Lecture 4 - Discrete Time Systems - Representations

July 7, 2015

Outline

Discrete Time Signal

Definition

Discrete time signals are sequences of values at the moments
. . . ,-2T,-T,0,T,2T,. . . .
$x[k]$ is the value of a signal at the moment $t = kT$

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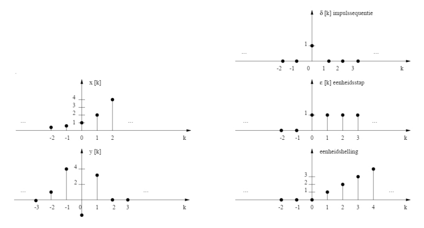
Example



Discrete Time System

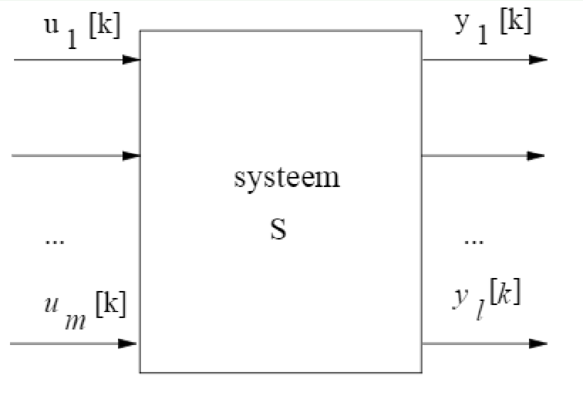
Definition

A linear time-invariant (LTI) discrete time system processes an inputvector $u[k]$ to an outputvector $y[k]$.
Such a system has:

- A vector of input $u[k]$
- A vector of output $y[k]$
- A vector of states $x[k]$

Discrete Time System

Example



How to represent a system?

- Block-diagram

How to represent a system?

- Block-diagram
- State space representation

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- Difference/differential equation

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- Impulse response

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- Transferfunctions

Outline

Block diagram

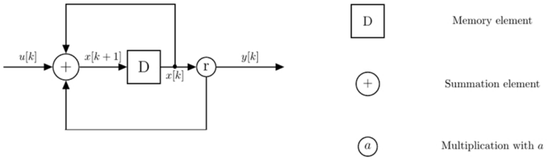


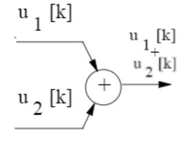
Figure: An example of a discrete time system

Definition

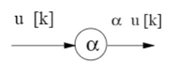
A block diagram is a visual representation of a system. All LTIs (Linear Time Invariant) systems can be constructed using 3 building blocks(Memory element, summation element, multiplication element). Note that every memory element

Building blocks

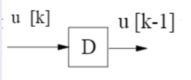
Adder



Constant Multiplier

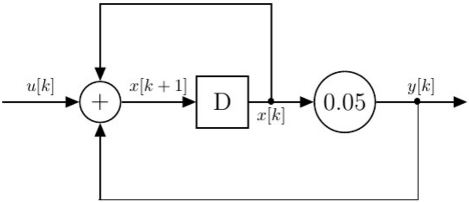


Delay element



Example: compond interes

- $u[k]$:The deposits and withdrawals from the bank account
- $x[k]$:The current saldo on bank account(before deposit and interest)
- $y[k]$: The acquired interest of that year
- $x[k+1]$: The saldo on the next year = current saldo + interest + deposits



$u[k]$	$x[k+1]$	$x[k]$	$y[k]$
50	50	0	0
0	52.5	50	2.5
-25	30.13	52.5	2.62
0	31.63	30.13	1.51
0	33.21	31.63	1.58
30	64.87	33.21	1.66
0	68.12	64.87	3.24
0	71.52	68.12	3.41

Bad block diagrams

Delay-free loops

The issue is that this leads to an implicit connection $u[k]$ depends on $y[k]$,which is not yet known You can easily rewrite this in an allowd shape $y[k] = u[k] + 3y[k] \iff y[k] = -\frac{1}{2}u[k]$

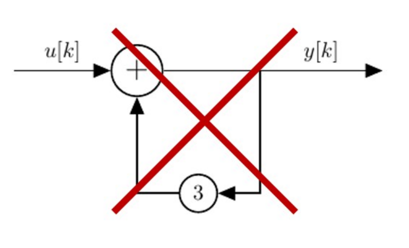
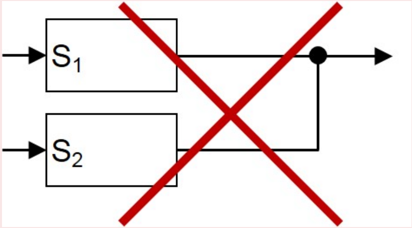


Figure: An example of a delay free loop

Bad block diagrams

Connecting two outputs without using a sum

The issue is that this can lead to inconsistencies. According to this block diagram the output of the systems S1 and S2 are equal.There is no way to get around this.



