# Outline

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# Numerical Integration

## General approach

For a given continuous-time integrator

$$H(s) = \frac{U(s)}{E(s)} = \frac{1}{s} \quad \Leftrightarrow \quad \dot{u}(t) = e(t) \quad \Leftrightarrow \quad u(t) = \int_0^t e(\tau)\mathrm{d}\tau$$

its output at $t = kT_s$ can be written as follows:

$$u(kT_s) = \int_0^{(k-1)T_s} e(\tau)\mathrm{d}\tau + \int_{(k-1)T_s}^{kT_s} e(\tau)\mathrm{d}\tau$$

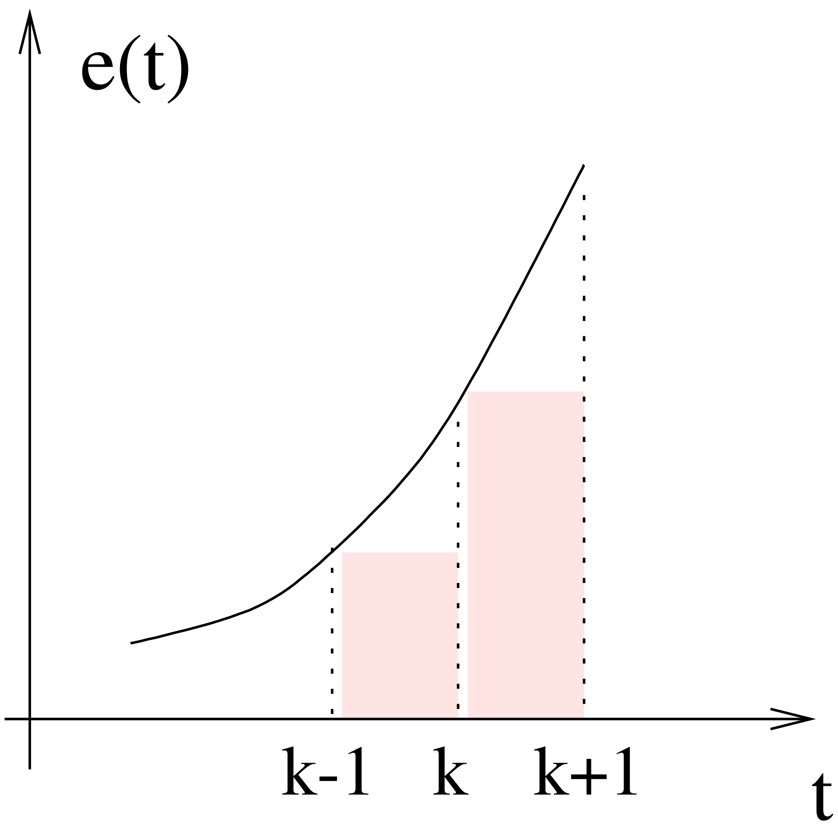
$$u(kT_s) = u((k-1)T_s) + \begin{cases} \text{area of } e(\tau) \\ \text{over } (k-1)T_s \le \tau < kT_s \end{cases} \tag{8.1}$$

where $T_s$ is the sampling time.

# Forward rectangular rule (=Forward Euler)

## General approach

The area is approximated by the rectangle looking **forward** from $(k-1)$ toward $k$ with an amplitude equal to the value of the function at $(k-1)$.



# Forward rectangular rule

## Mathematical approach

From equation (8.1) we have that

$$u(kT_s) = u\big((k-1)T_s\big) + T_s e\big((k-1)T_s\big)$$

By taking the $\mathcal{Z}$-transform we obtain the discrete equivalent of H(s),

$$U(z) = z^{-1}U(z) + T_s z^{-1}E(z) \quad \Leftrightarrow$$
$$(1-z^{-1})U(z) = T_s z^{-1}E(z) \quad \Leftrightarrow \quad \tfrac{z-1}{T_s}U(z) = E(z)$$

