

## Chapter 6: Frequency response of dynamical systems

July 9, 2015

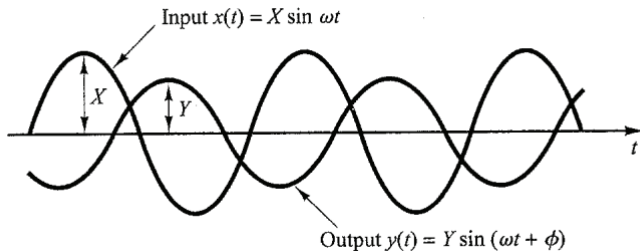
# Outline

- 1 The Frequency Response
- 2 What Is The Bode Plot
- 3 How To Construct A Bode Plot (by hand)
- 4 Constructing The Bode Plot In Matlab
- 5 Introduction To Nyquist Plots

# What is the frequency response of a system?

- Frequency response = steady state response of a system to a sinusoidal input.
- Assume an input  $x(t) = X \sin(\omega t)$ .
- The output in a linear system is then also sinusoidal, with a change in the magnitude and phase, i.e.  $y(t) = Y \sin(\omega t + \phi)$
- It can be shown that:  $Y = X \cdot |H(j\omega)|$  and  $\phi = \angle H(j\omega)$
- $H(j\omega)$  is therefore also called the sinusoidal transfer function.

# Relation of sinusoidal input/output in a linear system



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# The bode plot

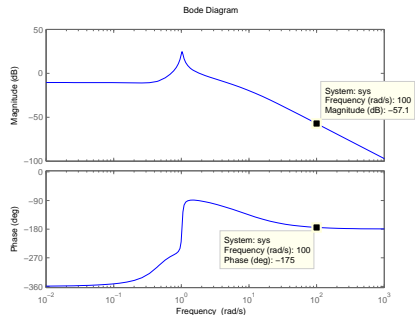
- A bode plot is a graphical representation of the sinusoidal transfer function  $H(j\omega)$
- It consists of two separate plots, a magnitude and a phase plot
- In this way, we can see the relation of a sinusoidal input with a given frequency  $\omega$  to a linear system and its output
- After all, the relation between the amplitudes is given by  $|H(j\omega)|$ , and the phase shift by  $\angle H(j\omega)$

# Example bode plot

$$H(s) = \frac{14s^2 + 7s + 3}{s^4 + 10s^3 + 10s^2 + 10s + 10}$$

- Say we use an input  $x(t) = 2 \sin(100t)$  in this system.
- The steady state response would be

$$y(t) = |H(j100)| \cdot 2 \sin(100t + \angle H(j100))$$



# The magnitude plot

Convention:

- for the ordinate (y-axis) we use  $20 \log_{10} |H(j\omega)|$  with the unit  $dB$
- for the abscissa (x-axis) we use a logarithmic plot of  $\omega$

This is thus a bi-log plot

The reason for using the logarithm of the modulus of  $H(j\omega)$  will become clear later



# The phase plot

Convention:

- for the ordinate (y-axis) we use  $\angle H(j\omega)$  in degrees
- for the abscissa (x-axis) we use a logarithmic plot of  $\omega$

This is thus a semi-log plot

# Discrete time systems

- The transfer function is a function of  $z$ , i.e.  $H(z)$
- In contrast to continuous time systems, we do not use  $H(j\omega)$ . Instead,  $H(e^{j\omega T_s})$  is now the sinusoidal transfer function.
- $T_s$  is the sample time, i.e. the amount of time in between each sample.

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# A new representation of the transfer function

From before:

$$H(s) = \frac{\beta_0 s^r + \beta_1 s^{r-1} + \dots + \beta_r}{s^n + \alpha_1 s^{n-1} + \dots + \alpha_n}$$

Factorization in zeros and poles

$$\Rightarrow H(s) = \frac{\beta_0 (s - n_1)(s - n_2) \dots (s - n_r)}{(s - p_1)(s - p_2) \dots (s - p_n)}$$

This is the usual representation. Now however, we will look for factors  $(1 + \frac{s}{s_i})$ , with  $s_i$  a so-called breakpoint.

