieve all the specifications.
5. From this difference between K_c and K_c', we make the lag compensator. Now, we can see the pole and the zero of the compensator.

Remark: we assume that the lag compensator achieved the transient response specifications. If not: make a lead-lag-compensator.

Steps to make a lag compensator

6. Draw the root locus of the compensated closed-loop system. Determine the locations of the dominant closed loop poles on the root-locus.
7. Adjust K_c from the magnitude conditions such that the closed-loop poles lie at the desired location. $(K_c \approx 1)$

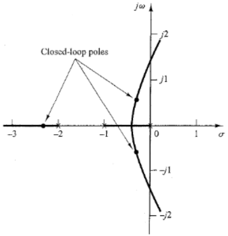
Some notes:

- The ratio of the value of gain required in the specifications and the gain found in the uncompensated system is equal to the ratio between the distance of the zero from the origin and that of the pole from the origin.
- If the angle of the lag compensator is very small, then the original root loci and the new root loci are almost equal.
- Sometimes can be used both a lead or a lag compensator.

Example 2: Lag compensators

Given:

- Feed-forward transfer function: $G(s) = \frac{1.06}{s(s+1)(s+2)}$
- Root locus plot of the uncompensated open-loop system:



Specifications:

- Increase of the static velocity error constant K_v to $5s^{-1}$
- Don't change the dominant closed-loop poles

Example 2: Lag compensators

Solution:

1. General calculations:
   - The transfer function of the closed-loop system:

$$\frac{Y(s)}{R(s)} = \frac{1.06}{s(s+1)(s+2)+1.06}$$

   with poles: $s = -0.3307 \pm j0.5864$
   - Damping ratio: $\zeta = 0.491$
   - Undamped natural frequency: 0.673 rad/s
   - Static velocity error constant: $K_v = 0.53s^{-1}$

Example 2: Lag compensators

Solution:

2. We assume a lag compensator C(s):

$$C(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{T\beta}}$$

3. To achieve the specifications, we choose:
   - $\beta = 10$
   - zero: s=-0.05
   - pole: s=-0.005
4. C(s) becomes:

$$C(s) = K_c \frac{s + 0.05}{s + 0.005}$$

Example 2: Lag compensators

Solution:

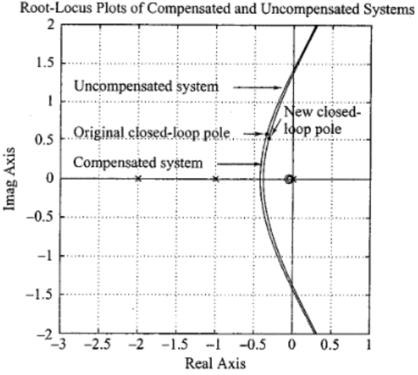
5. Determine K_c:
   The open-loop transfer function of the compensated system becomes:
   $C(s)G(s) = K_c \frac{s+0.05}{s+0.005} \frac{1.06}{s(s+1)(s+2)} \Rightarrow K = 1.06K_c$
   Holding the damping ratio the same, we find on the root locus plot (next slide) the poles: $s_{1,2} = -0.31 \pm 0.55j$.
   The open-loop gain is:

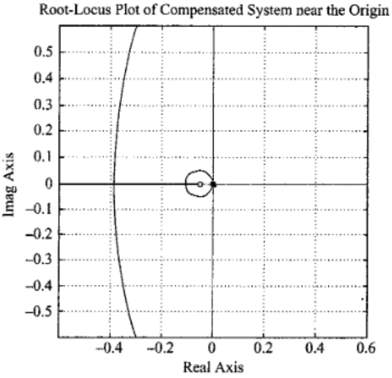
$$K = \left| \frac{s(s+0.005)(s+1)(s+2)}{s+0.05} \right|_{s=-0.31+0.55j} = 1.0235$$

$$\Rightarrow K_c = \frac{K}{1.06} = 0.9656$$

Example 2: Lag compensators

Solution:





