

Overdamped system

$$s_1 = (\zeta + \sqrt{\zeta^2 - 1})\omega_n \text{ and } s_2 = (\zeta - \sqrt{\zeta^2 - 1})\omega_n$$

Thus $y(t)$ includes two decaying exponential terms

- When $\zeta \gg 1$, one of the two decreases much faster than the other, the faster decaying exponential may be neglected;
- If $-s_2$ is located much closer to the $j\omega$ axis than $-s_1$ ($|s_2| \gg |s_1|$), then $-s_1$ may be neglected;
- Once the faster decaying exponential term has disappeared, the response is similar to that of a first-order system:

$$H(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}} = \frac{s_2}{s + s_2}.$$

Overdamped system

With the approximate transfer function, the unit-step response becomes:

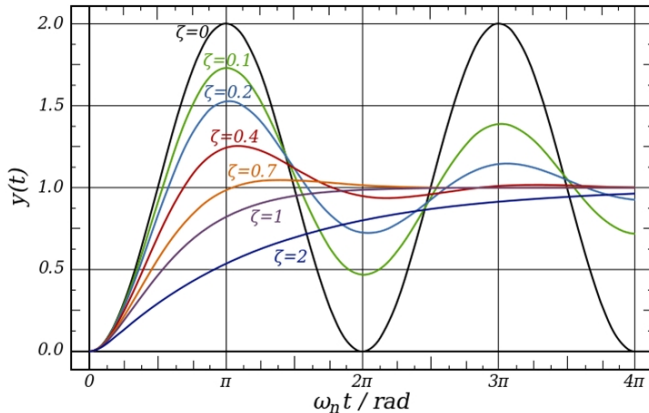
$$Y(s) = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}}{(s + \zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1})s}$$

The time response for the approximate transfer function is then given as:

$$y(t) = 1 - e^{-(\zeta - \sqrt{\zeta^2 - 1})\omega_n t}, \text{ for } t \geq 0$$

Second order systems - unit step response curves

Response on a step function



Second order systems - characteristics

The following formulas are only valid when $0 < \zeta < 1$!

- **Overshoot:** It is the maximum peak value of the response curve measured from unity.

$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

- **Rise time:** Time needed for the response to rise from 10% to 90%, or 0% to 100% of its final value.

$$t_r = \frac{1.8}{\omega_n}$$

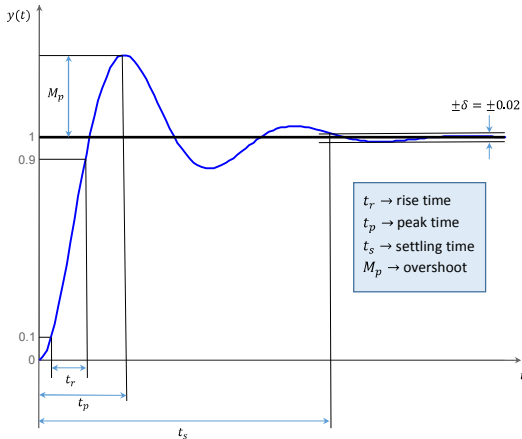
- **Peak Time:** Time required for the response to reach the first peak of the overshoot.

$$t_p = \frac{\pi}{\omega_d}$$

- **Settling Time:** Time required for the response curve to reach and stay within a range about the final value.

$$t_s = \frac{4}{\zeta\omega_n} \text{ (2\% criterion), } t_s = \frac{4.6}{\zeta\omega_n} \text{ (1\% criterion)}$$

Second order systems - characteristics



Example

Given: $\delta = \frac{0.02}{\sqrt{1-\zeta^2}}$

We find a settling time of:

$$e^{-\zeta\omega_n t_s} < 0.02$$

$$t_s = \frac{4}{\omega_n \zeta}$$

Second order systems - resonance

The resonance frequency is the frequency at which the systems output has a larger amplitude than at other frequencies. This happens when underdamped functions oscillate at a greater magnitude than the input. An input with this frequency can sometime have catastrophic effects.

A different view on the Tacoma bridge disaster:

<https://www.youtube.com/watch?v=6ai2QFxEtXo>

In fact the collapse was a result of a number of effects like Aerodynamic flutter and vortices. Read the full article here:
<http://www.ketchum.org/billah/Billah-Scanlan.pdf>

Second order systems - resonance

The resonance frequency is: $\omega_r = \omega_n \sqrt{1 - \zeta^2}$.

Systems with a damping > 0.707 do not resonate. The resonance frequency and the natural frequency are equal when a system has no damping.

Another phenomenon with bridges and resonance is that many people marching with the same rhythm can cause a bridge to start resonating like the Angers bridge in 1850. A more recent example is the Millennium bridge in London which started resonating.

Second order systems - damping

When we want a system with no resonance, we choose one with damping < 0.707 . This means a pole between 135° and 225° :

$$\arctan\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) = +135^\circ$$

We mostly want a short settling time ($< 4s$). This results in another restriction on the poles of the system:

$$\tau_n = \frac{4}{\omega\zeta} < 4s$$
$$\omega_n\zeta > 1$$

Second order systems - damping

