Chapter 4: Discrete-time Systems

August 18, 2015

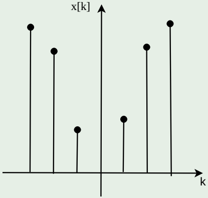
Outline

Discrete Time Signal

Definition

Discrete-time signals are defined only at discrete instants of time.
The value (amplitude) can be either continuous or discrete.
$x[k]$ is the value of a signal at the moment $t = kT$

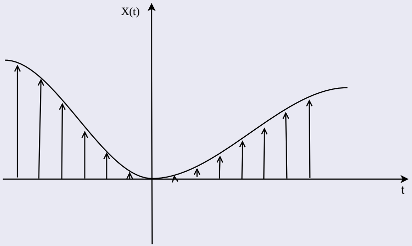
Example



Discrete Time System

Sampling

The sampling of a continuous time signal replaces the original continuous signal by a sequence of values at discrete time points.

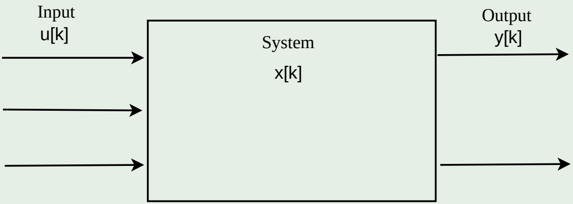


Discrete Time System

Definition

A linear time-invariant (LTI) discrete time system processes an input vector $u[k]$ to an output vector $y[k]$.

Example



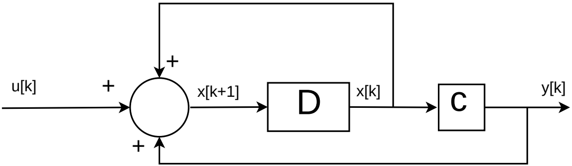
Discrete Time System

How to represent a discrete time system?

- Block-diagram
- State space representation
- Difference equation
- Impulse response
- Transfer function

Outline

Block-diagram

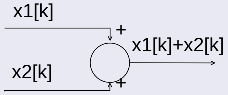


Definition

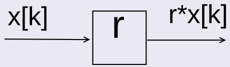
A block-diagram is a visual representation of a system. All LTI's (Linear Time Invariant) systems can be constructed using 3 building blocks (memory element, summation element and multiplication element). Note that every memory element corresponds to one state variable.

Building blocks

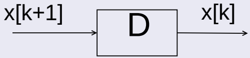
Adder



Constant Multiplier

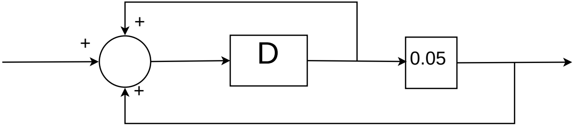


Delay element



Example: compound interest

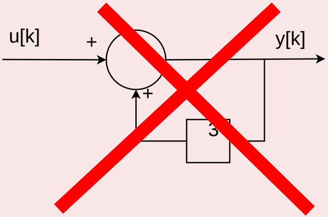
- $u[k]$:The deposits and withdrawals from the bank account
- $x[k]$:The current balance of the bank account(before deposit and interest)
- $y[k]$: The acquired interest of that year
- $x[k+1]$: The 'next year' balance of the bank account $=$ current balance $+$ interest $+$ deposits - withdrawals



Bad block-diagrams

Delay-free loops

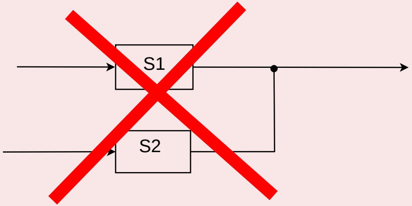
The issue is that this leads to an implicit connection. $y[k]$ depends on $y[k]$,which is not yet known. You can easily rewrite this in an allowed shape $y[k] = u[k] + 3y[k] \iff y[k] = -\frac{1}{2}u[k]$



Bad block diagrams

Connecting two outputs without using a sum

The issue is that this can lead to inconsistencies. According to this block diagram the output of the systems S1 and S2 are equal.



Outline

State space representation

Definition

State space representation of a LTI system:

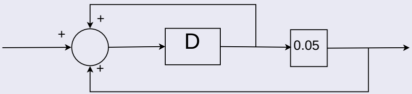
$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k] \end{aligned}$$

State space representation of a LTV system:

$$\begin{aligned} x[k+1] &= A[k]x[k] + B[k]u[k] \\ y[k] &= C[k]x[k] + D[k]u[k] \end{aligned}$$

From block-diagram to state space representation

Block-diagram



State space representation

1. Let the output of the memory elements be $x_i[k]$.
2. So the input of the memory elements are $x_i[k+1]$.
3. Trace back to retrieve equations for $x_i[k+1]$ and $y_i[k]$.

This results in:

$$x[k+1] = 1.05x[k] + u[k]$$
$$y[k] = 0.05x[k]$$

From state space representation to block-diagram

Example

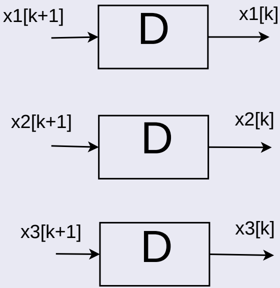
$$\begin{bmatrix} x_1[k+1] \\ x_2[k+1] \\ x_3[k+1] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1[k+1] \\ x_2[k+1] \\ x_3[k+1] \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} u[k]$$

$$y[k] = \begin{bmatrix} 5 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \end{bmatrix} u[k]$$

From state space space representation to block-diagram

Step 1

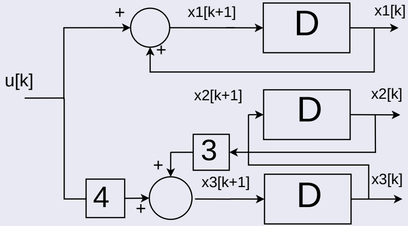
First add a delay element for every state $x_i[k]$.



