System Modeling, part 2

Marc Claesen

February 18, 2015

1. Nonlinear systems & linearization

2. System identification (cont)
   - Grey box identification
   - Black box identification

Outline

1. Nonlinear systems & linearization
2. System identification (cont)
   - Grey box identification
   - Black box identification

Nonlinear systems

In this course we focus on the linear state-space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \qquad \begin{cases} x[k+1] & = Ax[k] + Bu[k], \\ \quad y[k] & = Cx[k] + Du[k]. \end{cases}$$

Nonlinear systems

In this course we focus on the linear state-space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \qquad \begin{cases} x[k+1] & = Ax[k] + Bu[k], \\ \quad y[k] & = Cx[k] + Du[k]. \end{cases}$$

Most real life systems involve nonlinearity:

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big), \\ y(t) = g\big(x(t), u(t)\big), \end{cases}$$

Nonlinear systems

In this course we focus on the linear state-space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \qquad \begin{cases} x[k+1] & = Ax[k] + Bu[k], \\ y[k] & = Cx[k] + Du[k]. \end{cases}$$

Most real life systems involve nonlinearity:

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big), \\ y(t) = g\big(x(t), u(t)\big), \end{cases}$$

where f and/or g contain some nonlinearity, such as:

Nonlinear systems

In this course we focus on the linear state-space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \qquad \begin{cases} x[k+1] & = Ax[k] + Bu[k], \\ \quad y[k] & = Cx[k] + Du[k]. \end{cases}$$

Most real life systems involve nonlinearity:

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big), \\ y(t) = g\big(x(t), u(t)\big), \end{cases}$$

where f and/or g contain some nonlinearity, such as:

- *powers*: e.g. $\dot{x}(t) = Ax(t) + Bu(t) + \gamma u(t)^2$,

Nonlinear systems

In this course we focus on the linear state-space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \qquad \begin{cases} x[k+1] & = Ax[k] + Bu[k], \\ \quad y[k] & = Cx[k] + Du[k]. \end{cases}$$

Most real life systems involve nonlinearity:

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big), \\ y(t) = g\big(x(t), u(t)\big), \end{cases}$$

where f and/or g contain some nonlinearity, such as:

- *powers*: e.g. $\dot{x}(t) = Ax(t) + Bu(t) + \gamma u(t)^2$,
- *interactions*: e.g. $\dot{x}(t) = Ax(t) + Bu(t) + \gamma x(t)u(t)$,

Nonlinear systems

In this course we focus on the linear state-space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \qquad \begin{cases} x[k+1] & = Ax[k] + Bu[k], \\ \quad y[k] & = Cx[k] + Du[k]. \end{cases}$$

Most real life systems involve nonlinearity:

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big), \\ y(t) = g\big(x(t), u(t)\big), \end{cases}$$

where f and/or g contain some nonlinearity, such as:

- *powers*: e.g. $\dot{x}(t) = Ax(t) + Bu(t) + \gamma u(t)^2$,
- *interactions*: e.g. $\dot{x}(t) = Ax(t) + Bu(t) + \gamma x(t)u(t)$,
- *clipping*: e.g. $\alpha \leq x(t) \leq \beta$.

Linearization around equilibrium point

Linearization around equilibrium point

Nonlinear systems have (several) equilibrium points x_e, u_e, y_e:

$$\begin{cases} \dot{x}_e = f\left(x_e, u_e\right) = 0, \\ y_e = g\left(x_e, u_e\right). \end{cases}$$

Linearization around equilibrium point

Nonlinear systems have (several) equilibrium points x_e, u_e, y_e:

$$\begin{cases} \dot{x}_e = f\left(x_e, u_e\right) = 0, \\ y_e = g\left(x_e, u_e\right). \end{cases}$$

Linearizing in the region of (x_e, u_e, y_e):

$$x = x_e + \Delta x, \quad u = u_e + \Delta u, \quad y = y_e + \Delta y,$$

Linearization around equilibrium point

Nonlinear systems have (several) equilibrium points x_e, u_e, y_e:

$$\begin{cases} \dot{x}_e = f\left(x_e, u_e\right) = 0, \\ y_e = g\left(x_e, u_e\right). \end{cases}$$

Linearizing in the region of (x_e, u_e, y_e):

$$x = x_e + \Delta x, \quad u = u_e + \Delta u, \quad y = y_e + \Delta y,$$

with Δx, Δu and Δy *sufficiently* small.

