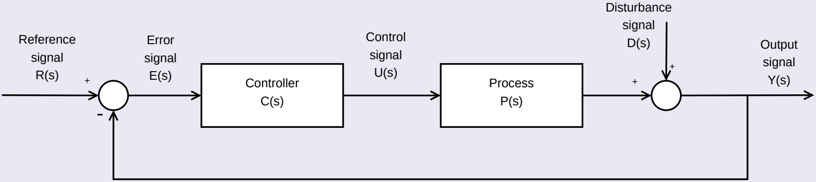
Chapter 12: Lead and Lag Compensators

August 5, 2015

Outline

Lead Compensator vs Lag Compensator

Schematical representation



Transfer functions

Lead compensator : $C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with $0<\alpha<1$

Lag compensator : $C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}}$ with $\beta>1$

Lead Compensator vs Lag Compensator: zeros and poles

Transfer functions

Lead compensator : $C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with $0<\alpha<1$

Lag compensator : $C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}}$ with $\beta>1$

Zeros and poles

Zeros: $s = -\frac{1}{\tau}$
Poles: $s = -\frac{1}{\alpha\tau}$ or $s = -\frac{1}{\beta\tau}$
For lead compensators the pole lies more to the left in the complex plane than the zero and vice versa for lag compensators.

Outline

Lead compensators

Transfer function

$C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with $0<\alpha<1$

Bode Diagram

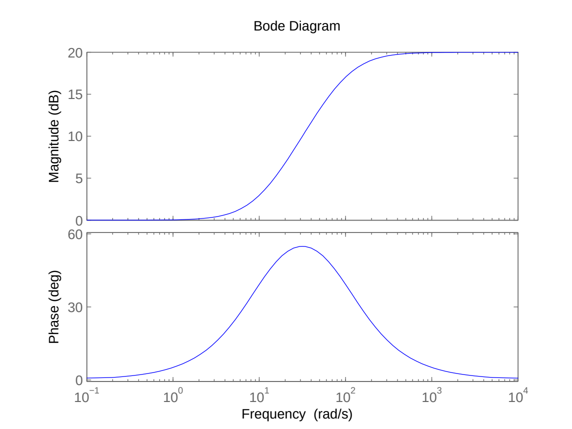
Example with K $= 10$ and $\alpha = 0.1$ (see next slide)
Magnitude of the lead compensator:

- becomes unity ($= 0$ dB) for small fequencies
- becomes 10 ($= 20$ dB) for high frequencies

$\Rightarrow$ Lead compensator is high-pass filter.

Lead compensators

Bode Diagram



Lead compensators

Impact

- Push the pole of the closed loop system to the left.
  - Stabilisation of the system (see root locus)
  - Increase response speed (lead compensator will stimulate some larger frequencies)
- Increase of the phase margin: the phase of the lead compensator is positive for every frequency, and will hence only increase the phase.
- Thanks to the presence of a pole, the high frequencies (where most of the unwanted noise is located) are less amplified. Again, a lead compensator is a high-pass filter.

Lead compensators

Design with Bode plots

- Design process: tuning of the phase margin, with as a surplus (because we will have one extra degree of freedom) the tuning of the steady state error.
- Compensate for the excessive phase lag that is a result of the components of P(s).
- Increase in phase at gain crossover frequency (GCF) if GCF is around pole and zero of the lead compensator.
- Gain is impacted by the lead compensator: the GCF of P(s).C(s) is not equal to the GCF of P(s).
- Required increase in phase gain: ϕ
- K will be used to tune the steady state error.

Lead Compensation Techniques Based on the Frequency-Response Approach

Step 1

- Remember the steady state error for references of the shape: $\frac{At^n\epsilon(\tau)}{n!}$ with $\epsilon(t)$ the step function.
- Translate your steady state requirement into another one:
  - $K_p = \lim_{s\to 0} P(s)C(s)$ (n = 0, proportional)
  - $K_v = \lim_{s\to 0} sP(s)C(s)$ (n = 1, linear)
  - $K_a = \lim_{s\to 0} s^2P(s)C(s)$ (n = 2, accelerating)
  - or another error constant
- With this $K_p/K_v/K_a$ and $\lim_{s\to 0} P(s)$, we can determine $\lim_{s\to 0} C(s) = \lim_{s\to 0} K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}} = K\alpha$.
- Verify whether a proportional controller with gain $K\alpha$ would suffice.

