Outline

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Transient Response

The time response of a control system can be written as:

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

where $y_{tr}(t)$ is the transient response and $y_{ss}(t)$ is the steady state response.

Definition

The transient response of a system is the time-difference between the change of the inputs and the change of the outputs: when the input of a system changes, the output does not change immediately but takes time to go to steady state.

First order systems: stability

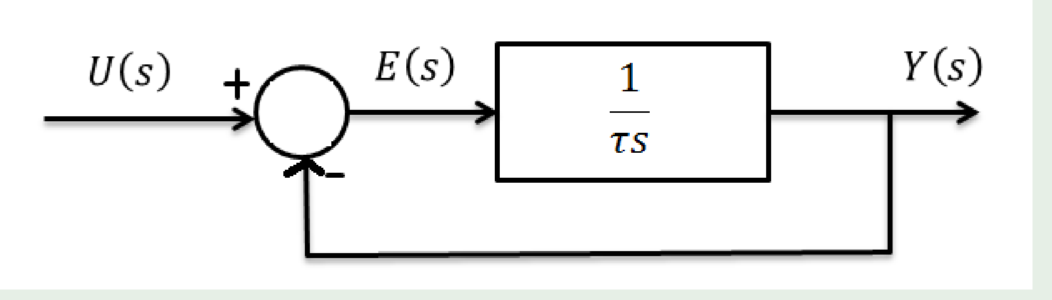
The most important characteristic of a dynamic system is absolute stability.

- A system is stable when it returns to equilibrium, if subject to initial condition
- A system is critically stable when oscillations of the output continue forever
- A system is unstable when the output diverges without bound from equilibrium, if subject to initial condition

First order systems

Example

Unit step response of RC circuit, thermal system, ...



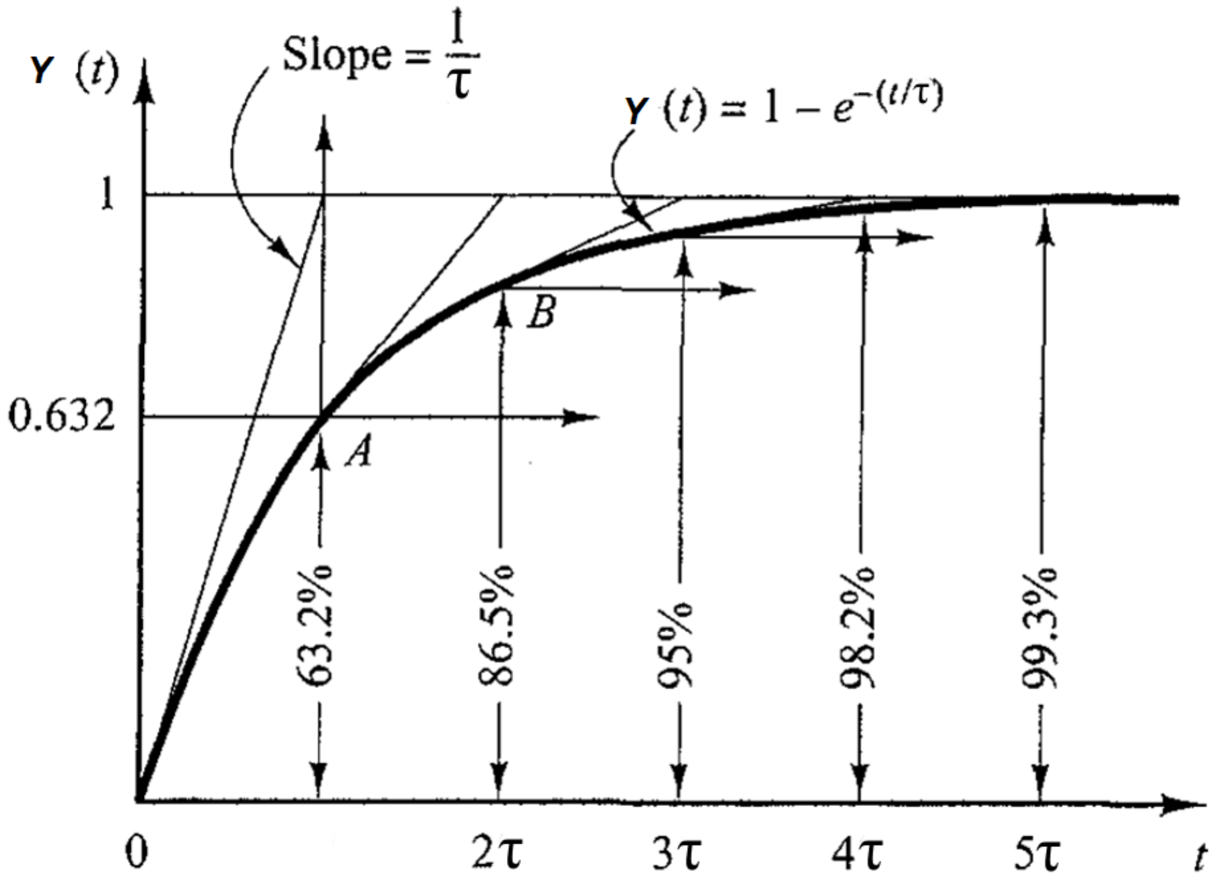
The transfer function is given by: $\frac{Y(s)}{U(s)} = \frac{1}{\tau s+1}$

