Chapter 3: System Modeling

August 26, 2015

Introduction

We can derive the mathematical model of a dynamic system in **two ways** mainly:

1. Physical Modeling:
   Applying the laws of physics, chemistry, thermodynamics,...
   Also called modeling from *First Principles*
   - Sometimes these are non-linear. Lots of methods of this course require linear systems. Therefore **linearization** is needed.
     e.g. $\sin(\theta) \sim \theta, \theta \to 0$
2. System identification or *Empirical Modeling*:
   Developing models from observed or collected data

Main classes of System identification methods

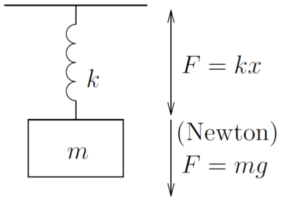
White box modeling: based on first principles.
$\rightarrow$ known equations (structure) & parameters (coefficients).

Grey box identification: first principles & experimentation.
$\rightarrow$ known equations, unknown/uncertain parameters.

Black box identification: based on experimentation.
$\rightarrow$ unknown equations & unknown parameters.

Most popular approaches are forms of black box identification.

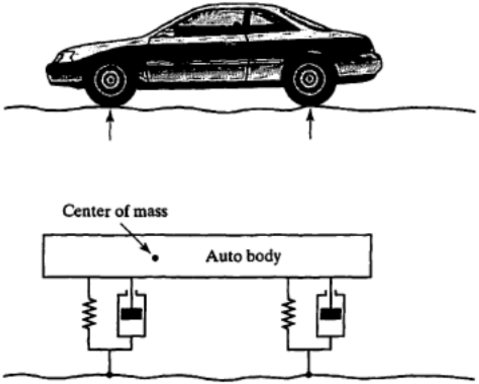
Example 1: Mass-Spring System

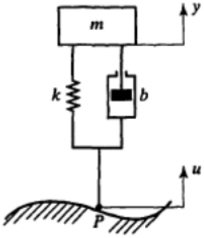


If spring is at rest at $x = 0$:

$$m \cdot \frac{d^2x}{dt^2} + k \cdot x = m \cdot g$$

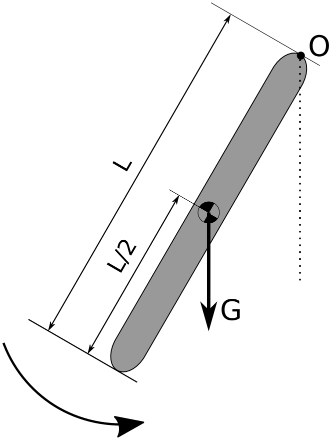
Example 2: Mass-Spring Damped





Force exerted by damper: $F = b\dot{x}$
Differential equation can be found by writing force equilibrium and moment equilibrium around center of mass

Example 3: Pendulum

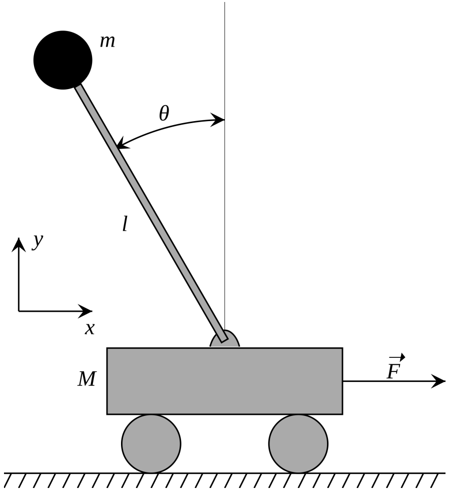


Dynamic equilibrium:

$$I\ddot{\theta}(t) = -mg\frac{L}{2}\sin(\theta(t)) \text{ with } I = \frac{mL^2}{3}$$
$$\ddot{\theta}(t) = -\frac{3g}{2L}\sin(\theta(t))$$

