Chapter 7: Sampling and reconstruction of signals

August 6, 2015

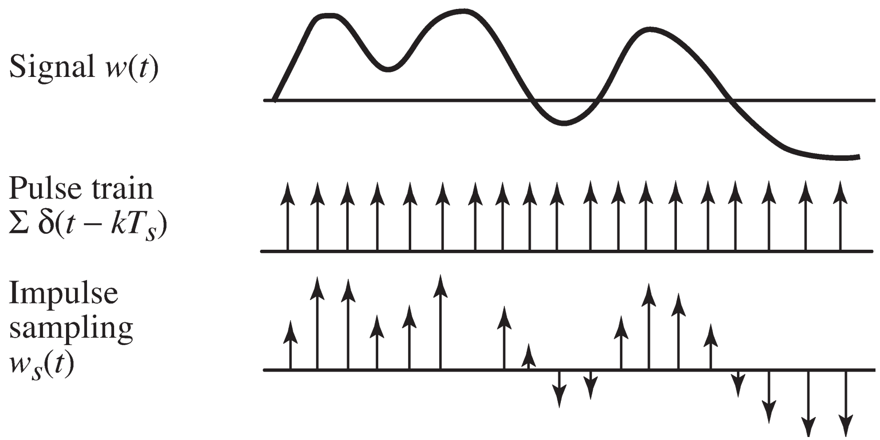
Outline

Discretization and reconstruction of signals

Definition

The use of digital logic and computers to calculate a control action for a continuous system introduces the operation of sampling. Samples are taken from the continuous signals and used in the computer to calculate the controls to be applied.

The role of sampling and the conversion from continuous to discrete and vice versa are important to the understanding of the complete response of digital control.



Types of signals

- **Continuous-time signal**: A signal defined over a continuous range of time.
- **Discrete-time signal**: A signal defined only at discrete instants of time (the independent variable t is quantized)
- **Analog signal**: A signal defined over a continuous range of time whose amplitude can assume a continuous range of values.
- **Quantized signal**: A signal in which the amplitude may assume only a finite number of distinct values.

