Discretization and reconstruction of signals

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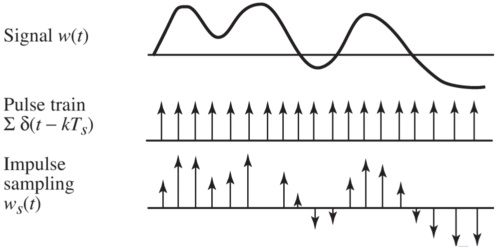
Discretization and reconstruction of signals

The use of digital logic and computers to calculate a control action for a continuous system introduces the operation of sampling.

Samples are taken from the continuous signals and used in the computer to calculate the controls to be applied.

The role of sampling and the conversion from continuous to discrete and vice versa are important to the understanding of the complete response of digital control.



Outline

Analysis of the sample and hold

To get samples of a continuous signal, we use an analog-to-digital converter. The conversion always takes a non-zero time.

To give the computer an accurate representation of the signal exactly at the sampling instants kT, the converter is preceded by a sample-and-hold circuit.
The sample-and-hold will take the impulses that are produced by the mathematical sampler and produce the piecewise constant output of the device.

Sampling operation

The sampling operation is represented by impulse modulation. Its role is to give a mathematical representation of taking periodic samples from $r(t)$ to produce $r(kT)$.

The sampler takes as input $r(t)$ and returns as output $r^*(t) = \sum_{k=-\infty}^{\infty} r(t)\delta(t - kT)$.

The Laplace transform of $r^*(t)$ can be computed as

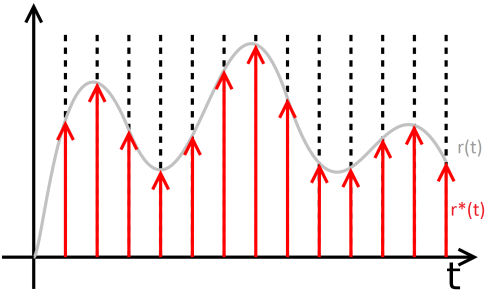
$$\mathcal{L}\{r^*(t)\} = \int_{-\infty}^{\infty} r^*(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} r(\tau)\delta(\tau - kT)e^{-s\tau}d\tau$$

Sampling operation

Using $\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)$ we obtain

$$\mathcal{L}\{r^*(t)\} = R^*(s) = \sum_{k=-\infty}^{\infty} r(kT)e^{-skT}$$

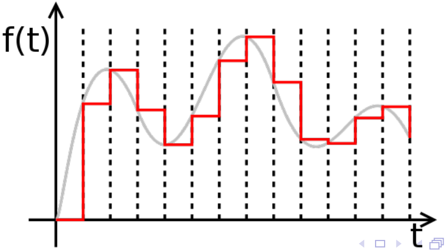
If the signal r(t) is shifted a small amount, then different samples will be selected by the sampling process for the output, proving that sampling is not a time-invariant process.



Hold operation

The hold operation is represented as a linear filter. It is defined by means whereby the impulses are extrapolated to the piecewise constant signal $r_h(t) = r(kT)$ with $kT \leq t < (k+1)T$.

A general technique is to use a polynomial fit to the past samples. If the extrapolation is done by a constant (a zero-order polynomial), then the extrapolator is called a zero-order hold and its transfer function is denoted $ZOH(s)$.



Zero-order hold

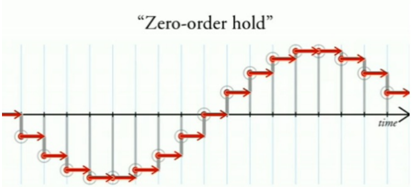
We compute the transfer function as the transform of its impulse response.

If $r^*(t) = \delta(t)$ then $r_h(t)$ is a pulse of height 1 and duration T:

$$r_h(t) = 1(t) - 1(t-T)$$

with Laplace transform

$$ZOH(s) = \mathcal{L}\{p(t)\} = \int_0^\infty [1(t) - 1(t-T)]e^{-s\tau}d\tau = \frac{1-e^{-sT}}{s}$$



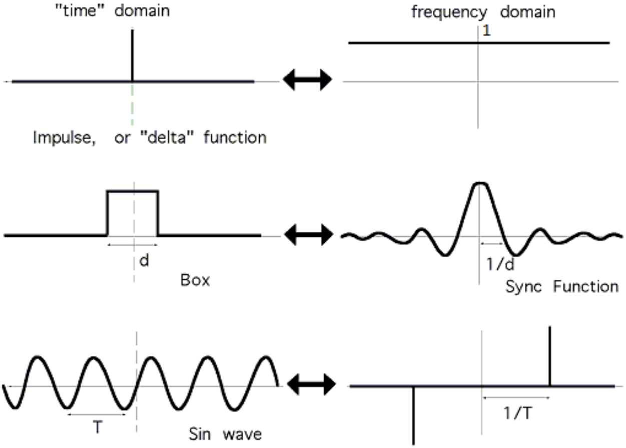
Outline

Fourier transform

$$f(t) \xrightarrow[F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt]{} F(j\omega)$$

$$f(t) \xleftarrow[f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega]{} F(j\omega)$$

