

Transient response analysis of first order and second order systems

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Transient Response

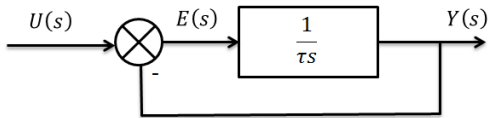
- The time response of a control system may be written as:

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

- Where $y_{tr}(t)$ is the transient response and $y_{ss}(t)$ is the steady state response.
- Most important characteristic of dynamic system is absolute stability.
 - System is stable when returns to equilibrium if subject to initial condition
 - System is critically stable when oscillations of the output continue forever
 - System is unstable when output diverges without bound from equilibrium if subject to initial condition
- Transient response: when input of system changes, output does not change immediately but takes time to go to steady state

First-order systems

- E.g. RC circuit, thermal system, ...

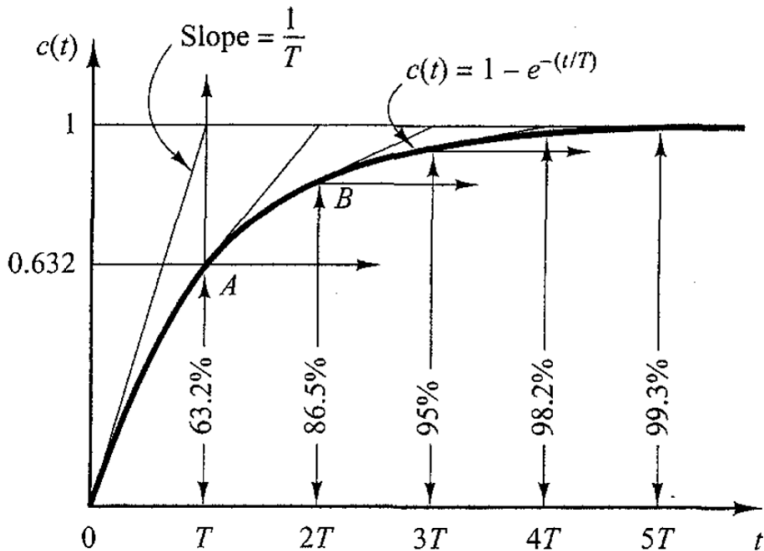


- Transfer function is given by: $\frac{Y(s)}{U(s)} = \frac{1}{\tau s + 1}$
- Unit step response
 - Laplace of unit-step is $\frac{1}{s} \rightarrow$ substituting $U(s) = \frac{1}{s}$ into equation $Y(s) = \frac{1}{s} \frac{1}{\tau s + 1}$
 - Expanding into partial fractions gives
$$Y(s) = \frac{1}{s} - \frac{\tau}{\tau s + 1} = \frac{1}{s} - \frac{1}{s + \frac{1}{\tau}}$$

Unit step transient response

- $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s + 1} = \frac{1}{s} - \frac{1}{s + \frac{1}{\tau}}$
- Taking the inverse Laplace transform
$$y(t) = 1 - e^{-\frac{t}{\tau}}, \text{ for } t \geq 0$$
- At $t = 0$, the output $c(t) = 0$
- At $t = \tau$, the output $c(t) = 0.632$, or $c(t)$ has reached 63.2% of it's total change $y(\tau) = 1 - e^{-1} = 0.632$
- Slope at time $t = 0$ is $\frac{1}{\tau}$
$$\left. \frac{dy}{dt} \right|_{t=0} = \left. \frac{1}{\tau} e^{-\frac{t}{\tau}} \right|_{t=0} = \frac{1}{\tau}$$
- Where τ is called the system's time constant

