The design of lead and lag compensators

September 2, 2015

Outline

1. Introduction
2. General design
3. Lead copensations
4. Lag compensation
5. Lag-lead compensators
6. Parallel compensation

Definitions

The main objective of this chapter: design and compensation of single-input-sigle-output linear time-invariant control systems.

- Compensation: the modification of the system dynamics to satisfy the given specifications.
- Specifications (transient response and steady-state requirements): given before the design.
- Design by root-locus method: making a new root locus by adding poles and zeros to the system's open-loop transfer function.
- Compensator: an other system inserted in parallel or in cascade with the system for the purpose of satisfying the specifications of the original system (e.g. lead, lag, lag-lead compensator's or PID controllers).

Compensators

A sinusoidal input is applied to the input of a network. We got a:

- lead network: if the steady-state output has a phase lead.
- lag network: if the steady-state output has a phase lag.
- lag-lead network: if we have phase lag and phase lead in the output but in different frequentcy regions (lag when the input has low frequency and lead in high frequentcy).

The amount of lag/lead is a function of the frequency.

A compensator with characteristic of a lead network, lag network, or lag-lead network is called a lead compensator, lag compensator, or lag-lead compensator.

Remark: with trail and error we find the optimal compensator

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Controllers

Scetching the problem:

1. We got a plant that not achieve all the specifications (and we cannot change the parameters of the original system).
2. We have to change other parameters such that the system will achieve all the specifications.
3. These other parameters can be changed by changing the root loci of the closed-loop system.
4. So we search the root loci of a compensator such that the overall system achieve the specifications.
5. We let the original system interact (cascade or parallel) with the compensator.

Remark: we discuss only continu time systems.

Root locus approach

The method:
Given: the root loci of the open-loop system (this are the parameters).
Now, the method consist of graphical determining the root loci of the closed-loop system.

Example: We want a certain gain of a system:

- The system is not stable at that gain.
- We have to change the root loci (and make the system stable) by designing a compensator
- Now, we got different root loci and the system is stable at the certain gain. So, the specifications are achieved.

In **general**: we want to have the root loci of the system on the good locations.

Addition of poles/zeros

Addition of poles:

- Pulls the root locus to the right.
- Lowers the system's relative stability.
- Slows down the setting of the response.

Addition of zeros:

- Pulls the root locus to the left.
- Increase the system's relative stability.
- Speeds up the settling of the response.
- Speeds up the transient response.
- Increases the anticipation of the system.

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Lead compensators

There are 3 ways to make a lead compensator:

1. Electronic networks using operational amplifiers.
2. Electrical RC networks.
3. Mechanical spring-dashpot systems.

When to use the root locus approach for lead compensators:
$\rightarrow$ When the specifications are given in terms of time-domain quantities. Examples:

- damping ratio
- rise time
- setting time
- maximum overshoot
- undamped natural frequency

Lead compensators

Steps to make a lead compensator (in cascade with the original system):

1. Determine the locations of the desired dominant poles (from the specifications).
2. Draw the root locus of the uncompensated system. If we change the gain, then the poles will change. If we can achieve the good locations of the poles on this way, then there is no need for a lead compensator. Else, if we cannot achieve the good locations, then we have to make a lead compensator.
3. Assume the lead compensator of this form:
   $C(s) = K_c \alpha \frac{Ts+1}{\alpha Ts+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$ with $0 < \alpha < 1$.

Lead compensators

$$C(s) = K_c\alpha\frac{Ts+1}{\alpha Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \text{ with } 0 < \alpha < 1$$

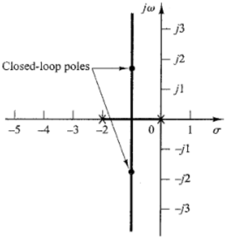
- Determining α and T:
  α and T are determined by the angle ϕ. This is the angle $\angle(C(s))$ must be equal to the difference between the angle of the desired pole and the angle of the original transfer function.
  Result: the closed-loop pole has the same angle as the desired angle (angle of the desired pole). (And by determine T and α we determine the poles and zeros of the compensator.)
- Determine K_c (=the open-loop gain) from the magnitude specifications.
  Result: the closed-loop pole has the desired magnitude.

Remark: if there is some freedom about the parameter α, then take α as high as possible.

