Chapter 2 - State Space Models

September 18, 2015

Outline

Outline

Definition

This is valid for any nonlinear causal system

$$\text{CT: } \dot{x} = f(x, u) \qquad DT : x_{k+1} = f(x_k, u_k)$$
$$y = h(x, u) \qquad y_k = h(x_k, u_k)$$

where

$$x = \text{the state of the system, } n \times 1 \text{ vector}$$
$$u = \text{the input of the system, } m \times 1 \text{ vector}$$
$$y = \text{the output of the system, } p \times 1 \text{ vector}$$
$$f = \text{state equation vector function}$$

Definition

This is valid for any nonlinear causal system

$$\text{CT: } \begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \qquad DT: \begin{aligned} x_{k+1} &= f(x_k,u_k) \\ y_k &= h(x_k,u_k) \end{aligned}$$

where

$$\begin{aligned} h &= \text{output equation vector function} \\ n &= \text{number of states} \Rightarrow n\text{-th order system} \\ m &= \text{number of inputs} \\ p &= \text{number of outputs} \end{aligned}$$

Outline

Based on the number of inputs and outputs

1. **SISO**: Single Input Single Output: $m = p = 1$
2. **SIMO**: Single Input Multiple Output: $m, p > 1$
3. **MISO**: Multiple Input Single Output: $m > 1, p = 1$
4. **MIMO**: Multiple Input Multiple Output: $m = 1, p > 1$

Outline

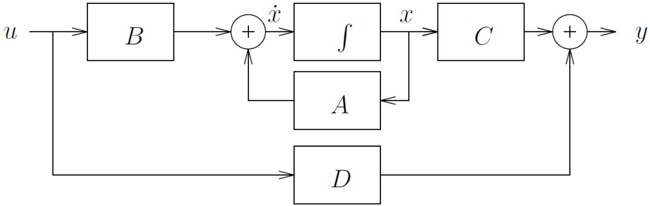
Outline

Definition: Continuous-time

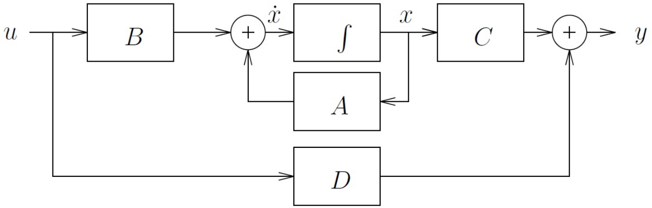
Only valid for linear causal systems

Continuous-time system:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$



Definition: Continuous-time



A: $n \times n$ system matrix

B: $n \times m$ input matrix

C: $p \times n$ output matrix

D: $p \times m$ direct transmission matrix

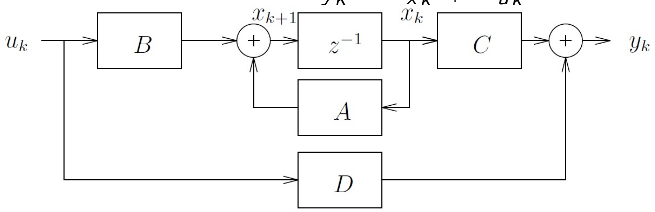
Definition: Discrete-time

Only valid for linear causal systems

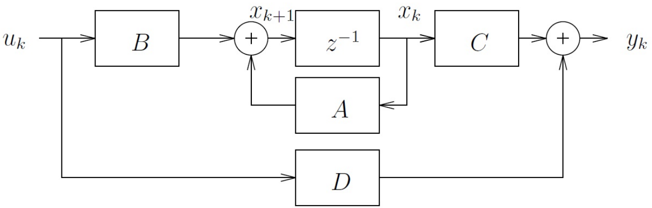
Discrete-time system:

$$x_{k+1} = A_{x\,k} + B_{u\,k}$$

$$y_k = C_{x\,k} + D_{u\,k}$$



Definition: Discrete-time



A: $n \times n$ system matrix

B: $n \times m$ input matrix

C: $p \times n$ output matrix

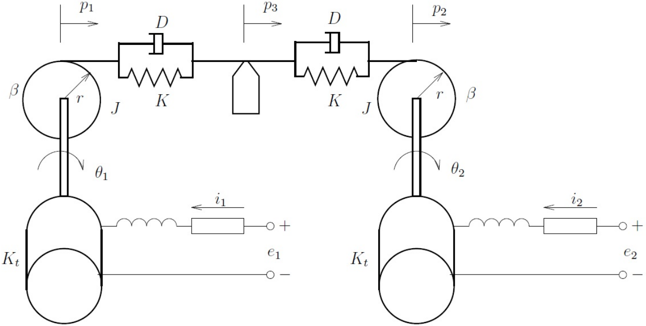
D: $p \times m$ direct transmission matrix

Outline

Example of linear state space modeling

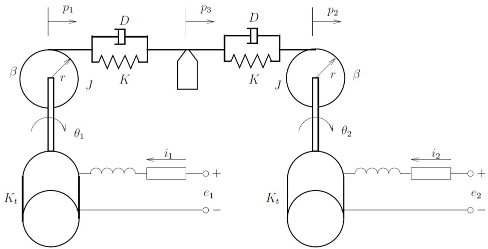
Example: Tape drive control - state space modeling

Process description:



Example of linear state space modeling

The drive motor on each end of the tape is independently controllable by voltage resources e1 and e2. The tape is modeled as a linear spring with a small amount of viscous damping near to the static equilibrium with tape tension 6N. The variables are defined as deviations from this equilibrium point.