Example 2: Lag compensators

Solution:

The transfer function of the lag compensator:

$$C(s) = 0.9656\frac{s + 0.05}{s + 0.005}$$

The transfer function of the compensated system:

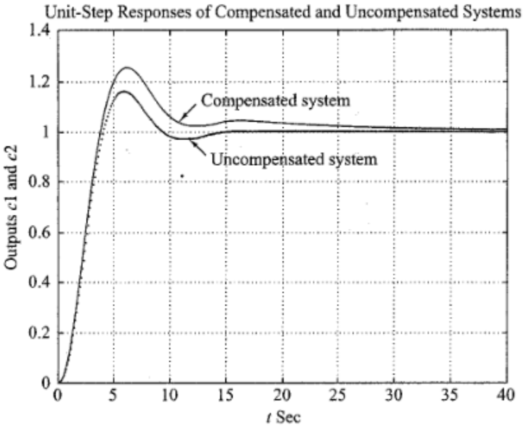
$$G(s) = \frac{1.0235(s + 0.05)}{s(s + 0.005)(s + 1)(s + 2)}$$

The static velocity error constant:

$$K_v = \lim_{s \to 0} sG(s) = 5.12s^{-1}$$

Example 2: Lag compensators

Solution:



Example 2: Lag compensators

Conclusions:

- A pole and zero $\begin{cases} \text{close to each other} \\ \text{close to the origin} \end{cases}$ don't change the root-locus.
- The compensator increases the order of the system from 3 to 4 by adding a pole.
- If a pole is further away from the jw axis, the pole is less dominant and it will have less effect on the transient response.
- If the undamped natural frequency is low, then:
  - the speed of the transient response will be less (the response will take a longer time to settle down);
  - the overshoot of the step response will increase.

Resume

Effect:

- Lead compensator's increase the gain by high frequencies and so increases the band with.
- Lag comp. reduce bandwidth.

Advantages:

- Lead comp. increase the dynamical response.
- Lag comp. reduce high frequent noise.

Disadvantages:

- Lead comp. require extra gain and are sensitive to high frequencies (because the big bandwidth).
- Lag comp. worsens the transient response.

Usage:

- Lead comp. for quick transient response.
- Lag comp. for specified steady-state error.

Outline

Lag-lead compensators

General:

Lead compensator's:

- speeds up the response;
- increase stability of the system.

Lag compensator's:

- improves steady-state accuracy;
- reduces the speed of the response.

Lag-lead compensator:

- If both transient response and steady-state must be improved.
- When 1 global component is more economical then both a lead and a lag component.

The lag-lead compensator has the advantages of both compensator's. It has 2 poles and zeros, so the system has order 2.

Lag-lead compensators

The lag-lead compensator C(s) has this transfer function:

$$C(s) = K_c \frac{\beta(T_1 s + 1)(T_2 s + 1)}{\gamma(\frac{T_1}{\gamma}s + 1)(\beta T_2 s + 1)} = K_c(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}})(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}})$$

with $\beta > 1$ and $\gamma > 1$.

Notes:

- The value βT_2 may not be to large, it must be physical realizable.
- There are two cases of this type compensator: $\gamma \neq \beta$ and $\gamma = \beta$.

Lag-lead compensators

Case 1:$\gamma \neq \beta$, the design process is a combination of the design of the lead compensator and that of the lag compensator:

1. Determine the location of the closed-loop poles (from the specifications).
2. The lead part of the compensator must contribute the angle ϕ: $\phi = \angle(\text{desired pole}) - \angle(\text{uncompensated open-loop system})$.
3. Calculate the parameters:
   - Take T_2 as large as possible.
   - Determine T_1 and γ such that: $\angle(\frac{s_1+\frac{1}{T_1}}{s_1+\frac{\gamma}{T_1}}) = \phi$
   - Determine K_c from the condition: $|K_c \frac{s_1+\frac{1}{T_1}}{s_1+\frac{\gamma}{T_1}} G(s)| = 1$ with G(s) the open loop transfer function.

