

Lecture 4 - Discrete Time Systems - Representations

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Outline

- 1 Introduction
- 2 Block-diagram
- 3 State Space representation
- 4 Diffence equations
- 5 Impulse response and convolution
- 6 Z-transform

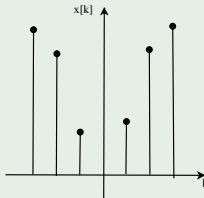
Discrete Time Signal

Definition

Discrete-time signals is defined only at discrete instants of time. The value (amplitude) can be either continuous or discrete.

$x[k]$ is the value of a signal at the moment $t = kT$

Example

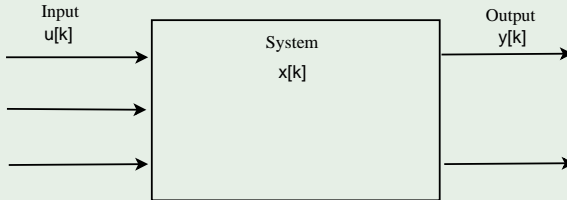


Discrete Time System

Definition

A linear time-invariant (LTI) discrete time system processes an inputvector $u[k]$ to an outputvector $y[k]$.

Example



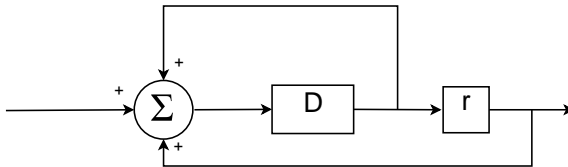
How to represent a system?

- Block-diagram
- State space representation
- Difference/differential equation
- Impulse response
- Transferfunctions

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Block diagram

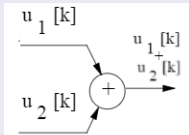


Definition

A block diagram is a visual representation of a system. All LTIs (Linear Time Invariant) systems can be constructed using 3 building blocks (Memory element, summation element, multiplication element). Note that every memory element corresponds to one state variable.

Building blocks

Adder



Constant Multiplier



Delay element

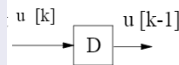
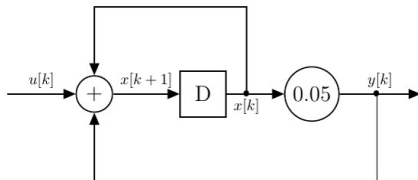


Figure:

Example: compound interest

- $u[k]$: The deposits and withdrawals from the bank account
- $x[k]$: The current saldo on bank account (before deposit and interest)
- $y[k]$: The acquired interest of that year
- $x[k + 1]$: The saldo on the next year = current saldo + interest + deposits

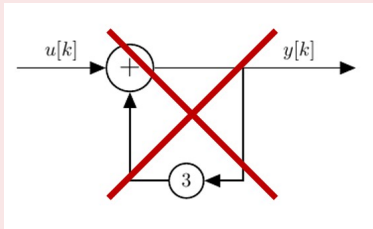


| $u[k]$ | $x[k + 1]$ | $x[k]$ | $y[k]$ |
|--------|------------|--------|--------|
| 50 | 50 | 0 | 0 |
| 0 | 52.5 | 50 | 2.5 |
| -25 | 30.13 | 52.5 | 2.62 |
| 0 | 31.63 | 30.13 | 1.51 |
| 0 | 33.21 | 31.63 | 1.58 |
| 30 | 64.87 | 33.21 | 1.66 |
| 0 | 68.12 | 64.87 | 3.24 |
| 0 | 71.52 | 68.12 | 3.41 |

Bad block diagrams

Delay-free loops

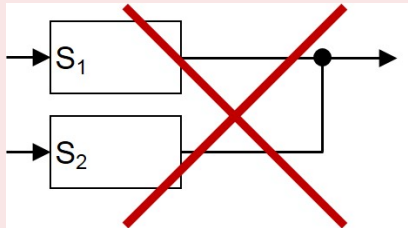
The issue is that this leads to an implicit connection $u[k]$ depends on $y[k]$, which is not yet known. You can easily rewrite this in an allowed shape $y[k] = u[k] + 3y[k] \iff y[k] = -\frac{1}{2}u[k]$



Bad block diagrams

Connecting two outputs without using a sum

The issue is that this can lead to inconsistencies. According to this block diagram the output of the systems S_1 and S_2 are equal. There is no way to get around this.



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State space representation

Definition

State space representation

$$\begin{aligned}x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k]\end{aligned}$$

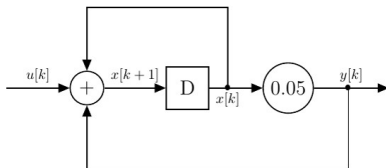
This state space representation is specific to LTI systems:

Linear: its easy to see these systems are linear

Time-invariant: the matrices A,B,C,D do not depend on time, if it were to be a time-variant system the matrices would be replaced by $A[k]$, $B[k]$, $C[k]$ and $D[k]$.

