Chapter 5: Continuous time systems

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Outline

Linear differential equations (LDE) are of the following form:

$$L[y(t)] = f(t),$$

where L is some linear operator.

Linear differential equations: definitions 1/2

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The **order of a LDE** is the index of the highest derivative of y.

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- if $A_{0:n}(t)$ are constants (ie. not functions of time), the LDE is said to have **constant coefficients**

Example: radioactive decay 1/2

Let $N(t)$ be the number of radioactive atoms at time t, then:

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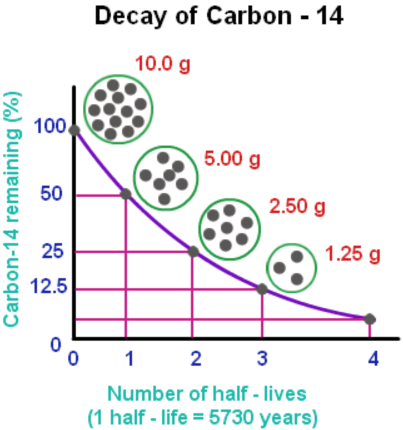

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Dividing by e^{zt} yields the nth order **characteristic polynomial**:

$$F(z) = \sum_{i=0}^{n} A_i z^{n-i} = 0.$$

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The specific linear combination depends on initial conditions.

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These zeros correspond to the following basis functions t:

$$e^{jt}, \quad e^{-jt}, \quad e^{t}, \quad te^{t}.$$

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The Laplace transform

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The final value theorem states $f(\infty) = \lim_{s \to 0} sF(s)$,
if all poles of $sF(s)$ are in the left half plane (ie. real part < 0).

Important properties of the Laplace transform

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linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$

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with $u(t) = \int_\infty^t \delta(t)dt$ (Heaviside) and $\delta(t)$ the Dirac delta.

Inverse Laplace transform

The inverse Laplace transform converts s-domain to time domain:

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Practically, the inverse Laplace transform takes two steps:

1. write $F(s)$ in terms of partial fractions
2. transform each term in the partial fraction based on tables of s/t-domain pairs (course notes p 4.32-4.33)

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Via induction, the Laplace transform of the nth order derivative:

$$\mathcal{L}\{f^{(n)}\} = s^nF(s) - \sum_{i=1}^{n} s^{n-i} f^{(n-i)}(0)$$

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Expanding Eq. (2) into (1) yields:

$$Y(s) \sum_{i=0}^{n} A_i s^i - \sum_{i=1}^{n} \sum_{j=1}^{i} A_i s^{i-j} y^{j-1}(0) = F(s)$$

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The solution in the time domain is obtained via the inverse Laplace transform: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

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This holds for linear, time-invariant systems with n states if:

$$rank(\mathcal{O}) = n, \quad \mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}, \quad \mathcal{O} : \textbf{observability matrix}$$

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A linear, time-invariant system with n states is controllable if:

$$rank(\mathcal{C}) = n, \quad \mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \dots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix},$$

where $\mathcal{C}$ is called the **controllability matrix**.

Outline

Transfer function

The transfer function of input i to output j is defined as:

$$H_{i,j}(s) = \frac{Y_j(s)}{U_i(s)}, \quad \mathbf{U}(s) = \mathcal{L}\{u(t)\}, \quad \mathbf{Y}(s) = \mathcal{L}\{y(t)\}.$$

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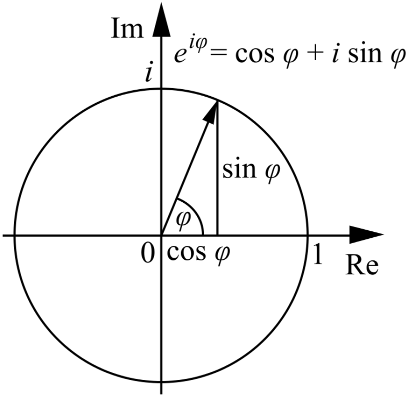
The frequency response of a system can be analyzed via $\mathbf{H}(j\omega)$:

$$e^{\sigma + j\omega} = e^{\sigma}(\cos\omega + j\sin\omega).$$

Illustration of Euler's formula



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Poles and zeros may cancel, ie. if $D(s) = N(s) = 0$ for some s.