Lead Compensation Techniques Based on the Frequency-Response Approach

Step 2

Determine ϕ, the amount with which you want to increase the PM; if the PM is OK, you dont need a lead compensator; a proportional controller with gain $K\alpha$ suffices.

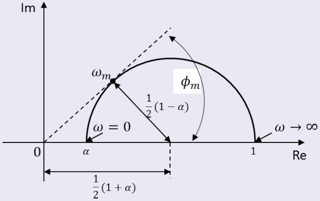
Step 3

Add 5°, to get $\phi_m = \phi + 5^\circ$ (if $\phi_m > 60^\circ$, you will need more than one lead compensator). The addition of the lead compensator shifts the gain crossover frequency to the right and decreases the phase margin.

Lead Compensation Techniques Based on the Frequency-Response Approach

Step 4

Determine α making use of the polar plot of $\frac{\alpha.(j\omega\tau+1)}{j\omega\alpha\tau+1}$



$\sin\phi_m = \frac{\frac{1}{2}.(1-\alpha)}{\frac{1}{2}.(1+\alpha)} = \frac{1-\alpha}{1+\alpha} \Rightarrow \alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m}$ Usually, $\alpha \geqslant 0.05$.
We know α and we know $K\alpha$ (step 1), so we can calculate K.

Lead Compensation Techniques Based on the Frequency-Response Approach

Step 5

Use the gain crossover frequency of P(s)C(s) as ω_m:

$$| P(j\omega_m)C(j\omega_m) |= 1$$

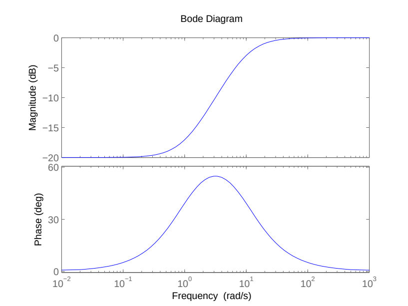
$$| P(j\omega_m) | K \frac{\sqrt{\frac{1}{\alpha\tau^2}+\frac{1}{\tau^2}}}{\sqrt{\frac{1}{\alpha\tau^2}+\frac{1}{\alpha^2\tau^2}}} =| P(j\omega_m) | K\sqrt{\alpha} = 1$$

$$20\log | P(j\omega_m) |= -20\log(K\sqrt{\alpha})$$

The value of the tangent point ω_m can be determined from P(s)'s Bode plot, because we know K and α from step 4.

Lead compensators

Step 6



The tangent point ω_m is the geometric mean of the two corner frequencies, so $\log \omega_m = \frac{1}{2}(\log \frac{1}{\tau} + \log \frac{1}{\alpha\tau})$ with $\tau = \frac{1}{\omega}$
$\Rightarrow \omega_m = \frac{1}{\sqrt{\alpha\tau}} \Rightarrow \tau = \frac{1}{\sqrt{\alpha}\omega_m}$.

Lead Compensation Techniques Based on the Frequency-Response Approach

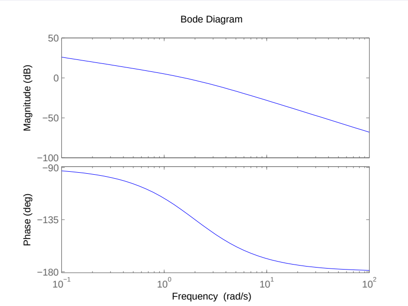
Step 7

Verify if the system behaves as desired. Check the gain margin for you to be sure it is satisfactory. If not, repeat the design process by modifying the pole-zero location of the compensator until a satisfactory result is obtained.

Example

Example

Given the system $P(s) = \frac{4}{s(s+2)}$. We want a phase margin of at least 50° and a steady state error for slope reference of maximal $\frac{A}{20}$.