Example of linear state space modeling

The equations of motion of the system are:

$$
\begin{aligned}
\text{capstan } 1: J\frac{d\omega_1}{dt} &= Tr - \beta\omega_1 + K_t i_1 \\
\text{speed of } x_1: \dot{p}_1 &= r\omega_1 \\
\text{motor } 1: L\frac{di_1}{dt} &= -Ri_1 - K_e\omega_1 + e_1 \\
\text{capstan } 2: J\frac{d\omega_2}{dt} &= Tr - \beta\omega_2 + K_t i_2 \\
\text{speed of } x_2: \dot{p}_2 &= r\omega_2
\end{aligned}
$$

Example of linear state space modeling

$$\text{motor 2}: L\frac{di_2}{dt} = -Ri_2 - K_e\omega_2 + e_2$$

$$\text{Tension of tape}: T = \frac{K}{2}(p_2 - p_1) + \frac{D}{2}(\dot{p}_2 - \dot{p}_1)$$

$$\text{Position of the head}: p_3 = \frac{p_1 + p_1}{2}$$

Example of linear state space modeling

Description of variables and constants:

$$\begin{aligned}
\mathrm{D} &= \text{damping in the tape-stretch motion} = 20 \text{ N/msec} \\
e_{1,2} &= \text{applied voltage, V} \\
i_{1,2} &= \text{current into the capstan motor} \\
\mathrm{J} &= \text{inertia of the wheel and the motor} \\
&= 4\text{x}10^{-5}\text{kg.m} \\
\beta &= \text{capstan rotational friction, 400 kgm} \\
\mathrm{K} &= \text{spring constant of the tape, } 4x10^{4}\text{N/m} \\
K_e &= \text{electrical constant of the motors} = 0.03 \text{ Vsec} \\
K_t &= \text{torque constant of the motors} = 0.03 \text{ Nm/A}
\end{aligned}$$

Example of linear state space modeling

Description of variables and constants:

$$
\begin{aligned}
L &= \text{armature inductance} = 10^{-3}\ \text{H} \\
R &= \text{armature resistance} = 1\ \Omega \\
r &= \text{radius of the take-up wheels, } 0.02\ \text{m} \\
T &= \text{tape tension at the read/write head, N} \\
p_{1,2,3} &= \text{tape position at capstan 1,2 and the head} \\
\dot{p}_{1,2,3} &= \text{tape velocity at capstan 1,2 and the head} \\
\theta_{1,2} &= \text{angular displacement of capstan 1,2} \\
\omega_{1,2} &= \text{speed of drive wheels} = \dot{\theta}_{1,2}
\end{aligned}
$$

Example of linear state space modeling

With a time scaling factor of 10^3 and a position scaling factor 10^5 for numerical reasons, the state equations become:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where

$$x = \begin{bmatrix} p_1 \\ w_1 \\ p_2 \\ w_2 \\ i_1 \\ i_2 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ -0.1 & -0.35 & 0.1 & 0.1 & 0.75 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0.1 & 0.1 & -0.1 & -0.35 & 0 & 0.75 \\ 0 & -0.03 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -0.03 & 0 & -1 \end{bmatrix}$$

Example of linear state space modeling

With a time scaling factor of 10^3 and a position scaling factor 10^5 for numerical reasons, the state equations become:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ -0.2 & -0.2 & 0.2 & 0.2 & 0 & 0 \end{bmatrix}$$

Example of linear state space modeling

With a time scaling factor of 10^3 and a position scaling factor 10^5 for numerical reasons, the state equations become:

$$\dot{x} = \mathrm{A}\mathrm{x} + \mathrm{B}\mathrm{u}$$
$$\mathrm{y} = \mathrm{C}\mathrm{x} + \mathrm{D}\mathrm{u}$$

where

$$\mathrm{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathrm{y} = \begin{bmatrix} p_3 \\ T \end{bmatrix}, \ \mathrm{u} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Outline

State-space model, transfer matrix and impulse response

Continuous-time system:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \xrightarrow[\;]{\substack{Laplace \\ x(0)=0}} \underbrace{G(s) = \frac{Y(s)}{U(s)} = D + C(sI - A)^{-1}B}_{\text{transfer matrix}}$$

in practice: $D = 0$

$$\Downarrow$$

$$\underbrace{G(t) = Ce^{At}B}_{\text{impulse response matrix}} \xrightarrow{\text{Laplace}}$$

Outline

Outline

Title 1

Consider a general nonlinear system in continuous time :

$$\frac{dx}{dt} = f(x, u)$$
$$y = h(x, u)$$

For small deviations about an equilibrium point (x_e, u_e, y_e) for which

$$f(x_e, u_e) = 0$$
$$y_e = h(x_e, u_e)$$

we define

$$x = x_e + \Delta x, u = u_e + \Delta u, y = y_e + \Delta y$$

Title 1

and obtain

$$\frac{dx}{dt} = \frac{d\Delta x}{dt} = f(x_e + \Delta x, u_e + \Delta u)$$

and

$$y_e + \Delta y = h(x, u) = h(x_e + \Delta x, u_e + \Delta u)$$

Title 1

By first order approximation we obtain a linear state space model from Δu to Δy :

$$\frac{d\Delta x}{dt} = f(x_e + \Delta x, u_e + \Delta u)$$

$$\Downarrow f(x_e, u_e) = 0$$

$$\frac{d\Delta x}{dt} \overset{(1)}{=} \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x_e, u_e}}_{\text{n x n}} \Delta x + \underbrace{\left.\frac{\partial f}{\partial u}\right|_{x_e, u_e}}_{\text{n x m}} \Delta u$$

$$\Downarrow \qquad\qquad \Downarrow$$

Title 1

and

$$y_e + \Delta y = h(x_e + \Delta x, u_e + \Delta u)$$
$$\Downarrow y_e = h(x_e, u_e)$$

$$\Delta y \stackrel{(1)}{=} \underbrace{\left.\frac{\partial h}{\partial x}\right|_{x_e, u_e}}_{\text{p x n}} \Delta x + \underbrace{\left.\frac{\partial h}{\partial u}\right|_{x_e, u_e}}_{\text{p x m}} \Delta u$$

Outline

Example

Consider a decalcification plant which is used to reduce the concentration of calcium hydroxide in water by forming a calcium carbonate precipitate.

