Outline

1. Concept of Root Locus
2. How To Sketch the Root Locus
   - General Approach
   - Summary of the rules for sketching the root locus
3. Design criteria
4. Root Locus and MATLAB
   - Root Locus
   - SISOTOOL

Outline

General approach

Now look at the equation: $G(s) = -\frac{1}{K}$. If K is to be real and positive, $G(s)$ must be real and negative. This means that if we arrange G(s) in polar form as magnitude and phase, $G(s)$ must have the opposite phase of K in order to satisfy the equation above. We can thus define a root locus in terms of this phase condition:

Definition II

The root locus of $G(s)$ is the set of points in the s-plane where the phase of $G(s)$ is 180°.

General approach

Since the phase remains unchanged if an integral multiple of 360° is added, we can express Definition II as: $\angle G(s) = 180^\circ + 360^\circ l$, where l is any integer. While it is very difficult to solve a high-order polynomial, computing the phase is relatively easy.

The usual case is when K is real and positive; we call this case the **positive or** 180° **locus**. When K is real and negative, $G(s)$ must be real and positive of s to be on the locus. Therefore, the phase of $G(s)$ must be 0°. This special case is called a **negative or** 0° **locus**.

Although measuring the phase is easy, measuring the phase at every point in the s-plane is hardly practical. It would be better if there would exist some general guidelines we can use for determining where the root locus is situated.

Guidelines in practice

We will now sketch the root locus of a system with transfer function:

$$G(s) = \frac{1}{s[(s+4)^2+16]} \tag{5}$$

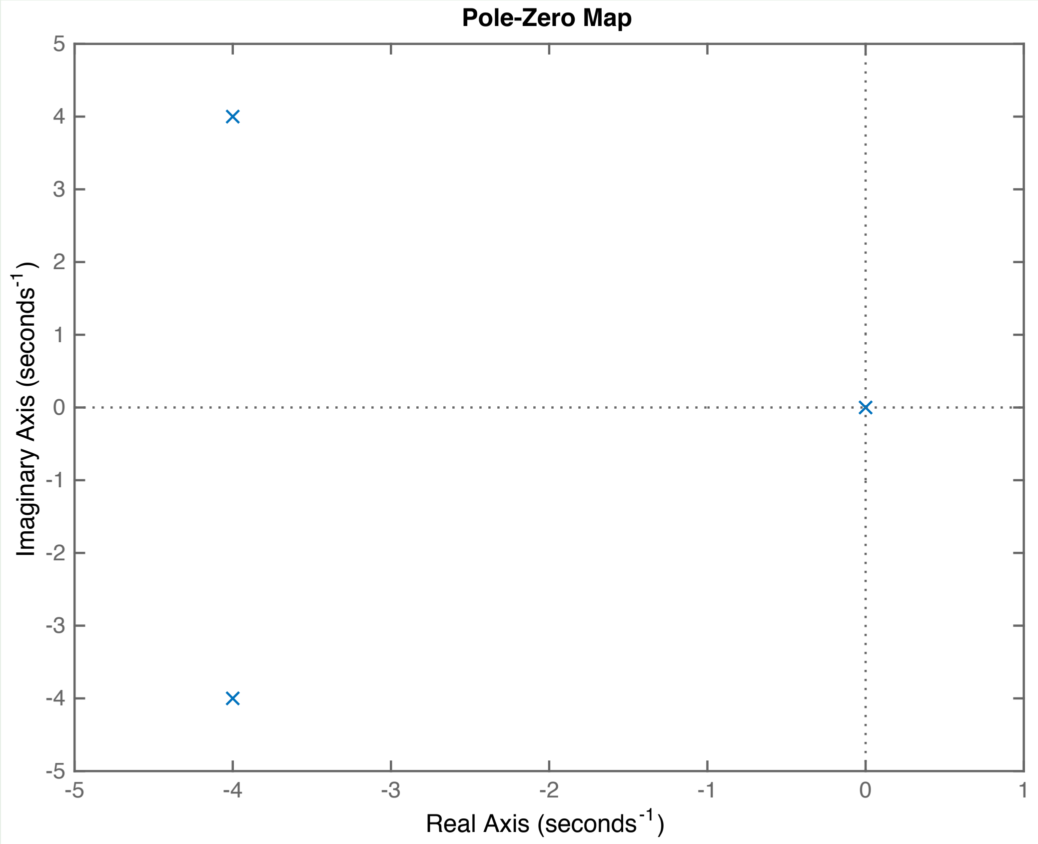
Using Definition II, we can check whether a point s_0 lies on the root locus for some value of K by checking if the following expression is valid:

$$\angle 1 - \angle s_0 - \angle[(s_0+4)^2+16] = 180^\circ + 360^\circ l \tag{6}$$

but as already mentioned, it is not practical to do this for every point. So we will now use the general guidelines to sketch the root locus.