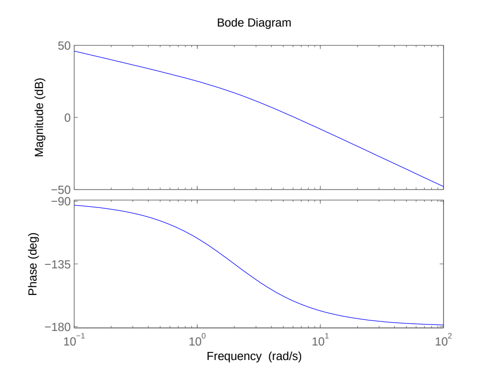
Example

Step 1

- Steady state requirement: $K_v = \frac{20}{s}$
  So, $\lim_{s\to 0} sP(s)C(s) = \lim_{s\to 0} s\frac{4}{s(s+2)}K\alpha = 2K\alpha = 20.$
  $\Rightarrow K\alpha = 10$
- Would a proportional controller with gain $K\alpha$ suffice? We have a look at the Bode Diagram of $K\alpha P(s)$ (see next slide) $\Rightarrow$ does not suffice! The phase margin is obviously smaller than 50°.

Example

Bode Diagram $K\alpha P(s)$



Example

Step 2

- Phase margin of $K\alpha P(s) = 18^\circ$ (see last slide)
- Calculation of phase margin without phase diagram:
  We need the frequency where the magnitude is 0 dB.
  So, $20 \log \mid K\alpha P(s) \mid = 0 \Rightarrow \mid K\alpha P(s) \mid = 1 \Rightarrow \mid P(s) \mid = 0.1$
  When substituting $s = j\omega$ and calculating the modulus of the complex number at the left side, the equation becomes:
  $\frac{-4\omega}{\omega(4+\omega^2)}^2 + \frac{-8}{\omega(4+\omega^2)}^2 = 0.01.$
  This equation has just one real positive solution in ω, $\omega = 6.168$.
  Now, you have the right frequency. You find the phase margin by calculating the difference between -180° and the phase of $K\alpha P(6.168j)$.
- We want a phase margin of at least $50^\circ \rightarrow \phi = 32^\circ$

Example

Step 3

$\phi_m = \phi + 5^\circ = 37^\circ$

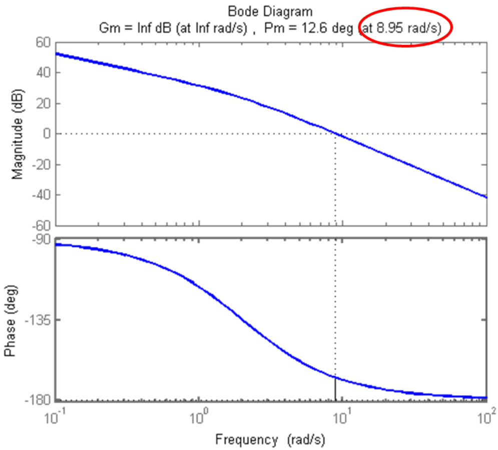
Step 4

$\alpha = \frac{1-\sin\phi_m}{1+\sin\phi_m} = 0.24$
From step 1, we know that $K = \frac{\alpha}{10} = 42$

Step 5

Find ω_m, the frequency at which the gain is $-20\log(K\sqrt{\alpha})$ dB.
$GCF(P(s)K\sqrt{\alpha}) = GCF(P(s)C(s)) \Rightarrow \omega_m = 9\frac{rad}{s}$
(see Bode Diagram next slide)

Example


Bode Diagram
Gm = Inf dB (at Inf rad/s) , Pm = 12.6 deg (at 8.95 rad/s)
Magnitude (dB)
60
40
20
0
-20
-40
-60
Phase (deg)
-90
-135
-180
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)

Example

Step 6

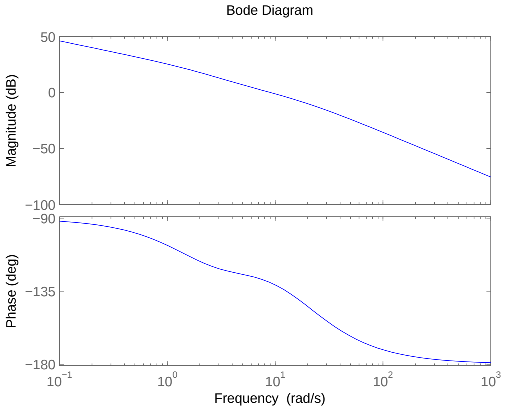
$\tau = \frac{1}{\omega_m \sqrt{\alpha}} = 0.23$

