

Discretization and reconstruction of signals

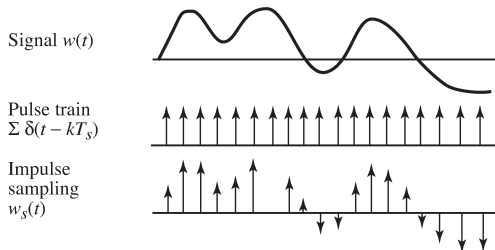
July 15, 2015

Discretization and reconstruction of signals

The use of digital logic and computers to calculate a control action for a continuous system introduces the operation of sampling.

Samples are taken from the continuous signals and used in the computer to calculate the controls to be applied.

The role of sampling and the conversion from continuous to discrete and vice versa are important to the understanding of the complete response of digital control.



Outline

- 1 Analysis of the sample and hold
- 2 Fourier transform
- 3 Spectrum of a sampled signal
 - Aliasing
 - Sampling theorem
 - Hidden oscillations
- 4 Data extrapolation (reconstruction)

Analysis of the sample and hold

To get samples of a continuous signal, we use an analog-to-digital converter. The conversion always takes a non-zero time.

To give the computer an accurate representation of the signal exactly at the sampling instants kT , the converter is preceded by a sample-and-hold circuit.

The sample-and-hold will take the impulses that are produced by the mathematical sampler and produce the piecewise constant output of the device.

Sampling operation

The sampling operation is represented by impulse modulation. Its role is to give a mathematical representation of taking periodic samples from $r(t)$ to produce $r(kT)$.

The sampler takes as input $r(t)$ and returns as output $r^*(t) = \sum_{k=-\infty}^{\infty} r(t)\delta(t - kT)$.

The Laplace transform of $r^*(t)$ can be computed as

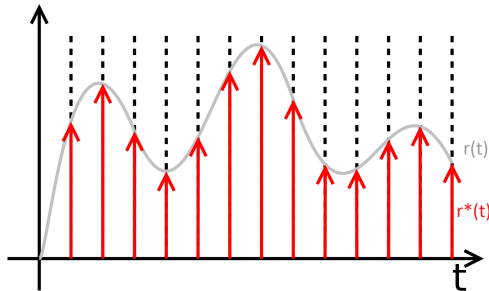
$$\mathcal{L}\{r^*(t)\} = \int_{-\infty}^{\infty} r^*(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} r(\tau)\delta(\tau - kT)e^{-s\tau}d\tau$$

Sampling operation

Using $\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$ we obtain

$$\mathcal{L}\{r^*(t)\} = R^*(s) = \sum_{k=-\infty}^{\infty} r(kT)e^{-skT}$$

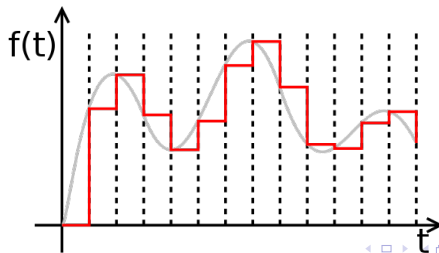
If the signal $r(t)$ is shifted a small amount, then different samples will be selected by the sampling process for the output, proving that sampling is not a time-invariant process.



Hold operation

The hold operation is represented as a linear filter. It is defined by means whereby the impulses are extrapolated to the piecewise constant signal $r_h(t) = r(kT)$ with $kT \leq t < (k+1)T$.

A general technique is to use a polynomial fit to the past samples. If the extrapolation is done by a constant (a zero-order polynomial), then the extrapolator is called a zero-order hold and its transfer function is denoted $ZOH(s)$.



Zero-order hold

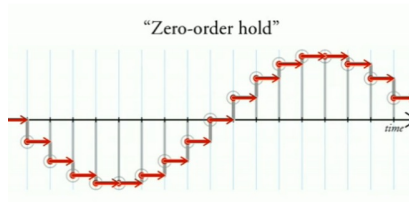
We compute the transfer function as the transform of its impulse response.