From block diagram to state space

Blockdiagram



State space representation

- 1 Let the output of the memory elements be $x_i[k]$.
- 2 So the input of the memory elements are x_{k+1}
- 3 Trace back to retrieve equations for $x_i[k+1]$ and $y_i[k]$

This results in:

$$\begin{aligned}x[k+1] &= u[k] + 1.05x[k] \\ y[k] &= 0.05x[k]\end{aligned}$$

From state space to block diagram

Example

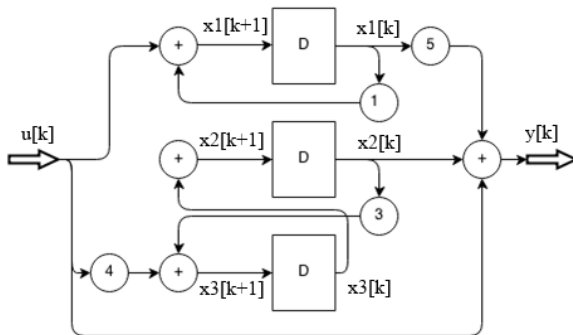
$$\begin{aligned}x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k]\end{aligned}$$

with $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$, $C = [5 \quad 1 \quad 0]$ and $D = [1]$

From state space to block diagram

- 1 First add a delay element for every state $x_i[k]$
- 2 Determine the input for every state $x[k+1]$ from the matrixes A and B, as a combination of the states $x[k]$ and inputs $u[k]$
- 3 Determine the outputs $y[k]$ in the same way with the matrixes C and D

From state space to block diagram



Different state space representations

State space representation is not unique

Take the following system, which connects $u[k]$ to $y[k]$:

$$\begin{aligned}x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k]\end{aligned}$$

Now take a non-singular square matrix T and the following system. The relation between $u[k]$ and $y[k]$ will be the same.

$$\begin{aligned}Tx[k+1] &= TAT^{-1}Tx[k] + TBu[k] \\ y[k] &= CT^{-1}Tx[k] + Du[k]\end{aligned}$$

With $x' = Tx$, $A' = TAT^{-1}$, $B' = TB$, $C' = CT^{-1}$ and $D' = D$, we have found a different state space representation for this system.

Solving state space equation

$$x[k+1] = Ax[k] + Bu[k]$$

$$y[k] = Cx[k] + Du[k]$$

We express $x[1], x[2], \dots$ in function of $x[0]$:

$$x[1] = Ax[0] + Bu[0]$$

$$x[2] = Ax[1] + Bu[1] = A^2x[0] + ABu[0] + Bu[1]$$

\vdots

$$x[k] = A^k x[0] + \sum_{i=0}^{k-1} A^{k-1-i} Bu[i]$$

The output is $y[k]$:

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Difference equations

Definition

Similar to differential equations, but for discrete time.

General form:
$$\sum_{i=0}^n a_i y[k+i] = \sum_{i=0}^n b_i u[k+i]$$

With n the order of the system.

Solution in 2 parts

- 1 Homogenous: solution from input zero
- 2 Particular: solution derived as a response from the input

Homogenous difference equations

Definition

General form: $\sum_{i=0}^n a_i y[k+i] = 0$

Example

$$y[k+1] - ay[k] = 0$$

$$y[k+1] = ay[k]$$

$$y[1] = ay[0]$$

$$y[2] = ay[1] = a^2 y[0]$$

$$\vdots$$

$$y[n] = a^n y[0]$$

Homogenous difference equations

solution

- Expected form of solution: r^k
- Substitution of the expected solution in the difference equation: $\sum_{i=0}^n a_i r^{k+i} = 0$
- Division by r^k leads to the characteristic equation: $\sum_{i=0}^n a_i r^i = 0$
- Solutions of the characteristic equation: r_1, r_2, r_3, \dots
- Homogenous solution to the difference equation:

$$y[k] = c_1 r_1^k + c_2 r_2^k + c_3 r_3^k + \dots = \sum_{i=1}^n c_i r_i^k$$

Example

- Homogeneous recurrence relations:
 $y[k + 2] - 5y[k + 1] + 6y[k] = 0$
- Initial value: $y[0] = 1, y[1] = 1$
- Characteristic polynomial: $r^2 - 5r + 6 = 0$
- Roots: 2 and 3
- General solution: $c_1 2^k + c_2 3^k$
- Initial values:

$$\begin{cases} 1 = c_1 + c_2 \\ 1 = 2c_1 + 3c_2 \end{cases}$$
$$\begin{cases} 2 = c_1 \\ -1 = c_2 \end{cases}$$

- Result: $y[k] = 2^{k+1} - 3^k$

Example: Fibonacci sequence



Figure: Leonardo Bonacci (c. 1170 c. 1250) known as Fibonacci was an Italian mathematician, considered to be "the most talented Western mathematician of the Middle Ages".

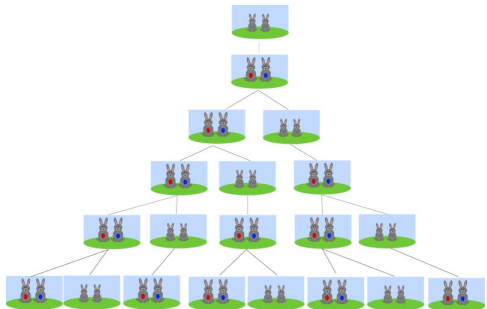


Figure: Fibonacci sequence

Example: Fibonacci sequence

Example

- Homogeneous recurrence relations: $y[k+2] = y[k+1] + y[k]$
- Initial value: $y[0] = 1, y[1] = 1$
- Characteristic polynomial: $r^2 - r - 1 = 0$
- Roots: $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$
- General solution: $y[k] = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$
- Initial values:

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 \frac{1+\sqrt{5}}{2} + c_2 \frac{1-\sqrt{5}}{2} = 1 \end{cases}$$
$$\begin{cases} c_1 = \frac{5+\sqrt{5}}{10} \\ c_2 = \frac{5-\sqrt{5}}{10} \end{cases}$$

- Result: $y[k] = \left(\frac{5+\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^k$

Multiple roots and Complex roots

Multiple roots

For a multiple root r_i with multiplicity m add $r_i^k, kr_i^k, \dots, k^{m-1}r_i^k$

Complex roots

Complex will result in oscillating behavior. If the difference equations and starting conditions are both real the complex roots can only be present in conjugate pairs, the constants will also be in conjugate pairs.

$$r_i = Re^{j\phi} \quad r_{i+1} = Re^{-j\phi}$$

$$c_i = R_0 e^{j\phi_0} \quad c_{i+1} = R_0 e^{-j\phi_0}$$

$$c_i r_i^k + c_{i+1} r_{i+1}^k = R_0 Re^{jk\phi + j\phi_0} + R_0 Re^{-(jk\phi + j\phi_0)}$$

This can be converted into a cosine using Eulers formula:

$$y[k] = R_0 R (\cos(k\phi + \phi_0) + \sin(k\phi + \phi_0)) + R_0 R (\cos(k\phi + \phi_0) - \sin(k\phi + \phi_0))$$

Eulers formula

Theorem

$$e^{j\phi} = \cos(\phi) + \sin(\phi)j$$

Proof.

Using power series:



Non-homogeneous difference equations

Definition

$$\sum_{i=0}^n a_i y[k+i] = \sum_{i=0}^n b_i u[k+i]$$

A linear combination of inputs results in the same linear combination of the outputs resulting from each input individually.

Solution

The equation can thus be solved for each input individually and the results added together afterwards.

The resulting particular solutions can then be added to the general form of the homogenous solution.

Particular solutions to difference equations

| Input $u[k]$ | Suggested solution $y[k]$ |
|-----------------------|--|
| k | $\alpha_1 k + \alpha_0$ |
| k^n | $\sum_{i=0}^n \alpha_i k^i$ |
| a^k | αa^k |
| $k^n a^k$ | $(\sum_{i=0}^n \alpha_i k^i) a^k$ |
| $\cos(k\phi)$ | $\alpha \cos(k\phi + \phi_0)$ |
| $a^k \cos(k\phi)$ | $\alpha a^k \cos(k\phi + \phi_0)$ |
| $k^n a^k \cos(k\phi)$ | $(\sum_{i=0}^n \alpha_i k^i) a^k \cos(k\phi + \phi_0)$ |

Example

Example

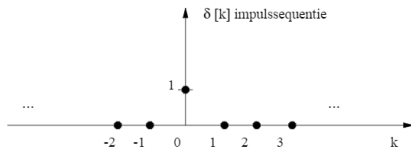
- Difference equation : $y[k + 2] - 5y[k + 1] + 6y[k] = (-1)^k$
- Initial value: $y[1] = \frac{1}{4}, y[0] = \frac{1}{12}$
- Homogeneous difference equation:
$$y[k + 2] - 5y[k + 1] + 6y[k] = 0$$
- Characteristic polynomial: $r^2 - 5r + 6 = 0$
- Homogeneous solution: $y_{hom}[k] = c_1 2^k + c_2 3^k$
- Particular solution: $y_{par}[k] = \alpha(-1)^k$
- Substitution: $\alpha(-1)^{k+2} - 5\alpha(-1)^{k+1} + 6\alpha(-1)^k = (-1)^k$
- $\alpha = \frac{1}{12}$
- General solution: $y[k] = c_1 2^k + c_2 3^k + \frac{1}{12}(-1)^k$
- Initial values: $c_1 = -\frac{1}{3}, c_2 = \frac{1}{3}$
- Result: $y[k] = -\frac{1}{3}2^k + \frac{1}{3}3^k + \frac{1}{12}(-1)^k$

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Impulse responses

Definition



Theorem

You can decompose any signal in a sum of impulse responses:

$$f[k] = \sum_{i=-\infty}^{i=\infty} \delta[k-i]f[i] = \delta[k] * f[k]$$

Convolution

Definition

$$w[k] = u[k] * v[k] = \sum_{i=-\infty}^{\infty} u[i]v[k-i]$$

Solve

- 1 Flip $v[i]$ around vertical axis($v[-i]$).
- 2 Slide to the right over k steps($v[k-i]$).
- 3 Multiply $u[i]$ and $v[k-i]$
- 4 Sum all the values.

