Overdamped system

A system is overdamped ($\zeta > 1$) when the two poles are negative, real and unequal. For a unit-step $R(s) = \frac{1}{s}$, $Y(s)$ can be written as

$$Y(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n^2\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n^2\sqrt{\zeta^2 - 1})}.$$

The inverse Laplace transform is

$$y(t) = 1 + \frac{w_n}{2\sqrt{\zeta^2 - 1}}(\frac{e^{-s_1 t}}{s1} - \frac{e^{-s_2 t}}{s2}), \text{ for } t \geq 0.$$

Where

$$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n \text{ and } s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n.$$

Overdamped system

$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$
Thus $y(t)$ includes two decaying exponential terms

- When $\zeta >> 1$, one of the two decreases much faster than the other, the faster decaying exponential may be neglected;
- If $-s_2$ is located much closer to the $j\omega$ axis than $-s_1$ ($|s_2| >> |s_1|$), then $-s_1$ may be neglected;
- Once the faster decaying exponential term has disappeared, the response is similar to that of a first-order system:

$$H(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} = \frac{s_2}{s + s_2}.$$

Overdamped system

With the approximate transfer function, the unit-step response becomes:

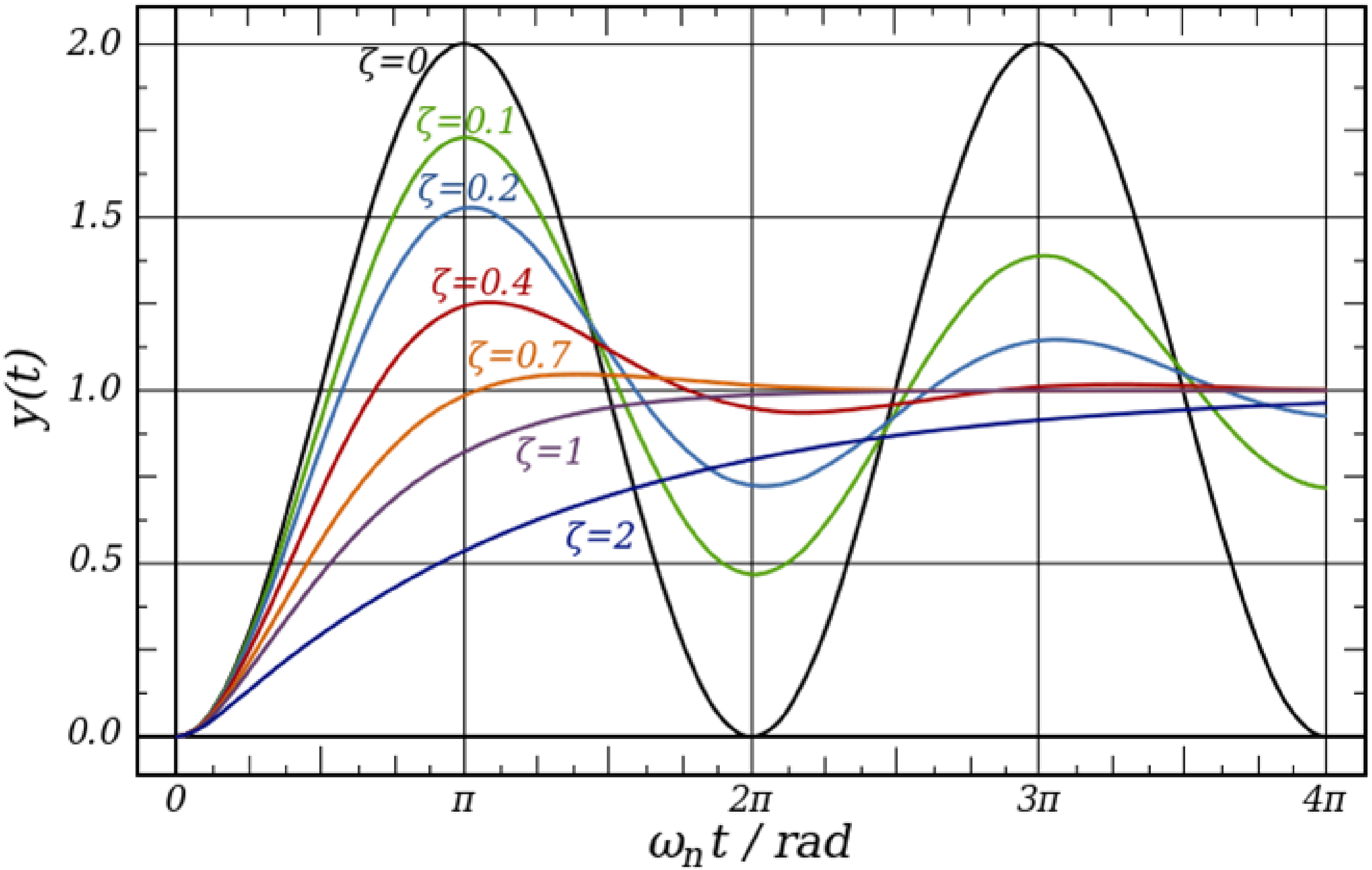
$$Y(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

The time response for the approximate transfer function is then given as:

$$y(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}, \text{ for } t \leq 0$$

