Chapter 9: Introduction to Control

July 23, 2015

Outline

1. Basics
   - Control Theory
   - Demo: Inverted Pendulum
2. Control Goals
   - Stability
   - Disturbance rejection
   - Reference tracking
   - Exercise
3. Closed-loop systems
   - Sensitivity
   - Robustness
   - Types of systems and Steady State Error
   - Noise- and disturbance rejection

An introduction to control

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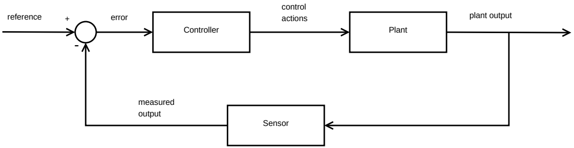
What is control theory?

Control theory is an interdisciplinary branch of engineering and mathematics that deals with the behavior of dynamical systems with inputs, and how their behavior is modified by feedback. The usual objective of control theory is to design a controller to produces inputs to the plant so its output follows a desired reference signal which may be a fixed or changing value. - wikipedia



Controllers

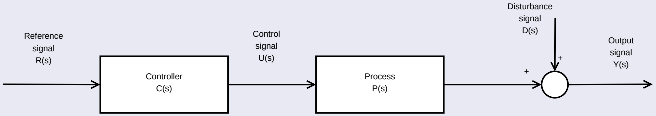
Types of controllers

- On-off controller
  - Thermostate at home
- **PID controllers, Lead and lag compensators (this course)**
  - Cruise-control in your car
- More advanced controllers
  - STATE-space feedback controllers
  - Model Predictive Controller (MPC)
  - Fuzzy Control
  - Neuro-fuzzy Control
  - ...

Open-loop System

Definition

In an open loop control system the actual output signal $Y(s)$ has no effect on the control action.



Reference signal R(s)

Controller C(s)

Control signal U(s)

Process P(s)

Disturbance signal D(s)

+

+

Output signal Y(s)

$$Y(s) = P(s)U(s) = P(s)C(s)R(s)$$

Example: pouring a glass of water

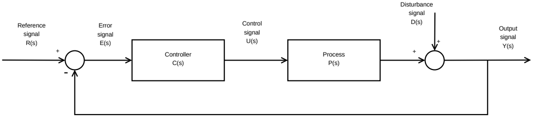
Take for example the following system:

- You are pouring a glass of water, but you **cannot look at the glass**.
- The desired output is a full glass of water within a reasonable time.
- The input can have two values: on or off (assume a quite primitive tap).
- It will not be easy to do this successfully.

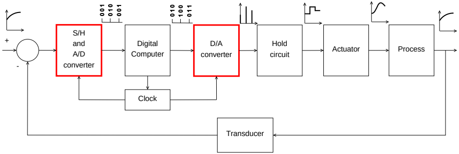
The solution is evident: look at the glass while pouring!

A general set-up of a closed loop system

- We will focus on closed loop control systems



- Digital Control Loop:



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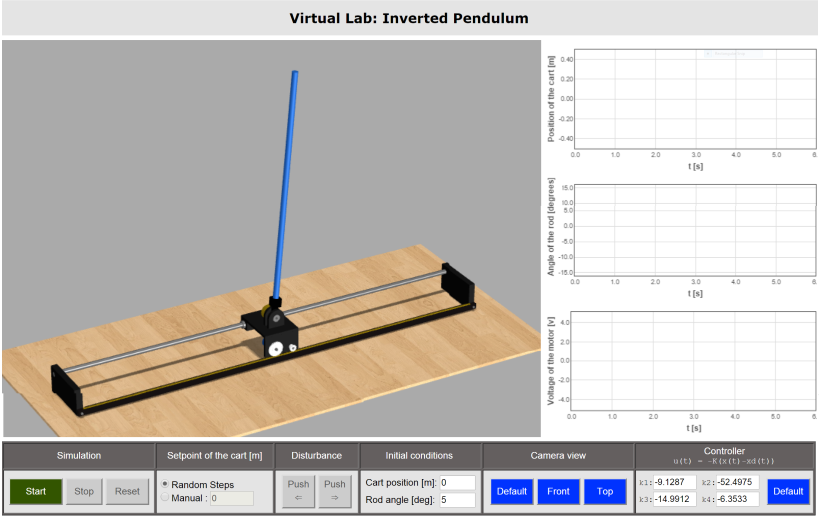

Virtual Lab: Inverted Pendulum

Simulation	Setpoint of the cart [m]	Disturbance	Initial conditions	Camera view	Controller $u(t) = -K(x(t)-xd(t))$
Start Stop Reset	● Random Steps ○ Manual : 0	Push ⇐ Push ⇒	Cart position [m]: 0 Rod angle [deg]: 5	Default Front Top	k1: -9.1287 k2: -52.4975 k3: -14.9912 k4: -6.3533 Default

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What is good control?