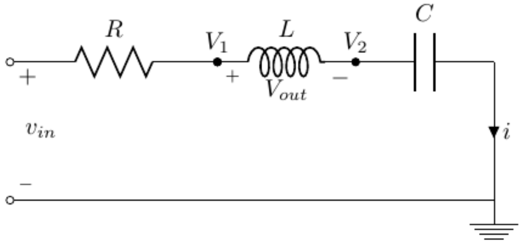
Small deviation of $\theta(t)$:
$$\ddot{\theta}(t) = -\tfrac{3g}{2L}\theta(t)$$
Solving the differential equation yields the general solution:
$$\theta(t) = A\cos(\omega_0 + \phi) \text{ with } \omega_0 = \sqrt{\tfrac{3g}{2L}}$$
and ϕ & A to be determined with the initial condition

Example 4: Inverted Pendulum



Analysis can be done with Newton like former example, but less tedious is using energy-methods (Lagrange)

Example 5: RLC Circuit



Besides input v_{in}, two internal variables are needed to determine output $\Rightarrow$ Second-order System

Inputs	Ouputs	Choosen States
v_{in}	v_{out}	V_2
		i

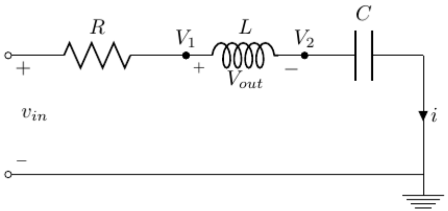
Example 5: RLC Circuit

Equations for each component:

$$i = \frac{V_{in} - V_1}{R} \qquad \text{(Ohm's law)}$$

$$V_1 - V_2 = L \cdot \frac{di}{dt} \qquad \text{(Coil)}$$

$$i = C \cdot \frac{dV_2}{dt} \qquad \text{(Capacitor)}$$



Example 5: RLC Circuit

- Writing derivatives of state variables in function of state variables and inputs: $\begin{cases} \frac{di}{dt} = \frac{V_1 - V_2}{L} = \frac{V_{in} - R \cdot i - V_2}{L} \\ \frac{dV_2}{dt} = \frac{i}{C} \end{cases}$
- Writing output in function of state variables and inputs: $V_{out} = V_1 - V_2 = V_{in} - Ri - V_2$

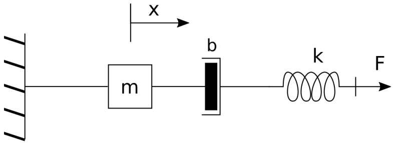
State Space Representation

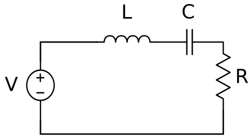
This yields the **State Space Representation** of the dynamic system. In Matrix form:

$$\begin{bmatrix} \frac{dV_2}{dt} \\ \frac{di}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} V_2 \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_{in}$$

$$V_{out} = \begin{bmatrix} -1 & -R \end{bmatrix} \begin{bmatrix} V_2 \\ i \end{bmatrix} + V_{in}$$

Force-Voltage Analogy





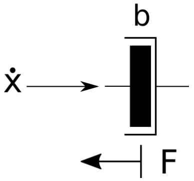
Let:

$$
\begin{array}{ccc}
F & \leftrightarrow & V \\
\dot{x} & \leftrightarrow & i \\
x & \leftrightarrow & q
\end{array}
$$

Force-Voltage Analogy

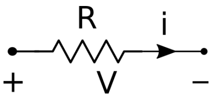
The analogy between the other quantities follows from comparing the physical laws.

Damping:



$F = b\dot{x}$

Resistance:

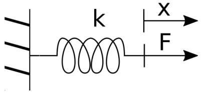


$V = Ri$

$$\boxed{b \leftrightarrow R}$$

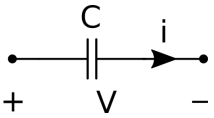
Force-Voltage Analogy

Spring:



$$F = kx$$
$$\Rightarrow \frac{dF}{dt} = k\frac{dx}{dt}$$

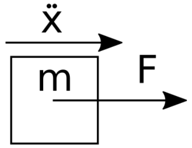
Capacitor:



$$\frac{dV}{dt} = \frac{i}{C}$$

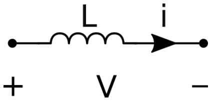
$$\boxed{k \leftrightarrow \frac{1}{C}}$$