Linearization around equilibrium point

Nonlinear systems have (several) equilibrium points x_e, u_e, y_e:

$$\begin{cases} \dot{x}_e = f\left(x_e, u_e\right) = 0, \\ y_e = g\left(x_e, u_e\right). \end{cases}$$

Linearizing in the region of (x_e, u_e, y_e):

$$x = x_e + \Delta x, \quad u = u_e + \Delta u, \quad y = y_e + \Delta y,$$

with Δx, Δu and Δy *sufficiently* small.

Linearizing is done via first order Taylor expansions:

$$\begin{cases} \frac{dx}{dt} = \frac{d\Delta x}{dt} = f(x, u) = f(x_e + \Delta x, u_e + \Delta u), \\ y_e + \Delta y = g(x, u) = g(x_e + \Delta x, u_e + \Delta u). \end{cases}$$

Example: decalcification plant

Used to reduce concentration of calcium hydroxide in water:

- chemical reaction: $Ca(OH)_2 + CO_2 \rightarrow CaCO_3 + H_2O$
- reaction speed: $r = c[Ca(OH)_2][CO_2]$
- rate of change of concentration:

$$\frac{d[Ca(OH)_2]}{dt} = \frac{k}{V} - \frac{r}{V},$$
$$\frac{d[CO_2]}{dt} = \frac{u}{V} - \frac{r}{V},$$

  with inflow rates k and u in mol/s and tank volume V in L.
- input u: inflow of CO_2, output: $[Ca(OH)_2]$

Nonlinear model and equilibrium point

Nonlinear model for the given reactor:

Nonlinear model and equilibrium point

Nonlinear model for the given reactor:

$$\frac{d[Ca(OH)_2]}{dt} = \frac{k}{V} - \frac{c}{V}[Ca(OH)_2][CO_2],$$

$$\frac{d[CO_2]}{dt} = \frac{u}{V} - \frac{c}{V}[Ca(OH)_2][CO_2],$$

$$y = [Ca(OH)_2],$$

with two state variables: $x_1 = [Ca(OH)_2]$ and $x_2 = [CO_2]$.

Nonlinear model and equilibrium point

Nonlinear model for the given reactor:

$$\frac{d[Ca(OH)_2]}{dt} = \frac{k}{V} - \frac{c}{V}[Ca(OH)_2][CO_2],$$
$$\frac{d[CO_2]}{dt} = \frac{u}{V} - \frac{c}{V}[Ca(OH)_2][CO_2],$$
$$y = [Ca(OH)_2],$$

with two state variables: $x_1 = [Ca(OH)_2]$ and $x_2 = [CO_2]$.

The equilibrium point $(k_{eq}, u_{eq}, x_{1,eq}, x_{2,eq}, y_{eq})$ of this system is:

Nonlinear model and equilibrium point

Nonlinear model for the given reactor:

$$\frac{d[Ca(OH)_2]}{dt} = \frac{k}{V} - \frac{c}{V}[Ca(OH)_2][CO_2],$$
$$\frac{d[CO_2]}{dt} = \frac{u}{V} - \frac{c}{V}[Ca(OH)_2][CO_2],$$
$$y = [Ca(OH)_2],$$

with two state variables: $x_1 = [Ca(OH)_2]$ and $x_2 = [CO_2]$.

The equilibrium point $(k_{eq}, u_{eq}, x_{1,eq}, x_{2,eq}, y_{eq})$ of this system is:

$$\frac{k_{eq}}{V} - \frac{c}{V}[Ca(OH)_2]_{eq}[CO_2]_{eq} = 0,$$
$$\frac{u_{eq}}{V} - \frac{c}{V}[Ca(OH)_2]_{eq}[CO_2]_{eq} = 0.$$

Linearization of the decalcification plant

For small deviations near the equilibrium:

$$\begin{aligned}
\frac{d\Delta x_1}{dt} &= -\frac{c}{V}[CO_2]_{eq}\Delta x_1 - \frac{c}{V}[Ca(OH)_2]_{eq}\Delta x_2, \\
\frac{d\Delta x_2}{dt} &= -\frac{c}{V}[CO_2]_{eq}\Delta x_1 - \frac{c}{V}[Ca(OH)_2]_{eq}\Delta x_2 + \frac{\Delta u}{V}, \\
\Delta y &= \Delta x_1.
\end{aligned}$$