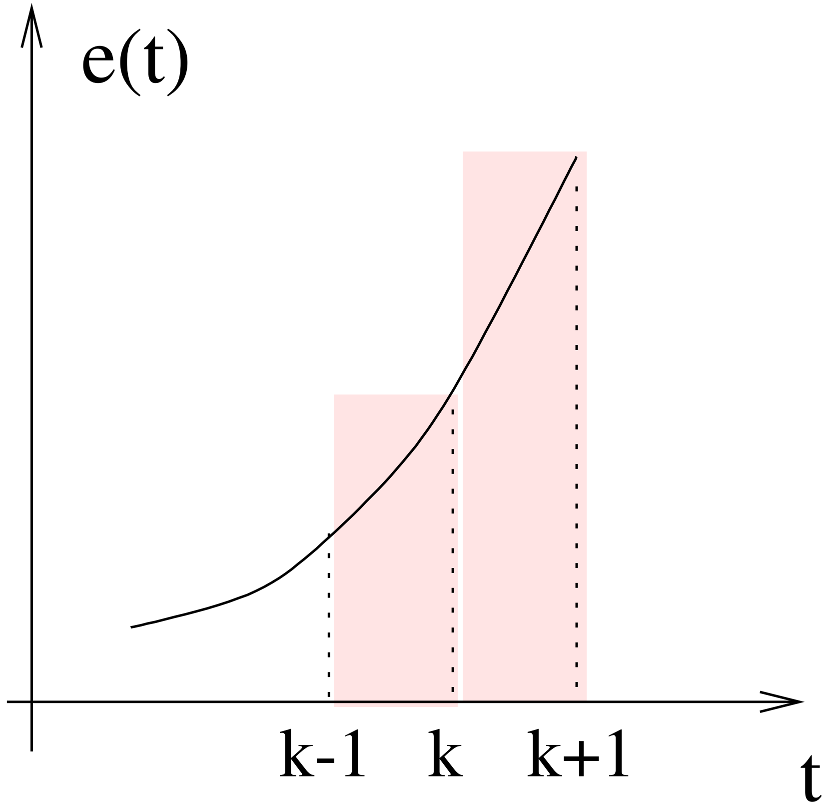
This means that we need to apply the following substitution in order to discretize a given continuous-time transfer function:

$$s \leftarrow \tfrac{z-1}{T_s} \ (8.2)$$

# Backward rectangular rule (=Backward Euler)

**General approach**

The area is approximated by the rectangle looking **backward** from $k$ toward $(k-1)$ with an amplitude equal to the value of the function at $k$.



# Backward rectangular rule

## Mathematical approach

From equation (8.1) we have that

$$u(kT_s) = u\big((k-1)T_s\big) + T_s e\big(kT_s\big)$$

By taking the $\mathcal{Z}$-transform we obtain the discrete equivalent of H(s),

$$U(z) = z^{-1}U(z) + T_s E(z) \quad \Leftrightarrow$$
$$(1 - z^{-1})U(z) = T_s E(z) \quad \Leftrightarrow \quad \tfrac{z-1}{zT_s}U(z) = E(z)$$

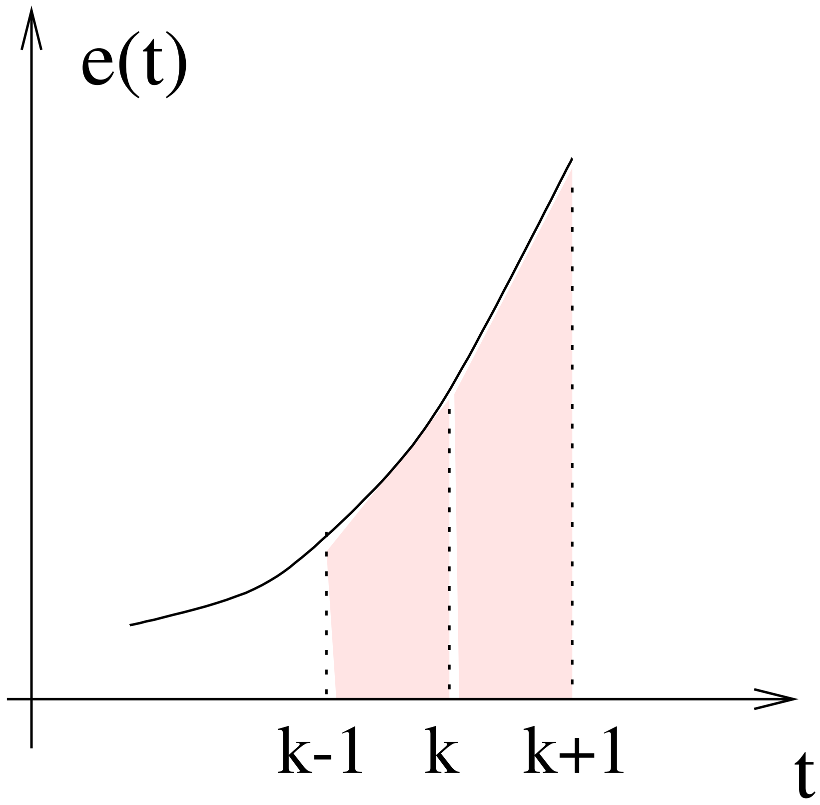
This means that we need to apply the following substitution in order to discretize a given continuous-time transfer function:

$$s \leftarrow \tfrac{z-1}{zT_s} \ (8.3)$$

# Bilinear rule (= trapezoidal or Tustin rule)

## General approach

This method makes use of the area of the **trapezoid** formed by the average of the selected rectangles used in the forward and backward rectangular rule.Thus the amplitude equal to the value of the function at $(k-1)$ and the amplitude equal to the value of the function at $(k)$ are connected by a line as shown in the illustration.



# Bilinear transformation

## Mathematical approach

From equation (8.1) we have that

$$u(kT_s) = u\big((k-1)T_s\big) + T_s \frac{e(kT_s)+e\big((k-1)T_s\big)}{2}$$

By taking the $\mathcal{Z}$-transform we obtain the discrete equivalent of H(s),

$$U(z) = z^{-1}U(z) + \tfrac{T_s}{2}\left(E(z) + z^{-1}E(z)\right) \quad \Leftrightarrow$$

$$(1 - z^{-1})U(z) = \tfrac{T_s}{2}(1 + z^{-1})E(z) \quad \Leftrightarrow \quad \tfrac{2}{T_s}\tfrac{z-1}{z+1}U(z) = E(z)$$

This means that we need to apply the following substitution in order to discretize a given continuous-time transfer function:

$$s \leftarrow \tfrac{2}{T_s}\tfrac{z-1}{z+1} \ (8.4)$$

# Bilinear rule

## Example

Given:

$$H(s) = \frac{s+1}{0.1s+1}$$

We now apply substitution (8.4):

$$H(z) = \frac{(2+T)(T-2)z^{-1}}{(0.2+T)+(T-0.2)z^{-1}}$$

Using T=0.25s, this results in:

$$H(z) = \frac{5(z-0.7778)}{z+0.1111}$$

# Discretization of state-space models

## General approach applied on Forward Euler

Given the following continuous-time model in state-space form:

$$\dot{x} = Ax + Bu \qquad sX = AX + BU$$
$$y = Cx + Du \qquad Y = CX + DU$$

If we use the Forward Euler method, we have that $s$ is replaced by $\frac{z-1}{T_s}$, so we can find the discrete-time equivalent as follows:

$$\frac{z-1}{T_s}X = AX + BU$$
$$Y = CX + DU$$

# Discretization of state-space models

## General approach applied on Forward Euler

Which leads to the following

$$zX = (I - AT_s)X + BT_sU$$
$$x(k+1) = (I - AT_s)x(k) + (BT_s)u(k)$$
$$x(k+1) = A_dx(k) + B_du(k)$$

The output equation $Y = CX + DU$ remains.