Example

The following equations hold (simplified model) :

- chemical reaction : $Ca(OH)_2 + CO_2 \quad CaCO_3 + H_2O$
- reaction speed : r = c$[Ca(OH)_2][CO_2]$
- rate of change of concentration :

$$\frac{d[Ca(OH)_2]}{dt} = \frac{k}{V} - \frac{r}{V}$$

$$\frac{d[CO_2]}{dt} = \frac{u}{V} - \frac{r}{V}$$

k and u are the inflow rates in MOL/s of calcium hydroxide and carbon dioxide respectively. V is the tank volume in liters. Let the inflow rate of carbon dioxide be the input and the concentration of calcium hydroxide be the output.

Example

A nonlinear state space model for this reactor is :

$$\frac{d[Ca(OH)_2]}{dt} = \frac{k}{V} - \frac{c}{V}[Ca(OH)_2][CO_2]$$
$$\frac{d[CO_2]}{dt} = \frac{u}{V} - \frac{c}{V}[Ca(OH)_2][CO_2]$$
$$y = [Ca(OH)_2]$$

The state variables are $x_1 = [Ca(OH)_2]$ and $x_2 = [CO_2]$.

Example

In equilibrium we have :

$$\frac{k}{V} - \frac{c}{V}[Ca(OH)_2]_{eq}[CO_2]_{eq} = \frac{k}{V} - \frac{c}{V}X_1X_2 = 0$$

$$\frac{u_{eq}}{V} - \frac{c}{V}[Ca(OH)_2]_{eq}[CO_2]_{eq} = \frac{U}{V} - \frac{c}{V}X_1X_2 = 0$$

$$Y = [Ca(OH)_2]_{eq} = X_1$$

For small deviations about the equilibrium :

$$\frac{d\Delta x_1}{dt} = -\frac{c}{V}X_2\Delta x_1 - \frac{c}{V}X_1\Delta x_2$$

$$\frac{d\Delta x_2}{dt} = -\frac{c}{V}X_2\Delta x_1 - \frac{c}{V}X_1\Delta x_2 + \frac{1}{V}\Delta u$$

Example

so,

$$A = \begin{bmatrix} -\frac{cX_2}{V} & -\frac{cX_1}{V} \\ -\frac{cX_2}{V} & -\frac{cX_1}{V} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{V} \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

If

$$k = 0.1 \ \frac{\text{mole}}{\text{sec}}, c = 0.5 \ \frac{l^2}{\text{sec.mole}}, U = 0.1 \ \frac{\text{mole}}{\text{sec}},$$
$$X_1 = 0.25 \ \frac{\text{mole}}{\text{l}}, X_2 = 0.8 \ \frac{\text{mole}}{\text{l}}, V = 5 \text{ l}$$

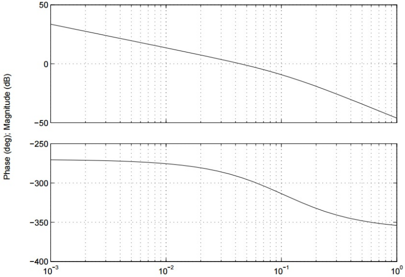
Then
$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} -0.08 & -0.025 & 0 \\ -0.08 & -0.025 & 0.2 \\ \hline 1 & 0 & 0 \end{array}\right]$$

Example

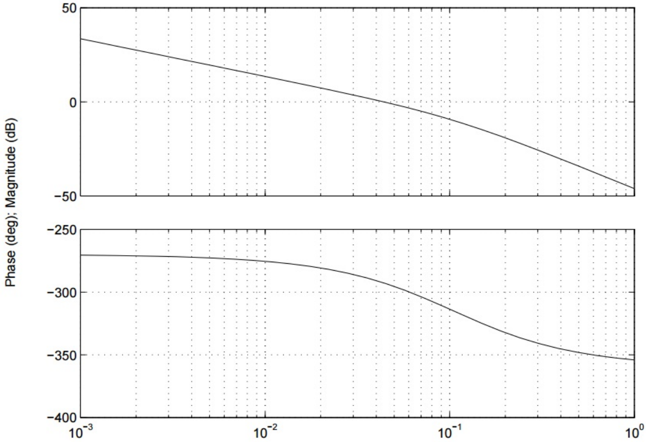
The corresponding transfer function is

$$\frac{\Delta y(s)}{\Delta u(s)} = \frac{-0.005}{s^2 + 0.105s}$$

and its bode plot



Example



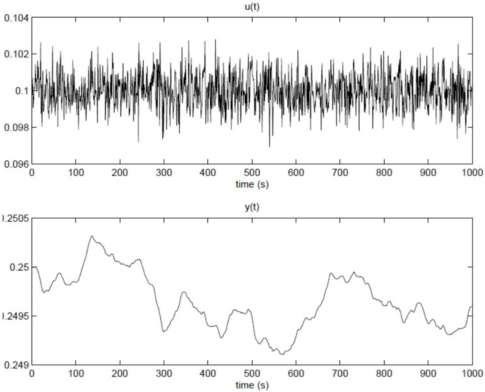
Example

One can also obtain a linear state space model for this chemical plant from linear system identification.
For small deviations about the equilibrium point the dynamics can be described fairly well by a linear (A, B, C, D)-model.

