Outline

Margins

The Nyquist plot allows us to determine the stability of the system
We know the stability changes when $1 + P(s)C(s)$ has an imaginary root (then the system is marginally stable).

We can see such a root in the Nyquist plot of $P(s)C(s)$.
After all, the Nyquist plot is the image of the imaginary axis, so if there is a root on the imaginary axis, the Nyquist plot of $1 + P(s)C(s)$ would go through 0 and the Nyquist plot of $P(s)C(s)$ would go through -1.

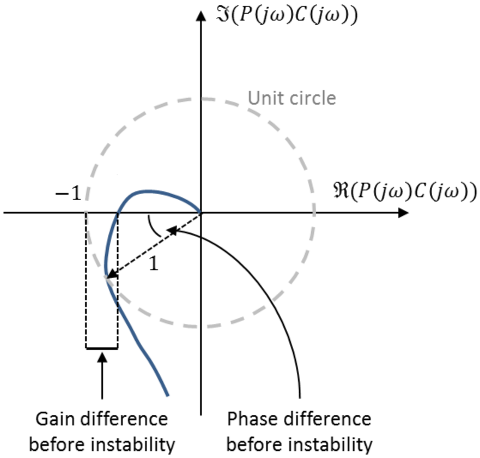
Margins

This explains why we can use the 'distance to -1' as a measure of stability.

We will see two different stability margins, which can be easily read off the Nyquist diagram:

- **The gain margin**: the amount of extra gain you can allow before instability occurs (in dB)
- **The phase margin**: the amount of phase you can add before instability occurs

Margins

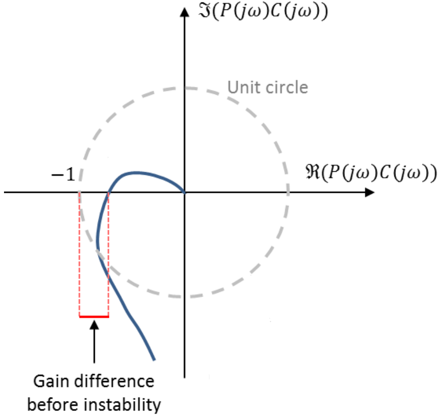

$\Im(P(j\omega)C(j\omega))$
Unit circle
-1
$\Re(P(j\omega)C(j\omega))$
1
Gain difference before instability
Phase difference before instability

Outline

Gain margin

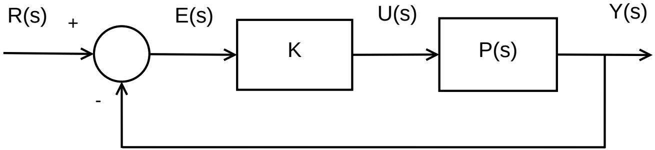
The gain margin is the amount of extra gain allowed before the system becomes unstable (or how much larger the gain has to be, before the system becomes unstable).

The gain margin is **multiplicative**, so it is the factor you have to multiply the gain with, so the Nyquist plot goes through -1 in the w-plane.
It will be expressed in dB ($20log_{10}K$).



Gain margin

Let's look at it the following way:



The stability margin of $P(s)$ with unity feedback is the K for which the system above is marginally stable.

Gain margin

So $KP(s)$ should equal -1 for an imaginary $s = j\omega$.

This requires $\angle(KP(j\omega)) = \angle P(j\omega)$ to equal $-180°$. This ω_π is called the **Gain Crossover Frequency (GCF)**.
K then has to be set such that $\left|KP(j\omega_\pi)\right| = 1$.

So a large gain can lead to instability and this risk only exists when there exists a ω_π for which $\angle P(j\omega) = -180°$.

We will illustrate this with an example.