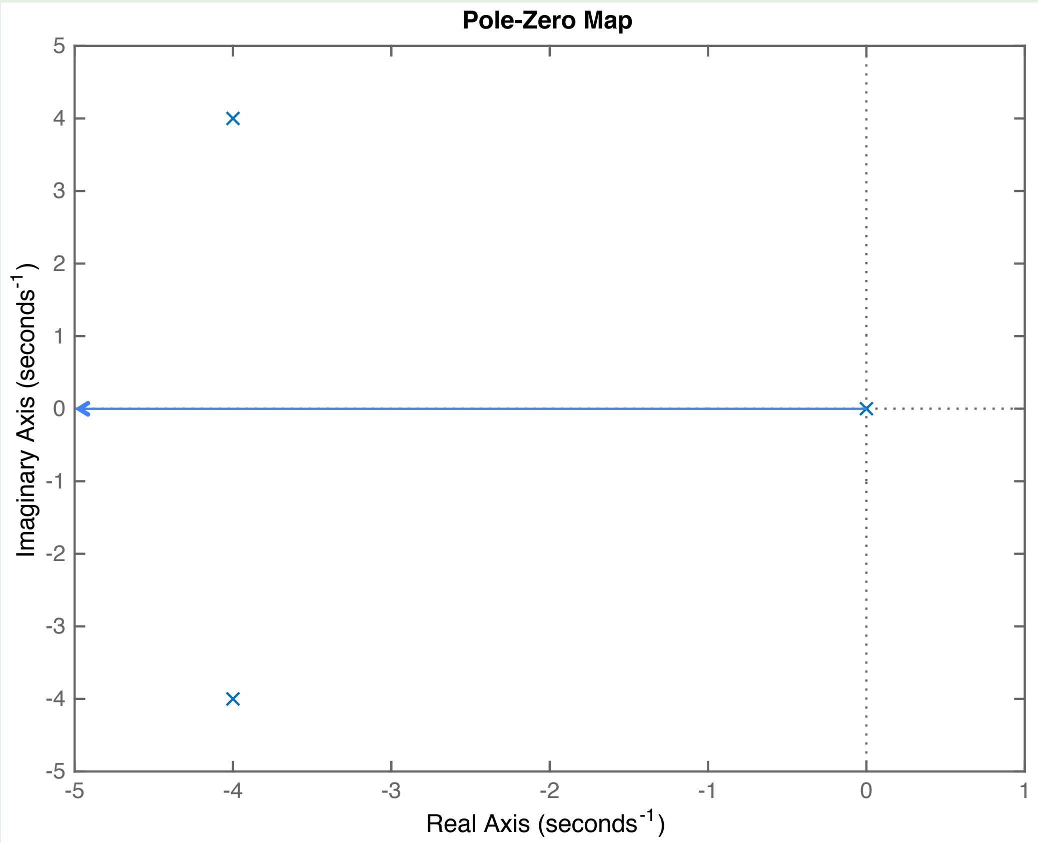
STEP 1: The open-loop $\times's$ and $\circ's$

Calculate and plot the open-loop poles and zero's:



STEP 2: Locus on the real axis

The portion of the real axis to the left of an odd number (counted from the right) of open loop poles and zeros are part of the locus:



STEP 3: Asymptotes for large values of K

As K approaches ∞, the equation $G(s) = -\frac{1}{K}$ can be satisfied only if $G(s) = 0$ where $G(s) = \frac{b(s)}{a(s)}$.

We can now substitute this in $1 + KG(s) = 0$, resulting in the following equation:

$$1 + K\frac{b(s)}{a(s)} = 1 + K\frac{s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n} = 0 \tag{7}$$

Since $n < m$, $G(s) = 0$ can occur when:

- $b(s) = 0$;
- $s \to \infty$.

