Chapter 6: Frequency response of dynamical systems

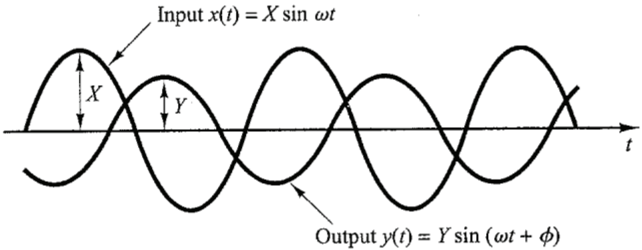
July 13, 2015

Outline

What is the frequency response of a system?

- Frequency response = steady state response of a system to a sinusoidal input.
- Assume an input $x(t) = X\sin(\omega t)$.
- The output in a linear system is then also sinusoidal, with a change in the magnitude and phase, i.e. $y(t) = Y\sin(\omega t + \phi)$
- It can be shown that: $Y = X \cdot |H(j\omega)|$ and $\phi = \angle H(j\omega)$
- $H(j\omega)$ is therefore also called the sinusoidal transfer function.

Relation of sinusoidal input/output in a linear system



Outline

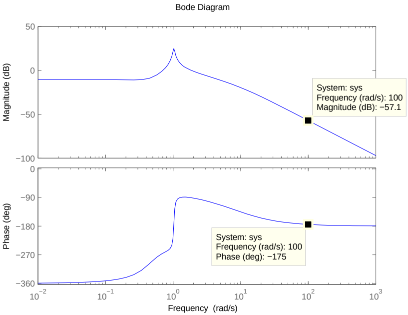
The bode plot

- A bode plot is a graphical representation of the sinusoidal transfer function $H(j\omega)$
- It consists of two seperate plots, a magnitude and a phase plot
- In this way, we can see the relation of a sinusoidal input with a given frequency ω to a linear system and its output
- After all, the relation between the amplitudes is given by $|H(j\omega)|$, and the phase shift by $\angle H(j\omega)$

Example bode plot

$$H(s) = \frac{14s^2+7s+3}{s^4+10s^3+10s^2+10s+10}$$

- Say we use an input $x(t) = 2\sin(100t)$ in this system.
- The steady state response would be



$$y(t) = |H(j100)| \cdot 2\sin(100t + \angle H(j100))$$

The magnitude plot

Convention:

- for the ordinate (y-axis) we use $20\log_{10}|H(j\omega)|$ with the unit *dB*
- for the abscissa (x-axis) we use a logarithmic plot of ω

This is thus a bi-log plot

The reason for using the logarithm of the modulus of $H(j\omega)$ will become clear later

The phase plot

Convention:

- for the ordinate (y-axis) we use $\angle H(j\omega)$ in degrees
- for the abscissa (x-axis) we use a logarithmic plot of ω

This is thus a semi-log plot

Discrete time systems

- The transfer function is a function of z, i.e. $H(z)$
- In contrast to continuous time systems, we do not use $H(j\omega)$. Instead, $H(e^{j\omega T_s})$ is now the sinusoidal transfer function.
- T_s is the sample time, i.e. the amount of time in between each sample.

Outline

A new representation of the transfer function

From before:

$$H(s) = \frac{\beta_0 s^r + \beta_1 s^{r-1} + \ldots + \beta_r}{s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n}$$

Factorization in zeros and poles

$$\Rightarrow \quad H(s) = \frac{\beta_0 (s - n_1)(s - n_2)\ldots(s - n_r)}{(s - p_1)(s - p_2)\ldots(s - p_n)}$$

This is the usual representation. Now however, we will look for factors $(1 + \frac{s}{s_i})$, with s_i a so-called breakpoint.

A new representation of the transfer function

We can do this by bringing all the zeros and poles not equal to zero outside the brackets, as follows:

$$H(s) = \beta_0 \frac{\prod(-n_i)}{\prod(-p_j)} \frac{(1+\frac{s}{-n_1})(1+\frac{s}{-n_2})\ldots(1+\frac{s}{-n_i})}{s^l(1+\frac{s}{-p_1})(1+\frac{s}{-p_2})\ldots(1+\frac{s}{-p_j})}$$

Replacing the constants by K, and setting

$$r_k = -n_k$$

$$s_k = -p_k$$

A new representation of the transfer function

We ultimately get:

$$H(s) = K\frac{(1+\frac{s}{r_1})(1+\frac{s}{r_2})\dots(1+\frac{s}{r_i})}{s^l(1+\frac{s}{s_1})(1+\frac{s}{s_2})\dots(1+\frac{s}{s_j})}$$

Now we are able to construct the bode plot of each different factor of $H(s)$. Afterwards we can just add up these plots using the calculation rules of complex numbers.