- Laplace of unit-step is $\frac{1}{s}$ $\rightarrow$ substituting $U(s) = \frac{1}{s}$:
  $Y(s) = \frac{1}{s}\frac{1}{\tau s+1}$;
- Expanding into partial fractions gives
  $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$.

Unit step transient response

1. $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$;
2. Taking the inverse Laplace transform
   $y(t) = 1 - e^{-\frac{t}{\tau}}$, for $t \geq 0$;
3. At $t = 0$, the output $y(t) = 0$;
4. At $t = \tau$, the output $y(t) = 0.632$, or $y(t)$ has reached 63.2% of its total change $y(\tau) = 1 - e^{-1} = 0.632$;
5. Slope at time $t = 0$ is $\frac{1}{\tau}$
   $\frac{dy}{dt}\big|_{t=0} = \frac{1}{\tau}e^{-\frac{t}{\tau}}\big|_{t=0} = \frac{1}{\tau}$,
   where τ is called the system time constant.

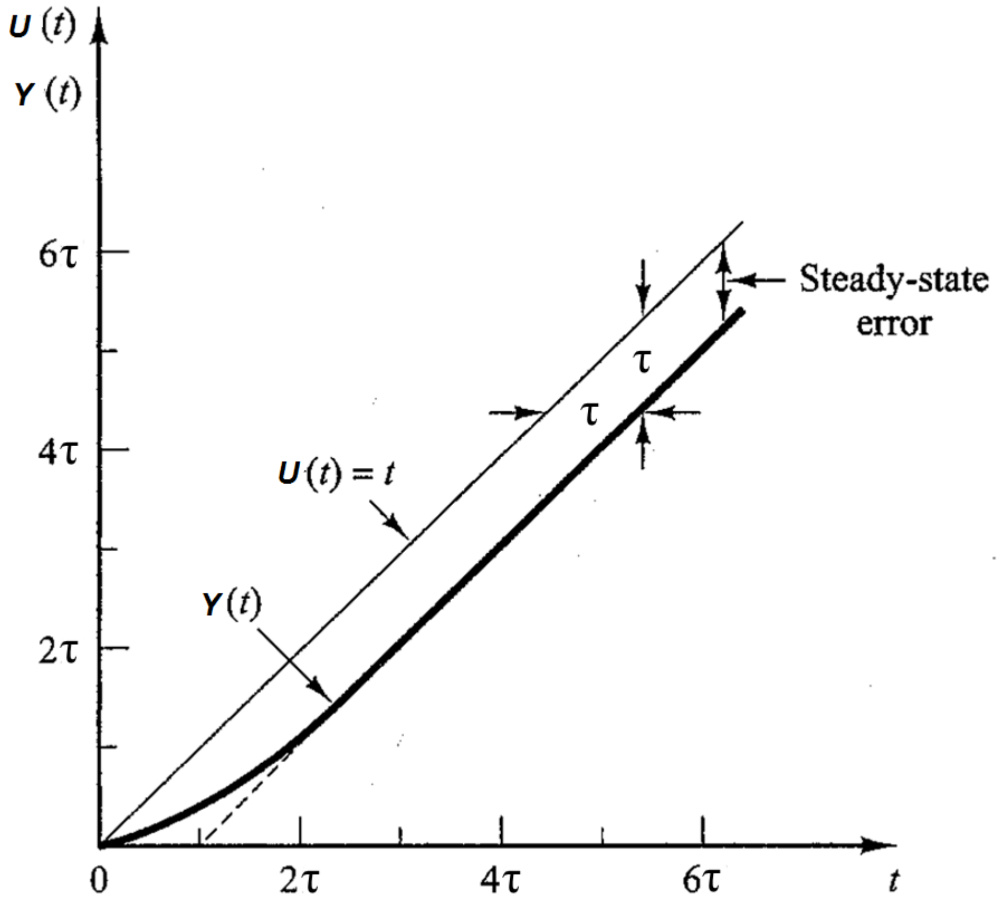
Unit step transient response