Hence, a small white noise disturbance u was generated and was added to the equilibrium value U.
We applied this signal to the input of a nonlinear model of the chemical reactor (Simulink model for instance),
i.e. $\mathrm{u(t) = U + \Delta u}$.

Example

The following input-output set was obtained :



Example

By applying a linear system identification algorithm (N4SID), the following 2nd-order model was obtained :

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} 0.0015 & 0.0589 & 0.0867 \\ 0.0027 & 0.1066 & 0.1713 \\ \hline 0.2546 & 0.129 & 0 \end{array}\right]$$

The corresponding transfer function is

$$\frac{\Delta y(s)}{\Delta u(s)} = \frac{-1.299.10^{-7}s - 0.05}{s^2 + 0.1051s + 1.346.10^{-8}}$$

It has 2 poles at $1.281.10^7$ and 1.051.

Example

As $1.281.10^7$ lies close to 0, and taking in account the properties of the manually derived linear model of page 45, we conclude that the plant has one integrator pole.

Hence, it might be better to fix one pole at $s = 0$. In this way it is guaranteed that the linear model obtained by system identification is stable.

The transfer function which was obtained using this modified identification procedure is

$$\frac{\Delta y(s)}{\Delta u(s)} = \frac{-1.68.10^{-8}s - 0.005}{s^2 + 0.1051s}$$

Example

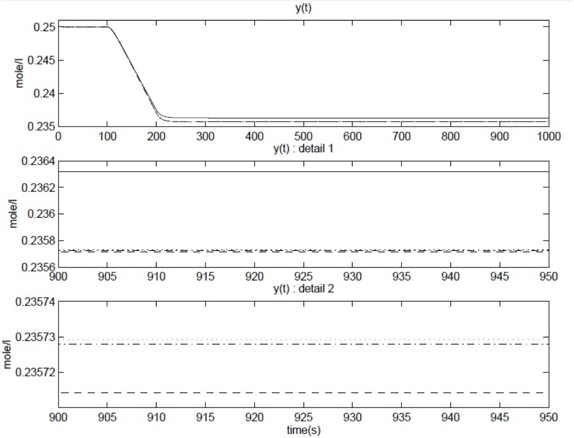
The different models are validated by comparing their response to the following input :

$$\begin{aligned} \Delta u(t) &= 0.003 \text{ if } 100 < t < 200 \\ &= 0 \text{ elsewhere} \end{aligned}$$

Four responses are shown :

- the nonlinear system (–)
- the linearised model obtained by hand (page 45) (- -)
- the 2nd-order linear model obtained from N4SID (. . .)
- the linear model obtained from N4SID having a fixed integrator pole by construction (-.)

Example

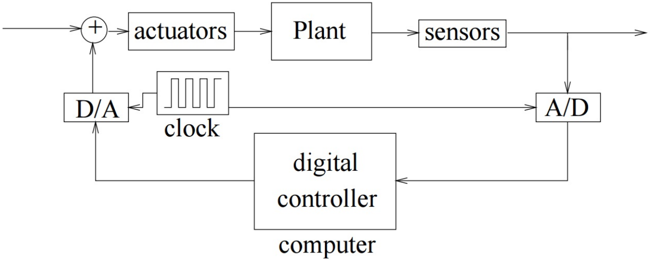

y(t)
mole/l
0.25
0.245
0.24
0.235
0
100
200
300
400
500
600
700
800
900
1000
y(t) : detail 1
0.2364
0.2362
0.236
0.2358
0.2356
900
905
910
915
920
925
930
935
940
945
950
y(t) : detail 2
0.23574
0.23573
0.23572
time(s)

Outline

Outline

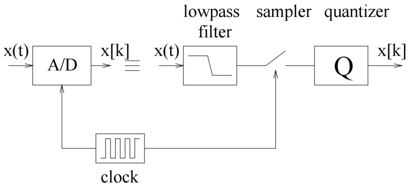
Digitization

in this course we want to control physical plants using a digital computer :



Digitization

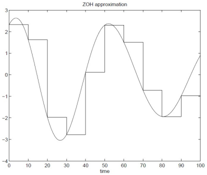
- analog-to-digital converter :



  - the analog anti-aliasing filter filters out high frequent components (> Nyquist frequency)
  - the filtered analog signal is sampled and quantized in order to obtain a digital signal. For quantization 10 to 12 bits are common.

