Outline

Impulse-invariant method

Practical rule

This method converts a continuous-time system into a discrete one by matching the impulse response using these transformations:

$H(s)$	$H(z)$, $a = e^{bT_s}$
$\frac{c}{s-b}$	$\frac{T_s c}{1-az^{-1}}$
$\frac{c}{(s-b)^2}$	$T_s^2 \frac{caz^{-1}}{(1-az^{-1})^2}$
$\frac{c}{(s-b)^3}$	$\frac{T_s^3 caz^{-1}(1+az^{-1})}{2(1-az^{-1})^3}$
$\frac{c}{(s-b)^4}$	$\frac{T_s^4 caz^{-1}(1+4az^{-1}+a^2z^{-2})}{6(1-az^{-1})^4}$

Outline

Zero-pole equivalent

General approach

The map $z = e^{sT}$ is applied to the poles as well as to the zeros of the continuous system. The following rules must be followed:

1. If $s = -a$ is a pole of H(s), then $z = e^{-aT}$;
2. All finite zeros $s = -b$ are mapped by $z = e^{-bT}$;
3. Zeros at ∞ are mapped to $z = -1$;
4. The DC gain is set such that $\lim_{s \to 0} G(s) = \lim_{z \to 1} G_d(z)$.

Zero-pole equivalent

Example

Given:

$$H(s) = \frac{s+1}{0.1s+1}$$

Pole at $s = -10$ and zero at $s = -1$.
New discrete transfer function with equivalent poles and zeros:

$$H(z) = K\frac{z-e^{-T}}{z-e^{-10T}}$$

K is chosen so that $\mid H(z) \mid_{z=1} = \mid H(s) \mid_{s=0} \rightarrow K = 4.150$
Using $T = 0.25$, this results in:

$$H(z) = 4.150\frac{z-0.7788}{z-0.0821}$$

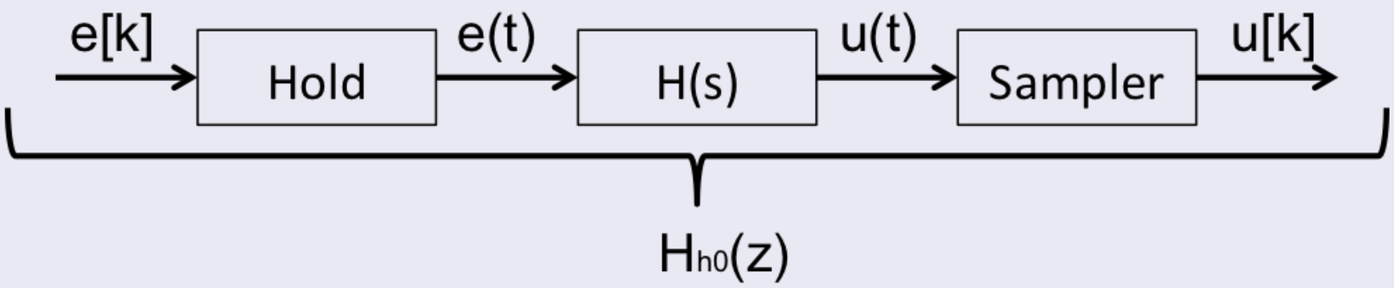
Outline

Hold equivalent

General approach

This method uses a discrete system consisting of 3 subsystems, each with its own purpose.

1. Hold: approximating $e(t)$ from the samples $e[k]$
2. H(s): putting the $e(t)$ through the given transfer function H(s) of the continuous system, resulting in u(t)
3. Sampler: sampling u(t)



There are many techniques for holding a sequence of samples.

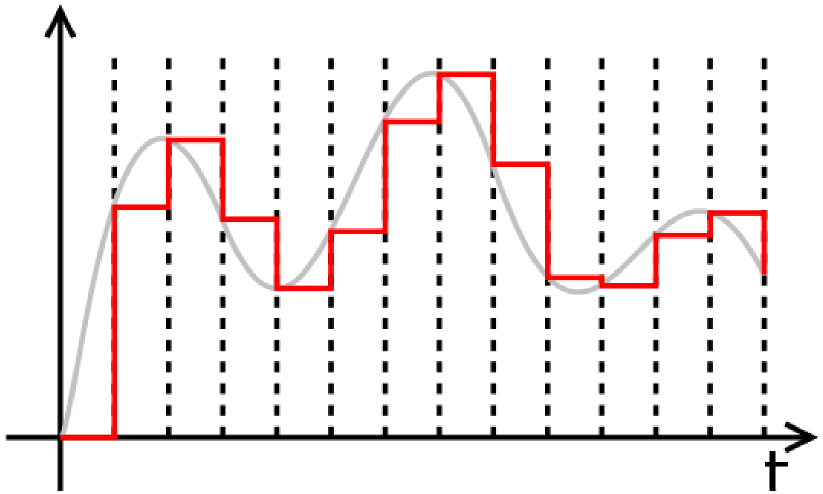
Zero-order hold equivalent (ZOH)

Practical rule

The zero-order hold equivalent transfer function $H_{zoh}(z)$ can be found by computing the following:

$$H_{zoh}(z) = (1 - z^{-1})\mathcal{Z}\{\frac{H(s)}{s}\}$$

This is obtained by holding e(t) constant at e(k) over the interval kT_s to $(k+1)T_s$ and $\mathcal{Z}$-transforming it.