Depends on the application

- Stability
- Disturbance rejection
- Reference tracking
- Sensitivity to errors in the model
- ...

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Examples: stability

Space shuttles are like inverted pendulums. Control systems make sure they do not flip over.

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Examples: Disturbance rejection

- Your body will try to keep your internal temperature as constant as possible, no matter how hot/cold it is outside.

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Examples: Reference tracking

Audi has a system for automatic driving in traffic jams. The audi will follow the car in front of him at an appropriate distance.
https://www.youtube.com/watch?v=Qa_ZSRj0WM0

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Exercise: name the correct property

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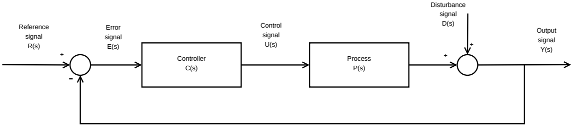
Transfer function of a closed-loop system



$$Y(s) - D(s) = P(s)U(s)$$

$$\text{with } U(s) = C(s)E(s)$$

$$Y(s) - D(s) = P(s)C(s)E(s)$$

$$\text{with } E(s) = R(s) - Y(s)$$

$$Y(s) - D(s) = P(s)C(s)(R(s) - Y(s))$$

$$Y(s) - D(s) = P(s)C(s)R(s) - P(s)C(s)Y(s)$$

$$\Rightarrow Y(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}R(s) + \frac{1}{1 + P(s)C(s)}D(s)$$

Transfer function from $R(s)$ to $Y(s)$

We define S as the transfer function from $R(s)$ to $Y(s)$

$$S(s) \triangleq \frac{P(s)C(s)}{1 + P(s)C(s)}$$

This transfer function $S(s)$ will help us to evaluate tracking
Almost perfect tracking: the output $Y(s)$ will follow $R(s)$ very closely $\Rightarrow S(s) \approx 1$

Transfer function from $D(s)$ to $Y(s)$

We define $T(s)$ as the transfer function from $D(s)$ to $Y(s)$.

$$T(s) \triangleq \frac{1}{1 + P(s)C(s)}$$

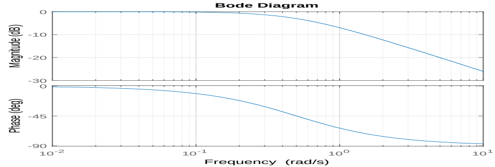
If the disturbance rejection is very good the disturbances will have almost no effect on the output $\Rightarrow T \approx 0$

$$\begin{cases} |S(j\omega)| \cong 1 \\ |T(j\omega)| \cong 0 \end{cases}$$

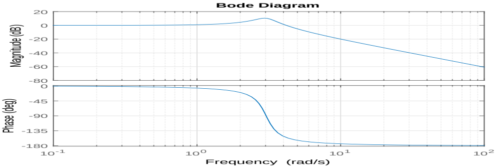
$\Rightarrow |P(j\omega)C(j\omega)|$ **(open loop gain)** is very large.
! a large open loop amplification might lead to an unstable system

Example: Which controller do you prefer

The transfer function from $R(s)$ to $Y(s)$ for two different controllers for high precision surgery



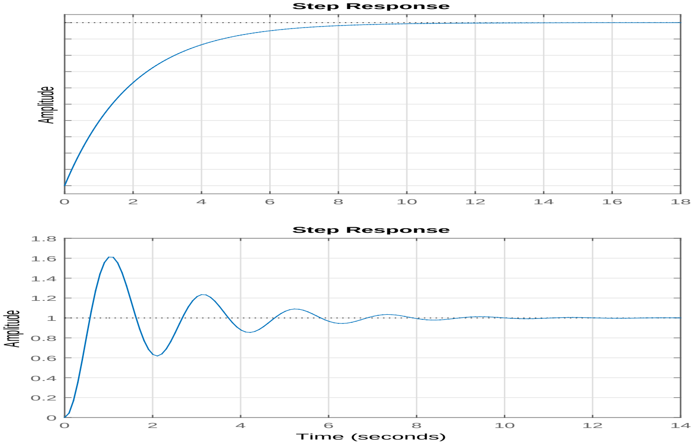
$$\frac{0.5}{s+0.5}$$



$$\frac{10}{1.09s^2+s+10}$$

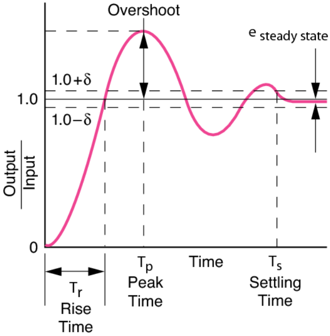
Example: step response of both controllers

The ouput $Y(s)$ when $R(s)$ is a step function.