From state space space representation to block-diagram

Step 2

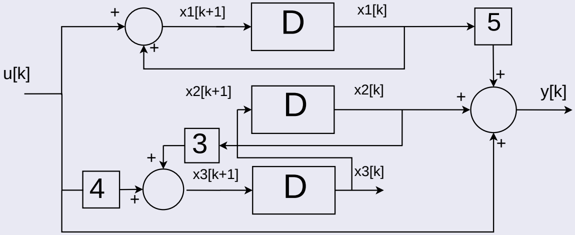
Determine the input for every state $x[k+1]$ from the matrices A and B, as a combination of the states $x_i[k]$ and inputs $u[k]$.



From state space space representation to block-diagram

Step 3

Determine the outputs y[k] in the same way with the matrices C and D.



Different state space representations

State space representation is not unique!

Take the following system, which connects u[k] to y[k]:

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

Now take a non-singular square matrix T and the following system.

$$Tx[k+1] = TAT^{-1}Tx[k] + TBu[k]$$
$$y[k] = CT^{-1}Tx[k] + Du[k]$$

The relation between u[k] and y[k] will be the same. With $x' = Tx, A' = TAT^{-1}, B' = TB, C' = CT^{-1}$ and $D' = D$, we have found a different state space representation for the original system.

Solving state space equation

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

We express $x[1], x[2], \ldots$ in function of $x[0]$ and $u[k]$:

$$\begin{aligned} x[1] &= Ax[0] + Bu[0] \\ x[2] &= Ax[1] + Bu[1] = A^2x[0] + ABu[0] + Bu[1] \\ &\vdots \\ x[k] &= A^k x[0] + \sum_{i=0}^{k-1} A^{k-1-i} Bu[i] \end{aligned}$$

Solving state space equation

The output is $y[k]$:

$$y[k] = \begin{cases} Cx[0] + Du[0] & \text{if } k = 0 \\ CA^k x[0] + \sum_{i=0}^{k-1} CA^{k-1-i} Bu[i] + Du[k] & \text{if } k > 0 \end{cases}$$

Outline

Difference equations

Definition

Similar to differential equations, but for discrete time.

General form: $\sum_{i=0}^{n} a_i y[k+i] = \sum_{i=0}^{n} b_i u[k+i]$

With n the order of the system.

Solution in 2 parts

1. Homogeneous: solution with no input
2. Particular: solution derived as a response from the input

Homogenous difference equations

Definition

General form: $\sum_{i=0}^{n} a_i y[k+i] = 0$

Example

$$y[k+1] - ay[k] = 0$$
$$y[k+1] = ay[k]$$
$$\text{So: } y[1] = ay[0]$$
$$y[2] = ay[1] = a^2 y[0]$$
$$\vdots$$
$$y[n] = a^n y[0]$$

Homogeneous difference equations

Solution

- Expected form of solution: r^k
- Substitution of the expected solution in the difference equation: $\sum_{i=0}^{n} a_i r^{k+i} = 0$
- Division by r^k leads to the characteristic equation: $\sum_{i=0}^{n} a_i r^i = 0$
- Solutions of the characteristic equation: $r_1, r_2, r_3, \ldots$
- Homogeneous solution to the difference equation: $y[k] = c_1 r_1^k + c_2 r_2^k + c_3 r_3^k + \cdots = \sum_{i=1}^{n} c_i r_i^k$

Evariste Galois

Figure: Evariste Galois (25/10/1811 - 31/5/1832) was a French mathematician born in Bourg-la-Reine. While still in his teens, he was able to determine a necessary and sufficient condition for a polynomial to be solvable by radicals.

Galois Theory

Galois theory provides a answer to the question why there is no formula for the roots of a fifth (or higher) degree polynomial equation in terms of the coefficients of the polynomial, using only the usual algebraic operations (addition, subtraction, multiplication, division) and application of radicals (square roots, cube roots, etc). It also explains why it is possible to solve equations of degree four or lower in the above manner, and why their solutions take the form that they do.

Example

- Homogeneous recurrence relations: $y[k+2] - 5y[k+1] + 6y[k] = 0$
- Initial value: $y[0] = 1$, $y[1] = 1$
- Characteristic polynomial: $r^2 - 5r + 6 = 0$
- Roots: 2 and 3
- General solution: $c_1 2^k + c_2 3^k$
- Using the initial values:

$$\begin{Bmatrix} 1 = c_1 + c_2 \\ 1 = 2c_1 + 3c_2 \end{Bmatrix}$$

$$\begin{Bmatrix} 2 = c_1 \\ -1 = c_2 \end{Bmatrix}$$

- Result: $y[k] = 2^{k+1} - 3^k$

Example: Fibonacci sequence

Figure: Leonardo Bonacci (c. 1170 - c. 1250) known as Fibonacci was an Italian mathematician, considered to be "the most talented Western mathematician of the Middle Ages".

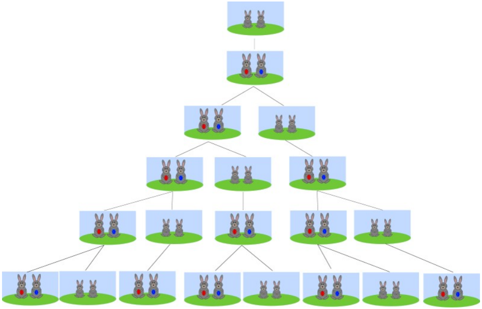


Figure: Fibonacci sequence

Example: Fibonacci sequence

Example

- Homogeneous recurrence relations: $y[k+2] = y[k+1] + y[k]$
- Initial value: $y[0] = 1, y[1] = 1$
- Characteristic polynomial: $r^2 - r - 1 = 0$
- Roots: $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$
- General solution: $y[k] = c_1(\frac{1+\sqrt{5}}{2}) + c_2(\frac{1-\sqrt{5}}{2})$
- Initial values:

$$\begin{Bmatrix} c_1 + c_2 = 1 \\ c_1\frac{1+\sqrt{5}}{2} + c_2\frac{1-\sqrt{5}}{2} = 1 \end{Bmatrix}$$

$$\begin{Bmatrix} c_1 = \frac{5+\sqrt{5}}{10} \\ c_2 = \frac{5-\sqrt{5}}{10} \end{Bmatrix}$$