Example 1: Lead compensators

Given:

- Feed-forward transfer function: $G(s) = \frac{4}{s(s+2)}$
- Root locus plot of the uncompensated open-loop system:



Asked:

- Undamped frequency: $w_n = 4$ rad/s
- Don't change the damping ratio

Example 1: Lead compensators

Solution:

1. Find the closed-loop poles:
   - The closed-loop transfer function: $\frac{C(s)}{R(s)} = \frac{4}{s^2+2s+4}$ with poles $s = -1 \pm \sqrt{3}j$
   - Damping ratio: $\zeta = 0.5$
   - undamped natural frequency: 2 rad/s
   - Static velocity constant: $K_v = 2s^{-1}$

$$\begin{cases} \zeta \text{ must be the same} \rightarrow \text{let the angle } (60^\circ) \text{ of the poles constant} \\ \text{Undamped frequency} = 4\text{rad/s} \rightarrow \text{determine magnitude} \end{cases}$$

The desired poles are: $s = -2 \pm j2\sqrt{3}$

Note: in some cases, the desired poles can be obtained by adjustment of the gain. But in this case not, so we have to

Example 1: Lead compensators

Solution:

2. Find the sum of the angles at the desired location of one of the dominant closed-loop poles with the open-loop poles and zeros of the original system, and determine the necessary angle ϕ to be added so that the total sum of the angles is equal to $\pm 180^\circ(2k+1)$. The lead compensator must contribute this angle ϕ.
3. Transfer function of the open-loop system:

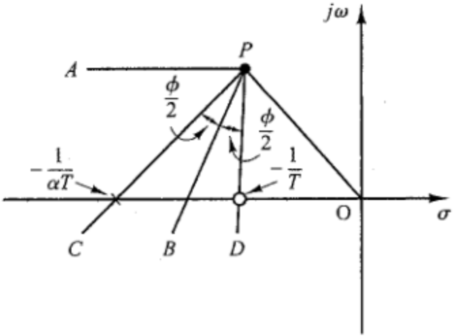
$$C(s)G(s) = (K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}})G(s) \text{ with } 0 < \alpha < 1$$

Remark: there are many solutions for T and α.

Example 1: Lead compensators

Solution:

4. Determine the pole and zero of the compensator (there are many possibilities): we use an algorithm to find the the pole and zero and to get an α as big as possible:



Example 1: Lead compensators

Algorithm:

1. Draw a horizontal line trough the desired dominant closed-loop pole P $\rightarrow$ PA
2. Connect the origin with P $\rightarrow$ PO
3. Bisect the two previous lines $\rightarrow$ PB
4. Draw two lines that make an angle $\phi/2$ with the line PB $\rightarrow$ PC, PD
5. The intersections of the negative real axis and the lines PC and PD are the zero and pole of the compensator.

Example 1: Lead compensators

Solution

5. We determine the angle ϕ:

$$\angle\left(\frac{4}{s(s+2)}\right)\Big|_{s=-2+j2\sqrt{3}} = -210^\circ$$

And so we find $\phi = 30^\circ$. From the algorithm we get: zero: -2.9 and pole: -5.4.

$$T = \frac{1}{2.9} = 0.345, \ \alpha T = \frac{1}{5.4} = 0.185$$

So, we find α: $\alpha = \frac{1}{5.4T} = 0.537$.

Example 1: Lead compensators

Solution:

5. Determine K_c from the magnitude condition.

Example 1: Lead compensators

Solution:

5. The open-loop transfer function of the compensated system becomes:
$C(s)G(s) = K_c \frac{s+2.9}{s+5.4} \frac{4}{s(s+2)} = \frac{K(s+2.9)}{s(s+2)(s+5.4)}$ with K=4K_c.

The magnitude condition:

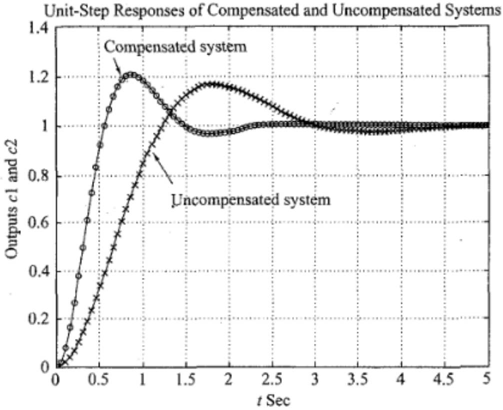
$$|\frac{K(s+2.9)}{s(s+2)(s+5.4)}|_{s=-2+j2\sqrt{3}} = 1 \Rightarrow K = 18.7 \Rightarrow K_c = \frac{K}{4} = 4.68$$

And we get the transfer function of the lead compensator:

$$C(s) = 4.68 \frac{s+2.9}{s+5.4}$$

Example 1: Lead compensators

Solution:



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Lag compensators

Problem: A system that satisfy the desired transient response but don't satisfy the steady-state.
$\rightarrow$ We got to compensate the system in such a way that it also satisfy the steady-state specifications (we have to make a lag compensator in cascade with the original system).