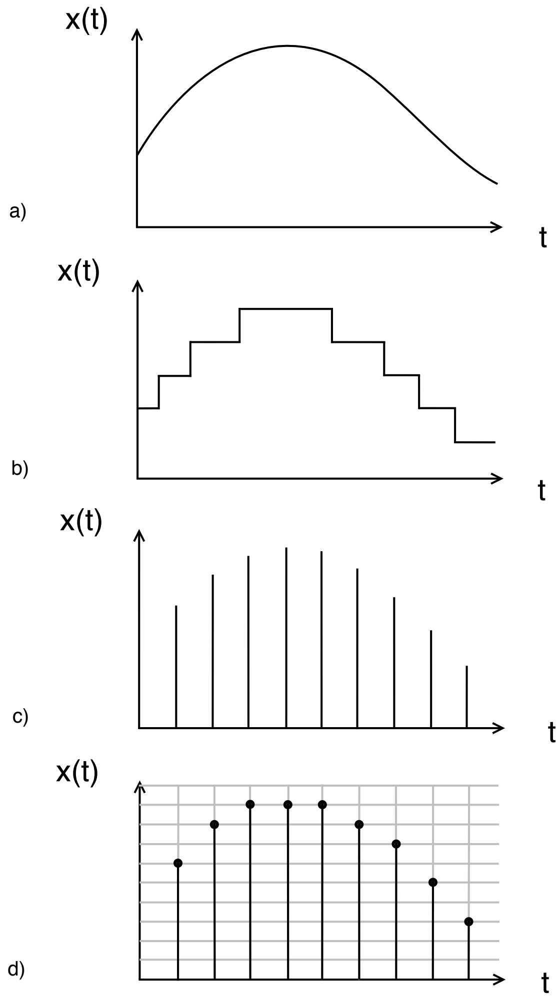
Types of signals

a. Continuous-time analog signal

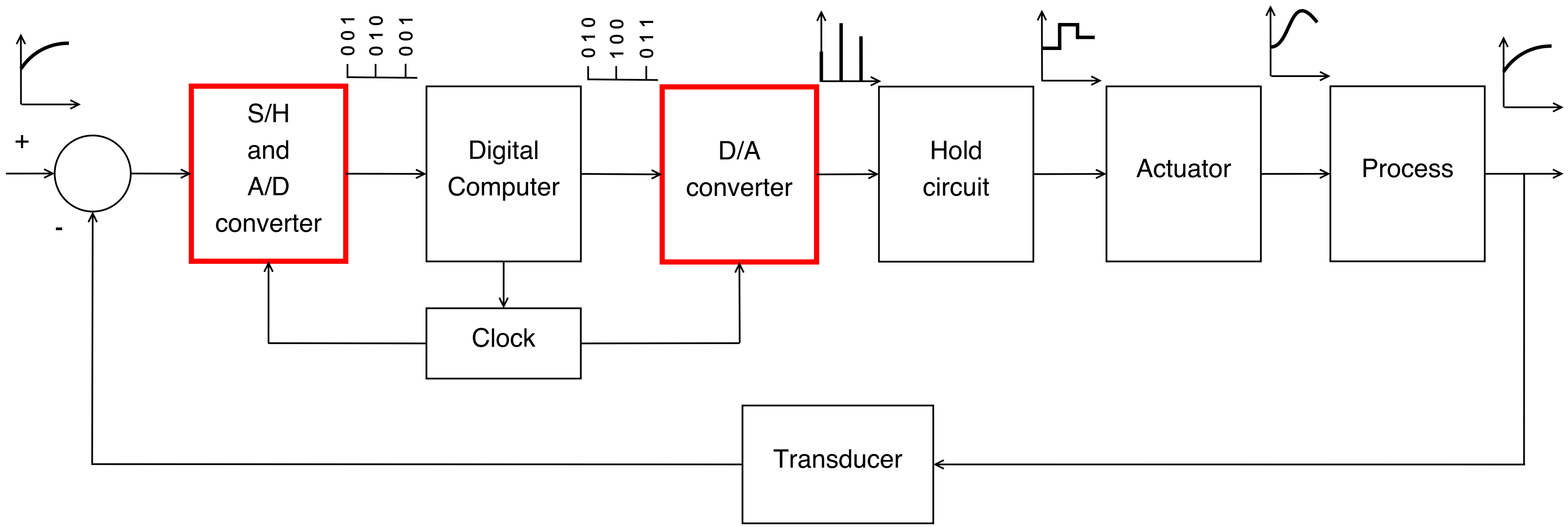
b. Continuous-time quantized signal

c. Sampled-data signal (discrete-time analog signal)

d. Digital signal (discrete-time quantized signal)



Digital control system



Definitions

Sample and hold

A circuit that receives an analog input signal and holds this signal at a constant value for a specified period of time.

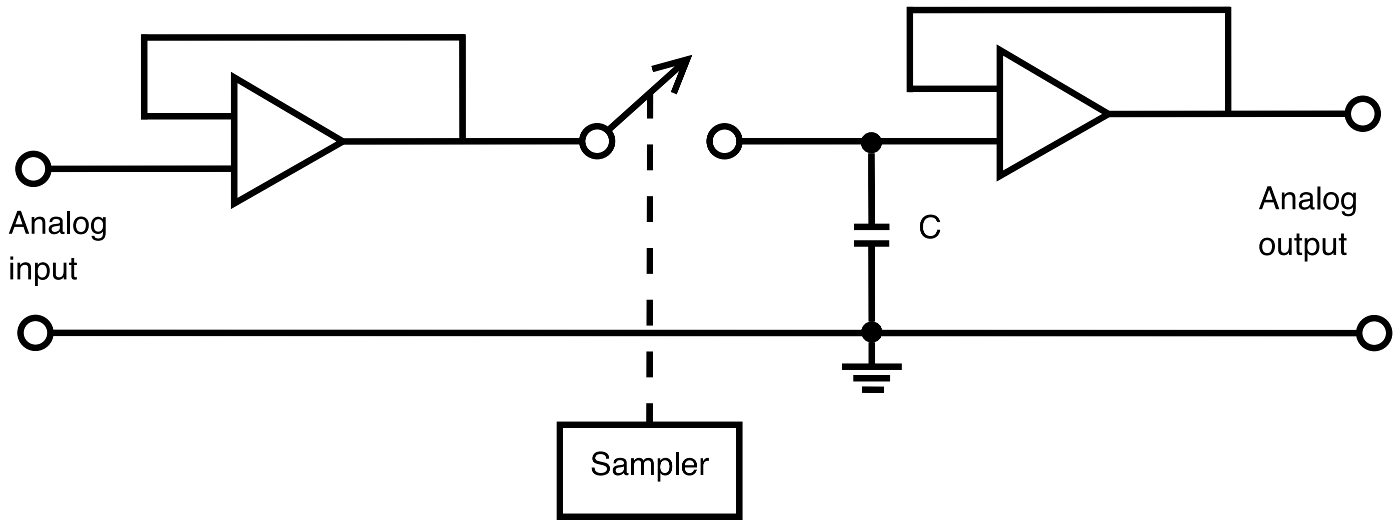
A/D converter

An analog-to-digital converter, also called an encoder, is a device that converts an analog signal into a digital signal.

D/A converter

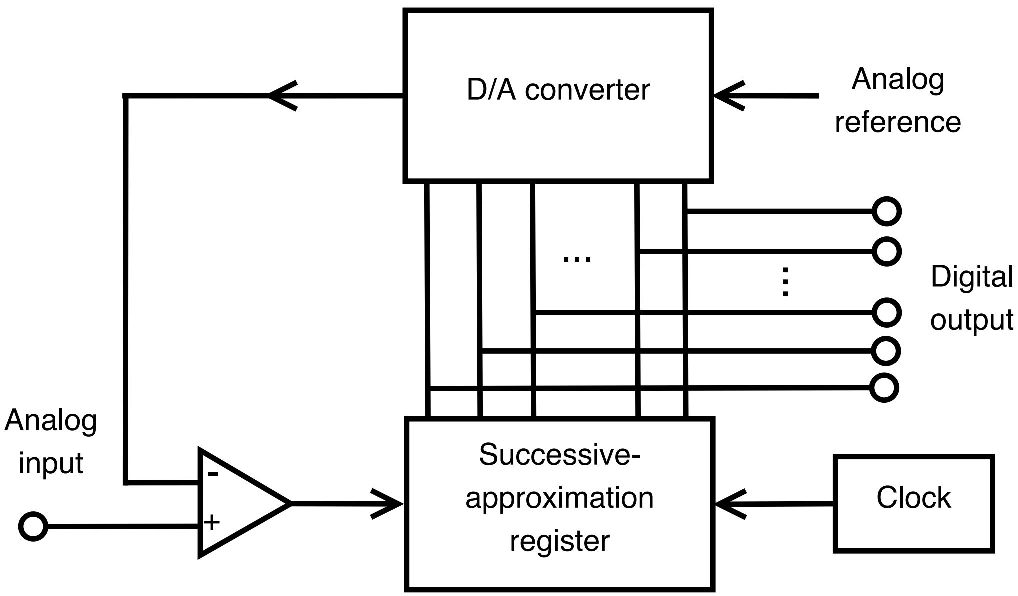
A digital-to-analog converter, also called a decoder, is a device that converts a digital signal into an analog signal.