Unit step transient response



Unit ramp transient response

- Laplace transform of unit ramp is $\frac{1}{s^2}$

$$Y(s) = \frac{1}{\tau s + 1} \frac{1}{s^2}$$

- Expanding into partial fractions gives

$$Y(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s + 1}$$

- Taking the inverse Laplace transform

$$y(t) = t - \tau + \tau e^{-\frac{t}{\tau}}, \text{ for } t \geq 0$$

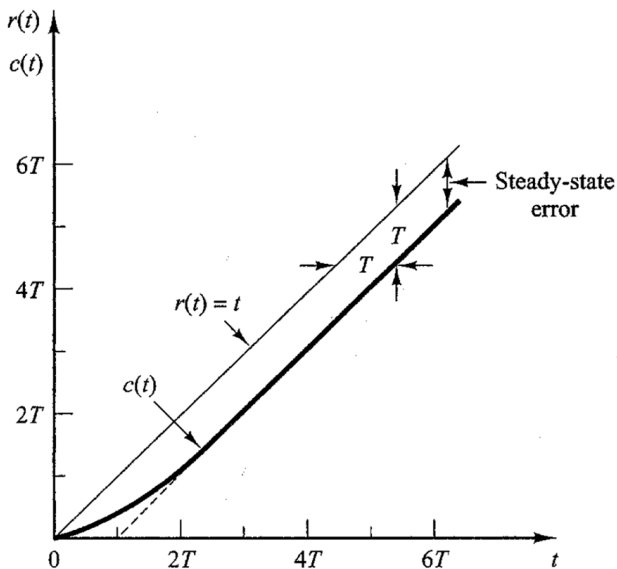
- The error signal $e(t)$ is then

$$e(t) = r(t) - c(t) = \tau(1 - e^{-\frac{t}{\tau}})$$

- For t approaching infinity, $e(t)$ approaches τ

$$e(\infty) = \tau$$

Unit ramp transient response



Unit-Impulse Response

- For a unit-impulse input, $U(s) = 1$ and the output is

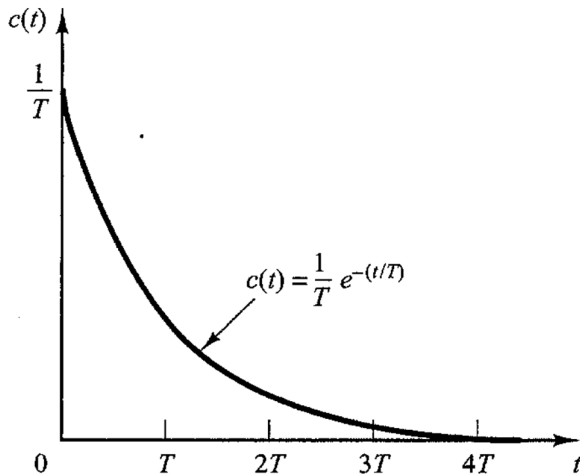
$$Y(s) = \frac{1}{\tau s + 1}$$

- The inverse Laplace transform gives

$$y(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}, \text{ for } t \geq 0$$

- For $t \rightarrow +\infty$, $y(t) \rightarrow 0$

Unit-Impulse Response



Second order systems

- A second order system can generally be written as:

$$\frac{Y(s)}{U(s)} = H(s) = \frac{as^2 + bs + c}{ds^2 + es + f}$$

- A system where the closed-loop transfer function possesses two poles is called a second-order system
- If the transfer function has two real poles, the frequency response can be found by combining the effects of both poles

- Sometimes the transfer function has two complex conjugate poles. In that case we have to find a different solution for finding the frequency response.
- In order to study the transient behaviour, let us first consider the following simplified example of a second order system

$$H(s) = \frac{c}{ds^2 + es + c}$$

Step response second order system

- $H(s) = \frac{c}{ds^2 + es + c}$
- The transfer function can be rewritten as:

$$\begin{aligned} H(s) &= \frac{\frac{c}{d}}{s^2 + \frac{e}{d}s + \frac{c}{d}} \\ &= \frac{\frac{c}{d}}{\left[s + \frac{e}{2d} + \sqrt{\left(\frac{e}{2d}\right)^2 - \frac{c}{d}}\right]\left[s + \frac{e}{2d} - \sqrt{\left(\frac{e}{2d}\right)^2 - \frac{c}{d}}\right]} \end{aligned}$$

- The poles are complex conjugates if

$$e^2 - 4dc < 0$$

- The poles are real if

$$e^2 - dc \geq 0$$

Step response second order system

- To simplify the transient analysis, it is convenient to write

$$\frac{f}{d} = \omega_n^2, \quad \frac{e}{d} = 2\zeta\omega_n = 2\sigma$$

- Where

σ is the attenuation

ω_n is the undamped natural frequency

ζ is the damping ratio

- The transfer function can now be rewritten as

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Which is called the standard form of the second-order system.