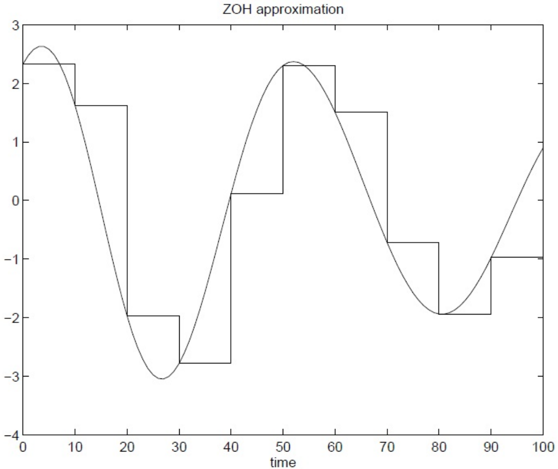
Digitization

- digital-to-analog converter :
  - zero-order hold



    - introduces a delay
    - frequency spectrum deformation
  - first-order hold
  - n-th order polynomial (more general case) : fit a $n - th$ order polynomial through the $n + 1$ most recent samples and extrapolate to the next time instance

Digitization



Outline

Numerical Integration Rules

1. Discretization by applying numerical integration rules: a continuous-time integrator

$$\dot{x}(t) = e(t)$$
$$\Leftrightarrow$$
$$sX(s) = E(s)$$

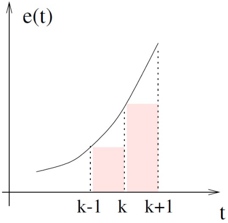
can be approximated using

Euler's method

- the forward rectangular rule or Eulers method

$$\frac{x_{k+1} - x_k}{T_s} = e_k \text{ with } x_k = (kT_s) \Leftrightarrow$$

$$\frac{z-1}{T_s} X(z) = E(z)$$

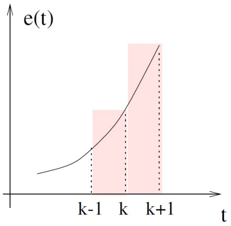


Backward rectangular rule

- the backward rectangular rule

$$\frac{x_{k+1} - x_k}{T_s} = e_{k+1} \Leftrightarrow$$

$$\frac{z-1}{zT_s} X(z) = E(z)$$

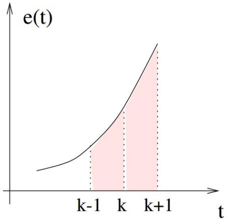


Trapezoid rule or bilinear transformation

- the trapezoid rule or bilinear transformation

$$\frac{x_{k+1} - x_k}{T_s} = \frac{e_{k+1} + e_k}{2} \Leftrightarrow$$

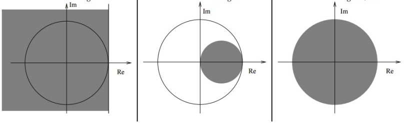
$$\frac{2}{T_s}\frac{z-1}{z+1}X(z) = E(z)$$



Numerical Integration Rules

Change a continuous model $G(s)$ into a discrete model $G_d(z)$ by replacing all integrators with their discrete equivalents:

Euler	backward rect.	bilinear transf.
$s \longrightarrow \frac{z-1}{T_s}$	$s \longrightarrow \frac{z-1}{zT_s}$	$s \longrightarrow \frac{2}{T_s}\frac{z-1}{z+1}$



Except for the forward rectangular rule stable continuous poles (grey zone) are guaranteed to be placed in stable discrete areas, i.e. within the unit circle.

The bilinear transformation maps stable poles $\rightarrow$ stable poles and unstable poles $\rightarrow$ unstable poles.

Numerical Integration Rules

The following continuous-time model is given :

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \leftrightarrow \quad \begin{aligned} sX &= AX + BU \\ y &= CX + DU \end{aligned}$$

following Eulers method s is replaced by $\frac{z-1}{T_s}$, so

$$\frac{z-1}{Ts} X = AX + BU$$

or

$$zX = I + AT_s)X + BT_s U$$

Numerical Integration Rules

the output equation $Y = CX + DU$ remains. A similar calculation can be done for the backward rectangular rule and the bilinear transformation resulting in the following table :

	Euler	backward rect.	bilinear transf.
A_d	$I + AT_s$	$(I - AT_s)^{-1}$	$(I - \frac{AT_s}{2})^{-1}(I + \frac{AT_s}{2})$
B_d	BT_s	$(I - AT_s)^{-1}BT_s$	$(I - \frac{AT_s}{2})^{-1}BT_s$
C_d	C	$C(I - AT_s)^{-1}$	$C(I - \frac{AT_s}{2})^{-1}$
D_d	D	$D + C(I - AT_s)^{-1}BT_s$	$D + C(I - \frac{AT_s}{2})^{-1}\frac{BT_s}{2}$

Outline

Zero-order hold

2. Discretization by assuming zero-order hold ...

$$\dot{x} = Ax + Bu$$

$\Rightarrow$

$$x(t) = e^{At}x(0) + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)}_{\text{convolution integral}}$$

Let the sampling time be T_s, then

$$\begin{aligned} x(t+T_s) &= e^{At+T_s}x(0) + \int_0^{t+T_s} e^{A(t+T_s-\tau)}Bu(\tau) \\ &= e^{AT_s}x(t) + e^{AT_s}\underbrace{\int_t^{t+T_s} e^{A(t-\tau)}Bu(\tau)}_{\text{approximated}} \end{aligned}$$

Zero-order hold

$$\xrightarrow{\text{ZOH}} x_{k+1} \overset{t=kT_s}{=} e^{AT_s}x_k + e^{AT_s}A^{-1}(I - e^{-AT_s})Bu_k$$
$$= \underbrace{e^{AT_s}}_{A_d}x_k + \underbrace{A^{-1}(e^{AT_s} - I)B}_{B_d}u_k$$

One can prove that $G_d(z)$ can be expressed as

$$G_d(z) \overset{ZOH}{=} (1 - z^{-1})\mathcal{Z}\left\{\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}\right\}$$

For this reason this discretization method is sometimes called step invariance mapping.

Outline

Zero-pole mapping

3. discretization by zero-pole mapping :

Following the previous method the poles of $G_d(z)$ are related to the poles of $G(s)$ according to $z = e^{sT_s}$.