Second order systems unit step response curves

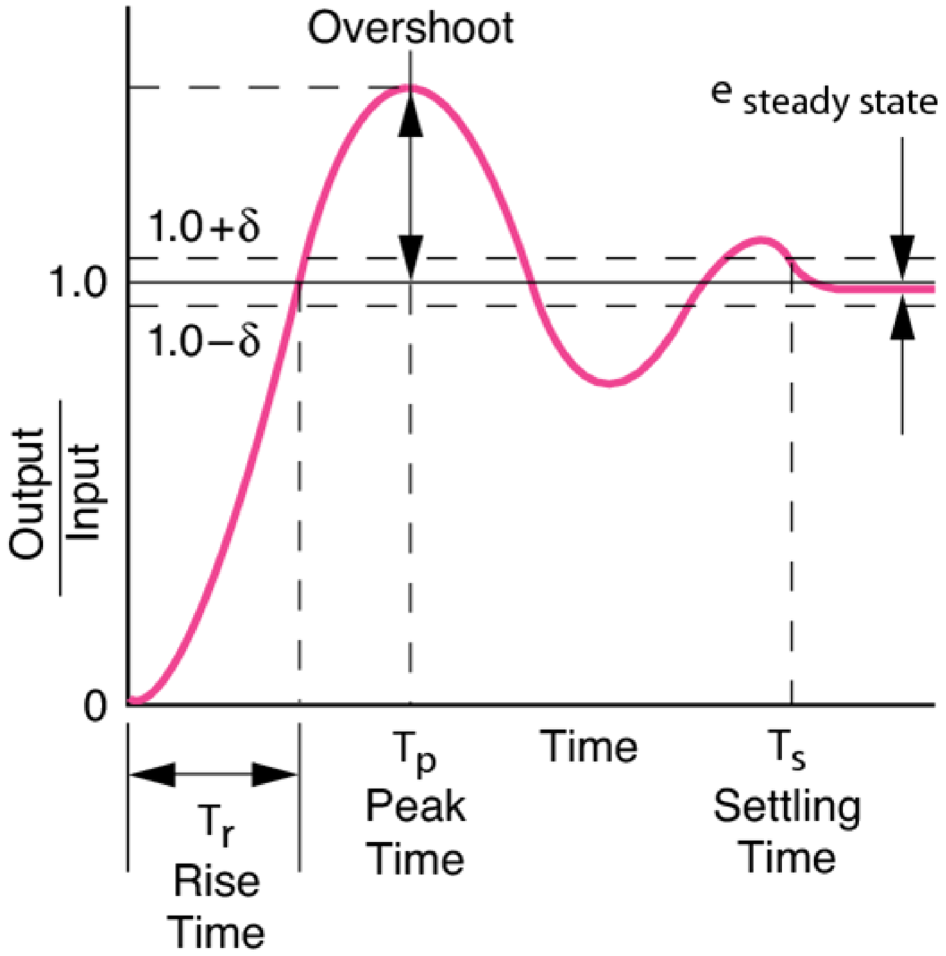
Response on a step function



Second order systems - characteristics

- Overshoot: Highest amplitude above steady state:
  $M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$;
- Rise Time: Time needed to reach the steady state for the first time. $t_r = \frac{1.8}{\omega_n}$;
- Peak Time: Time to reach overshoot.
  $t_p = \frac{\pi}{\omega_d}$;
- Settling Time: Time needed to approximate the steady state: $t_s = \frac{4.6}{\zeta\omega_n}$. Important note: this formulas can only be used when $0 < \zeta < 1$!

Second order systems - characteristics



Example

Given: $\delta = \frac{0.02}{\sqrt{1-\zeta^2}}$

We find a settling time of:

$$e^{-\zeta\omega_n t_s} < 0.02$$

$$t_s = \frac{4}{\omega_n \zeta}$$

Second order systems - resonance

The resonance frequency is the frequency at which the systems output has a larger amplitude than at other frequencies. This happens when underdamped functions oscillate at a greater magnitude than the input. An input with this frequency can sometime have catastrophic effects.

A different view on the Tacoma bridge disaster:
`https://www.youtube.com/watch?v=6ai2QFxStxo`

In fact the collapse was a result of a number of effects like Aerodynamic flutter and vortices. Read the full article here:
`http://www.ketchum.org/billah/Billah-Scanlan.pdf`

Second order systems - resonance

The resonance frequency is: $\omega_r = \omega_n \sqrt{1 - \zeta^2}$.

Systems with a damping > 0.707 do not resonate. The resonance frequency and the natural frequency are equal when a system has no damping.

Another phenomenon with bridges and resonance is that many people marching with the same rhythm can cause a bridge to start resonating like the Angers bridge in 1850. A more recent example is the Millennium bridge in London which started resonating.

Second order systems - damping

When we want a system with no resonance, we choose one with damping < 0.707. This means a pole between $135°$ and $225°$:

$$\arctan\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = +135°$$

We mostly want a short settling time ($<$ 4s). This results in another restriction on the poles of the system:

$$\tau_n = \frac{4}{\omega\zeta} < 4\text{s}$$

$$\omega_n\zeta > 1$$

Second order systems - damping