Chapter 7: Sampling and reconstruction of signals

August 7, 2015

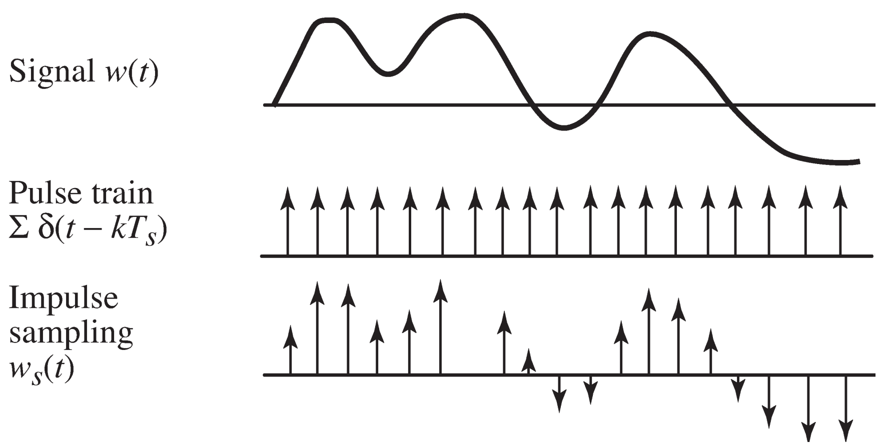
Outline

Discretization and reconstruction of signals

Definition

The use of digital logic and computers to calculate a control action for a continuous system introduces the operation of sampling. Samples are taken from the continuous signals and used in the computer to calculate the controls to be applied.

The role of sampling and the conversion from continuous to discrete and vice versa are important to the understanding of the complete response of digital control.



Types of signals

- **Continuous-time signal**: A signal defined over a continuous range of time.
- **Discrete-time signal**: A signal defined only at discrete instants of time (the independent variable t is quantized)
- **Analog signal**: A signal defined over a continuous range of time whose amplitude can assume a continuous range of values.
- **Quantized signal**: A signal in which the amplitude may assume only a finite number of distinct values.

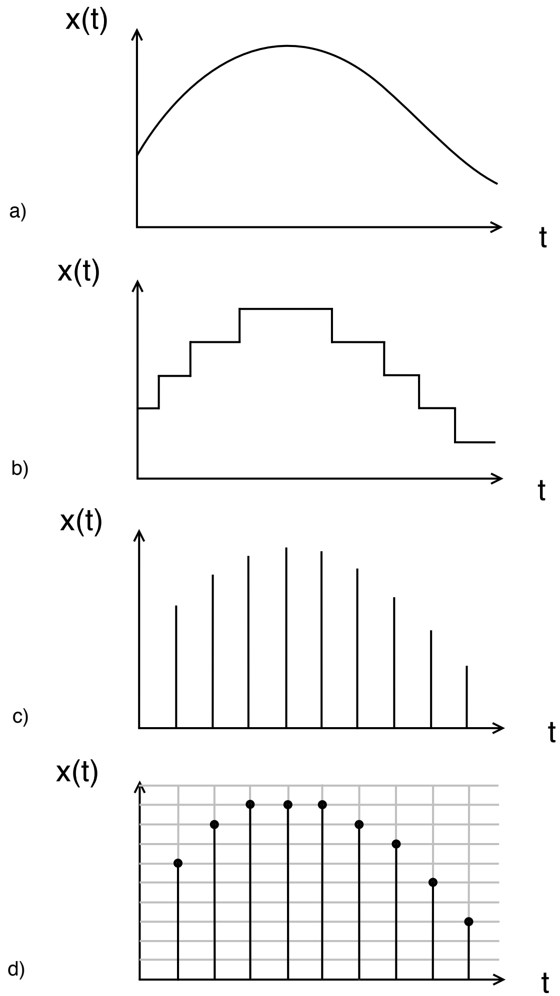
Types of signals

a. Continuous-time analog signal

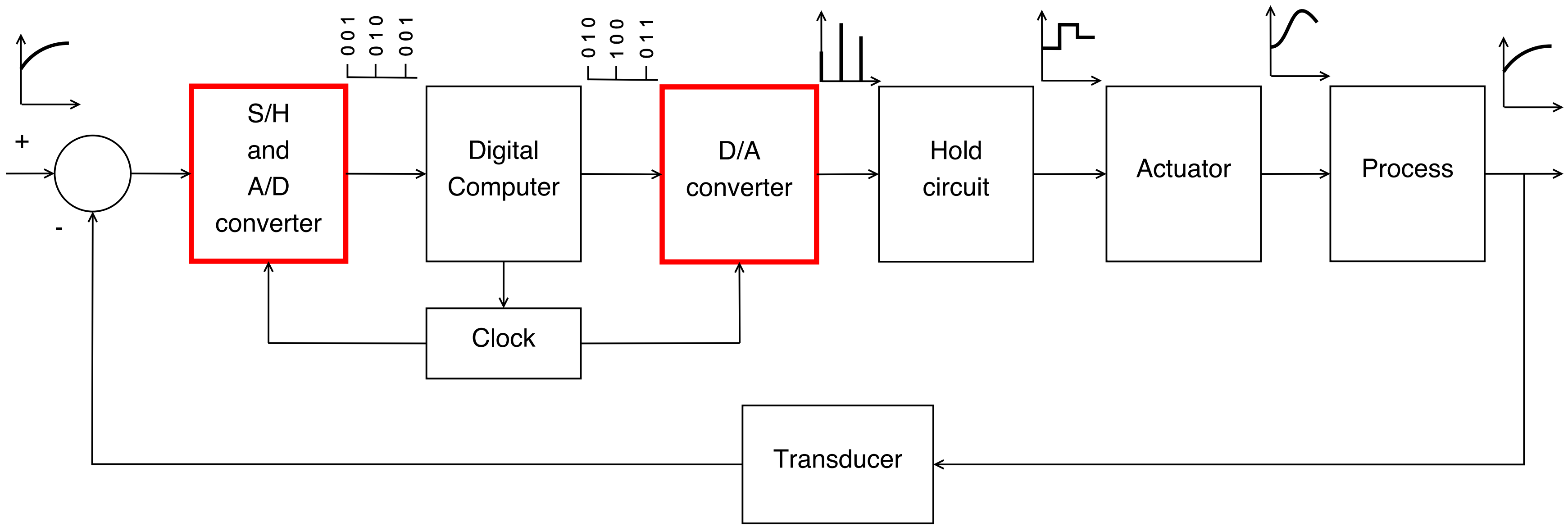
b. Continuous-time quantized signal

c. Sampled-data signal (discrete-time analog signal)

d. Digital signal (discrete-time quantized signal)



Digital control system



Definitions

Sample and hold

A circuit that receives an analog input signal and holds this signal at a constant value for a specified period of time.

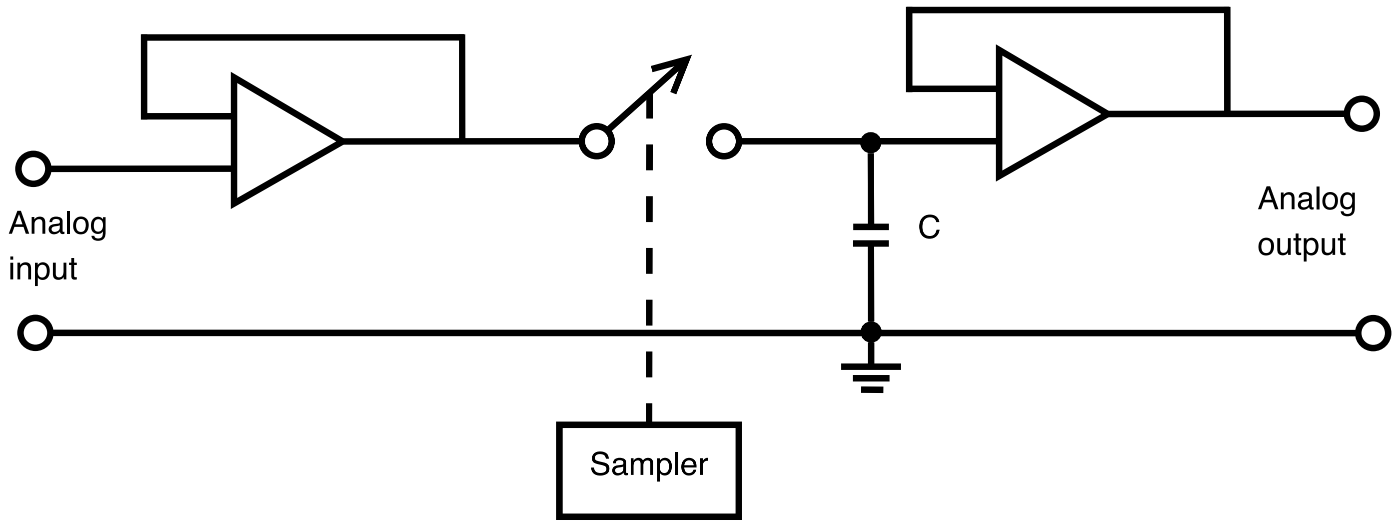
A/D converter

An analog-to-digital converter, also called an encoder, is a device that converts an analog signal into a digital signal.