Newton:



$$F = m\ddot{x}$$
$$= m\frac{d\dot{x}}{dt}$$

Coil:



$$V = L\frac{di}{dt}$$

$$\boxed{m \leftrightarrow L}$$

Example 6: Hoover dam

Define:

- Inflow of water: $u(t)$
- Current volume of water: $x(t)$
- Outflow of water: $y(t)$
- Water level: $h(t)$

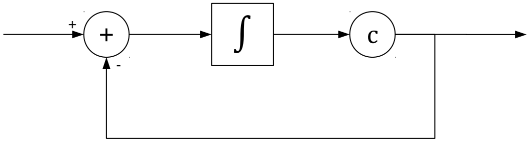
Assume that $x(t) = c_1 \cdot h(t)$

What will happen when we open the gate?

Example 6: Hoover dam

- Outflow depends on height:
  $$y(t) = c_2 \cdot h(t)$$
- The state of the system is defined by the contained volume of water:
  $$\dot{x}(t) = u(t) - y(t) = u(t) - c_2 \cdot h(t)$$
- Thus a **State Space Representation** is, with $c \triangleq \frac{c_2}{c_1}$:

$$\dot{x}(t) = u(t) - c \cdot x(t)$$
$$y(t) = c \cdot x(t)$$



Nonlinear systems

In this course we focus on the linear state-space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \qquad \begin{cases} x[k+1] & = Ax[k] + Bu[k], \\ \quad y[k] & = Cx[k] + Du[k]. \end{cases}$$

Most real life systems involve nonlinearity:

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big), \\ y(t) = g\big(x(t), u(t)\big), \end{cases}$$

where f and/or g contain some nonlinearity, such as:

- *powers*: e.g. $\dot{x}(t) = Ax(t) + Bu(t) + \gamma u(t)^2$,
- *interactions*: e.g. $\dot{x}(t) = Ax(t) + Bu(t) + \gamma x(t)u(t)$,
- *clipping*: e.g. $\alpha \leq x(t) \leq \beta$.

Linearization around equilibrium points

Nonlinear systems have (several) equilibrium points x_e, u_e, y_e:

$$\begin{cases} \dot{x}_e = f\left(x_e, u_e\right) = 0, \\ y_e = g\left(x_e, u_e\right). \end{cases}$$

Linearizing in the region of (x_e, u_e, y_e):

$$x = x_e + \Delta x, \quad u = u_e + \Delta u, \quad y = y_e + \Delta y,$$

with Δx, Δu and Δy *sufficiently* small.

Linearizing is done via **first order Taylor expansions**.

Linearization around equilibrium points

Linearizing is done via **first order Taylor expansions**.

$$\begin{cases} \frac{dx}{dt} = \frac{d(x_e+\Delta x)}{dt} = \frac{d\Delta x}{dt} = f(x,u) = f(x_e+\Delta x, u_e+\Delta u), \\ y_e + \Delta y = g(x,u) = g(x_e+\Delta x, u_e+\Delta u). \end{cases}$$

We write the *vectors* x and u in their individual components to simplify interpretation:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, ..., x_n, u_1, .., u_l) \\ &\vdots \\ \dot{x}_n &= f_1(x_1, ..., x_n, u_1, .., u_l) \\ \dot{y}_1 &= h_1(x_1, ..., x_n, u_1, .., u_l) \\ &\vdots \\ \dot{y}_l &= h_l(x_1, ..., x_n, u_1, .., u_l) \end{aligned}$$

Linearization around equilibrium points

The first order Taylor expansion of $f()$ around (x_e, u_e) is described by the **Jacobian Matrix**:

$$\frac{dx}{dt} = f(x_e, u_e) + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_l} \\ \vdots & & & & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_l} \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \\ \Delta u_1 \\ \vdots \\ \Delta u_l \end{bmatrix}$$

