System Modeling - Part 1

July 8, 2015

Outline

Introduction

We can derive the mathematical model of a dynamic system in two ways mainly:

- Physical Modeling:
  Applying the laws of physics, chemistry, thermodynamics,...
  Also called modeling from *First Principles*

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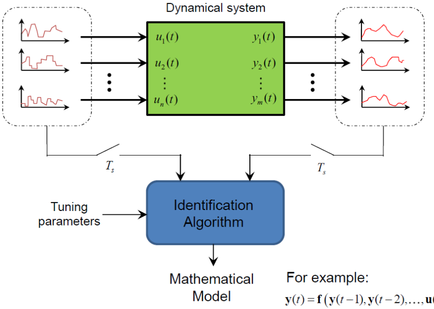
- Physical Modeling:
  Applying the laws of physics, chemistry, thermodynamics,...
  Also called modeling from *First Principles*
- System identification or *Empirical Modeling*:
  Developing models from observed or collected data

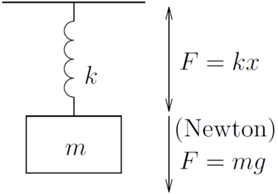


For example:

$$\mathbf{y}(t) = \mathbf{f}\left(\mathbf{y}(t-1), \mathbf{y}(t-2), \ldots, \mathbf{u}(t), \mathbf{u}(t-1), \ldots\right)$$

Outline

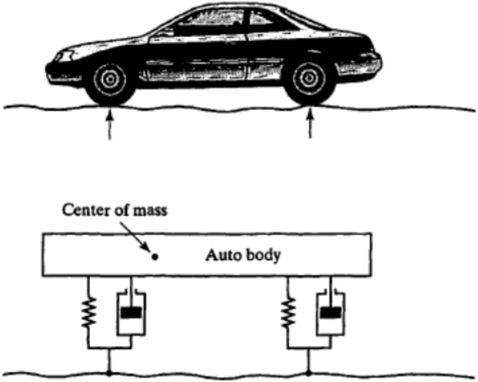
Example 1: Mass-Spring System

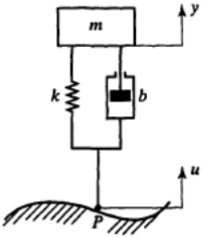


If spring is at rest at $x = 0$:

$$m \cdot \frac{d^2x}{dt^2} + k \cdot x = m \cdot g$$

Example 2: Mass-Spring Damped

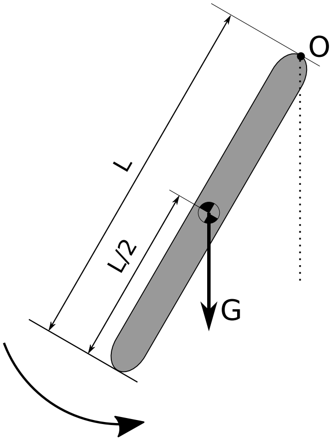




Force excerted by damper: $F = b\dot{x}$

Differential equation can be found by writing force equilibrium and moment equilibrium around center of mass

Example 3: Pendulum



Dynamic equilibrium:

$$I\ddot{\theta}(t) = -mg\frac{L}{2}\sin(\theta(t)) \text{ with } I = \frac{mL^2}{3}$$

$$\ddot{\theta}(t) = -\frac{3g}{2L}\sin(\theta(t))$$

Small deviation of $\theta(t)$:

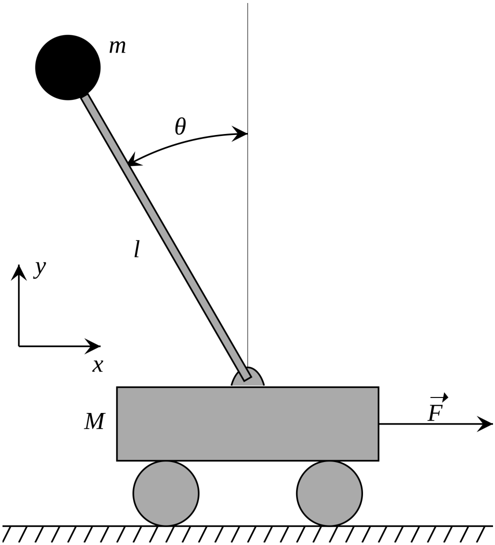
$$\ddot{\theta}(t) = -\tfrac{3g}{2L}\theta(t)$$

Solving the differential equation yields the general solution:

$$\theta(t) = A\cos(\omega_0 + \phi) \text{ with } \omega_0 = \sqrt{\tfrac{3g}{2L}}$$

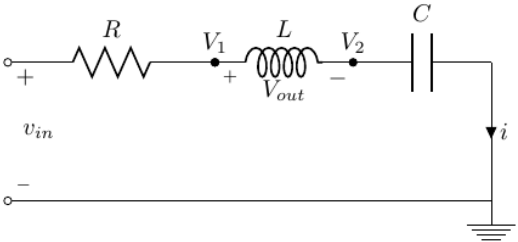
and ϕ & A to be determined with the initial condition

