# Frequency response to dynamical systems

July 7, 2015

# Outline

# The transfer function

From previous lectures:

$$
\begin{aligned}
y(t) &= h(t) \otimes u(t) \\
\Rightarrow \quad \mathscr{L}\{y(t)\} &= \mathscr{L}\{h(t) \otimes u(t)\} \\
\Rightarrow \quad Y(s) &= H(s) \cdot U(s) \\
H(s) &= \frac{Y(s)}{U(s)}
\end{aligned}
$$

This is the transfer function, the relation between input and output in the Laplace domain (continuous time systems)

# Plot of H(s)?

- $s$ and $H(s)$ are both complex $\rightarrow$ 4D-graph?
- No, we will substitute $s$ for $j\omega$ with $\omega$ the angular frequency [rad/s]. (We will often use frequency to indicate $\omega$ but keep in mind that $\omega = 2\pi f$)
- We will also split $H(s) = H(j\omega)$ using its polar representation in two, an amplitude and a phase plot
- Remember: $H(j\omega) = |H(j\omega)| \exp \angle(H(j\omega))$
- The amplitude plot and the phase plot of $H(j\omega)$ are called the bode plot

# Outline

# The amplitude plot

Convention:

- for the ordinate (y-axis) we use $20\log_{10}|H(j\omega)|$ with the special unit *dB*
- for the abscissa (x-axis) we use a logarithmic plot of $\omega$

This is thus a bi-log plot
The reason for using the logarithm of the modulus of $H(j\omega)$ will become clear later

# The phase plot

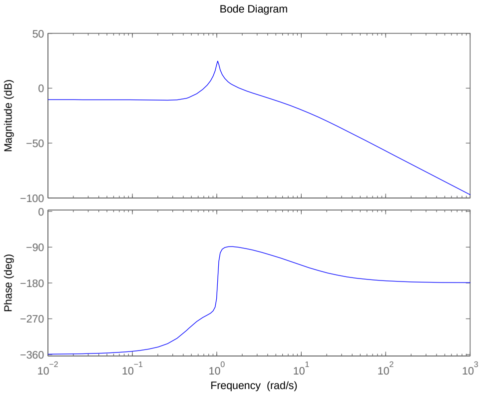
Convention:

- for the ordinate (y-axis) we use $\angle H(j\omega)$ in degrees
- for the abscissa (x-axis) we use a logarithmic plot of $\omega$

This is thus a semi-log plot

# Example bode plot

$$H(s) = \frac{14s^2+7s+3}{s^4+10s^3+10s^2+10s+10}$$



# Outline

# A new representation of the transfer function

From before:

$$H(s) = \frac{\beta_0 s^r + \beta_1 s^{r-1} + \ldots + \beta_r}{s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n}$$

Factorization in zeros and poles

$$\Rightarrow \quad H(s) = \frac{\beta_0 (s - n_1)(s - n_2) \ldots (s - n_r)}{(s - p_1)(s - p_2) \ldots (s - p_n)}$$

This is the usual representation. Now however, we will look for factors $(1 + \frac{s}{s_i})$, with $s_i$ a so-called breakpoint.

# A new representation of the transfer function

We can do this by bringing all the zeros and poles not equal to zero outside the brackets, as follows:

$$H(s) = \beta_0 \frac{\prod(-n_i)}{\prod(-p_j)} \frac{(1+\frac{s}{-n_1})(1+\frac{s}{-n_2})\dots(1+\frac{s}{-n_i})}{s^l(1+\frac{s}{-p_1})(1+\frac{s}{-p_2})\dots(1+\frac{s}{-p_j})}$$

Replacing the constants by $K$, and setting

$$r_k = -n_k$$

$$s_k = -p_k$$

# A new representation of the transfer function

We ultimately get:

$$H(s) = K\frac{(1+\frac{s}{r_1})(1+\frac{s}{r_2})\ldots(1+\frac{s}{r_i})}{s^l(1+\frac{s}{s_1})(1+\frac{s}{s_2})\ldots(1+\frac{s}{s_j})}$$

Now we are able to construct the bode plot of each different factor of $H(s)$. Afterwards we can just add up these plots using the calculation rules of complex numbers.

