

# Outline

- 1 Introduction
- 2 Main Approaches
  - Numerical Integration
  - Impulse Invariant Method
  - Zero-pole Equivalent
  - Hold Equivalent
- 3 Sampling Time
- 4 Discretization and MATLAB
  - Commands
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# Impulse-invariant method

## Practical rule

This method converts a continuous-time system into a discrete one by matching the impulse response using these transformations:

$H(s)$	$H(z), a = e^{bT_s}$
$\frac{c}{s-b}$	$\frac{T_s c}{1-az^{-1}}$
$\frac{c}{(s-b)^2}$	$T_s^2 \frac{caz^{-1}}{(1-az^{-1})^2}$
$\frac{c}{(s-b)^3}$	$\frac{T_s^3 caz^{-1}(1+az^{-1})}{2(1-az^{-1})^3}$
$\frac{c}{(s-b)^4}$	$\frac{T_s^4 caz^{-1}(1+4az^{-1}+a^2z^{-2})}{6(1-az^{-1})^4}$

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# Zero-pole equivalent

## General approach

The map  $z = e^{sT}$  is applied to the poles as well as to the zeros of the continuous system. The following rules must be followed:

- 1 If  $s = -a$  is a pole of  $H(s)$ , then  $z = e^{-aT}$ ;
- 2 All finite zeros  $s = -b$  are mapped by  $z = e^{-bT}$ ;
- 3 Zeros at  $\infty$  are mapped to  $z = -1$ ;
- 4 The DC gain is set such that  $\lim_{s \rightarrow 0} G(s) = \lim_{z \rightarrow 1} G_d(z)$ .

# Zero-pole equivalent

## Example

Given:

$$H(s) = \frac{s+1}{0.1s+1}$$

Pole at  $s = -10$  and zero at  $s = -1$ .

New discrete transfer function with equivalent poles and zeros:

$$H(z) = K \frac{z - e^{-T}}{z - e^{-10T}}$$

K is chosen so that  $|H(z)|_{z=1} = |H(s)|_{s=0} \rightarrow K = 4.150$

Using  $T = 0.25$ , this results in:

$$H(z) = 4.150 \frac{z - 0.7788}{z - 0.0821}$$

# Outline

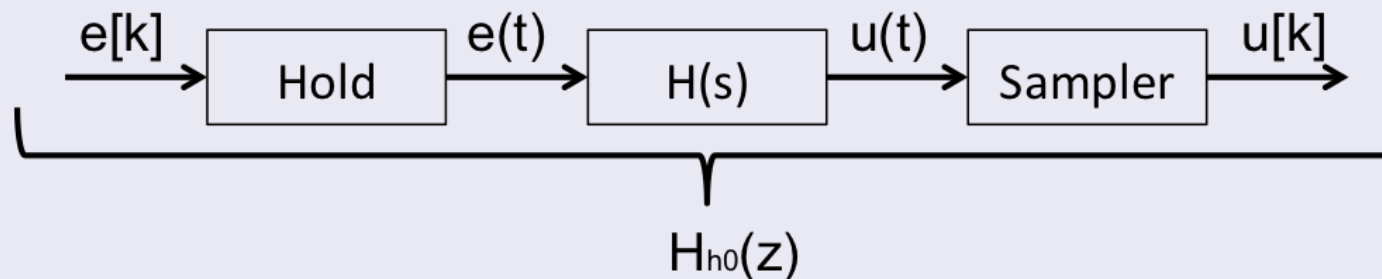
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# Hold equivalent

## General approach

This method uses a discrete system consisting of 3 subsystems, each with its own purpose.

- 1 Hold: approximating  $e(t)$  from the samples  $e[k]$
- 2  $H(s)$ : putting the  $e(t)$  through the given transfer function  $H(s)$  of the continuous system, resulting in  $u(t)$
- 3 Sampler: sampling  $u(t)$



There are many techniques for holding a sequence of samples.

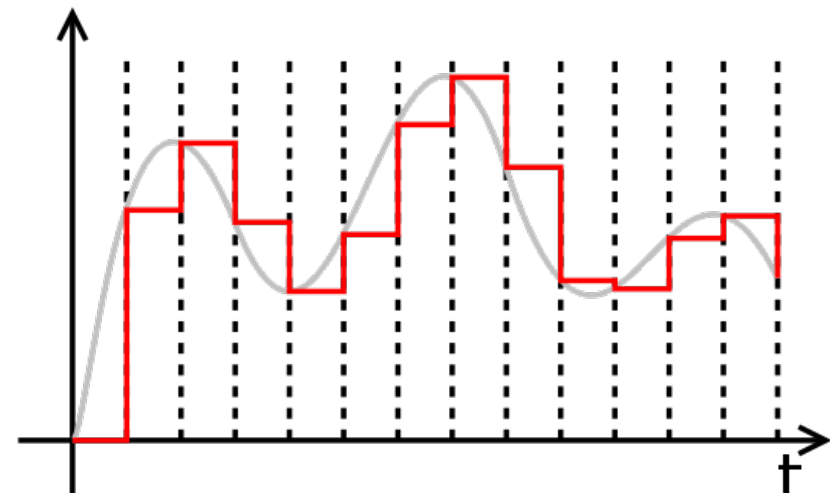
# Zero-order hold equivalent (ZOH)

