Continuous-time systems 2

1. Properties of state-space representation

2. Transfer functions
    - Impulse response and time constant
    - Relationship between state space and transfer functions

Outline

Observability

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This holds for linear, time-invariant systems with n states if:

$$rank(\mathcal{O}) = n, \quad \mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}, \quad \mathcal{O} : \textbf{observability matrix}$$

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A linear, time-invariant system with n states is controllable if:

$$rank(\mathcal{C}) = n, \quad \mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \ldots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix},$$

where $\mathcal{C}$ is called the **controllability matrix**.

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Transfer function

The transfer function of input i to output j is defined as:

$$H_{i,j}(s) = \frac{Y_j(s)}{U_i(s)}, \quad \mathbf{U}(s) = \mathcal{L}\{u(t)\}, \quad \mathbf{Y}(s) = \mathcal{L}\{y(t)\}.$$

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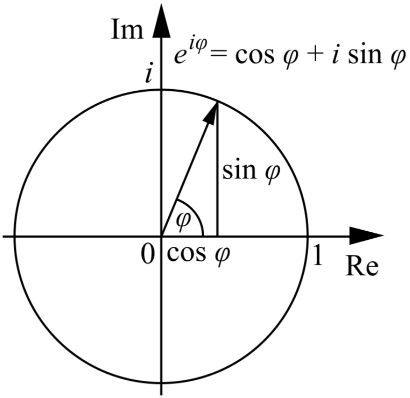
The frequency response of a system can be analyzed via $\mathbf{H}(j\omega)$:

$$e^{\sigma + j\omega} = e^{\sigma}(\cos\omega + j\sin\omega).$$

Illustration of Euler's formula



Poles and zeros

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Poles and zeros may cancel, ie. if $D(s) = N(s) = 0$ for some s.

Steady-state response

The output of a linear time-invariant system yields consists of:

- a steady-state output $y_{ss}(t)$, which similar periodicity to $u(t)$
  $\rightarrow$ y_{ss} comprises the same frequencies as $u(t)$
- a transient output $y_{tr}(t)$
  $\rightarrow$ if the system is stable, then $\lim_{t\rightarrow\infty} y_{tr}(t) = 0$
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The steady-state output $y_{ss}(t)$ of a linear time invariant system:

- consists of signals of same frequencies as the input signal $u(t)$
- which may have been magnified and/or phase changed

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For stable continuous time systems the impulse response always converges to 0:

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The speed of convergence depends on the position of the poles.

Time constant

The transfer function of first order systems can be written as:

$$H(s) = \frac{K}{\tau s + 1} \quad \leftrightarrow \quad h(t) = \frac{K}{\tau} e^{-t/\tau},$$

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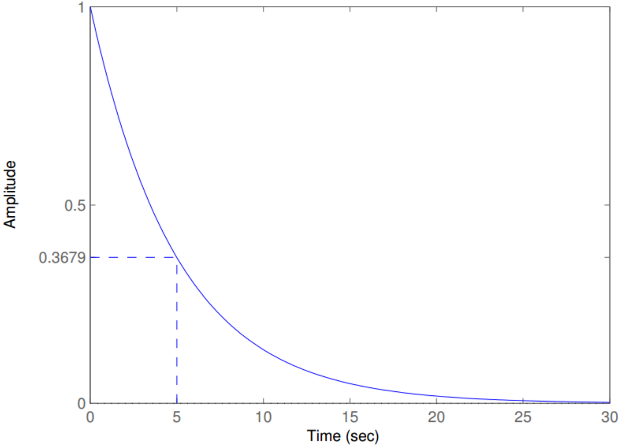
The time constant summarizes the speed of a system's dynamics:

- after τ seconds, the impulse response reaches $h(0)/e$.
- after τ seconds, the step response has reached $1 - e^{-1} \approx 63\%$ of its regime value.

Impulse response $H(s) = 5/(5s+1) \leftrightarrow h(t) = exp(-t/5)$



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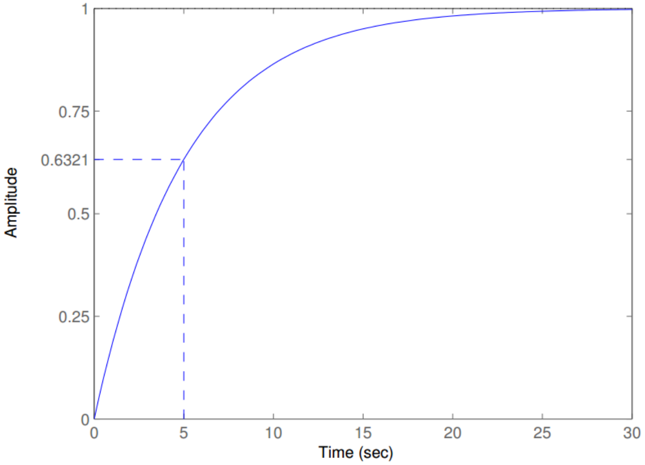

Outline

From state-space to transfer functions

We start from the linear state-space representation:

time domain Laplace domain

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \quad \leftrightarrow \quad \begin{cases} s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{cases}$$

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$$\begin{aligned} (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{B}\mathbf{U}(s) \\ \mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \\ \Rightarrow \mathbf{Y}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) \\ \Rightarrow \mathbf{H}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \end{aligned}$$

Relationship between poles and eigenvalues of $\mathbf{A}$ 1/2

Poles are zeros of the denominator of $\mathbf{H}(s)$, e.g. those values of s for which $\mathbf{H}(s)$ is singular.

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$\rightarrow$ all poles of $\mathbf{H}(s)$ are eigenvalues of $\mathbf{A}$

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Consider the following SISO system with 2 states:

$$\begin{bmatrix} sX_1(s) \\ sX_2(s) \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} + \begin{bmatrix} \beta \\ 2 \end{bmatrix} U(s)$$

$$Y(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$$

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The transfer function $H(s) = \frac{\beta}{s-\alpha}$ has only one pole ($s_1 = \alpha$).
$\rightarrow$ not all eigenvalues of **A** are poles in transfer functions $\mathbf{H}(s)$.