Lag-lead compensators

Case 1:

4. Determine β if K_v is given:

$$K_v = \lim_{s \to 0} sC(s)G(s) = \lim_{s \to 0} sK_c\left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right)\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right)G(s)$$

$$= \lim_{s \to 0} sK_c G(s)\frac{\beta}{\gamma}$$

5. Determine T_2 from:

$$\left|\frac{s_1+\frac{1}{T_2}}{s_1+\frac{1}{\beta T_2}}\right| \approx 1 \text{ and } -5^\circ < \angle\left(\frac{s+\frac{1}{T_2}}{s_1+\frac{1}{\beta T_2}}\right) < 0^\circ$$

Lag-lead compensators

Case 2: $\gamma = \beta$:

1. Determine the location of the closed-loop poles (from the specifications)
2. If K_v is specified, determine K_c from:

$$K_v = \lim_{s \to 0} sC(s)G(s) = \lim_{s \to 0} sK_c\left(\frac{s + \frac{1}{T_1}}{s + \frac{\beta}{T_1}}\right)\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}}\right)G(s)$$

$$= \lim_{s \to 0} sK_c G(s)$$

3. Determine the deficiency ϕ, that must be contributed by the lead part of the lag-lead compensator, such that we get the dominant closed-loop poles on the desired location.

Lag-lead compensators

Case 2: $\gamma = \beta$:

4. Determine T_1 andβ from:

$$|K_c(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}})G(s_1)| = 1 \text{ and } \angle(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}) = \phi$$

5. Determine T_2 from:

$$|\frac{s_1+\frac{1}{T_2}}{s_1+\frac{1}{\beta T_2}}| \approx 1 \text{ and } -5^\circ < \angle(\frac{s+\frac{1}{T_2}}{s_1+\frac{1}{\beta T_2}}) < 0^\circ$$

Example 3: Lag-lead compensators

Given:

- Feed-forward transfer function:

$$G(s) = \frac{4}{s(s+0.5)}$$

Specifications:

- The damping ratio of the dominant closed-loop poles must be: $\zeta = 0.5$
- Increase the undamped natural frequency to 5 rad/s
- Increase the static velocity error constant K_v to 80 s^{-1}

Example 3: Lag-lead compensators

Solution:

1. General calculations:
   - The closed loop poles: $s = -0.2500 \pm 1.9843j$
   - Damping ratio: $\zeta = 0.125$
   - Undamped natural frequency: 2 rad/s
   - Static velocity error constant: $K_v = 8s^{-1}$
2. We use a lag-lead compensator with transfer function:

$$C(s) = K_c\left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}}\right)\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}}\right) \text{ with } \gamma > 1, \beta > 1, \gamma \neq \beta$$

Example 3: Lag-lead compensators

Solution:

3. The desired locations of the closed loop poles (from specifications) are: $s = -2.5 \pm 4.33j$
4. Determine the angle of deficiency ϕ:

$$\angle\left(\frac{4}{s(s+0.5)}\right)\Big|_{s=-2.5+4.33j} = -235^\circ = -180^\circ - 55^\circ \Rightarrow \phi = 55^\circ$$

This angle must be contributed by the lead part of the compensator, such that the root locus passes trough the desired closed-loop poles.

