

System Modeling - Part 1

July 8, 2015

Outline

1 Introduction

2 First Principles Modeling

Introduction

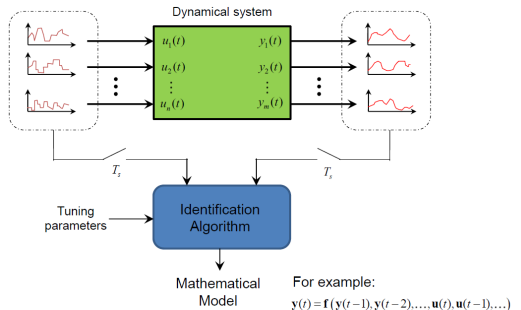
We can derive the mathematical model of a dynamic system in two ways mainly:

- Physical Modeling:
Applying the laws of physics, chemistry, thermodynamics,...
Also called modeling from *First Principles*

Introduction

We can derive the mathematical model of a dynamic system in two ways mainly:

- Physical Modeling:
Applying the laws of physics, chemistry, thermodynamics,...
Also called modeling from *First Principles*
- System identification or *Empirical Modeling*:
Developing models from observed or collected data

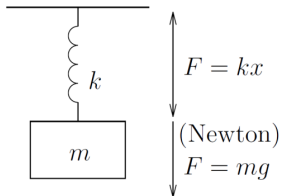


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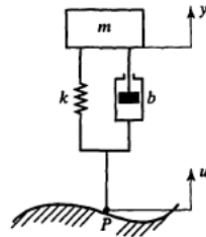
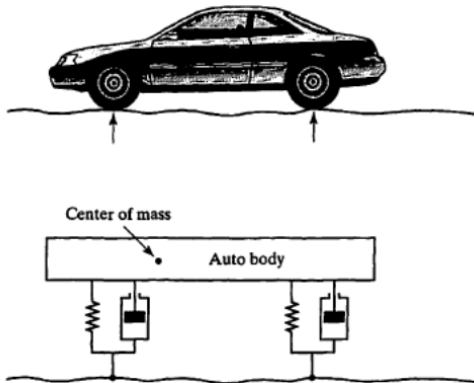
Example 1: Mass-Spring System



If spring is at rest at $x = 0$:

$$m \cdot \frac{d^2 x}{dt^2} + k \cdot x = m \cdot g$$

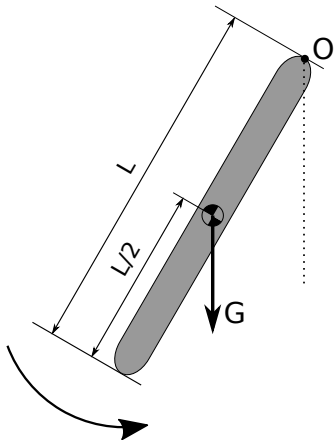
Example 2: Mass-Spring Damped



Force exerted by damper: $F = b\dot{x}$

Differential equation can be found by writing force equilibrium and moment equilibrium around center of mass

Example 3: Pendulum



Dynamic equilibrium:

$$I\ddot{\theta}(t) = -mg\frac{L}{2}\sin(\theta(t)) \text{ with } I = \frac{mL^2}{3}$$

$$\ddot{\theta}(t) = -\frac{3g}{2L}\sin(\theta(t))$$

Small deviation of $\theta(t)$:

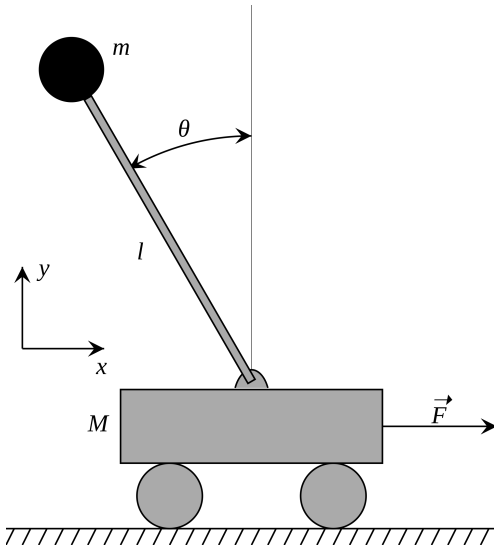
$$\ddot{\theta}(t) = -\frac{3g}{2L}\theta(t)$$

Solving the differential equation yields the general solution:

$$\theta(t) = A\cos(\omega_0 t + \phi) \text{ with } \omega_0 = \sqrt{\frac{3g}{2L}}$$

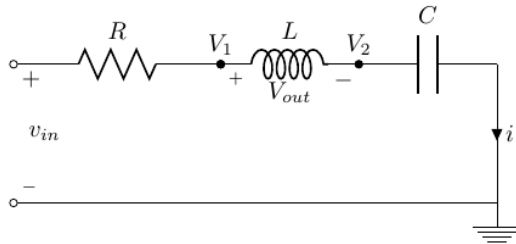
and ϕ & A to be determined with the initial condition