D/A converter

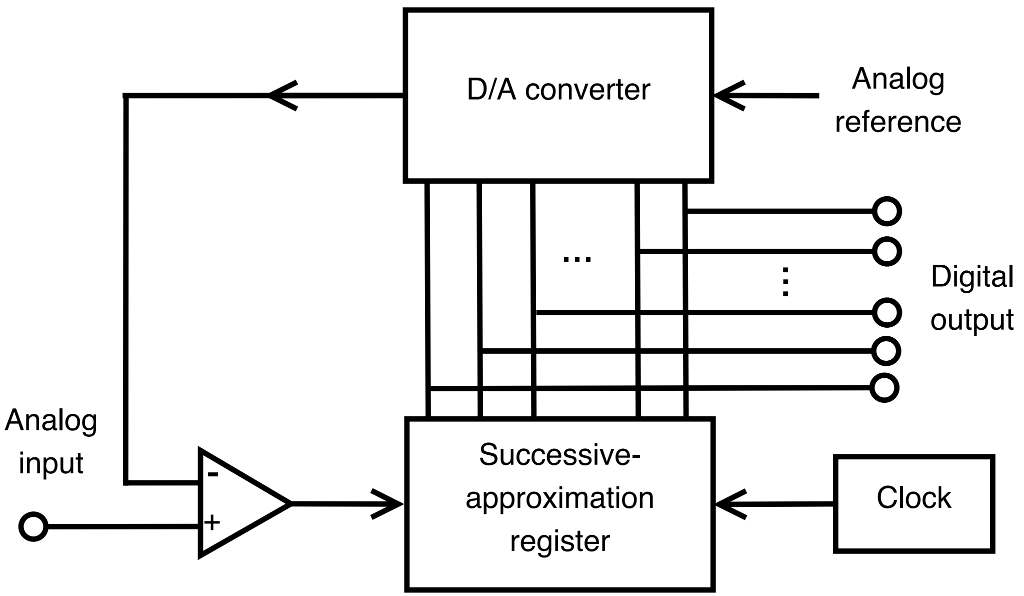
A digital-to-analog converter, also called a decoder, is a device that converts a digital signal into an analog signal.

Sample and hold



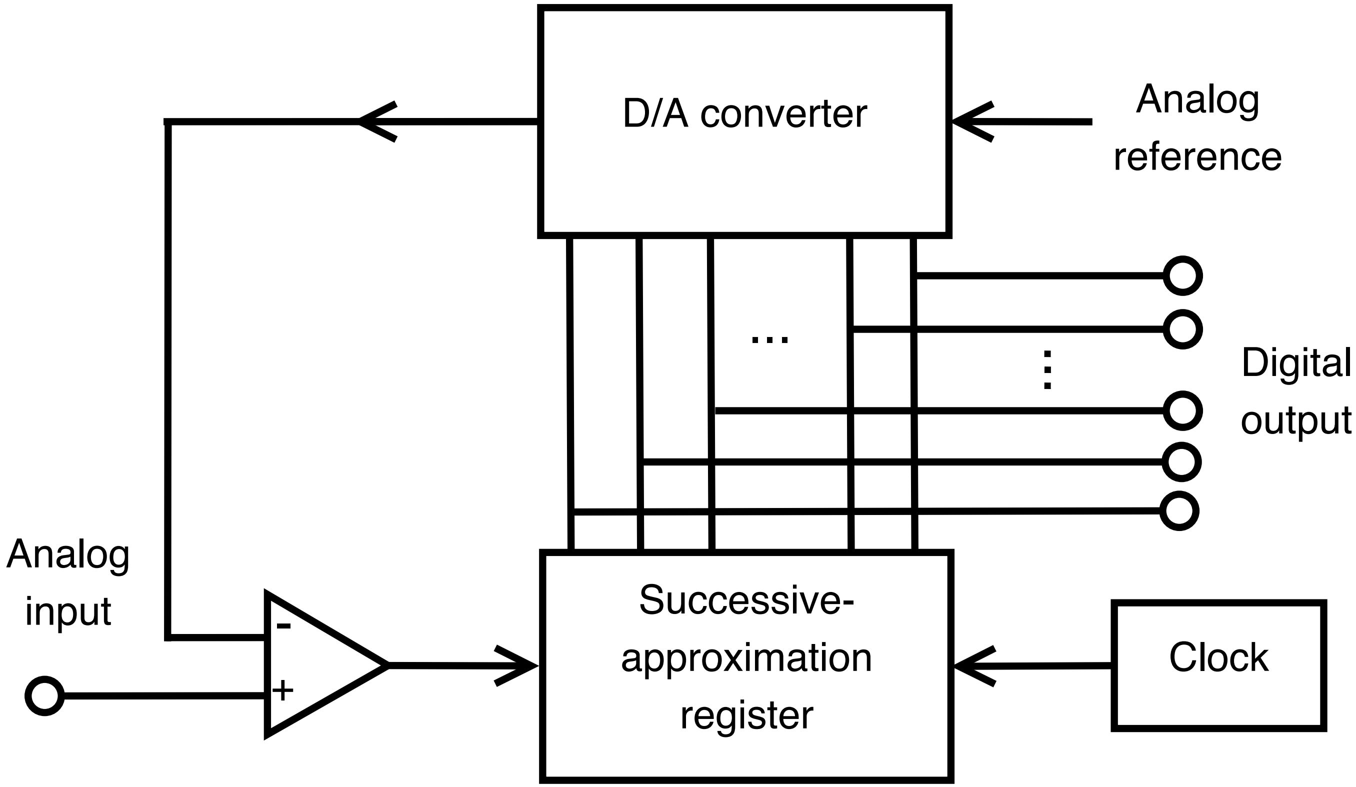
When the switch is closed, the input signal is connected and the circuit is in **tracking mode**. The charge on the capacitor in the circuit tracks the input voltage.
When the switch is open, the input signal is disconnected and the circuit is in **hold mode**. The capacitor voltage holds constant for a specified time period.

A/D converter



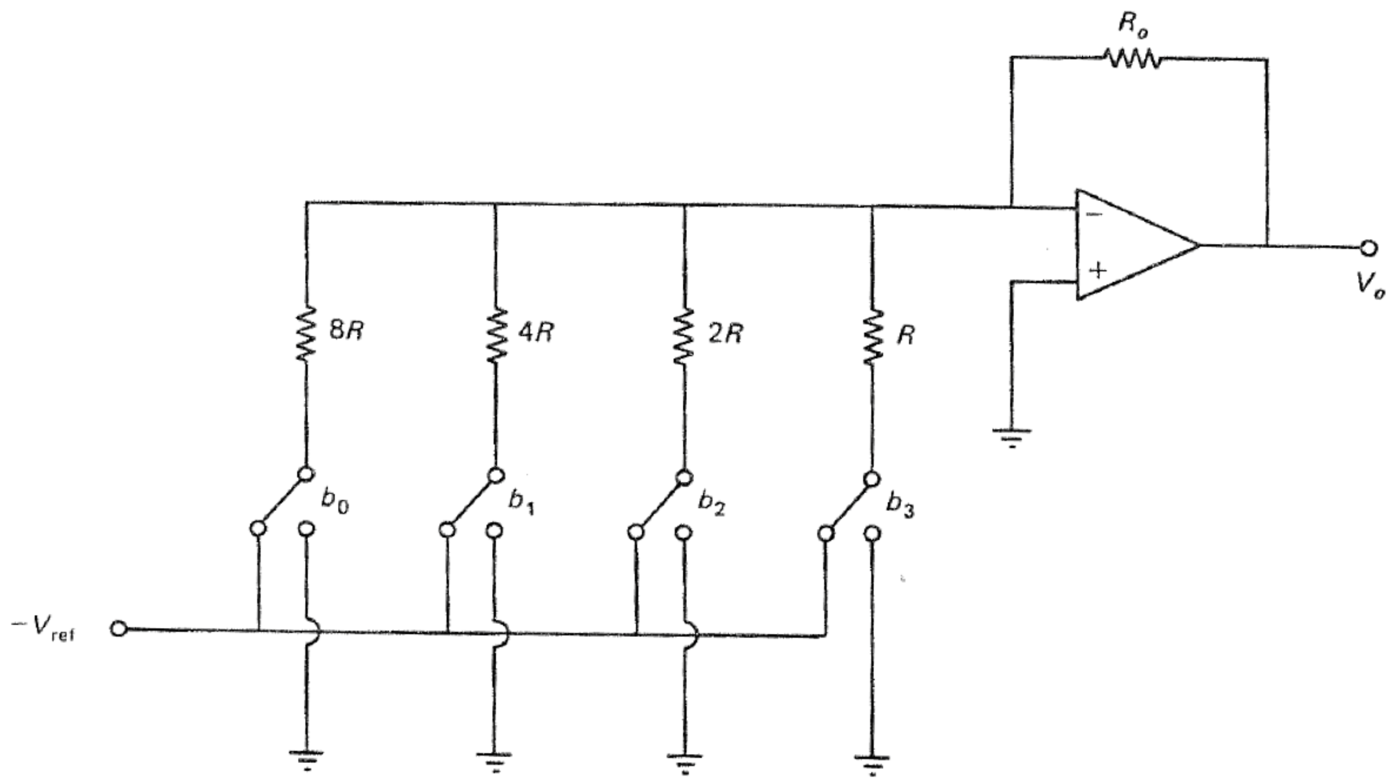
The successive-approximation register (SAR) is most frequently used. It turns on the most significant bit and compares it with the analog input. The comparator decides whether to leave the bit on or turn it off (if the analog input voltage is larger, the most significant bit is set on). Then the register continues with bit 2.

