

## Chapter 5: Continuous time systems

July 10, 2015

- 1 Linear differential equations
- 2 Laplace transform
- 3 Solving LDEs with the Laplace transform
- 4 Properties of state-space representation
- 5 Transfer functions
  - Impulse response and time constant
  - Relationship between state space and transfer functions
- 6 Transient response analysis of first order and second order systems
  - First order systems
  - Second order systems

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# Linear differential equations: definitions 1/2

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The **order of a LDE** is the index of the highest derivative of  $y$ .

$$L_n(y) = \sum_{i=0}^n A_i(t) \frac{d^{n-i}y}{dt^{n-i}} = f(t).$$

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- if  $A_{0:n}(t)$  are constants (ie. not functions of time), the LDE is said to have **constant coefficients**

## Example: radioactive decay 1/2

Let  $N(t)$  be the number of radioactive atoms at time  $t$ , then:

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for some constant  $k > 0$ .

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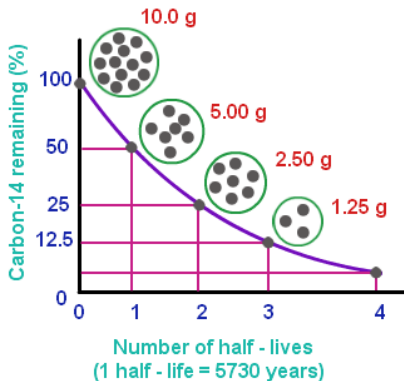
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This is a first order homogeneous LDE with constant coefficients.

# Example: radioactive decay 2/2

## Decay of Carbon - 14



# Solving homogeneous LDEs with constant coefficients 1/3

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Dividing by  $e^{zt}$  yields the  $n$ th order **characteristic polynomial**:

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The specific linear combination depends on initial conditions.

Example:

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These zeros correspond to the following basis functions  $t$ :

$$e^{jt}, \quad e^{-jt}, \quad e^t, \quad te^t.$$

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# The Laplace transform

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The final value theorem states  $f(\infty) = \lim_{s \rightarrow 0} sF(s)$ ,  
if all poles of  $sF(s)$  are in the left half plane (ie. real part  $< 0$ ).



# Important properties of the Laplace transform

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with  $u(t) = \int_{-\infty}^t \delta(\tau)d\tau$  (Heaviside) and  $\delta(t)$  the Dirac delta.

# Inverse Laplace transform

The inverse Laplace transform converts s-domain to time domain:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{j2\pi} \int_{\gamma-jT}^{\gamma+jT} e^{st} F(s) ds.$$



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Practically, the inverse Laplace transform takes two steps:

- 1 write  $F(s)$  in terms of partial fractions
- 2 transform each term in the partial fraction based on tables of  $s/t$ -domain pairs (course notes p 4.32-4.33)

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Via induction, the Laplace transform of the  $n$ th order derivative:

$$\mathcal{L}\{f^{(n)}\} = s^n F(s) - \sum_{i=1}^n s^{n-i} f^{(n-i)}(0)$$

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Via the linearity of the Laplace transform:

$$\sum_{i=0}^n A_i \mathcal{L}\{y^{(n-i)}(t)\} = \mathcal{L}\{f(t)\}$$



# Solving LDEs with the Laplace transform 3/3

$$\sum_{i=0}^n A_i \mathcal{L}\{y^{(n-i)}(t)\} = \mathcal{L}\{f(t)\} \quad (1)$$

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Expanding Eq. (2) into (1) yields:

$$Y(s) \sum_{i=0}^n A_i s^i - \sum_{i=1}^n \sum_{j=1}^i A_i s^{i-j} y^{j-1}(0) = F(s)$$

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The solution in the time domain is obtained via the inverse Laplace transform:  $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ .

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This holds for linear, time-invariant systems with  $n$  states if:

$$\text{rank}(\mathcal{O}) = n, \quad \mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}, \quad \mathcal{O} : \text{observability matrix}$$

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A linear, time-invariant system with  $n$  states is controllable if:

$$\text{rank}(\mathcal{C}) = n, \quad \mathcal{C} = [\mathbf{B} \quad \mathbf{AB} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}],$$

where  $\mathcal{C}$  is called the **controllability matrix**.