STEP 3: Asymptotes for large values of K

We now look closer to the second condition: $G(s) = 0$ when $s \to \infty$.

For very large values of s, the highest-order power of s in Eq.(8) predominates. We can divide $a(s)$ by $b(s)$ and match the dominant two terms (highest powers in s) to the expansion $(s-\alpha)^{n-m}$, resulting in the following approximation:

$$1 + K\frac{1}{(s-\alpha)^{n-m}} \tag{8}$$

We need to find the locus for the asymptotic system (Eq.(8)) and α.

STEP 3: Asymptotes for large values of K

To find the locus, we choose $s_0 = Re^{j\phi}$ for some large value of R and some variable ϕ.

Since all poles of Eq.(9) are in the same place, the angle of its transfer function is 180° if all $n - m$ angles (ϕ_l) sum to 180°. Therefore, ϕ_l is given by:

$$\phi_l = \frac{180^\circ + 360^\circ l}{n - m}, l = 1, 2, ..., n - m \tag{9}$$

For our system: $n - m = 3$, thus $\phi_1, \phi_2, \phi_3 = 60^\circ, 180^\circ, 300^\circ$. All the lines of the asymptotic locus come from $s_0 = \alpha$.

STEP 3: Asymptotes for large values of K

To determine α we use a simple property of polynomials. Suppose we have a monic polynomial with coefficients a_i and roots p_i and we equate the polynomial form with the factored form:

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n = (s - p_1)(s - p_2)\ldots(s - p_n).$$

If we multiply out the factors on the right side of this equation, we can see that the coefficient of s^{n-1} is $-p_1 - p_2 - \ldots - p_n$. On the left side of the equation we see that this term is a_1.

Consequently $a_1 = -\Sigma p_i$, the coefficient of the second-highest term in a monic polynomial is the negative sum of its roots (the poles of $G(s)$). The same can be concluded for $b_1 = -\Sigma z_i$.

STEP 3: Asymptotes for large values of K

Applying these results in the closed-loop characteristic polynomial, leads to:

$$s^n + a_1 s^{n-1} + \ldots + a_n + K(s^m + b_1 s^{m-1} + \ldots + b_m) = 0.$$

The negative of the sum of the poles is the coefficient of s^{n-1} and is independent of K if $m < n - 1$. However, since this is the closed-loop characteristic equation, this coefficient is the negative of the sum of the roots of the closed-loop system Σr_i, hence

- the center point of the roots does not change with K if $m < n - 1$;
- $-\Sigma r_i = -\Sigma p_i$.

STEP 3: Asymptotes for large values of K

For large values of K, m of the roots r_i are approximately equal to the zeros z_i and $n-m$ of the roots are from the asymptotic $\frac{1}{(s-\alpha)^{n-m}}$ system, whose poles add up to $(n-m)\alpha$.

Combining these results we conclude that the sum of all the roots equals the sum of those roots that go to infinity plus the sum of those roots that go to the zeros of $G(s)$:

$$+\Sigma r_i = +(n-m)\alpha + \Sigma z_i = +\Sigma p_i.$$

Solving for α we get

$$\alpha = \frac{\Sigma p_i - \Sigma z_i}{n-m}.$$

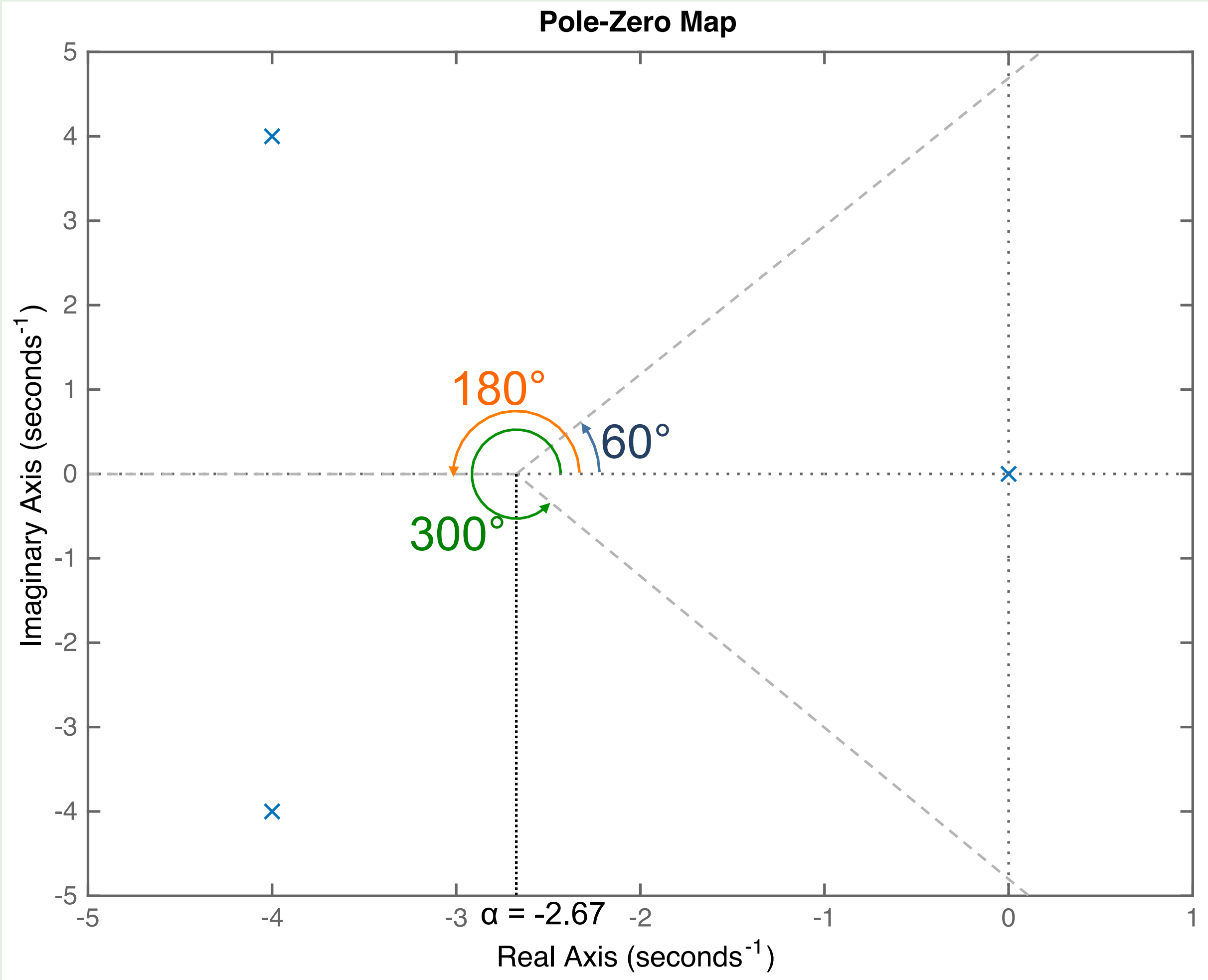
STEP 3: Asymptotes for large values of K

Complex poles and zeros always occur in complex conjugate pairs, consequently in the sums Σp_i and Σz_i the imaginary parts will always add to zero.

For our system: $\alpha = \frac{-4-4+0}{3-0} = -2.67$.

We can now use the results for ϕ_1, ϕ_2, ϕ_3 and α to draw the asymptotes.