A/D converter



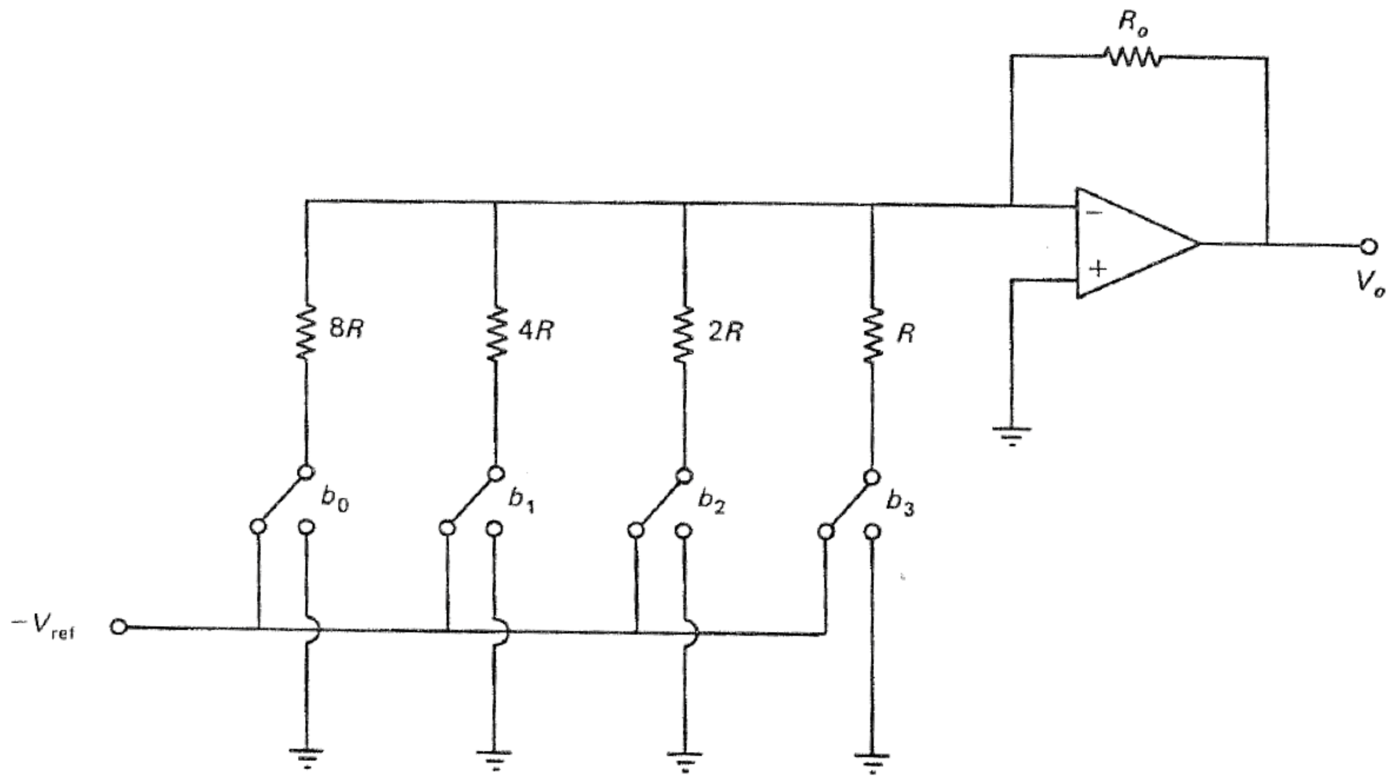
After n comparisons, the digital output of the register indicates all those bits that remain on and produces the desired digital code.

D/A converter



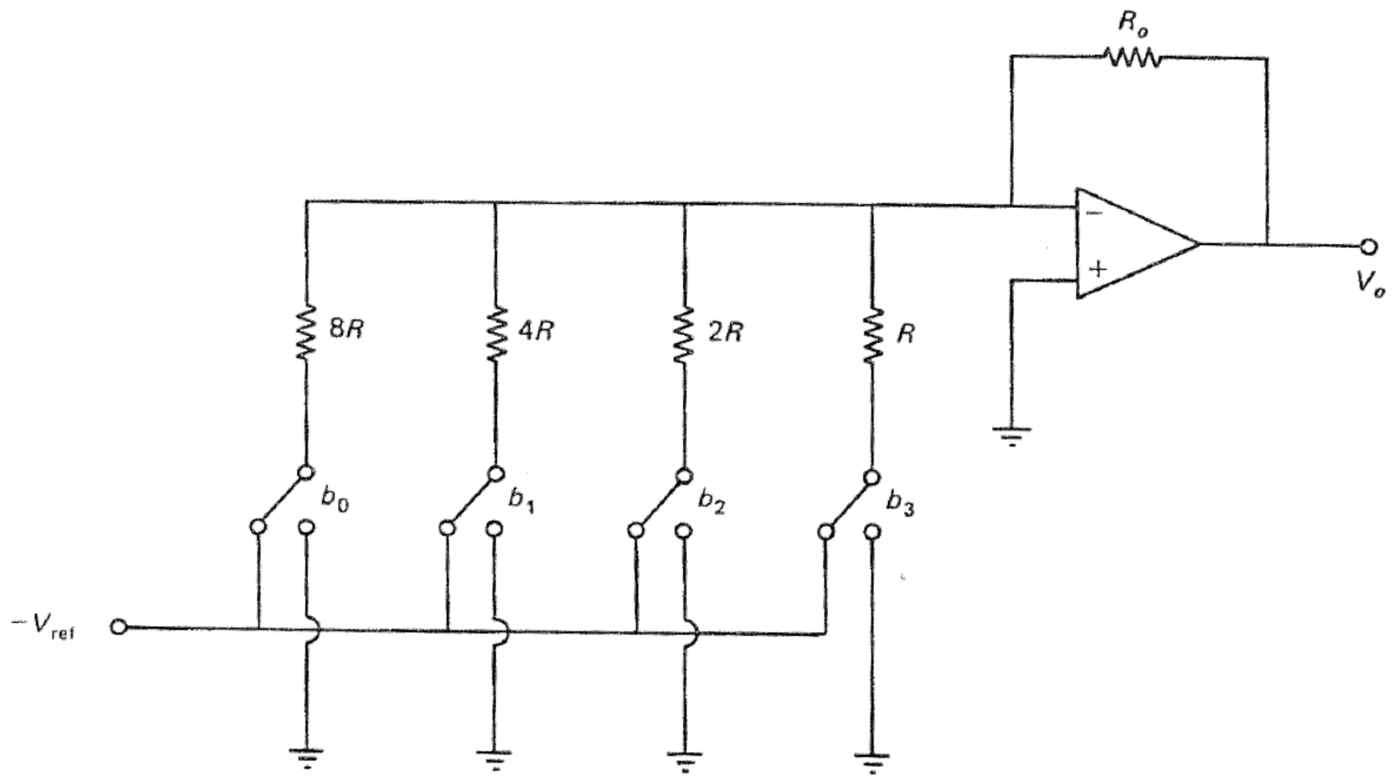
The input resistor of the operational amplifier have their resistance values weighted in a binary fashion.

D/A converter



When the logic circuit receives binary 1, the switch connects the resistor to the reference voltage. When the logic circuit receives binary 0, the switch connects the resistor to ground.

D/A converter



If the binary number (here) is $b_3 b_2 b_1 b_0$, each of the b's can be 0 or 1, the output is $V_o = \frac{R_o}{R}\left(b_3 + \frac{b_2}{2} + \frac{b_1}{4} + \frac{b_0}{8}\right) V_{ref}$

Outline

Analysis of the sample and hold

To get samples of a continuous signal, we use an analog-to-digital converter. The conversion always takes a non-zero time.

To give the computer an accurate representation of the signal exactly at the sampling instants kT_s, the converter is preceded by a sample-and-hold circuit.

The sample-and-hold will take the impulses that are produced by the mathematical sampler and produce the piecewise constant output of the device.

Sampling operation

The sampling operation is represented by impulse modulation. Its role is to give a mathematical representation of taking periodic samples from $r(t)$ to produce $r(kT_s)$.

The sampler takes as input $r(t)$ and returns as output

$$r^*(t) = \sum_{k=-\infty}^{\infty} r(t)\delta(t - kT_s).$$

The Laplace transform of $r^*(t)$ can be computed as

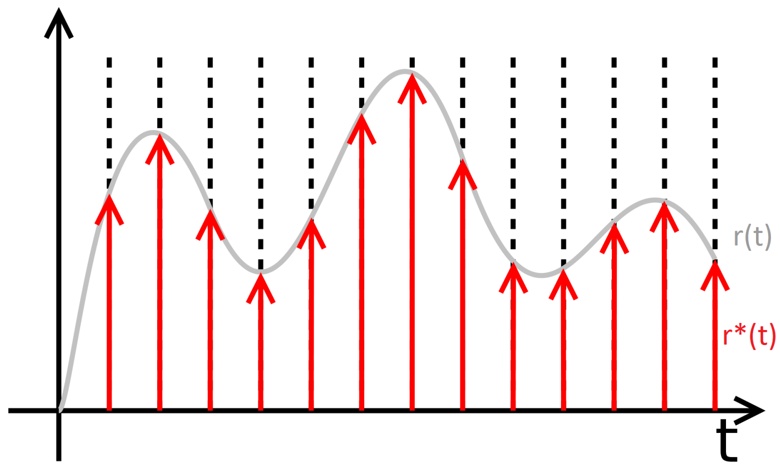
$$\mathcal{L}\{r^*(t)\} = \int_{-\infty}^{\infty} r^*(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} r(\tau)\delta(\tau - kT_s)e^{-s\tau}d\tau$$

Sampling operation

Using $\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$ we obtain

$$\mathcal{L}\{r^*(t)\} = R^*(s) = \sum_{k=-\infty}^{\infty} r(kT_s)e^{-skT_s}$$

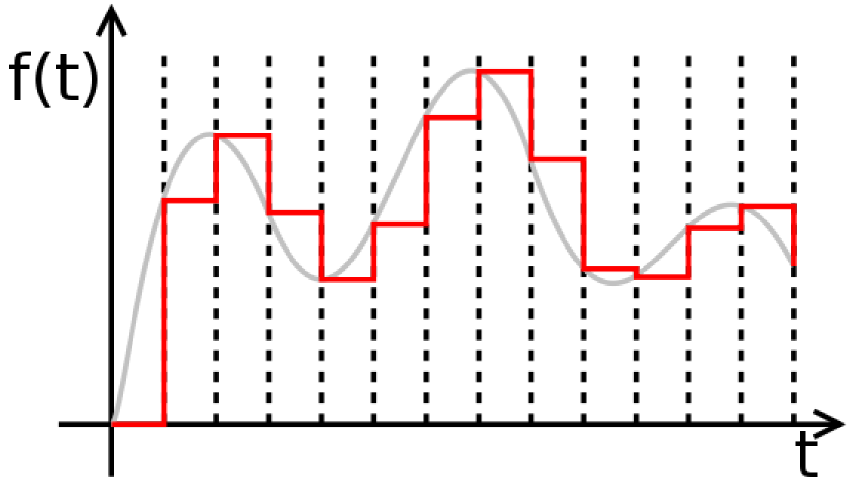
If the signal r(t) is shifted a small amount, then different samples will be selected by the sampling process for the output, proving that sampling is not a time-invariant process.