Sample and hold



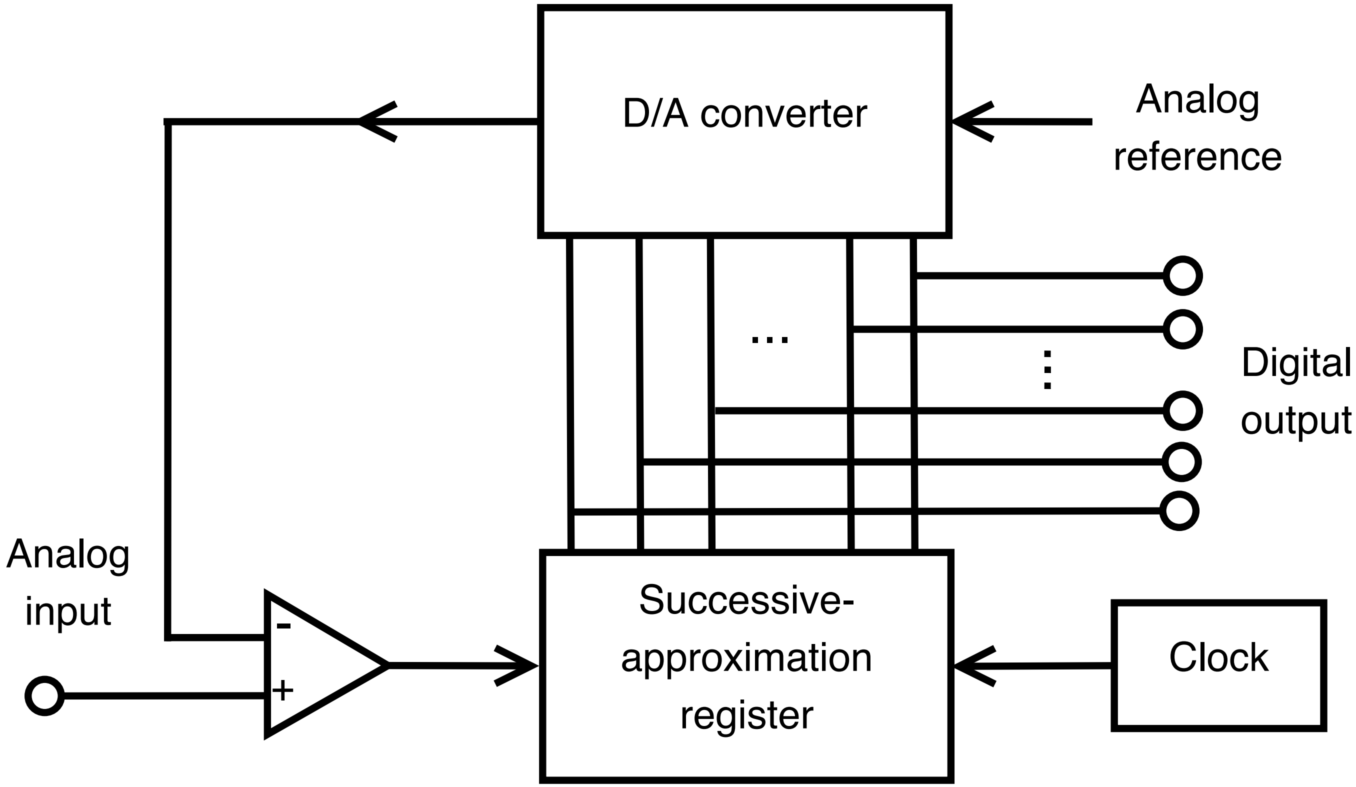
When the switch is closed, the input signal is connected and the circuit is in **tracking mode**. The charge on the capacitor in the circuit tracks the input voltage.
When the switch is open, the input signal is disconnected and the circuit is in **hold mode**. The capacitor voltage holds constant for a specified time period.