Quality of reference tracking

- Look at the step response from $R(s)$ to $Y(s)$. All the known criteria can help you determine the quality.
    - Rise-time
    - Settling time
    - Steady state error
    - Overshoot
    - ...



Model errors

- In practice, the transfer function $P(s)$ is often unknown. It is important to know how model errors affect the result. Sensitivity and robustness are key concepts to evaluate these effects.
- Sensitivity
    - Quantifies the effect of small model errors on the output.
- Robustness
    - Refers to bigger changes of the model. A controller is robust if it works properly over a given set of parameters.

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Sensitivity

- Sensitivity is a measure of the effect of a (small) disturbance on the model (e.g. variations in the process parameters)

$$Y(s)+\Delta Y(s)=\frac{(P(s)+\Delta P(s))C(s)}{1+(P(s)+\Delta P(s))C(s)}R(s)+\frac{1}{1+(P(s)+\Delta P(s))C(s)}D(s)$$

- Look at the effect on the system without disturbances ($D(s){=}0$)

$$\begin{aligned}\Delta Y &= \frac{(P+\Delta P)C}{1+(P+\Delta P)C}R-\frac{PC}{1+PC}R\\ &= \frac{(P+\Delta P)C(1+PC)-PC-PC(P+\Delta P)C}{(1+(P+\Delta P)C)(1+PC)}R\\ &= \frac{\Delta PC}{(1+(P+\Delta P)C)(1+PC)}R\\ &= Y\frac{\Delta P}{P}\frac{1}{1+(P+\Delta P)C}\end{aligned}$$

Sensitivity

- Now take the relative change due to this disturbance of the model and take the limit for $\partial x \to 0$; this gives the following (measure of the) sensitivity:

$$S_P^Y(s) = \frac{\frac{\partial Y}{Y}(s)}{\frac{\partial P}{P}(s)} = \frac{1}{1 + P(s)C(s)}$$

- Again, a very large $|P(s)C(s)|$ looks like a good choice, but again there is a risk for instability!
- Note that the sensitivity can be determined for any parameter

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Example 1: robustness

- Suppose we want to stabilize the system $P(s) = \frac{1}{(s-a)}$. We only know that $a \in [0.20; 0.80]$. We propose to use a proportional controller $C(s) = K$. The closed loop transfer function becomes

$$S(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{\frac{1}{s-a}K}{1 + \frac{1}{s-a}K} = \frac{K}{s - a + K}$$

The controller stabilizes the system if $-a + K > 0$. If we choose K very large the system will be robust against changes in a.

Example 2: robustness

Suppose we want to stabilize the system $P(s) = \frac{s}{(s-a)}$ with $a \in [0.20; 0.80]$. If we choose the control law $C(s) = \frac{K}{s}$ this results in the same transfer function.

$$S(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} = \frac{\frac{s}{s-a}\frac{K}{s}}{1 + \frac{s}{s-a}\frac{K}{s}} = \frac{K}{s - a + K}$$

Again we choose $K > a$ to ensure stability. But how does the controller perform on the slightly perturbed system $\tilde{P}(s) = \frac{(s+\epsilon)}{(s-a)}$? The transfer function from $R(s)$ to $Y(s)$ becomes

$$S(s) = \frac{\frac{s+\epsilon}{s-a}\frac{K}{S}}{1 + \frac{s+\epsilon}{s-a}\frac{K}{s}} = \frac{(s+\epsilon)K}{s^2 + (K-a)s + \epsilon K}$$

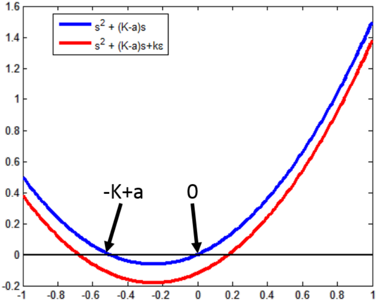
Example 2: robustness

$$\tilde{S}(s) = \frac{(s+\epsilon)K}{s^2 + (K-a)s + \epsilon K}$$

The figure on the right plots

$$s^2 + (K-a)s$$
$$s^2 + (K-a)s + \epsilon K$$

For ϵ negative the system becomes unstable. The control system is not robust and useless in practical situations. You shouldnt use control laws which rely on a pole-zero cancellation.