Hold operation

The hold operation is represented as a linear filter. It is defined by means whereby the impulses are extrapolated to the piecewise constant signal $r_h(t) = r(kT_s)$ with $kT_s \leq t < (k+1)T_s$.

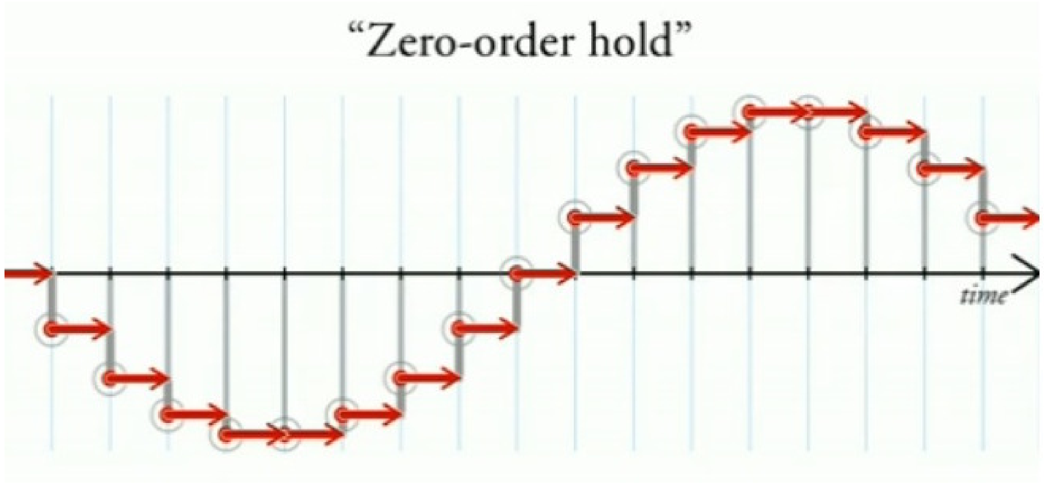
A general technique is to use a polynomial fit to the past samples.If the extrapolation is done by a constant (a zero-order polynomial), then the extrapolator is called a zero-order hold and its transfer function is denoted $ZOH(s)$.



Zero-order hold

We compute the transfer function as the transform of its impulse response. If $r^*(t) = \delta(t)$ then $r_h(t)$ is a pulse of height 1 and duration T: $r_h(t) = 1(t) - 1(t - T_s)$. Which has the following Laplace transform:

$$ZOH(s) = \mathcal{L}\{p(t)\} = \int_0^\infty [1(t) - 1(t - T_s)]e^{-s\tau} d\tau = \frac{1-e^{-sT_s}}{s}$$



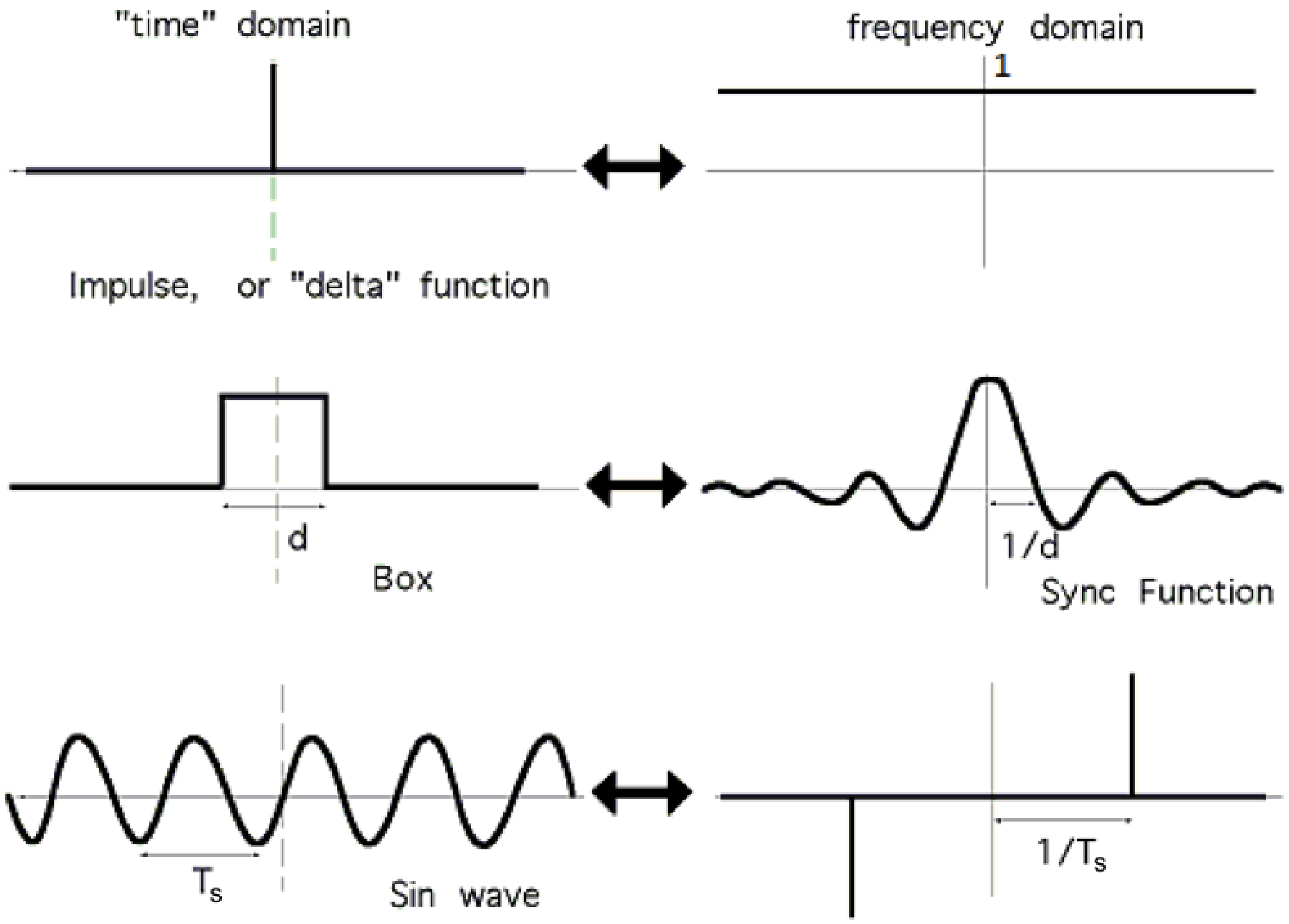
Outline

Fourier transform

$$f(t) \xrightarrow[F(j\omega)=\int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt]{} F(j\omega)$$

$$f(t) \xleftarrow[f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)e^{j\omega t}d\omega]{} F(j\omega)$$

Fourier transform



Fourier transform: properties

- **Linearity**

$$\begin{cases} f_1(t) \leftrightarrow F_1(j\omega) \\ f_2(t) \leftrightarrow F_2(j\omega) \end{cases} \Rightarrow \quad af_1(t) + bf_2(t) \leftrightarrow aF_1(j\omega) + bF_2(j\omega)$$

- **Time-scaling**

$$f(at) \leftrightarrow (\frac{1}{|a|})F(\frac{j\omega}{a})$$

- **Translation/Time-shifting**

$$f(t - t_0) \leftrightarrow e^{-j\omega t_0}F(j\omega)$$

- **Modulation/Frequency-shifting**

$$e^{j\omega_0 t}f(t) \leftrightarrow F(j(\omega - \omega_0))$$

Fourier transform: properties

- **Reciprocity**

$$F(-jt) \leftrightarrow 2\pi f(\omega)$$

- **Derivative in t**

$$\frac{df(t)}{dt} \leftrightarrow j\omega F(j\omega) \qquad \frac{d^n f(t)}{dt^n} \leftrightarrow (j\omega)^n F(j\omega)$$

- **Derivative in ω**

$$(-jt)^n f(t) \leftrightarrow \frac{d^n F(j\omega)}{d\omega^n} \qquad \frac{f(t)}{-jt} \leftrightarrow \int_{-\infty}^{\infty} F(j\Omega) d\Omega \; if \; f(0) = 0$$

- **Convolution**

$$y(t) = h(t) * u(t) \leftrightarrow Y(j\omega) = H(j\omega)U(j\omega)$$
$$v(t) = h(t)u(t) \leftrightarrow V(j\omega) = \frac{1}{2\pi} H(j\omega) * U(j\omega)$$

Outline

Spectrum of a sampled signal

$r^*(t)$ is the product of $r(t)$ and a train of impulses. The latter series is periodic and can be represented by a Fourier series:

$$\sum_{k=-\infty}^{\infty} \delta(t - kT_s) = \sum_{n=-\infty}^{\infty} C_n e^{j(2\pi n/T_s)} t,$$

where the Fourier coefficients C_n are given by:

$$C_n = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \sum_{k=-\infty}^{\infty} \delta(t - kT_s) e^{-jn(2\pi t/T_s)} dt.$$

Spectrum of a sampled signal

The only term in the sum of impulses that is in the range of the integral is the $\delta(t)$ at the origin, so the integral reduces to:

$$C_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2}\delta(t)e^{-jn(2\pi t/T_s)}dt = \frac{1}{T_s}.$$

We derived the Fourier series of the sum of impulses:

$$\sum_{k=-\infty}^{\infty}\delta(t-kT_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty}e^{j(2\pi n/T_s)t}.$$

We define $\omega_s = \frac{2\pi}{T_s}$ as the sampling frequency (rad/s).