Example 3: Lag-lead compensators

Solution:

5. Determine the zero and pole of the compensator (many possibilities):
   We take a zero at -0.5 (this will cancel the pole -0.5 of the plant) and a pole at -5.021 (such that the angle ϕ is contributed). The lead part becomes:

$$K_c \frac{s+0.5}{s+5.021}$$

   and we find:
   - $T_1 = 2$
   - $\gamma = 5.021/0.5 = 10.04$
   - From the magnitude condition: $|K_c \frac{4(s+0.5)}{(s+5.021)s(s+0.5)}|_{s=-2.5+4.33j} = 1 \Rightarrow K_c = 6.26$

Example 3: Lag-lead compensators

Solution:

6. Determine the lag part of the compensator:
   Determine β:

$$K_v = \lim_{s\to 0} sC(s)G(s) = \lim_{s\to 0} sK_c\frac{\beta}{\gamma}G(s) = 4.988\beta = 80 \Rightarrow \beta = 16.04$$

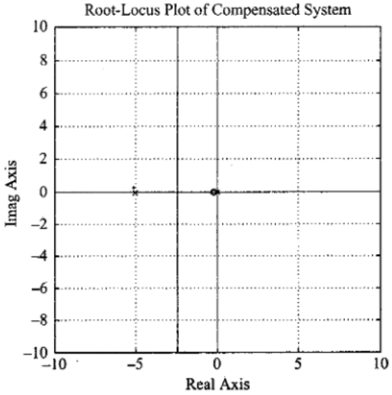
Determine T_2: T_2 must satisfy:

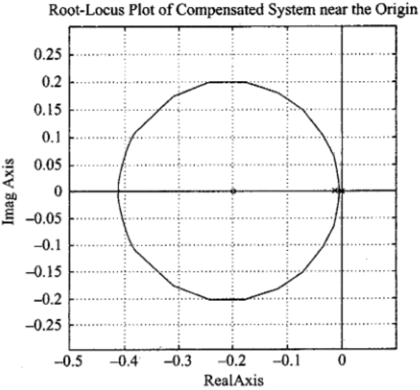
$$\left|\frac{s+\frac{1}{T_2}}{s+\frac{1}{16.04T_2}}\right|_{s=s_1} \approx 1 \text{ and } -5^\circ < \angle\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{16.04T_2}}\right)\Big|_{s=s_1} < 0^\circ$$

$\Rightarrow T_2 = 5$ with $(s_1 = -2.5 + 4.33j)$

Example 3: Lag-lead compensators

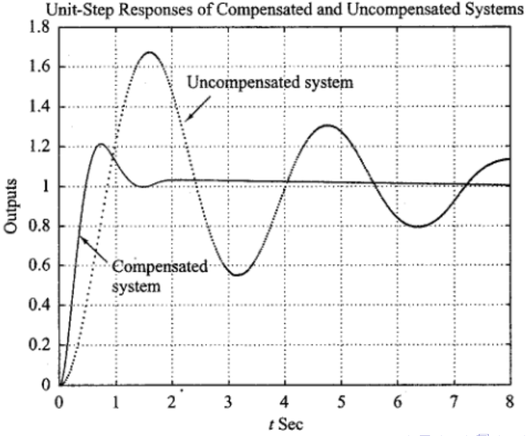
Results:





Example 3: Lag-lead compensators

Results:



Example 4: Lag-lead compensators

Given:

- Feed-forward transfer function:

$$G(s) = \frac{4}{s(s+0.5)}$$

Specifications:

- The damping ratio of the dominant closed-loop poles must be: $\zeta = 0.5$
- Increase the undamped natural frequency to 5 rad/s
- Increase the static velocity error constant K_v to 80 s^{-1}

Example 4: Lag-lead compensators

Solution:

1. General calculations:
   - The closed loop poles: $s = -0.2500 \pm 1.9843j$
   - Damping ratio: $\zeta = 0.125$
   - Undamped natural frequency: 2 rad/s
   - Static velocity error constant: $K_v = 8s^{-1}$
2. We use a lag-lead compensator with transfer function $(\beta = \gamma)$:

$$C(s) = K_c\left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}}\right)\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}}\right) \text{ with } \beta > 1$$

3. The desired locations of the dominant closed-loop poles: $s = -2.5 \pm 4.33j$

Example 4: Lag-lead compensators

Solution:

4. The compensated open-loop transfer function:

$$C(s)G(s) = K_c\left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right)\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right)\frac{4}{s(s+0.5)}$$