Example 3: Robustness of steady-state error

- The steady state error is defined as follows:

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{t\to\infty} (r(t) - y(t)) \\ &= \lim_{s\to 0} s(R(s) - Y(s)) \quad \text{(final value theorem)}\end{aligned}$$

- A very small steady state error (preferably zero) indicates that the controller tracks the reference very well
- For the open loop system with a step reference

$$e_{ol}(\infty) = \lim_{s\to 0} s(1 - C(s)P(s))R(s) = 1 - C(0)P(0)$$

  - So the open loop controller can be free of a steady state error (for a step reference $\epsilon(t)$) by calibrating the controller such that $C(0)P(0) = 1$
  - $\Rightarrow$ a precise calibration of the DC gain

Example 3: Steady state error

- Closed loop system:

$$e_{cl}(\infty) = \lim_{s\to 0} s\left(1 - \frac{C(s)P(s)}{1+C(s)P(s)}\right)R(s)$$
$$= \lim_{s\to 0} \frac{s}{1+C(s)P(s)}R(s) = \frac{1}{1+C(0)P(0)}$$

  - The steady state error is small if $C(0)P(0)$ is very large
  - Again, calibrating the DC gain
  - The difference is that here we only need a large gain, which is far less demanding than having to make it equal to 1

An open loop controller is not robust

- We can now show how this results in a great advantage of the closed loop strategy over the open loop strategy:
  - If P changes slightly (for instance due to a factor that has not been taken up into the model) to $P + \Delta P$
  - Then making $e_{ol}(\infty)$ small would require to calibrate anew
  - Whereas $e_{cl}(\infty)$ would remain small, as long as $(P(0) + \Delta P(0))C(0)$ remains large
- Hence an open loop controller can not control the output **robustly** against changes in P

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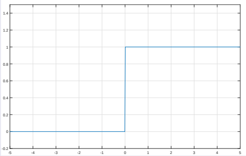
Steady state error: step

Definition

$$\epsilon(t) = \begin{cases} 0, & t \leq 0 \\ \frac{1}{2}, & t = 0 \\ 1, & t \geqslant 0 \end{cases}$$



- For a step reference $r(t) = A\epsilon(t) \Rightarrow R(s) = \frac{A}{s}$

$$e_{cl}(\infty) = \frac{A}{1 + C(0)P(0)}$$

So $e_{cl}(\infty) = 0$ if C(s)P(s) has a least one pole in zero $(C(0)P(0) \to \infty)$

Steady state error: ramp

- For a ramp reference function $r(t) = At\epsilon(t) \Rightarrow R(s) = \frac{A}{s^2}$, we have

$$e_{cl}(\infty) = \lim_{s\to 0} \frac{s}{1 + C(s)P(s)} \frac{A}{s^2}$$

So $e_{cl}(\infty) = 0$ if $C(s)P(s)$ has at least two poles in zero. $\left(\lim_{s\to 0} sC(s)P(s) = \infty\right)$

Type of a system

- The type of a system is determined by the number of poles $P(s)C(s)$ has in zero, and hence it is linked to what type of references it can track perfectly
- Write $P(s)C(s)$ as $\frac{K\prod_{k=1}^{m}(s-z_k)}{s^l\prod_{k=1}^{n-l}(s-p_k)}$ with l the amount of poles in zero
- We say this system is of type l
    - It will be able to track references of shape $At^0\epsilon(t)$ up to $At^{(l-1)}\epsilon(t)$ perfectly (for $t \to \infty$, $\lim_{t\to\infty} e_{cl}(t) = 0$)
    - A reference of shape $At^l\epsilon(t)$ will be missed by a finite factor (for $t \to \infty$, $\lim_{t\to\infty} e_{cl}(t) = C$)
    - A reference of shape $At^{l+\cdots}\epsilon(t)$ will be missed by infinity (for $t \to \infty$, $\lim_{t\to\infty} e_{cl}(t) = \infty$)