Spectrum of a sampled signal

We take the Laplace transform of the output of the sampler,

$$\begin{aligned} R^*(s) &= \int_{-\infty}^{\infty} r(t)\left\{\frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}\right\}e^{-st}\,dt \\ &= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} r(t)e^{jn\omega_s t}e^{-st}\,dt \\ &= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} r(t)e^{-(s-jn\omega_s)t}\,dt. \end{aligned} \tag{1}$$

Spectrum of a sampled signal

Definition

Since the integral is the Laplace transform of $r(t)$ with only a change of variable where the frequency goes, the result can be written as:

$$R^*(s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} R(s - jn\omega_s).$$

Outline

Aliasing

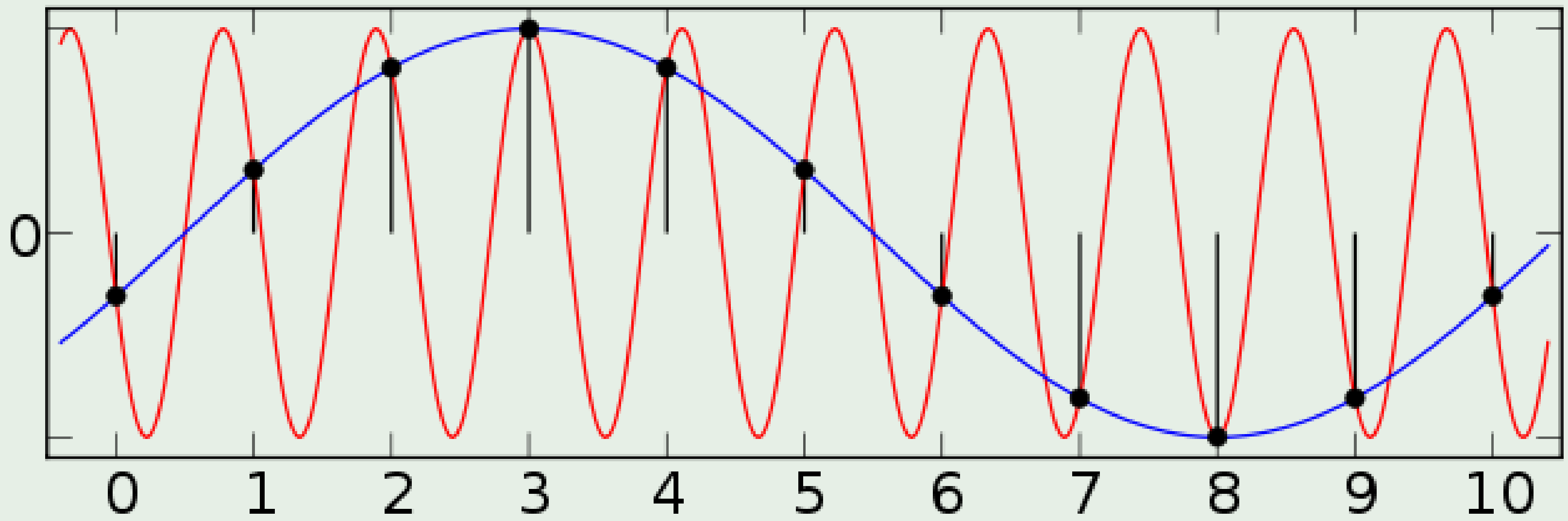
Definition

Aliasing is an effect that causes different signals to become indistinguishable when sampled. Frequencies that are too high to be sampled are folded onto lower frequencies. We cannot distinguish them based on their samples alone.

Aliasing

Example

The red sine wave is being sampled at just over its bandwidth, however the blue sine wave will be recreated as it also fits all data points and is within the expected bandwidth.



Aliasing

As a direct result of the sampling operation, when data is sampled at a frequency $\frac{2\pi}{T_s}$, the total harmonic content at a given frequency ω_1 is to be found not only from the original signal at ω_1, but also from all those frequencies that are aliases of ω_1, namely $\omega_1 + n2\pi/T_s = \omega_1 + n\omega_s$.

The errors caused by aliasing can be very severe if a substantial quantity of high-frequency components is contained in the signal to be sampled. To minimize this error, the sampling operation is preceded by a low-pass anti-aliasing filter that will remove all spectral content above the half-sampling frequency (π/T_s).

Outline

Shannon-Nyquist sampling theorem

If all content above the half-sampling frequency is removed, no aliasing is introduced by sampling. Also the signal spectrum is not distorted, even though it is repeated endlessly, centered at $n2\pi/T_s$.

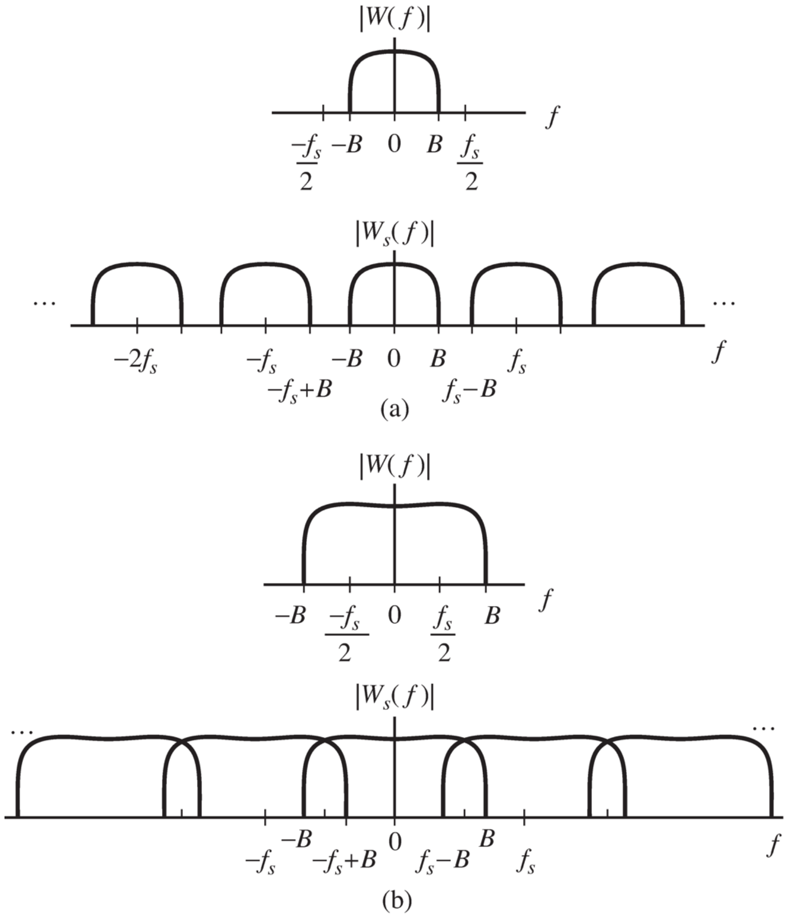
This critical frequency, π/T_s, is called the **Nyquist frequency**. Band-limited signals that have no components above the Nyquist frequency are represented unambiguously by their samples.

This is the **sampling theorem**: One can recover a signal from its samples if the sampling frequency ($\omega_s = 2\pi/T_s$) is at least twice the highest frequency (π/T) in the signal. This maximum frequency is also called the **bandwidth** B.

Shannon-Nyquist sampling theorem

The signal can be fully reconstructed if there are no overlaps in the frequency domain. If the sampling frequency is at least twice the bandwidth B, then the signal can be reconstructed without a problem (no overlap). (fig. a)
If the sampling frequency is too low then information will be lost (overlap). (fig. b)

Sampling frequency $f_s \geq 2B$



Outline

Hidden oscillations

Definition

There is the possibility that a signal contains some frequencies that the samples do not show at all.

Such signals, when they occur in digital control systems, are called **hidden oscillations**.