STEP 3: Asymptotes for large values of K

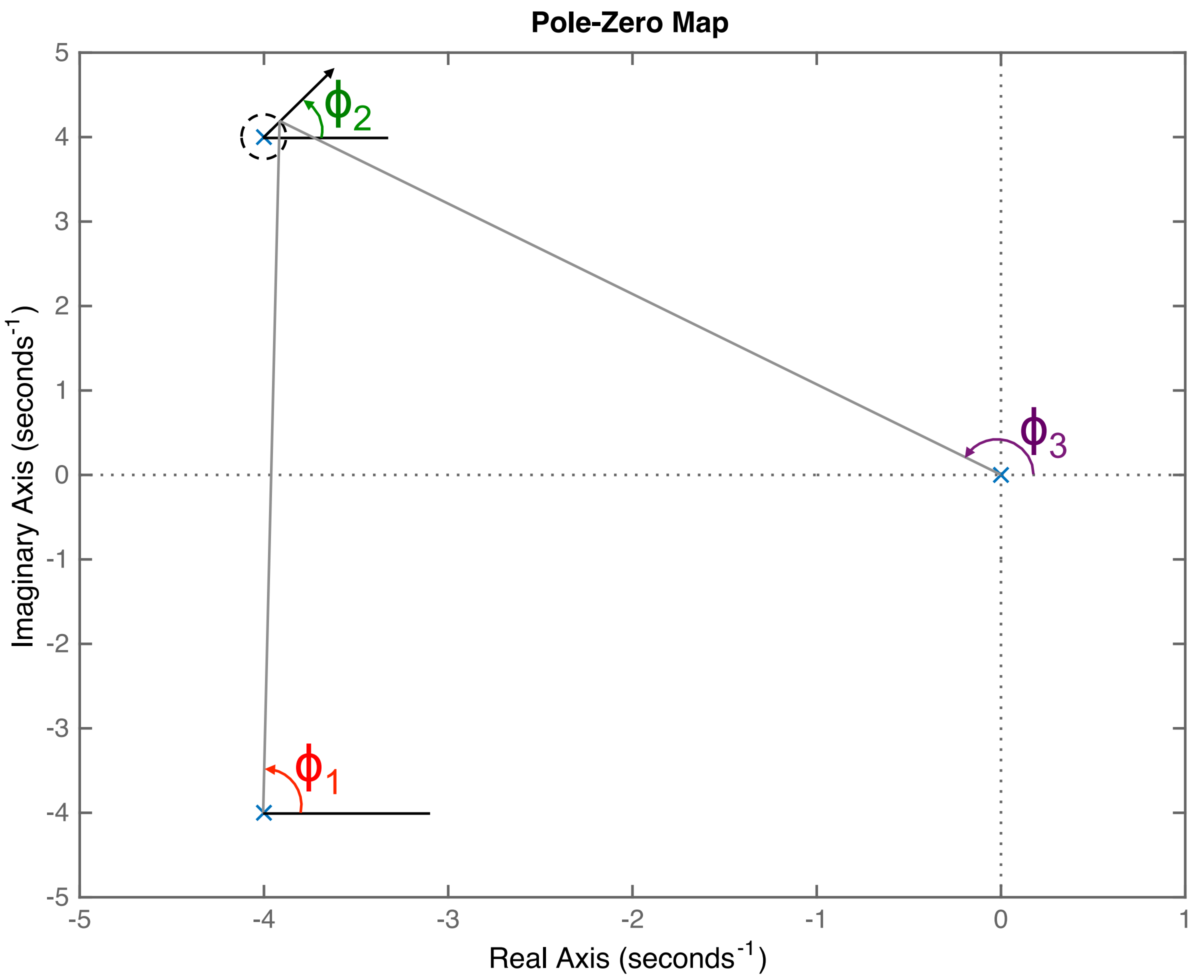


STEP 4: Departure and arrival angles

We know that the locus begins at the $\times' s$ and that it goes to either $\circ' s$ or to infinity along the radial asymptotic lines.

We next compute the angle by which a branch of the locus departs from one of the poles. We take a test point s_0 very near pole 2 at $-4+4j$ and compute the angle of $G(s_0)$ (for illustration see next slide). We select the test point close enough to pole 2 that the angles ϕ_1 and ϕ_3 to the test point can be considered the same as those angles to pole 2: $\phi_1 = 90^\circ$ and $\phi_3 = 135^\circ$.