- Result: $y[k] = (\frac{5+\sqrt{5}}{10})(\frac{1+\sqrt{5}}{2})^k + (\frac{5-\sqrt{5}}{10})(\frac{1-\sqrt{5}}{2})^k$

Multiple roots and complex roots

Multiple roots

For a multiple root r_i with multiplicity m add $r_i^k, kr_i^k, \ldots, k^{m-1}r_i^k$

Complex roots

Complex roots will result in oscillating behaviour. If the difference equations and initial conditions are both real the complex roots can only be present in conjugate pairs, the constants will also be in conjugate pairs.

$$r_i = Re^{j\phi} \qquad r_{i+1} = Re^{-j\phi}$$
$$c_i = R_0 e^{j\phi_0} \qquad c_{i+1} = R_0 e^{-j\phi_0}$$
$$c_i r_i^k + c_{i+1} r_{i+1}^k = R_0 R e^{jk\phi + j\phi_0} + R_0 R e^{-(jk\phi + j\phi_0)}$$

This can be converted into a cosine and sine using Eulers formula:

$$y[k] = R_0 R\big(\cos(k\phi + \phi_0) + \sin(k\phi + \phi_0)\big) + R_0 R\big(\cos(k\phi + \phi_0) - \sin(k\phi + \phi_0)\big)$$
$$= 2R_0 R\big(\cos(k\phi + \phi_0)\big)$$

Eulers formula

Theorem

$e^{j\phi} = \cos(\phi) + \sin(\phi)j$

Proof.

Using power series:

$$\begin{aligned} e^{\phi j} &= 1 + jx - \frac{x^2}{2!} - \frac{jx^3}{3!} + \frac{x^4}{4!} + \frac{jx^5}{5!} + \dots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right) + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots\right)j \\ &= \cos(\phi) + \sin(\phi j) \end{aligned}$$

$\square$

Leonhard Euler (15/4/1707 - 18/9/1783) was a pioneering Swiss mathematician and physicist. He made important discoveries in fields as diverse as infinitesimal calculus and graph theory. He also introduced much of the modern mathematical terminology and notation, particularly for mathematical analysis, such as the notion of a mathematical function.

Non-homogeneous difference equations

Definition

$$\sum_{i=0}^{n} a_i y[k+i] = \sum_{i=0}^{n} b_i u[k+i]$$

A linear combination of inputs results in the same linear combination of the outputs resulting from each input individually.

Solution

The equation can thus be solved for each input individually and the results added together afterwards.
The resulting particular solutions can then be added to the general form of the homogeneous solution.

Particular solutions to difference equations

Input $u[k]$	Suggested solution $y[k]$
k	$\alpha_1 k + \alpha_0$
k^n	$\sum_{i=0}^{n} \alpha_i k^i$
a^k	αa^k
$k^n a^k$	$(\sum_{i=0}^{n} \alpha_i k^i) a^k$
$cos(k\phi)$	$\alpha cos(k\phi + \phi_0)$
$a^k cos(k\phi)$	$\alpha a^k cos(k\phi + \phi_0)$
$k^n a^k cos(k\phi)$	$(\sum_{i=0}^{n} \alpha_i k^i \alpha a^k cos(k\phi + \phi_0)$

Example

Example

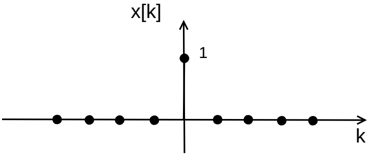
- Difference equation : $y[k+2] - 5y[k+1] + 6y[k] = (-1)^k$
- Initial value: $y[1] = \frac{1}{4}, y[0] = \frac{1}{12}$
- Homogeneous difference equation: $y[k+2] - 5y[k+1] + 6y[k] = 0$
- Characteristic polynomial: $r^2 - 5r + 6 = 0$
- Homogeneous solution: $y_{hom}[k] = c_1 2^k + c_2 3^k$
- Particular solution: $y_{par}[k] = \alpha(-1)^k$
- Substitution: $\alpha(-1)^{k+2} - 5\alpha(-1)^{k+1} + 6\alpha(-1)^k = (-1)^k$
- $\alpha = \frac{1}{12}$
- General solution: $y[k] = c_1 2^k + c_2 3^k + \frac{1}{12}(-1)^k$
- Using the initial values: $c_1 = -\frac{1}{3}, c_2 = \frac{1}{3}$
- Result: $y[k] = -\frac{1}{3}2^k + \frac{1}{3}3^k + \frac{1}{12}(-1)^k$

Outline

Impulse response

Definition

$$\delta[k] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$



Theorem

You can decompose any signal as follows:

$$f[k] = \sum_{i=-\infty}^{i=\infty} \delta[k-i] f[i] = \delta[k] * f[k]$$

Convolution

Definition

$$w[k] = u[k] * v[k] = \sum_{i=-\infty}^{\infty} u[i]v[k-i]$$

Solve

1. Flip v[i] around vertical axis(v[-i]).
2. Slide to the right over k steps(v[k-i]).
3. Multiply $u[i]$ and $v[k-i]$
4. Sum all the values.