## Practical rule

The zero-order hold equivalent transfer function  $H_{zoh}(z)$  can be found by computing the following:

$$H_{zoh}(z) = (1 - z^{-1}) \mathcal{Z} \left\{ \frac{H(s)}{s} \right\}$$

This is obtained by holding  $e(t)$  constant at  $e(k)$  over the interval  $kT_s$  to  $(k+1)T_s$  and  $\mathcal{Z}$ -transforming it.





# Zero-order hold equivalent

## Example

$$H(s) = \frac{0.1}{s+0.1}$$

Step 1: multiply by  $1/s$  and perform partial fraction expansion

$$\frac{H(s)}{s} = \frac{0.1}{s*(s+0.1)} = \frac{1}{s} - \frac{1}{s+0.1}$$

Step 2: perform  $\mathcal{Z}$ -transformation

$$\mathcal{Z}\left\{\frac{H(s)}{s}\right\} = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-0.1T}z^{-1}}$$

Step3: simplify and multiply by  $(1 - z^{-1})$

$$H_{zoh}(z) = \frac{1-e^{-0.1T}}{z-e^{-0.1T}}$$

# Zero-order hold equivalent

## Step Invariant Transformation

This rule is also called the step-invariant method because it matches the step response of the continuous and the discrete system. Next transformations may come in handy:

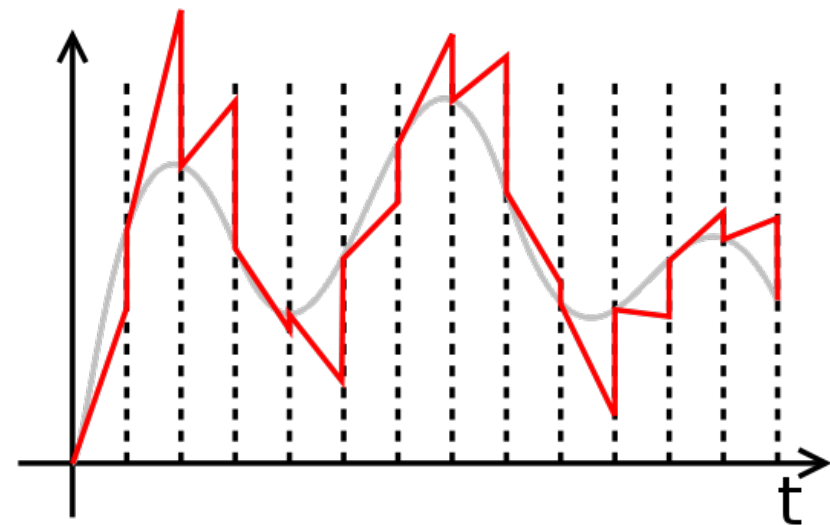
$f(t)$	$F(s)$	$f(k)$	$F(z)$
$u(t)$ , unit step	$\frac{1}{s}$	$u(k)$ , unit step	$\frac{z}{z-1}$
$tu(t)$	$\frac{1}{s^2}$	$kTu(k)$	$\frac{Tz}{(z-1)^2}$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$(e^{-aT})^k u(k)$	$\frac{z}{z-e^{-aT}}$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$kT(e^{-aT})^k u(k)$	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
$\sin(\omega t)u(t)$	$\frac{\omega}{s^2 + \omega^2}$	$\sin(k\omega T)u(k)$	$\frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$
$\cos(\omega t)u(t)$	$\frac{s}{s^2 + \omega^2}$	$\cos(k\omega T)u(k)$	$\frac{z(z - \cos \omega T)}{z^2 - 2z \cos \omega T + 1}$

# First-order hold equivalent

## Practical rule

The non-causal first-order hold equivalent transfer function  $H_{foh}(z)$  can be found by computing the following:

$$H_{foh}(z) = \frac{(z-1)^2}{T_z} \mathcal{Z} \left\{ \frac{H(s)}{s^2} \right\}$$



# First-order hold equivalent

## Example

$$H(s) = \frac{1}{s^2}$$

Step 1: multiply by  $1/s^2$  and perform partial fraction expansion

$$H(s) = \frac{1}{s^4}$$

Step 2: perform  $\mathcal{Z}$ -transformation

$$\mathcal{Z}\left\{\frac{H(s)}{s^2}\right\} = \frac{T^3}{6} \frac{(z^2+4z+1)z}{(z-1)^4}$$

Step 3: simplify and multiply by  $\frac{(z-1)^2}{Tz}$

$$H_{foh}(z) = \frac{T^2}{6} \frac{z^2+4z+1}{(z-1)^2}$$