Step 7

We verify whether or not our solution is correct. We ask the Bode diagram of $K \frac{4}{s(s+2)} \frac{s+\frac{1}{\tau}}{s+\frac{1}{\alpha\tau}}$ with α, τ and K the results of our calculations. (see next slide)
We see that:

- the phase margin is indeed more than 32°
- the new tangent point is indeed about $9 \frac{rad}{s}$

Example

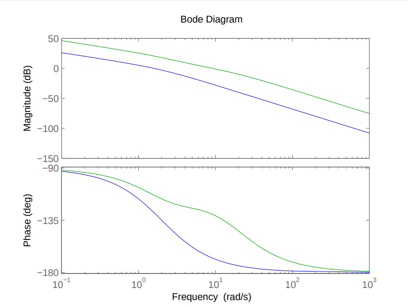

Bode Diagram
Magnitude (dB)
50
0
−50
−100
Phase (deg)
−90
−135
−180
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/s)

Example

Comparing compensated system vs non-compensated system

Blue: non-compensated system

Green: compensated system



Summary lead compensators

Evaluation of impact

- Pushing the poles to the left: this is not directly visible here, but is linked to the increased band width.
- The increase in bandwidth (this is linked to the response speed) and the increase in the phase margin were apparent in the Bode plot of P(s)C(s).
- A (small) decrease in the steady-state error occurs, since we designed it as such.
  Why small? The steady-state error decreases when the DC gain gets larger, but a lead compensators impact on the gain is not really built to increase the DC gain, the shape of a lag compensator is much more fit for this.

Summary lead compensators

Design with root locus

Design lead compensators with root locus for time-domain quantities - use dominant pole locations to fulfill overshoot, rise time, settling time, damping ratio and other requirements.

Outline

Lag compensators

Transfer function

$$C(s) = K.\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}} \text{ with } \beta>1$$

Bode Diagram

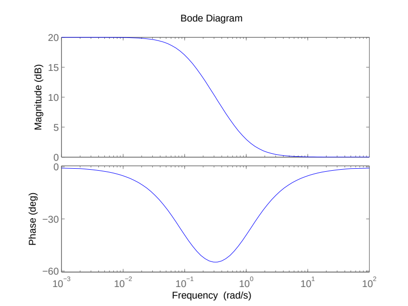
Example with K $=$ 1 and $\beta = 10$ (see next slide)
Magnitude of the lag compensator:

- becomes 10 ($=$ 20 dB) for small frequencies
- becomes unity ($=$ 0 dB) for high frequencies

$\Rightarrow$ Lag compensator is low-pass filter.

Lag compensators

Bode Diagram of example

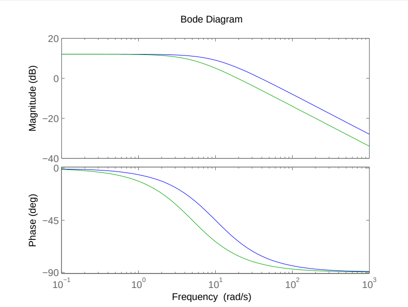


Lag compensators

Impact of lag compensators: Bode diagram

Blue: non-compensated system

Green: compensated system



Lag compensators

Impact of lag compensators: Bode diagram

- Lead compensators: increase the stability and tune the steady-state error by increasing the phase at the crossover frequency.
- Impact lag compensator = lead compensator, but different approach! By decreasing the gain, the gain crossover frequency comes down to a frequency at which the corresponding phase is higher.

Lag compensators

Impact of lag compensators

Large difference between lead and lag: their effect on the bandwidth of the system and hence on its speed of response.