- The dynamic behavior of the second-order system can then be described in terms of only two parameters ζ and ω_n

Step response second order system

- If $0 < \zeta < 1$, the poles are complex conjugates and lie in the left-half s -plane
 - The system is then called **underdamped**
 - The **transient response is oscillatory**
- If $\zeta = 0$, the **transient response doesn't die out**
- If $\zeta = 1$, the system is called **critically damped**
- If $\zeta > 1$, the system is called **overdamped**
- We will now look at the unit step response for each of these cases

Underdamped system

- For the underdamped case ($0 < \zeta < 1$), the transfer function can be written as:

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

- Where ω_d is called the damped natural frequency

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

- For a unit-step input we can write

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}$$

- Which can be rewritten as

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} - \frac{\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

Underdamped system

- It can be shown that

$$\mathcal{L}^{-1}\left[\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\right] = e^{-\zeta\omega_n t} \cos(\omega_d t)$$

$$\mathcal{L}^{-1}\left[\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right] = e^{-\zeta\omega_n t} \sin(\omega_d t)$$

- Therefore:

$$\begin{aligned}\mathcal{L}^{-1}[Y(s)] &= y(t) \\ &= 1 - e^{-\zeta\omega_n t} \left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d t) \right)\end{aligned}$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)$$

- It can be seen that the frequency of the transient oscillation is the damped natural frequency ω_d and thus varies with the damping ratio ζ

Underdamped system

- The error signal is the difference between input and output

$$\begin{aligned} e(t) &= y(t) - u(t) \\ &= e^{-\zeta\omega_n t} \left(\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d t) \right) \end{aligned}$$

- The error signal exhibits a damped sinusoidal oscillation
- At steady state, or at $t = \infty$, the error goes to zero

Underdamped system

- If damping $\zeta = 0$, the response becomes undamped
 - Oscillations continue indefinitely
 - Filling in $\zeta = 0$ into the equation for $y(t)$ gives us $y(t) = 1 - \cos(\omega_n t)$, for $t \geq 0$
 - We see that the system now oscillates at the natural frequency ω_n
 - If a linear system has any amount of damping, the undamped natural frequency cannot be observed experimentally, only ω_d can be observed
 - ω_d is always lower than ω_n

Critically damped system

- If the two poles of the system are equal, the system is critically damped and $\zeta = 1$
- For a unit-step, $R(s) = \frac{1}{s}$ and we can write

$$Y(s) = \frac{\omega_n^2}{(s + \omega_n)^2 s}$$

- The inverse Laplace transform gives us

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t) \text{ for } t \geq 0$$

Overdamped system

- A system is overdamped ($\zeta > 1$) when the two poles are negative, real and unequal

- For a unit-step $R(s) = \frac{1}{s}$, $Y(s)$ can be written as

$$Y(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n^2\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n^2\sqrt{\zeta^2 - 1})}$$

- The inverse Laplace transform is

$$y(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2} \right) \text{ for } t \geq 0$$

- Where

$$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n \text{ and } s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$$

Overdamped system

$$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n \text{ and } s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$$

- Thus $y(t)$ includes two decaying exponential terms
 - When $\zeta \gg 1$, one of the two decreases much faster than the other, and then the faster decaying exponential may be neglected
 - If $-s_2$ is located much closer to the $j\omega$ axis than $-s_1$ ($|s_2| \gg |s_1|$), then $-s_1$ may be neglected
 - Once the faster decaying exponential term has disappeared, the response is similar to that of a first-order system
 - In that case, $H(s)$ can be approximated by