Convolution theorem

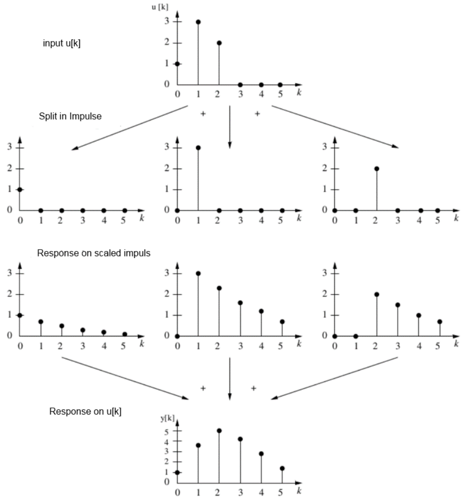
Theorem

$y[k] = u[k] * h[k]$

Proof.

$$\delta[k] \rightarrow h[k]$$
$$\delta[k+i] \rightarrow h[k+i]$$
$$u[k] = \sum_{i=-\infty}^{i=\infty} \delta[k-i]u[i] \rightarrow \sum_{i=-\infty}^{i=\infty} h[k-i]u[i]$$
$$u[k] \rightarrow u[k] * h[k] = y[k]$$

$\square$


input u[k]
u [k]
Split in Impulse
+
+
Response on scaled impuls
+
+
Response on u[k]
y[k]
k

Impulse response

Definition (Impulse response)

The impulse response of a dynamic system is the output when the input is a unit-impulse.

Impulse response

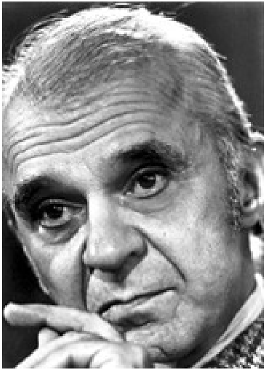
$$h[k] = \begin{cases} 0 & \text{if } k < 0 \\ D & \text{if } k = 0 \\ CA^{k-1}B & \text{if } k > 0 \end{cases}$$

Examples of Dirac-deltas

Popping balloons for acoustic measurements

Example: Leontief model of a planned economy

- Won the Nobel prize in 1973
- A simple model that assigns values to different sectors
- For simplicity we choose a planned economy. But today governments all over the world are using similar models to model their economy.



Example: Leontief model of a planned economy

Leontief divided the economy in sectors who buy from each other. The optimal production of some sector this month depends on the demand (external and internal) of the following month. So the system is non causal.

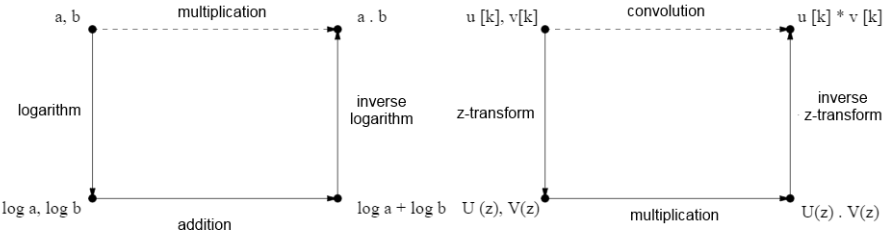
$$x[k-1] = Ax[k] + Bu[k]$$
$$y[k] = Ix[k]$$

Outline

Z-transform

Definition

- Discrete equivalent to the Laplace-transform
- Converts time-dependent descriptions of systems to the time-independent Z-domain.
- Simplifies many calculations:
  - Convolution theorem $\rightarrow$ convolution becomes multiplication
  - Linear difference equations become simple algebraic expressions
  - ...


a, b
multiplication
a . b
logarithm
inverse
logarithm
log a, log b
addition
log a + log b
u [k], v[k]
convolution
u [k] * v [k]
z-transform
inverse
z-transform
U (z), V(z)
multiplication
U(z) . V(z)

Z-transform

2 forms

- Bilateral: Requires knowledge of x for all values of k, including negative values. Can be used for non-causal systems $X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$
- Unilateral: Only requires knowledge of x for positive values of k. Can only be used for causal systems without loss of information $X(z) = \sum_{k=0}^{\infty} x[k]z^{-k}$

Z-transform

Example

$$x[k] = \begin{Bmatrix} 1 & -1 & 0 & 2 & 4 \\ & & & \uparrow & \end{Bmatrix}$$

$$X(z) = \sum_{k=-3}^{1} x[k]z^{-k} = z^3 - z^2 + 2 + 4z^{-1}$$

Properties Unilateral Z-transform

Property	Time Domain	Z-domain
Linearity	$af_1[n] + bf_2[n] + \dots$	$aF_1(Z) + bF_2(Z) + \dots$
Right Shift(m>0)	$f[k-m]$	$z^{-m}F(Z)$
Left Shift (m>0)	$f[k+m]$	$z^m\left(F(z) - \sum_{i=0}^{m-1} f[i]z^{-i}\right)$
Convolution	$f[k] * g[k]$	$F(z)G(z)$
Multiplication by a^k	$a^k f[k]$	$F(a^{-1}z)$
Summation in time	$\sum_{i=0}^{k} f[i]$	$\frac{z}{z-1}F(Z)$
Differentiation in z	$k^m f[k]$	$\left(-z\frac{d}{d}\right)^m F(z)$
Periodic Sequence	$f[k] = f[k+N]$	$F(z) = \frac{z^N}{z^{N-1}} \sum_{k=0}^{N-1} f[k]z^{-1}$
Initial Value	f[0]	$\lim_{\lvert z\rvert \to \infty} F(z)$
Final value	$f[\infty]$	$\lim_{z \to 1}(z-1)F(z)$

List of common Z-transform pairs

$\delta[k]$	1
$\delta[k-m]$	z^{-m}
1	$\frac{z}{(z-1)}$
a^k	$\frac{z}{(z-a)}$
ka^k	$\frac{az}{(z-a)^2}$
k^2a^k	$\frac{az(z+a)}{(z-a)^3}$
k^3a^k	$\frac{az(z^2+4az+a^2)}{(z-a)^4}$
k^4a^k	$\frac{az(z^3+11az^2+11a^2z+a^3)}{(z-a)^5}$
k^5a^k	$\frac{az(z^4+26az^3+66a^2z^2+26a^3z+a^4)}{(z-a)^6}$
$\frac{(k+1)(k+2)\ldots(k+m)a^k}{m!}$	$\frac{z^{m+1}}{(z-a)^{m+1}}$