Intermezzo complex numbers

- The magnitude of the product of complex numbers is equal to the product of the magnitudes of these numbers
- The phase of the product of complex numbers is equal to the sum of the phases of these numbers
- The logarithm of a product of numbers is equal to the sum of the logarithms of these numbers

This comes down to

$$20\log_{10}|H(j\omega)| = \sum 20\log_{10}|\text{factors}|$$

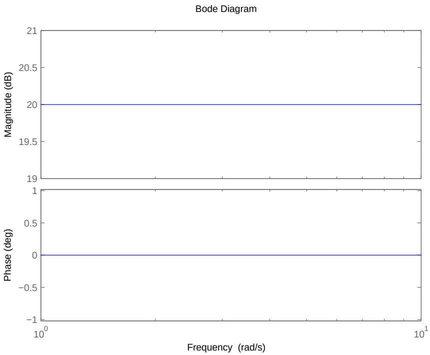
$$\angle H(j\omega) = \sum(\angle\text{factors})$$

Next we will quickly go over the simple bodeplots of the different factors of $H(s)$

The constant K

- $20\log_{10}|K| = \text{constant}$
- $\angle K = 0^\circ$ or $\pm 180^\circ$ (resp $K > 0$ and $K < 0$)

Example: $K = 10$



$(1 + \frac{j\omega}{r_i})$ in the numerator

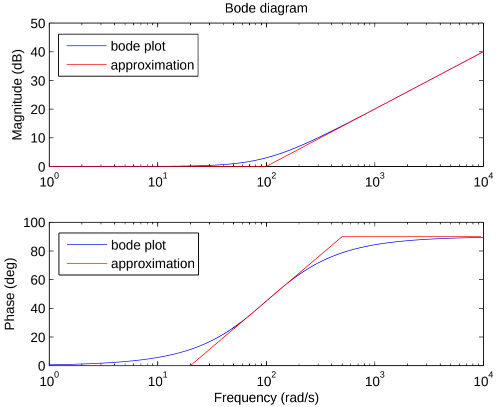
(Assume $r_i > 0$)

- What if $\omega \to 0$? $\quad (1 + \frac{j\omega}{r_i}) \to 1$
  - $20\log_{10}|1| = 0$
  - $\angle 1 = 0^\circ$
- What if $\omega \to \infty$? $\quad (1 + \frac{j\omega}{r_i}) \to j\infty$
  - $20\log_{10}|j\infty| = \infty$
  - $\angle j\infty = 90^\circ$
- The two terms balance each other out for $\omega = r_i$ (remember, this is called a breakpoint).
  - $20\log_{10}|1 + j| = 20\log_{10}(\sqrt{2}) \approx 3\text{dB}$
  - $\angle(1 + j) = 45^\circ$

A breakpoint is therefore also called a 3dB point

$(1 + \frac{j\omega}{r_i})$ in the numerator

Example: $r_i = 100$



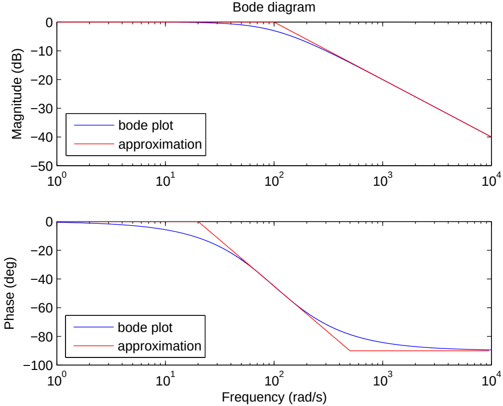
$(1+\frac{j\omega}{s_i})$ in the denominator

This factor is equivalent to the previous one. The only difference is the sign change in both plots:

- What if $\omega \to 0$? $\quad \frac{1}{1+\frac{j\omega}{s_i}} \to 1$
  - $20\log_{10}|1| = 0$
  - $\angle 1 = 0^\circ$
- What if $\omega \to \infty$? $\quad \frac{1}{1+\frac{j\omega}{s_i}} \to \frac{1}{j\infty} \to -j0$
  - $20\log_{10}|-j0| = -\infty$
  - $\angle -j0 = -90^\circ$
- The two terms balance each other out for $\omega = s_i$
  - $20\log_{10}\frac{1}{|1+j|} = 20\log_{10}(\frac{1}{\sqrt{2}}) \approx -3\text{dB}$
  - $\angle\frac{1}{(1+j)} = -45^\circ$

$(1 + \frac{j\omega}{s_i})$ in the denominator

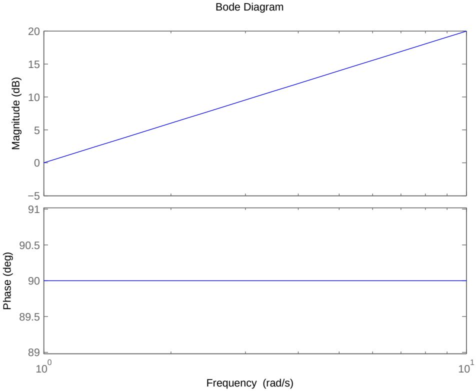
Example $s_i = 100$



$j\omega$ in the numerator

- This is simply a (ascending) straight line in the magnitude plot, with a slope of 20 dB/decade
- Constant phase of 90°
- What if $\omega \to 0$? $\quad j\omega \to j0$
  - $20 \log_{10} |j0| = -\infty$
  - $\angle j0 = 90^\circ$
- What if $\omega \to \infty$? $\quad j\omega \to j\infty$
  - $20 \log_{10} |j\infty| = \infty$
  - $\angle j\infty = 90^\circ$

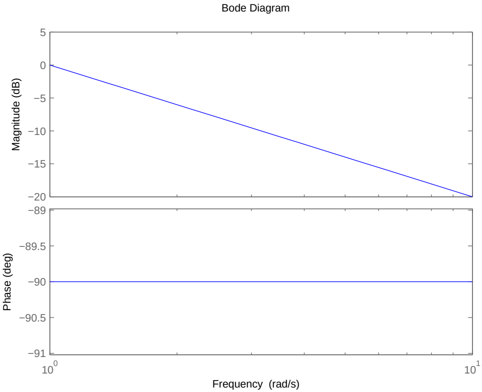
$j\omega$ in the numerator



$j\omega$ in the denominator

- This is simply a (descending) straight line in the magnitude plot, with a slope of -20 dB/decade
- Constant phase of -90°
- What if $\omega \to 0$? $\frac{1}{j\omega} \to \frac{1}{j0} \to -j\infty$
  - $20\log_{10}|-j\infty| = \infty$
  - $\angle -j\infty = -90^\circ$
- What if $\omega \to \infty$? $\frac{1}{j\omega} \to \frac{1}{j\infty} \to -j0$
  - $20\log_{10}|-j0| = -\infty$
  - $\angle -j0 = -90^\circ$

$j\omega$ in the denominator



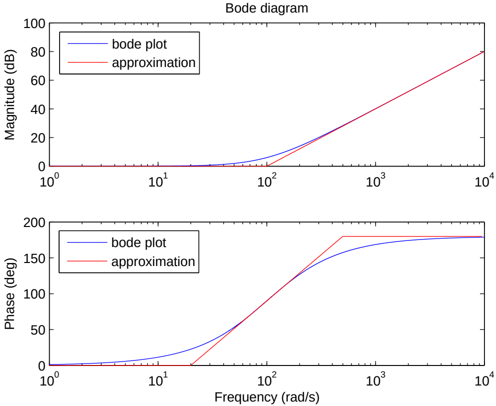
What if the multiplicity is higher than 1?

