Outline

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Outline

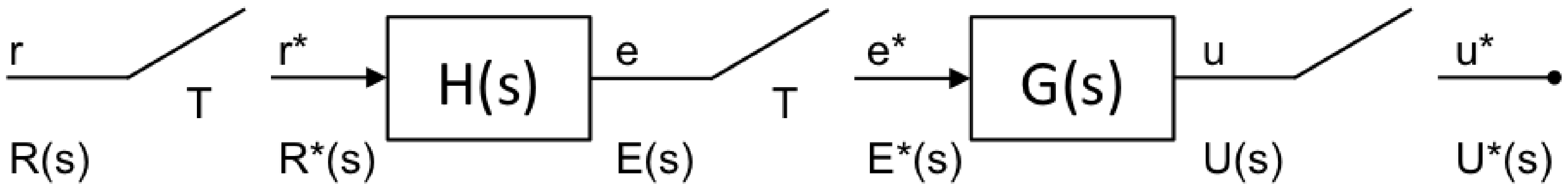
Problem Statement

To analyse a feedback system that contains a digital controller, we need to be able to compute the transforms of output signals of systems that contain sampling operations in various places, including feedback loops, in the block diagram.

The technique presented in this chapter is a simple extension of the block-diagram analysis of systems that are all continuous or all discrete. However a couple of rules need to be carefully observed.

Block-diagram analysis

We represent the process of sampling a continuous signal and holding it by impulse modulation followed by low-pass filtering. For example the following system:



leads to:

$$\begin{aligned} E(s) &= R^*(s)H(s) \\ U(s) &= E^*(s)G(s) \end{aligned} \tag{2}$$

Block-diagram analysis

Relation I

If the transform of the signal to be sampled is a product of a transform that is already periodic of period $\frac{2\pi}{T}$, and one that is not, as in $U(s) = E^*(s)G(s)$, where $E^*(s)$ is periodic and $G(s)$ is not, we can show that $E^*(s)$ comes out as a factor, resulting in the following rule:

$$U^*(s) = \left(E^*(s)G(s)\right)^* = E^*(s)G^*(s) \tag{3}$$

We will prove this in the frequency domain, using $R^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} R(s - jn\omega_s)$.

Proof of $U^*(s) = \big(E^*(s)G(s)\big)^* = E^*(s)G^*(s)$

If $U(s) = E^*(s)G(s)$, then by definition we have:

$$U^*(s) = \frac{1}{T}\sum_{n=-\infty}^{\infty} E^*(s - jn\omega_s)G(s - jn\omega_s), \tag{4}$$

but $E^*(s)$ is

$$E^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} E(s - jk\omega_s), \tag{5}$$

so that

$$E^*(s - jn\omega_s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} E(s - jk\omega_s - jn\omega_s). \tag{6}$$

Proof of $U^*(s) = \left(E^*(s)G(s)\right)^* = E^*(s)G^*(s)$

Now in Eq.(6) we can let $k = l - n$ to get

$$\begin{aligned} E^*(s - jn\omega_s) &= \frac{1}{T}\sum_{l=-\infty}^{\infty} E(s - jl\omega_s) \\ &= E^*(s) \end{aligned} \tag{7}$$

In other words, because E^* is already periodic, shifting it an integral number of periods leaves it unchanged. Substituting Eq.(7) into Eq.(4) yiels

$$\begin{aligned} U^*(s) &= E^*(s)\frac{1}{T}\sum_{-\infty}^{\infty} G(s - jn\omega_s) \\ &= E^*(s)G^*(s) \end{aligned} \tag{8}$$

Block diagram analysis

Note what is not true

If $U(s) = E(s)G(s)$, then $U^*(s) \neq E^*(s)G^*(s)$ but rather $U^*(s) = (EG)^*(s)$. The periodic character of E^* in Eq.(3) is crucial.