If $r^*(t) = \delta(t)$ then $r_h(t)$ is a pulse of height 1 and duration T :

$$r_h(t) = 1(t) - 1(t - T)$$

with Laplace transform

$$ZOH(s) = \mathcal{L}\{p(t)\} = \int_0^\infty [1(t) - 1(t - T)]e^{-s\tau} d\tau = \frac{1 - e^{-sT}}{s}$$



Outline

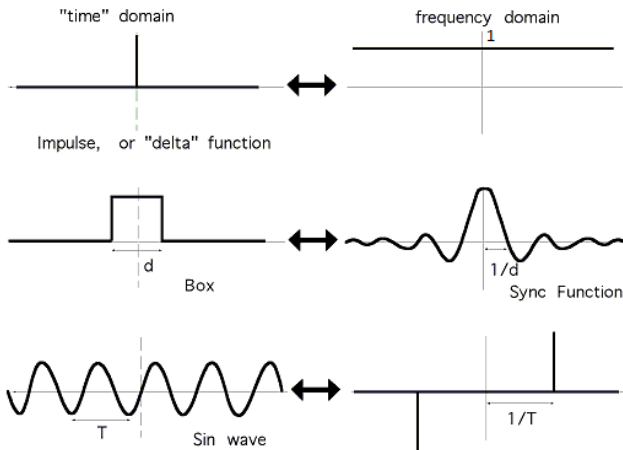
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Fourier transform

$$f(t) \xrightarrow{F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt} F(j\omega)$$

$$f(t) \xleftarrow{f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)e^{j\omega t} d\omega} F(j\omega)$$

Fourier transform



Fourier transform: properties

- **Linearity**

$$\begin{cases} f_1(t) \leftrightarrow F_1(j\omega) \\ f_2(t) \leftrightarrow F_2(j\omega) \end{cases} \Rightarrow af_1(t) + bf_2(t) \leftrightarrow aF_1(j\omega) + bF_2(j\omega)$$

- **Time-scaling**

$$f(at) \leftrightarrow \left(\frac{1}{|a|}\right)F\left(\frac{j\omega}{a}\right)$$

- **Translation/Time-shifting**

$$f(t - t_0) \leftrightarrow e^{-j\omega t_0} F(j\omega)$$

- **Modulation/Frequency-shifting**

$$e^{j\omega_0 t} f(t) \leftrightarrow F(j(\omega - \omega_0))$$

Fourier transform: properties

- **Reciprocity**

$$F(-jt) \leftrightarrow 2\pi f(\omega)$$

- **Derivative in t**

$$\frac{df(t)}{dt} \leftrightarrow j\omega F(j\omega)$$

$$\frac{d^n f(t)}{dt^n} \leftrightarrow (j\omega)^n F(j\omega)$$

- **Derivative in ω**

$$(-jt)^n f(t) \leftrightarrow \frac{d^n F(j\omega)}{d\omega^n}$$

$$\frac{f(t)}{-jt} \leftrightarrow \int_{-\infty}^{\infty} F(j\Omega) d\Omega \text{ if } f(0) = 0$$

- **Convolution**

$$y(t) = h(t) * u(t) \leftrightarrow Y(j\omega) = H(j\omega)U(j\omega)$$

$$v(t) = h(t)u(t) \leftrightarrow V(j\omega) = \frac{1}{2\pi} H(j\omega) * U(j\omega)$$

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Spectrum of a sampled signal

$r^*(t)$ is the product of $r(t)$ and a train of impulses. The latter series is periodic and can be represented by a Fourier series

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) = \sum_{n=-\infty}^{\infty} C_n e^{j(2\pi n/T)t},$$

where the Fourier coefficients C_n are given by:

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \delta(t - kT) e^{-jn(2\pi t/T)} dt.$$

Spectrum of a sampled signal

The only term in the sum of impulses that is in the range of the integral is the $\delta(t)$ at the origin, so the integral reduces to

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jn(2\pi t/T)} dt = \frac{1}{T}$$

We derived the Fourier series of the sum of impulses

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j(2\pi n/T)t}$$

We define $\omega_s = \frac{2\pi}{T}$ as the sampling frequency (rad/s).