Fourier transform



Fourier transform: properties

- **Linearity**

$$\begin{cases} f_1(t) \leftrightarrow F_1(j\omega) \\ f_2(t) \leftrightarrow F_2(j\omega) \end{cases} \quad \Rightarrow \quad af_1(t) + bf_2(t) \leftrightarrow aF_1(j\omega) + bF_2(j\omega)$$

- **Time-scaling**

$$f(at) \leftrightarrow \left(\frac{1}{|a|}\right) F\left(\frac{j\omega}{a}\right)$$

- **Translation/Time-shifting**

$$f(t - t_0) \leftrightarrow e^{-j\omega t_0} F(j\omega)$$

- **Modulation/Frequency-shifting**

$$e^{j\omega_0 t} f(t) \leftrightarrow F(j(\omega - \omega_0))$$

Fourier transform: properties

- **Reciprocity**

  $$F(-jt) \leftrightarrow 2\pi f(\omega)$$

- **Derivative in t**

  $$\frac{df(t)}{dt} \leftrightarrow j\omega F(j\omega) \qquad \frac{d^n f(t)}{dt^n} \leftrightarrow (j\omega)^n F(j\omega)$$

- **Derivative in ω**

  $$(-jt)^n f(t) \leftrightarrow \frac{d^n F(j\omega)}{d\omega^n} \qquad \frac{f(t)}{-jt} \leftrightarrow \int_{-\infty}^{\infty} F(j\Omega) d\Omega \; if \; f(0) = 0$$

- **Convolution**

  $$y(t) = h(t) * u(t) \leftrightarrow Y(j\omega) = H(j\omega)U(j\omega)$$
  $$v(t) = h(t)u(t) \leftrightarrow V(j\omega) = \frac{1}{2\pi} H(j\omega) * U(j\omega)$$

Outline

Spectrum of a sampled signal

$r^*(t)$ is the product of $r(t)$ and a train of impulses. The latter series is periodic and can be represented by a Fourier series

$$\sum_{k=-\infty}^{\infty} \delta(t-kT) = \sum_{n=-\infty}^{\infty} C_n e^{j(2\pi n/T)} t,$$

where the Fourier coefficients C_n are given by:

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} \delta(t-kT) e^{-jn(2\pi t/T)} dt.$$

Spectrum of a sampled signal

The only term in the sum of impulses that is in the range of the integral is the $\delta(t)$ at the origin, so the integral reduces to

$$C_n = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jn(2\pi t/T)} dt = \frac{1}{T}$$

We derived the Fourier series of the sum of impulses

$$\sum_{k=-\infty}^{\infty} \delta(t - kT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{j(2\pi n/T)t}$$

We define $\omega_s = \frac{2\pi}{T}$ as the sampling frequency (rad/s).

Spectrum of a sampled signal

We take the Laplace transform of the output of the sampler,

$$\begin{aligned} R^*(s) &= \int_{-\infty}^{\infty} r(t)\Big\{\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}\Big\}e^{-st}\,dt \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} r(t)e^{jn\omega_s t}e^{-st}\,dt \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} r(t)e^{-(s-jn\omega_s)t}\,dt. \end{aligned} \tag{1}$$

Since the integral is the Laplace transform of $r(t)$ with only a change of variable where the frequency goes, the result can be written as:

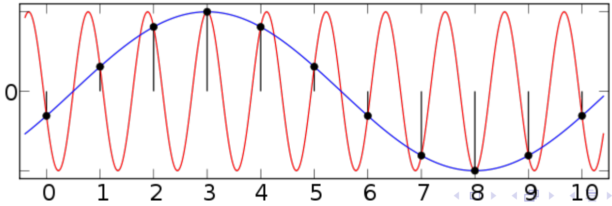
$$R^*(s) = \tfrac{1}{T}\textstyle\sum_{n=-\infty}^{\infty} R(s - jn\omega_s).$$

Outline

Aliasing

Aliasing is an effect that causes different signals to become indistinguishable when sampled. Frequencies that are too high to be sampled are folded onto lower frequencies. We cannot distinguish between them based on their samples alone.

Take for example the image below.
The red sine wave is being sampled at just over its bandwidth, however the blue sine wave will be recreated as it also fits all data points and is within the expected bandwidth.



Aliasing

As a direct result of the sampling operation, when data are sampled at a frequency $\frac{2\pi}{T}$, the total harmonic content at a given frequency ω_1 is to be found not only from the original signal at ω_1, but also from all those frequencies that are aliases of ω_1, namely $\omega_1 + n2\pi/T = \omega_1 + n\omega_s$.

The errors caused by aliasing can be very severe if a substantial quantity of high-frequency components is contained in the signal to be sampled.

To minimize this error, the sampling operation is preceded by a low-pass anti-aliasing filter that will remove all spectral content above the half-sampling frequency (π/T).