- A lead compensator increases the bandwidth/speed of response
  - good if you want the system to react fast
- A lag compensator decreases the bandwidth/speed of response
  - good if your model if bad at high frequencies
  - good to reduce the impact of (mostly high-frequency) noise

Lag compensators

Design with Bode plots

We have three degrees of freedom:

- one to have a sufficient drop in gain
- one to push the drop in the phase to lower frequencies (that way we can use $\measuredangle P(s)$ as an approximation of $\measuredangle P(s)C(s)$ reliably to some extent
- one to tune the steady state error

Lag compensators

Design with Bode plots

- Increase of phase margin $\Rightarrow$ decrease of the magnitude at some higher frequencies
- Decrease of the steady state error $\Rightarrow$ increase of the magnitude at DC

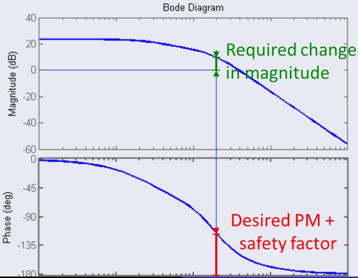
A lag compensator can realize both conditions.

- At DC value, the gain becomes: $\lim_{s\to 0} K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}} = K\beta$
- At high frequencies, the gain becomes: $\lim_{s\to\infty} K\frac{s+\frac{1}{\tau}}{s+\frac{1}{\beta\tau}} = K$

Lag compensators

Design with Bode plots

K has to be such that the drop in magnitude is sufficient, the value of β has make the steady state error decrease enough and the value of τ has to be such that the transfer between from $K\beta$ to τ occurs at the right frequency.



Lag Compensation Techniques Based on the Frequency-Response Approach

Step 1

- Determine $K\beta$ in a similar way as we found $K\alpha$ for the lead compensator.
- Translate your steady-state requirement into a requirement on:
  - $K_p = \lim_{s\to 0} P(s)C(s)$
  - $K_v = \lim_{s\to 0} sP(s)C(s)$
  - $K_a = \lim_{s\to 0} s^2P(s)C(s)$
  - or another error constant
- With this $K_p/K_v/K_a$ and $\lim_{s\to 0} P(s)$, we can determine $\lim_{s\to 0} C(s) = K\beta$.
- Verify whether a proportional controller with gain $K\beta$ would suffice.

Lag Compensation Techniques Based on the Frequency-Response Approach

Step 2

- Take the zero one decade smaller than the frequency (ω) at which P(s) had the desired phase (-180° + the desired phase margin + a safety factor of 10°). The addition of 10° compensates for the phase lag of the lag compensator.
- Verify the effect of a single zero at a frequency one decade smaller than ω.
  - The drop in magnitude is as good as complete.
  - The drop in phase cannot be more than -5.7°.
- Compute $\tau = \frac{10}{\omega}$. τ has to be large enough such that the magnitude is almost entirely dropped, and the phase drop has almost disappeared.

Lag Compensation Techniques Based on the Frequency-Response Approach

Step 3

- Determine K from the Bode plot.
  How to do this? Find the frequency (ω) with desired phase margin (+ safety factor), then find the magnitude at that frequency; which is equal to the required change in magnitude (= Q).
  $\Rightarrow K = \frac{1}{Q}$
- Safety factor about 10°
  - the drop in magnitude will not be complete (this is very marginal)
  - the lag compensator influences the phase plot

Lag Compensation Techniques Based on the Frequency-Response Approach

Step 4

We just have calculated K (step 3) and we know $K\beta$ (step 1), so it's possible to determine β.

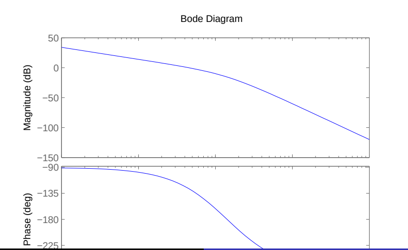
Step 5

Verify the behavior of the resulting system.

Example

Example

Given the system $P(s) = \frac{1}{s(s+1)(s+2)}$. The system has a PM of 53.4° at a frequency of $0.446\frac{rad}{s}$ and a GM of $15.6dB$ at a frequency of $1.41\frac{rad}{s}$. We want that a ramp input $A\epsilon(t)$ results in a steady state error of at most $\frac{A}{5}$, or $K_v = \frac{5}{s}$ and a PM of at least 40°.