## A new representation of the transfer function

We can do this by bringing all the zeros and poles not equal to zero outside the brackets, as follows:

$$H(s) = \beta_0 \frac{\prod(-n_i) (1 + \frac{s}{-n_1})(1 + \frac{s}{-n_2}) \dots (1 + \frac{s}{-n_i})}{\prod(-p_j) s^l (1 + \frac{s}{-p_1})(1 + \frac{s}{-p_2}) \dots (1 + \frac{s}{-p_j})}$$

Replacing the constants by  $K$ , and setting

$$r_k = -n_k$$

$$s_k = -p_k$$

# A new representation of the transfer function

We ultimately get:

$$H(s) = K \frac{(1 + \frac{s}{r_1})(1 + \frac{s}{r_2}) \dots (1 + \frac{s}{r_i})}{s^l (1 + \frac{s}{s_1})(1 + \frac{s}{s_2}) \dots (1 + \frac{s}{s_j})}$$

Now we are able to construct the bode plot of each different factor of  $H(s)$ . Afterwards we can just add up these plots using the calculation rules of complex numbers.

## Intermezzo complex numbers

- The magnitude of the product of complex numbers is equal to the product of the magnitudes of these numbers
- The phase of the product of complex numbers is equal to the sum of the phases of these numbers
- The logarithm of a product of numbers is equal to the sum of the logarithms of these numbers

This comes down to

$$20 \log_{10} |H(j\omega)| = \sum 20 \log_{10} |\text{factors}|$$

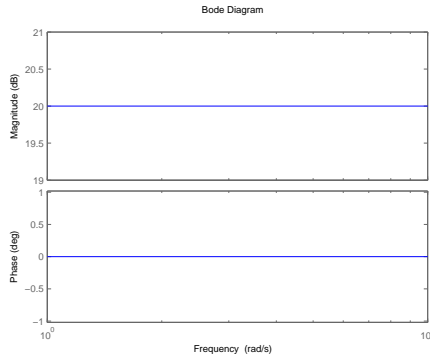
$$\angle H(j\omega) = \sum (\angle \text{factors})$$

Next we will quickly go over the simple bodeplots of the different factors of  $H(s)$

## The constant K

- $20 \log_{10} |K| = \text{constant}$
- $\angle K = 0^\circ$  or  $\pm 180^\circ$  (resp  $K > 0$  and  $K < 0$ )

Example:  $K = 10$





## $(1 + \frac{j\omega}{r_i})$ in the numerator

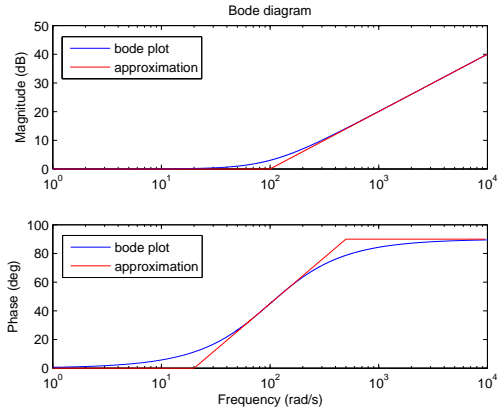
(Assume  $r_i > 0$ )

- What if  $\omega \rightarrow 0$  ?  $(1 + \frac{j\omega}{r_i}) \rightarrow 1$ 
  - $20 \log_{10} |1| = 0$
  - $\angle 1 = 0^\circ$
- What if  $\omega \rightarrow \infty$  ?  $(1 + \frac{j\omega}{r_i}) \rightarrow j\infty$ 
  - $20 \log_{10} |j\infty| = \infty$
  - $\angle j\infty = 90^\circ$
- The two terms balance each other out for  $\omega = r_i$  (remember, this is called a breakpoint).
  - $20 \log_{10} |1 + j| = 20 \log_{10}(\sqrt{2}) \approx 3\text{dB}$
  - $\angle(1 + j) = 45^\circ$

A breakpoint is therefore also called a 3dB point

# $(1 + \frac{j\omega}{r_i})$ in the numerator

Example:  $r_i = 100$



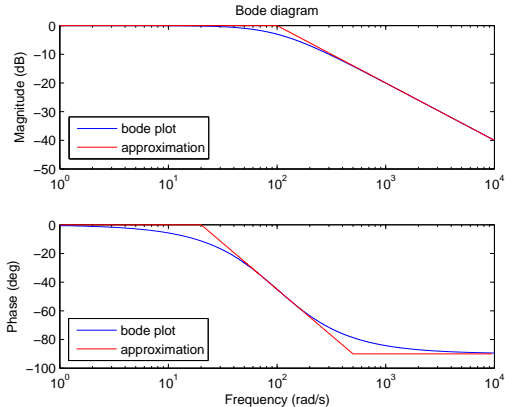
## $(1 + \frac{j\omega}{s_i})$ in the denominator

This factor is equivalent to the previous one. The only difference is the sign change in both plots:

- What if  $\omega \rightarrow 0$  ?  $\frac{1}{1 + \frac{j\omega}{s_i}} \rightarrow 1$ 
  - $20 \log_{10} |1| = 0$
  - $\angle 1 = 0^\circ$
- What if  $\omega \rightarrow \infty$  ?  $\frac{1}{1 + \frac{j\omega}{s_i}} \rightarrow \frac{1}{j\infty} \rightarrow -j0$ 
  - $20 \log_{10} |-j0| = -\infty$
  - $\angle -j0 = -90^\circ$
- The two terms balance each other out for  $\omega = s_i$ 
  - $20 \log_{10} \frac{1}{|1+j|} = 20 \log_{10}(\frac{1}{\sqrt{2}}) \approx -3\text{dB}$
  - $\angle \frac{1}{(1+j)} = -45^\circ$

# $(1 + \frac{j\omega}{s_i})$ in the denominator

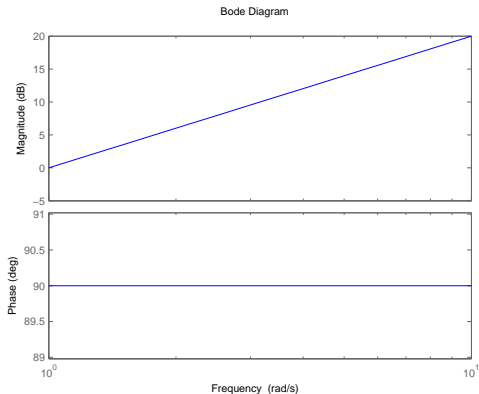
Example  $s_i = 100$



## $j\omega$ in the numerator

- This is simply a (ascending) straight line in the magnitude plot, with a slope of 20 dB/decade
- Constant phase of  $90^\circ$
- What if  $\omega \rightarrow 0$  ?  $j\omega \rightarrow j0$ 
  - $20 \log_{10} |j0| = -\infty$
  - $\angle j0 = 90^\circ$
- What if  $\omega \rightarrow \infty$  ?  $j\omega \rightarrow j\infty$ 
  - $20 \log_{10} |j\infty| = \infty$
  - $\angle j\infty = 90^\circ$

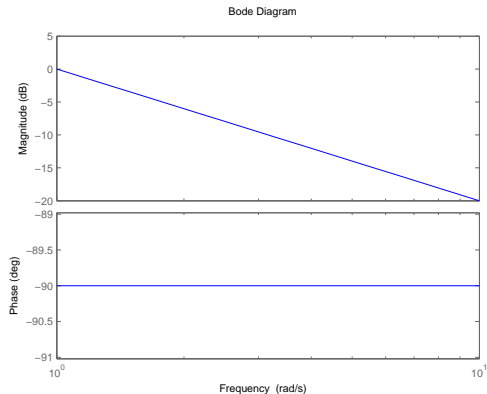
## $j\omega$ in the numerator



## $j\omega$ in the denominator

- This is simply a (descending) straight line in the magnitude plot, with a slope of -20 dB/decade
- Constant phase of  $-90^\circ$
- What if  $\omega \rightarrow 0$  ?  $\frac{1}{j\omega} \rightarrow \frac{1}{j0} \rightarrow -j\infty$ 
  - $20 \log_{10} |-j\infty| = \infty$
  - $\angle -j\infty = -90^\circ$
- What if  $\omega \rightarrow \infty$  ?  $\frac{1}{j\omega} \rightarrow \frac{1}{j\infty} \rightarrow -j0$ 
  - $20 \log_{10} |-j0| = -\infty$
  - $\angle -j0 = -90^\circ$

## $j\omega$ in the denominator





# What if the multiplicity is higher than 1?

Take for example multiplicity 2, a second order factor.

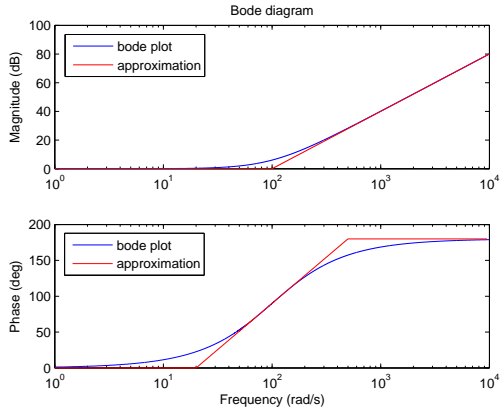
A second order factor has twice the effect of a first order factor.