Gain margin: example

Consider the process

$$P(s) = \frac{1}{s(s+2)}.$$

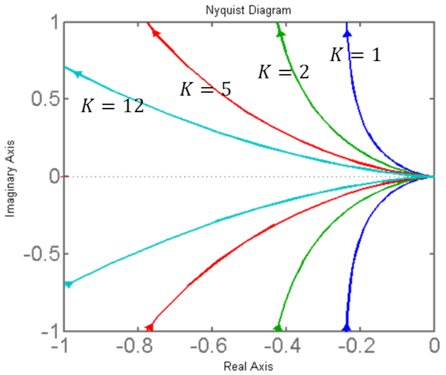
We derive the argument as a function of ω:

$$\angle P(j\omega) = -tan^{-1}(\frac{\omega}{0}) - tan^{-1}(\frac{\omega}{2}) = -90^\circ - tan^{-1}(\frac{\omega}{2}).$$

For this to equal to -180°, it requires $\omega_\pi \to \infty$.
This makes $P(j\omega) = 0$ and this means the gain margin is infinite.

Gain margin: example

This solution is shown in the Nyquist plot below:

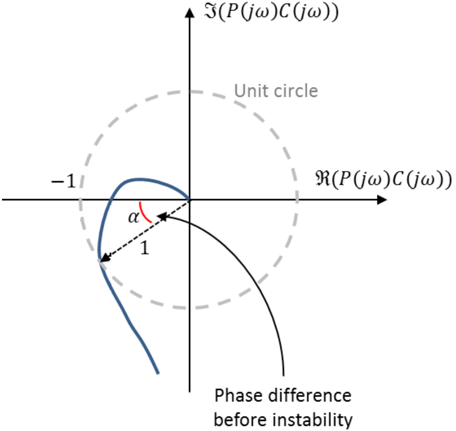


The only crossover with the real axis occurs at $\omega = 0$ and that does not change with increasing K.

Outline

Phase margin

The phase margin is that amount of additional phase lag to bring the system to the verge of instability.



Phase margin

This can be interpreted as multiplying $P(s)$ with $e^{j\theta}$ until -1 is crossed, so $P(s)e^{j\theta} = -1$ for an imaginary $s = j\omega$.

This requires $\left|P(j\omega)e^{j\theta}\right| = \left|P(j\omega)\right|$ to equal 1. This ω_0 is called the **Phase Crossover Frequency (PCF)**.
θ then has to be set such that

$$\angle(P(s)e^{j\theta}) = \angle P(s) + \theta = -180^\circ.$$

The gain margin is defined as positive, but that doesn't matter, because of the symmetry with respect to the real axis.
If a rotation of θ degrees results in a crossing of -1, then a rotation of $-\theta$ does the same.

Phase margin: example

Consider the process

$$P(s) = \frac{1}{s(s+2)}.$$

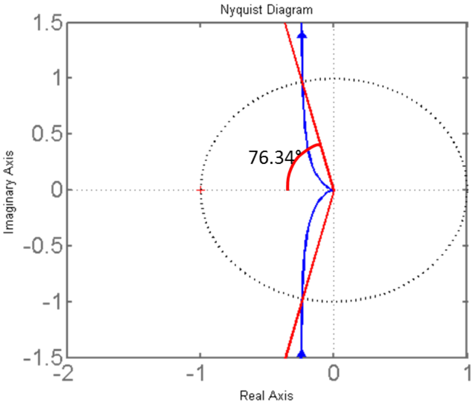
We derive the modulus as a function of ω:

$$\left|P(j\omega)\right| = \frac{1}{\omega}\frac{1}{\sqrt{\omega^2+4}}.$$

For this to equal to 1, we find $\sqrt{\omega^4+4\omega^2} = 1$ or $\omega_0 = \sqrt{\sqrt{5}-2} = 0.486$.
Now we need to find θ so $\angle(P(j\omega_0)e^{j\theta}) = -180^\circ$.
This results in $\theta = -180^\circ + tan^{-1}(\frac{\omega_0}{0}) + tan^{-1}(\frac{\omega_0}{2}) = -76.34^\circ$.
This means the phase margin is 76.34°.