List of common Z-transform pairs

$a^k \sin k\omega T$	$\frac{az \sin \omega T}{z^2 - 2az \cos \omega T + a^2}$
$a^k \cos k\omega T$	$\frac{z(z - a \cos \omega T)}{z^2 - 2az \cos \omega T + a^2}$
$\cos(k\omega T + \theta)$	$\frac{z(z \cos \theta - \cos(\omega T - \theta))}{z^2 - 2az \cos \omega T + 1}$
$ka^k \sin k\omega T$	$\frac{z(z-a)(z+a) a \sin \omega T}{(z^2 - 2az \cos \omega T + a^2)^2}$
$ka^k \cos k\omega T$	$\frac{az(z^2 \cos \omega T - 2az + a^2 \cos \omega T)}{(z^2 - 2az \cos \omega T + a^2)^2}$
$a^k \sinh k\omega T$	$\frac{az \sinh \omega T}{z^2 - 2az \cosh \omega T + a^2}$
$a^k \cosh k\omega T$	$\frac{z(z - a \cosh \omega T)}{z^2 - 2az \cosh \omega T + a^2}$
$\frac{a^k}{k!}$	$e^{\frac{a}{z}}$
$\frac{n!}{(n-k)!k!} a^k b^{n-k}$	$\frac{(bz+a)^n}{z^n}$
$\frac{1}{k}$	$\ln\left(\frac{z}{z-1}\right)$
$\frac{k(k-1)(k-2)\ldots(k-m+1)}{m!}$	$\frac{z}{(z-1)^{m+1}}$

Region of convergence

Z-transform is not unique

Two different signals could have the same Z-transform over a different region of convergence.

Definition

The region of convergence is the set of complex numbers z for which:

$$\sum_{k=-\infty}^{\infty} \mid x[k]z^{-k} \mid < \infty$$

- We will look at convergence separately for positive and negative k, splitting the convergence criterion in 2:

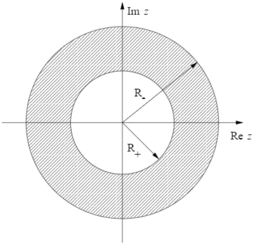
$$k < 0 : \mid x[k] \mid \leq M_- R_-^k$$
$$k \geq 0 : \mid x[k] \mid \leq M_+ R_+^k$$

- Using $z = re^{j\theta}$ with R+ as small as possible and R- as large as possible we get:

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \mid x[k]z^{-k} \mid &= \sum_{k=-\infty}^{\infty} \mid x[k] \mid r^{-k} \\ &= \sum_{k=1}^{\infty} \mid x[-k] \mid r^{k} + \sum_{k=0}^{\infty} \mid x[k] \mid r^{-k} \\ &\leq M_- \sum_{k=1}^{\infty} (R_-^{-1} r)^k + M_+ \sum_{k=0}^{\infty} (R_+ r^{-1})^k\end{aligned}$$

Region of convergence

The sums are finite if $R_{-}^{-1}r < 1$ and $R_{+}r^{-1} < 1$ Region of convergence: $R_{+} < R_{-}$ R+¡ R-: Ring R-¡ R+: No ROC Causal system, for negative k: $x[k] = 0 \Rightarrow R_{-} = +\infty$ cannot contain any poles of the system ROC of a stable system always contains the unit circle.



Inverse Z-transform

Inverse Z-transform

1. Split the function into partial fractions
2. Use the table to transform each partial fraction individually to the time domain

Partial fraction decomposition

1. Factorizing the denominator
2. If all poles(zeros of the denominator) have multiplicity 1:

$$F(z) = \frac{\sum_{i=0}^{n} b_i z^i}{a_n(z-p_1)(z-p_2)\ldots(z-p_n)}$$

$$= \alpha_0 + \alpha_1\left(\frac{z}{z-p_1}\right) + \cdots + \alpha_n\left(\frac{z}{z-p_n}\right)$$

3. The coefficients can be calculated by:

$$\alpha_0 = F(0) \qquad \alpha_i = \left[\tfrac{z-p_i}{z}F(z)\right]_{z=p_i}$$

Inverse Z-transform

Partial fraction decomposition

4. If there are poles with multiplicity higher than 1:

$$F(z) = \frac{\sum_{i=0}^{n} b_i z^i}{a_n(z-p_1)^{n_1}(z-p_2)^{n_2}\dots}$$
$$= \alpha_0 + \alpha_1\left(\frac{z}{z-p_1}\right) + \alpha_2\left(\frac{z}{z-p_1}\right)^2 + \dots + \alpha_{n_1}\left(\frac{z}{z-p_1}\right)^{n_1}$$
$$+ \beta_1\left(\frac{z}{z-p_2}\right) + \beta_2\left(\frac{z}{z-p_2}\right)^2 + \dots \beta_{n_2}\left(\frac{z}{z-p_2}\right)^{n_2}$$

5. Where the highest coefficient for each pole can be calculated by:

$$\alpha_0 = F(0)\ \alpha_1 = \left[\left(\frac{z-p_1}{z}\right)^{n_1} F(z)\right]_{z=p_1} \quad \beta_1 = \left[\left(\frac{z-p_2}{z}\right)^{n_2} F(z)\right]_{z=p_2}$$

Inverse Z-transform

Partial fraction decomposition

6. Any remaining coefficients can be found by evaluating the equation for a number of values of z.

$F(z)$	$f[k]$
1	$\delta[k]$
$\frac{z}{z-a}$	a^k
$\frac{z^{m+1}}{(z-a)^{m+1}}$	$\frac{(k+1)(k+2)+\ldots(k+m)a^k}{m!}$

Inverse Z-transform

Example

$$F(z) = \frac{z^3+2z^2+z+1}{z^3-z^2-8z+12}$$

- The denominator has a zero in 2 (m=2) and -3
- Partial fraction decomposition:

$$\alpha_0 + \alpha_1\left(\frac{z}{z-2}\right) + \alpha_2\left(\frac{z}{z-2}\right)^2 + \beta_1\left(\frac{z}{z+3}\right)$$