Outline

State space representation

Definition

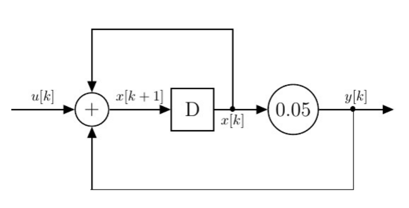
State space representation

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

This state space representation is specific to LTI systems:
Linear: its easy to see these systems are linear
Time-invariant: the matrices A,B,C,D do not depend on time, if it were to be a time-variant system the matrices would be replaced by $A[k], B[k], C[k] and D[k]$.

From block diagram to state space

Blockdiagram



State space representation

1. Let the output of the memory elements be $x_i[k]$.
2. So the input of the memory elements are x_{k+1}
3. Trace back to retrieve equations for $x_i[k+1]$ and $y_i[k]$

This results in:

$$x[k+1] = u[k] + 1.05x[k]$$
$$y[k] = 0.05x[k]$$

From state space to block diagram

State space representation

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

with $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, $C = \begin{bmatrix} 5 & 1 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 \end{bmatrix}$

Blockdiagram

1. First add a delay element for every state $x_i[k]$
2. Determine the input for every state x[k+1] from the matrixes A and B, as a combination of the states x[k] and inputs u[k]
3. Determine the outputs y[k] in the same way with the matrixes C and D

From state space to block diagram (DT)

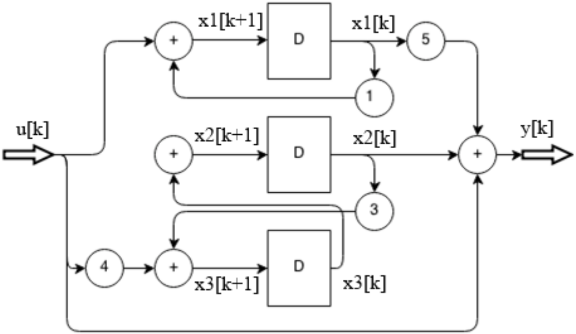


Figure:

Different state space representations

State space representation is not unique

Take the following system, which connects u[k] to y[k]:

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

Now take a non-singular square matrix T and the following system. The relation between u[k] and y[k] will be the same.

$$Tx[k+1] = TAT^{-1}Tx[k] + TBu[k]$$
$$y[k] = CT^{-1}Tx[k] + Du[k]$$

With $x' = Tx, A' = TAT^{-1}, B' = TB, C' = CT^{-1}$ and $D' = D$, we have found a different state space representation for this system.

Solving state space equation

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

We express $X[1], x[2], \dots$ in function of $x[0]$:

$$\begin{aligned} \mathrm{x}[1] &= Ax[0] + Bu[0] \\ \mathrm{x}[2] &= Ax[1] + Bu[1] = A^2x[0] + ABu[0] + Bu[1] \\ &\vdots \\ \mathrm{x}[\mathrm{k}] &= A^kx[0] + \sum_{i=0}^{k-1} A^{k-1-i}Bu[i] \end{aligned} \tag{1}$$

The output is $y[k]$:

$$y[k] = \begin{cases} Cx[0] + Du[0] & \text{if k} = 0 \\ CA^kx[0] + \sum_{i=0}^{k-1} CA^{k-1-i}Bu[i] + Du[k] & \text{if k} > 0 \end{cases}$$

Outline

Diffence equations

Definition

Similar to differential equations, but for discrete time.

General form: $\sum_{i=0}^{n} a_i y[k+i] = \sum_{i=0}^{n} b_i u[k+i]$

With n the order of the system.