Convolution theorem (DT)

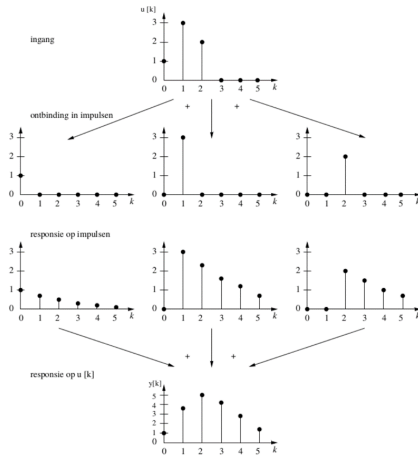
Theorem

$$y[k] = u[k] * h[k]$$

Proof.

$$\begin{aligned}\delta[k] &\rightarrow h[k] \\ \delta[k+i] &\rightarrow h[k+i] \\ \sum_{i=-\infty}^{i=\infty} \delta[k-i]u[i] &\rightarrow \sum_{i=-\infty}^{i=\infty} h[k-i]u[i] \\ u[k] &\rightarrow u[k] * h[k] = y[k]\end{aligned}$$





Impulse response

Definition

The impulse response of a dynamic system is its output when presented with a brief input signal, called an impulse.

Impulse response

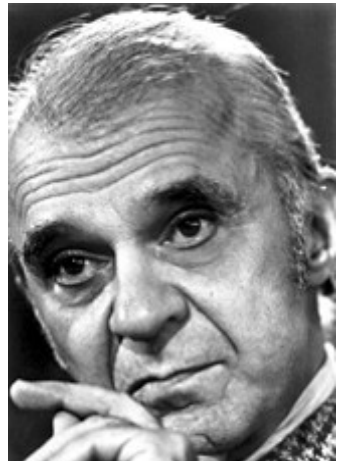
Examples of Dirac-deltas

Popping balloons for acoustic measurements



Example: Leontief model of a planned economy

- Won the nobel prize in 1973
- A simple model that assigns values to different sectors
- For simplicity we choose a planned economy. But today governments all over the world are using similar models to model their economy.



Example: Leontief model of a planned economy

Leontief divided the economy in sectors who buy from each other.
To produce one unit of industry 0.40 units of agriculture are required
TABEL IMPORTEREN

Example: Leontief model of a planned economy

$x_i[k]$: production of sector i in month k
 $l_i[k]$: the demand to goods from sector i in the next month
Note: in planned economy demand can be steered, economists can decide how many rations they give.

The model $x[k+1] = Ax[k] + lu[k]$

$$y[k] = lx[k]$$

It is anti-causal: a subdivision of non-causal, for which only future values have to be known to know the current value

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Z-transform

Definition

- Discrete equivalent to the Laplace-transform
- Converts time dependent descriptions of systems to the time-independent Z-domain.
- Simplifies many calculations:
 - Convolution theorem convolution becomes multiplication
 - Linear difference equations become simple algebraic expressions
 - ...

Z-transform

2 forms

- Bilateral: Requires knowledge of h for all values of k , including negative values Can be used for non-causal systems

$$X(z) = \sum_{k=-\infty}^{\infty} x[k]z^{-k}$$

- Unilateral: Only requires knowledge of h for positive values of k Can only be used for causal systems without loss of

information
$$X(z) = \sum_{k=0}^{\infty} x[k]z^{-k}$$

Z-transform

Example

$$x[k] = \left\{ \begin{array}{ccccc} 1 & -1 & 0 & 2 & 4 \\ & & & \uparrow & \end{array} \right\}$$

$$X(z) = \sum_{k=-3}^1 x[k]z^{-k} = z^3 - z^2 + 2 + 4z^{-1}$$

Properties Unilateral Z-transform

| Property | Time Domain | Z-domain |
|-------------------------|-----------------------------|--|
| Linearity | $af_1[n] + bf_2[n] + \dots$ | $aF_1(Z) + bF_2(Z) + \dots$ |
| Right Shift($m>0$) | $f[k - m]$ | $z^{-m}F(Z)$ |
| Left Shift ($m>0$) | $f[k + m]$ | $z^m \left(F(z) - \sum_{i=0}^{m-1} f[i]z^{-i} \right)$ |
| Convolution | $f[k] * g[k]$ | $F(z)G(z)$ |
| Multiplication by a^k | $a^k f[k]$ | $F(a^{-1}z)$ |
| Summation in time | $\sum_{i=0}^k f[i]$ | $\frac{z}{z-1} F(Z)$ |
| Differentiation in z | $k^m f[k]$ | $\left(-z \frac{d}{dz} \right)^m F(z)$ |
| Periodic Sequence | $f[k] = f[k + N]$ | $F(z) = \frac{z^N}{z^N - 1} \sum_{k=0}^{N-1} f[k]z^{-1}$ |
| Initial Value | $f[0]$ | $\lim_{ z \rightarrow \infty} F(z)$ |
| Final value | $f[\infty]$ | $\lim_{z \rightarrow 1} (z-1)F(z)$ |

List of common Z-transform pairs

| | |
|--------------------------------------|--|
| $\delta[k]$ | 1 |
| $\delta[k - m]$ | z^{-m} |
| 1 | $\frac{z}{(z-1)}$ |
| a^k | $\frac{z}{(z-a)}$ |
| ka^k | $\frac{az}{(z-a)^2}$ |
| $k^2 a^k$ | $\frac{az(z+a)}{(z-a)^3}$ |
| $k^3 a^k$ | $\frac{az(z^2+4az+a^2)}{(z-a)^4}$ |
| $k^4 a^k$ | $\frac{az(z^3+11az^2+11a^2z+a^3)}{(z-a)^5}$ |
| $k^5 a^k$ | $\frac{az(z^4+26az^3+66a^2z^2+26a^3z+a^4)}{(z-a)^6}$ |
| $\frac{(k+1)(k+2)\dots(k+m)a^k}{m!}$ | $\frac{z^{m+1}}{(z-a)^{m+1}}$ |