Spectrum of a sampled signal

We take the Laplace transform of the output of the sampler,

$$\begin{aligned}
 R^*(s) &= \int_{-\infty}^{\infty} r(t) \left\{ \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \right\} e^{-st} dt \\
 &= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} r(t) e^{jn\omega_s t} e^{-st} dt \\
 &= \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} r(t) e^{-(s-jn\omega_s)t} dt.
 \end{aligned} \tag{1}$$

Since the integral is the Laplace transform of $r(t)$ with only a change of variable where the frequency goes, the result can be written as:

$$R^*(s) = \frac{1}{T} \sum_{n=-\infty}^{\infty} R(s - jn\omega_s).$$

Outline

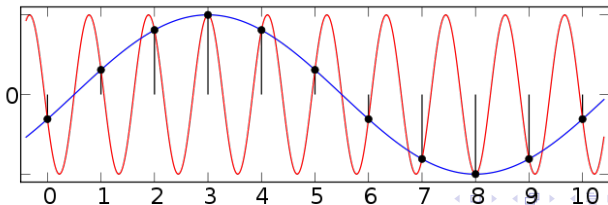
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Aliasing

Aliasing is an effect that causes different signals to become indistinguishable when sampled. Frequencies that are too high to be sampled are folded onto lower frequencies. We cannot distinguish between them based on their samples alone.

Take for example the image below.

The red sine wave is being sampled at just over its bandwidth, however the blue sine wave will be recreated as it also fits all data points and is within the expected bandwidth.



Aliasing

As a direct result of the sampling operation, when data are sampled at a frequency $\frac{2\pi}{T}$, the total harmonic content at a given frequency ω_1 is to be found not only from the original signal at ω_1 , but also from all those frequencies that are aliases of ω_1 , namely $\omega_1 + n2\pi/T = \omega_1 + n\omega_s$.

The errors caused by aliasing can be very severe if a substantial quantity of high-frequency components is contained in the signal to be sampled.

To minimize this error, the sampling operation is preceded by a low-pass anti-aliasing filter that will remove all spectral content above the half-sampling frequency (π/T).

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Shannon-Nyquist sampling theorem

If all content above the half-sampling frequency is removed, no aliasing is introduced by sampling. Also the signal spectrum is not distorted, even though it is repeated endlessly, centered at $n2\pi/T$.

This critical frequency, π/T , is called the **Nyquist frequency**. Band-limited signals that have no components above the Nyquist frequency are represented unambiguously by their samples.

This is the **sampling theorem**: One can recover a signal from its samples if the sampling frequency ($\omega_s = 2\pi/T$) is at least twice the highest frequency (π/T) in the signal. This maximum frequency is also called the **bandwidth B**.

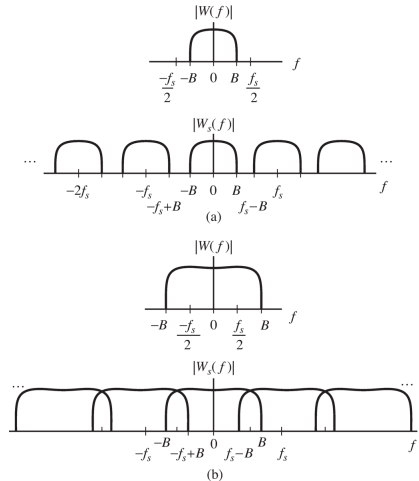
Shannon-Nyquist sampling theorem

The signal can be fully reconstructed if there are no overlaps in the frequency domain.

If the sampling frequency is at least twice the bandwidth B , then the signal can be reconstructed without a problem (no overlap). (fig. a)

If the sampling frequency is too low then information will be lost (overlap). (fig. b)

Sampling frequency $f_s \geq 2B$



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Hidden oscillations

There is the possibility that a signal could contain some frequencies that the samples do not show at all.

Such signals, when they occur in digital control systems, are called **hidden oscillations**.

They can only occur at multiples of the Nyquist frequency (π/T).

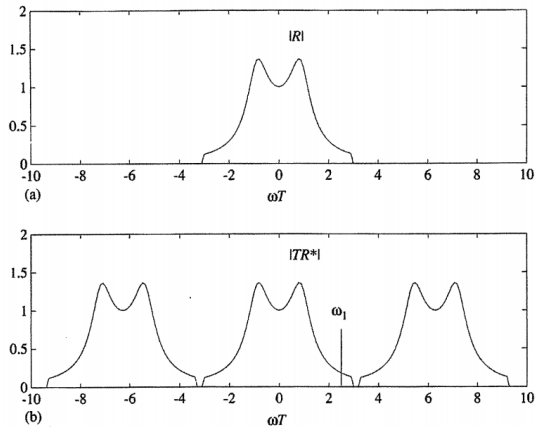
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Reconstruction

Sampling theorem: *under the right conditions* it is possible to recover a signal from its samples.

The figure to the right shows the spectrum of $R(j\omega)$. It is contained in the low-frequency part of $R^*(j\omega)$. Therefore, to recover $R(j\omega)$ we need to process $R^*(j\omega)$ through a low-pass filter and multiply by T .