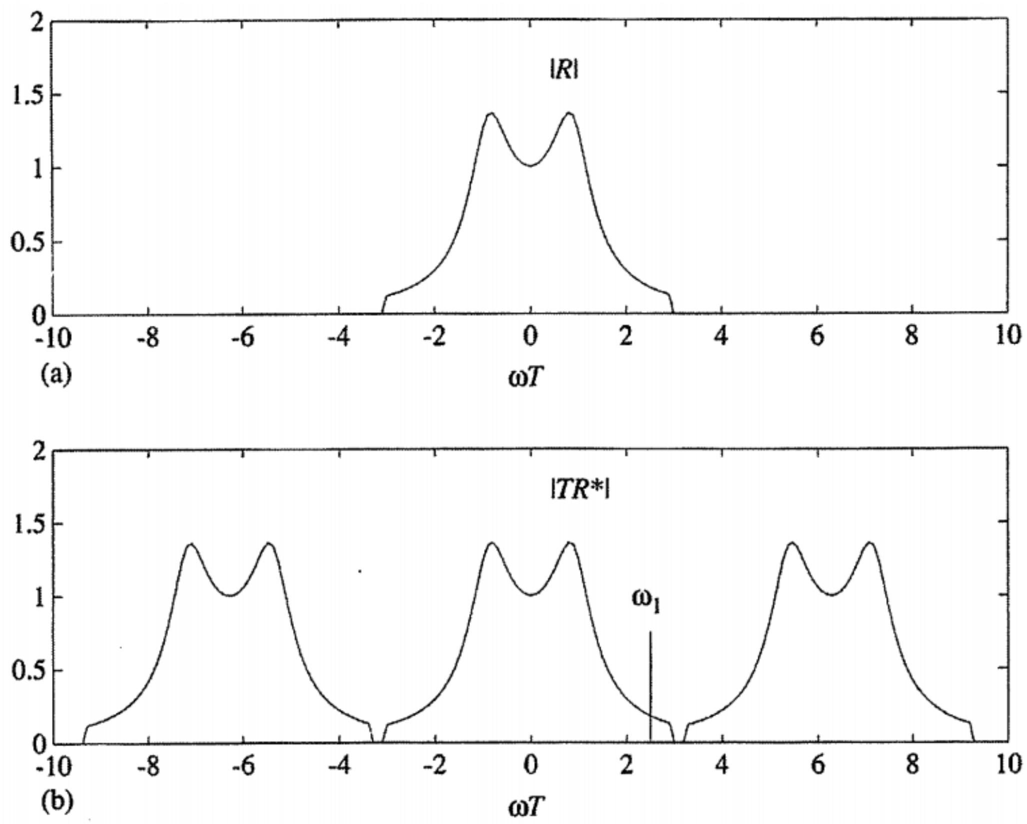
They can only occur at multiples of the Nyquist frequency (π/T_s).

Outline

Reconstruction

<u>Sampling theorem</u>: *under the right conditions* it is possible to recover a signal from its samples.

The figure to the right shows the spectrum of $R(j\omega)$. It is contained in the low-frequency part of $R^*(j\omega)$. Therefore, to recover $R(j\omega)$ we need to process $R^*(j\omega)$ through a low-pass filter and multiply by T_s.



Reconstruction

If $R(j\omega)$ has zero energy for frequencies in the bands above the Nyquist frequency, in other words R is band-limited, then an ideal low-pass filter with gain T_s for $-\pi/T_s \leq \omega \leq \pi/T_s$ and zero elsewhere would recover $R(j\omega)$ from $R^*(j\omega)$ exactly.

If we define the ideal low-pass filter characteristic as $L(j\omega)$, we have:

$$R(j\omega) = L(j\omega)R^*(j\omega).$$

The signal $r(t)$ is the inverse Fourier transform of $R(j\omega)$. Because $R(j\omega)$ is the *product* of two Fourier transforms, $r(t)$ is the *convolution* of the time functions $\ell(t)$ and $r^*(t)$.

$$r(t) = \ell(t) * r^*(t).$$

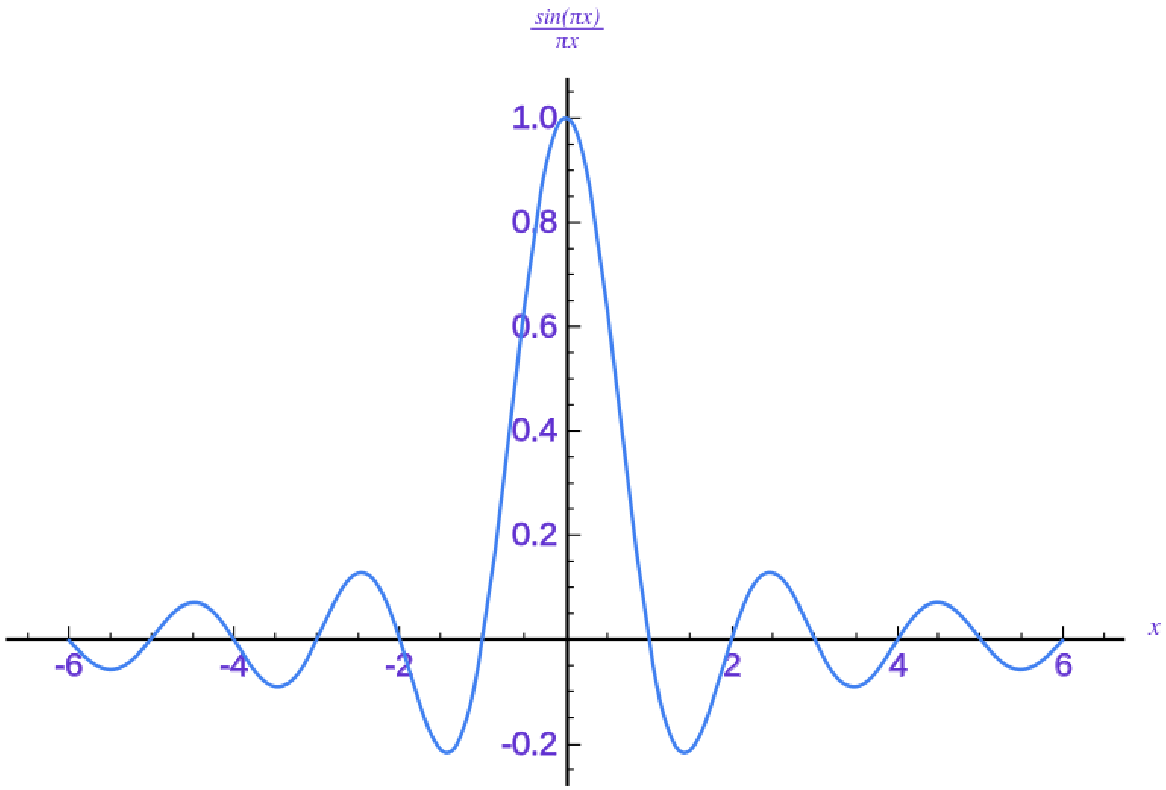
Ideal low-pass filter

The impulse response of the filter can be computed using this definition:

$$
\begin{aligned}
\ell(t) &= \frac{1}{2\pi}\int_{-\pi/T_s}^{\pi/T_s} T e^{j\omega t}\, d\omega \\
&= \frac{T_s}{2\pi}\frac{e^{j\omega t}}{jt}\Big|_{-\pi/T_s}^{\pi/T_s} \\
&= \frac{T_s}{2\pi jt}\left(e^{j(\pi t/T_s)} - e^{-j(\pi t/T_s)}\right) \\
&= \frac{sin(\pi t/T_s)}{\pi t/T_s} \\
&\triangleq sinc\frac{\pi t}{T_s}
\end{aligned}
$$

Ideal low-pass filter

The sinc functions are the interpolators that fill in the time gaps between samples with a signal that has no frequencies above π/T_s.



Reconstruction

Using the previous equations, we find:

$$r(t) = \int_{-\infty}^{\infty} r(\tau) \sum_{k=-\infty}^{\infty} \delta(\tau - kT_s) sinc \frac{\pi(t-\tau)}{T_s} d\tau.$$

Using the shifting property of the impulse, we obtain:

$$r(t) = \sum_{k=-\infty}^{\infty} r(kT_s) sinc \frac{\pi(t-kT_s)}{T_s}.$$

This filter is non-causal because $\ell(t)$ is nonzero for $t < 0$. $\ell(t)$ starts at $t = -\infty$ while the impulse that triggers it does not occur until $t = 0$. The non-causality can be overcome by adding a phase lag, $e^{-j\omega\lambda}$, to $L(j\omega)$, which adds a delay to the filter and to the signals processed through it.

Zero-order hold

The transfer function of the zero-order hold was introduced as

$$ZOH(j\omega) = \frac{1-e^{-j\omega T_s}}{j\omega}.$$

We express this function in magnitude and phase form, to discover the frequency properties of $ZOH(j\omega)$.

We factor out $e^{-j\omega T_s/2}$ and multiply and divide by $2j$:

$$\begin{aligned} ZOH(j\omega) &= e^{-j\omega T_s/2}\left\{\frac{e^{j\omega T_s/2} - e^{-j\omega T_s/2}}{2j}\right\}\frac{2j}{j\omega} \\ &= T_s e^{-j\omega T_s/2}\frac{sin(\omega T_s/2)}{\omega T_s/2} \\ &= e^{-j\omega T_s/2}T_s sinc(\omega T_s/2) \end{aligned}$$