# Intermezzo complex numbers

- The amplitude of the product of complex numbers is equal to the product of the amplitudes of these numbers
- The phase of the product of complex numbers is equal to the sum of the phases of these numbers
- The logarithm of a product of numbers is equal to the sum of the logarithms of these numbers

This comes down to

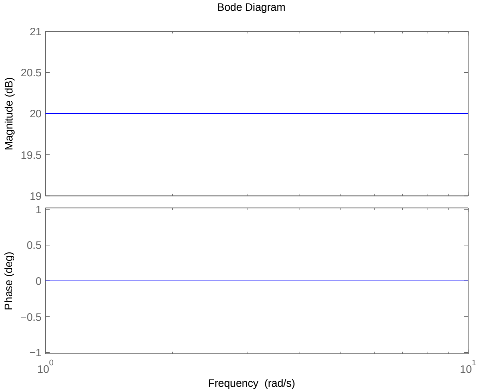
$$20\log_{10}|H(j\omega)| = \sum 20\log_{10}|\text{factors}|$$

$$\angle H(j\omega) = \sum(\angle\text{factors})$$

Next we will quickly go over the simple bodeplots of the different factors of $H(s)$

# The constant K

- $20 \log_{10} |K| = \text{constant}$
- $\angle K = 0^\circ$ or $\pm 180^\circ$ (resp $K > 0$ and $K < 0$)




Example: $K = 10$

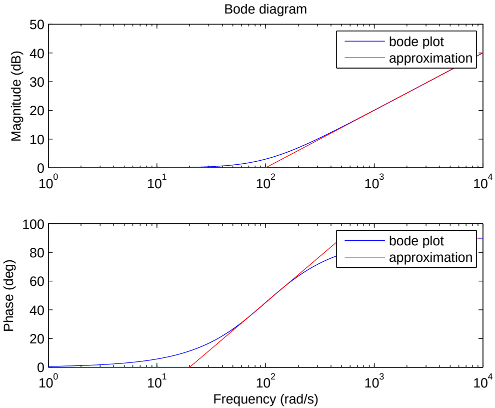
## $(1+\frac{j\omega}{r_i})$ in the numerator

(Assume $r_i > 0$)

- What if $\omega \to 0$ ?  $(1+\frac{j\omega}{r_i}) \to 1$
  - $20\log_{10}|1| = 0$
  - $\angle 1 = 0^\circ$
- What if $\omega \to \infty$ ?  $(1+\frac{j\omega}{r_i}) \to j\infty$
  - $20\log_{10}|j\infty| = \infty$
  - $\angle j\infty = 90^\circ$
- The two terms balance each other out for $\omega = r_i$ (remember, this is called a break point).
  - $20\log_{10}|1+j| = 20\log_{10}(\sqrt{2}) \approx 3\text{dB}$
  - $\angle(1+j) = 45^\circ$

A break point is therefore also called a 3dB point

# $(1 + \frac{j\omega}{r_i})$ in the numerator




Example: $r_i = 100$

# $(1 + \frac{j\omega}{s_i})$ in the denominator

This factor is equivalent to the previous one. The only difference is the sign change in both plots as:

- $\log|\frac{1}{z}| = -\log|z|$
- $\angle\frac{1}{z} = -\angle z$

# $(1 + \frac{j\omega}{s_i})$ in the denominator

# $j\omega$ in the numerator

- This is simply a (ascending) straight line in the amplitude plot, with a slope of 20 dB/decade
- Constant phase of $90^\circ$

# $j\omega$ in the denominator

- This is simply a (descending) straight line in the amplitude plot, with a slope of -20 dB/decade
- Constant phase of $-90^\circ$

# Second order factors

A second order factor has twice the effect of a first order factor.
Consider for example $(1 + \frac{j\omega}{100})^2$ in the numerator:
Similar for higher order factors

# Exception

Up until now we always considered $r_i$ and $s_i > 0$, but what if we had a factor $(1 - \frac{j\omega}{r_i})$ for example?

- The amplitude plot remains unchanged, as $|1 + \frac{j\omega}{r_i}| = |1 - \frac{j\omega}{r_i}|$
- The phase plot is reversed, as $\angle(1 + \frac{j\omega}{r_i}) = -\angle(1 - \frac{j\omega}{r_i})$

# Exception

# Outline

# Example

# Outline

# Conclusion

- Today we revised the basics about (constructing) the bode plot
- These techniques will be used in later lectures regarding controllers
- The importance of the bode plot will then also be shown