Concrete:

- Changing the steady-state by increasing the open-loop gain.
- Untouch the transient response by untouch the root locus in the neighbourhood of dominant closed-loop poles:
  a) poles and zeros close to each other;
  b) poles and zeros close to the origin;
  c) angle of the compensator must be small, $\angle(C(s)) < 5^\circ$.

Lag compensators

$C(s) = K_c\beta\frac{Ts+1}{\beta Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$ with $\beta > 1$

- Choose poles and zeros close together
- Choose K_c close to 1 (if K_c is exact 1, the transient response won't change)
- Take β as large as possible
- Take T as large as possible

Remark: by the choice of β and T, we have to take account of the reality (the physical realisation), so there is a limitation on the values.

The **downside** of the lag compensator: the settling time will increase because the pole and zero of the closed-loop system are close to the origin.

Lag compensators

Steps to make a lag compensator (in cascade with the original system):

1. Draw the root locus of the uncompensated open loop system and search the dominant closed loop poles on the root-locus (from the specifications).
2. The lag compensator is $C(s) = K_c\beta\frac{Ts+1}{\beta Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$.
3. Calculate the static error constant.
4. Calculate the new static error constant if we achieve all the specifications.
5. From this difference we make the lag compensator. Now, we can see the pole and the zero of the compensator.

Remark: we assume that the lag compensator achieved the transient response specifications. If not: make a

Lag compensators

6. Draw the root locus of the compensated closed-loop system. Determine the locations of the dominant closed loop poles on the root-locus.
7. Adjust K_c from the magnitude conditions such that the closed-loop poles lie at the desired location. $(K_c \approx 1)$

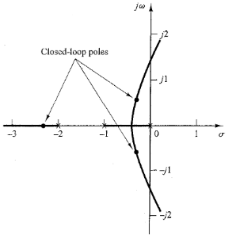
Some notes:

- The ratio of the value of gain required in the specifications and the gain found in the uncompensated system is equal to the ratio between the distance of the zero from the origin and that of the pole from the origin.
- If the angle of the lag compensator is very small, then the original root loci and the new root loci are almost equal.
- Sometimes can be used both a lead or a lag compensator.

Example 2: Lead compensators

Given:

- Feed-forward transfer function: $G(s) = \frac{1.06}{s(s+1)(s+2)}$
- Root locus plot of the uncompensated open-loop system:



Asked:

- Increase of the static velocity error constant K_v to $5s^{-1}$
- Don't change the dominant closed-loop poles

Example 2: Lead compensators

Solution:

1. General calculations:
   - The transfer function of the closed-loop system:

$$\frac{Y(s)}{R(s)} = \frac{1.06}{s(s+1)(s+2)+1.06}$$

   with poles: $s = -0.3307 \pm j0.5864$
   - Damping ratio: $\zeta = 0.491$
   - Undamped natural frequency: 0.673 rad/s
   - Static velocity error constant: $K_v = 0.53s^{-1}$

Example 2: Lead compensators

Solution:

2. We assume a lag compensator C(s):

$$C(s) = K_c \frac{s + \frac{1}{T}}{s + \frac{1}{T\beta}}$$

3. To achieve the specifications, we choose:
   - $\beta = 10$
   - zero: s=-0.05
   - pole: s=-0.005
4. C(s) becomes:

$$C(s) = K_c \frac{s + 0.05}{s + 0.005}$$

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Lag-lead compensators

General:

Lead compensators:

- speeds up the response;
- increase stability of the system.

Lag compensators:

- improves steady-state accuracy;
- reduces the speed of the response.

Lag-lead compensator:

- If both transient response and steady-state must be improved.
- When 1 global component is more economical then both a lead and a lag component.

The lag-lead compensator has the advantages of both compensator. It has 2 poles and zeros, so the system has order 2.

Lag-lead compensators

The lag-lead compensator C(s) has this form:
$C(s) = K_c \frac{\beta(T_1 s+1)(T_2 s+1)}{\gamma(\frac{T_1}{\gamma}s+1)(\beta T_2 s+1)} = K_c(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}})(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}})$ with $\beta > 1$ and $\gamma > 1$. Notes:

- The value βT_2 may not be to large, it must be fysical realizable.
- There are two cases of this type compensator: $\gamma \neq \beta$ or $\gamma = \beta$.