- $\alpha_0 = F(0) = \frac{1}{12}$

$$\alpha_2 = \left[\frac{F(z)(z-2)^2}{z^2}\right]_{z=2} = \left[\frac{z^3+2z+z+1}{z^2(z+3)}\right]_{z=2} = \frac{19}{20}$$

$$\beta_1 = \left[\frac{F(z)(z+3)}{z}\right]_{z=-3} = \left[\frac{z^3+2z+z+1}{z(z-2)^2}\right]_{z=-3} = \frac{11}{75}$$

Example

- By evaluating the function for z=1:
  $\frac{5}{4} = \alpha_0 - \alpha_1 + \alpha_2 + \beta/4$
  $\alpha_1 = -9/50$
- Result : $F(z) = \frac{1}{12} - \frac{9}{50}\frac{z}{z-2} + \frac{19}{20}\frac{z^2}{(z-2)^2} + \frac{11}{75}\frac{z}{z+3}$
- Inverse Z-transform:
  $f[k] = \delta[k]/12 + 2^k k \frac{19}{20} + 2^k \frac{77}{100} + (-3)^k \frac{11}{75}$

Inverse Z-transform

Another technique for calculating the inverse Z-transform is direct division.
The numerator of the transfer function is divided by the denominator via long division.

Teller

$$
\begin{array}{l}
z^3 + 2z^2 + z + 1 \\
\underline{z^3 - z^2 - 8z + 12} \\
\quad 3z^2 + 9z - 11 \\
\quad \underline{3z^2 - 3z - 24 + 36z^{-1}} \\
\qquad 12z + 13 - 36z^{-1} \\
\qquad \underline{12z - 12 - 96z^{-1} + 144z^{-2}} \\
\qquad\quad 25 + 60z^{-1} - 144z^{-2} \\
\qquad\qquad \ldots
\end{array}
$$

Noemer

$$
\begin{array}{|l}
z^3 - z^2 - 8z + 12 \\
\hline
1 + 3z^{-1} + 12z^{-2} + 25z^{-3} + \ldots = \sum_{k=0}^{\infty} f[k] z^{-k}
\end{array}
$$

Solving difference equations with the Z-transform

A system is described by a difference equation of the following form:

$$\sum_{i=0}^{n} a_i y[k+i] = \sum_{i=0}^{n} b_i u[k+i]$$

After the Z-transform:

$$a_0 Y(z) + \sum_{i=1}^{n} a_i z^i \left[Y(z) - \sum_{j=0}^{i-1} y[j] z^{-j} \right] =$$
$$b_0 U(z) + \sum_{i=1}^{n} b_i z_i \left[U(z) - \sum_{j=0}^{i-1} u[j] z^{-j} \right]$$

Solving difference equations with the Z-transform

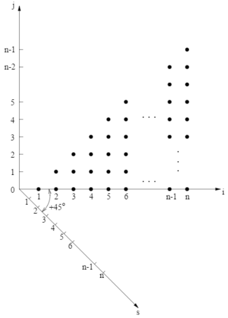
Rearranged:

$$Y(z) = \frac{\sum_{i=1}^{n} b_i z^i}{\sum_{i=1}^{n} a_i z^i} U(z) - \frac{\sum_{i=1}^{n} b_i z^i \left[\sum_{j=0}^{i-1} u[j] z^{-j} \right] - \sum_{i=1}^{n} a_i z^i \left[\sum_{j=0}^{i-1} y[j] z^{-j} \right]}{\sum_{i=1}^{n} a_i z^i}$$

Solving difference equations with the Z-transform

We'll apply the following transformation of the double summations:

$$\sum_{i=1}^{n} b_i z^i \sum_{j=0}^{i-1} u[j] z^{-j} = \sum_{s=1}^{n} \sum_{j=0}^{n-s} b_{s+j} u[j] z^s$$



Solving difference equations with the Z-transform

The final simplified result is:

$$Y(z) = \frac{\sum_{i=1}^{n} b_i z^i}{\sum_{i=1}^{n} a_i z^i} U(z) + \frac{\sum_{s=1}^{n} \left(\sum_{j=0}^{n-s} a_{s+j} y[j] - \sum_{j=0}^{n-s} b_{s+j} u[j] \right) z^s}{\sum_{i=1}^{n} a_i z^i}$$

With this result it is easy to find the resulting output from a given input or vice-versa given a difference equation.

Right-hand fraction = output resulting from initial conditions: will vanish with time = transient behaviour

Left-hand fraction = output resulting from input: will remain = steady state response.

$H(z) = \frac{\sum_{i=1}^{n} b_i z_i}{\sum_{i=1}^{n} a_i z_i}$ is the transfer function of the system. This is the z transform of $h[k]$ (the impulse response)

The transfer function can also be derived directly from the state space model of a system:

$$x[k+1] = Ax[k] + Bu[k]$$
$$y[k] = Cx[k] + Du[k]$$

The Z-transform gives:

$$z(X(z) - x[0]) = AX(z) + BU(z)$$
$$Y(z) = CX(z) + DU(z)$$

Rearranged to have X(z) in explicit form:

$$X(z) = (zI - A)^{-1} z x[0] + (zI - A)^{-1} BU(z)$$
$$Y(z) = C(zI - A)^{-1} z x[0] + \left[C(zI - A)^{-1} B + D \right] U(z)$$

If $x[0] = 0$ and $u[k] = \delta[k](U(z) = 1)$:
$H(z) = C(zI - A)^{-1} B + D$
$Y(z) = H(z)U(z)$

Z-Transform

Definition

A pole p_i of the is system is a point in the complex z-plane where $H(p_i) = \pm\infty$ i.e. the denominator becomes zero.

$$\sum_{i=0}^{n} a_i z^i = 0$$

Definition

A zero n_i is a point where $H(n_i) = 0$ i.e. the numerator becomes zero.

$$\sum_{i=0}^{n} b_i z^i = 0$$

Link between eigenvalues and poles

Are eigenvalues of A poles of H(z)?

- As z approaches an eigenvalue of A, $(zI - A)^{-1}$ is no longer defined.
- $C(zI - A)^{-1}B$ may still be defined depending on the values of C and B.

Rule 1

An Eigenvalue of A will sometimes, but not always, be a pole of H(z).

