

Outline

- 1 Introduction
- 2 Analysis of the sample and hold
- 3 Fourier transform
- 4 Spectrum of a sampled signal
 - Aliasing
 - Sampling theorem
 - Hidden oscillations
- 5 Data extrapolation (reconstruction)
- 6 Block-diagram analysis of sampled-data systems
 - General Approach
 - Examples

Hidden oscillations

Definition

There is the possibility that a signal contains some frequencies that the samples do not show at all.

Such signals, when they occur in digital control systems, are called **hidden oscillations**.

They can only occur at multiples of the Nyquist frequency (π/T_s).

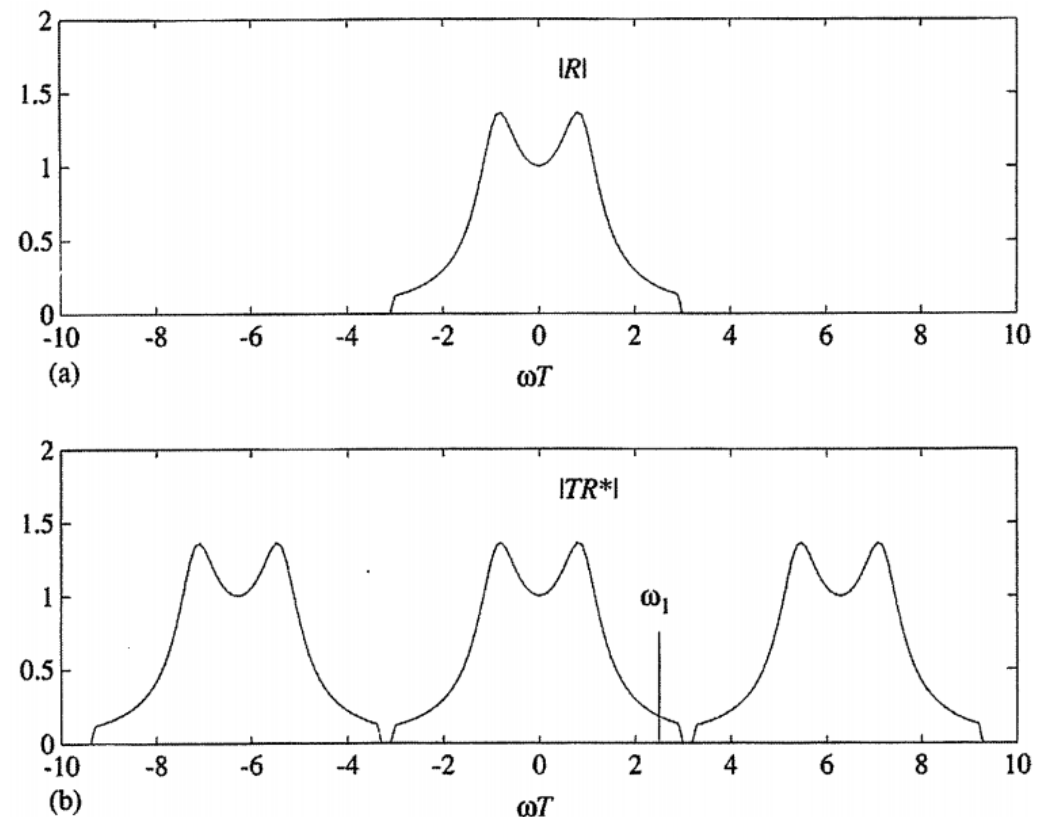
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Reconstruction

Sampling theorem: *under the right conditions* it is possible to recover a signal from its samples.

The figure to the right shows the spectrum of $R(j\omega)$. It is contained in the low-frequency part of $R^*(j\omega)$. Therefore, to recover $R(j\omega)$ we need to process $R^*(j\omega)$ through a low-pass filter and multiply by T_s .



Reconstruction

If $R(j\omega)$ has zero energy for frequencies in the bands above the Nyquist frequency, in other words R is band-limited, then an ideal low-pass filter with gain T for $-\pi/T \leq \omega \leq \pi/T$ and zero elsewhere would recover $R(j\omega)$ from $R^*(j\omega)$ exactly.

If we define the ideal low-pass filter characteristic as $L(j\omega)$, we have:

$$R(j\omega) = L(j\omega)R^*(j\omega).$$

The signal $r(t)$ is the inverse Fourier transform of $R(j\omega)$. Because $R(j\omega)$ is the *product* of two Fourier transforms, $r(t)$ is the *convolution* of the time functions $\ell(t)$ and $r^*(t)$.

