# Chapter 5: Continuous time systems Transient response analysis of first order and second order systems

Van Assche Jonah

July 9, 2015

# Transient Response

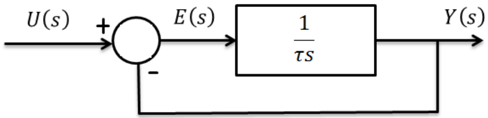
- The time response of a control system may be written as:

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

- Where $y_{tr}(t)$ is the transient response and $y_{ss}(t)$ is the steady state response.
- Most important characteristic of dynamic system is absolute stability.
  - System is stable when returns to equilibrium if subject to initial condition
  - System is critically stable when oscillations of the output continue forever
  - System is unstable when output diverges without bound from equilibrium if subject to initial condition
- Transient response: when input of system changes, output does not change immediately but takes time to go to steady state

# First-order systems

- E.g.RC circuit, thermal system, ...




- Transfer function is given by: $\frac{Y(s)}{U(s)} = \frac{1}{\tau s+1}$
- Unit step response
  - Laplace of unit-step is $\frac{1}{s}$ $\rightarrow$ substituting $U(s) = \frac{1}{s}$ into equation $Y(s) = \frac{1}{s}\frac{1}{\tau s+1}$
  - Expanding into partial fractions gives

    $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$

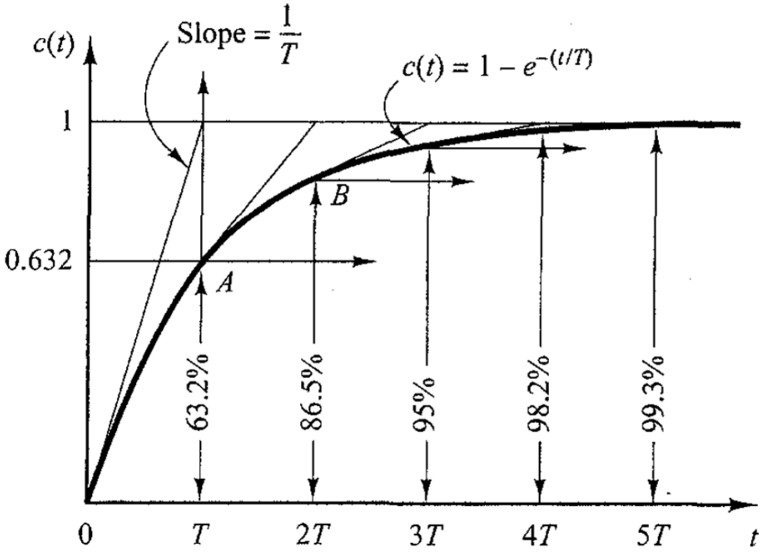
# Unit step transient response

- $Y(s) = \frac{1}{s} - \frac{\tau}{\tau s+1} = \frac{1}{s} - \frac{1}{s+\frac{1}{\tau}}$
- Taking the inverse Laplace transform

  $y(t) = 1 - e^{-\frac{t}{\tau}}$, for $t \geq 0$
- At $t = 0$, the output $c(t) = 0$
- At $t = \tau$, the output $c(t) = 0.632$, or $c(t)$ has reached 63.2% of it's total change $y(\tau) = 1 - e^{-1} = 0.632$
- Slope at time $t = 0$ is $\frac{1}{\tau}$

  $\frac{dy}{dt}|_{t=0} = \frac{1}{\tau}e^{-\frac{t}{\tau}}|_{t=0} = \frac{1}{\tau}$
- Where $\tau$ is called the system's time constant