# Discretization of state-space models

## State-space models

A similar calculation can be done for the backward rectangular rule and the bilinear transformation resulting in the following table:

| | Euler | backward rect. | bilinear transf. |
|---|---|---|---|
| $A_d$ | $I + AT_s$ | $(I - AT_s)^{-1}$ | $(I - \frac{AT_s}{2})^{-1}(I + \frac{AT_s}{2})$ |
| $B_d$ | $BT_s$ | $(I - AT_s)^{-1}BT_s$ | $(I - \frac{AT_s}{2})^{-1}BT_s$ |
| $C_d$ | $C$ | $C(I - AT_s)^{-1}$ | $C(I - \frac{AT_s}{2})^{-1}$ |
| $D_d$ | $D$ | $D + C(I - AT_s)^{-1}BT_s$ | $D + C(I - \frac{AT_s}{2})^{-1}\frac{BT_s}{2}$ |

# Stability of the numerical integration methods

## Stability

As already mentioned, a discrete system is stable when its poles lie within the unit circle of the z-plane and a continuous system is stable when its poles have a negative real part in the s-plane. Subsequently the $(s = j\omega)$-axis is the boundary between poles of stable and unstable continuous systems.

Each of the discretization methods can be considered as a map from the s-plane to the z-plane. It is interesting to know how the $j\omega$-axis is mapped by every rule and where the stable part of the s-plane appears in the z-plane. This can be realized by solving formulas (8.2-8.3) to z and replacing s by $j\omega$.

# Stability of the numerical integration methods

## Boudaries of the stable regions

Expressions of z in terms of s:

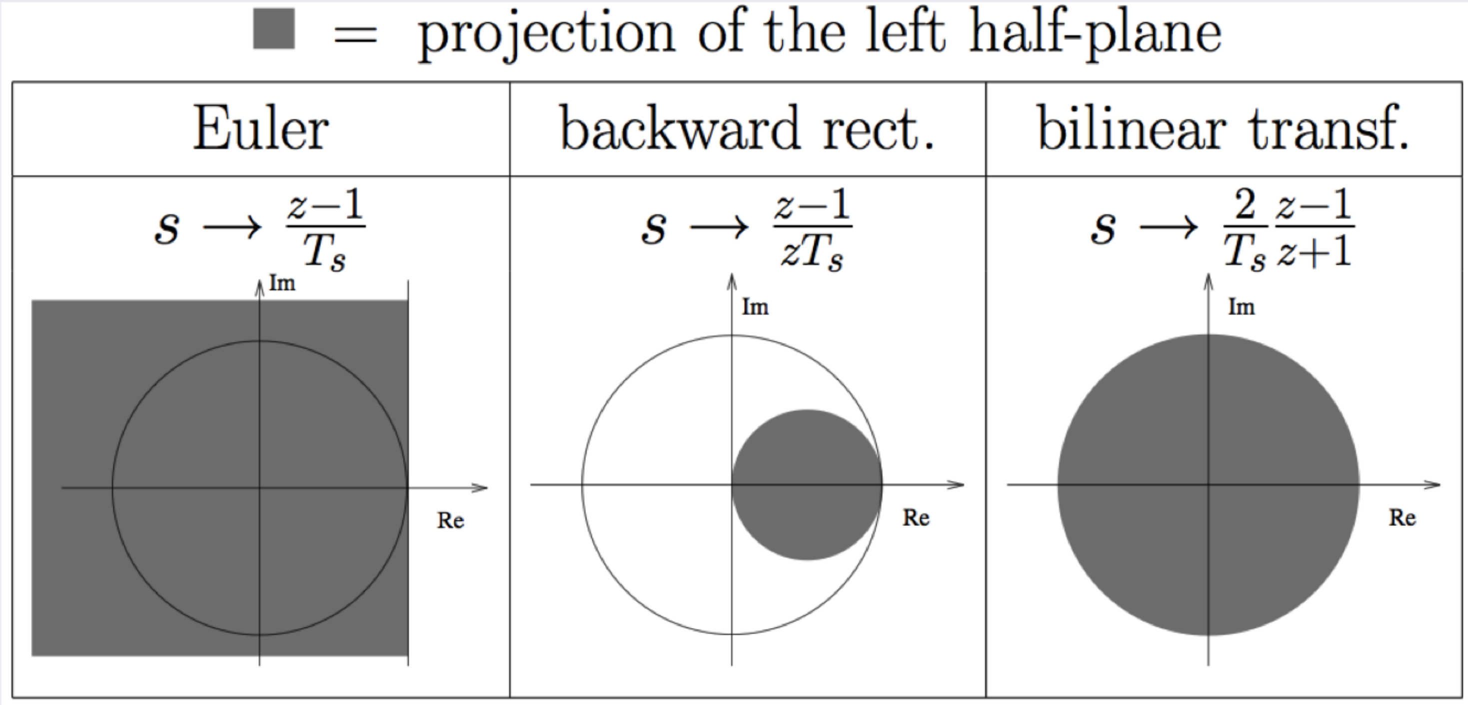
- $z = 1 + T_s s$
- $z = \frac{1}{1 - T_s s}$
- $z = \frac{1 + T_s \frac{s}{2}}{1 - T_s \frac{s}{2}}$