Zero-order hold

The magnitude function is

$$|ZOH(j\omega)| = T_s \left| sinc \frac{\omega T_s}{2} \right|$$

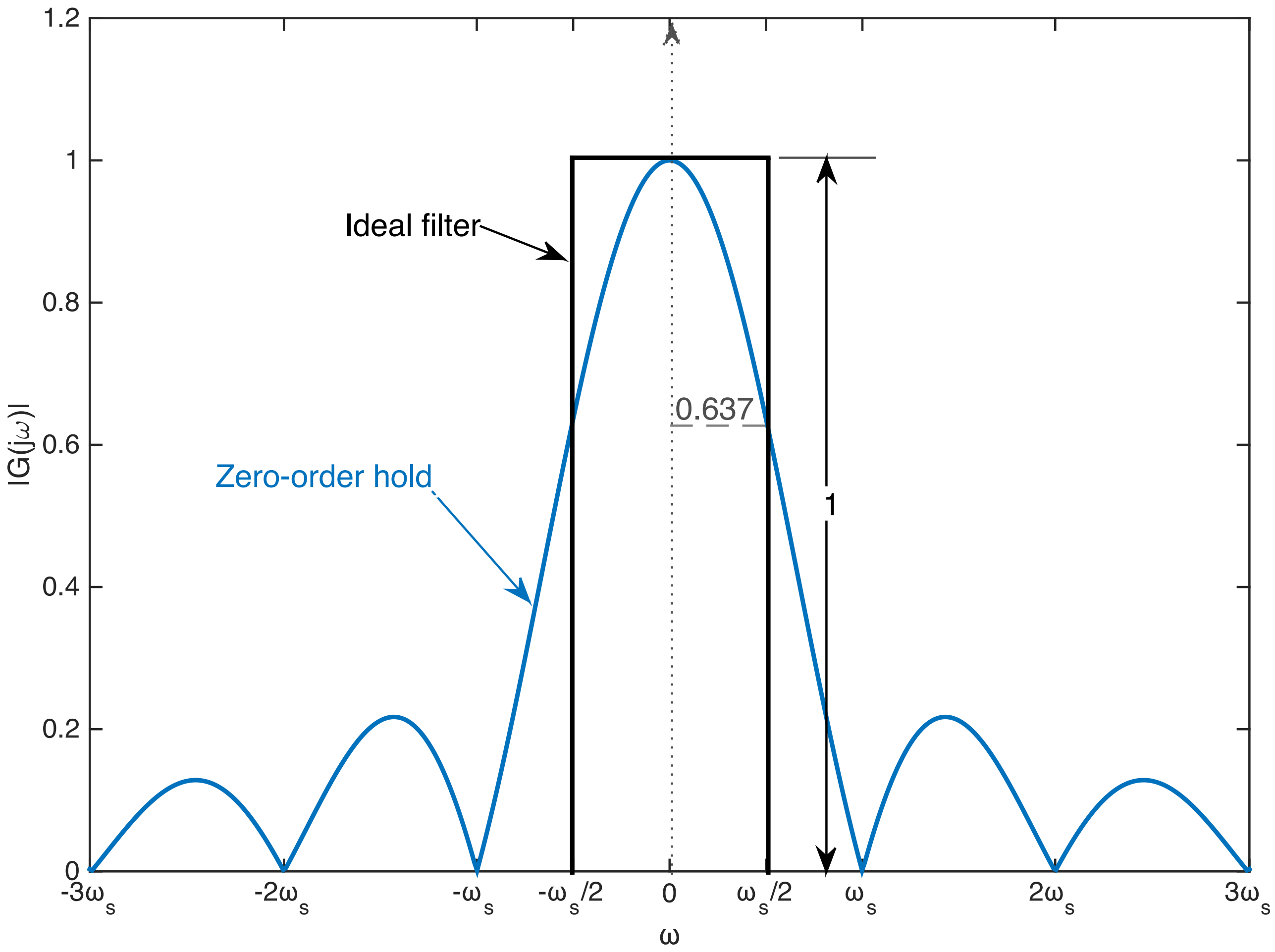
and the phase is

$$\angle ZOH(j\omega) = \frac{-\omega T_s}{2}$$

plus the 180° shifts where the sinc function changes sign.

Thus the effect of the zero-order hold is to introduce a phase shift of $\omega T_s/2$ (a time delay of $T_s/2$ seconds) and to multiply the gain by a function with the magnitude of $sinc(\omega T_s/2)$.

Zero-order hold filter vs ideal filter

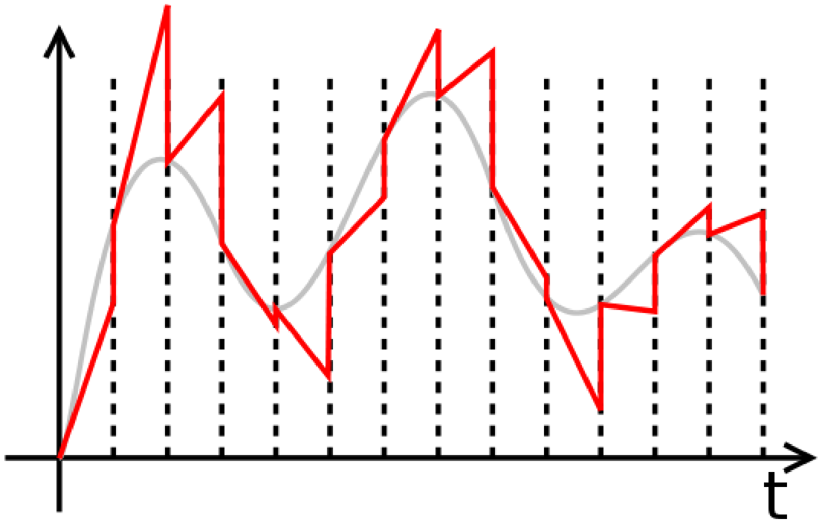


Note: for the sake of comparison, the magnitudes $|G(j\omega)|$ are normalized.

First-order hold

If the extrapolation is done by a first order polynomial, then the extrapolator is called a first-order hold and its transfer function is denoted $FOH(s)$.

$$FOH(s) = (1 - e^{-sT_s})\frac{sT_s+1}{sT_s^2}$$



Outline

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1. Introduction
2. Analysis of the sample and hold
3. Fourier transform
4. Spectrum of a sampled signal
   - Aliasing
   - Sampling theorem
   - Hidden oscillations
5. Data extrapolation (reconstruction)
6. Block-diagram analysis of sampled-data systems
   - General Approach
   - Examples

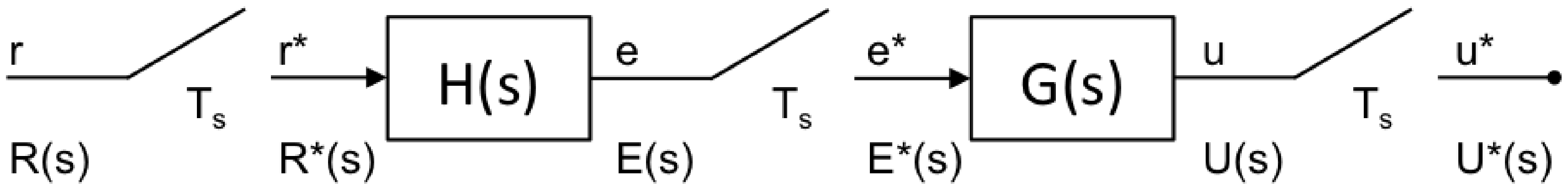
Problem Statement

To analyse a feedback system that contains a digital controller, we need to be able to compute the transforms of output signals of systems that contain sampling operations in various places, including feedback loops, in the block diagram.

The technique presented in this chapter is a simple extension of the block-diagram analysis of systems that are all continuous or all discrete. However a couple of rules need to be carefully observed.

Block-diagram analysis

We represent the process of sampling a continuous signal and holding it by impulse modulation followed by low-pass filtering. For example the following system:



leads to:

$$\begin{aligned} E(s) &= R^*(s)H(s) \\ U(s) &= E^*(s)G(s) \end{aligned} \tag{2}$$

Block-diagram analysis

Relation I

If the transform of the signal to be sampled is a product of a transform that is already periodic of period $\frac{2\pi}{T_s}$, and one that is not, as in $U(s) = E^*(s)G(s)$, where $E^*(s)$ is periodic and $G(s)$ is not, we can show that $E^*(s)$ comes out as a factor, resulting in the following rule:

$$U^*(s) = \left(E^*(s)G(s)\right)^* = E^*(s)G^*(s) \tag{3}$$

We will prove this in the frequency domain, using $R^*(s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} R(s - jn\omega_s)$.

Proof of $U^*(s) = \left(E^*(s)G(s)\right)^* = E^*(s)G^*(s)$

If $U(s) = E^*(s)G(s)$, then by definition we have:

$$U^*(s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} E^*(s - jn\omega_s)G(s - jn\omega_s), \tag{4}$$

but $E^*(s)$ is

$$E^*(s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} E(s - jk\omega_s), \tag{5}$$

so that

$$E^*(s - jn\omega_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} E(s - jk\omega_s - jn\omega_s). \tag{6}$$

Proof of $U^*(s) = \left(E^*(s)G(s)\right)^* = E^*(s)G^*(s)$

Now in Eq.(6) we can let $k = l - n$ to get

$$\begin{aligned} E^*(s - jn\omega_s) &= \frac{1}{T_s}\sum_{l=-\infty}^{\infty} E(s - jl\omega_s) \\ &= E^*(s) \end{aligned} \tag{7}$$

In other words, because E^* is already periodic, shifting it an integral number of periods leaves it unchanged. Substituting Eq.(7) into Eq.(4) yiels

$$\begin{aligned} U^*(s) &= E^*(s)\frac{1}{T_s}\sum_{-\infty}^{\infty} G(s - jn\omega_s) \\ &= E^*(s)G^*(s) \end{aligned} \tag{8}$$

Block diagram analysis

Note what is not true

If $U(s) = E(s)G(s)$, then $U^*(s) \neq E^*(s)G^*(s)$ but rather $U^*(s) = (EG)^*(s)$. The periodic character of E^* in Eq.(3) is crucial.