List of common Z-transform pairs

| | |
|--------------------------------------|---|
| $a^k \sin k\omega T$ | $\frac{az \sin \omega T}{z^2 - 2az \cos \omega T + a^2}$ |
| $a^k \cos k\omega T$ | $\frac{z(z - a \cos \omega T)}{z^2 - 2az \cos \omega T + a^2}$ |
| $\cos(k\omega T + \theta)$ | $\frac{z(z \cos \theta - \cos(\omega T - \theta))}{z^2 - 2az \cos \omega T + 1}$ |
| $ka^k \sin k\omega T$ | $\frac{z(z-a)(z+a)a \sin \omega T}{(z^2 - 2az \cos \omega T + a^2)^2}$ |
| $ka^k \cos k\omega T$ | $\frac{az(z^2 \cos \omega T - 2az + a^2 \cos \omega T)}{(z^2 - 2az \cos \omega T + a^2)^2}$ |
| $a^k \sinh k\omega T$ | $\frac{az \sinh \omega T}{z^2 - 2az \cosh \omega T + a^2}$ |
| $a^k \cosh k\omega T$ | $\frac{z(z - a \cosh \omega T)}{z^2 - 2az \cosh \omega T + a^2}$ |
| $\frac{a^k}{k!}$ | $e^{\frac{a}{z}}$ |
| $\frac{n!}{(n-k)!k!} a^k b^{n-k}$ | $\frac{(bz+a)^n}{z^n}$ |
| $\frac{1}{k}$ | $\ln\left(\frac{z}{z-1}\right)$ |
| $\frac{k(k-1)(k-2)\dots(k-m+1)}{m!}$ | $\frac{z}{(z-1)^{m+1}}$ |

Region of convergence

Z-transform not unique

Two different signals could have the same Z-transform over a different region of convergence.

Definition

The region of convergence is the set of complex numbers z for which:

$$\sum_{k=-\infty}^{\infty} |x[k]z^{-k}| < \infty$$

- We will look at convergence separately for positive and negative k , splitting the convergence criterion in 2:

$$k < 0 : |x[k]| \leq M_- R_-^k$$

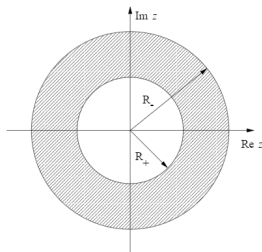
$$k \geq 0 : |x[k]| \leq M_+ R_+^k$$

- Using $z = re^{j\theta}$ with R_+ as small as possible and R_- as large as possible we get:

$$\begin{aligned} \sum_{k=-\infty}^{\infty} |x[k]z^{-k}| &= \sum_{k=-\infty}^{\infty} |x[k]| r^{-k} \\ &= \sum_{k=1}^{\infty} |x[-k]| r^k + \sum_{k=0}^{\infty} |x[k]| r^{-k} \\ &\leq M_- \sum_{k=1}^{\infty} (R_-^{-1}r)^k + M_+ \sum_{k=0}^{\infty} (R_+ r^{-1})^k \end{aligned}$$

Region of convergence

The sums are finite if $R_-^{-1}r < 1$ and $R_+r^{-1} < 1$ Region of convergence: $R_+ < R_-$ Ring R_- R_+ : No ROC Causal system, for negative k : $x[k] = 0 \Rightarrow R_- = +\infty$ cannot contain any poles of the system ROC of a stable system always contains the unit circle.



Inverse Z-transform

Inverse Z-transform

- 1 Split the function up in partial fractions
- 2 Use the table to transform each partial fraction individually to the t-domain

Partial fraction decomposition

- 1 Factorizing the denominator
- 2 If all poles(zeros of the denominator) have multiplicity 1:

$$F(z) = \frac{\sum_{i=0}^n b_i z^i}{a_n(z - p_1)(z - p_2) \dots (z - p_n)} \quad (1)$$

$$= \alpha_0 + \alpha_1 \left(\frac{z}{z - p_1} \right) + \alpha_2 \left(\frac{z}{z - p_2} \right) + \dots + \alpha_n \left(\frac{z}{z - p_n} \right) \quad (2)$$

- 3 The coefficients can be calculated by:

$$\alpha_0 = F(0) \quad \alpha_i = \left[\frac{z - p_i}{z} F(z) \right]_{z=p_i}$$

Inverse Z-transform

- 4 If there are poles with multiplicity higher than 1:

$$F(z) = \frac{\sum_{i=0}^n b_i z^i}{a_n(z - p_1)^{n_1}(z - p_2)^{n_2} \dots} \quad (3)$$

$$= \alpha_0 + \alpha_1 \left(\frac{z}{z - p_1} \right) + \alpha_2 \left(\frac{z}{z - p_1} \right)^2 + \dots + \alpha_{n_1} \left(\frac{z}{z - p_1} \right)^{n_1} \quad (4)$$

$$+ \beta_1 \left(\frac{z}{z - p_2} \right) + \beta_2 \left(\frac{z}{z - p_2} \right)^2 + \dots + \beta_{n_2} \left(\frac{z}{z - p_2} \right)^{n_2} \quad (5)$$

- 5 Where the highest coefficient for each pole can be calculated by:

$$\alpha_0 = F(0) \quad \alpha_1 = \left[\left(\frac{z - p_1}{z} \right)^{n_1} F(z) \right]_{z=p_1} \quad \beta_1 = \left[\left(\frac{z - p_2}{z} \right)^{n_2} F(z) \right]_{z=p_2}$$

Inverse Z-transform

- 6 Any remaining coefficients can be found by evaluating the equation for a number of values of z .

