Outline

Observability

Definition

A measure of how well a system's internal states $\mathbf{x}$ can be inferred by knowledge of its outputs $\mathbf{y}$.

Formally, a system is said to be observable if, for any possible sequence of state and control vectors, the current state can be determined in finite time using only the outputs.
This holds for linear, time-invariant systems with n states if:

$$rank(\mathcal{O}) = n, \quad \mathcal{O} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}, \quad \mathcal{O} : \textbf{observability matrix}$$

Controllability

Definition

A measure of the ability to move a system around in its entire configuration space using only certain admissible manipulations.

A system is controllable if its state can be moved from any initial state $\mathbf{x}_0$ to any final state $\mathbf{x}_f$ via some finite sequence of inputs $\mathbf{u}_0 \ldots \mathbf{u}_f$.
A linear, time-invariant system with n states is controllable if:

$$rank(\mathcal{C}) = n, \quad \mathcal{C} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \ldots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix},$$

where $\mathcal{C}$ is called the **controllability matrix**.

Outline

Transfer function

Definition

The transfer function of input i to output j is defined as:

$$H_{i,j}(s) = \frac{Y_j(s)}{U_i(s)}, \quad \mathbf{U}(s) = \mathcal{L}\{u(t)\}, \quad \mathbf{Y}(s) = \mathcal{L}\{y(t)\}.$$

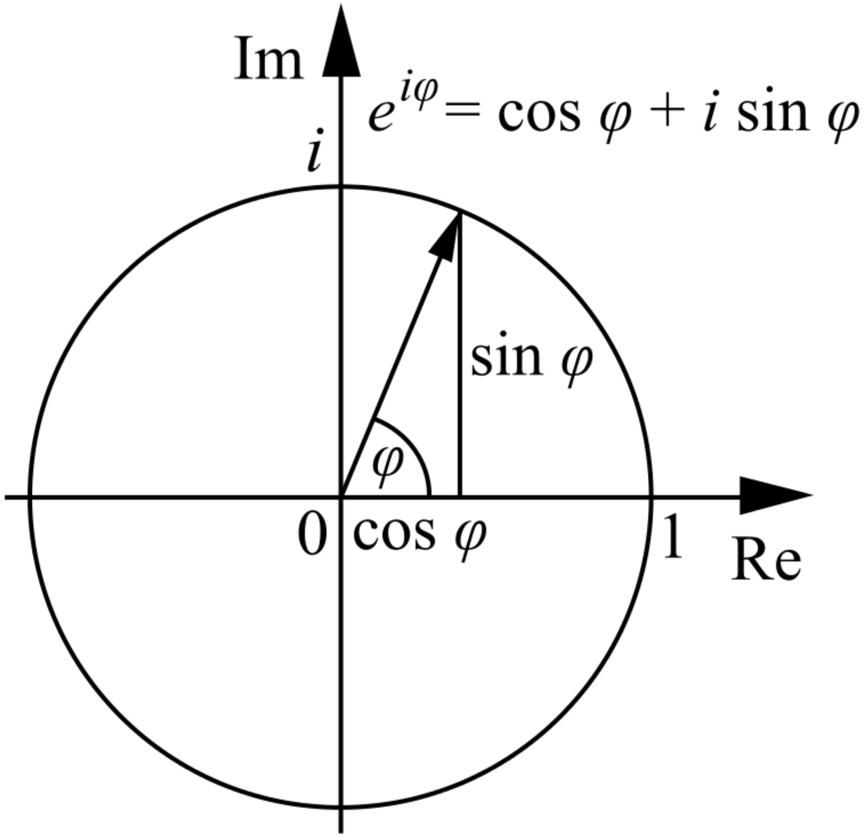
MIMO systems with n inputs and m outputs have $n \times m$ transfer functions, one for each input-output pair.

The complex Laplace variable can be rewritten: $s = \sigma + j\omega$.

The frequency response of a system can be analyzed via $\mathbf{H}(j\omega)$:

$$e^{\sigma + j\omega} = e^{\sigma}(\cos\omega + j\sin\omega).$$

Illustration of Euler's formula



Poles and zeros

In general, the transfer function can be written as:

$$H(s) = \frac{N(s)}{D(s)}.$$

The poles of $H(s)$ are zeros of $D(s)$, i.e. $\{s \ : \ D(s) = 0\}$.

- $|H(s)| = \infty$ if s is a pole.

The zeros of $H(s)$ are zeros of $N(s)$, i.e. $\{s \ : \ N(s) = 0\}$.

- $H(s) = 0$ if s is a zero.

Poles and zeros may cancel, i.e. if $D(s) = N(s) = 0$ for some s.

Steady state response

The output of a linear time-invariant system consists of:

- a steady-state output $y_{ss}(t)$, with the same period as $u(t)$
  $\rightarrow$ y_{ss} comprises the same frequencies as $u(t)$;
- a transient output $y_{tr}(t)$
  $\rightarrow$ if the system is stable, then $\lim_{t\to\infty} y_{tr}(t) = 0$
  $\rightarrow$ $y_{tr}(t)$ depends on the initial state $\mathbf{x}_0(t)$ of the system.

If we apply an input $u(t) = cos(\alpha t + \theta)$, then:

$$y_{ss}(t) = |H(j\alpha)|cos(\alpha t + \theta + \angle H(j\alpha)).$$

The steady-state output $y_{ss}(t)$ of a linear time invariant system:

- consists of signals of same frequencies as the input signal $u(t)$;
- which may have been magnified and/or phase changed.