Linearization of the decalcification plant

For small deviations near the equilibrium:

$$\frac{d\Delta x_1}{dt} = -\frac{c}{V}[CO_2]_{eq}\Delta x_1 - \frac{c}{V}[Ca(OH)_2]_{eq}\Delta x_2,$$
$$\frac{d\Delta x_2}{dt} = -\frac{c}{V}[CO_2]_{eq}\Delta x_1 - \frac{c}{V}[Ca(OH)_2]_{eq}\Delta x_2 + \frac{\Delta u}{V},$$
$$\Delta y = \Delta x_1.$$

The resulting linear state-space model is $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$:

Linearization of the decalcification plant

For small deviations near the equilibrium:

$$\begin{aligned}
\frac{d\Delta x_1}{dt} &= -\frac{c}{V}[CO_2]_{eq}\Delta x_1 - \frac{c}{V}[Ca(OH)_2]_{eq}\Delta x_2,\\
\frac{d\Delta x_2}{dt} &= -\frac{c}{V}[CO_2]_{eq}\Delta x_1 - \frac{c}{V}[Ca(OH)_2]_{eq}\Delta x_2 + \frac{\Delta u}{V},\\
\Delta y &= \Delta x_1.
\end{aligned}$$

The resulting linear state-space model is $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$:

$$\begin{bmatrix} \frac{d[Ca(OH)_2]}{dt} \\ \frac{d[CO_2]}{dt} \end{bmatrix} = -\begin{bmatrix} \frac{c}{V}[CO_2]_{eq} & \frac{c}{V}[Ca(OH)_2]_{eq} \\ \frac{c}{V}[CO_2]_{eq} & \frac{c}{V}[Ca(OH)_2]_{eq} \end{bmatrix} \begin{bmatrix} [Ca(OH)_2] \\ [CO_2] \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{V} \end{bmatrix} u(t)$$

$$y(t) = [Ca(OH)_2]$$

Outline

1. Nonlinear systems & linearization

2. System identification (cont)
   - Grey box identification
   - Black box identification

Main classes of identification methods

White box modeling: based on first principles.

Main classes of identification methods

White box modeling: based on first principles.
$\rightarrow$ known equations (structure) & parameters (coefficients).

Main classes of identification methods

White box modeling: based on first principles.
$\rightarrow$ known equations (structure) & parameters (coefficients).

Grey box identification: first principles & experimentation.

Main classes of identification methods

White box modeling: based on first principles.
$\rightarrow$ known equations (structure) & parameters (coefficients).

Grey box identification: first principles & experimentation.
$\rightarrow$ known equations, unknown/uncertain parameters.

Main classes of identification methods

White box modeling: based on first principles.
$\rightarrow$ known equations (structure) & parameters (coefficients).

Grey box identification: first principles & experimentation.
$\rightarrow$ known equations, unknown/uncertain parameters.

Black box identification: based on experimentation.

Main classes of identification methods

White box modeling: based on first principles.
$\rightarrow$ known equations (structure) & parameters (coefficients).

Grey box identification: first principles & experimentation.
$\rightarrow$ known equations, unknown/uncertain parameters.

Black box identification: based on experimentation.
$\rightarrow$ unknown equations & unknown parameters.

Main classes of identification methods

White box modeling: based on first principles.
$\rightarrow$ known equations (structure) & parameters (coefficients).

Grey box identification: first principles & experimentation.
$\rightarrow$ known equations, unknown/uncertain parameters.

Black box identification: based on experimentation.
$\rightarrow$ unknown equations & unknown parameters.

Most popular approaches are forms of black box identification.

Outline

1. Nonlinear systems & linearization
2. System identification (cont)
   - Grey box identification
   - Black box identification

Grey box identification: conceptual

Grey box identification: conceptual

Grey box identification starts from a known model structure but with unknown/uncertain parameters $\leftrightarrow$ **parametric statistics**.

Grey box identification: conceptual

Grey box identification starts from a known model structure but with unknown/uncertain parameters $\leftrightarrow$ **parametric statistics**.

