Continuous-time systems 1

March 2, 2015

Outline

Linear differential equations: definitions 1/2

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The **order of a LDE** is the index of the highest derivative of y.

Linear differential equations: definitions 2/2

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- if $A_{0:n}(t)$ are constants (ie. not functions of time), the LDE is said to have **constant coefficients**

Example: radioactive decay 1/2

Let $N(t)$ be the number of radioactive atoms at time t, then:

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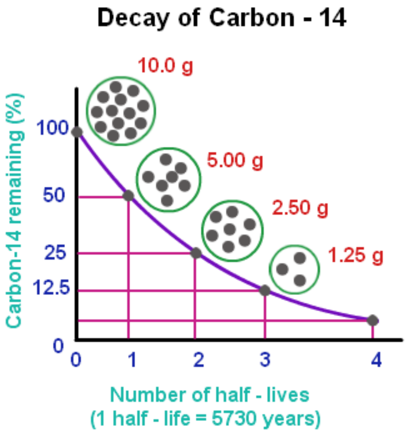

Solving homogeneous LDEs with constant coefficients 1/3

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Dividing by e^{zt} yields the nth order **characteristic polynomial**:

$$F(z) = \sum_{i=0}^{n} A_i z^{n-i} = 0.$$

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The specific linear combination depends on initial conditions.

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These zeros correspond to the following basis functions t:

$$e^{jt}, \quad e^{-jt}, \quad e^t, \quad te^t.$$

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The Laplace transform

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The final value theorem states $f(\infty) = \lim_{s\to 0} sF(s)$,
if all poles of $sF(s)$ are in the left half plane (ie. real part < 0).

Important properties of the Laplace transform

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with $u(t) = \int_{\infty}^{t} \delta(t)dt$ (Heaviside) and $\delta(t)$ the Dirac delta.

Inverse Laplace transform

The inverse Laplace transform converts s-domain to time domain:

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Practically, the inverse Laplace transform takes two steps:

1. write $F(s)$ in terms of partial fractions
2. transform each term in the partial fraction based on tables of s/t-domain pairs (course notes p 4.32-4.33)

Outline

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The Laplace transform can be used to solve LDEs with given initial conditions (the previous approach gave us the basis functions).

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Via induction, the Laplace transform of the nth order derivative:

$$\mathcal{L}\{f^{(n)}\} = s^nF(s) - \sum_{i=1}^{n} s^{n-i} f^{(n-i)}(0)$$

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Via the linearity of the Laplace transform:

$$\sum_{i=0}^{n} A_i \mathcal{L}\{y^{(n-i)}(t)\} = \mathcal{L}\{f(t)\}$$

Solving LDEs with the Laplace transform 3/3

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Solving LDEs with the Laplace transform 3/3

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Expanding Eq. (2) into (1) yields:

$$Y(s)\sum_{i=0}^{n} A_i s^i - \sum_{i=1}^{n}\sum_{j=1}^{i} A_i s^{i-j} y^{j-1}(0) = F(s)$$

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The solution in the time domain is obtained via the inverse Laplace transform: $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.