Steady state response

The output of a linear time-invariant system yields consists of:

- a steady-state output $y_{ss}(t)$, which similar periodicity to $u(t)$
  $\rightarrow$ y_{ss} comprises the same frequencies as $u(t)$
- a transient output $y_{tr}(t)$
  $\rightarrow$ if the system is stable, then $\lim_{t\rightarrow\infty} y_{tr}(t) = 0$
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The steady-state output $y_{ss}(t)$ of a linear time invariant system:

- consists of signals of same frequencies as the input signal $u(t)$
- which may have been magnified and/or phase changed

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The speed of convergence depends on the position of the poles.

Time constant

The transfer function of first order systems can be written as:

$$H(s) = \frac{K}{\tau s + 1} \quad \leftrightarrow \quad h(t) = \frac{K}{\tau} e^{-t/\tau},$$

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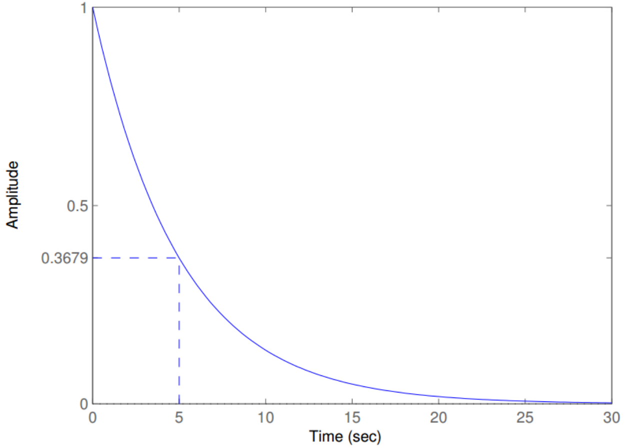
The time constant summarizes the speed of a system's dynamics:

- after τ seconds, the impulse response reaches $h(0)/e$.
- after τ seconds, the step response has reached $1 - e^{-1} \approx 63\%$ of its regime value.

Impulse response $H(s) = 5/(5s+1) \leftrightarrow h(t) = exp(-t/5)$



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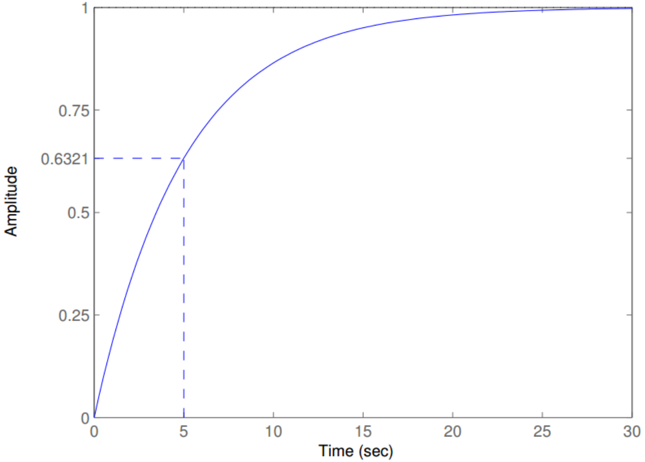

Outline

From state-space to transfer functions

We start from the linear state-space representation:

time domain — Laplace domain

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases} \leftrightarrow \begin{cases} s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \end{cases}$$

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$$\begin{aligned} (s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{B}\mathbf{U}(s) \\ \mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \\ \Rightarrow \mathbf{Y}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s) \\ \Rightarrow \mathbf{H}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \end{aligned}$$

Relationship between poles and eigenvalues of $\mathbf{A}$ 1/2

Poles are zeros of the denominator of $\mathbf{H}(s)$, e.g. those values of s for which $\mathbf{H}(s)$ is singular.