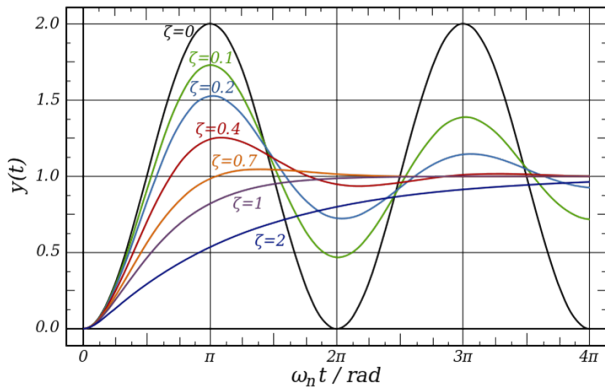
$$H(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} = \frac{s_2}{s + s_2}$$

Overdamped system

- With the approximate transfer function, the unit-step response becomes $Y(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$
- The time response for the approximate transfer function is then given as $c(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$, for $t \geq 0$

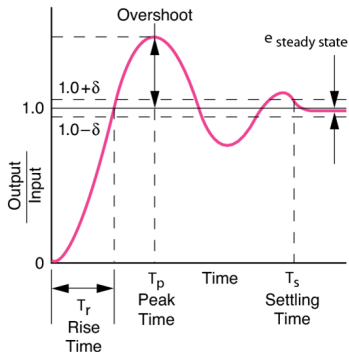
Second order systems unit step response curves

- Response on a step function



Second order systems - characteristics

- Overshoot: Highest amplitude above steady state.
- Rise Time: Time needed to reach the steady state for the first time.
- Peak Time: Time to reach overshoot.

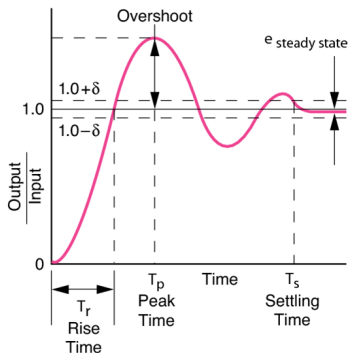


Second order systems - characteristics

- Settling Time: Time needed to approximate the steady state.
- For: $\delta = \frac{0.02}{\sqrt{1-\zeta^2}}$
- We find:

$$e^{-\zeta\omega_n\tau_s} < 0.02$$

$$\tau_s = \frac{4}{\omega_n\zeta}$$



Second order systems - resonance

- The resonance frequency is the frequency at which the systems output has a larger amplitude than at other frequencies. This happens when underdamped functions oscillate at a greater magnitude than the input.
- An input with this frequency can sometime have catastrophic effects.
- A different view on the Tacoma bridge disaster:
<https://www.youtube.com/watch?v=6ai2QFxStxo>
- In fact the collapse was a result of a number of effects like Aerodynamic flutter and vortices. Read the full article here:
<http://www.ketchum.org/billah/Billah-Scanlan.pdf>

Second order systems - resonance

- The resonance frequency is: $\omega_r = \omega_n \sqrt{1 - \zeta^2}$
- Systems with a damping > 0.707 do not resonate
- The resonance frequency and the natural frequency are equal when a system has no damping.
- Another phenomenon with bridges and resonance is that many people marching with the same rhythm can cause a bridge to start resonating like the Angers bridge in 1850. A more recent example is the Millennium bridge in London who started resonating (see video lecture 2).

Second order systems - damping

- When we want a system with no resonance, we choose one with damping < 0.707 . This means a pole between 135° and 225° :

$$\arctan\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = +135^\circ$$

- We mostly want a short settling time ($< 4s$). This results in another restriction on the poles of the system:

$$\tau_n = \frac{4}{\omega\zeta} < 4s$$
$$\omega_n\zeta > 1$$

Second order systems - damping