We assume linear, continuous time state space representation:

$$\begin{cases} \dot{x}(t) = A x(t) + B u(t), \\ y(t) = C x(t) + D u(t). \end{cases}$$

Grey box identification: conceptual

Grey box identification starts from a known model structure but with unknown/uncertain parameters $\leftrightarrow$ **parametric statistics**.

We assume linear, continuous time state space representation:

$$\begin{cases} \dot{x}(t) = {\color{blue}A}x(t) + {\color{blue}B}u(t), \\ y(t) = {\color{blue}C}x(t) + {\color{blue}D}u(t). \end{cases}$$

Given: states, inputs, outputs and guesstimates of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ & $\tilde{D}$.

Grey box identification: conceptual

Grey box identification starts from a known model structure but with unknown/uncertain parameters $\leftrightarrow$ **parametric statistics**.

We assume linear, continuous time state space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases}$$

Given: states, inputs, outputs and guesstimates of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ & $\tilde{D}$.
Task: estimate $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ adequately via experiments.

Grey box identification: conceptual

Grey box identification starts from a known model structure but with unknown/uncertain parameters $\leftrightarrow$ **parametric statistics**.

We assume linear, continuous time state space representation:

$$\begin{cases} \dot{x}(t) = {\color{blue}A}x(t) + {\color{blue}B}u(t), \\ y(t) = {\color{blue}C}x(t) + {\color{blue}D}u(t). \end{cases}$$

Given: states, inputs, outputs and guesstimates of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ & $\tilde{D}$.
Task: estimate $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ adequately via experiments.

"*All models are wrong, but some are useful.*" - George E. P. Box

Linear regression

Consider input matrix $\mathbf{X}$, output vector $\mathbf{y}$ and residuals ϵ:

$$\mathbf{X}\theta = \mathbf{y} + \epsilon.$$

The parameter vector θ must be estimated, given the observations.

Linear regression

Consider input matrix $\mathbf{X}$, output vector $\mathbf{y}$ and residuals ϵ:

$$\mathbf{X}\theta = \mathbf{y} + \epsilon.$$

The parameter vector θ must be estimated, given the observations.

A common estimation approach is ordinary least squares (OLS):

$$\begin{aligned}(\mathbf{X}^T\mathbf{X})\hat{\theta}_{OLS} &= \mathbf{X}^T\mathbf{y},\\ \hat{\theta}_{OLS} &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}.\end{aligned}$$

Linear regression

Consider input matrix $\mathbf{X}$, output vector $\mathbf{y}$ and residuals ϵ:

$$\mathbf{X}\theta = \mathbf{y} + \epsilon.$$

The parameter vector θ must be estimated, given the observations.

A common estimation approach is ordinary least squares (OLS):

$$\begin{aligned}(\mathbf{X}^T\mathbf{X})\hat{\theta}_{OLS} &= \mathbf{X}^T\mathbf{y},\\ \hat{\theta}_{OLS} &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}.\end{aligned}$$

The OLS estimate minimizes the sum-of-squares of errors, i.e.:

$$\hat{\theta}_{OLS} = \arg\min_{\theta} \sum_{i=1}^{N} \left(y(i) - \sum_{j=1}^{d} X(i,j)\theta(j)\right)^2$$

Linear regression with ordinary least squares

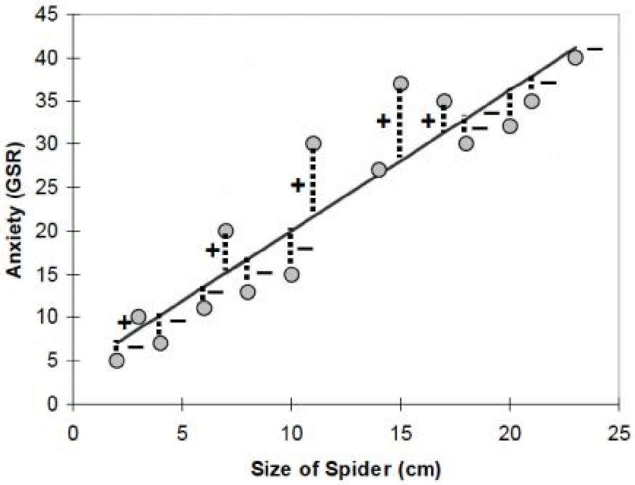


Image taken from `http://freakonometrics.hypotheses.org/2348`.