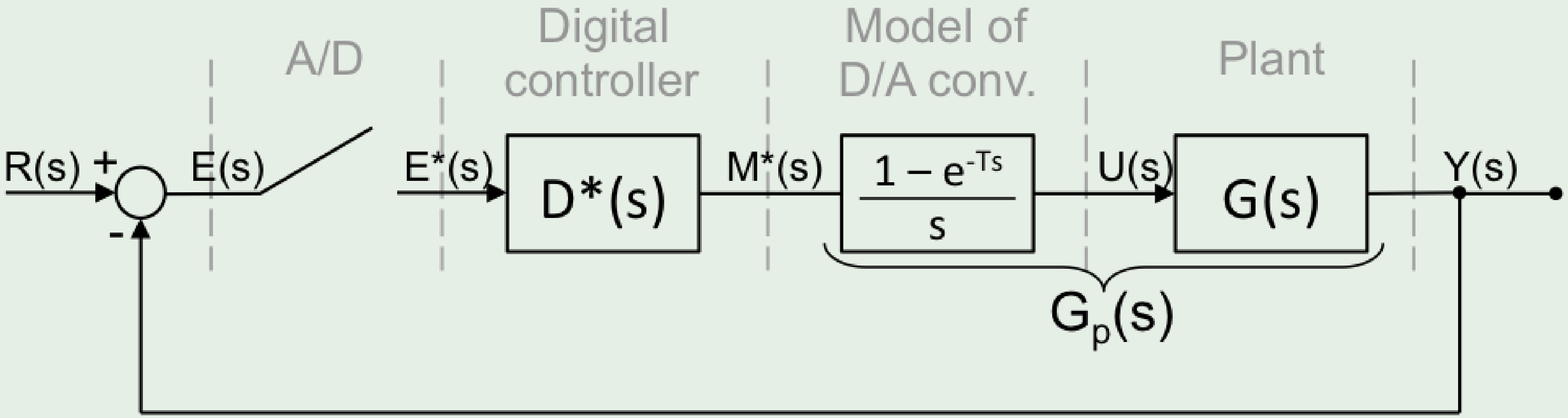
Relation II

Given a sampled-signal transform such as $U^*(s)$, we can find the corresponding $\mathcal{Z}$-transform simply by letting $e^{sT} = z$ or $U(z) = U^*(s)|_{e^{sT}=z}$.

Outline

Example 1 (1/4)

Let's compute the discrete-time representation of the following block diagram:



From the diagram we have that:

$$E(s) = R(s) - Y(s) \quad (9)$$

$$Y(s) = G_p(s)D^*(s)E^*(s) \quad (10)$$

Example 1 (2/4)

Applying Relation I on both Eq.(9) and Eq.(10) results in:

$$E^*(s) = R^*(s) - Y^*(s) \tag{11}$$

$$Y^*(s) = \left(G_p(s)D^*(s)E^*(s)\right)^* = G_p^*(s)D^*(s)E^*(s) \tag{12}$$

Next we insert Eq.(11) in Eq.(12):

$$Y^*(s) = G_p^*(s)D^*(s)\left(R^*(s) - Y^*(s)\right) \tag{13}$$

Example 1 (3/4)

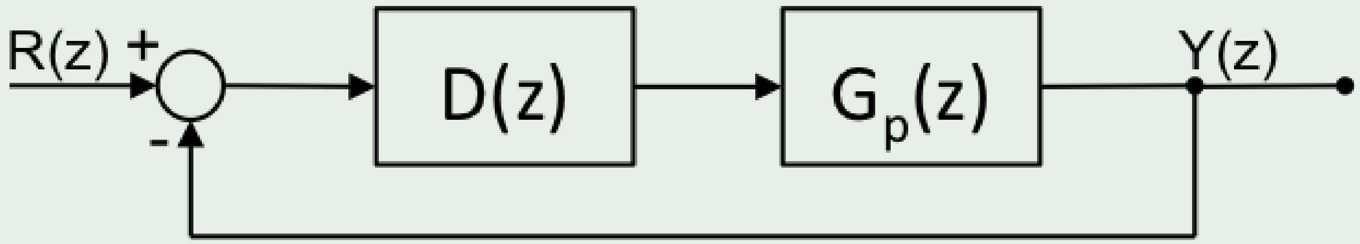
Finally we convert Eq.(13) into a transfer function:

$$
\begin{aligned}
Y^*(s) &= G_p^*(s)D^*(s)\big(R^*(s) - Y^*(s)\big) \\
Y^*(s) + G_p^*(s)D^*(s)Y^*(s) &= G_p^*(s)D^*(s)R^*(s) \\
Y^*(s)\big(1 + G_p^*(s)D^*(s)\big) &= G_p^*(s)D^*(s)R^*(s) \\
\frac{Y^*(s)}{R^*(s)} &= \frac{G_p^*(s)D^*(s)}{1 + G_p^*(s)D^*(s)} \\
\frac{Y(z)}{R(z)} &= \frac{G_p(z)D(z)}{1 + G_p^{(}z)D(z)}
\end{aligned}
$$

The last expression is obtained by applying Relation II.

Example 1 (4/4)