With the partial derivatives evaluated in x_e and u_e
$f(x_e, u_e) = \frac{dx_e}{dt} = 0$ because we *choose* x_e and u_e to be equilibrium points

Linearization around equilibrium points

This can be split up in a contribution by the state x and the input u:

$$\frac{d\Delta x}{dt} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}}_{A} \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_l} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_l} \end{bmatrix}}_{B} \begin{bmatrix} \Delta u_1 \\ \vdots \\ \Delta u_l \end{bmatrix}$$

Similarly C & D can be constructed from the Jacobian Matrix of $h(x, u)$

Example: decalcification plant

Used to reduce concentration of calcium hydroxide in water:

- chemical reaction: $Ca(OH)_2 + CO_2 \rightarrow CaCO_3 + H_2O$
- reaction speed: $r = c[Ca(OH)_2][CO_2]$
- rate of change of concentration:

$$\frac{d[Ca(OH)_2]}{dt} = \frac{k}{V} - \frac{r}{V},$$
$$\frac{d[CO_2]}{dt} = \frac{u}{V} - \frac{r}{V},$$

  with inflow rates k and u in mol/s and tank volume V in L.
- input u: inflow of CO_2, output: $[Ca(OH)_2]$

Nonlinear model and equilibrium point

Nonlinear model for the given reactor:

$$\frac{d[Ca(OH)_2]}{dt} = \frac{k}{V} - \frac{c}{V}[Ca(OH)_2][CO_2],$$
$$\frac{d[CO_2]}{dt} = \frac{u}{V} - \frac{c}{V}[Ca(OH)_2][CO_2],$$
$$y = [Ca(OH)_2],$$

with two state variables: $x_1 = [Ca(OH)_2]$ and $x_2 = [CO_2]$.

The equilibrium point $(k_{eq}, u_{eq}, x_{1,eq}, x_{2,eq}, y_{eq})$ of this system is:

$$\frac{k_{eq}}{V} - \frac{c}{V}[Ca(OH)_2]_{eq}[CO_2]_{eq} = 0,$$
$$\frac{u_{eq}}{V} - \frac{c}{V}[Ca(OH)_2]_{eq}[CO_2]_{eq} = 0.$$

Linearization of the decalcification plant

For small deviations near the equilibrium:

$$\begin{aligned}
\frac{d\Delta x_1}{dt} &= -\frac{c}{V}[CO_2]_{eq}\Delta x_1 - \frac{c}{V}[Ca(OH)_2]_{eq}\Delta x_2,\\
\frac{d\Delta x_2}{dt} &= -\frac{c}{V}[CO_2]_{eq}\Delta x_1 - \frac{c}{V}[Ca(OH)_2]_{eq}\Delta x_2 + \frac{\Delta u}{V},\\
\Delta y &= \Delta x_1.
\end{aligned}$$

The resulting linear state-space model is $\begin{cases}\dot{x}(t) = Ax(t) + Bu(t)\\ y(t) = Cx(t) + Du(t)\end{cases}$:

$$\begin{bmatrix}\frac{d[Ca(OH)_2]}{dt}\\ \frac{d[CO_2]}{dt}\end{bmatrix} = -\begin{bmatrix}\frac{c}{V}[CO_2]_{eq} & \frac{c}{V}[Ca(OH)_2]_{eq}\\ \frac{c}{V}[CO_2]_{eq} & \frac{c}{V}[Ca(OH)_2]_{eq}\end{bmatrix}\begin{bmatrix}[Ca(OH)_2]\\ [CO_2]\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{V}\end{bmatrix}u(t)$$

$$y(t) = [Ca(OH)_2]$$

Grey box identification: conceptual

Grey box identification starts from a known model structure but with unknown/uncertain parameters $\leftrightarrow$ **parametric statistics**.

We assume linear, continuous time state space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases}$$

Given: states, inputs, outputs and guesstimates of $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ & $\tilde{D}$.
Task: estimate $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ adequately via experiments.

"*All models are wrong, but some are useful.*" - George E. P. Box

Linear regression

Consider input matrix $\mathbf{X}$, output vector $\mathbf{y}$ and residuals ϵ:

$$\mathbf{X}\theta = \mathbf{y} + \epsilon.$$

The parameter vector θ must be estimated, given the observations.