$$r(t) = \ell(t) * r^*(t).$$

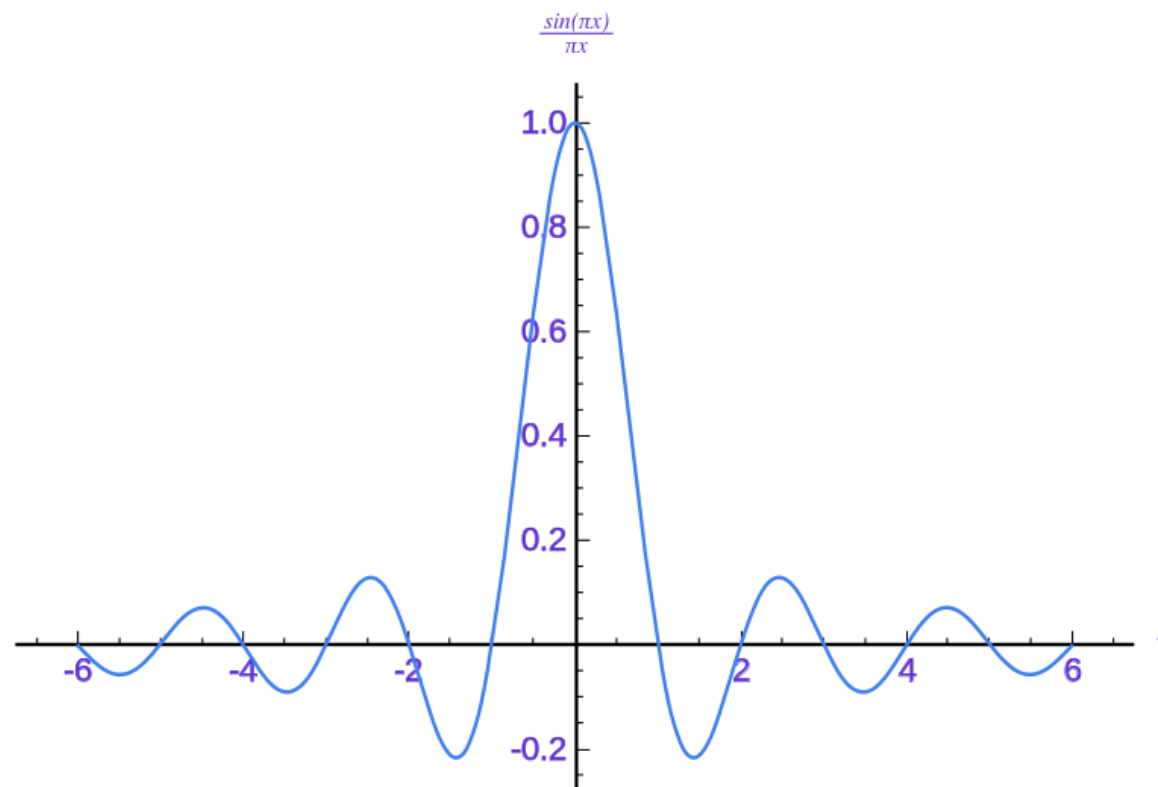
Ideal low-pass filter

The impulse response of the filter can be computed using this definition:

$$\begin{aligned}\ell(t) &= \frac{1}{2\pi} \int_{-\pi/T_s}^{\pi/T_s} T e^{j\omega t} d\omega \\ &= \frac{T_s}{2\pi} \frac{e^{j\omega t}}{jt} \bigg|_{-\pi/T_s}^{\pi/T_s} \\ &= \frac{T_s}{2\pi jt} (e^{j(\pi t/T_s)} - e^{-j(\pi t/T_s)}) \\ &= \frac{\sin(\pi t/T_s)}{\pi t/T_s} \\ &\triangleq \text{sinc} \frac{\pi t}{T_s}\end{aligned}$$

Ideal low-pass filter

The sinc functions are the interpolators that fill in the time gaps between samples with a signal that has no frequencies above π/T_s .



Reconstruction

Using the previous equations, we find:

$$r(t) = \int_{-\infty}^{\infty} r(\tau) \sum_{k=-\infty}^{\infty} \delta(\tau - kT_s) \text{sinc} \frac{\pi(t-\tau)}{T_s} d\tau.$$

Using the shifting property of the impulse, we obtain:

$$r(t) = \sum_{k=-\infty}^{\infty} r(kT_s) \text{sinc} \frac{\pi(t-kT_s)}{T_s}.$$

This filter is non-causal because $\ell(t)$ is nonzero for $t < 0$. $\ell(t)$ starts at $t = -\infty$ while the impulse that triggers it does not occur until $t = 0$. The non-causality can be overcome by adding a phase lag, $e^{-j\omega\lambda}$, to $L(j\omega)$, which adds a delay to the filter and to the signals processed through it.

Zero-order hold

The transfer function of the zero-order hold was introduced as

$$ZOH(j\omega) = \frac{1 - e^{-j\omega T_s}}{j\omega}.$$

We express this function in magnitude and phase form, to discover the frequency properties of $ZOH(j\omega)$.