If we assume by simplicity that also the zeros undergo this transformation, the following heuristic may be applied:

Zero-pole mapping

(a) map poles and zeros according to $z = e^{sT_s}$

(b) if the numerator is of lower order than the denominator, add discrete zeros at -1 until the order of the numerator is one less than the order of the denominator. A lower numerator order corresponds to zeros at ∞ in continuous time. By discretization they are put at -1.

(c) adjust the DC gain such that

$$\lim_{s \to 0} G(s) = \lim_{z \to 1} G_d(z)$$

Outline

Example

Example: given the following SISO system

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} -0.2 & 0.5 & 1 \\ 1 & 0 & 0 \\ \hline 1 & 0.4 & 0 \end{array}\right]$$

$$\Rightarrow G(s) = C(sI - A)^{-1}B + D = \frac{s + 0.4}{s^2 + 0.2s + 0.5}$$

Now compare the following discretization methods ($T_s = 1sec.$) :

1. bilinear transformation
2. ZOH
3. pole-zero mapping

Bilinear transformation

1. Bilinear transformation:

(a) A_d, B_d, C_d and D_d are calculated using the conversion table

(b) $G_{bilinear}(z) = C_d(zI - A_d)^{-1}B_d + D_d$

$$= \frac{0.4898+0.1633z^{-1}-0.3265z^{-2}}{1-1.4286z^{-1}+0.8367z^{-2}}$$

ZOH

2. Zero-order hold:

(a) $A_d = e^{AT_s}$, $B_d = A^{-1}(e^{AT_s} - 1)B$, $C_d = C$ and $D_d = D$

(b) $G_{ZOH}(z) = C_d(zI - A_d)^{-1}B_d + D_d$

$= \frac{1.0125z^{-1} - 0.6648z^{-2}}{1 - 1.3841z^{-1} + 0.8187z^{-2}}$

Pole-zero mapping

3. Pole-zero mapping:

(a) the poles of $G(s)$ are $-0.1 + j0.7$ and $-0.1 - j0.7$

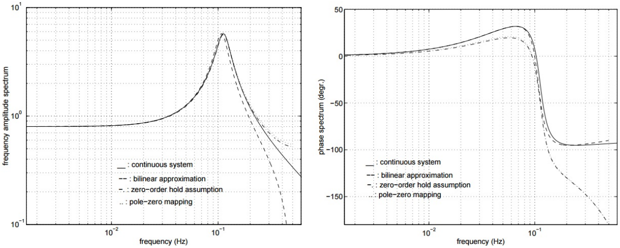
(b) there is one zero at -0.4

(c)

$$\begin{aligned}
G_{pz} &= K \frac{z - e^{-0.4}}{(z - e^{-0.1+j0.7})(z - e^{-0.1-j0.7})} \\
&= K \frac{z^{-1} - 0.6703z^{-2}}{1 - 1.3841z^{-1} + 0.8187z^{-2}} \\
\text{Hence } K &= \frac{\lim_{s\to 0} G(s)}{\lim_{z\to 1} G_{pz}(z)} = 1.0546
\end{aligned}$$

Bodeplots

Compare the bode plots :



Outline

Sampling rate selection

- the lower the sampling rate, the lower the implementation cost (cheap microcontroller and A/D converters), the rougher the response and the larger the delay between command changes and system response.
- an absolute lower bound is set by the specification to track a command input with a certain frequency :

$$f_s \geq 2BW_{cl}$$

f_s is the sampling rate and BW_{cl} is the closed-loop system bandwidth.

Sampling rate selection

- if the controller is designed for disturbance rejection,

$$f_s > 20BW_{cl}$$

- when the sampling rate is too high, finite-word effects show up in small word-size microcontrollers ($<$ 10 bits).
- for systems where the controller adds damping to a lightly damped mode with resonant frequency f_r,

$$f_s > 2f_r$$

In practice, the sampling rate is a factor 20 to 40 higher than BW_{cl}.

Outline

Advantages of state space models

- More general models: LTI and Nonlinear Time Varying (NTV).
- Geometric concepts: more mathematical tools (linear algebra).
- Internal and external descriptions: divide and conquer strategy.
- Unified framework: the same for SISO and MIMO.

Outline

Outline

Control canonical form (SISO):

$$\dot{x} = A_c x + B_c u, \;\; y = C_c x$$

$$A = \begin{bmatrix} -a_1 & -a_2 & \dots & \dots & -a_n \\ 1 & 0 & \dots & \dots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$C_c = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}$$

Transfer function

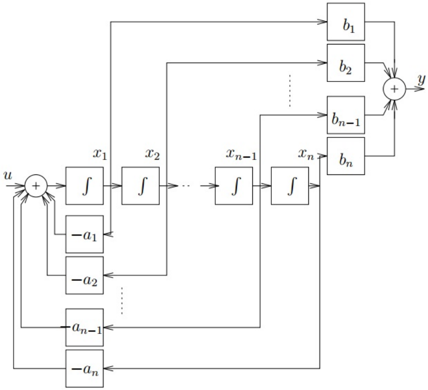
Corresponding transfer function:

$$G(s) = \frac{b(s)}{a(s)}$$

$$b(s) = b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n$$
$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n$$

Block diagram

Block diagram for the control canonical form



Modal canonical form

Modal canonical form:

$$\dot{x} = A_m x + B_m u, \;\; y = C_m x.$$

$$A_m = \begin{bmatrix} -s_1 & & & \\ & -s_2 & & \\ & & \ddots & \\ & & & -s_n \end{bmatrix}, B_m = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