Take for example multiplicity 2, a second order factor.
A second order factor has twice the effect of a first order factor.
Consider for example the effect of a double zero:

- Slope of 40dB/decade instead of 20dB/decade after the breakpoint
- Phase shift of 180° instead of 90°

Second order factor

$$(1 + \frac{j\omega}{100})^2$$



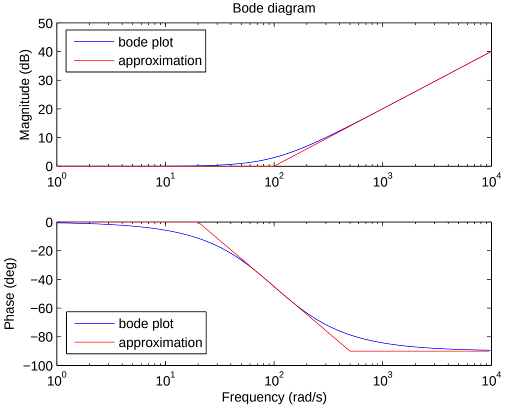
Exception

Up until now we always considered r_i and $s_i > 0$, but what if we have a factor $(1 - \frac{j\omega}{r_i})$ for example?

- The magnitude plot remains unchanged, as $|1 + \frac{j\omega}{r_i}| = |1 - \frac{j\omega}{r_i}|$
- The phase plot is reversed, as $\angle(1 + \frac{j\omega}{r_i}) = -\angle(1 - \frac{j\omega}{r_i})$
- If we have such a factor in the denominator, the system will be unstable!

Exception

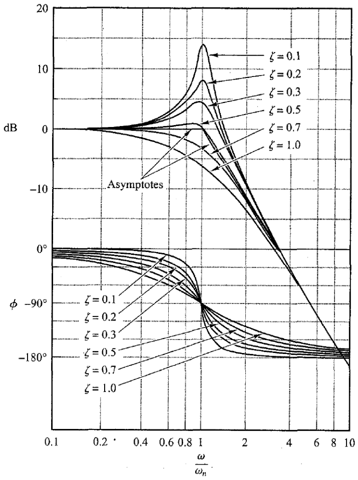
Example $(1 - \frac{j\omega}{100})$ in the numerator



Quadratic factors and resonance

- Also possible is a quadratic factor $[1 + 2\zeta(j\frac{\omega}{\omega_n}) + (j\frac{\omega}{\omega_n})^2]^{\pm 1}$
- ζ is called the damping factor, ω_n is the natural frequency
- If $\zeta > 1$, this quadratic factor can be expressed as two first order factors with real zeros/poles
- But if $0 < \zeta < 1$, this quadratic factor is the product of two complex-conjugate factors
- For low ζ, the asymptotic approximations are not accurate. Instead, a peak occurs in the magnitude plot around ω_n
- This peak is a phenomenon known as resonance.

Quadratic factors and resonance



Example

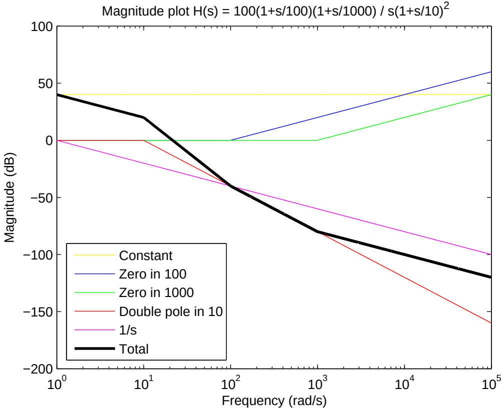
Suppose we want to construct (by hand) the bode plot of

$$H(s) = \frac{s^2 + 1100s + 100000}{10s^3 + 200s^2 + 1000s}$$

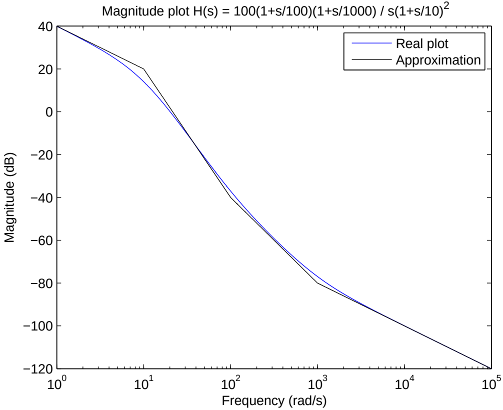
The first step is to find the representation with breakpoints.

$$\begin{aligned} H(s) &= \frac{s^2 + 1100s + 100000}{10s^3 + 200s^2 + 1000s} \\ &= \frac{(s+100)(s+1000)}{10s(s+10)^2} \\ &= \frac{100000(1+\frac{s}{100})(1+\frac{s}{1000})}{1000s(1+\frac{s}{10})^2} = \frac{100(1+\frac{s}{100})(1+\frac{s}{1000})}{s(1+\frac{s}{10})^2} \end{aligned}$$

