Example 1

Problem statement and solution

Suppose we want to stabilize the system $P(s) = \frac{1}{(s-a)}$. We only know that $a \in [0.20; 0.80]$. We propose to use a proportional controller $C(s) = K$. The closed-loop transfer function becomes

$$H(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{\frac{1}{s-a}K}{1 + \frac{1}{s-a}K} = \frac{K}{s - a + K}$$

The controller stabilizes the system if $-a + K > 0$. If we choose K very large the system will be robust against changes in a.

Example 2

Problem statement and solution

Suppose we want to stabilize the system $P(s) = \frac{s}{(s-a)}$ with $a \in [0.20; 0.80]$. If we choose the control law $C(s) = \frac{K}{s}$ this results in the same transfer function.

$$H(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{\frac{s}{s-a}\frac{K}{s}}{1 + \frac{s}{s-a}\frac{K}{s}} = \frac{K}{s - a + K}$$

Again we choose $K > a$ to ensure stability. But how does the controller perform on the slightly perturbed system $\tilde{P}(s) = \frac{(s+\epsilon)}{(s-a)}$? The transfer function from $R(s)$ to $Y(s)$ becomes

$$\tilde{H}(s) = \frac{\frac{s+\epsilon}{s-a}\frac{K}{S}}{1 + \frac{s+\epsilon}{s-a}\frac{K}{s}} = \frac{(s+\epsilon)K}{s^2 + (K-a)s + \epsilon K}$$

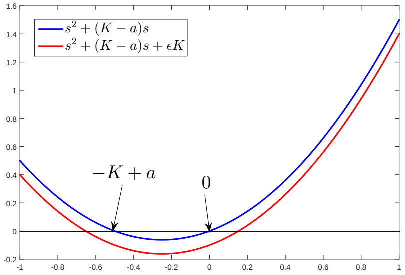
Example 2

$$\tilde{S}(s) = \frac{(s+\epsilon)K}{s^2 + (K-a)s + \epsilon K}$$

The figure on the right plots

$$s^2 + (K-a)s$$
$$s^2 + (K-a)s + \epsilon K$$

For ϵ negative the system becomes unstable. The control system is not robust and useless in practical situations. You should not use control laws which rely on a pole-zero cancellation.



Sensitivity - Robustness

Steady-state error

The steady state error is defined as follows:

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{t\to\infty} (r(t) - y(t)) \\ &= \lim_{s\to 0} s(R(s) - Y(s)) \qquad \text{(final value theorem)}\end{aligned}$$

A very small steady state error (preferably zero) indicates that the controller tracks the reference very well.

Sensitivity - Robustness

Open-loop system

For an open loop system with a step reference, the steady-state error is given by

$$e_{ol}(\infty) = \lim_{s \to 0} s(1 - C(s)P(s))R(s) = 1 - C(0)P(0)$$

- So, the open loop system has no steady state error (for a step reference) if the controller is calibrated such that $C(0)P(0) = 1$
- $\Rightarrow$ a precise calibration of the DC gain (not robust)

Sensitivity - Robustness

Closed-loop system

The steady-state error for a step reference is given by

$$e_{cl}(\infty) = \lim_{s \to 0} s \left(1 - \frac{C(s)P(s)}{1 + C(s)P(s)}\right) R(s)$$
$$= \lim_{s \to 0} \frac{s}{1 + C(s)P(s)} R(s) = \frac{1}{1 + C(0)P(0)}$$

- The steady state error is small if $C(0)P(0)$ is very large
- Again, calibrating the DC gain
- The difference is that we only need a large gain, which is far less demanding than having to make it equal to 1

An open loop controller is not robust

We can now show how this results in a great advantage of the closed-loop strategy over the open loop strategy:

- If P changes slightly (for instance due to a factor that has not been taken up into the model) to $P + \Delta P$
- Then making $e_{ol}(\infty)$ small would require to calibrate anew
- Whereas $e_{cl}(\infty)$ would remain small, as long as $(P(0) + \Delta P(0))C(0)$ remains large

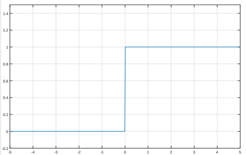
Hence an open loop configuration can not control the output **robustly** against changes in P