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# Transfer function

The transfer function of input  $i$  to output  $j$  is defined as:

$$H_{i,j}(s) = \frac{Y_j(s)}{U_i(s)}, \quad \mathbf{U}(s) = \mathcal{L}\{u(t)\}, \quad \mathbf{Y}(s) = \mathcal{L}\{y(t)\}.$$

MIMO systems with  $n$  inputs and  $m$  outputs have  $n \times m$  transfer functions, one for each input-output pair.

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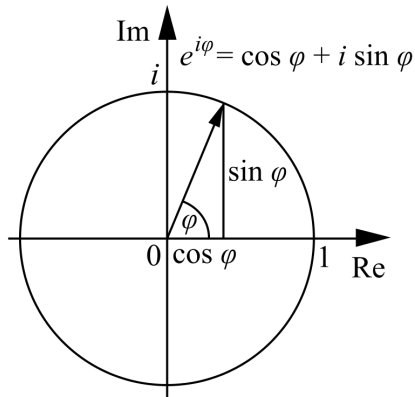
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The complex Laplace variable can be rewritten:  $s = \sigma + j\omega$ .

The frequency response of a system can be analyzed via  $\mathbf{H}(j\omega)$ :

$$e^{\sigma+j\omega} = e^{\sigma}(\cos \omega + j \sin \omega).$$

# Illustration of Euler's formula



In general, the transfer function can be written as:

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Poles and zeros may cancel, ie. if  $D(s) = N(s) = 0$  for some  $s$ .

# Steady state response

The output of a linear time-invariant system yields consists of:

- a steady-state output  $y_{ss}(t)$ , which similar periodicity to  $u(t)$   
→  $y_{ss}$  comprises the same frequencies as  $u(t)$
- a transient output  $y_{tr}(t)$   
→ if the system is stable, then  $\lim_{t \rightarrow \infty} y_{tr}(t) = 0$   
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If we apply an input  $u(t) = \cos(\alpha t + \theta)$ , then:

$$y_{ss}(t) = |H(j\alpha)| \cos(\alpha t + \theta + \angle H(j\alpha))$$

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→  $y_{tr}(t)$  depends on the initial state  $\mathbf{x}_0(t)$  of the system

If we apply an input  $u(t) = \cos(\alpha t + \theta)$ , then:

$$y_{ss}(t) = |H(j\alpha)| \cos(\alpha t + \theta + \angle H(j\alpha))$$

The steady-state output  $y_{ss}(t)$  of a linear time invariant system:

- consists of signals of same frequencies as the input signal  $u(t)$
- which may have been magnified and/or phase changed

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For stable continuous time systems the impulse response always converges to 0:

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The speed of convergence depends on the position of the poles.

The transfer function of first order systems can be written as:

$$H(s) = \frac{K}{\tau s + 1} \quad \leftrightarrow \quad h(t) = \frac{K}{\tau} e^{-t/\tau},$$

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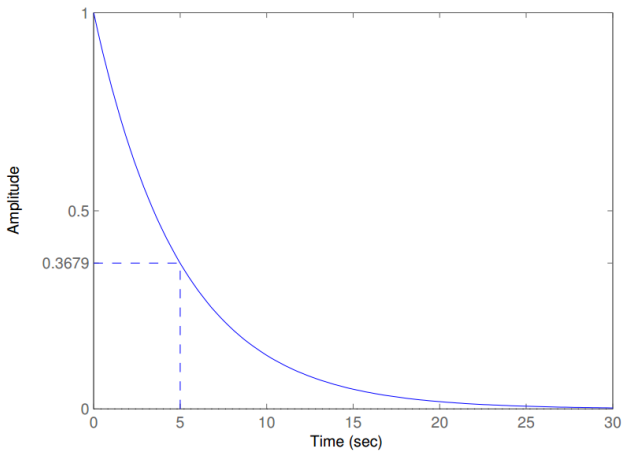
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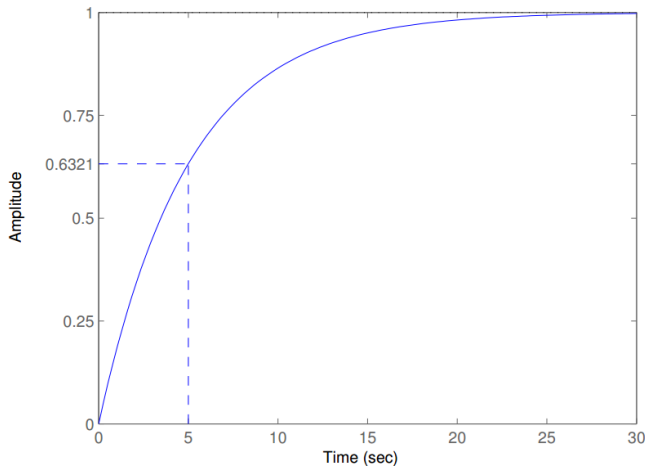
The time constant summarizes the speed of a system's dynamics:

- after  $\tau$  seconds, the impulse response reaches  $h(0)/e$ .
- after  $\tau$  seconds, the step response has reached  $1 - e^{-1} \approx 63\%$  of its regime value.

# Impulse response $H(s) = 5/(5s + 1) \leftrightarrow h(t) = \exp(-t/5)$



# Step response $H(s) = 5/(5s + 1) \leftrightarrow h(t) = \exp(-t/5)$



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# From state-space to transfer functions

We start from the linear state-space representation:

time domain

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

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$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\Rightarrow \mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) + \mathbf{D}\mathbf{U}(s)$$

$$\Rightarrow \mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

# Relationship between poles and eigenvalues of **A** 1/2

Poles are zeros of the denominator of  $\mathbf{H}(s)$ , e.g. those values of  $s$  for which  $\mathbf{H}(s)$  is singular.

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Consider the following SISO system with 2 states:

$$\begin{bmatrix} sX_1(s) \\ sX_2(s) \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} + \begin{bmatrix} \beta \\ 2 \end{bmatrix} U(s)$$
$$Y(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix}$$

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The transfer function  $H(s) = \frac{\beta}{s-\alpha}$  has only one pole ( $s_1 = \alpha$ ).

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# Transient Response

The time response of a control system may be written as:

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

Where  $y_{tr}(t)$  is the transient response and  $y_{ss}(t)$  is the steady state response. Most important characteristic of dynamic system is absolute stability.

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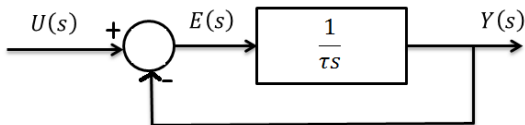
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Transient response: when input of system changes, output does not change immediately but takes time to go to steady state

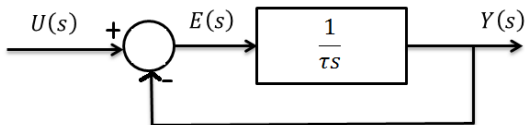
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E.g. RC circuit, thermal system, ...



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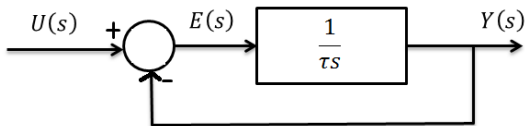
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Unit step response

- Laplace of unit-step is  $\frac{1}{s} \rightarrow$  substituting  $U(s) = \frac{1}{s}$  into equation  $Y(s) = \frac{1}{s} \frac{1}{\tau s + 1}$
- Expanding into partial fractions gives

$$Y(s) = \frac{1}{s} - \frac{\tau}{\tau s + 1} = \frac{1}{s} - \frac{1}{s + \frac{1}{\tau}}$$

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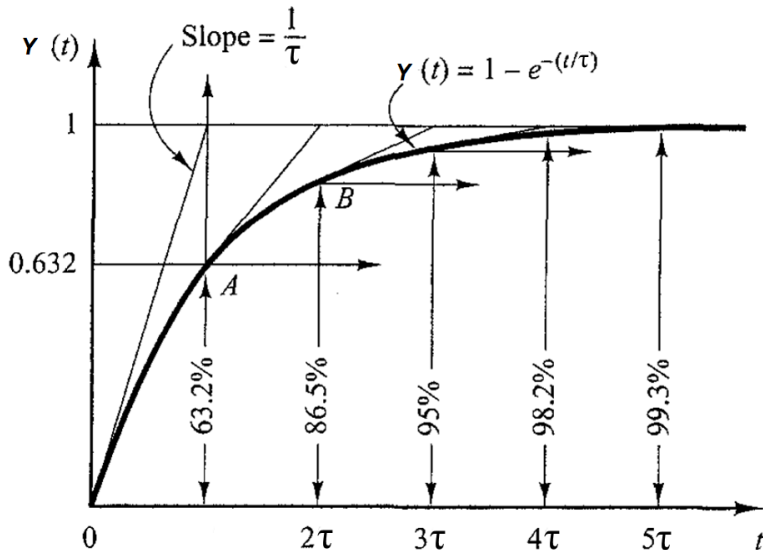
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- ⑤ Slope at time  $t = 0$  is  $\frac{1}{\tau}$

$$\left. \frac{dy}{dt} \right|_{t=0} = \left. \frac{1}{\tau} e^{-\frac{t}{\tau}} \right|_{t=0} = \frac{1}{\tau}$$