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Consider the following SISO system with 2 states:

$$\begin{bmatrix} sX_1(s) \\ sX_2(s) \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} + \begin{bmatrix} \beta \\ 2 \end{bmatrix} U(s)$$

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The transfer function $H(s) = \frac{\beta}{s-\alpha}$ has only one pole ($s_1 = \alpha$).
$\rightarrow$ not all eigenvalues of **A** are poles in transfer functions $\mathbf{H}(s)$.

Outline

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Transient Response

The time response of a control system may be written as:

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

Where $y_{tr}(t)$ is the transient response and $y_{ss}(t)$ is the steady state response. Most important characteristic of dynamic system is absolute stability.

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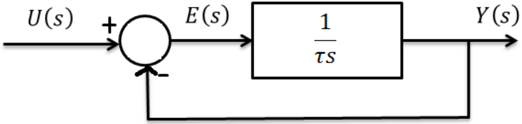
Transient response: when input of system changes, output does not change immediately but takes time to go to steady state

First order systems

E.g.RC circuit, thermal system, ...



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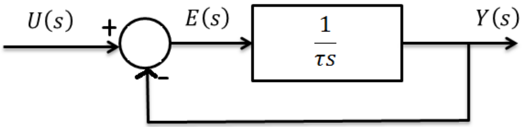

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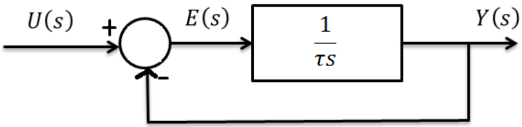

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Unit step response

- Laplace of unit-step is $\frac{1}{s}$ $\rightarrow$ substituting $U(s) = \frac{1}{s}$ into equation $Y(s) = \frac{1}{s}\frac{1}{\tau s+1}$
- Expanding into partial fractions gives $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$

Unit step transient response

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4. At $t = \tau$, the output $y(t) = 0.632$, or $y(t)$ has reached 63.2%

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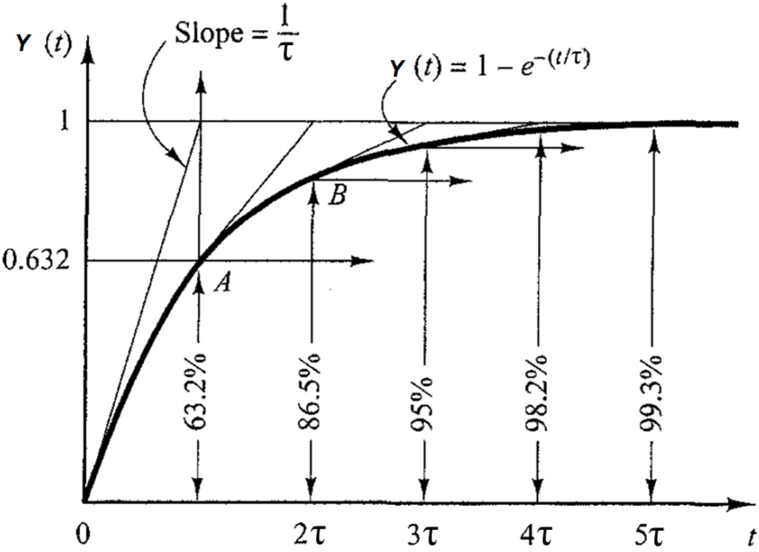
5. Slope at time $t = 0$ is $\frac{1}{\tau}$

   $\frac{dy}{dt}|_{t=0} = \frac{1}{\tau}e^{-\frac{t}{\tau}}|_{t=0} = \frac{1}{\tau}$

   Where τ is called the system time constant

Unit step transient response



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1. Laplace transform of unit ramp is $\frac{1}{s^2}$