Maximum likelihood estimation

The maximum likelihood estimate $\hat{\theta}_{ML}$ is the parameter vector that maximizes the likelihood $\mathcal{L}(\cdot)$ of observing the (known) outputs $\mathbf{y}$, given the (known) inputs $\mathbf{X}$:

$$\hat{\theta}_{ML} = \arg\max_{\theta} \mathcal{L}\left(\mathbf{y}, \mathbf{X} \mid \theta\right)$$

Maximum likelihood estimation

The maximum likelihood estimate $\hat{\theta}_{ML}$ is the parameter vector that maximizes the likelihood $\mathcal{L}(\cdot)$ of observing the (known) outputs $\mathbf{y}$, given the (known) inputs $\mathbf{X}$:

$$\hat{\theta}_{ML} = \arg\max_{\theta} \mathcal{L}\big(\mathbf{y}, \mathbf{X} \mid \theta\big)$$

For some structures, ML estimate can be obtained in closed form.

Maximum likelihood estimation

The maximum likelihood estimate $\hat{\theta}_{ML}$ is the parameter vector that maximizes the likelihood $\mathcal{L}(\cdot)$ of observing the (known) outputs $\mathbf{y}$, given the (known) inputs $\mathbf{X}$:

$$\hat{\theta}_{ML} = \arg\max_{\theta} \mathcal{L}\big(\mathbf{y}, \mathbf{X} \mid \theta\big)$$

For some structures, ML estimate can be obtained in closed form.

Example: least squares estimators are the maximum likelihood estimators if the associated residuals ϵ are normally distributed.

Maximum a posteriori (MAP) estimation

Bayesian: maximum likelihood estimation with a *prior* $p(\theta)$.
$\rightarrow$ MAP estimation is a regularization of ML estimation

Maximum a posteriori (MAP) estimation

Bayesian: maximum likelihood estimation with a *prior* $p(\theta)$.
$\rightarrow$ MAP estimation is a regularization of ML estimation

Bayes' theorem: $P(A \mid B) = P(B \mid A) \cdot P(A) \; / \; P(B)$.

Maximum a posteriori (MAP) estimation

Bayesian: maximum likelihood estimation with a *prior* $p(\theta)$.
$\rightarrow$ MAP estimation is a regularization of ML estimation

Bayes' theorem: $P(A \mid B) = P(B \mid A) \cdot P(A) \ / \ P(B)$.

If a prior distribution $p(\cdot)$ is available for θ, then the posterior distribution for θ becomes:

$$\theta \mapsto \mathcal{L}(\theta \mid \mathbf{y}, \mathbf{X}) = \frac{\mathcal{L}(\mathbf{y}, \mathbf{X} \mid \theta) p(\theta)}{\int_{\vartheta} \mathcal{L}(\mathbf{y}, \mathbf{X} \mid \vartheta) p(\vartheta) d\vartheta}.$$

Maximum a posteriori (MAP) estimation

Bayesian: maximum likelihood estimation with a *prior* $p(\theta)$.
$\rightarrow$ MAP estimation is a regularization of ML estimation

Bayes' theorem: $P(A \mid B) = P(B \mid A) \cdot P(A) \ / \ P(B)$.

If a prior distribution $p(\cdot)$ is available for θ, then the posterior distribution for θ becomes:

$$\theta \mapsto \mathcal{L}(\theta \mid \mathbf{y}, \mathbf{X}) = \frac{\mathcal{L}(\mathbf{y}, \mathbf{X} \mid \theta) p(\theta)}{\int_{\vartheta} \mathcal{L}(\mathbf{y}, \mathbf{X} \mid \vartheta) p(\vartheta) d\vartheta}.$$

The MAP estimate is the mode of the posterior distribution of θ:

$$\hat{\theta}_{MAP} = \arg\max_{\theta} \mathcal{L}(\mathbf{y}, \mathbf{X} \mid \theta) p(\theta).$$

Errors-in-variables approach

Additionally accounts for measurement errors in inputs.
$\leftrightarrow$ standard regression only accounts for errors in *outputs*

Errors-in-variables approach

Additionally accounts for measurement errors in inputs.
$\leftrightarrow$ standard regression only accounts for errors in *outputs*

Typically described via *latent variables*:

$$\begin{cases} x = x^\star + \eta, \\ y = y^\star + \epsilon, \\ y^\star = g(x^\star \mid \theta), \end{cases}$$

with x, y the observed inputs, outputs and latent variables $x^\star$, $y^\star$.

