# System Modeling, part 2

Marc Claesen

July 9, 2015

1. Nonlinear systems & linearization

2. System identification (cont)
    - Grey box identification
    - Black box identification

# Outline

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# Nonlinear systems

In this course we focus on the linear state-space representation:

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Linearizing is done via first order Taylor expansions:

$$\begin{cases} \frac{dx}{dt} = \frac{d\Delta x}{dt} = f(x, u) = f(x_e + \Delta x, u_e + \Delta u), \\ y_e + \Delta y = g(x, u) = g(x_e + \Delta x, u_e + \Delta u). \end{cases}$$

# Example: decalcification plant

Used to reduce concentration of calcium hydroxide in water:

- chemical reaction: $Ca(OH)_2 + CO_2 \to CaCO_3 + H_2O$
- reaction speed: $r = c[Ca(OH)_2][CO_2]$
- rate of change of concentration:

$$\frac{d[Ca(OH)_2]}{dt} = \frac{k}{V} - \frac{r}{V},$$
$$\frac{d[CO_2]}{dt} = \frac{u}{V} - \frac{r}{V},$$

  with inflow rates $k$ and $u$ in mol/s and tank volume $V$ in L.
- input $u$: inflow of $CO_2$, output: $[Ca(OH)_2]$

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$$\frac{k_{eq}}{V} - \frac{c}{V}[Ca(OH)_2]_{eq}[CO_2]_{eq} = 0,$$
$$\frac{u_{eq}}{V} - \frac{c}{V}[Ca(OH)_2]_{eq}[CO_2]_{eq} = 0.$$

# Linearization of the decalcification plant

For small deviations near the equilibrium:

$$\begin{aligned}
\frac{d\Delta x_1}{dt} &= -\frac{c}{V}[CO_2]_{eq}\Delta x_1 - \frac{c}{V}[Ca(OH)_2]_{eq}\Delta x_2,\\
\frac{d\Delta x_2}{dt} &= -\frac{c}{V}[CO_2]_{eq}\Delta x_1 - \frac{c}{V}[Ca(OH)_2]_{eq}\Delta x_2 + \frac{\Delta u}{V},\\
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$$y(t) = [Ca(OH)_2]$$

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Most popular approaches are forms of black box identification.

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"*All models are wrong, but some are useful.*" - George E. P. Box

# Linear regression

Consider input matrix $\mathbf{X}$, output vector $\mathbf{y}$ and residuals $\epsilon$:

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The OLS estimate minimizes the sum-of-squares of errors, i.e.:

$$\hat{\theta}_{OLS} = \arg\min_{\theta} \sum_{i=1}^{N} \Big(y(i) - \sum_{j=1}^{d} X(i,j)\theta(j)\Big)^2$$

# Linear regression with ordinary least squares

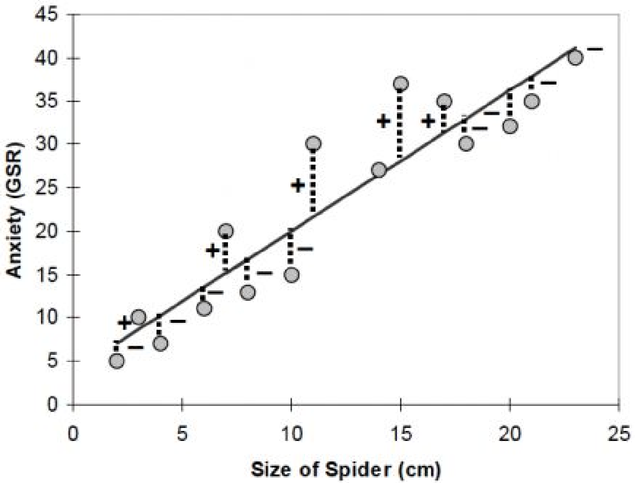

Image taken from `http://freakonometrics.hypotheses.org/2348`.

# Maximum likelihood estimation

The maximum likelihood estimate $\hat{\theta}_{ML}$ is the parameter vector that maximizes the likelihood $\mathcal{L}(\cdot)$ of observing the (known) outputs $\mathbf{y}$, given the (known) inputs $\mathbf{X}$:

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**Example**: least squares estimators are the maximum likelihood estimators if the associated residuals $\epsilon$ are normally distributed.