Definition

If every eigenvalue of A is also a pole of H(z) then a minimal number of internal states has been achieved.

Link between eigenvalues and poles

Are poles of H(z) eigenvalues of A?

$H(z) = C(zI - A)^{-1}B + D$

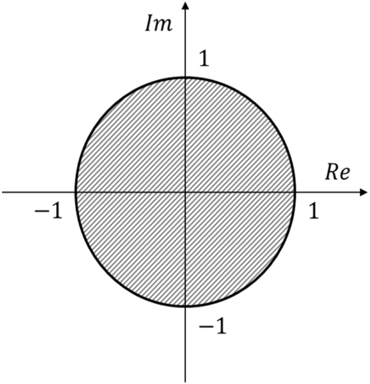
B, C and D are matrices with properly defined values. If H(z) is undefined then $(zI - A)^{-1}$ must be the cause. So, z must be an eigenvalue of A

Rule 2

Poles of H(z) are always eigenvalues of A

Stability

- Internal Stability
  - All possible internal states return to zero after a finite time in the absence of an input.
  - All eigenvalues of the matrix A are contained within the a circle of radius 1 around zero in the complex plane.
- BIBO-Stability (Bounded-Input Bounded-Output)
  - Every bounded input results in a bounded output
  - All poles are contained within the a circle of radius 1 around zero in the complex plane
  - BIBO-Stability follows from Internal Stability, but the inverse is not necessarily true.


Im
1
Re
−1
1
−1

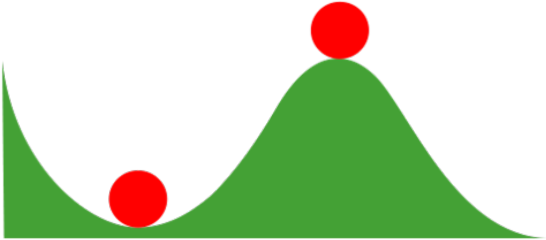
Can unstable systems exist?

According to the mathematical models we have discussed unstable systems need an infinite amount of energy.

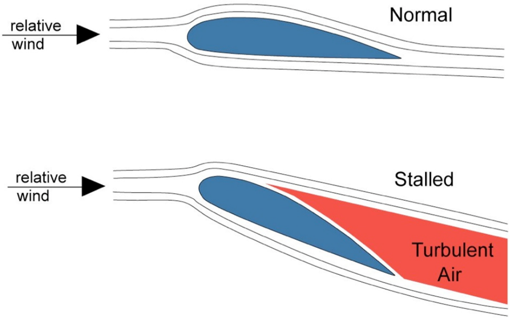
What happens in the real world?

- The system enters a state in which the current linear model is no longer valid (Non linear behaviour) .
- Smaller unaccounted effects become more prominent
- The system malfunctions and may cause damage to itself or its surroundings.
- Something else bad happens

Stability



Airplane stall



- Airplanes generate lift using the Venturi effect.
- Faster moving air has a lower pressure.
- Eddy currents may be created due to a too slow airspeed or too sharp ascent.
- Turbulent airflow causes a loss of the lift generated by the Venturi effect.
- Without the necessary lift an air plane becomes an unstable system.

Buses and other tall vehicles have a tendency to roll when taking turns too quickly. A London bus is loaded with sandbags and must be able to lean at an angle of at least 28° while still returning all tires to the ground. Modern day car manufacturers have to pass multiple tests for stability while manoeuvring

Steady state behavior via Z-transform

Starting from the previous result:

$$Y(z) = \frac{\sum\limits_{i=1}^{n} b_i z^i}{\sum\limits_{i=1}^{n} a_i z^i} U(z) + \frac{\sum\limits_{s=1}^{n}\left(\sum\limits_{j=0}^{n-s} a_{s+j} y[j] - \sum\limits_{j=0}^{n-s} b_{s+j} u[j]\right) z^s}{\sum\limits_{i=1}^{n} a_i z^i}$$

We wish to find the resulting output from the input:

$u[k] = \cos(k\alpha + \theta)$

To simplify derivation, we use:

$u[k] = e^{j(k\alpha+\theta)}$

With Z-transform: $U(z) = \frac{ze^{j\theta}}{z-e^{j\alpha}}$

Steady state behavior via Z-transform

Filling in U(z) and splitting into partial fractions:

$$Y(z) = \frac{\sum_{i=1}^{n} b_i z^i}{\sum_{i=1}^{n} a_i z^i} \frac{z e^{j\theta}}{z - e^{j\alpha}} + \frac{\sum_{s=1}^{n} \left(\sum_{j=0}^{n-s} a_{s+j} y[j] - \sum_{j=0}^{n-s} b_{s+j} u[j] \right) z^s}{\sum_{i=1}^{n} a_i z^i}$$

$$= \frac{cz}{z - e^{j\alpha}} + \frac{d_1 z}{z - p_1} + \cdots + \frac{d_n z}{z - p_n} + g$$

Calculating the coefficient c:

$$c = \left[Y(z) \frac{z - e^{j\alpha}}{z} \right]_{z = e^{j\alpha}} = \left[H(z) e^{j\theta} \right]_{z = e^{j\alpha}} = H(e^{j\alpha}) e^{j\theta}$$

Steady state behavior via Z-transform

After the inverse Z-transform:

$$\begin{aligned} y[k] &= H(e^{j\alpha})e^{j(k\alpha+\theta)} + d_1 p_1^k + \cdots + d_n p_n^k + g\delta[k] \\ &= \mid H(e^{j\alpha}) \mid e^{j(k\alpha+\theta+\angle H(e^{j\alpha}))} + d_1 p_1^k + \cdots + d_n p_n^k + g\delta[k] \end{aligned}$$

Because of linearity we can ignore the imaginary component, leading to the result:

$$y[k] = \underbrace{\mid H(e^{j\alpha}) \mid cos(k\alpha + \theta + \angle H(e^{j\alpha}))}_{\text{staedy state}} + \underbrace{Re(d_1 p_1^k + \cdots + d_n p_n^k + g\delta[k])}_{\text{transient behavior}}$$