Where  $\tau$  is called the system time constant

# Unit step transient response



# Unit ramp transient response

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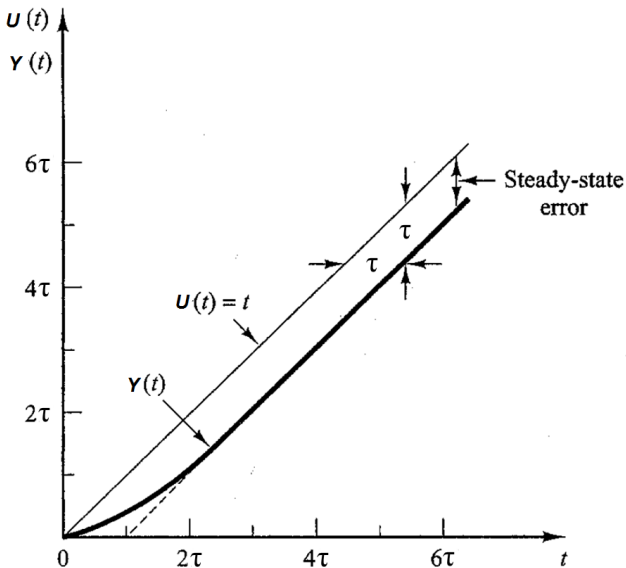
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$$e(\infty) = \tau$$

## Unit ramp transient response





# Unit-Impulse Response

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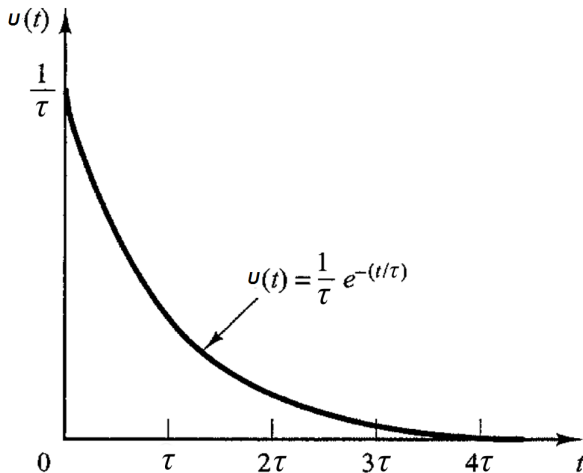
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For  $t \rightarrow +\infty$ ,  $y(t) \rightarrow 0$

# Unit-Impulse Response



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# Second order systems

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If the transfer function has two real poles, the frequency response can be found by combining the effects of both poles



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In order to study the transient behaviour, let us first consider the following simplified example of a second order system

$$H(s) = \frac{c}{ds^2 + es + c}$$

# Step response second order system

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④ The poles are real if

$$e^2 - dc \geq 0$$

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To simplify the transient analysis, it is convenient to write

$$\frac{f}{d} = \omega_n^2, \quad \frac{e}{d} = 2\zeta\omega_n = 2\sigma$$

Where

$\sigma$  is the attenuation

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Which is called the standard form of the second-order system.

The dynamic behavior of the second-order system can then be described in terms of only two parameters  $\zeta$  and  $\omega_n$ .