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If a prior distribution $p(\cdot)$ is available for $\theta$, then the posterior distribution for $\theta$ becomes:

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Additionally accounts for measurement errors in inputs.
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Typically described via *latent variables*:

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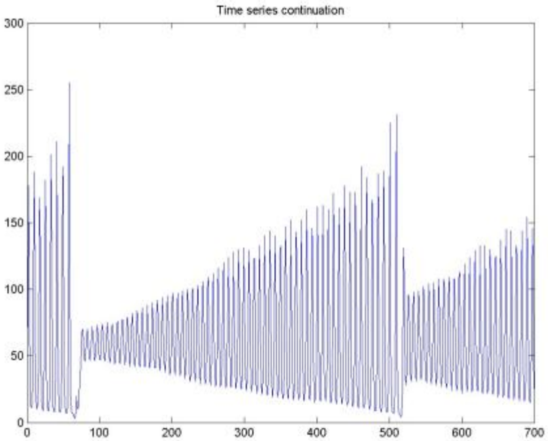
- unknown parameters (values in $A$, $B$, $C$, $D$)

# Time series: Santa Fe laser



# Modelling the Santa Fe laster

This laser can be treated as an autonomous discrete time system:

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→ lets say linear, i.e. $y = \mathbf{X}\theta$ → **unknown parameters** $\theta \in \mathbb{R}^N$.

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Treat it as a regression problem with $N$ inputs: $y = f\big(X_1, \ldots, X_N\big)$.
→ lets say linear, i.e. $y = \mathbf{X}\theta$ → **unknown parameters** $\theta \in \mathbb{R}^N$.
→ for given $N$, we can estimate $\theta$ via grey box methods.

# Modelling the Santa Fe laster

This laser can be treated as an autonomous discrete time system:

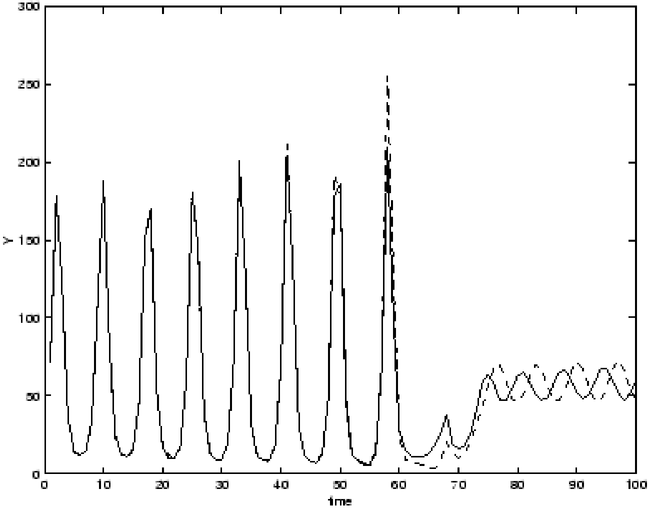
$$\begin{cases} x[k+1] = f\big(x[k-N+1], \ldots, x[k]\big), \\ y[k] = x[k]. \end{cases}$$

The output depends on the past $N$ states & no inputs.
→ how large is $N$? → **unknown structure**

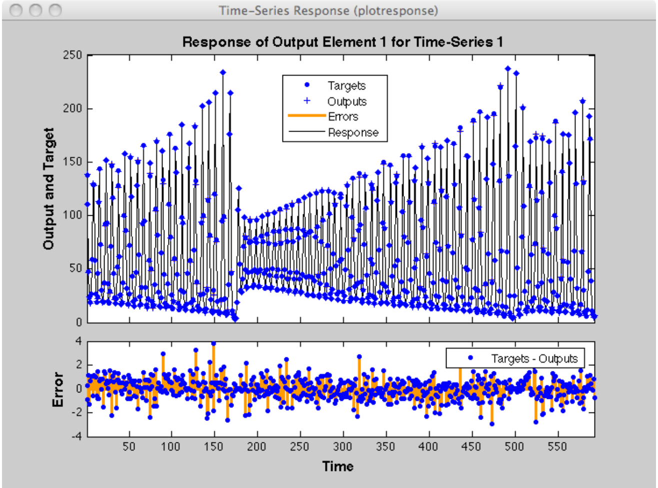
Treat it as a regression problem with $N$ inputs: $y = f\big(X_1, \ldots, X_N\big)$.
→ lets say linear, i.e. $y = \mathbf{X}\theta$ → **unknown parameters** $\theta \in \mathbb{R}^N$.
→ for given $N$, we can estimate $\theta$ via grey box methods.