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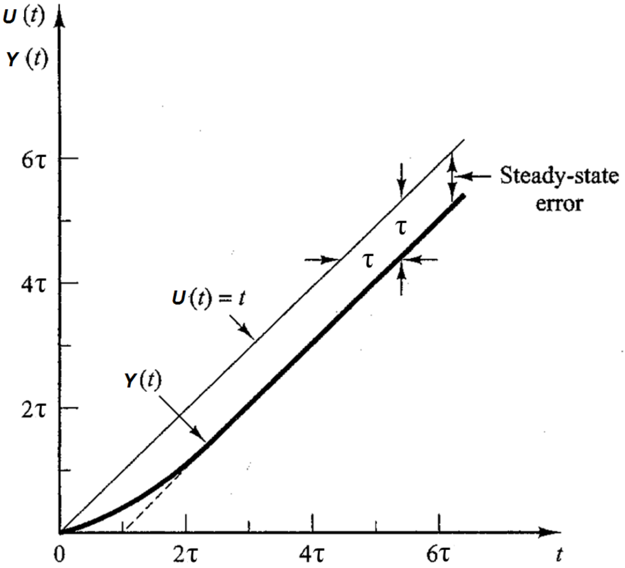
   $e(\infty) = \tau$

Unit ramp transient response



Unit-Impulse Response

For a unit-impulse input, $U(s) = 1$ and the output is

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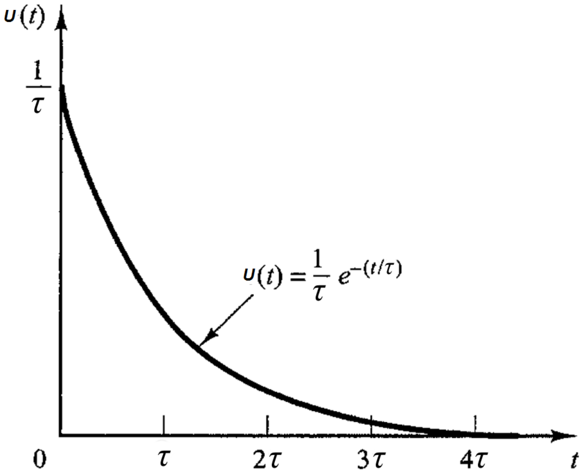

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Second order systems

A second order system can generally be written as:

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If the transfer function has two real poles, the frequency response can be found by combining the effects of both poles

Sometimes the transfer function has two complex conjugate poles. In that case we have to find a different solution for finding the frequency response.

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In order to study the transient behaviour, let us first consider the following simplified example of a second order system

$$H(s) = \frac{c}{ds^2 + es + c}$$

Step response second order system

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$$= \frac{\frac{c}{d}}{[s + \frac{e}{2d} + \sqrt{(\frac{e}{2d})^2 - \frac{c}{d}}][s + \frac{e}{2d} - \sqrt{(\frac{e}{2d})^2 - \frac{c}{d}}]}$$

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4. The poles are real if

$$e^2 - dc \geq 0$$

Step response second order system

To simplify the transient analysis, it is convenient to write

$$\frac{f}{d} = \omega_n^2,\ \frac{e}{d} = 2\zeta\omega_n = 2\sigma$$

Where
σ is the attenuation
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The dynamic behavior of the second-order system can then be described in terms of only two parameters ζ and ω_n

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- The system is then called **underdamped**
- The **transient response is oscillatory**

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If $\zeta > 1$, the system is called **overdamped**

We will now look at the unit step response for each of these cases

Underdamped system

For the underdamped case ($0 < \zeta < 1$), the transfer function can be written as:

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

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$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

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$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

For a unit-step input we can write

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}$$

Which can be rewritten as partial fractions

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} - \frac{\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

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It can be shown that

$$\mathcal{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t} cos(\omega_d t)$$
$$\mathcal{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] = e^{-\zeta\omega_n t} sin(\omega_d t)$$