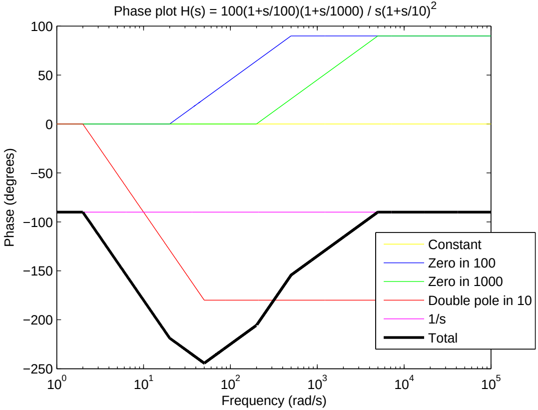
Example Magnitude plot



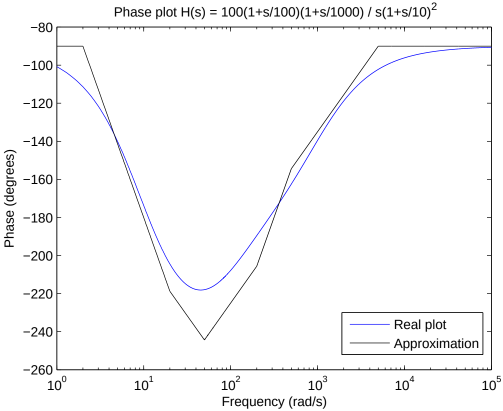
Example Magnitude plot



Example Phase plot



Example Phase plot



Outline

Basic commands in Matlab

In Matlab it is very easy to draw the bode plot.

- First, define the system using one of the following commands:
  - tf(num,den) (num and den are respectively the numerator and denominator of the transfer function)
  - zpk(z,p,K) (using the zeros (z), the poles (p) and the gain (K) of the transfer function)
  - ss(A,B,C,D) (using the matrices of the state-space model)
- In case of a discrete time system, Ts (the sample time) is also needed as a last parameter in these commands
- Next, use the command bode(sys)

Matlab example

```
%Examples for creating the bode plot in Matlab
%Say we have the transfer function
%H(s) = (5s^2 - 10s + 5)/(s^2 + 5s + 4)
num = [5 -10 5];
den = [1 5 4];
sys = tf(num,den);
bode(sys)
figure

%Using the same system, we first find the factorization
%H(s) = 5*(s-1)^2/[(s+1)(s+4)]
z = [1 1];
p = [-1 -4];
K = 5;
```

Matlab example

```
sys = zpk(z,p,K);
bode(sys)

figure
%If we had a discrete time system with the same transfer
%function
%H(z) = (5z^2 - 10z + 5)/(z^2 + 5z + 4)
%and sampling time Ts = 1/2 of a second

sys = tf(num,den,0.5);
bode(sys)
```

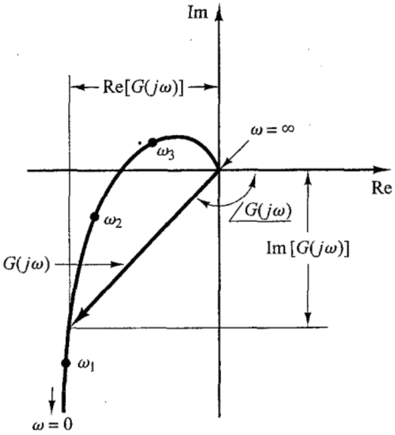
Outline

Nyquist plot

A Nyquist plot is also called a polar plot, and is another way to plot $H(j\omega)$.
In a polar plot, as ω is varied from 0 to ∞, $H(j\omega)$ is plotted as a point in the complex plane.

- $|H(j\omega)|$ is the distance between the origin and the point
- $\angle H(j\omega)$ is the angle between the vector to the point and the positive real axis, measured counterclockwise

Nyquist plot



Nyquist plot in matlab

Similar to constructing the bode plot in matlab, we first have to define the system using tf, zpk or ss.
Then we use the command nyquist(sys).

```
%How to create a Nyquist plot in matlab
sys = tf([14 7 3],[1 10 10 10 10]);
nyquist(sys)
```

Bode and Nyquist plot