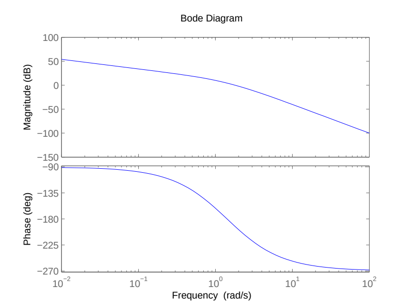
Example

Step 1

- Steady state requirement: $K_v = \frac{5}{s}$
  $\Rightarrow \lim_{s\to 0} sP(s)C(s) = \frac{1}{2}\lim_{s\to 0} C(s) = \frac{5}{s} \Rightarrow K\beta = 10$
- Would a proportional controller with gain $K\beta$ suffice? To answer this, we must have a look at the Bode plot of $K\beta P(s)$ (see next slide).
  $\Rightarrow$ Does not suffice! Adding a gain of $10 = 20dB$ to get the right steady state error, the phase gain would become negative. In other words, at the frequency where the magnitude of $K\beta P(s)$ equals 0 dB, the phase is less than -180° which means the system would become unstable.

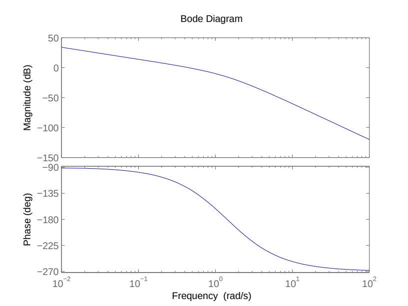
Example

Step 1 continued



Example

Step 2



Determine required phase

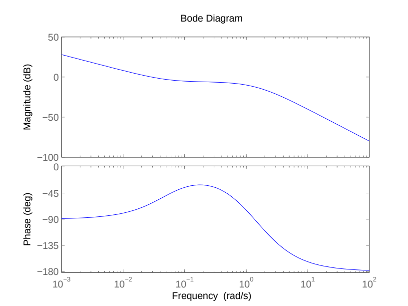
$phase = -180\,^\circ + 40\,^\circ + 10\,^\circ = -130\,^\circ$

At a phase of $-130\,^\circ$, $\omega = 0.5\frac{rad}{s} \Rightarrow \tau = \frac{10}{0.5} = 20$

Example

Step 2 continued

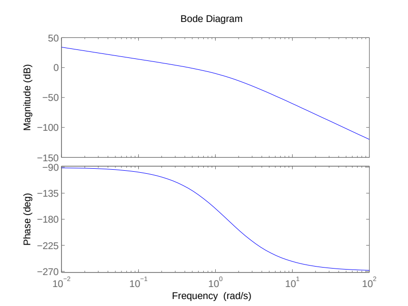
Verify the effect of a single zero at a frequency one decade smaller than ω.



Example

Step 3

At $\omega = 0.5\frac{rad}{s}$, the magnitude equals 0 dB $= 1$, so Q $= 1$.
$K = \frac{1}{Q} = 1$



Example

Step 4

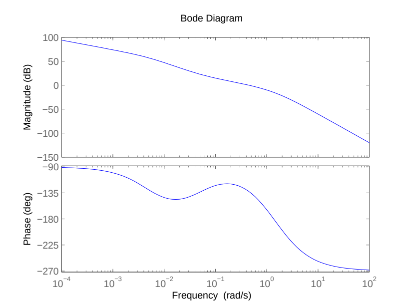
From the results of step 1 and step 3, we find $\beta = 10$.

Step 5

We find $C(s) = \frac{s+0.05}{s+0.05}$.
We verify the behavior of $P(s)C(s) = \frac{1}{s(s+1)(s+2)} \frac{s+0.05}{s+0.05}$ on its Bode Diagram (see next slide).
The phase margin is indeed greater than $40\,^\circ$.