By substituting s by $j\omega$ the boundaries of the regions in the z-plane, which originate from the stable portion of the s-plane, are obtained.

# Stability of the numerical integration methods

## Graphical representation

Stable s-plane poles map onto the shaded regions in the z-plane. The unit circle is shown for reference.



# Stability of the numerical integration methods

## Mapping of the left-hand-s-plane

- Forward Euler: a stable continuous-time system may become unstable after discretization;
- Backward Euler: a stable continuous-time system will stay stable after discretization, but the number of degrees of freedom is restricted;
- Bilinear transformation: the entire left-hand-plane is mapped into the unit circle.

# Bilinear rule with prewarping

## Distortion

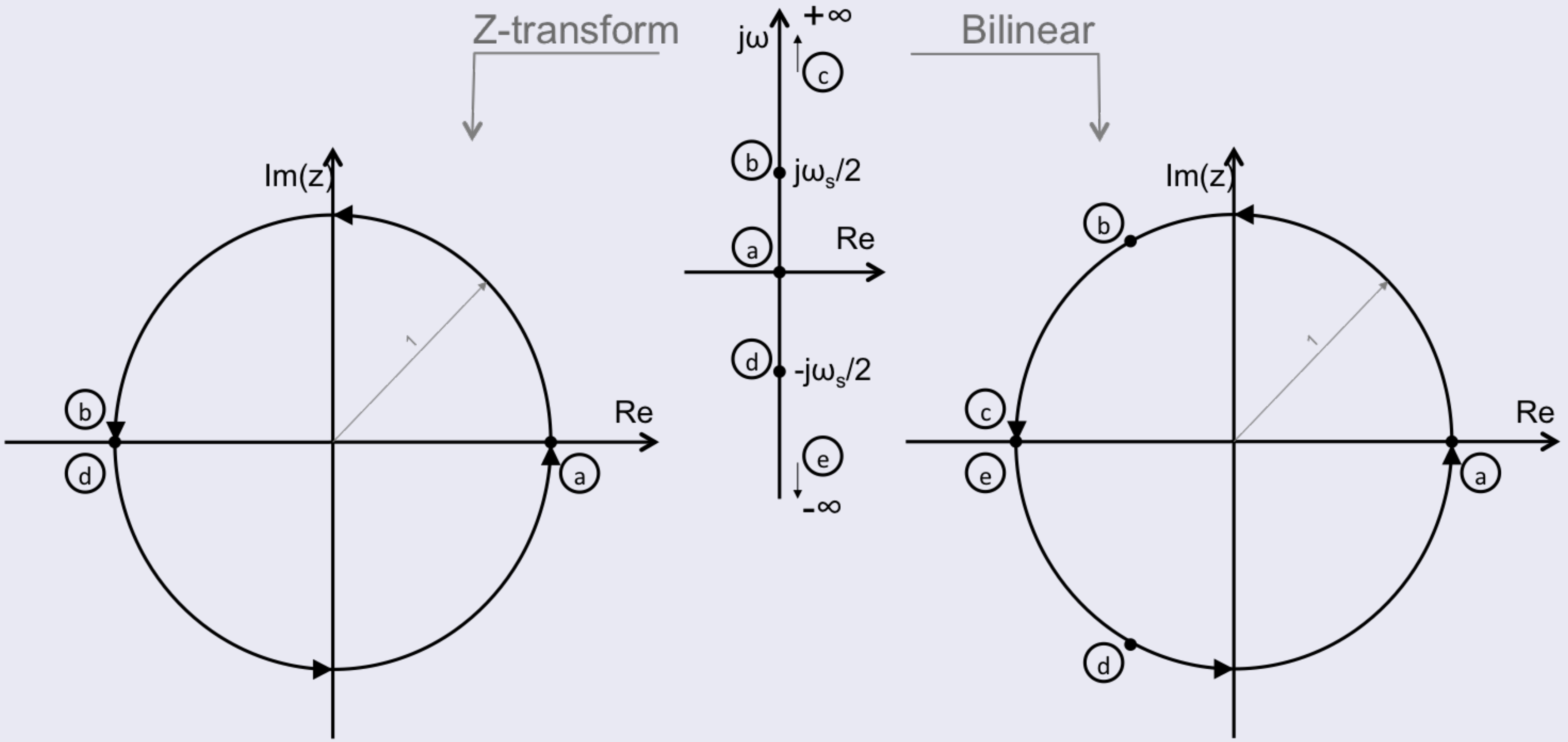
Looking at the graphical representation of the stability on the previous slide, you can see that the bilinear rule maps the stable region of the s-plane into the stable region of the z-plane. The entire $j\omega$-axis is compressed into the $2\pi$-length of the unit circle, causing a frequency distortion.