STEP 4: Departure and arrival angles



STEP 4: Departure and arrival angles

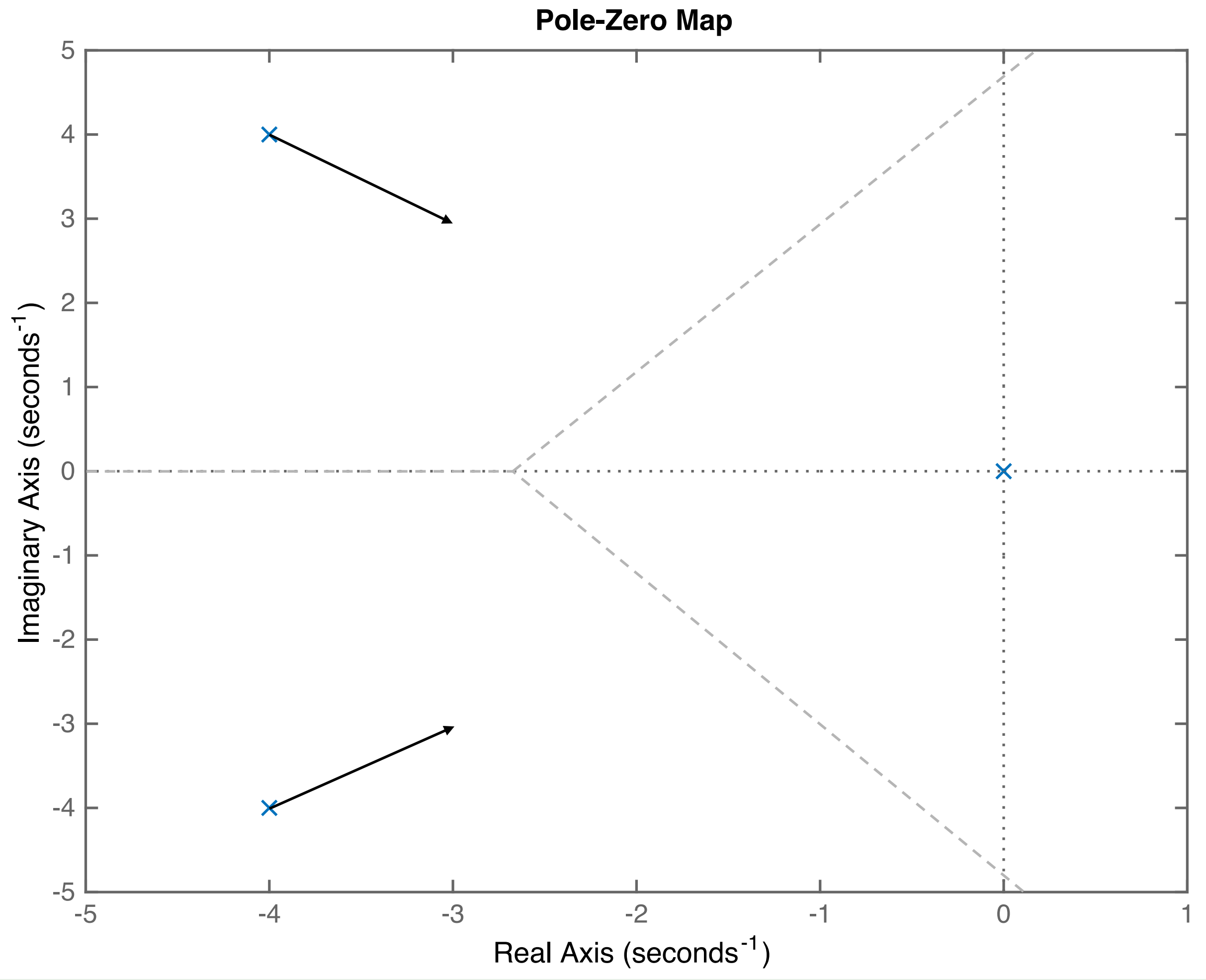
We can calculate ϕ_2 from the angle condition:

$$-90^\circ - \phi_2 - 135^\circ = +180^\circ + 360^\circ l$$

where l is chosen so that $-180^\circ < \phi_2 < +180^\circ$. If we take $l = -1$, then $\phi_2 = -45^\circ$.

Due to the complex conjugate symmetry of the plots, the angle of departure of the locus near pole 1 at $-4 - 4j$ will be $+45^\circ$. The angle of departure from pole 3 at the origin is 180°.