Example 4: Inverted Pendulum



Analysis can be done with Newton like former example, but less tedious is using energy-methods (Lagrange)

Example 5: RLC Circuit



Besides input v_{in}, two internal variables are needed to determine output $\Rightarrow$ Second-order System

Inputs	Ouputs	Choosen States
v_{in}	v_{out}	V_2
		i

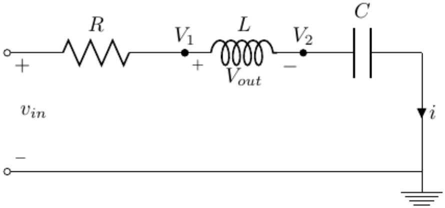
Example 5: RLC Circuit

Equations for each component:

$$i = \frac{V_{in} - V_1}{R}$$

$$V_1 - V_2 = L \cdot \frac{di}{dt}$$

$$i = C \cdot \frac{dV_2}{dt}$$



Example 5: RLC Circuit

- Writing derivatives of state variables in function of state variables and inputs: $\begin{cases} \frac{di}{dt} = \frac{V_1 - V_2}{L} = \frac{V_{in} - R \cdot i - V_2}{L} \\ \frac{dV_2}{dt} = \frac{i}{C} \end{cases}$
- Writing output in function of state variables and inputs: $V_{out} = V_1 - V_2 = V_{in} - Ri - V_2$

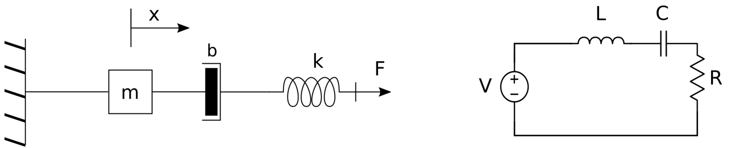
State Space Representation

This yields the **State Space Representation** of the dynamic system. In Matrix form:

$$\begin{bmatrix} \frac{dV_2}{dt} \\ \frac{di}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} V_2 \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_{in}$$

$$V_{out} = \begin{bmatrix} -1 & -R \end{bmatrix} \begin{bmatrix} V_2 \\ i \end{bmatrix} + V_{in}$$

Force-Voltage Analogy



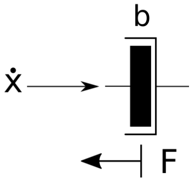
Let:

$$\begin{array}{ccc} F & \leftrightarrow & V \\ \dot{x} & \leftrightarrow & i \\ x & \leftrightarrow & q \end{array}$$

Force-Voltage Analogy

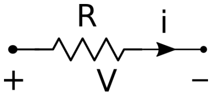
The analogy between the other quantities follows from comparing the physical laws.

Damping:



$$F = b\dot{x}$$

Resistance:

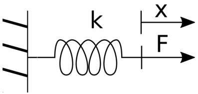


$$V = Ri$$

$$\boxed{b \leftrightarrow R}$$

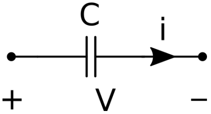
Force-Voltage Analogy

Spring:



$$F = kx$$
$$\Rightarrow \frac{dF}{dt} = k\frac{dx}{dt}$$

Capacitor:

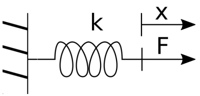


$$\frac{dV}{dt} = \frac{i}{C}$$

$$\boxed{k \leftrightarrow \frac{1}{C}}$$

Force-Voltage Analogy

Spring:



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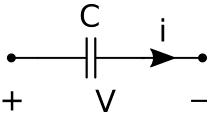

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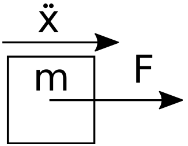
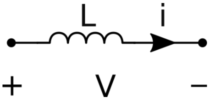
Newton:



$$F = m\ddot{x}$$
$$= m\frac{d\dot{x}}{dt}$$

Coil:



$$V = L\frac{di}{dt}$$

$$\boxed{m \leftrightarrow L}$$

Example 6: Hoover dam

Define:

- Inflow of water: $u(t)$
- Current volume of water: $x(t)$
- Outflow of water: $y(t)$
- Water level: $h(t)$

Assume that $x(t) = c_1 \cdot h(t)$

What will happen when we open the gate?

Example 6: Hoover dam

- Outflow depends on height:
$$y(t) = c_2 \cdot h(t)$$
- The state of the system is defined by the contained volume of water:
$$\dot{x}(t) = u(t) - y(t) = u(t) - c_2 \cdot h(t)$$
- Thus a **State Space Representation** is, with $c \triangleq \frac{c_2}{c_1}$:

$$\dot{x}(t) = u(t) - c \cdot x(t)$$
$$y(t) = c \cdot x(t)$$