Errors-in-variables approach

Additionally accounts for measurement errors in inputs.
$\leftrightarrow$ standard regression only accounts for errors in *outputs*

Typically described via *latent variables*:

$$\begin{cases} x = x^\star + \eta, \\ y = y^\star + \epsilon, \\ y^\star = g(x^\star \mid \theta), \end{cases}$$

with x, y the observed inputs, outputs and latent variables $x^\star$, $y^\star$.
Assumption: latent variables $x^\star$ and $y^\star$ exist which follow the true functional relationship $g(\cdot)$.

Errors-in-variables approach

Additionally accounts for measurement errors in inputs.
$\leftrightarrow$ standard regression only accounts for errors in *outputs*

Typically described via *latent variables*:

$$\begin{cases} x = x^\star + \eta, \\ y = y^\star + \epsilon, \\ y^\star = g(x^\star \mid \theta), \end{cases}$$

with x, y the observed inputs, outputs and latent variables $x^\star$, $y^\star$.
Assumption: latent variables $x^\star$ and $y^\star$ exist which follow the true functional relationship $g(\cdot)$.

Task: estimate θ.

Outline

1. Nonlinear systems & linearization

2. System identification (cont)
   - Grey box identification
   - Black box identification

Black box identification

Start from unknown equations & unknown parameters.
$\rightarrow$ related to **machine learning** and **nonparametric statistics**.

Black box identification

Start from unknown equations & unknown parameters.
→ related to **machine learning** and **nonparametric statistics**.

If we assume a linear state space system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \qquad \begin{cases} x[k+1] & = Ax[k] + Bu[k], \\ \quad y[k] & = Cx[k] + Du[k]. \end{cases}$$

Black box identification

Start from unknown equations & unknown parameters.
$\rightarrow$ related to **machine learning** and **nonparametric statistics**.

If we assume a linear state space system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \qquad \begin{cases} x[k+1] & = Ax[k] + Bu[k], \\ y[k] & = Cx[k] + Du[k]. \end{cases}$$

Black box identification deals with:

Black box identification

Start from unknown equations & unknown parameters.
$\rightarrow$ related to **machine learning** and **nonparametric statistics**.

If we assume a linear state space system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \qquad \begin{cases} x[k+1] & = Ax[k] + Bu[k], \\ y[k] & = Cx[k] + Du[k]. \end{cases}$$

Black box identification deals with:

- unknown states, both in number & physical interpretation

Black box identification

Start from unknown equations & unknown parameters.
$\rightarrow$ related to **machine learning** and **nonparametric statistics**.

If we assume a linear state space system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \qquad \begin{cases} x[k+1] & = Ax[k] + Bu[k], \\ y[k] & = Cx[k] + Du[k]. \end{cases}$$

Black box identification deals with:

- unknown states, both in number & physical interpretation
  $\rightarrow$ dimensions of A, B & C unknown

Black box identification

Start from unknown equations & unknown parameters.
→ related to **machine learning** and **nonparametric statistics**.

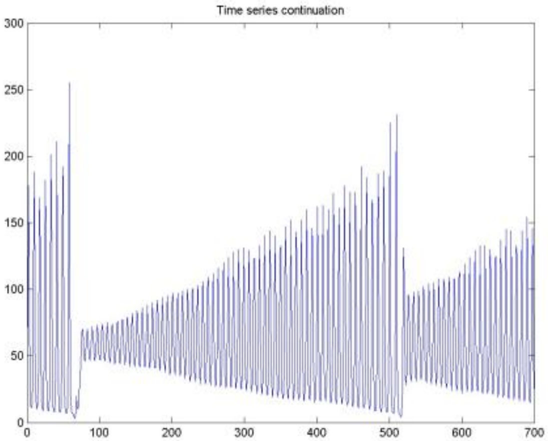
If we assume a linear state space system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \qquad \begin{cases} x[k+1] & = Ax[k] + Bu[k], \\ y[k] & = Cx[k] + Du[k]. \end{cases}$$

Black box identification deals with:

- unknown states, both in number & physical interpretation
  → dimensions of A, B & C unknown
- unknown parameters (values in A, B, C, D)

Time series: Santa Fe laser



Modelling the Santa Fe laster

This laser can be treated as an autonomous discrete time system:

$$\begin{cases} x[k+1] = f\left(x[k-N+1], \ldots, x[k]\right), \\ y[k] = x[k]. \end{cases}$$

The output depends on the past N states & no inputs.