# Unit step transient response



# Unit ramp transient response

- Laplace transform of unit ramp is $\frac{1}{s^2}$

  $Y(s) = \frac{1}{\tau s+1}\frac{1}{s^2}$

- Expanding into partial fractions gives

  $Y(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s+1}$

- Taking the inverse Laplace transform

  $y(t) = t - \tau + \tau e^{-\frac{t}{\tau}}$, for $t \geq 0$

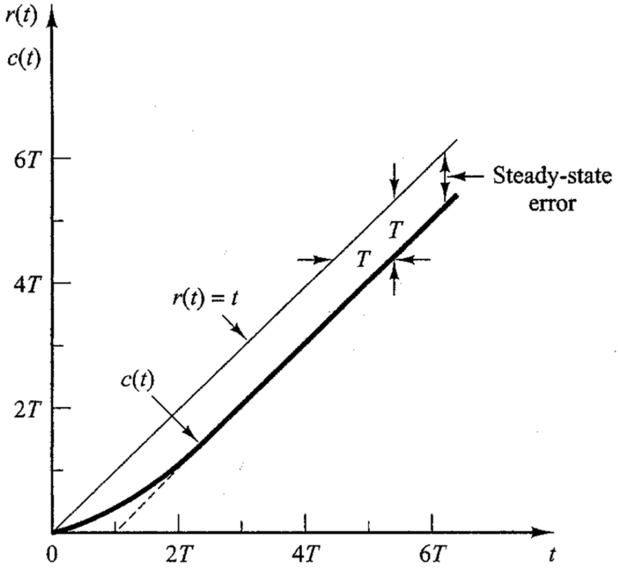
- The error signal $e(t)$ is then

  $e(t) = r(t) - c(t) = \tau(1 - e^{-\frac{t}{\tau}})$

- For $t$ approaching infinity, $e(t)$ approaches $\tau$

  $e(\infty) = \tau$

# Unit ramp transient response



# Unit-Impulse Response

- For a unit-impulse input, $U(s) = 1$ and the output is

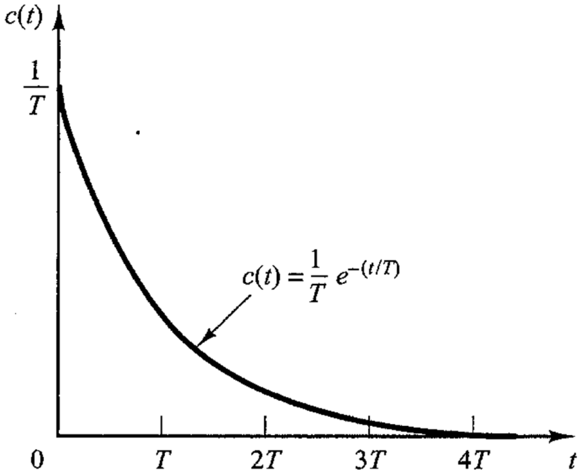
$$Y(s) = \frac{1}{\tau s + 1}$$

- The inverse Laplace transform gives

$$y(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}}, \text{ for } t \geq 0$$

- For $t \to +\infty$, $y(t) \to 0$

# Unit-Impulse Response



# Second order systems

- A second order system can generally be written as:

$$\frac{Y(s)}{U(s)} = H(s) = \frac{as^2 + bs + c}{ds^2 + es + f}$$

- A system where the closed-loop transfer function possesses two poles is called a second-order system
- If the transfer function has two real poles, the frequency response can be found by combining the effects of both poles

- Sometimes the transfer function has two complex conjugate poles. In that case we have to find a different solution for finding the frequency response.

- In order to study the transient behaviour, let us first consider the following simplified example of a second order system

$$H(s) = \frac{c}{ds^2 + es + c}$$

# Step response second order system

- $H(s) = \frac{c}{ds^2+es+c}$
- The transfer function can be rewritten as:

$$H(s) = \frac{\frac{c}{d}}{s^2 + \frac{e}{d}s + \frac{c}{d}}$$
$$= \frac{\frac{c}{d}}{[s + \frac{e}{2d} + \sqrt{(\frac{e}{2d})^2 - \frac{c}{d}}][s + \frac{e}{2d} - \sqrt{(\frac{e}{2d})^2 - \frac{c}{d}}]}$$

- The poles are complex conjugates if

$$e^2 - 4dc < 0$$

- The poles are real if

$$e^2 - dc \geq 0$$

# Step response second order system

- To simplify the transient analysis, it is convenient to write

$$\frac{f}{d} = \omega_n^2,\ \frac{e}{d} = 2\zeta\omega_n = 2\sigma$$

- Where
  $\sigma$ is het attenuation
  $\omega_n$ is the undamped natural frequency
  $\zeta$ is the damping ratio

- The transfer function can now be rewritten as

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega s + \omega_n^2}$$

  Which is called the standard form of the second-order system.