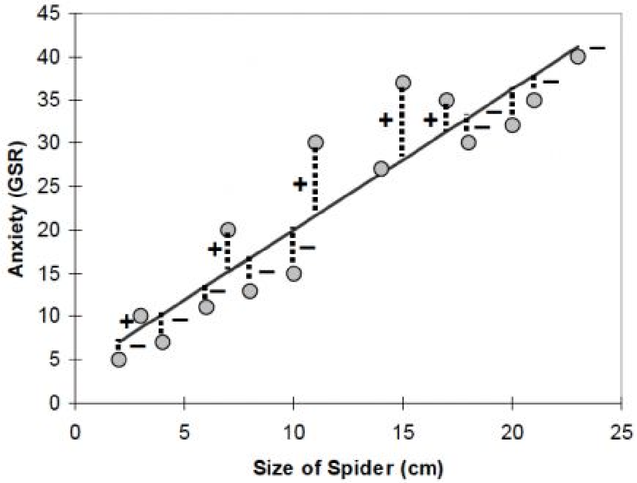
A common estimation approach is ordinary least squares (OLS):

$$\begin{aligned}(\mathbf{X}^T\mathbf{X})\hat{\theta}_{OLS} &= \mathbf{X}^T\mathbf{y},\\ \hat{\theta}_{OLS} &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}.\end{aligned}$$

The OLS estimate minimizes the sum-of-squares of errors, i.e.:

$$\hat{\theta}_{OLS} = \arg\min_{\theta} \sum_{i=1}^{N} \left(y(i) - \sum_{j=1}^{d} X(i,j)\theta(j)\right)^2$$

Linear regression with ordinary least squares



Maximum likelihood estimation

The maximum likelihood estimate $\hat{\theta}_{ML}$ is the parameter vector that maximizes the likelihood $\mathcal{L}(\cdot)$ of observing the (known) outputs $\mathbf{y}$, given the (known) inputs $\mathbf{X}$:

$$\hat{\theta}_{ML} = \arg\max_{\theta} \mathcal{L}\big(\mathbf{y}, \mathbf{X} \mid \theta\big)$$

For some structures, ML estimate can be obtained in closed form.

Example: least squares estimators are the maximum likelihood estimators if the associated residuals ϵ are normally distributed.

Maximum a posteriori (MAP) estimation

Bayesian: maximum likelihood estimation with a *prior* $p(\theta)$.
$\rightarrow$ MAP estimation is a regularization of ML estimation

Bayes' theorem: $P(A \mid B) = P(B \mid A) \cdot P(A) \ / \ P(B)$.

If a prior distribution $p(\cdot)$ is available for θ, then the posterior distribution for θ becomes:

$$\theta \mapsto \mathcal{L}(\theta \mid \mathbf{y}, \mathbf{X}) = \frac{\mathcal{L}(\mathbf{y}, \mathbf{X} \mid \theta) p(\theta)}{\int_{\vartheta} \mathcal{L}(\mathbf{y}, \mathbf{X} \mid \vartheta) p(\vartheta) d\vartheta}.$$

The MAP estimate is the mode of the posterior distribution of θ:

$$\hat{\theta}_{MAP} = \arg\max_{\theta} \mathcal{L}(\mathbf{y}, \mathbf{X} \mid \theta) p(\theta).$$

Errors-in-variables approach

Additionally accounts for measurement errors in inputs.
$\leftrightarrow$ standard regression only accounts for errors in *outputs*

Typically described via *latent variables*:

$$\begin{cases} x = x^\star + \eta, \\ y = y^\star + \epsilon, \\ y^\star = g(x^\star \mid \theta), \end{cases}$$

with x, y the observed inputs, outputs and latent variables $x^\star$, $y^\star$.
Assumption: latent variables $x^\star$ and $y^\star$ exist which follow the true functional relationship $g(\cdot)$.

Task: estimate θ.

Black box identification

Start from unknown equations & unknown parameters.
→ related to **machine learning** and **nonparametric statistics**.