Therefore:

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$$= 1 - e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t))$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} sin(\omega_d t + tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta}))$$

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$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}sin(\omega_d t + tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta}))$$

It can be seen that the frequency of the transient oscillation is the damped natural frequency ω_d and thus varies with the damping ratio ζ

The error signal is the difference between input and output

$$
\begin{aligned}
e(t) &= y(t) - u(t) \\
&= e^{-\zeta\omega_n t}\left(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t)\right)
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The error signal exhibits a damped sinusoidal oscillation

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e(t) &= y(t) - u(t) \\
&= e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t))
\end{aligned}
$$

The error signal exhibits a damped sinusoidal oscillation

At steady state, or at $t = \infty$, the error goes to zero

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Where

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- In that case, $H(s)$ can be approximated by

$$H(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} = \frac{s_2}{s + s_2}$$

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With the approximate transfer function, the unit-step response becomes

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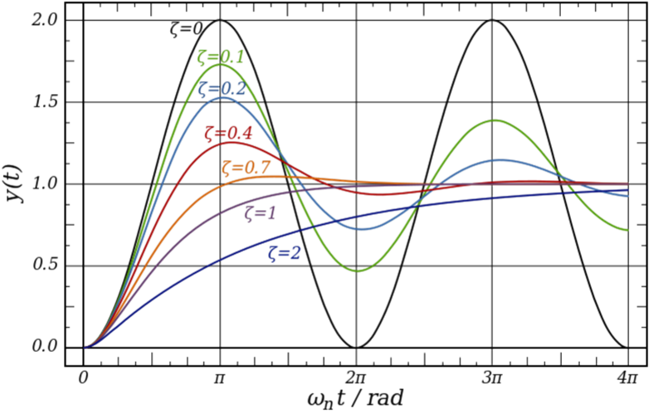
The time response for the approximate transfer function is then given as

$$y(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}, \text{ for } t \leq 0$$

Response on a step function



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$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

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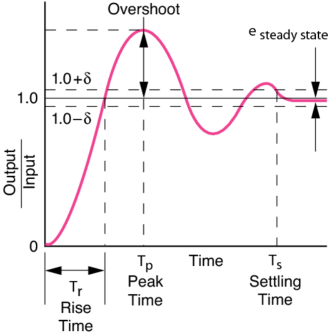
Peak Time: Time to reach overshoot.

$t_p = \frac{\pi}{\omega_d}$



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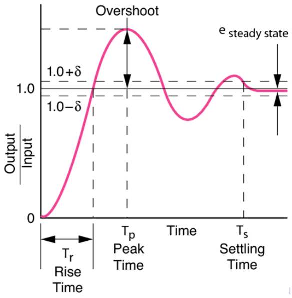
For: $\delta = \frac{0.02}{\sqrt{1-\zeta^2}}$

We find:

$$e^{-\zeta\omega_n T_s} < 0.02$$

$$T_s = \frac{4}{\omega_n \zeta}$$



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A different view on the Tacoma bridge disaster:
`https://www.youtube.com/watch?v=6ai2QFxStxo`

In fact the collapse was a result of a number of effects like Aerodynamic flutter and vortices. Read the full article here:
`http://www.ketchum.org/billah/Billah-Scanlan.pdf`

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Another phenomenon with bridges and resonance is that many people marching with the same rhythm can cause a bridge to start resonating like the Angers bridge in 1850. A more recent example is the Millennium bridge in London who started resonating.

When we want a system with no resonance, we choose one with damping < 0.707. This means a pole between 135° and 225°:

$$arctan(\frac{\sqrt{1-\zeta^2}}{\zeta}) = +135^\circ$$

Second order systems - damping

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We mostly want a short settling time (< 4s). This results in another restriction on the poles of the system:

$$\tau_n = \frac{4}{\omega\zeta} < 4\text{s}$$

$$\omega_n\zeta > 1$$

Second order systems - damping