- The dynamic behavior of the second-order system can then be described in terms of only two parameters $\zeta$ and $\omega_n$

# Step response second order system

- If $0 < \zeta < 1$, the poles are complex conjugates and lie in the left-half $s$-plane
  - The system is then called **underdamped**
  - The **transient response is oscillatory**
- If $\zeta = 0$, the **transient response doesn't die out**
- If $\zeta = 1$, the system is called **critically damped**
- If $\zeta > 1$, the system is called **overdamped**
- We will now look at the unit step response for each of these cases

# Underdamped system

- For the underdamped case ($0 < \zeta < 1$), the transfer function can be written as:

$$H(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + j\omega_d)(s + \zeta\omega_n - j\omega_d)}$$

- Where $\omega_d$ is called the damped natural frequency

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

- For a unit-step input we can write

$$Y(s) = \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)s}$$

- Which can be rewritten as

$$Y(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$
$$= \frac{1}{s} - \frac{s + \zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} - \frac{\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- It can be shown that

$$\mathcal{L}^{-1}[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}] = e^{-\zeta\omega_n t} cos(\omega_d t)$$
$$\mathcal{L}^{-1}[\frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}] = e^{-\zeta\omega_n t} sin(\omega_d t)$$

- Therefore:

$$\mathcal{L}^{-1}[Y(s)] = y(t)$$

$$= 1 - e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t))$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} sin(\omega_d t + tan^{-1}(\frac{\sqrt{1-\zeta^2}}{\zeta}))$$

- It can be seen that the frequency of the transient oscillation is the damped natural frequency $\omega_d$ and thus varies with the damping ratio $\zeta$

# Underdamped system

- The error signal is the difference between input and output

$$
\begin{aligned}
e(t) &= y(t) - u(t) \\
&= e^{-\zeta\omega_n t}(cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}} sin(\omega_d t))
\end{aligned}
$$

- The error signal exhibits a damped sinusoidal oscillation

- At steady state, or at $t = \infty$, the error goes to zero

# Underdamped system

- If damping $\zeta = 0$, the response becomes undamped
  - Oscillations continue indefinitely
  - Filling in $\zeta = 0$ into the equation for y(t) gives us $y(t) = 1 - cos(_n t)$, for $t \geq 0$
  - We see that the system now oscillates at the natural frequency $\omega_n$
  - If a linear system has any amount of damping, the undamped natural frequency cannot be observed experimentally, only $\omega_d$ can be observed
  - $\omega_d$ is always lower than $\omega_n$

# Critically damped system

- If the two poles of the system are equal, the system is critically damped and $\zeta = 1$
- For a unit-step, $R(s) = \frac{1}{s}$ and we can write

$$Y(s) = \frac{\omega_n^2}{(s+\omega_n)^2 s}$$

- The inverse Laplace transform gives us

$$y(t) = 1 - e^{-\omega_n t}(1+\omega_n t) \text{ for } t \geq 0$$

# Overdamped system

- A system is overdamped ($\zeta > 1$) when the two poles are negative, real and unequal
- For a unit-step $R(s) = \frac{1}{s}$, $Y(s)$ can be written as

$$Y(s) = \frac{\omega_n^2}{(s + \zeta\omega_n + \omega_n^2\sqrt{\zeta^2 - 1})(s + \zeta\omega_n - \omega_n^2\sqrt{\zeta^2 - 1})}$$

- The inverse Laplace transform is

$$y(t) = 1 + \frac{w_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{-s_1 t}}{s1} - \frac{e^{-s_2 t}}{s2}\right), \text{ for } t \geq 0$$

- Where

$$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n \text{ and } s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$$