A/D converter



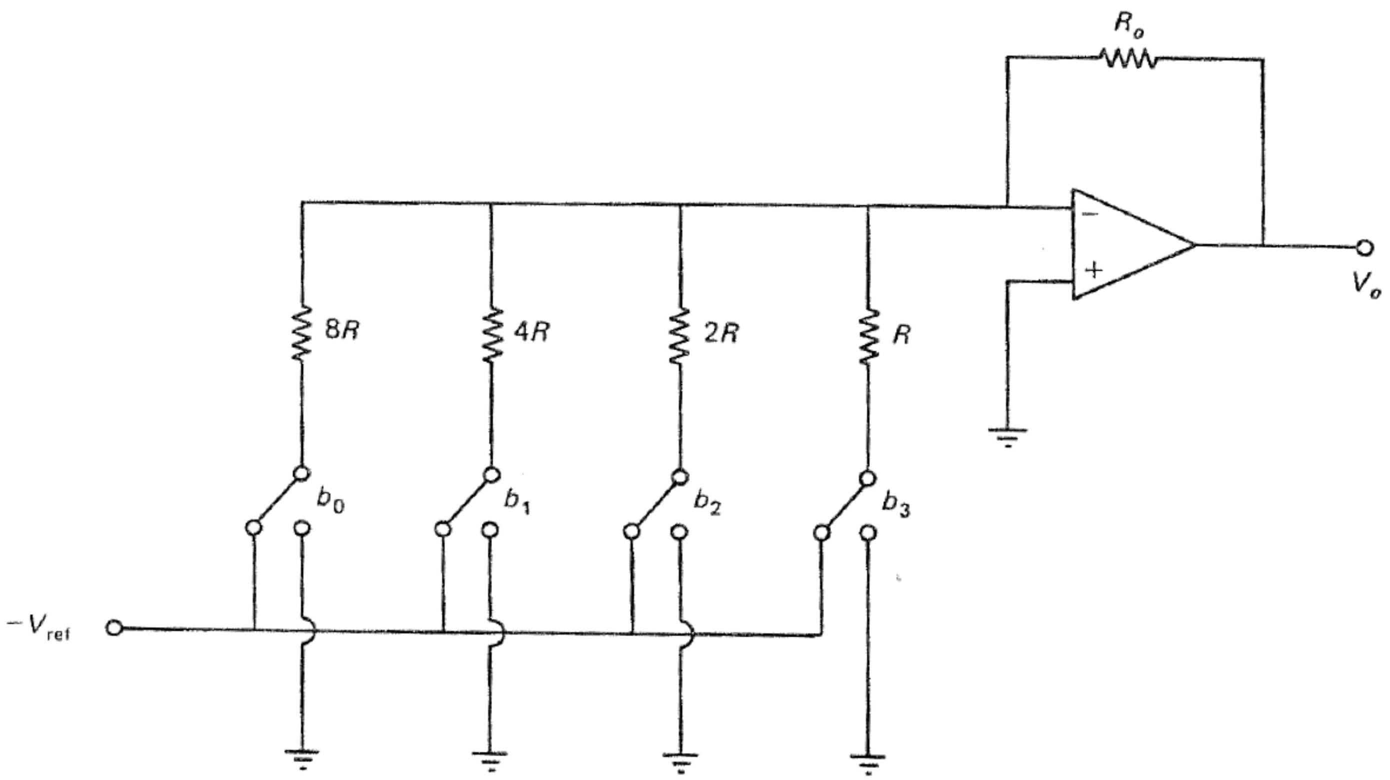
The successive-approximation register (SAR) is most frequently used. It turns on the most significant bit and compares it with the analog input. The comparator decides whether to leave the bit on or turn it off (if the analog input voltage is larger, the most significant bit is set on). Then the register continues with bit 2.

A/D converter



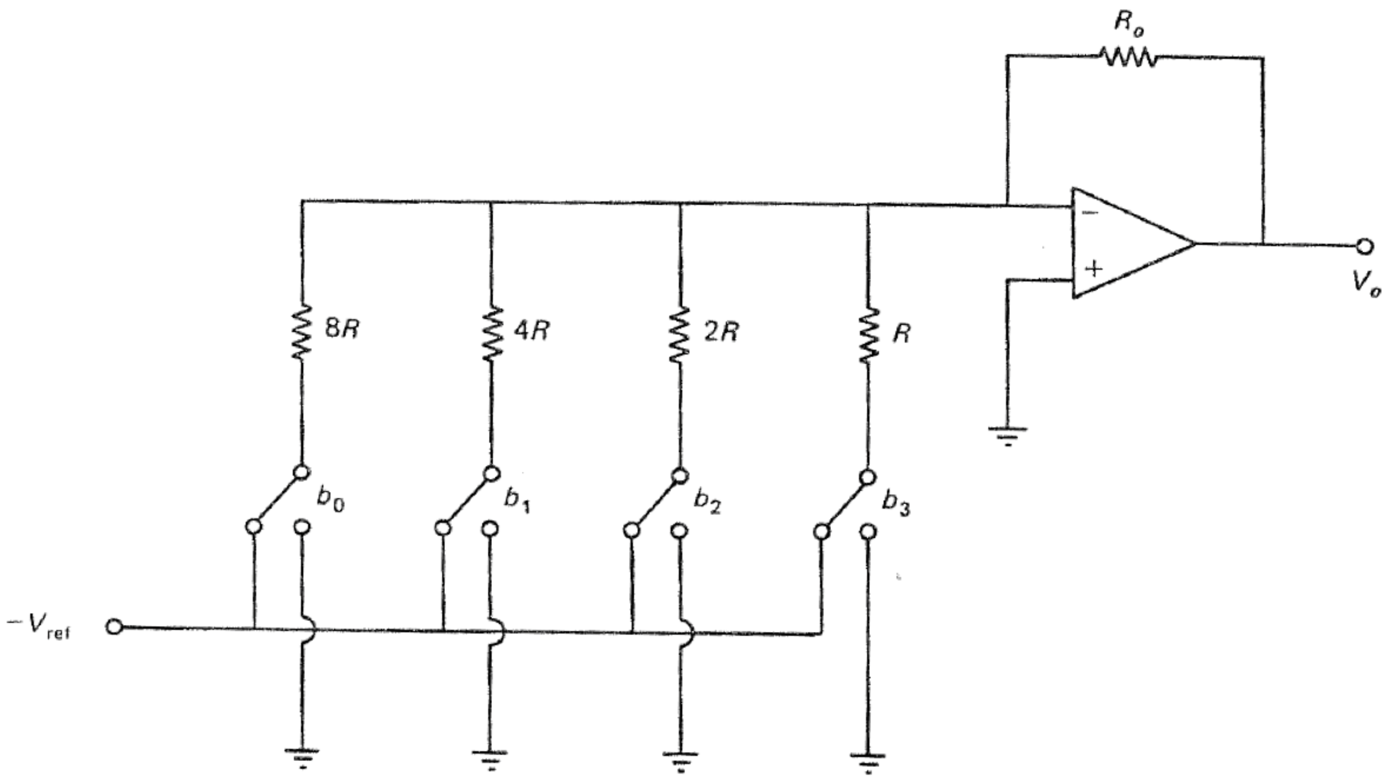
After n comparisons, the digital output of the register indicates all those bits that remain on and produces the desired digital code.

