Chapter 5: Continuous-time systems

July 27, 2015

Outline

1. Linear differential equations
2. Laplace transform
3. Solving LDEs with the Laplace transform
4. Properties of state-space representation
5. Transfer functions
   - Impulse response and time constant
   - Relationship between state space and transfer functions
6. Transient response analysis of first order and second order systems
   - First order systems
   - Second order systems

Linear differential equations: definitions 1/2

Linear differential equations (LDE) are of the following form:

$$L[y(t)] = f(t),$$

where L is some linear operator.
The linear operator L is of the following form:

$$L_n(y) = \sum_{i=0}^{n} A_i(t) \frac{d^{n-i}y}{dt^{n-i}},$$

with given functions $A_{1:n}$.
The **order of a LDE** is the index of the highest derivative of y.

Linear differential equations: definitions 2/2

$$L_n(y) = \sum_{i=0}^{n} A_i(t) \frac{d^{n-i} y}{dt^{n-i}} = f(t).$$

- y is a scalar function $\rightarrow$ **ordinary differential equation** (ODE);
- y is a vector function $\rightarrow$ **partial differential equation** (PDE);
- $f = 0 \rightarrow$ **homogeneous equation**
  $\rightarrow$ solutions are called **complementary functions**;
- if $A_{0:n}(t)$ are constants (i.e. not functions of time), the LDE is said to have **constant coefficients**.

Example: radioactive decay 1/2

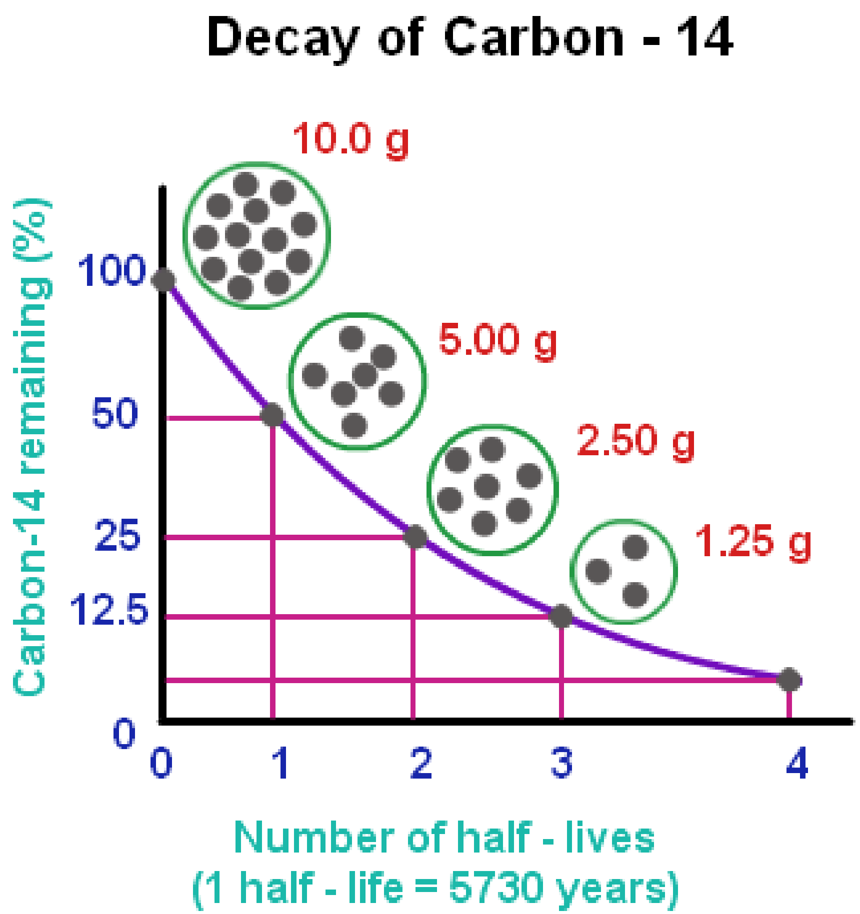
Let $N(t)$ be the number of radioactive atoms at time t, then:

$$\frac{dN(t)}{dt} = -kN(t),$$

for some constant $k > 0$.