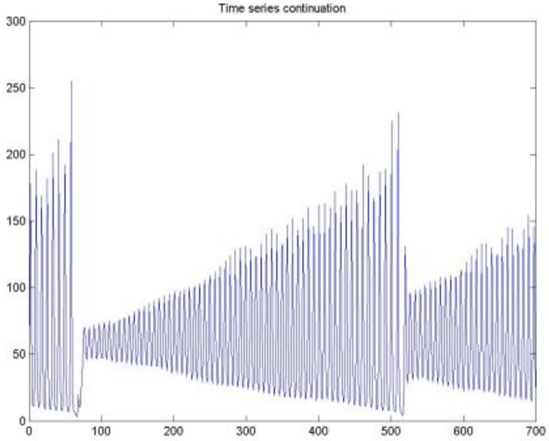
If we assume a linear state space system:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases} \qquad \begin{cases} x[k+1] & = Ax[k] + Bu[k], \\ y[k] & = Cx[k] + Du[k]. \end{cases}$$

Black box identification deals with:

- unknown states, both in number & physical interpretation
  → dimensions of A, B & C unknown
- unknown parameters (values in A, B, C, D)

Time series: Santa Fe laser



Modelling the Santa Fe laser

This laser can be treated as an autonomous discrete time system:

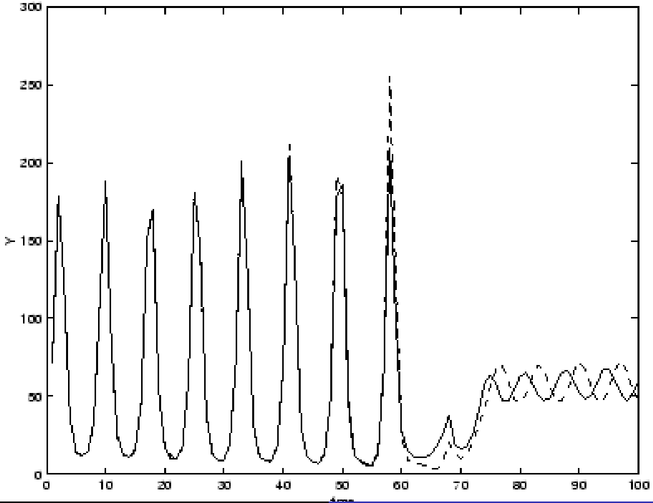
$$\begin{cases} x[k+1] = f\big(x[k-N+1], \ldots, x[k]\big), \\ y[k] = x[k]. \end{cases}$$

The output depends on the past N states & no inputs.
$\rightarrow$ how large is N? $\rightarrow$ **unknown structure**

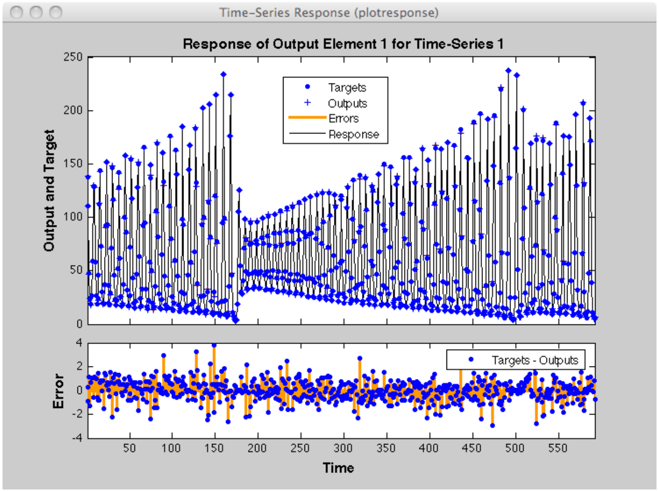
Treat it as a regression problem with N inputs: $y = f\big(X_1, \ldots, X_N\big)$.
$\rightarrow$ lets say linear, i.e. $y = \mathbf{X}\theta$ $\rightarrow$ **unknown parameters** $\theta \in \mathbb{R}^N$.
$\rightarrow$ for given N, we can estimate θ via grey box methods.