Example

Step 5 continued

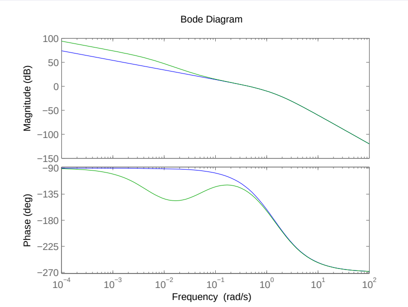


Example

Comparing compensated system vs non-compensated system

Blue: non-compensated system

Green: compensated system



Summary lead compensators

Goal

- To decrease the magnitude in order to shift the gain crossover frequency to a frequency with a larger phase margin; the extra degree of freedom is then used to tune the steady state error.
- In this example: lag compensator to tune the steady state error with a minimal impact on the phase margin.
  So this time we used a lag compensator to increase the DC gain but leave the gain at higher frequencies unaltered.

Summary lead compensators

Impact, using root locus

- We can use lag compensators to reduce the steady state error significantly but with a marginal impact on (the relevant part of) the root locus.
- On top of that, we still have one degree of freedom that allows us to pick any position on that root locus!
- This is useful if the desired closed loop poles are already on the root locus, but if a proportional controller would give a too large steady state error.

Outline

Comparision lead vs lag compensators

Method

- Lead: realize phase lead in phase diagram at ω_m to guarantee suficient phase margin
- Lag: realize weakening in amplitude diagram to move ω_m at values with sufficient phase margin

Effect

- Lead: increases reinforcement at higher frequencies $\Rightarrow$ increases band width
- Lag: decreases band width

Comparision lead vs lag compensators

Advantages

- Lead: increases dynamic response
- Lag: reduces high-frequented noise

Disadvantages

- Lead: needs reinforcement and is sensitive at high-frequented noise because of greater band width
- Lag: worsens transient respons

Comparision lead vs lag compensators

Use

- Lead: if you need fast transient respons
- Lag: for specificated regime errors

Don't use

- Lead: if phase decreases fast at the tangent point ω_m
- Lag: if there is no low-frequented area with sufficient phase margin

Outline

Lag-lead compensators

Transfer function

$C(s) = K \frac{s+\frac{1}{\tau_1}}{s+\frac{1}{\alpha\tau_1}} \frac{s+\frac{1}{\tau_2}}{s+\frac{1}{\beta\tau_2}}$ with $\beta>1$, $0<\alpha<1$, $\tau_1<\tau_2$

Usually, we take $\beta = \frac{1}{\alpha}$, but that's not necessary.

- The term $\frac{s+\frac{1}{\tau_1}}{s+\frac{1}{\alpha\tau_1}}$ produces the effect of the lead network.
- The term $\frac{s+\frac{1}{\tau_2}}{s+\frac{1}{\beta\tau_2}}$ produces the effect of the lag network.
- Zeros: $\frac{1}{\tau_1}$ and $\frac{1}{\tau_2}$
- Poles: $\frac{1}{\alpha\tau_1}$ and $\frac{1}{\beta\tau_2}$

Lag-lead compensators

Bode Diagram

Example with K $= 10$, $\alpha = 0.1$, $\beta = 10$, $\tau_1 = 0.01$, $\tau_2 = 10$ (see next slide)

Magnitude of lag-lead compensator:

- becomes 10 ($=$ 20 dB) at low frequencies
- becomes unity ($=$ 0 dB) at frequencies of about $1\,\frac{rad}{s}$ to $10\,\frac{rad}{s}$
- becomes 10 ($=$ 20 dB) at high frequencies

$\Rightarrow$ Lag-lead compensator is a band stop filter

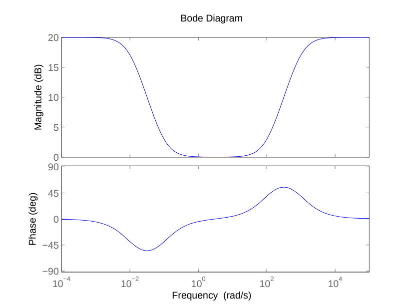
Explanation: $\beta\tau_2 > \tau_2 > \tau_1 > \alpha\tau_1$

$\Rightarrow \frac{1}{\beta\tau_2} < \frac{1}{\tau_2} < \frac{1}{\tau_1} < \frac{1}{\alpha\tau_1}$

$\Rightarrow pole_1 < zero_1 < zero_2 < pole_2$

Lag-Lead compensators

Bode diagram



Lag-lead compensators