Modelling the Santa Fe laster

This laser can be treated as an autonomous discrete time system:

$$\begin{cases} x[k+1] = f\big(x[k-N+1], \ldots, x[k]\big), \\ y[k] = x[k]. \end{cases}$$

The output depends on the past N states & no inputs.
$\rightarrow$ how large is N? $\rightarrow$ **unknown structure**

Modelling the Santa Fe laster

This laser can be treated as an autonomous discrete time system:

$$\begin{cases} x[k+1] = f\big(x[k-N+1], \dots, x[k]\big), \\ y[k] = x[k]. \end{cases}$$

The output depends on the past N states & no inputs.
$\rightarrow$ how large is N? $\rightarrow$ **unknown structure**

Treat it as a regression problem with N inputs: $y = f\big(X_1, \dots, X_N\big)$.

Modelling the Santa Fe laster

This laser can be treated as an autonomous discrete time system:

$$\begin{cases} x[k+1] = f\big(x[k-N+1], \ldots, x[k]\big), \\ y[k] = x[k]. \end{cases}$$

The output depends on the past N states & no inputs.
$\rightarrow$ how large is N? $\rightarrow$ **unknown structure**

Treat it as a regression problem with N inputs: $y = f\big(X_1, \ldots, X_N\big)$.
$\rightarrow$ lets say linear, i.e. $y = \mathbf{X}\theta$ $\rightarrow$ **unknown parameters** $\theta \in \mathbb{R}^N$.

Modelling the Santa Fe laster

This laser can be treated as an autonomous discrete time system:

$$\begin{cases} x[k+1] = f\big(x[k-N+1], \dots, x[k]\big), \\ y[k] = x[k]. \end{cases}$$

The output depends on the past N states & no inputs.
$\rightarrow$ how large is N? $\rightarrow$ **unknown structure**

Treat it as a regression problem with N inputs: $y = f\big(X_1, \dots, X_N\big)$.
$\rightarrow$ lets say linear, i.e. $y = \mathbf{X}\theta$ $\rightarrow$ **unknown parameters** $\theta \in \mathbb{R}^N$.
$\rightarrow$ for given N, we can estimate θ via grey box methods.

Modelling the Santa Fe laster

This laser can be treated as an autonomous discrete time system:

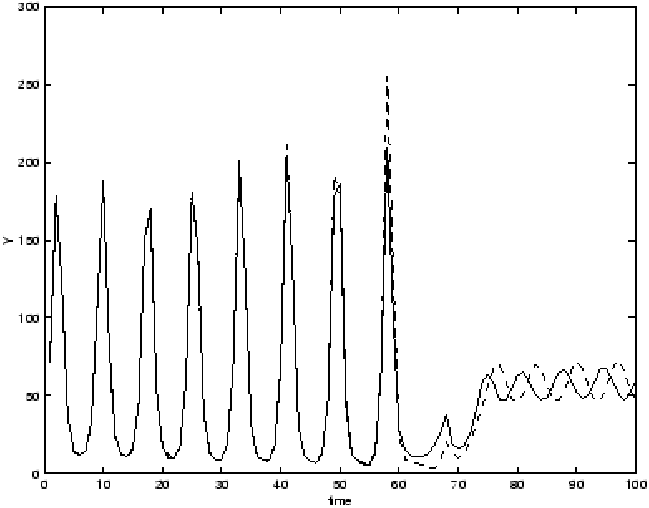
$$\begin{cases} x[k+1] = f\big(x[k-N+1], \ldots, x[k]\big), \\ y[k] = x[k]. \end{cases}$$

The output depends on the past N states & no inputs.
$\rightarrow$ how large is N? $\rightarrow$ **unknown structure**