Reconstruction

If $R(j\omega)$ has zero energy for frequencies in the bands above the Nyquist frequency, in other words R is band-limited, then an ideal low-pass filter with gain T for $-\pi/T \leq \omega \leq \pi/T$ and zero elsewhere would recover $R(j\omega)$ from $R^*(j\omega)$ exactly.

If we define the ideal low-pass filter characteristic as $L(j\omega)$, we have:

$$R(j\omega) = L(j\omega)R^*(j\omega).$$

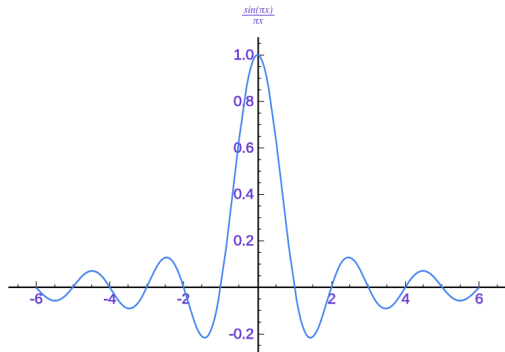
The signal $r(t)$ is the inverse Fourier transform of $R(j\omega)$. Because $R(j\omega)$ is the *product* of two Fourier transforms, $r(t)$ is the *convolution* of the time functions $\ell(t)$ and $r^*(t)$.

$$r(t) = \ell(t) * r^*(t)$$

Ideal low-pass filter

The impulse response of the filter can be computed using this definition

$$\begin{aligned}
 \ell(t) &= \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} T e^{j\omega t} d\omega \\
 &= \frac{T}{2\pi} \frac{e^{j\omega t}}{jt} \Big|_{-\pi/T}^{\pi/T} \\
 &= \frac{T}{2\pi jt} (e^{j(\pi t/T)} - e^{-j(\pi t/T)}) \\
 &= \frac{\sin(\pi t/T)}{\pi t/T} \\
 &\triangleq \text{sinc} \frac{\pi t}{T}
 \end{aligned}$$



The sinc functions are the interpolators that fill in the time gaps between samples with a signal that has no frequencies above π/T .

Reconstruction

Using the previous equations, we find:

$$r(t) = \int_{-\infty}^{\infty} r(\tau) \sum_{k=-\infty}^{\infty} \delta(\tau - kT) \text{sinc} \frac{\pi(t-\tau)}{T} d\tau.$$

Using the shifting property of the impulse, we have:

$$r(t) = \sum_{k=-\infty}^{\infty} r(kT) \text{sinc} \frac{\pi(t - kT)}{T}$$

This equation is a constructive statement of the sampling theorem. There is one disadvantage. Because $\ell(t)$ is nonzero for $t < 0$, this filter is noncausal. $\ell(t)$ starts at $t = -\infty$ while the impulse that triggers it does not occur until $t = 0$. The noncausality can be overcome by adding a phase lag, $e^{-j\omega\lambda}$, to $L(j\omega)$, which adds a delay to the filter and to the signals processed through it.

Zero-order hold

The transfer function of the zero-order hold was introduced as

$$ZOH(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega}.$$

We express this function in magnitude and phase form, to discover the frequency properties of $ZOH(j\omega)$.

We factor out $e^{-j\omega T/2}$ and multiply and divide by $2j$:

$$\begin{aligned} ZOH(j\omega) &= e^{-j\omega T/2} \left\{ \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j} \right\} \frac{2j}{j\omega} \\ &= T e^{-j\omega T/2} \frac{\sin(\omega T/2)}{\omega T/2} \\ &= e^{-j\omega T/2} T \operatorname{sinc}(\omega T/2) \end{aligned}$$

Zero-order hold

The magnitude function is

$$|ZOH(j\omega)| = T \left| \text{sinc} \frac{\omega T}{2} \right|$$

and the phase is

$$\angle ZOH(j\omega) = \frac{-\omega T}{2}$$

plus the 180° shifts where the sinc function changes sign.

Thus the effect of the zero-order hold is to introduce a phase shift of $\omega T/2$ (a time delay of $T/2$ seconds) and to multiply the gain by a function with the magnitude of $\text{sinc}(\omega T/2)$.

Zero-order hold

Spectrum of ideal filter



Spectrum of ZOH