5. Calculation of the parameters:
   K_v must be 80 s^{-1}:

$$K_v = \lim_{s\to 0} sG(s)C(s) = \lim_{s\to 0} K_c\frac{4}{0.5} = 8K_c = 80 \Rightarrow K_c = 10$$

Example 4: Lag-lead compensators

Solution:

5. From

$$\left|\frac{s+\frac{1}{T_1}}{s+\frac{\beta}{T_1}}\right|\left|\frac{40}{s(s+0.5)}\right|_{s=s_1} = 1 \text{ and } \angle\left(\frac{s+\frac{1}{T_1}}{s+\frac{\beta}{T_1}}\right)\Big|_{s=s_1} = 55^\circ (= \phi)$$

we get: $T_1 = 0.420$ and $\beta = 3.503$.
For T_2 we may choose 10.

Results:
The transfer function of the compensator:

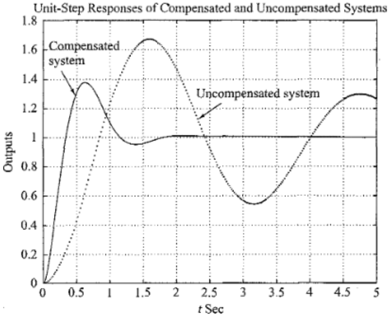
$$C(s) = 10\frac{(s+2.38)(s+0.1)}{(s+8.34)(s+0.0285)}$$

Example 4: Lag-lead compensators

Results:

The transfer function of the compensated open-loop system:

$$C(s)G(s) = \frac{40(s+2.38)(s+0.1)}{(s+8.34)(s+0.0285)s(s+0.5)}$$

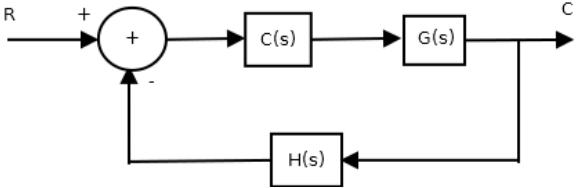


Outline

Serial compensation

$\rightarrow$ Up to now, we have only discussed compensator's in cascade with the system.

Derivation of the characteristic equation:
Serial:

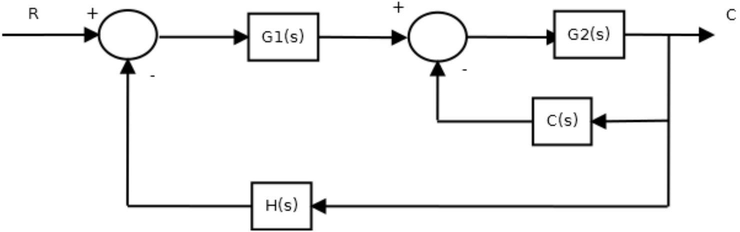


The closed-loop transfer function: $\frac{C}{R} = \frac{GC}{1+GCH}$.

The characteristic equation is: $1 + GCH = 0$

Parallel compensation

Derivation of the characteristic equation:
Parallel:



The closed-loop transfer function: $\frac{C}{R} = \frac{G_1 G_2}{1+CG_2+G_1 G_2 H}$.

The characteristic equation is: $1 + G_1 G_2 H + G_2 C = 0$

Parallel compensation

Derivation of the characteristic equation:

Parallel:

If we divide the characteristic equation by $1 + G_1 G_2 H$ we obtain:

$$1 + \frac{CG_2}{1 + G_1 G_2 H} = 0$$

Now, we define: $G_f = \frac{G_2}{1+G_1 G_2 H}$.

So, we become a characteristic equation: $1 + CG_f = 0$

$\rightarrow$ This has the same form as the serial characteristic equation.

So, the same methods can be applied.

Notes:

- A parallel compensator system is called a velocity feedback system.
- The compensator can be saw as a gain element.