Inverse Z-transform

| | |
|-------------------------------|--------------------------------------|
| $F(z)$ | $f[k]$ |
| 1 | $\delta[k]$ |
| $\frac{z}{z-a}$ | a^k |
| $\frac{z^{m+1}}{(z-a)^{m+1}}$ | $\frac{(k+1)(k+2)\dots(k+m)a^k}{m!}$ |

Inverse Z-transform

Example

$$F(z) = \frac{z^3 + 2z^2 + z + 1}{z^3 - z^2 - 8z + 12}$$

- The denominator has a zero in 2 (m=2) and -3
- Partial fraction decomposition:

$$\alpha_0 + \alpha_1 \left(\frac{z}{z-2} \right) + \alpha_2 \left(\frac{z}{z-2} \right)^2 + \beta_1 \left(\frac{z}{z+3} \right)$$

$$\bullet \alpha_0 = F(0) = \frac{1}{12}$$

$$\alpha_2 = \left[\frac{F(z)(z-2)^2}{z^2} \right]_{z=2} = \left[\frac{z^3 + 2z + z + 1}{z^2(z+3)} \right]_{z=2} = \frac{19}{20}$$

$$\beta_1 = \left[\frac{F(z)(z+3)}{z} \right]_{z=-3} = \left[\frac{z^3 + 2z + z + 1}{z(z-2)^2} \right]_{z=-3} = \frac{11}{75}$$

Example

- By evaluating the function for $z=1$:

$$\frac{5}{4} = \alpha_0 - \alpha_1 + \alpha_2 + \beta/4$$

$$\alpha_1 = -9/50$$

- Result : $F(z) = \frac{1}{12} - \frac{9}{50} \frac{z}{z-2} + \frac{19}{20} \frac{z^2}{(z-2)^2} + \frac{11}{75} \frac{z}{z+3}$

- Inverse Z-transform:

$$f[k] = \gamma[k]/12 + 2^k k \frac{19}{20} + 2^k \frac{77}{100} + (-3)^k \frac{11}{75}$$

Inverse Z-transform

Another technique for calculating the inverse Z-transform is direct division.

The numerator of the transfer function is divided by the denominator via long division.

Teller

$$z^3 + 2z^2 + z + 1$$

$$\underline{z^3 - z^2 - 8z + 12}$$

$$3z^2 + 9z - 11$$

$$\underline{3z^2 - 3z - 24 + 36z^{-1}}$$

$$12z + 13 - 36z^{-1}$$

$$\underline{12z - 12 - 96z^{-1} + 144z^{-2}}$$

$$25 + 60z^{-1} - 144z^{-2}$$

...

Noemer

$$z^3 - z^2 - 8z + 12$$

$$1 + 3z^{-1} + 12z^{-2} + 25z^{-3} + \dots = \sum_{k=0}^{\infty} f[k]z^{-k}$$

Solving difference equations with the Z-transform

A system is described by a difference equation of the following form:

$$\sum_{i=0}^n a_i y[k+i] = \sum_{i=0}^n b_i u[k+i]$$

After the Z-transform:

$$a_0 Y(z) + \sum_{i=1}^n a_i z_i \left[Y(z) - \sum_{j=0}^{i-1} y[j] z^{-j} \right] =$$

$$b_0 U(z) + \sum_{i=1}^n b_i z_i \left[U(z) - \sum_{j=0}^{i-1} u[j] z^{-j} \right]$$

Solving difference equations with the Z-transform

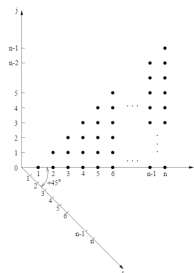
Rearranged:

$$Y(z) = \frac{\sum_{i=1}^n b_i z^i}{\sum_{i=1}^n a_i z^i} U(z) - \frac{\sum_{i=1}^n b_i z^i \left[\sum_{j=0}^{i-1} u[j] z^{-j} \right] - \sum_{i=1}^n a_i z^i \left[\sum_{j=0}^{i-1} y[j] z^{-j} \right]}{\sum_{i=1}^n a_i z^i}$$

Solving difference equations with the Z-transform

We'll apply the following transformation of the double summations:

$$\sum_{i=1}^n b_i z^i \sum_{j=0}^{i-1} u[j] z^{-j} = \sum_{s=1}^n \sum_{j=0}^{n-s} b_{s+j} u[j] z^s$$



Solving difference equations with the Z-transform

The final simplified result is:

With this result it is easy to find the resulting output from a given input or vice-versa given a difference equation. Right-hand fraction = output resulting from starting conditions: will vanish with time = transient behavior Left-hand fraction = output resulting from input: will remain = steady state response.