Nonlinear models can be obtained via machine learning methods.
→ neural networks, support vector machine, random forest, ...

# Predictions of a least-squares support vector machine



# Predictions of an artificial neural network



# Neural network: biological

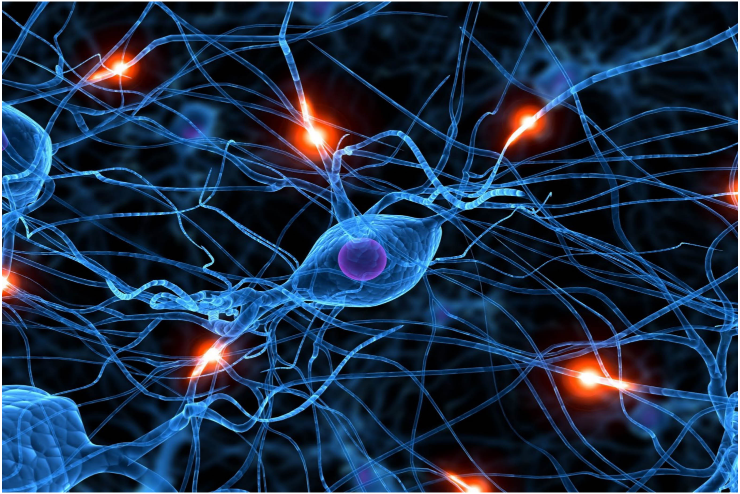


Image taken from http://www.extremetech.com/wp-content/uploads/2013/09/340.jpg.

# Structure of a single neuron

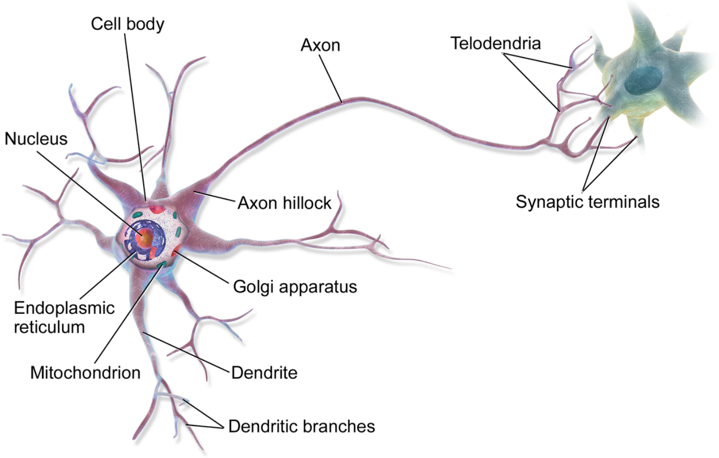
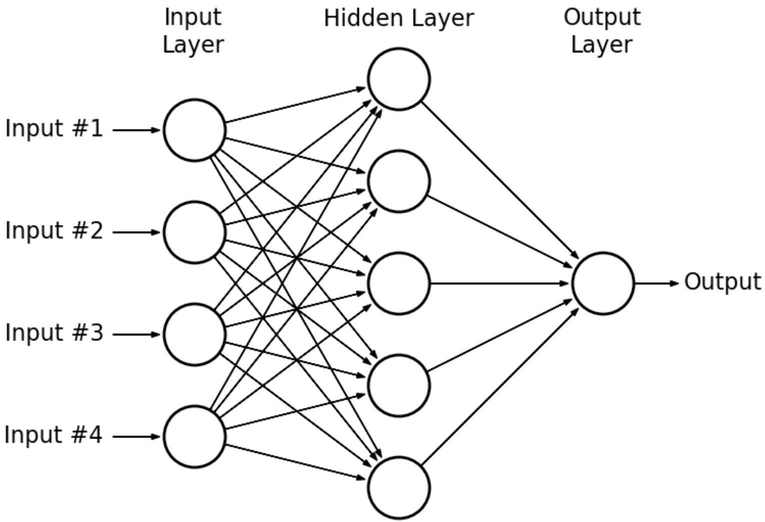


Image taken from `http://en.wikipedia.org/wiki/File:Blausen_0657_MultipolarNeuron.png`.

# Neural network: artificial


Input
Layer
Hidden Layer
Output
Layer
Input #1
Input #2
Input #3
Input #4
Output