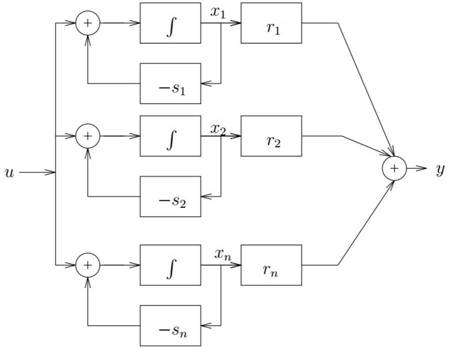
$$C_m = \begin{bmatrix} r_1 & r_2 & \dots & r_n \end{bmatrix}$$

Corresponding transfer function:

$$G(s) = \sum_{i=1}^{n} \frac{r_i}{s + s_i}$$

Block diagram

Block diagram for the modal canonical form:



Outline

A state space model (A, B, C,D) is not unique for a physical system. Let

$$x = T\bar{x}, \ \det(T) \neq 0,$$

$$\bar{A} = T^{-1}AT, \ \bar{B} = T^{-1}B, \ \bar{C} = CT, \ \bar{D} = D$$

$\rightarrow$

$$\begin{aligned} \dot{\bar{x}} &= T^{-1}AT\bar{x} + T^{-1}Bu = \bar{A}\bar{x} + \bar{B}u, \\ y &= CT\bar{x} + Du = \bar{C}\bar{x} + Du \end{aligned}$$

T is chosen to give the most convenient state-space description for a given problem (e.g.control or modal canonical forms).

Example

Example: Eigenvalue decomposition of A:

$$A = X \Lambda X^1, \Lambda = \begin{bmatrix} \alpha_1 & \beta_1 & & & \\ -\beta_1 & \alpha_1 & & & \\ & & \ddots & & \\ & & & \lambda_1 & \\ & & & & \ddots \end{bmatrix}$$

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \xrightarrow{X^{-1}x=z} \quad \begin{aligned} \dot{z} &= \Lambda z + X^{-1}Bu \\ y &= CX_z + Du \end{aligned}$$

Outline

Controllability

An n-th order system is called controllable if one can reach any state x from any given initial state x_0 in a finite time.

Controllability matrix:

$$C = \begin{bmatrix} B & AB & \dots & A^{n-1} & B \end{bmatrix}$$
$$rank(C) = n \iff (A, B) \text{ controllable}$$

Explanation: Consider a linear (discrete) system of the form

$$x_{k+1} = Ax_k + Bu_k$$

Controllability

Then

$$x_{k+1} = Ax_k + Bu_k$$

such that

$$x_{k+1} - A^{k+1}x_0 = \begin{bmatrix} B & AB & \dots & A^k & B \end{bmatrix} \begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$$

Controllability

Note that

$$rank \begin{bmatrix} B & AB & \dots & A^k & B \end{bmatrix} = rank \begin{bmatrix} B & AB & \dots & A^{n-1} & B \end{bmatrix}$$

for $k \geq n-1$ (Cayley-Hamilton Theorem).

There exists always a vector $\begin{bmatrix} u_k \\ u_{k-1} \\ \vdots \\ u_0 \end{bmatrix}$ if rank(C)=n.

Remarks: Controllability matrices for a continuous time system and a discrete time system are of the same form.

Outline

Observability

An n-th order system is called observable if knowledge of the input u and the output y over a finite time interval is sufficient to determine the state x.

Observability matrix:

$$O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$$rank(O) = n \iff (A, C) \text{ observable}$$

Observability

Explanation:

Consider an autonomous system

$$x_{k+1} = Ax_k$$
$$y_k = Cx_k$$

Then we find easily

$$y_k = CA_k x_0$$

and

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^k \end{bmatrix} x_0$$

Observability

Note that

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^k \end{bmatrix} = rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

for $k \geq n-1$.

One can always determine x_0 if $\text{rank}(O) = n$.

Remarks: Observability matrices for a continuous time system and a discrete time system are of the same form.

Outline

PBH controllability test

(A, B) is not controllable if and only if there exists a left eigenvector $q \neq 0$ of A such that

$$A^T q = q\lambda$$
$$B^T q = 0$$

In other words, (A, B) is controllable if and only if there is no left eigenvector q of A that is orthogonal to B. If there is a λ and q satisfying the PBH test, then we say that the mode corresponding to eigenvalue λ is uncontrollable (uncontrollable mode), otherwise it is controllable (controllable mode).

PBH controllability test

How to understand (for SISO) ? Put A in the modal canonical form:

$$A = \begin{bmatrix} * & & \\ & \lambda_i & \\ & & * \end{bmatrix}, B = \begin{bmatrix} * \\ 0 \\ * \end{bmatrix}, \Rightarrow q = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Uncontrollable mode: $\dot{x}_i = \lambda_i x_i$

PBH observability test

(A, C) is not observable if and only if there exists a right eigenvector $p \neq 0$ of A such that

$$A_p = p\lambda$$
$$C_p = 0$$

If there is a λ and p satisfy the PBH test, then we say that the mode corresponding to eigenvalue λ is unobservable (unobservable mode), otherwise it is observable (observable mode).

How to understand for SISO ? Put A in a modal canonical form.

Outline

Stability/Stabilizability/Detectability

Stability:
A system with system matrix A is unstable if A has an eigenvalue λ with real(λ)≥ 0 for a continuous time system and $|\lambda| \geq 1$ for a discrete time system.

Stabilizability:
(A, B) is stabilizable if all unstable modes are controllable.

Detectability:
(A, C) is detectable if all unstable modes are observable.