Unit ramp transient response

1. Laplace transform of unit ramp is $\frac{1}{s^2}$
   $Y(s) = \frac{1}{\tau s+1}\frac{1}{s^2}$;
2. Expanding into partial fractions gives
   $Y(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s+1}$;
3. Taking the inverse Laplace transform
   $y(t) = t - \tau + \tau e^{-\frac{t}{\tau}}$, for $t \geq 0$;
4. The error signal $e(t)$ is then
   $e(t) = u(t) - y(t) = \tau(1 - e^{-\frac{t}{\tau}})$;
5. For t approaching infinity, $e(t)$ approaches τ
   $e(\infty) = \tau$.

Unit ramp transient response



Unit-Impulse Response

For a unit-impulse input, $U(s) = 1$ and the output is:

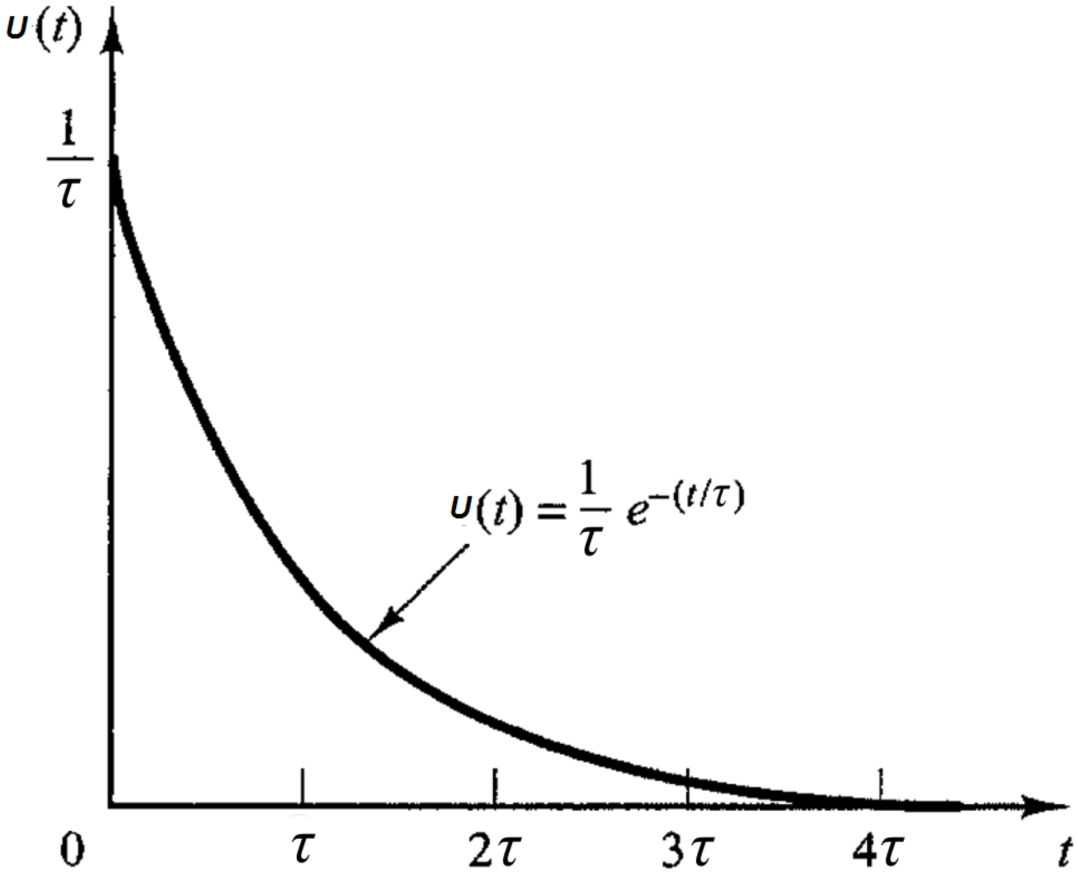
$$Y(s) = \frac{1}{\tau s + 1}.$$

The inverse Laplace transform gives:

$$y(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}, \text{ for } t \geq 0.$$

For $t \to +\infty$, $y(t) \to 0$.