Consider for example the effect of a double zero:

- Slope of 40dB/decade instead of 20dB/decade after the breakpoint
- Phase shift of  $180^\circ$  instead of  $90^\circ$

## Second order factor

$$\left(1 + \frac{j\omega}{100}\right)^2$$



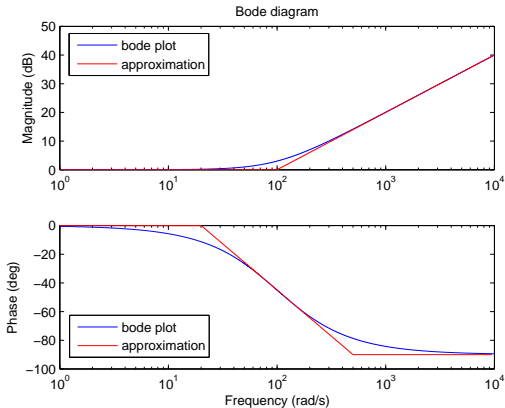
## Exception

Up until now we always considered  $r_i$  and  $s_i > 0$ , but what if we have a factor  $(1 - \frac{j\omega}{r_i})$  for example?

- The magnitude plot remains unchanged, as  $|1 + \frac{j\omega}{r_i}| = |1 - \frac{j\omega}{r_i}|$
- The phase plot is reversed, as  $\angle(1 + \frac{j\omega}{r_i}) = -\angle(1 - \frac{j\omega}{r_i})$
- If we have such a factor in the denominator, the system will be unstable!

# Exception

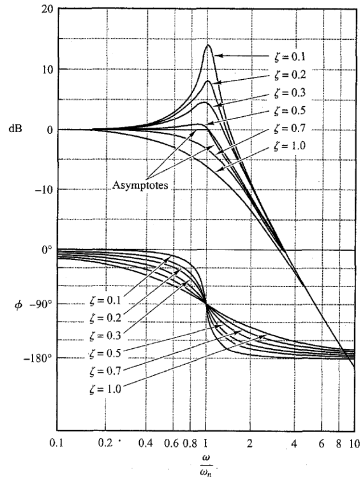
Example  $(1 - \frac{j\omega}{100})$  in the numerator



## Quadratic factors and resonance

- Also possible is a quadratic factor  $[1 + 2\zeta(j\frac{\omega}{\omega_n}) + (j\frac{\omega}{\omega_n})^2]^{\pm 1}$
- $\zeta$  is called the damping factor,  $\omega_n$  is the natural frequency
- If  $\zeta > 1$ , this quadratic factor can be expressed as two first order factors with real zeros/poles
- But if  $0 < \zeta < 1$ , this quadratic factor is the product of two complex-conjugate factors
- For low  $\zeta$ , the asymptotic approximations are not accurate. Instead, a peak occurs in the magnitude plot around  $\omega_n$
- This peak is a phenomenon known as resonance.

## Quadratic factors and resonance



## Example

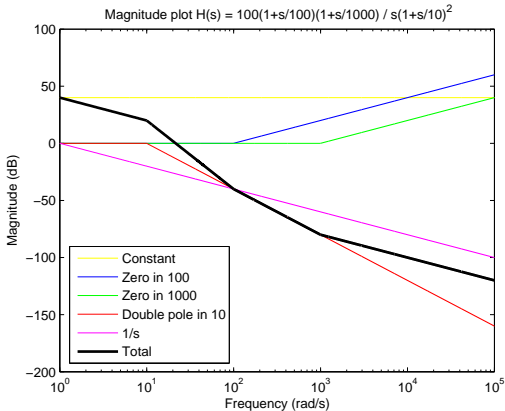
Suppose we want to construct (by hand) the bode plot of

$$H(s) = \frac{s^2 + 1100s + 100000}{10s^3 + 200s^2 + 1000s}$$

The first step is to find the representation with breakpoints.

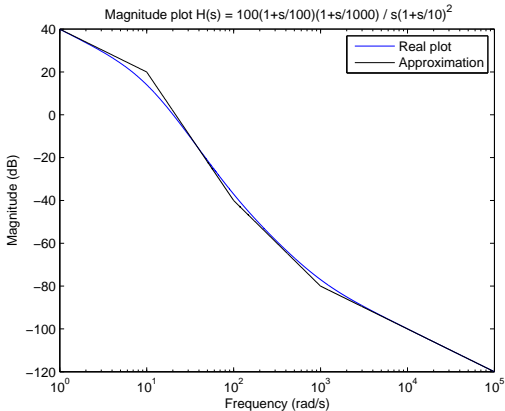
$$\begin{aligned} H(s) &= \frac{s^2 + 1100s + 100000}{10s^3 + 200s^2 + 1000s} \\ &= \frac{(s + 100)(s + 1000)}{10s(s + 10)^2} \\ &= \frac{100000(1 + \frac{s}{100})(1 + \frac{s}{1000})}{1000s(1 + \frac{s}{10})^2} = \frac{100(1 + \frac{s}{100})(1 + \frac{s}{1000})}{s(1 + \frac{s}{10})^2} \end{aligned}$$