Input	Output
$cos(k\alpha + \theta)$	$\mid H(e^{j\alpha}) \mid cos(k\alpha + \theta + \angle H(e^{j\alpha}))$

Steady state behavior via Z-transform

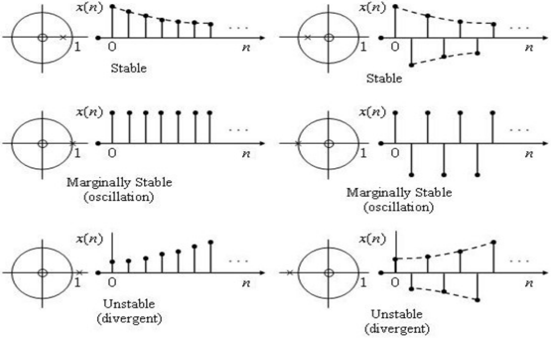
Example

- $u[k] = cos(3k + \pi) \Rightarrow \alpha = 3 \quad \theta = pi$
- $H(z) = \frac{z^2+4}{(z^2+6)(z-1)}$
- $H(e^{3j}) = 0.357e^{0.055j}$
- The resulting output has been reduced to a third in amplitude and has undergone a small phase shift.
  $y[k] = 0.357\cos(3k + \pi + 0.055)$

Complex Eigenvalues

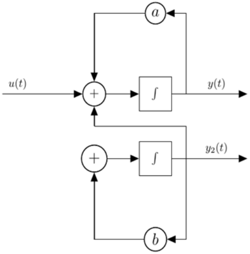
As with the roots to the characteristic equation in difference equations, complex and/or negative eigenvalues for A create oscillation. $\|\lambda\| < 1$: The oscillation will decrease in magnitude: stable $\|\lambda\| > 1$: The oscillation will increase in magnitude: unstable $\|\lambda\| = 1$: The oscillation will maintain the same magnitude indefinitely: unstable
The smallest achievable period is 2 times the step time, for negative real eigenvalues.

Complex Eigenvalues

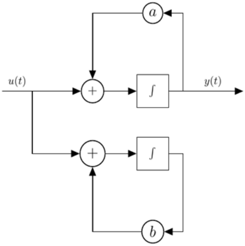


Outline

Observability and controllability



Not controllable



Not observable

Observability

A system is observable if every initial state $x[0]$ can be determined from the observation of y[k].
The state space model without inputs gives us:
$x[k+1] = Ax[k] \qquad y[k] = Cx[k]$
Now we can determine a set of vector equations in $x[0]$
$x[1] = Ax[0], x[2] = A^2x[0], \ldots, x[n-1] = A^{n-1}x[0]$
If the system has n internal states then n equations are needed:
$y[0] = Cx[0], y[1] = Cx[1] = CAx[0], \ldots, y[n-1] = CA^{n-1}x[0]$

$$\begin{bmatrix} y[0] \\ y[1] \\ \vdots \\ y[n-1] \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \begin{bmatrix} x_1[0] \\ x_2[0] \\ \vdots \\ x_n[0] \end{bmatrix}$$

Observability

The system is observable if

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank n.

Controllability

A system is controllable if it can be brought to a desired state using the inputs in a finite time.
Again we start from the state space model:
The following equations can be derived:

$$x[k+1] = Ax[k] + Bu[k]$$
$$x[1] = Ax[0] + Bu[0]$$
$$x[2] = A^2x[0] + ABu[0] + Bu[1]$$
$$x[3] = A^3x[0] + A^2Bu[0] + ABu[1] + Bu[2]$$
$$\vdots$$
$$x[n] = A^nx[0] + A^{n-1}Bu[0] + \cdots + Bu[n-1]$$

Controllability

This last equation can be rewritten as:

$$x[n] - A^n x[0] = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \begin{bmatrix} u[n-1] \\ u[n-2] \\ \vdots \\ u[1] \\ u[0] \end{bmatrix}$$

For a given x[0] and a desired x[n] the required inputs can be found by solving this system. $\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$ is called the controllability matrix of the system. A system is said to be controllable if the set of equations can be solved for a given x[0] and any desired x[n]. This is the case if the controllability matrix has rank n.

Detectability and Stabilizability

Observability and controllability are important terms in control theory.

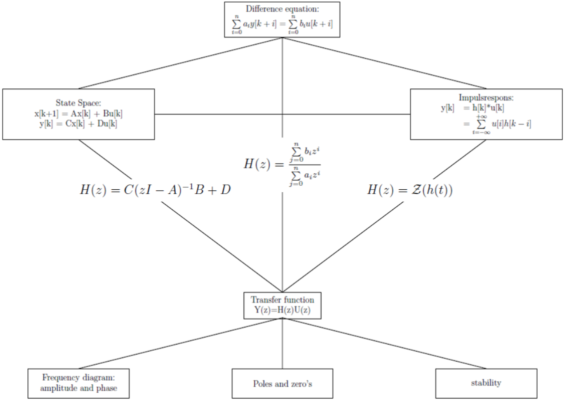
Detectability and stabilizability are also often used as weaker constraints.

A system is detectable if all unstable states are observable.

A system is stabilizable if all unstable states are controllable.

Detectability and stabilizability are also important terms in control theory.

Overview


Difference equation:
$\sum_{i=0}^{n} a_i y[k+i] = \sum_{i=0}^{n} b_i u[k+i]$
State Space:
x[k+1] = Ax[k] + Bu[k]
y[k] = Cx[k] + Du[k]
Impulsrespons:
y[k] = h[k]*u[k]
$= \sum_{i=-\infty}^{+\infty} u[i]h[k-i]$
$H(z) = C(zI-A)^{-1}B + D$
$H(z) = \frac{\sum_{j=0}^{n} b_j z^i}{\sum_{j=0}^{n} a_j z^i}$
$H(z) = \mathcal{Z}(h(t))$
Transfer function
Y(z)=H(z)U(z)
Frequency diagram:
amplitude and phase
Poles and zero's
stability