Solution in 2 parts

1. Homogenous: solution from input zero
2. Particular: solution derived as a response from the input

Homogenous difference equations

Definition

General form: $\sum_{i=0}^{n} a_i y[k+i] = 0$

Example

$$y[k+1] - ay[k] = 0$$
$$y[k+1] = ay[k]$$
$$y[1] = ay[0]$$
$$y[2] = ay[1] = a^2 y[0]$$
$$\vdots$$
$$y[n] = a^n y[0]$$

Homogenous difference equations

solution

- Expected form of solution: r^k
- Substitution of the expected solution in the difference equation: $\sum_{i=0}^{n} a_i r^{k+i} = 0$
- Division by r^k leads to the characteristic equation: $\sum_{i=0}^{n} a_i r^i = 0$
- Solutions of the characteristic equation: $r_1, r_2, r_3, \ldots$
- Homogenous solution to the difference equation:

$$y[k] = c_1 r_1^k + c_2 r_2^k + c_3 r_3^k + \cdots = \sum_{i=1}^{n} c_i r_i^k$$

Example

- Homogeneous recurrence relations:
  $y[k+2] - 5y[k+1] + 6y[k] = 0$
- Initial value: $y[0] = 1$, $y[1] = 1$
- Characteristic polynomial: $r^2 - 5r + 6 = 0$
- Roots: 2 and 3
- General solution: $c_1 2^k + c_2 3^k$
- Intital values:

$$\begin{Bmatrix} 1 = c_1 + c_2 \\ 1 = 2c_1 + 3c_2 \end{Bmatrix}$$

$$\begin{Bmatrix} 2 = c_1 \\ -1 = c_2 \end{Bmatrix}$$

- Result: $y[k] = 2^{k+1} - 3^k$

Example: Fibonacci sequence

Figure: Leonardo Bonacci (c. 1170 c. 1250)known as Fibonacci was an Italian mathematician, considered to be "the most talented Western mathematician of the Middle Ages".

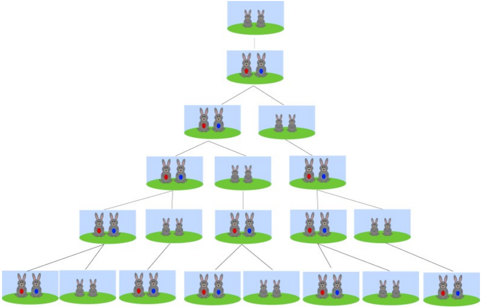


Figure: Fibonacci sequence

Example: Fibonacci sequence

Example

- Homogeneous recurrence relations: $y[k+2] = y[k+1] + y[k]$
- Initial value: $y[0] = 1, y[1] = 1$
- Characteristic polynomial: $r^2 - r - 1 = 0$
- Roots: $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$
- General solution: $y[k] = c_1(\frac{1+\sqrt{5}}{2}) + c_2(\frac{1-\sqrt{5}}{2})$
- Intital values:

$$\begin{Bmatrix} c_1 + c_2 = 1 \\ c_1\frac{1+\sqrt{5}}{2} + c_2\frac{1-\sqrt{5}}{2} = 1 \end{Bmatrix}$$

$$\begin{Bmatrix} c_1 = \frac{5+\sqrt{5}}{10} \\ c_2 = \frac{5-\sqrt{5}}{10} \end{Bmatrix}$$

- Result: $y[k] = (\frac{5+\sqrt{5}}{10})(\frac{1+\sqrt{5}}{2}) + (\frac{5-\sqrt{5}}{10})(\frac{1-\sqrt{5}}{2})$

Multiple roots and Complex roots

Multiple roots

For a multiple root r_i with multiplicity m add r_i^k,kr_i^k,...,$k^{m-1}r_i^k$

Complex roots

Complex will result in oscillating behavior. If the difference equations and starting conditions are both real the complex roots can only be present in conjugate pairs, the constants will also be in conjugate pairs.

$$r_i = Re^{j\phi} \; r_{i+1} = Re^{-j\phi}$$

$$c_i = R_0 e^{j\phi_0} \; c_{i+1} = R_0 e^{-j\phi_0}$$

$$c_i r_i^k + c_{i+1} r_{i+1}^k = R_0 R e^{jk\phi + j\phi_0} + R_0 R e^{-(jk\phi + j\phi_0)}$$

This can be converted into a cosine using Eulers formula:

$$y[k] = R_0 R\Big(\cos(k\phi + \phi_0) + \sin(k\phi + \phi_0)\Big) + R_0 R\Big(\cos(k\phi + \phi_0) - \sin(k\phi + \phi_0)\Big)$$

Eulers formula

Theorem

$e^{j\phi} = \cos(\phi) + \sin(\phi)j$

Proof.