Example of a type 0 system

- Step reference $r(t) = A\epsilon(t) \Rightarrow R(s) = \frac{A}{s}$

$$e_{cl}(\infty) = \lim_{s\to 0} \frac{s}{1 + C(s)P(s)} \frac{A}{s}$$
$$= \lim_{s\to 0} \frac{A}{1 + \frac{K\prod_{k=1}^{m}(-z_k)}{\prod_{k=1}^{n-l}(-p_k)}} = \frac{A}{1+K_p}$$

  with K_p the static position constant

- Ramp reference $r(t) = At\epsilon(t) \Rightarrow R(s) = \frac{A}{s^2}$

$$e_{cl}(\infty) = \lim_{s\to 0} \frac{s}{1 + C(s)P(s)} \frac{A}{s^2}$$
$$= \lim_{s\to 0} \frac{1}{1 + \frac{K\prod_{k=1}^{m}(-z_k)}{\prod_{k=1}^{n-l}(-p_k)}} = \infty$$

Steady state errors - type of a system

- With:

$$\begin{aligned} K_p &= \lim_{s\to 0} P(s)C(s) & K_p &= \text{Static position constant} \\ K_v &= \lim_{s\to 0} sP(s)C(s) & K_v &= \text{Static velocity constant} \\ K_a &= \lim_{s\to 0} s^2P(s)C(s) & K_a &= \text{Static acceleration constant} \end{aligned}$$

- And the respective steady state errors for different system types are:

Type I	**Step $\mathbf{A\epsilon(t)}$**	**Ramp $\mathbf{At\epsilon(t)}$**	**Parabola $\mathbf{\frac{At^2\epsilon(t)}{2}}$**
0	$\frac{A}{1+K_p}$	∞	∞
1	0	$\frac{A}{K_v}$	∞
2	0	0	$\frac{A}{K_a}$

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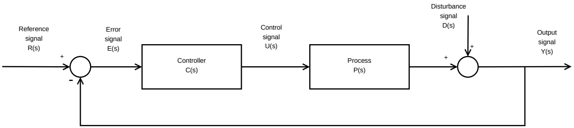
Noise rejection and disturbance rejection

- Both refer to rejection of the input D but are not the same
- Measurement noise is often modelled by white noise. (High frequency signal)
- Disturbances are actual changes to the state of the system



Noise versus Disturbances

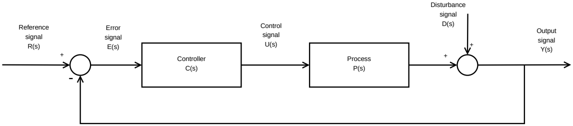
- Cruise control:
  Measurement errors on the speed are noise
  A change in slope is a disturbance

Noise versus Disturbances

- Remember from the start of the lecture

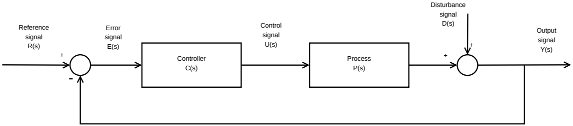
$$Y(s) = \frac{P(s)C(s)}{1+P(s)C(s)}R(s) + \frac{1}{1+P(s)C(s)}D(s)$$

- Choosing $T(s) = \frac{1}{1+P(s)C(s)}$ sufficiently small results in the rejection of D. This can be achieved by choosing $C(s)$ sufficiently high



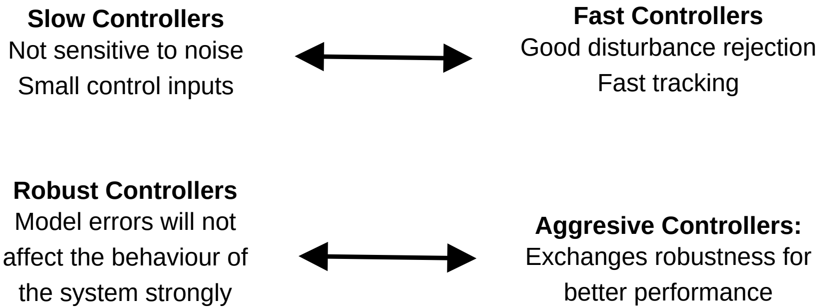
Disturbance vs noise

- But what happens to measurement noise
  - The amplified measurement noise will be applied to the input of the plant.
  - In reality measurement noise requires no control actions. Good noise rejection means that the controller ignores this noise.



Disturbance versus noise rejection

- Good disturbance rejection requires fast action to bring the system back to the desired state
- Good noise rejection requires that measurement noise will be ignored
- Note that a controller can not see the difference between measurement noise and disturbances. Slow controllers will be less sensitive to measurement noise. Fast system will have better disturbance rejection



Slow Controllers
Not sensitive to noise
Small control inputs

Fast Controllers
Good disturbance rejection
Fast tracking

Robust Controllers
Model errors will not
affect the behaviour of
the system strongly

Aggresive Controllers:
Exchanges robustness for
better performance