Lag-lead compensators

Case 1:$\gamma \neq \beta$, the design process is a combination of the design of the lead compensator and that of the lag compensator:

1. Determine the location of the closed-loop poles (from the specifications)
2. The lead part of the compensator must contribute the angle ϕ: $\phi = \angle$(desired pole)$-\angle$(uncompensated open-loop system)
3. - Take T_2 as large as possible.
   - Determine T_1 and γ such that: $\angle(\frac{s_1+\frac{1}{T_1}}{s_1+\frac{\gamma}{T_1}}) = \phi$
   - Determine K_c from the condition: $|K_c\frac{s_1+\frac{1}{T_1}}{s_1+\frac{\gamma}{T_1}}G(s)| = 1$ with G(s) the open loop transfer function.

Lag-lead compensators

Case 1: $\gamma \neq \beta$, the design process is a combination of the design of the lead compensator and that of the lag compensator:

4. Determine β if K_v is given:

$$K_v = \lim_{s \to 0} sC(s)G(s) = \lim_{s \to 0} sK_c\left(\frac{s + \frac{1}{T_1}}{s + \frac{\gamma}{T_1}}\right)\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}}\right)G(s)$$

$$= \lim_{s \to 0} sK_c G(s)\frac{\beta}{\gamma}$$

5. Determine T_2 from:
$\left|\frac{s_1 + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}}\right| \approx 1$ and $-5^\circ < \angle\left(\frac{s + \frac{1}{T_2}}{s_1 + \frac{1}{\beta T_2}}\right) < 0^\circ$

Lag-lead compensators

Case 2: $\gamma = \beta$:

1. Determine the location of the closed-loop poles (from the specifications)
2. If K_v is specified, determine K_c from:

$$K_v = \lim_{s \to 0} sC(s)G(s) = \lim_{s \to 0} sK_c\left(\frac{s + \frac{1}{T_1}}{s + \frac{\beta}{T_1}}\right)\left(\frac{s + \frac{1}{T_2}}{s + \frac{1}{\beta T_2}}\right)G(s)$$

$$= \lim_{s \to 0} sK_c G(s)$$

3. Determine the deficiency ϕ, that must be contributed by the lead part of the lag-lead compensator, such that we get the dominant closed-loop poles on the desired location.

Lag-lead compensators

Case 2: $\gamma = \beta$:

3. Determine T_1 andβ from:
   $|K_c(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}})G(s_1)| = 1$ and $\angle(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}) = \phi$
4. Determine T_2 from:
   $|\frac{s_1+\frac{1}{T_2}}{s_1+\frac{1}{\beta T_2}}| \approx 1$ and $-5^\circ < \angle(\frac{s+\frac{1}{T_2}}{s_1+\frac{1}{\beta T_2}}) < 0^\circ$

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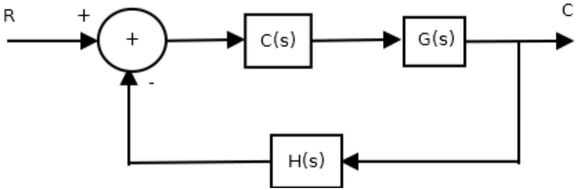
Serial compensation

$\rightarrow$ Up to now, we have only disused serial compensator's.

Derivation of the characteristic equation:
Serial:



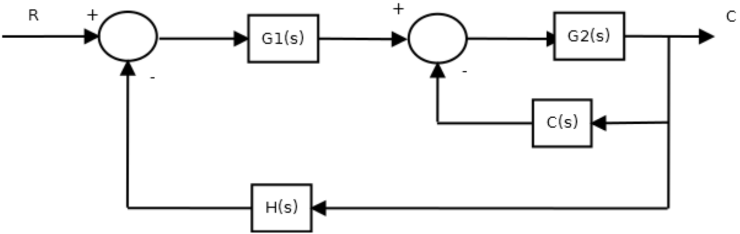
The closed-loop transfer function: $\frac{C}{R} = \frac{GC}{1+GCH}$.

The characteristic equation is: $1 + GCH = 0$

Parallel compensation

Derivation of the characteristic equation:

Parallel:



The closed-loop transfer function: $\frac{C}{R} = \frac{G_1 G_2}{1 + CG_2 + G_1 G_2 H}$.

The characteristic equation is: $1 + G_1 G_2 H + G_2 C = 0$

Parallel compensation

Derivation of the characteristic equation:

Parallel:

If we divide the characteristic equation by $1 + G_1 G_2 H$ we obtain:

$$1 + \frac{CG_2}{1 + G_1 G_2 H} = 0$$

Now, we define: $G_f = \frac{G_2}{1+G_1 G_2 H}$.

So, we become a characteristic equation: $1 + CG_f = 0$

$\rightarrow$ This has the same form as the serial characteristic equation.

So, the same methods can be applied.

Notes:

- A paralle compensator system is called a velocity feedback system.
- The compensator can be saw as a gain element.