Using power series:

$$\begin{aligned} e^{\phi j} &= 1 + jx - \frac{x^2}{2!} - \frac{jx^3}{3!} + \frac{x^4}{4!} + \frac{jx^5}{5!} + \dots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)j \qquad (2) \\ &= \cos(\phi) + \sin(\phi)j \end{aligned}$$

□

Non-homogeneous difference equations

Definition

$$\sum_{i=0}^{n} a_i y[k+i] = \sum_{i=0}^{n} b_i u[k+i]$$

A linear combination of inputs results in the same linear combination of the outputs resulting from each input individually.

Solution

The equation can thus be solved for each input individually and the results added together afterwards.
The resulting particular solutions can then be added to the general form of the homogenous solution.

Particular solutions to difference equations

Input $u[k]$	Suggested solution $y[k]$
k	$\alpha_1 k + \alpha_0$
k^n	$\sum_{i=0}^{n} \alpha_i k^i$
a^k	αa^k
$k^n a^k$	$(\sum_{i=0}^{n} \alpha_i k^i) a^k$
$cos(k\phi)$	$\alpha cos(k\phi + \phi_0)$
$a^k cos(k\phi)$	$\alpha a^k cos(k\phi + \phi_0)$
$k^n a^k cos(k\phi)$	$(\sum_{i=0}^{n} \alpha_i k^i \alpha a^k cos(k\phi + \phi_0))$

Example

Example

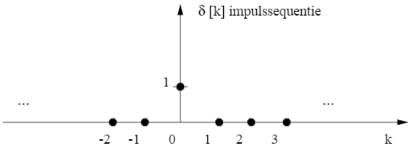
- Difference equation : $y[k+2] - 5y[k+1] + 6y[k] = (-1)^k$
- Initial value: $y[1] = \frac{1}{4}, y[0] = \frac{1}{12}$
- Homogeneous difference equation:
  $y[k+2] - 5y[k+1] + 6y[k] = 0$
- Characteristic polynomial: $r^2 - 5r + 6 = 0$
- Homogeneous solution: $y_{hom}[k] = c_1 2^k + c_2 3^k$
- Particular solution: $y_{par}[k] = \alpha(-1)^k$
- Subsitution: $\alpha(-1)^{k+2} - 5\alpha(-1)^{k+1} + 6\alpha(-1)^k = (-1)^k$
- $\alpha = \frac{1}{12}$
- General solution: $y[k] = c_1 2^k + c_2 3^k + \frac{1}{12}(-1)^k$
- Initial values: $c_1 = -\frac{1}{3}, c_2 = \frac{1}{3}$
- Result: $y[k] = -\frac{1}{3}2^k + \frac{1}{3}3^k + \frac{1}{12}(-1)^k$

Outline

Impulse responses

Definition

$$\delta[k] = \begin{cases} 1 & \text{if k} = 0 \\ 0 & \text{otherwise} \end{cases}$$



Theorem

You can decompose any signal in a sum of impulse response:s

$$f[k] = \sum_{i}^{i=\infty} \delta[k-i]f[i] = \delta[k] * f[k]$$

Convolution

Definition

$$w[k] = u[k] * v[k] = \sum_{i=-\infty}^{\infty} u[i]v[k-i]$$

Solve

1. Flip v[i] around vertical axis(v[-i]).
2. Slide to the right over k steps(v[k-i]).
3. Multiply $u[i]$ and $v[k-i]$
4. Sum all the vaules.

Convolution theorem (DT)

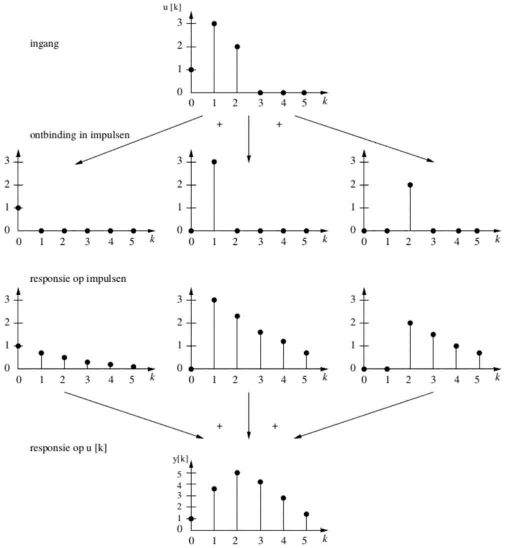
Theorem

$y[k] = u[k] * h[k]$

Proof.

$$\delta[k] \rightarrow h[k]$$
$$\delta[k+i] \rightarrow h[k+i]$$
$$\sum_{i=-\infty}^{i=\infty} \delta[k-i]u[i] \rightarrow \sum_{i=-\infty}^{i=\infty} h[k-i]u[i]$$
$$u[k] \rightarrow u[k] * h[k] = y[k]$$

$\square$


ingang
u [k]
ontbinding in impulsen
responsie op impulsen
responsie op u [k]
y[k]
k

Impulse response

Definition

The impulse response of a dynamic system is its output when presented with a brief input signal, called an impulse.