D/A converter



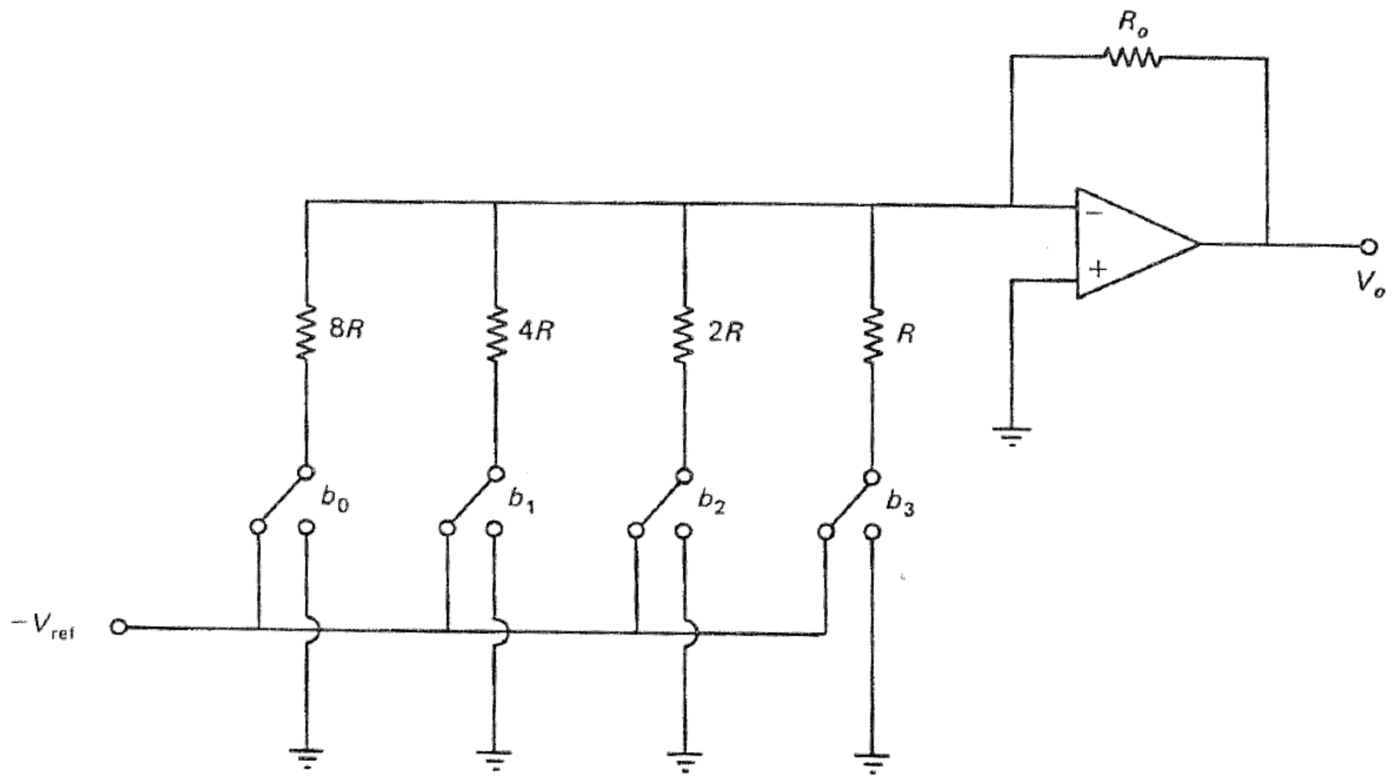
The input resistor of the operational amplifier have their resistance values weighted in a binary fashion.

D/A converter



When the logic circuit receives binary 1, the switch connects the resistor to the reference voltage. When the logic circuit receives binary 0, the switch connects the resistor to ground.

D/A converter



If the binary number (here) is $b_3 b_2 b_1 b_0$, each of the b's can be 0 or 1, the output is $V_o = \frac{R_o}{R}\left(b_3 + \frac{b_2}{2} + \frac{b_1}{4} + \frac{b_0}{8}\right) V_{ref}$

Outline

Analysis of the sample and hold

To get samples of a continuous signal, we use an analog-to-digital converter. The conversion always takes a non-zero time.

To give the computer an accurate representation of the signal exactly at the sampling instants kT_s, the converter is preceded by a sample-and-hold circuit.

The sample-and-hold will take the impulses that are produced by the mathematical sampler and produce the piecewise constant output of the device.

Sampling operation

The sampling operation is represented by impulse modulation. Its role is to give a mathematical representation of taking periodic samples from $r(t)$ to produce $r(kT_s)$.