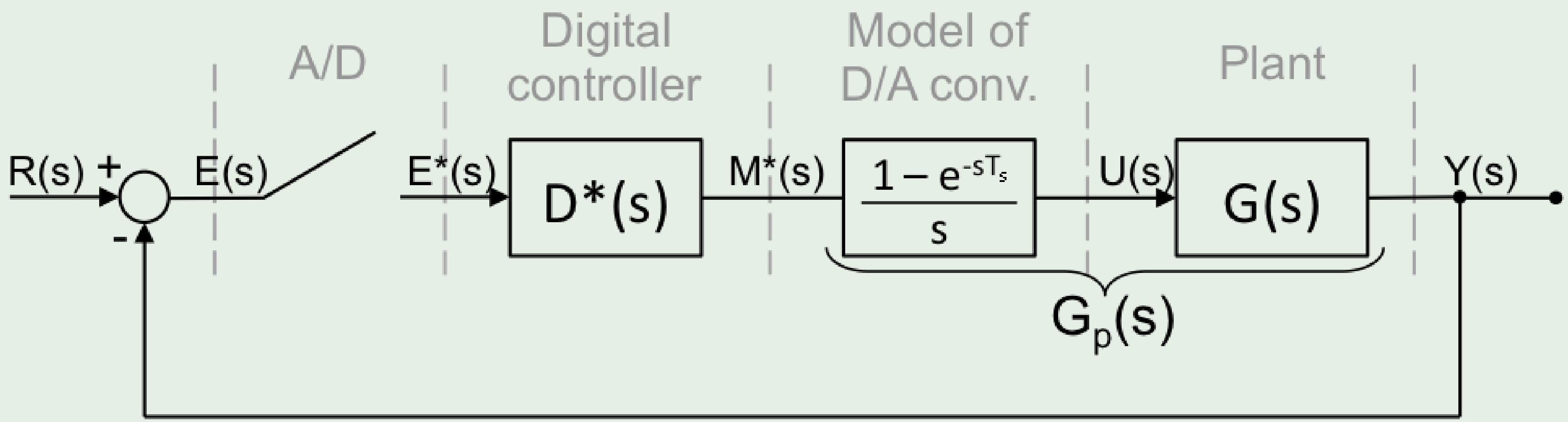
Relation II

Given a sampled-signal transform such as $U^*(s)$, we can find the corresponding $\mathcal{Z}$-transform simply by letting $e^{sT_s} = z$ or $U(z) = U^*(s)|_{e^{sT_s}=z}$.

Outline

Example 1 (1/4)

Let's compute the discrete-time representation of the following block diagram:



From the diagram we have that:

$$E(s) = R(s) - Y(s) \tag{9}$$

$$Y(s) = G_p(s)D^*(s)E^*(s) \tag{10}$$

Example 1 (2/4)

Applying Relation I on both Eq.(9) and Eq.(10) results in:

$$E^*(s) = R^*(s) - Y^*(s) \tag{11}$$

$$Y^*(s) = \left(G_p(s)D^*(s)E^*(s)\right)^* = G_p^*(s)D^*(s)E^*(s) \tag{12}$$

Next we insert Eq.(11) in Eq.(12):

$$Y^*(s) = G_p^*(s)D^*(s)\left(R^*(s) - Y^*(s)\right) \tag{13}$$

Example 1 (3/4)

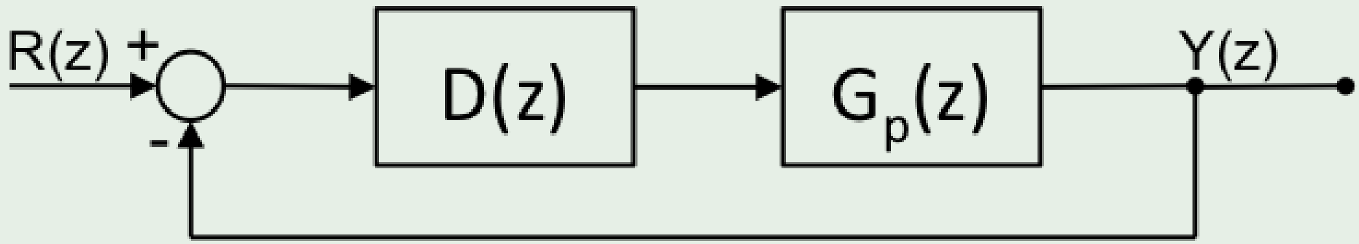
Finally we convert Eq.(13) into a transfer function:

$$
\begin{aligned}
Y^*(s) &= G_p^*(s)D^*(s)\big(R^*(s) - Y^*(s)\big) \\
Y^*(s) + G_p^*(s)D^*(s)Y^*(s) &= G_p^*(s)D^*(s)R^*(s) \\
Y^*(s)\big(1 + G_p^*(s)D^*(s)\big) &= G_p^*(s)D^*(s)R^*(s) \\
\frac{Y^*(s)}{R^*(s)} &= \frac{G_p^*(s)D^*(s)}{1 + G_p^*(s)D^*(s)} \\
\frac{Y(z)}{R(z)} &= \frac{G_p(z)D(z)}{1 + G_p^(z)D(z)}
\end{aligned}
$$

The last expression is obtained by applying Relation II.

Example 1 (4/4)

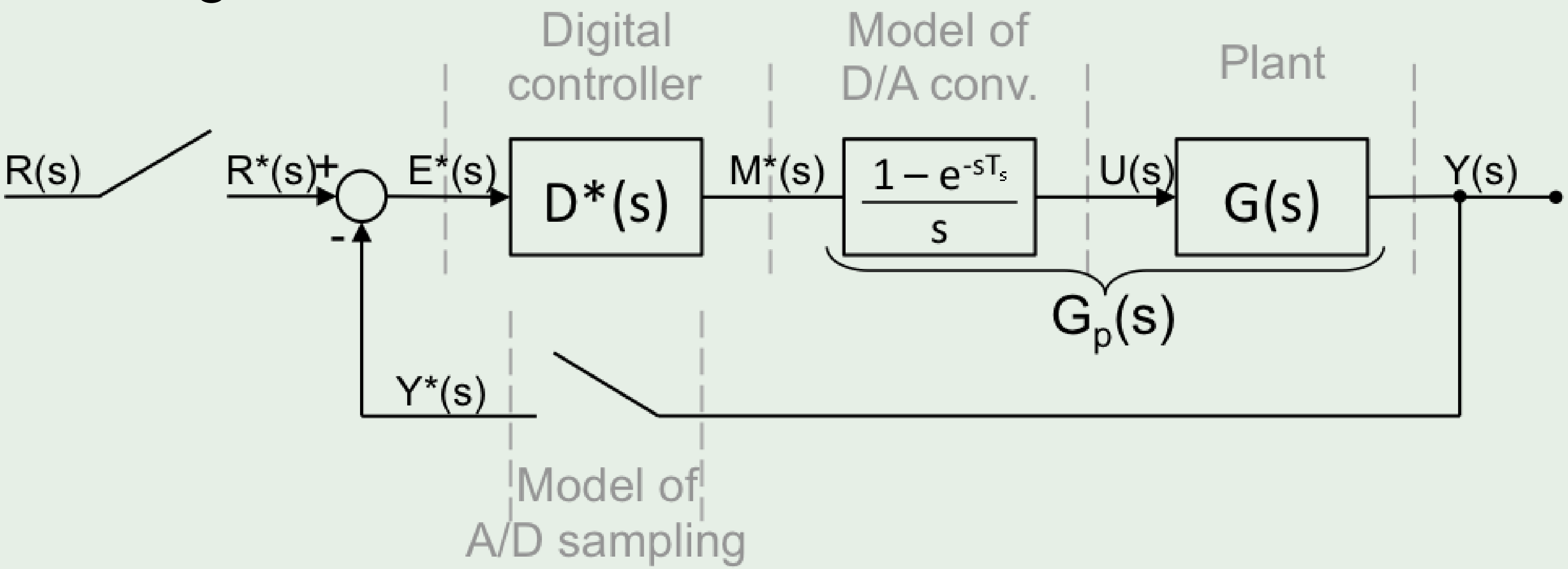
Notice that the resulting transfer function is also the transfer function of the following discrete-time system:



Consequently, we have found a discrete-time equivalent of a system which contains continuous-time and sampled signals.

Example 2 (1/3)

Let's compute the discrete-time representation of the following block diagram:



From the diagram we have that:

$$E^*(s) = R^*(s) - Y^*(s) \tag{14}$$

$$Y(s) = G_p(s)D^*(s)E^*(s) \tag{15}$$

Example 2 (2/3)

Applying Relation I on Eq.(15), yields:

$$Y^*(s) = \left(G_p(s)D^*(s)E^*(s)\right)^* = G_p^*(s)D^*(s)E^*(s) \tag{16}$$

Next we insert Eq.(14) in Eq.(16):

$$Y^*(s) = G_p^*(s)D^*(s)\left(R^*(s) - Y^*(s)\right) \tag{17}$$

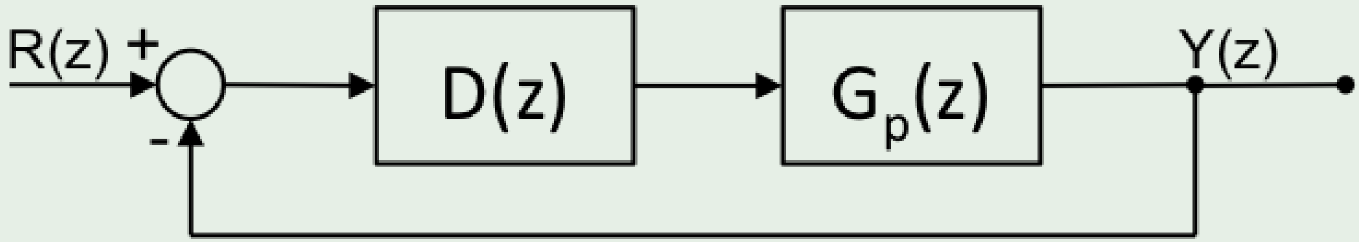
This is the exact same expression as we have for example 1. We will find the same transfer function from Eq.(17).

Example 2 (3/3)

The transfer function:

$$\frac{Y(z)}{R(z)} = \frac{G_p(z)D(z)}{1+G_p^{(}z)D(z)}$$

is also the transfer function of the following discrete-time system:



Consequently, we have found a discrete-time equivalent of a system which contains continuous-time and sampled signals.

NOTE

$$G_p^*(s) = \left[\frac{(1-e^{-sT_s})}{s} G(s)\right]^*$$

Taking out the periodic parts, which are those in which s appears only as e^{sT_s}, we have that

$$G_p^*(s) = (1 - e^{-sT_s})\left[\frac{G(s)}{s}\right]^*$$

Letting $e^{sT_s} = z$, we have that:

$$G_p(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$