# Step response second order system

If  $0 < \zeta < 1$ , the poles are complex conjugates and lie in the left-half  $s$ -plane

- The system is then called **underdamped**
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If  $\zeta > 1$ , the system is called **overdamped**

We will now look at the unit step response for each of these cases

# Underdamped system

For the underdamped case ( $0 < \zeta < 1$ ), the transfer function can be written as:

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

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For a unit-step input we can write

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}$$



# Underdamped system

Which can be rewritten as partial fractions

$$\begin{aligned} Y(s) &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} - \frac{\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \end{aligned}$$

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It can be shown that

$$\begin{aligned} \mathcal{L}^{-1}\left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] &= e^{-\zeta\omega_nt} \cos(\omega_d t) \\ \mathcal{L}^{-1}\left[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right] &= e^{-\zeta\omega_nt} \sin(\omega_d t) \end{aligned}$$

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$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)\right)$$

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It can be seen that the frequency of the transient oscillation is the damped natural frequency  $\omega_d$  and thus varies with the damping ratio  $\zeta$

# Underdamped system

The error signal is the difference between input and output

$$\begin{aligned} e(t) &= y(t) - u(t) \\ &= e^{-\zeta\omega_n t} \left( \cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin(\omega_d t) \right) \end{aligned}$$

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The error signal exhibits a damped sinusoidal oscillation

At steady state, or at  $t = \infty$ , the error goes to zero



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Where

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- Once the faster decaying exponential term has disappeared, the response is similar to that of a first-order system
- In that case,  $H(s)$  can be approximated by

$$H(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} = \frac{s_2}{s + s_2}$$

# Overdamped system

With the approximate transfer function, the unit-step response becomes

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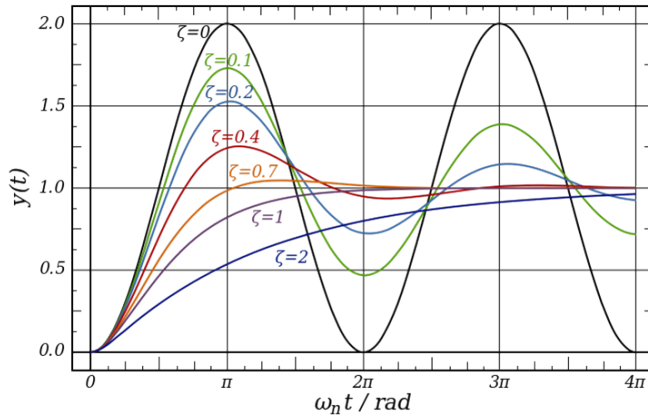
$$Y(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

The time response for the approximate transfer function is then given as

$$y(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}, \text{ for } t \geq 0$$

# Second order systems unit step response curves

Response on a step function



# Second order systems - characteristics

Overshoot: Highest amplitude above steady state.

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

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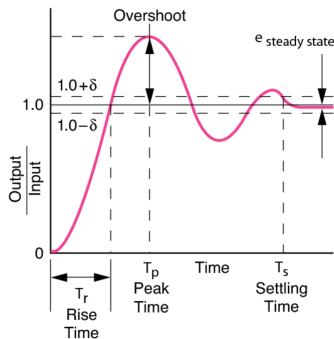
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Rise Time: Time needed to reach the steady state for the first time.  $t_r = \frac{1.8}{\omega_n}$

Peak Time: Time to reach overshoot.

$$t_p = \frac{\pi}{\omega_d}$$



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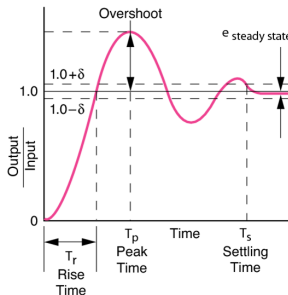
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$$\text{For: } \delta = \frac{0.02}{\sqrt{1-\zeta^2}}$$

We find:

$$e^{-\zeta \omega_n T_s} < 0.02$$

$$T_s = \frac{4}{\omega_n \zeta}$$



## Second order systems - resonance

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A different view on the Tacoma bridge disaster:

<https://www.youtube.com/watch?v=6ai2QFxStxo>

In fact the collapse was a result of a number of effects like Aerodynamic flutter and vortices. Read the full article here:

<http://www.ketchum.org/billah/Billah-Scanlan.pdf>



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Another phenomenon with bridges and resonance is that many people marching with the same rhythm can cause a bridge to start resonating like the Angers bridge in 1850. A more recent example is the Millennium bridge in London who started resonating.

# Second order systems - damping

When we want a system with no resonance, we choose one with damping  $< 0.707$ . This means a pole between  $135^\circ$  and  $225^\circ$ :

$$\arctan\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = +135^\circ$$

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We mostly want a short settling time ( $< 4s$ ). This results in another restriction on the poles of the system:

$$\tau_n = \frac{4}{\omega\zeta} < 4s$$
$$\omega_n\zeta > 1$$

# Second order systems - damping