We factor out $e^{-j\omega T_s/2}$ and multiply and divide by $2j$:

$$\begin{aligned} ZOH(j\omega) &= e^{-j\omega T_s/2} \left\{ \frac{e^{j\omega T_s/2} - e^{-j\omega T_s/2}}{2j} \right\} \frac{2j}{j\omega} \\ &= T_s e^{-j\omega T_s/2} \frac{\sin(\omega T_s/2)}{\omega T_s/2} \\ &= e^{-j\omega T_s/2} T_s \text{sinc}(\omega T_s/2) \end{aligned}$$

Zero-order hold

The magnitude function is

$$|ZOH(j\omega)| = T_s \left| \text{sinc} \frac{\omega T_s}{2} \right|$$

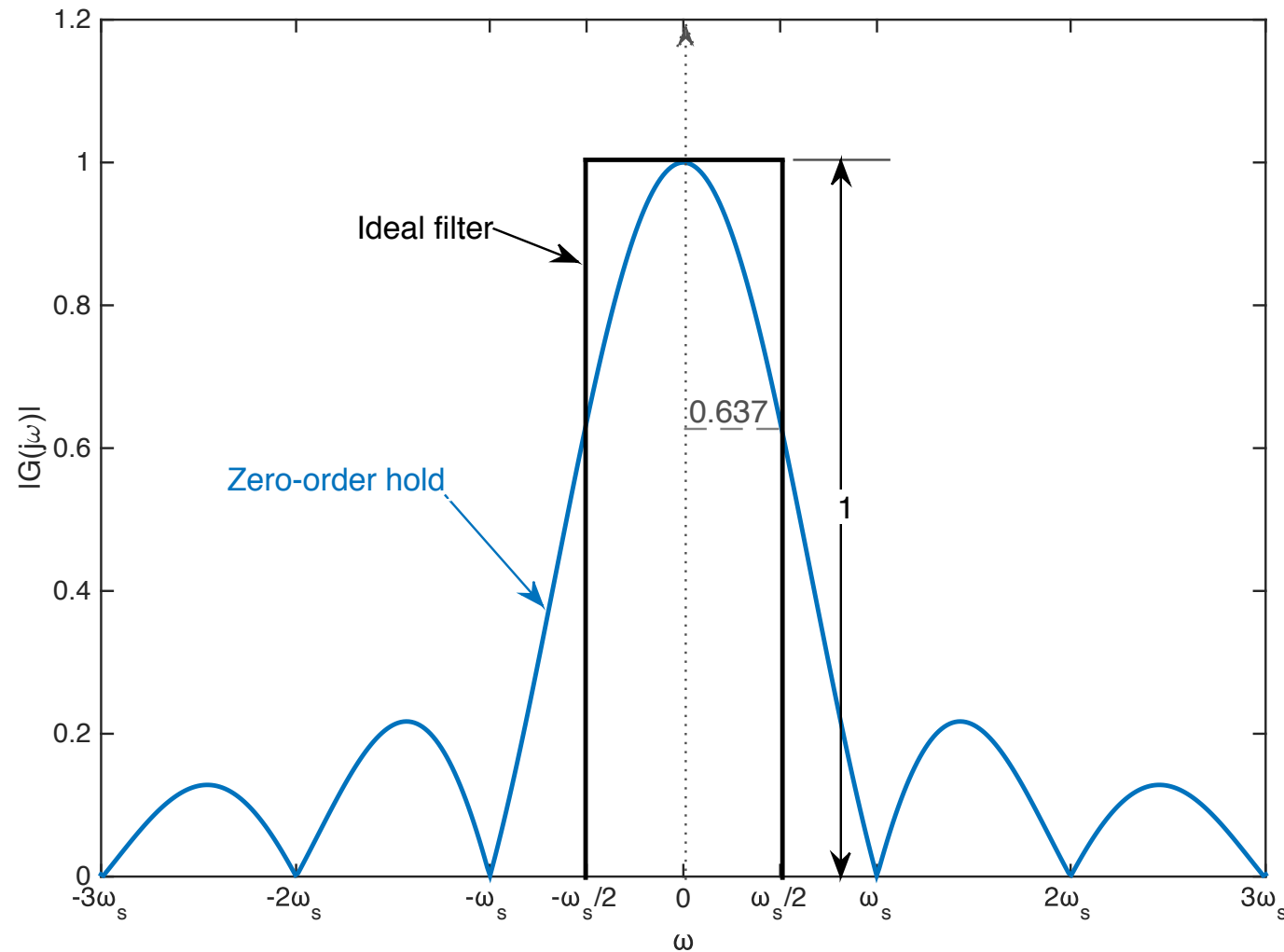
and the phase is

$$\angle ZOH(j\omega) = \frac{-\omega T_s}{2}$$

plus the 180° shifts where the sinc function changes sign.

Thus the effect of the zero-order hold is to introduce a phase shift of $\omega T_s/2$ (a time delay of $T_s/2$ seconds) and to multiply the gain by a function with the magnitude of $\text{sinc}(\omega T_s/2)$.

Zero-order hold filter vs ideal filter



First-order hold

If the extrapolation is done by a first order polynomial, then the extrapolator is called a first-order hold and its transfer function is denoted $FOH(s)$.

$$FOH(s) = (1 - e^{-sT_s}) \frac{sT_s + 1}{sT_s^2}$$