## Origin of prewarping

On the next slide you can see that when we compare the $\mathcal{Z}$-transform with the bilinear transformation, points $b$ and $d$ in the s-plane are mapped onto different points in the z-plane. $(-1+0j)$ can only be reached by the bilinear rule when $\omega$ goes to infinity.

# Bilinear rule with prewarping

## Origin of prewarping



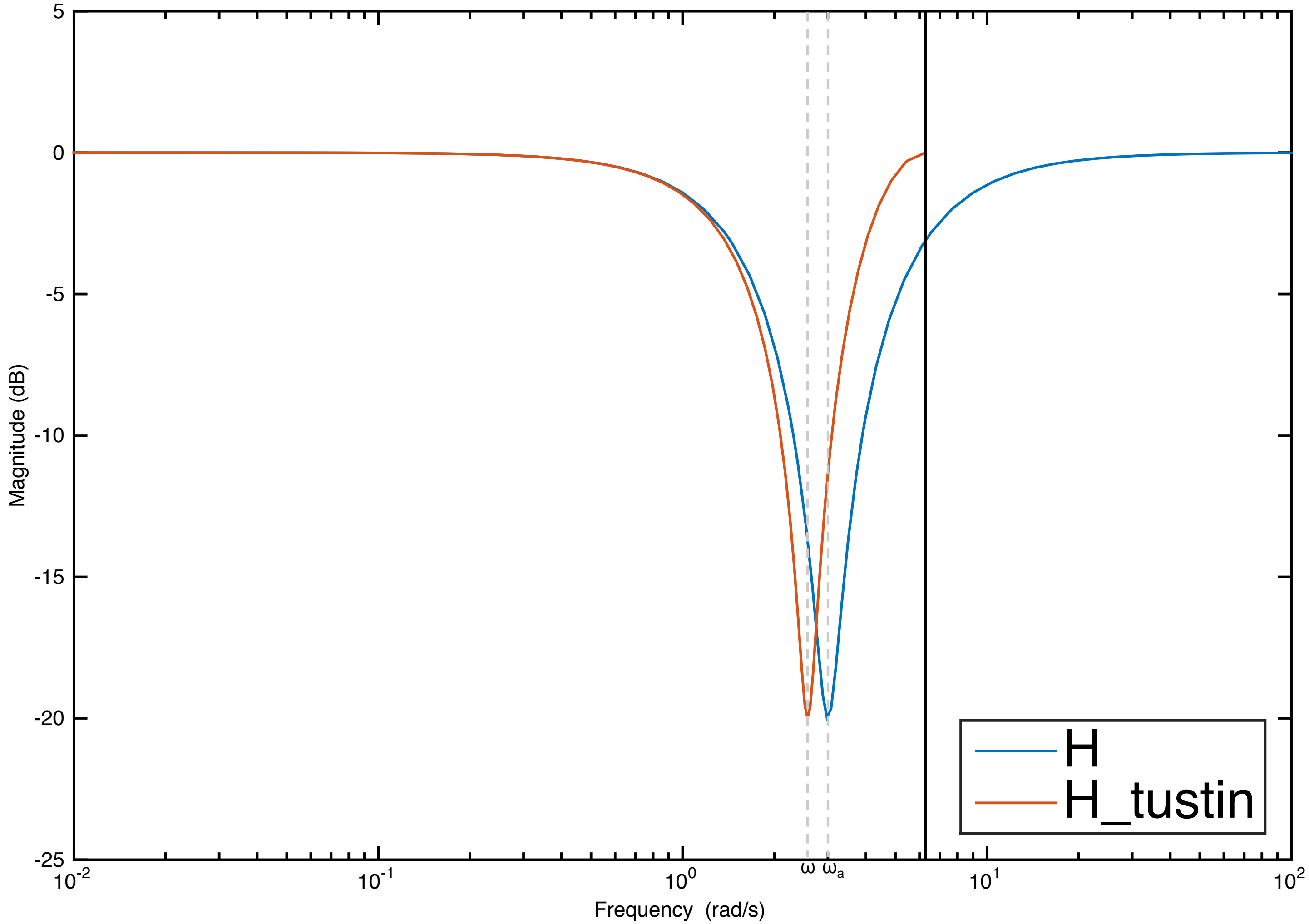
# Bilinear rule with prewarping

## Distortion: frequency domain

The bilinear rule causes every feature that is visible in the frequency response of the continuous-time filter to also be visible in the discrete-time filter, but at a different frequency. However, at low frequencies, the same behaviour of the function is guaranteed. This is illustrated by the figure on the next slide.

When the actual frequency of $\omega$ is input to the discrete-time filter designed by use of the bilinear transform, it is desired to know at what frequency, $\omega_a$, for the continuous-time filter that this $\omega$ is mapped to.

# Bilinear rule with prewarping



# Bilinear rule with prewarping

## $\omega_a$ with input $\omega$

We now look for a relation between $\omega_a$ and $\omega$, using the 's-domain'-'z-domain' relation $z = e^{sT}$ and the substitution $s = j\omega$:

$$\begin{aligned}
\left(\frac{2}{T_s}\frac{z-1}{z+1}\right)_{z=e^{j\omega T_s}} &= \frac{2}{T_s}\frac{e^{j\omega T_s}-1}{e^{j\omega T_s}+1} \\