Phase margin: example

This solution is shown graphically in the Nyquist plot below:



What should the margins be?

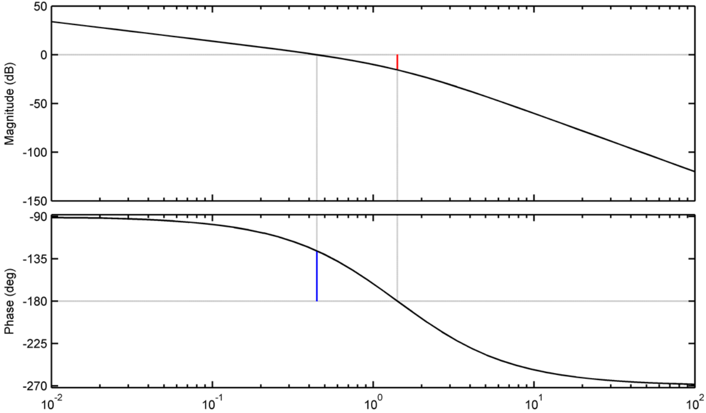
Phase margin:

- This is more subtle than the gain margin
- If it is too small, instability might occur due to practicalities, the model is not perfect
- If it is too small, we get large overshoots and large oscillations that fade away very slowly
- Sometimes a good value is 60°, but it is highly case-dependent

A good margin does not offer certainty about the stability, whereas a bad phase margin (very large or very small) does give certainty about instability.

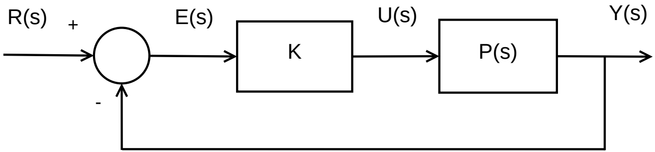
Margins using Bode plots

We can also easily derive the gain margin and phase margin from the Bode plot of P(s)C(s):



Outline

Margins: design example



Consider the following process

$$P(s) = \frac{1 - s}{(1 + s)^2}.$$

Design a simple proportional controller K with a phase margin of 45°.

Margins: design example

We find ω by demanding

$$\begin{aligned}
\angle(P(j\omega)e^{\frac{j\pi}{4}}) &= -180^\circ \\
45^\circ + tan^{-1}(-\omega) - tan^{-1}(\omega) - tan^{-1}(\omega) &= -180^\circ \\
-3tan^{-1}(\omega) &= -225^\circ \\
\omega &= tan(75^\circ) = 3.73
\end{aligned}$$

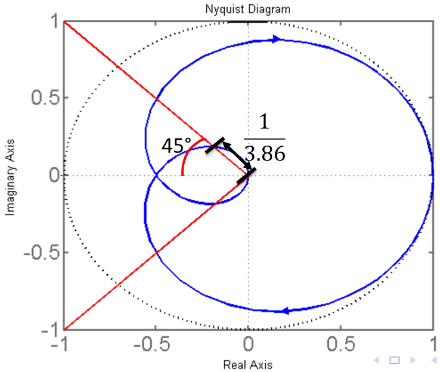
Now we can find K (which doesn't influence the argument) by setting $\left|KP(j\omega)\right| = 1$:

$$K\frac{\sqrt{\omega^2+1}}{\sqrt{\omega^2+1}^2} = \frac{K}{\sqrt{\omega^2+1}} = \frac{K}{3.86} = 1 \rightarrow K = 3.86 = 5.87dB$$

Margins: design example

We can also do this graphically with the Nyuist plot.

First determine the point that corresponds to $\theta = 45^\circ$.
Then determine the modulus, K is the inverse of that modulus.



Summary

- The Nyquist stability criterion came to existence as a cheap alternative to determine stability of a closed loop system with unity feedback
- It also shows the phase margin and the gain margin, which are used to measure the stability of the system
- It is relevant as a design tool, as shown in the last example