$H(z) = \frac{\sum_{i=1}^n b_i z^i}{\sum_{i=1}^n a_i z^i}$ is the transfer function of the system. This is the z transform of $h[k]$ (the impulse response)

The transfer function can be derived directly from the state space model of a system:

$$\begin{aligned}x[k + 1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k]\end{aligned}$$

The Z-transform gives:

$$\begin{aligned}z(X(z) - x[0]) &= AX(z) + BU(z) \\ Y(z) &= CX(z) + DU(z)\end{aligned}$$

Rearranged to have $X(z)$ in explicit form:

$$\begin{aligned}X(z) &= (zI - A)^{-1}zx[0] + (zI - A)^{-1}BU(z) \\ Y(z) &= C(zI - A)^{-1}zx[0] + \left[C(zI - A)^{-1}B + D \right] U(z)\end{aligned}$$

If $x[0] = 0$ and $u[k] = \gamma[k](U(z) = 1)$:

$$H(z) = C(zI - A)^{-1}B + D$$

$$Y(z) = H(z)U(z)$$

Z-Transform

Definition

A pole p_i of the system is a point in the complex z -plane where $H(p_i) = \pm\infty$

$$\sum_{i=0}^n a_i z^i = 0$$

Definition

A zero n_i is a point where $H(n_i) = 0$.

$$\sum_{i=0}^n b_i z^i = 0$$

Link between Eigenvalues and poles

Are Eigenvalues of A poles of $H(z)$?

- As z approaches an Eigenvalue of A , $(zI - A)^{-1}$ is no longer defined.
- $C(zI - A)^{-1}B$ may still be defined depending on the values of C and B .

Rule 1

An Eigenvalue of A will sometimes, but not always, be a pole of $H(z)$.

Definition

If every eigenvalue of A is also a pole of $H(z)$ then a minimal number of internal states has been achieved.

Link between Eigenvalues and poles

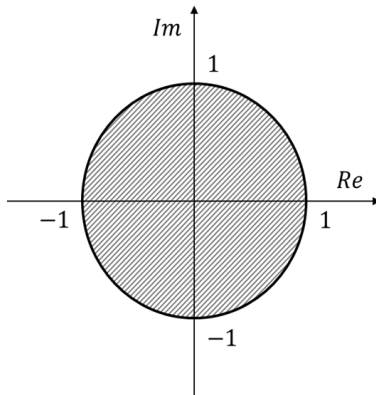
Are poles of $H(z)$ Eigenvalues of A ? $H(z) = C(zI - A)^{-1}B + D$
 B , C and D are matrixes with properly defined values If $H(z)$ is undefined then $(zI - A)^{-1}$ must be the cause z must be an Eigenvalue of A

Rule 2

Poles of $H(z)$ are always Eigenvalues of A

Stability

- Internal Stability
 - All possible internal states return to zero after a finite time in the absence of an input.
 - All eigenvalues of the matrix A are contained within the a circle of radius 1 around zero in the complex plane.
- BIBO-Stability (Bounded-Input Bounded-Output)
 - Every bounded input results in a bounded output
 - All poles are contained within the a circle of radius 1 around zero in the complex plane
 - BIBO-Stability follows from Internal Stability, but the inverse is not necessarily true.



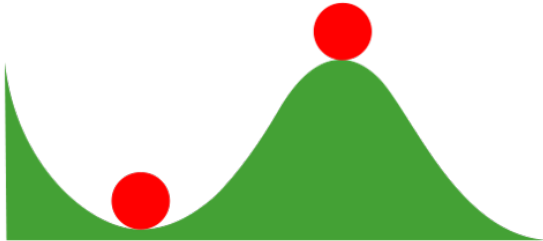
Can unstable systems exist?

According to the mathematical models we have discussed unstable systems need an infinite amount of energy.

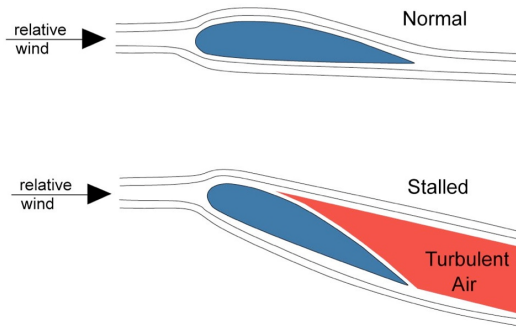
What happens in the real world?

- The system enters a state in which the current linear model is no longer valid(Non linear behavior) .
- Smaller unaccounted effects become more prominent
- The system malfunctions and may cause damage to itself or its surroundings.
- Something else bad happens

Stability



Airplane stall



- Airplanes generate lift using the Venturi effect.
- Faster moving air has a lower pressure.
- Eddy currents may be created due to a too slow airspeed or too sharp ascent.
- Turbulent airflow causes a loss of the lift generated by the Venturi effect.
- Without the necessary lift an airplane becomes an unstable system.