The sampler takes as input $r(t)$ and returns as output

$$r^*(t) = \sum_{k=-\infty}^{\infty} r(t)\delta(t - kT_s).$$

The Laplace transform of $r^*(t)$ can be computed as

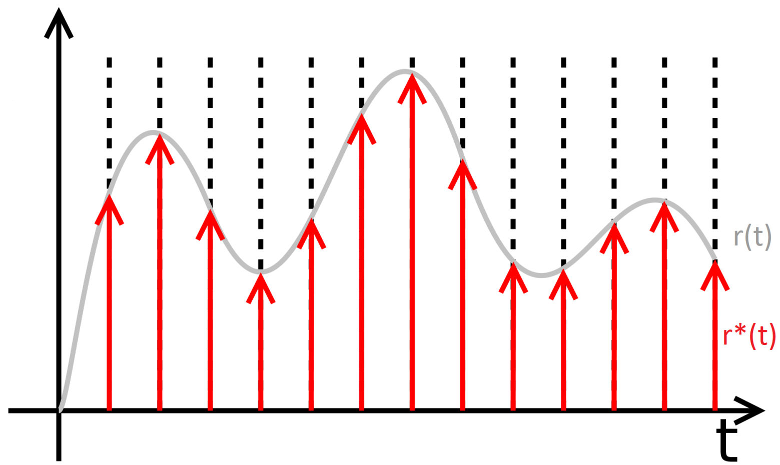
$$\mathcal{L}\{r^*(t)\} = \int_{-\infty}^{\infty} r^*(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} r(\tau)\delta(\tau - kT_s)e^{-s\tau}d\tau$$

Sampling operation

Using $\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$ we obtain

$$\mathcal{L}\{r^*(t)\} = R^*(s) = \sum_{k=-\infty}^{\infty} r(kT_s)e^{-skT_s}$$

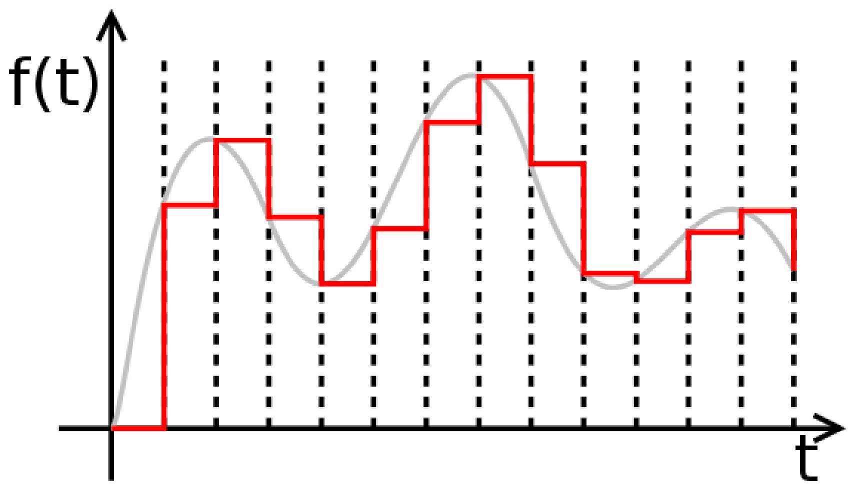
If the signal r(t) is shifted a small amount, then different samples will be selected by the sampling process for the output, proving that sampling is not a time-invariant process.



Hold operation

The hold operation is represented as a linear filter. It is defined by means whereby the impulses are extrapolated to the piecewise constant signal $r_h(t) = r(kT_s)$ with $kT_s \leq t < (k+1)T_s$.

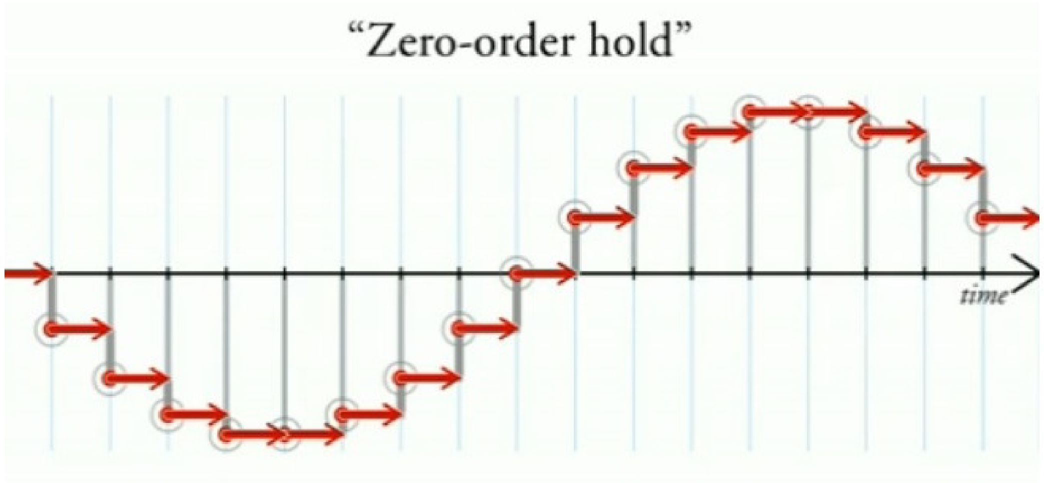
A general technique is to use a polynomial fit to the past samples.If the extrapolation is done by a constant (a zero-order polynomial), then the extrapolator is called a zero-order hold and its transfer function is denoted $ZOH(s)$.