## Example Magnitude plot

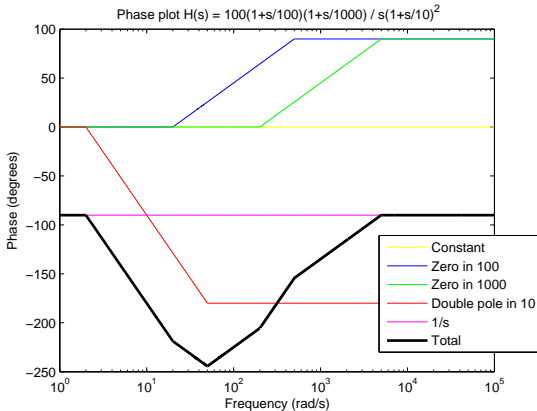




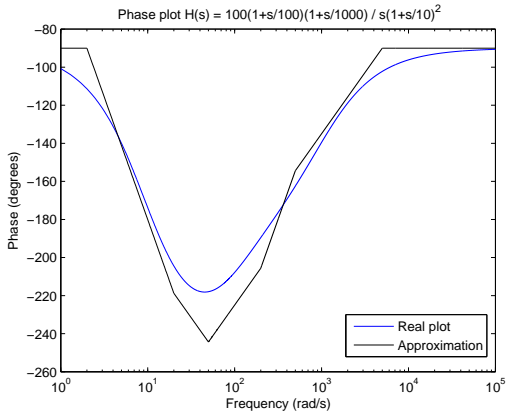
## Example Magnitude plot



## Example Phase plot



## Example Phase plot



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## Basic commands in Matlab

In Matlab it is very easy to draw the bode plot.

- First, define the system using one of the following commands:
  - `tf(num,den)` (`num` and `den` are respectively the numerator and denominator of the transfer function)
  - `zpk(z,p,K)` (using the zeros (`z`), the poles (`p`) and the gain (`K`) of the transfer function)
  - `ss(A,B,C,D)` (using the matrices of the state-space model)
- In case of a discrete time system, `Ts` (the sample time) is also needed as a last parameter in these commands
- Next, use the command `bode(sys)`

## Matlab example

%Examples for creating the bode plot in Matlab

%Say we have the transfer function

$H(s) = (5s^2 - 10s + 5)/(s^2 + 5s + 4)$

num = [5 -10 5];

den = [1 5 4];

sys = tf(num,den);

bode(sys)

figure

%Using the same system, we first find the factorization

$H(s) = 5*(s-1)^2/[(s+1)(s+4)]$

z = [1 1];

p = [-1 -4];

K = 5;

## Matlab example

```
sys = zpk(z,p,K);  
bode(sys)
```

```
figure
```

```
%If we had a discrete time system with the same transfer
```

```
%function
```

```
% $H(z) = (5z^2 - 10z + 5)/(z^2 + 5z + 4)$ 
```

```
%and sampling time  $T_s = 1/2$  of a second
```

```
sys = tf(num,den,0.5);  
bode(sys)
```

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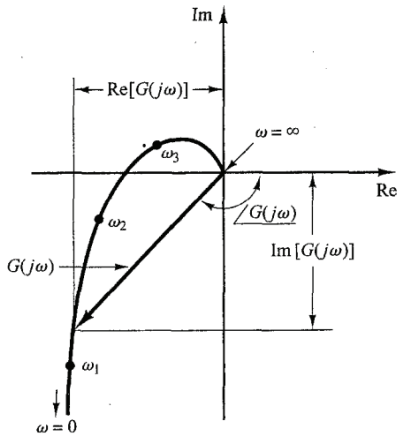
# Nyquist plot

A Nyquist plot is also called a polar plot, and is another way to plot  $H(j\omega)$ .

In a polar plot, as  $\omega$  is varied from 0 to  $\infty$ ,  $H(j\omega)$  is plotted as a point in the complex plane.

- $|H(j\omega)|$  is the distance between the origin and the point
- $\angle H(j\omega)$  is the angle between the vector to the point and the positive real axis, measured counterclockwise

# Nyquist plot



## Nyquist plot in matlab

Similar to constructing the bode plot in matlab, we first have to define the system using `tf`, `zpk` or `ss`.

Then we use the command `nyquist(sys)`.

```
%How to create a Nyquist plot in matlab  
sys = tf([14 7 3],[1 10 10 10 10]);  
nyquist(sys)
```

# Bode and Nyquist plot