STEP 4: Departure and arrival angles

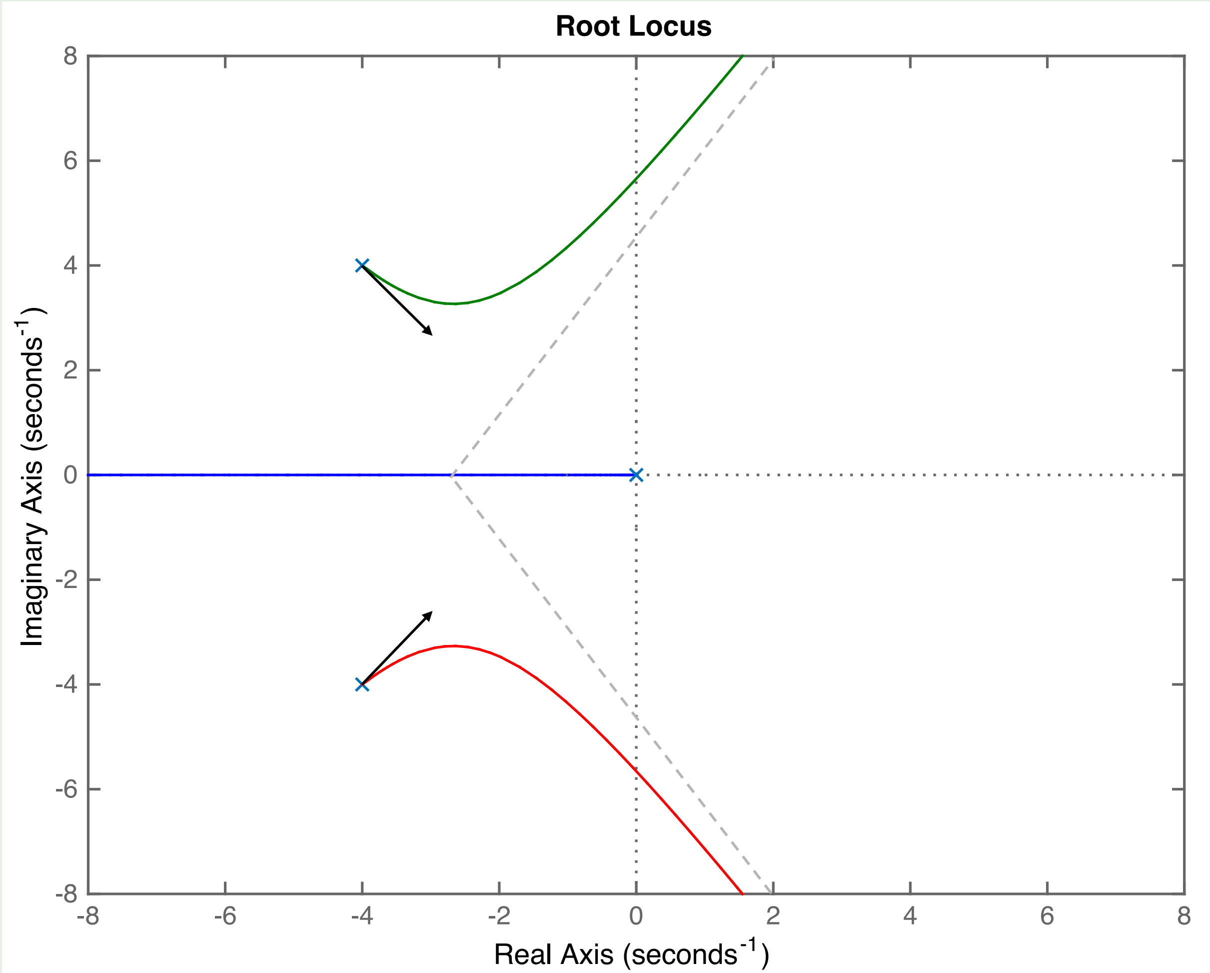


STEP 5: Location of multiple roots and their arrival and departure angles

- Two locus segments coming together at any point in the s-plane will always approach at at relative angle of 180° and then break away with a 90° change in direction;
- Three locus segments coming together will always approach at 120° angles that are rotated 60° relative to the arrival angles.

In our example, this step is not applied. But the first case (with two locus segments) can be seen in the first example from this chapter.

STEP 6: Complete the sketch



Outline

Overview of the guidelines

Drawing the root locus yourself can be done by following the next steps:

1. Mark poles with $\times$ and zeros with $\circ$;
2. Draw the locus on the real axis to the left of an odd number of real poles plus zeros;
3. Draw the asymptotes, centered at α and leaving at angles ϕ_l, where:

$$n - m = \#asymptotes$$

$$\alpha = \frac{\Sigma p_i - \Sigma z_i}{n - m}$$

$$\phi_l = \frac{180^\circ + 360^\circ (l - 1)}{n - m}, l = 1, 2, ..., n - m.$$

Overview of the guidelines

4. Compute locus departure angles from the poles and arrival angles at the zeros:

$$q\phi_{dep} = \Sigma\psi_i - \Sigma\phi_i - 180^\circ - 360^\circ l,$$
$$q\psi_{arr} = \Sigma\phi_i - \Sigma\psi_i + 180^\circ + 360^\circ l,$$

where q is the multiplicity of the pole or zero and l takes on q integer values so that the angles are between $\pm 180^\circ$. $\Sigma\phi_i$ is the sum of the angles of the (remaining) poles and $\Sigma\psi_i$ is the sum of the angles of the (remaining) zeros.