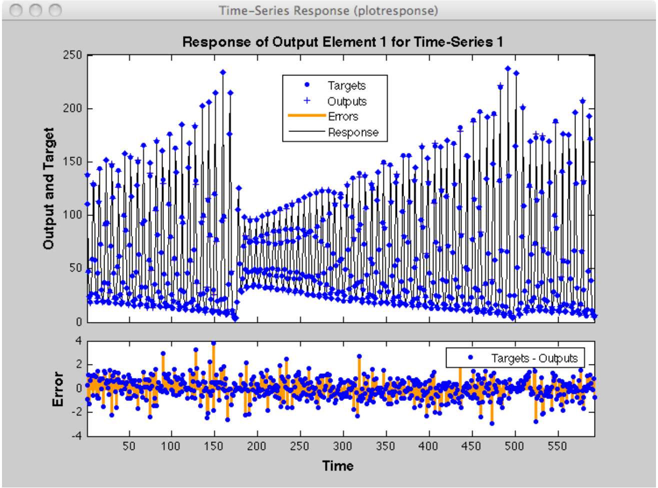
Treat it as a regression problem with N inputs: $y = f\big(X_1, \ldots, X_N\big)$.
$\rightarrow$ lets say linear, i.e. $y = \mathbf{X}\theta$ $\rightarrow$ **unknown parameters** $\theta \in \mathbb{R}^N$.
$\rightarrow$ for given N, we can estimate θ via grey box methods.

Nonlinear models can be obtained via machine learning methods.
$\rightarrow$ neural networks, support vector machine, random forest, ...

Predictions of a least-squares support vector machine



Predictions of an artificial neural network



Neural network: biological

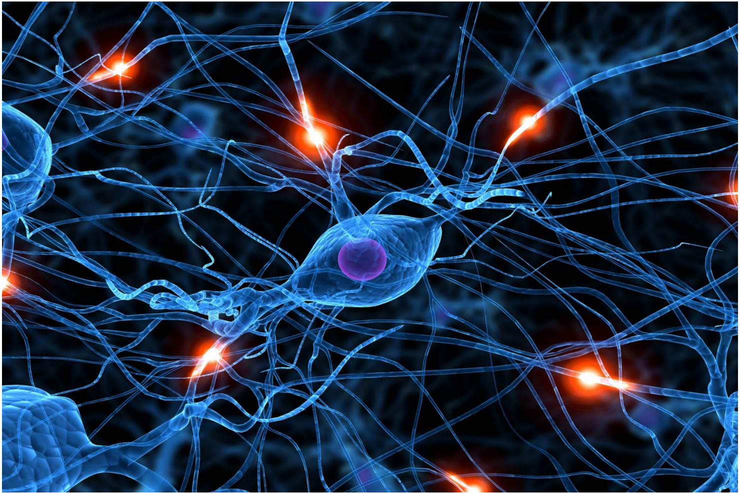


Image taken from http://www.extremetech.com/wp-content/uploads/2013/09/340.jpg.

Structure of a single neuron

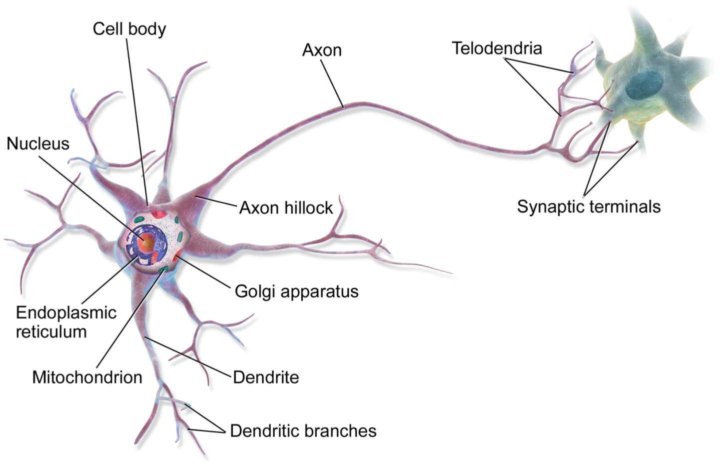


Image taken from http://en.wikipedia.org/wiki/File:Blausen_0657_MultipolarNeuron.png.

Neural network: artificial