Zero-order hold

We compute the transfer function as the transform of its impulse response. If $r^*(t) = \delta(t)$ then $r_h(t)$ is a pulse of height 1 and duration T: $r_h(t) = 1(t) - 1(t - T_s)$. Which has the following Laplace transform:

$$ZOH(s) = \mathcal{L}\{p(t)\} = \int_0^{\infty} [1(t) - 1(t - T_s)]e^{-s\tau} d\tau = \frac{1-e^{-sT_s}}{s}$$



Outline

Fourier transform

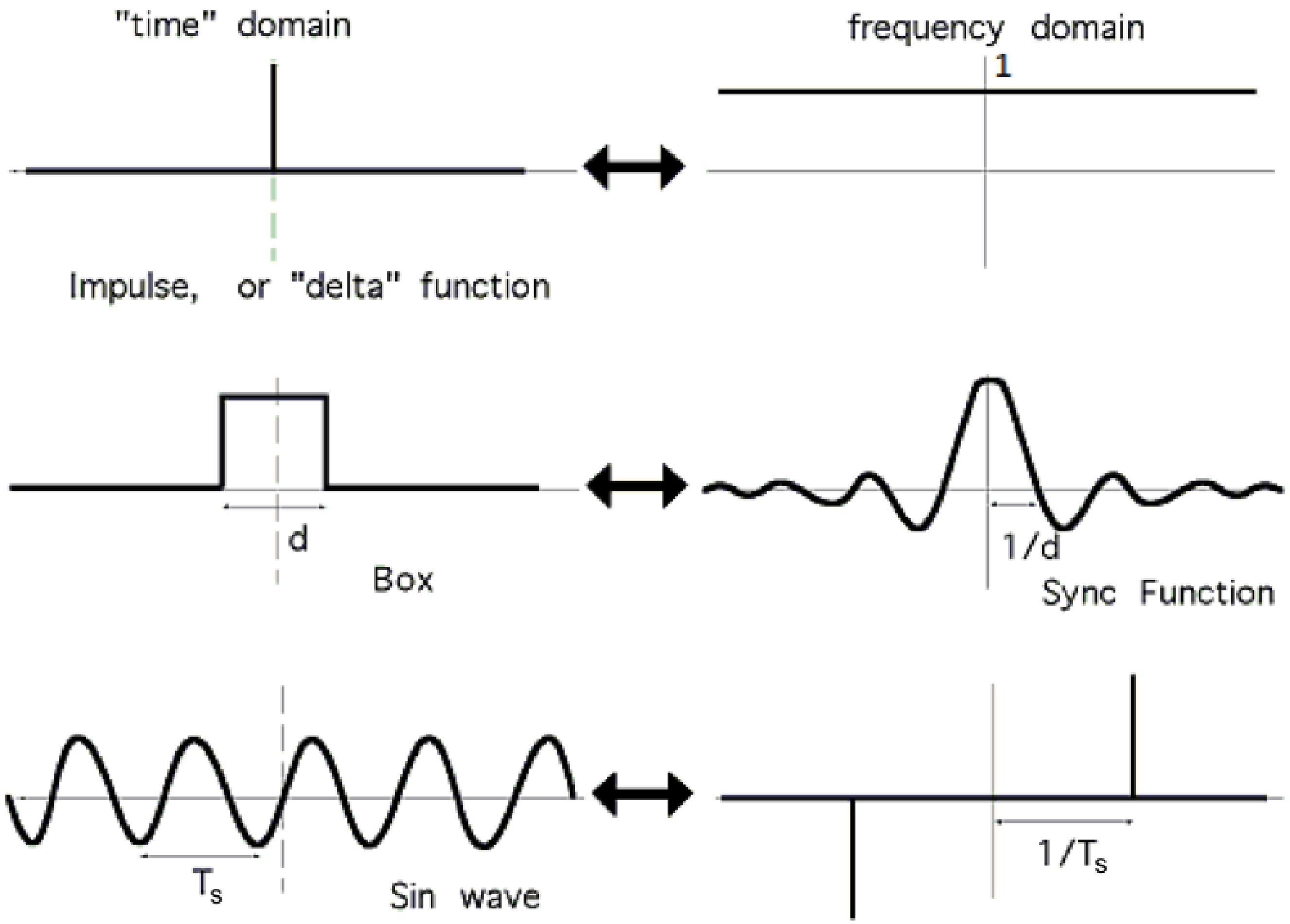
$$f(t) \longrightarrow F(j\omega)$$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) \longleftarrow F(j\omega)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega$$

Fourier transform



Fourier transform: properties

- **Linearity**

$$\begin{cases} f_1(t) \leftrightarrow F_1(j\omega) \\ f_2(t) \leftrightarrow F_2(j\omega) \end{cases} \Rightarrow \quad af_1(t) + bf_2(t) \leftrightarrow aF_1(j\omega) + bF_2(j\omega)$$

- **Time-scaling**

$$f(at) \leftrightarrow \left(\frac{1}{|a|}\right) F\left(\frac{j\omega}{a}\right)$$

- **Translation/Time-shifting**

$$f(t - t_0) \leftrightarrow e^{-j\omega t_0} F(j\omega)$$

- **Modulation/Frequency-shifting**

$$e^{j\omega_0 t} f(t) \leftrightarrow F(j(\omega - \omega_0))$$

Fourier transform: properties

- **Reciprocity**

  $$F(-jt) \leftrightarrow 2\pi f(\omega)$$

- **Derivative in t**

  $$\frac{df(t)}{dt} \leftrightarrow j\omega F(j\omega) \qquad \frac{d^n f(t)}{dt^n} \leftrightarrow (j\omega)^n F(j\omega)$$

- **Derivative in** ω

  $$(-jt)^n f(t) \leftrightarrow \frac{d^n F(j\omega)}{d\omega^n} \qquad \frac{f(t)}{-jt} \leftrightarrow \int_{-\infty}^{\infty} F(j\Omega) d\Omega \; if \; f(0) = 0$$