This is a first order homogeneous LDE with constant coefficients.

Example: radioactive decay 2/2



Solving homogeneous LDEs with constant cofficients 1/3

Solutions of LDEs must be of the form e^{zt} with $z \in \mathbb{C}$.
We assume an LDE with constant coefficients:

$$\sum_{i=0}^{n} A_i y^{(n-i)} = 0.$$

Replacing $y = e^{zt}$ leads to:

$$\sum_{i=0}^{n} A_i z^{n-i} e^{zt} = 0$$

Dividing by e^{zt} yields the nth order **characteristic polynomial**:

$$F(z) = \sum_{i=0}^{n} A_i z^{n-i} = 0.$$

Solving homogeneous LDEs with constant cofficients 2/3

Characteristic equation:

$$F(z) = \sum_{i=0}^{n} A_i z^{n-i} = 0.$$

1. Solving the polynomial $F(z)$ yields n zeros z_1 to z_n;
2. Substituting a given zero z_i into e^{zt} gives a solution $e^{z_i t}$.

Homogeneous LDEs obey the superposition position:
$\rightarrow$ any linear combination of solutions $e^{z_1 t}, \ldots, e^{z_n t}$ is a solution

Solving homogeneous LDEs with constant cofficients 2/3

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Homogeneous LDEs obey the superposition position:
$\rightarrow$ any linear combination of solutions $e^{z_1 t}, \dots, e^{z_n t}$ is a solution
$\rightarrow$ $e^{z_1 t}, \dots, e^{z_n t}$ form a basis of the solution space of the LDE
The specific linear combination depends on initial conditions.

Solving homogeneous LDEs with constant cofficients 3/3

Example

$$y^{(4)}(t) - 2y^{(3)}(t) + 2y^{(2)}(t) - 2y^{(1)}(t) + y(t) = 0.$$

This is a 4th order homogeneous LDE with constant coefficients. The corresponding characteristic equation:

$$F(z) = z^4 - 2z^3 + 2z^2 - 2z + 1 = 0.$$

The zeros of $F(z)$ are ($j = \sqrt{-1}$):

$$z_1 = j, \quad z_2 = -j, \quad z_{3,4} = 1.$$

These zeros correspond to the following basis functions t:

$$e^{jt}, \quad e^{-jt}, \quad e^t, \quad te^t.$$

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The Laplace transform

Definition

The Laplace transform of $f(t)$, for all real numbers $t \geq 0$:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt.$$

The parameter $s = \sigma + j\omega$ is the complex number frequency.

The initial value theorem states $f(0^+) = \lim_{s \to \infty} sF(s)$.
The final value theorem states $f(\infty) = \lim_{s \to 0} sF(s)$,
if all poles of $sF(s)$ are in the left half plane (i.e. real part < 0).

Important properties of the Laplace transform

property	**time domain**	**s-domain**
linearity	$af(t) + bg(t)$	$aF(s) + bG(s)$
differentiation	$f^{(1)}(t)$	$sF(s) - f(0)$
integration	$\int_0^t f(\tau)d\tau = (u * f)(t)$	$\frac{1}{s}F(s)$
convolution	$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$	$F(s) \cdot G(s)$
time scaling	$f(at)$	$\frac{1}{a}F(\frac{s}{a})$
time shifting	$f(t-a)u(t-a)$	$e^{-as}F(s)$

Inverse Laplace transform

Definition

The inverse Laplace transform converts s-domain to time domain:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{j2\pi} \int_{\gamma - jT}^{\gamma + jT} e^{st} F(s) ds.$$

Practically, the inverse Laplace transform takes two steps:

1. write $F(s)$ in terms of partial fractions
2. transform each term in the partial fraction based on tables of s/t-domain pairs (course notes p. 4.32-4.33)