# Overdamped system

- $s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$ and $s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$
- Thus $y(t)$ includes two decaying exponential terms
  - When $\zeta >> 1$, one of the two decreases much faster than the other, and then the faster decaying exponential may be neglected
  - If $-s_2$ is located much closer to the $j\omega$ axis than $-s_1$ ($|s_2| >> |s_1|$), then $-s_1$ may be neglected
  - Once the faster decaying exponential term has disappeared, the response is similar to that of a first-order system
  - In that case, $H(s)$ can be approximated by

$$H(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} = \frac{s_2}{s + s_2}$$

# Overdamped system

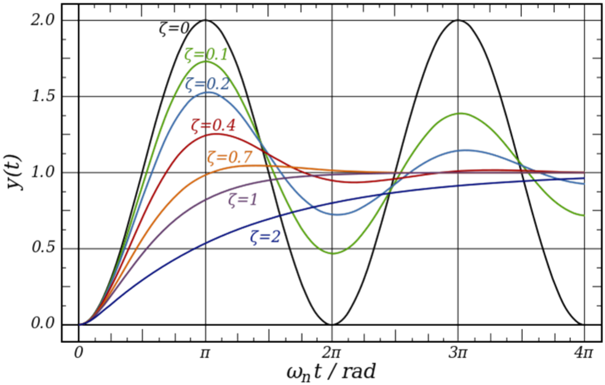
- With the approximate transfer function, the unit-step response becomes

$$Y(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

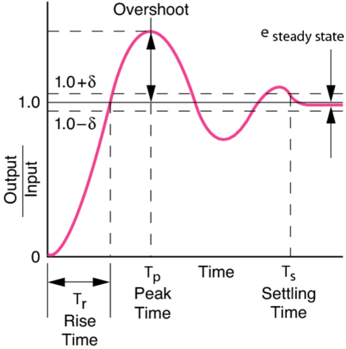
- The time response for the approximate transfer function is then given as

$$c(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}, \text{ for } t \leq 0$$

- Response on a step function



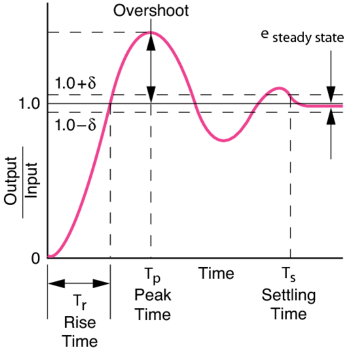
- Overshoot: Highest amplitude above steady state.
- Rise Time: Time needed to reach the steady state for the first time.
- Peak Time: Time to reach overshoot.



- Settling Time: Time needed to approximate the steady state.
- For: $\delta = \frac{0.02}{\sqrt{1-\zeta^2}}$
- We find:

$$e^{-\zeta\omega_n\tau_s} < 0.02$$

$$\tau_s = \frac{4}{\omega_n\zeta}$$



- The resonance frequency is the frequency at which the systems output has a larger amplitude than at other frequencies. This happens when underdamped functions oscillate at a greater magnitude than the input.
- An input with this frequency can sometime have catastrophic effects.
- A different view on the Tacoma bridge disaster: `https://www.youtube.com/watch?v=6ai2QFxStxo`
- In fact the collapse was a result of a number of effects like Aerodynamic flutter and vortices. Read the full article here: `http://www.ketchum.org/billah/Billah-Scanlan.pdf`

- The resonance frequency is: $\omega_r = \omega_n \sqrt{1 - \zeta^2}$
- Systems with a damping $> 0.707$ do not resonate
- The resonance frequency and the natural frequency are equal when a system has no damping.
- Another phenomenon with bridges and resonance is that many people marching with the same rhythm can cause a bridge to start resonating like the Angers bridge in 1850. A more recent example is the Millennium bridge in London who started resonating (see video lecture 2).

# Second order systems - damping

- When we want a system with no resonance, we choose one with damping $< 0.707$. This means a pole between $135^\circ$ and $225^\circ$:

$$arctan(\frac{\sqrt{1-\zeta^2}}{\zeta}) = +135^\circ$$

- We mostly want a short settling time ($< 4$s). This results in another restriction on the poles of the system:

$$\tau_n = \frac{4}{\omega\zeta} < 4\text{s}$$
$$\omega_n\zeta > 1$$

# Second order systems - damping