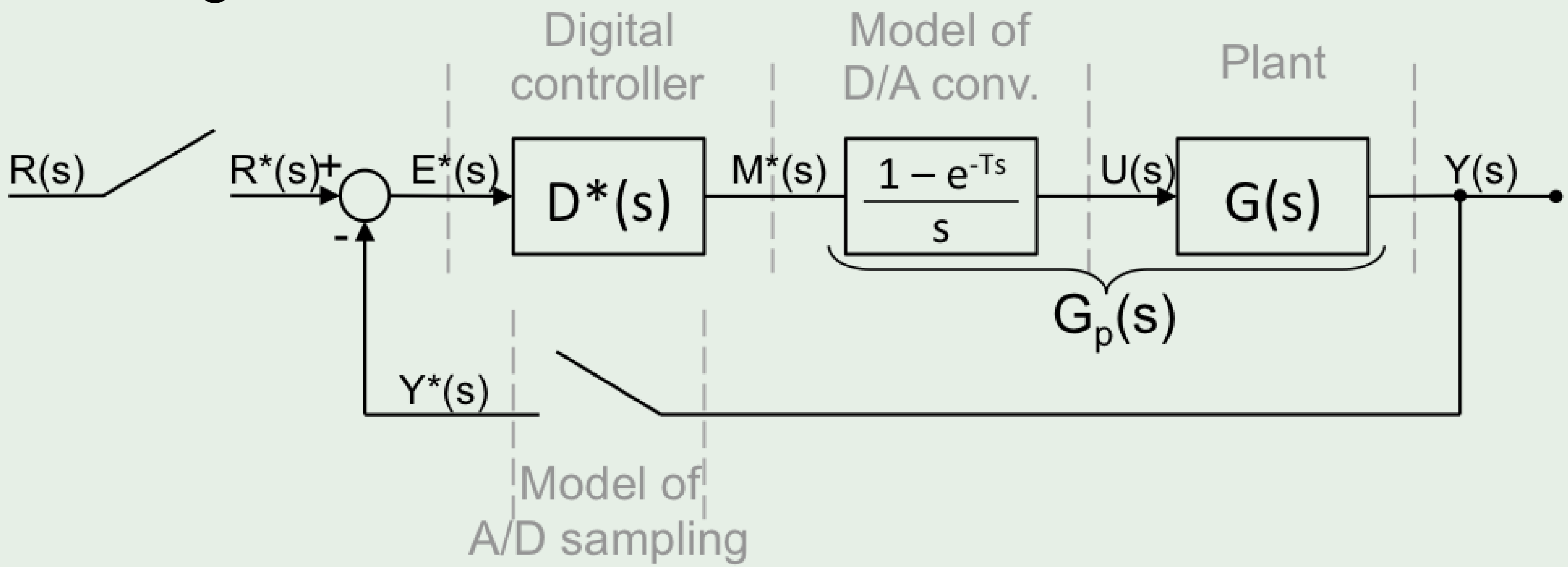
Notice that the resulting transfer function is also the transfer function of the following discrete-time system:



Consequently, we have found a discrete-time equivalent of a system which contains continuous-time and sampled signals.

Example 2 (1/3)

Let's compute the discrete-time representation of the following block diagram:



From the diagram we have that:

$$E^*(s) = R^*(s) - Y^*(s) \tag{14}$$

$$Y(s) = G_p(s)D^*(s)E^*(s) \tag{15}$$

Example 2 (2/3)

Applying Relation I on Eq.(15), yields:

$$Y^*(s) = \left(G_p(s)D^*(s)E^*(s)\right)^* = G_p^*(s)D^*(s)E^*(s) \tag{16}$$

Next we insert Eq.(14) in Eq.(16):

$$Y^*(s) = G_p^*(s)D^*(s)\left(R^*(s) - Y^*(s)\right) \tag{17}$$

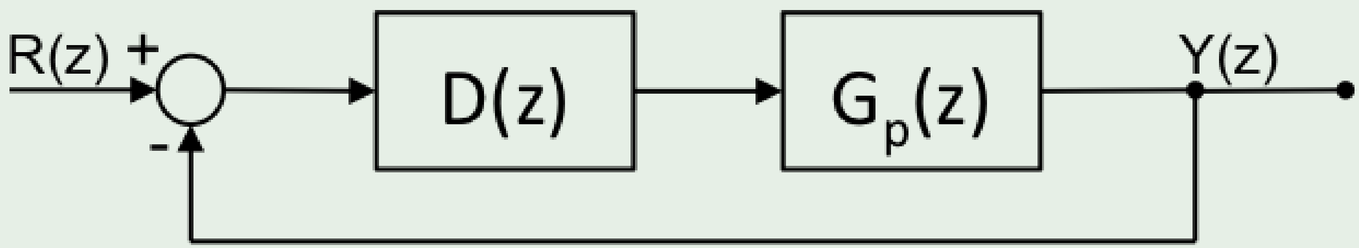
This is the exact same expression as we have for example 1. We will find the same transfer function from Eq.(17).

Example 2 (3/3)

The transfer function:

$$\frac{Y(z)}{R(z)} = \frac{G_p(z)D(z)}{1+G_p^{(}z)D(z)}$$

is also the transfer function of the following discrete-time system:



Consequently, we have found a discrete-time equivalent of a system which contains continuous-time and sampled signals.

NOTE

$$G_p^*(s) = \left[\frac{(1-e^{-Ts})}{s} G(s)\right]^*$$

Taking out the periodic parts, which are those in which s appears only as e^{Ts}, we have that

$$G_p^*(s) = (1 - e^{-Ts})\left[\frac{G(s)}{s}\right]^*$$

Letting $e^{Ts} = z$, we have that:

$$G_p(z) = (1 - z^{-1})\mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$