Unit-Impulse Response



Outline

Second order systems

A second order system can generally be written as:

$$\frac{Y(s)}{U(s)} = H(s) = \frac{as^2 + bs + c}{ds^2 + es + f}$$

A system where the closed-loop transfer function possesses two poles is called a second-order system.

If the transfer function has two real poles, the frequency response can be found by combining the effects of both poles

Second order systems

Sometimes the transfer function has two complex conjugate poles. In that case we have to find a different solution for finding the frequency response.

In order to study the transient behaviour, let us first consider the following simplified example of a second order system:

$$H(s) = \frac{c}{ds^2 + es + c}.$$

Step response second order system

1. $H(s) = \frac{c}{ds^2+es+c}$;
2. The transfer function can be rewritten as:

$$H(s) = \frac{\frac{c}{d}}{s^2 + \frac{e}{d}s + \frac{c}{d}}$$

$$= \frac{\frac{c}{d}}{\left[s + \frac{e}{2d} + \sqrt{\left(\frac{e}{2d}\right)^2 - \frac{c}{d}}\right]\left[s + \frac{e}{2d} - \sqrt{\left(\frac{e}{2d}\right)^2 - \frac{c}{d}}\right]};$$

3. The poles are complex conjugates if $e^2 - 4dc < 0$;
4. The poles are real if $e^2 - 4dc \geq 0$.

Step response second order system

To simplify the transient analysis, it is convenient to write:

- $\frac{f}{d} = \omega_n^2$,
- $\frac{e}{d} = 2\zeta\omega_n = 2\sigma$

where σ is the attenuation, ω_n is the natural frequency and ζ is the damping ratio.

The transfer function can now be rewritten as

$$H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega s+\omega_n^2} \text{ (= standard form).}$$

The dynamic behavior of the second-order system can then be described in terms of only two parameters ζ and ω_n.

Step response second order system

If $0 < \zeta < 1$, the poles are complex conjugates and lie in the left-half s-plane

- The system is then called **underdamped**
- The **transient response is oscillatory**

If $\zeta = 0$, the **transient response doesn't die out**. If $\zeta = 1$, the system is called **critically damped**. If $\zeta > 1$, the system is called **overdamped**. We will now look at the unit step response for each of these cases.

Underdamped system

For the underdamped case $(0 < \zeta < 1)$, the transfer function can be written as:

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

Where ω_d is called the damped natural frequency $\omega_d = \omega_n\sqrt{1 - \zeta^2}$.

For a unit-step input we can write

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}.$$

Underdamped system

Which can be rewritten as partial fractions

$$
\begin{aligned}
Y(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\
&= \frac{1}{s} - \frac{s + \zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} - \frac{\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}.
\end{aligned}
$$

It can be shown that

$$
\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] &= e^{-\zeta\omega_n t} cos(\omega_d t) \\
\mathcal{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] &= e^{-\zeta\omega_n t} sin(\omega_d t).
\end{aligned}
$$

Underdamped system

Therefore:

$$
\begin{aligned}
\mathcal{L}^{-1}\{Y(s)\} &= y(t) \\
&= 1 - e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t)) \\
&= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} sin(\omega_d t + tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta})).
\end{aligned}
$$

It can be seen that the frequency of the transient oscillation is the damped natural frequency ω_d and thus varies with the damping ratio ζ.

Underdamped system

The error signal is the difference between input and output

$$\begin{aligned} e(t) &= y(t) - u(t) \\ &= e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t)) \end{aligned}$$

The error signal exhibits a damped sinusoidal oscillation. At steady state, or at $t = \infty$, the error goes to zero.

Underdamped system

If damping $\zeta = 0$, the response becomes **undamped**

- Oscillations continue indefinitely;
- Filling in $\zeta = 0$ into the equation for y(t) gives us: $y(t) = 1 - cos(\omega_n t)$, for $t \geq 0$;
- We see that the system now oscillates at the natural frequency ω_n;
- If a linear system has any amount of damping, the undamped natural frequency cannot be observed experimentally, only ω_d can be observed;
- ω_d is always lower than ω_n.