Zero-order hold equivalent

Example

$$H(s) = \frac{0.1}{s+0.1}$$

Step 1: multiply by 1/s and perform partial fraction expansion

$$\frac{H(s)}{s} = \frac{0.1}{s*(s+0.1)} = \frac{1}{s} - \frac{1}{s+0.1}$$

Step 2: perform z-transformation

$$\mathcal{Z}\{\frac{H(s)}{s}\} = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-0.1T}*z^{-1}}$$

Step3: simplify and multiply by $(1 - z^1)$

$$H_{zoh}(z) = \frac{1-e^{-0.1T}}{z-e^{0.1T}}$$

Zero-order hold equivalent

Step Invariant Transformation

This rule is also called the step-invariant method because it matches the step response of the continuous and the discrete system. Next transformations may be useful:

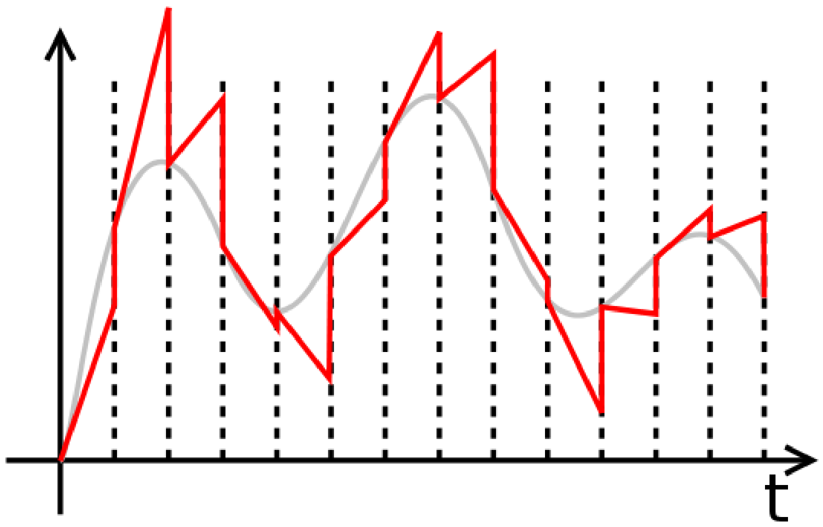
$f(t)$	$F(s)$	$f(k)$	$F(z)$
$u(t)$, unit step	$\frac{1}{s}$	$u(k)$, unit step	$\frac{z}{z-1}$
$tu(t)$	$\frac{1}{s^2}$	$kTu(k)$	$\frac{Tz}{(z-1)^2}$
$e^{-\mathrm{at}}u(t)$	$\frac{1}{s+a}$	$(e^{-\mathrm{aT}})^{\mathrm{k}}u(k)$	$\frac{z}{z-e^{-\mathrm{aT}}}$
$te^{-\mathrm{at}}u(t)$	$\frac{1}{(s+a)^2}$	$kT(e^{-\mathrm{aT}})^{\mathrm{k}}u(k)$	$\frac{Tze^{-\mathrm{aT}}}{(z-e^{-\mathrm{aT}})^2}$
$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$	$\sin(k\omega T)u(k)$	$\frac{z\sin\omega T}{z^2-2z\cos\omega T+1}$
$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$	$\cos(k\omega T)u(k)$	$\frac{z(z-\cos\omega T)}{z^2-2z\cos\omega T+1}$

First-order hold equivalent

Practical rule

The non-causal first-order hold equivalent transfer function $H_{foh}(z)$ can be found by computing the following:

$$H_{foh}(z) = \frac{(z-1)^2}{Tz}\mathcal{Z}\left\{\frac{H(s)}{s^2}\right\}$$



First-order hold equivalent

Example

$$H(s) = \frac{1}{s^2}$$

Step 1: multiply by $1/s^2$ and perform partial fraction expansion

$$H(s) = \frac{1}{s^4}$$

Step 2: perform z-transformation

$$\mathcal{Z}\{\frac{H(s)}{s^2}\} = \frac{T^3}{6}\frac{(z^2+4z+1)z}{(z-1)^4}$$

Step 3: simplify and multiply by $\frac{(z-1)^2}{T*z}$

$$H_{foh}(z) = \frac{T^2}{6}\frac{z^2+4z+1}{(z-1)^2}$$