&= \frac{2}{T_s}\frac{e^{j\omega T_s/2}(e^{j\omega T_s/2}-e^{-j\omega T_s/2})}{e^{j\omega T_s/2}(e^{j\omega T_s/2}+e^{-j\omega T_s/2})} \\
&= j\frac{2}{T_s}\frac{sin(\omega T_s/2)}{cos(\omega T_s/2)} \\
&= j\frac{2}{T_s}tan(\frac{\omega T_s}{2}) = j\omega_a
\end{aligned}$$

# Bilinear rule with prewarping

## Frequency warping

The discrete-time system has the same behavior at frequency $\omega$ as the continuous-time system at frequency $\omega_a = \frac{2}{T_s} tan(\frac{\omega T_s}{2})$.

Specifically, the gain and phase shift that the discrete-time filter has at frequency $\omega$ is the same gain and phase shift that the continuous-time filter has at frequency $\omega_a = \frac{2}{T_s} tan(\frac{\omega T_s}{2})$.

This effect of the non-linear relation between $\omega$ and $\omega_a$ is called **frequency warping**.

# Bilinear rule with prewarping

## Frequency prewarping

In certain situations, however, we really want the characteristics to be conserved during the discretization.

By setting $\omega_a = \frac{2}{T} tan(\frac{\omega T}{2})$ for every frequency specification that the designer has control over, the frequency warping will be compensated. This is called **frequency prewarping**.

The digital filter can be made to match the frequency response of the continuous filter at frequency $\omega_0$ if the following transformation is substituted into the continuous filter transfer function:

$$s \leftarrow \frac{\omega_0}{tan\left(\frac{\omega_0 T_s}{2}\right)} \frac{z-1}{z+1} \quad (8.4)$$

# Bilinear rule with prewarping