Example 4: Inverted Pendulum



Analysis can be done with Newton like former example, but less tedious is using energy-methods (Lagrange)

Example 5: RLC Circuit



Besides input v_{in} , two internal variables are needed to determine output \Rightarrow Second-order System

| Inputs | Outputs | Chosen States |
|----------|-----------|---------------|
| v_{in} | v_{out} | V_2 i |

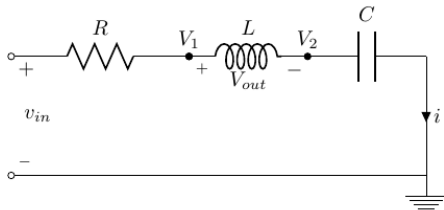
Example 5: RLC Circuit

Equations for each component:

$$i = \frac{V_{in} - V_1}{R}$$

$$V_1 - V_2 = L \cdot \frac{di}{dt}$$

$$i = C \cdot \frac{dV_2}{dt}$$



Example 5: RLC Circuit

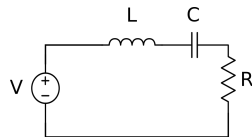
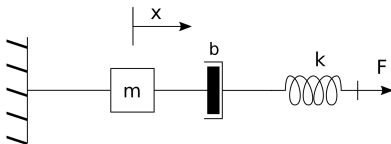
- Writing derivatives of state variables in function of state variables and inputs:
$$\begin{cases} \frac{di}{dt} = \frac{V_1 - V_2}{L} = \frac{V_{in} - R \cdot i - V_2}{L} \\ \frac{dV_2}{dt} = \frac{i}{C} \end{cases}$$
- Writing output in function of state variables and inputs:
$$V_{out} = V_1 - V_2 = V_{in} - Ri - V_2$$

State Space Representation

This yields the **State Space Representation** of the dynamic system. In Matrix form:

$$\begin{bmatrix} \frac{dV_2}{dt} \\ \frac{di}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} V_2 \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_{in}$$
$$V_{out} = \begin{bmatrix} -1 & -R \end{bmatrix} \begin{bmatrix} V_2 \\ i \end{bmatrix} + V_{in}$$

Force-Voltage Analogy



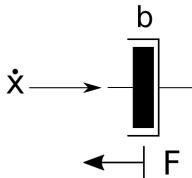
Let:

| | | |
|-----------|-------------------|-----|
| F | \leftrightarrow | V |
| \dot{x} | \leftrightarrow | i |
| x | \leftrightarrow | q |

Force-Voltage Analogy

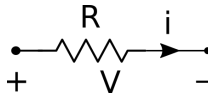
The analogy between the other quantities follows from comparing the physical laws.

Damping:



$$F = b\dot{x}$$

Resistance:

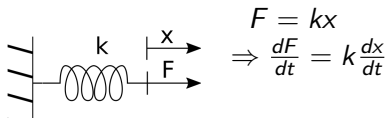


$$V = Ri$$

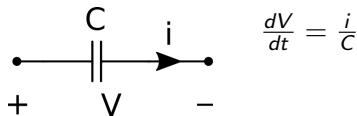
$$b \leftrightarrow R$$

Force-Voltage Analogy

Spring:



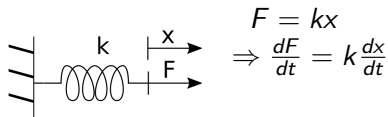
Capacitor:



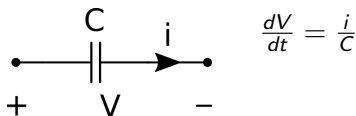
$$k \leftrightarrow \frac{1}{C}$$

Force-Voltage Analogy

Spring:

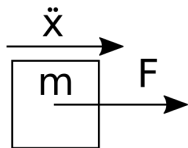


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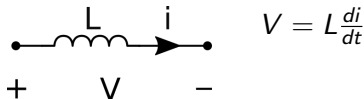


$$k \leftrightarrow \frac{1}{C}$$

Newton:



Coil:



$$m \leftrightarrow L$$

Example 6: Hoover dam

Define:

- Inflow of water: $u(t)$
- Current volume of water: $x(t)$
- Outflow of water: $y(t)$
- Water level: $h(t)$

Assume that $x(t) = c_1 \cdot h(t)$

What will happen when we open the gate?



Example 6: Hoover dam

- Outflow depends on height:

$$y(t) = c_2 \cdot h(t)$$

- The state of the system is defined by the contained volume of water:

$$\dot{x}(t) = u(t) - y(t) = u(t) - c_2 \cdot h(t)$$

- Thus a **State Space Representation** is, with $c \triangleq \frac{c_2}{c_1}$:

$$\dot{x}(t) = u(t) - c \cdot x(t)$$

$$y(t) = c \cdot x(t)$$