Overview of the guidelines

5. Use the results from the study of multiple roots to help in sketching how locus segments come together and break away: two segments come together at 180° and break away at $\pm 90^\circ$, three locus segments approach each other at relative angles of 120° and depart at angles rotated by 60°.
6. Complete the locus, using the facts developed in the previous steps and making reference to the illustrative loci for guidance. The branches start at the pole locations and end at the zero locations.

Overview of the guidelines

If further refinement is required at the stability boundary, you can also estimate the points where the root locus crosses the imaginary axis. This is done by using Routh's stability criterion. We do not discuss it in this course, but you can find an explanation for it in the additional material of this chapter.

If you want to do this, include this between step 4 and step 5.

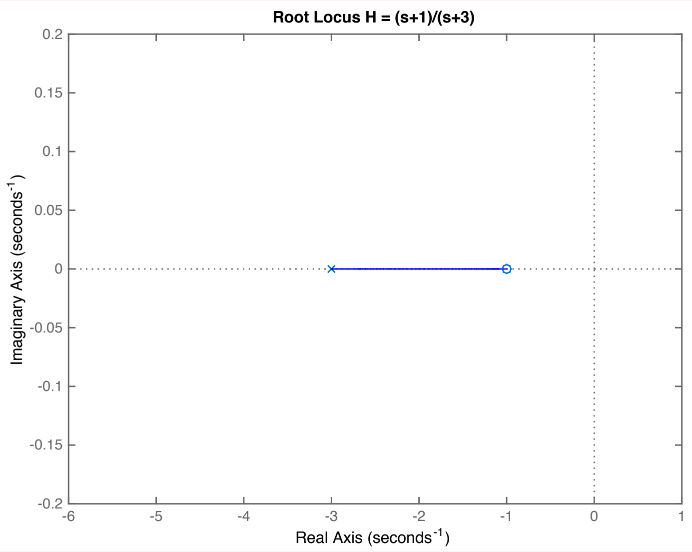
Important to remember!

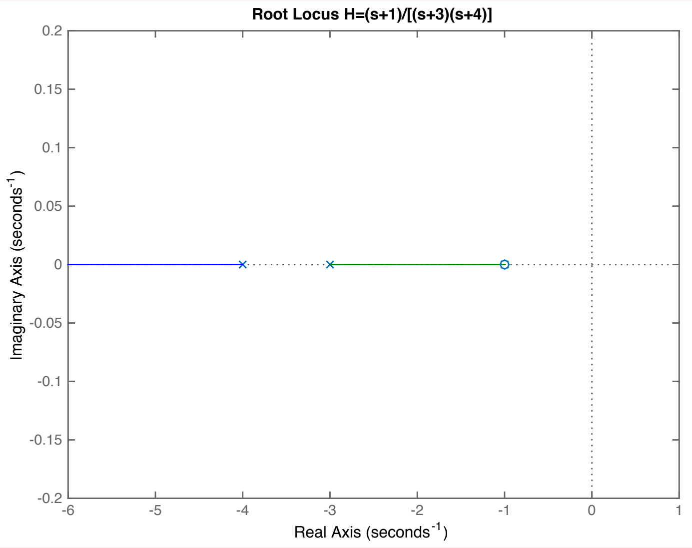
While you are sketching your root locus, keep in mind the following things:

1. The root locus has max(#poles of $G(s)$, # zeros of $G(s)$) branches;
2. Complex roots of $1+KG(s)$ (closed-loop poles) always occur in conjugate pairs;
3. A branch will never cross over itself;
4. Branches leave and enter the real axis at 90° (this is not a 100% general rule, but it is true in most cases);

Important to remember!

5. Each branch starts at an open loop pole of $G(s)$ (for $K = 0$) and ends at a closed loop zero of $G(s)$ (for $K \rightarrow \infty$). If #poles of $G(s) \neq$ #zeros of $G(s)$ the extra branches go to/come from a zero/pole in infinity.





Example 1

Example

Try to understand why the root locus of the following system, $H(s) = \frac{(s+0.2)(s+1)}{s(s+0.8)(s-1)}$, takes the shape as shown in the figure.