Outline

Steady state error: step

Definition

$$\epsilon(t) = \begin{cases} 0, & t \leqslant 0 \\ \frac{1}{2}, & t = 0 \\ 1, & t \geqslant 0 \end{cases}$$



For a step reference $r(t) = a\epsilon(t) \Rightarrow R(s) = \frac{a}{s}$

$$e_{cl}(\infty) = \frac{s}{1 + C(0)P(0)} \frac{a}{s}$$

So $e_{cl}(\infty) = 0$ if C(s)P(s) has a least one pole in zero $(C(0)P(0) \rightarrow \infty)$

Steady state error: ramp

For a ramp reference function $r(t) = at\epsilon(t) \Rightarrow R(s) = \frac{a}{s^2}$, we have

$$e_{cl}(\infty) = \lim_{s\to 0} \frac{s}{1 + C(s)P(s)} \frac{a}{s^2}$$

So, $e_{cl}(\infty) = 0$ if $C(s)P(s)$ has at least two poles in zero.
$\left(\lim_{s\to 0} sC(s)P(s) = \infty\right)$

Type of a system

- The type of a system is determined by the number of poles $P(s)C(s)$ has in zero, and hence it is linked to what type of references it can track perfectly
- Write $P(s)C(s)$ as $\frac{K\prod_{k=1}^{m}(s-z_k)}{s^i\prod_{k=1}^{n-i}(s-p_k)}$ with i the amount of poles in zero
- We say this system is of type i
    - It will be able to track references of shape $at^0\epsilon(t)$ up to $at^{i-1}\epsilon(t)$ perfectly for $t\to\infty$ ($\lim_{t\to\infty} e_{cl}(t)=0$)
    - A reference of shape $at^i\epsilon(t)$ will be missed by a finite factor for $t\to\infty$ ($\lim_{t\to\infty} e_{cl}(t)=C$)
    - A reference of shape $at^{i+1}\epsilon(t)$, $at^{i+2}\epsilon(t)$, ... will be missed by infinity for $t\to\infty$ ($\lim_{t\to\infty} e_{cl}(t)=\infty$)

Example of a type 0 system

Example

Step reference $r(t) = a\epsilon(t) \Rightarrow R(s) = \frac{a}{s}$

$$e_{cl}(\infty) = \lim_{s \to 0} \frac{s}{1 + C(s)P(s)} \frac{a}{s}$$
$$= \lim_{s \to 0} \frac{a}{1 + \frac{K \prod_{k=1}^{m}(-z_k)}{\prod_{k=1}^{n-i}(-p_k)}} = \frac{a}{1 + K_p}$$

where K_p is the static position error constant

Example of a type 0 system

Example

Ramp reference $r(t) = at\epsilon(t) \Rightarrow R(s) = \frac{a}{s^2}$

$$\begin{aligned} e_{cl}(\infty) &= \lim_{s\to 0} \frac{s}{1 + C(s)P(s)} \frac{a}{s^2} \\ &= \lim_{s\to 0} \frac{1}{1 + \frac{K\prod_{k=1}^{m}(-z_k)}{\prod_{k=1}^{n-i}(-p_k)}} \frac{a}{s} \\ &= \lim_{s\to 0} \frac{a}{s + s\frac{K\prod_{k=1}^{m}(-z_k)}{\prod_{k=1}^{n-i}(-p_k)}} = \frac{a}{K_v} = \infty \end{aligned}$$

where K_v is the static velocity error constant

Steady state errors - type of a system

$$K_p = \lim_{s \to 0} P(s)C(s) \qquad K_p = \text{Static position error constant}$$
$$K_v = \lim_{s \to 0} sP(s)C(s) \qquad K_v = \text{Static velocity error constant}$$
$$K_a = \lim_{s \to 0} s^2P(s)C(s) \qquad K_a = \text{Static acceleration error constant}$$

And the respective steady state errors for different system types are:

Type l	**Step $a\epsilon(t)$**	**Ramp $at\epsilon(t)$**	**Parabola $\frac{at^2\epsilon(t)}{2}$**
0	$\frac{a}{1+K_p}$	∞	∞
1	0	$\frac{a}{K_v}$	∞
2	0	0	$\frac{a}{K_a}$