Impulse response

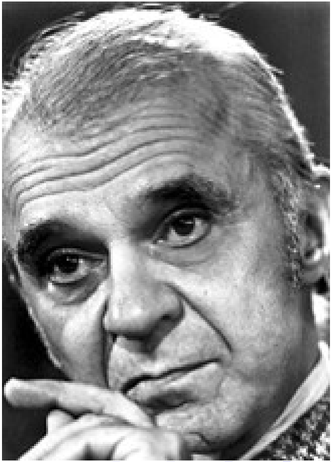
$$h[k] = \begin{cases} 0 & \text{if k} < 0 \\ D & \text{k} = 0 \\ CA^{k-1}B & \text{k} > 0 \end{cases}$$

Examples of Dirac-deltas

Popping balloons for acoustic measurements

Example: Leontief model of a planned economy

- Won the nobel prize in 1973
- A simple model that assigns values to different sectors
- For simplicity we choose a planned economy. But today governments all over the world are using similar models to model their economy.



Example: Leontief model of a planned economy

Leontief divided the economy in sectors who buy from eachother. To produce one unit of industry 0.40 units of agriculture are required TABEL INPORTEREN

Example: Leontief model of a planned economy

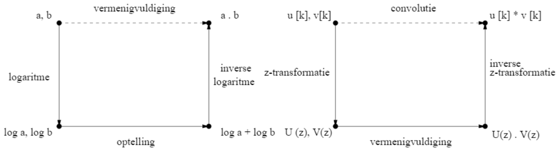
:production of sector i in month k :the demand to goods from sector i in the next month Note: in planned economy demand can be steered, economists can decide how many rations they give.
The model $x[k-1] = Ax[k] + Iu[k]$
y[k] = I x[k]
It is anti-causal: a subdivision of non-causal, for which only future values have to be known to know the current value More info

Outline

Z-transform

Definition

- Discrete equivalent to the Laplace-transform
- Converts time dependent descriptions of systems to the time-independent Z-domain.
- Simplifies many calculations:
  - Convolution theorem convolution becomes multiplication
  - Linear difference equations become simple algebraic expressions
  - ...



Z-transform

2 forms

- Bilateral: Requires knowledge of h for all values of k, including negative values Can be used for non-causal systems
  $$X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$$
- Unilateral: Only requires knowledge of h for positive values of k Can only be used for causal systems without loss of information $X(z) = \sum_{k=0}^{\infty} x[k]z^{-k}$

Z-transform

Example

$$x[k] = \begin{Bmatrix} 1 & -1 & 0 & 2 & 4 \\ & & & \uparrow & \end{Bmatrix}$$

$$X(z) = \sum_{k=-3}^{1} x[k]z^{-k} = z^3 - z^2 + 2 + 4z^{-1}$$

Properties Unilateral Z-transform

Propery	Time Domain	Z-domain
Linearity	$af_1[n] + bf_2[n] + \ldots$	$aF_1(Z) + bF_2(Z) + \ldots$
Right Shift(m¿0)	$f[k-m]$	$z^{-m}F(Z)$
Left Shif (m¿0)	$f[k+m]$	$z^m\left(F(z) - \sum_{i=0}^{m-1} f[i]z^{-i}\right)$
Convolution	$f[k] * g[k]$	$F(z)G(z)$
Multiplication by a^k	$a^k f[k]$	$F(a^{-1}z)$
Summation in time	$\sum_{i=0}^{k} f[i]$	$\frac{z}{z-1}F(Z)$
Differentation in z	$k^m f[k]$	$\left(-z\frac{d}{d}\right)^m F(z)$
Periodic Sequence	$f[k] = f[k+N]$	$F(z) = \frac{z^N}{z^{N-1}} \sum_{k=0}^{N-1} f[k]z^{-1}$
Initial Value	f[0]	$\lim_{\lvert z\rvert \to \infty} F(z)$
Final value	$f[\infty]$	$\lim_{z\to 1}(z-1)F(z)$