Nonlinear models can be obtained via machine learning methods.
$\rightarrow$ neural networks, support vector machine, random forest, ...

Predictions of a least-squares support vector machine



Predictions of an artificial neural network



Neural network: biological

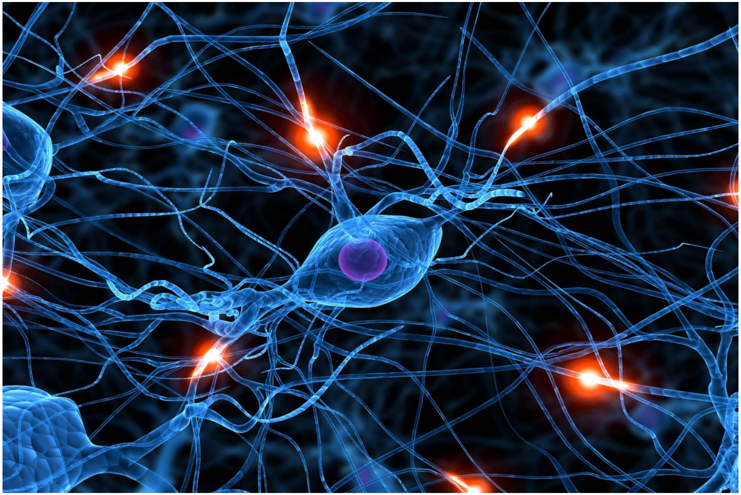


Image taken from `http://www.extremetech.com/wp-content/uploads/2013/09/340.jpg`.

Structure of a single neuron

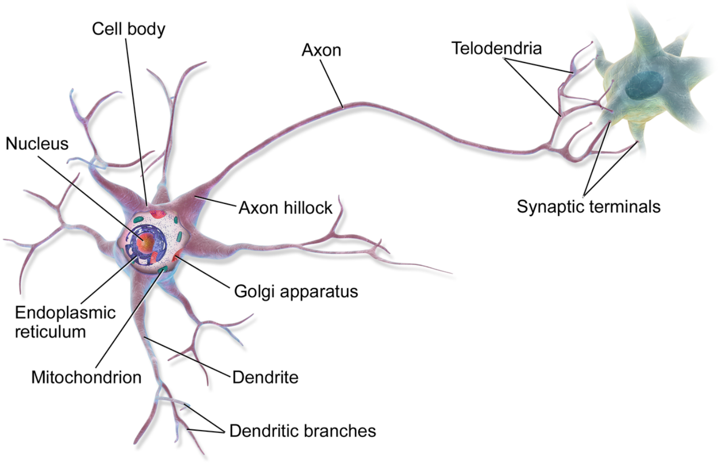
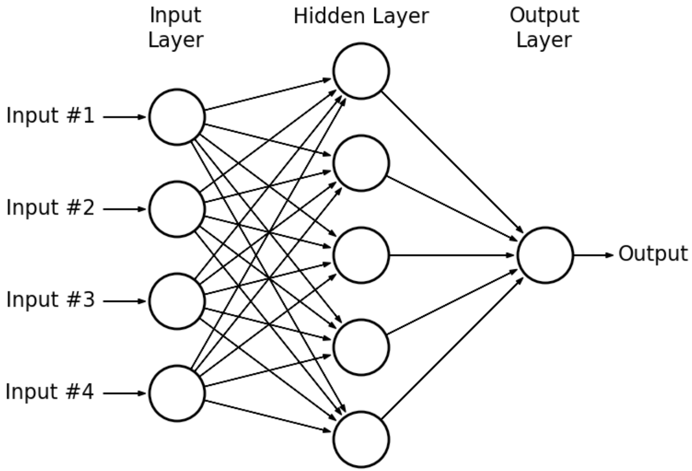


Image taken from `http://en.wikipedia.org/wiki/File:Blausen_0657_MultipolarNeuron.png`.

Neural network: artificial


Input Layer
Hidden Layer
Output Layer
Input #1
Input #2
Input #3
Input #4
Output