- **Convolution**

  $$y(t) = h(t) * u(t) \leftrightarrow Y(j\omega) = H(j\omega)U(j\omega)$$
  $$v(t) = h(t)u(t) \leftrightarrow V(j\omega) = \frac{1}{2\pi} H(j\omega) * U(j\omega)$$

Outline

Spectrum of a sampled signal

$r^*(t)$ is the product of $r(t)$ and a train of impulses. The latter series is periodic and can be represented by a Fourier series:

$$\sum_{k=-\infty}^{\infty} \delta(t - kT_s) = \sum_{n=-\infty}^{\infty} C_n e^{j(2\pi n/T_s)} t,$$

where the Fourier coefficients C_n are given by:

$$C_n = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \sum_{k=-\infty}^{\infty} \delta(t - kT_s) e^{-jn(2\pi t/T_s)} dt.$$

Spectrum of a sampled signal

The only term in the sum of impulses that is in the range of the integral is the $\delta(t)$ at the origin, so the integral reduces to:

$$C_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t) e^{-jn(2\pi t/T_s)} dt = \frac{1}{T_s}.$$

We derived the Fourier series of the sum of impulses:

$$\sum_{k=-\infty}^{\infty} \delta(t - kT_s) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{j(2\pi n/T_s)t}.$$

We define $\omega_s = \frac{2\pi}{T_s}$ as the sampling frequency (rad/s).

Spectrum of a sampled signal

We take the Laplace transform of the output of the sampler,

$$
\begin{aligned}
R^*(s) &= \int_{-\infty}^{\infty} r(t)\left\{\frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}\right\}e^{-st}\,dt \\
&= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} r(t)e^{jn\omega_s t}e^{-st}\,dt \\
&= \frac{1}{T_s}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} r(t)e^{-(s-jn\omega_s)t}\,dt.
\end{aligned}
\tag{1}
$$

Spectrum of a sampled signal

Definition

Since the integral is the Laplace transform of $r(t)$ with only a change of variable where the frequency goes, the result can be written as:

$$R^*(s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} R(s - jn\omega_s).$$

Outline

Aliasing

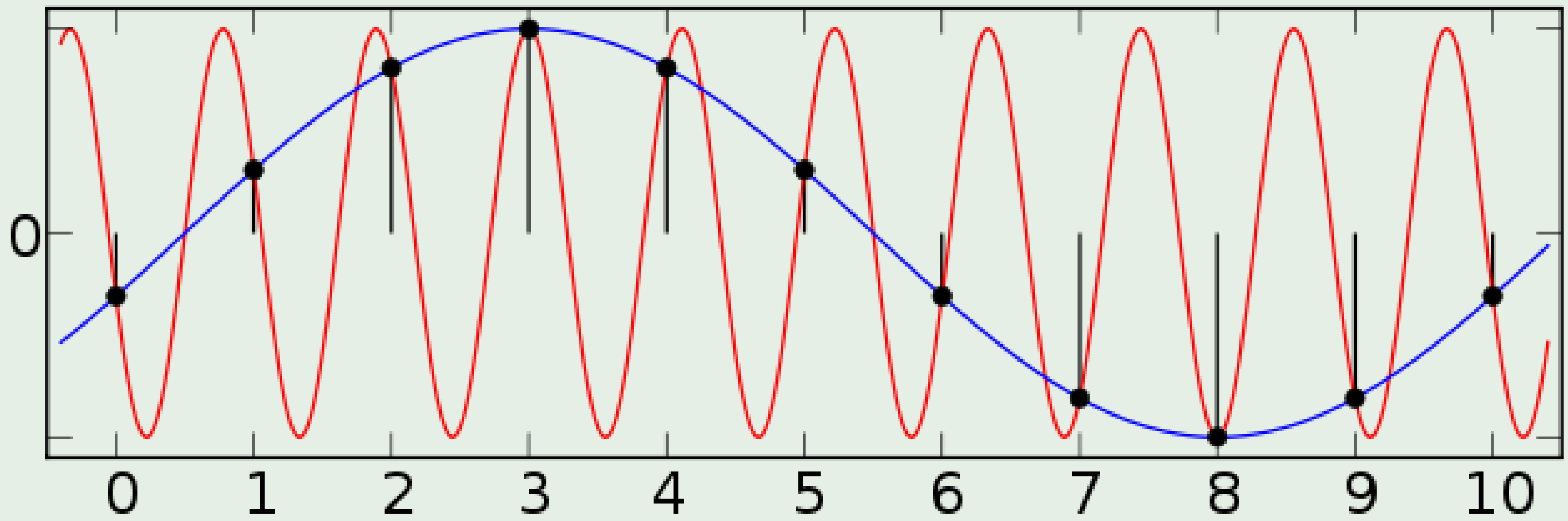
Definition

Aliasing is an effect that causes different signals to become indistinguishable when sampled. Frequencies that are too high to be sampled are folded onto lower frequencies. We cannot distinguish them based on their samples alone.

Aliasing

Example

The red sine wave is being sampled at just over its bandwidth, however the blue sine wave will be recreated as it also fits all data points and is within the expected bandwidth.



Aliasing

As a direct result of the sampling operation, when data is sampled at a frequency $\frac{2\pi}{T_s}$, the total harmonic content at a given frequency ω_1 is to be found not only from the original signal at ω_1, but also from all those frequencies that are aliases of ω_1, namely $\omega_1 + n2\pi/T_s = \omega_1 + n\omega_s$.

The errors caused by aliasing can be very severe if a substantial quantity of high-frequency components is contained in the signal to be sampled. To minimize this error, the sampling operation is preceded by a low-pass anti-aliasing filter that will remove all spectral content above the half-sampling frequency (π/T_s).