Busses and other tall vehicles have a tendency to roll when taking turns too quickly. A London bus is loaded with sandbags and must be able to lean at an angle of at least 28 while still returning all tires to the ground. Modern day car manufacturers have to pass multiple tests for stability while maneuvering



Steady state behavior via Z-transform

Starting from the previous result:

$$Y(z) = \frac{\sum_{i=1}^n b_i z^i}{\sum_{i=1}^n a_i z^i} U(z) + \frac{\sum_{s=1}^n \left(\sum_{j=0}^{n-s} a_{s+j} y[j] - \sum_{j=0}^{n-s} b_{s+j} u[j] \right) z^s}{\sum_{i=1}^n a_i z^i}$$

We wish to find the resulting output from the input:

$$u[k] = \cos(k\alpha + \theta)$$

To simplify derivation, we use:

$$u[k] = e^{j(k\alpha + \theta)}$$

With Z-transform: $U(z) = \frac{ze^{j\theta}}{z - e^{j\alpha}}$

Steady state behavior via Z-transform

Filling in $U(z)$ and splitting into partial fractions:

Calculating the coefficient c :

After the inverse Z-transform:

Steady state behavior via Z-transform

Because of linearity we can ignore the imaginary component, leading to the result:

$$y[k] = |H(e^{j\alpha})| \cos(k\alpha + \theta + \angle H(e^{j\alpha})) + \operatorname{Re}(d_1 p_1^k + \cdots + d_n p_n^k + g\gamma[k])$$

| Input | Output |
|--------------------------|---|
| $\cos(k\alpha + \theta)$ | $ H(e^{j\alpha}) \cos(k\alpha + \theta + \angle H(e^{j\alpha}))$ |

Steady state behavior via Z-transform

Example

- $u[k] = \cos(3k + \pi) \Rightarrow \alpha = 3 \& \theta = \pi$
- $H(z) = \frac{z^2 + 4}{(z^2 + 6)(z - 1)}$
- $H(e^{3j}) = 0.357e^{0.055j}$
- The resulting output has been reduced to a third in amplitude and has undergone a small phase shift.
 $y[k] = 0.357 \cos(3k + \pi + 0.055)$

Complex Eigenvalues

As with the roots to the characteristic equation in difference equations, complex and/or negative Eigenvalues for A create oscillation. The magnitude of the oscillation will grow/decline with

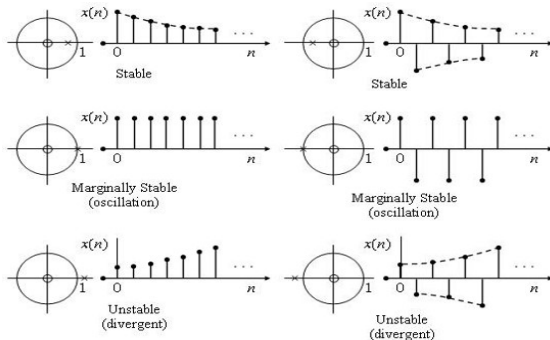
. $\|\lambda\| < 1$: The oscillation will decrease in magnitude: stable

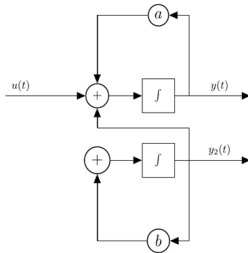
$\|\lambda\| > 1$: The oscillation will increase in magnitude: unstable

$\|\lambda\| = 1$: The oscillation will maintain the same magnitude indefinitely: unstable

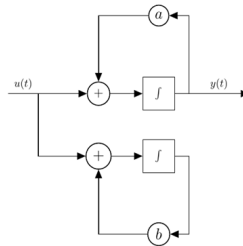
The smallest achievable period is 2 times the step time, for negative real eigenvalues.

Complex Eigenvalues





Not controllable



Not observable

Figure:

Observability

A system is observable if the current state can be determined in finite time by measuring the outputs.

The state space model without inputs gives us:

$$x[k+1] = Ax[k] \quad y[k] = Cx[k]$$

$$x[1] = Ax[0], x[2] = A^2x[0], \dots, x[n-1] = A^{n-1}x[0]$$

If $x[k]$ has n internal states then n equations are needed:

$$y[0] = Cx[0], y[1] = Cx[1] = CAx[0], \dots, y[n-1] = CA^{n-1}x[0]$$

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x[0] = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ \vdots \\ y[n-1] \end{bmatrix}$$

Observability

A system is controllable if it can be brought to a desired state using the inputs in a finite time.

Again we start from the state space model:

The following equations can be derived:

$$x[k+1] = Ax[k] + Bu[k]$$

$$x[1] = Ax[0] + Bu[0]$$

$$x[2] = A^2x[0] + ABu[0] + Bu[1]$$

$$x[3] = A^3x[0] + A^2Bu[0] + ABu[1] + Bu[2]$$

$$\vdots$$

$$x[n] = A^n x[0] + A^{n-1} Bu[0] + \dots + Bu[n-1]$$

Controllability

This last equation can be rewritten as:

$$x[n] - A^n x[0] = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \begin{bmatrix} u[n-1] \\ u[n-2] \\ \vdots \\ u[1] \\ u[0] \end{bmatrix} \quad \text{For a given}$$

$x[0]$ and a desired $x[n]$ the required inputs can be found by solving this system. $\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$ is called the controllability matrix of the system. A system is said to be controllable if the set of equations can be solved for a given $x[0]$ and any desired $x[n]$.

This is the case if the controllability matrix has a rank n .

Detectability and Stabilizability

Observability and controllability are important terms in control theory.

Detectability and stabilizability are also often used as weaker constraints.

A system is detectable if all unstable states are observable.

A system is stabilizable if all unstable states are controllable.

Detectability and stabilizability are also important terms in control theory.

Overview