Example

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k \\
y_k &= Cx_k + Du_k
\end{aligned}
$$

$$
\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccc|c} 1.1 & 0 & 0 & 0 & 1 \\ 0 & -0.5 & 0.6 & 0 & 2 \\ 0 & -0.6 & -0.5 & 0 & -1 \\ 0 & -0.5 & 0.6 & 0 & 2 \\ \hline 0 & 1 & 2 & 3 & 1 \end{array}\right]
$$

Example

Modes	Stable?	Stabilizable?	Detectable?
1.1	no	yes	no
$-0.5 \pm 0.6i$	yes	yes	yes
2	no	yes	yes

Outline

Kalman decomposition

Given a state space system $[A, B, C, D]$. Then we can always find an invertible similarity transformation T such that the transformed matrices have the structure

$$TAT^{1} = \text{MATRIX}$$

$$TB = \text{MATRIX}, CT^{-1} = \text{MATRIX}$$

where

$$r_1 = rank(OC),\ \ r_2 = rank(C) - r_1,$$
$$r_3 = rank(O) - r_1,\ \ r_4 = n - r_1 - r_2 - r_3.$$

Kalman decomposition

The subsystem

$$[A_{co}, B_{co}, C_{co}, D]$$

is controllable and observable.

The subsystem

$$\begin{pmatrix} A_{co} & 0 \\ A_{21} & A_{\bar{c}o} \end{pmatrix}, \begin{pmatrix} B_{co} \\ B_{\bar{c}o} \end{pmatrix}, \begin{pmatrix} C_{co} & 0 \end{pmatrix}, D$$

is controllable.

Kalman decomposition

The subsystem

$$\begin{pmatrix} A_{co} & A_{13} \\ 0 & A_{\bar{c}o} \end{pmatrix}, \begin{pmatrix} B_{co} \\ 0 \end{pmatrix}, \begin{pmatrix} C_{co} & C_{\bar{c}o} \end{pmatrix}, D$$

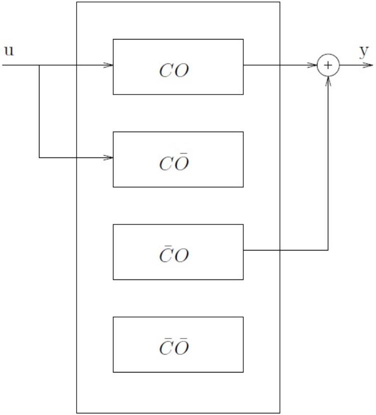
is observable.

The subsystem

$$[A_{\bar{c}o}, 0, 0, D]$$

is neither controllable nor observable.

Kalman decomposition



Outline

Minimal realization

A minimal realization is one that has the smallest-size A matrix for all triplets $[A, B, C]$ satisfying

$$G(s) = D + C(sI A)^1 B$$

where $G(s)$ is a given transfer matrix.

$[A, B, C, D]$ is minimal $\iff$ controllable and observable.

Minimal realization

Let $[A_i, B_i, C_i, D], i = 1, 2$, be two minimal realizations of a transfer matrix, then

$$\begin{array}{c}[A_1, B_1, C_1, D] \\ T \Downarrow\Uparrow T^{-1} \\ [A_2, B_2, C_2, D]\end{array}$$

with

$$T = C_1 C_2^T (C_1 C_2^T)^{-1}$$

Outline

Outline

Transfer matrix

General transfer matrix for a state space model :

$$G(\xi) = D + C(\xi I - A)^{-1}B$$

ξ can be s CT or z DT

Poles

Characteristic polynomial of matrix A:

$$\begin{aligned}\alpha(\xi) &= det(\xi I - A)\\ &= \alpha_n \xi^n + \alpha_{n-1}\xi^{n-1} + \ldots + \alpha_1 \xi + \alpha_0\end{aligned}$$

Characteristic equation: $$\alpha(\xi) = 0$$

The eigenvalues $\lambda_i = 1, \ldots, n$ of the system matrix A are called the poles of the system.
The pole polynomial is defined as

$$\pi(\xi) = \prod_{i=1}^{n}(\xi - \lambda_i)$$

Outline

Physical interpretation of the pole

Consider the following second order system :

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} \alpha & \beta & b_1 \\ -\beta & \alpha & b_2 \\ \hline c_1 & c_2 & 0 \end{array}\right]$$

The transfer matrix is given by (try to verify it) :

$$\begin{aligned} G(\xi) &= C(\xi I - A)^{-1}B + D \\ &= \frac{\xi(b_1c_1 + b_2c_2) + \beta(b_2c_1 + b_1c_2) - \alpha(b_1c_1 + b_2c_2)}{\xi^2 + 2\alpha\xi + \alpha^2 + \beta^2} \end{aligned}$$

Physical interpretation of the pole

There are 2 poles at $\alpha \pm j\beta$

There is a zero at

$$\frac{\alpha(b_1c_1 + b_2c_2) - \beta(b_2c_1 + b_1c_2)}{b_1c_1 + b_2c_2}$$

Continuous time

In continuous time the impulse response takes on the form:

$$g(t) = \mathcal{L}^{-1}\{G(s)\}$$

$$= \mathcal{L}^{-1}\{= \frac{s(b_1c_1 + b_2c_2) + \beta(b_2c_1 + b_1c_2) - \alpha(b_1c_1 + b_2c_2)}{s^2 + 2\alpha s + \alpha^2 + \beta^2}\}$$

$$= (A_m\cos(\beta t) + B_m\sin(\beta t))e^{\alpha t} = C_m e^{\alpha t}\cos(\beta t + \gamma)$$

$$\text{with } A_m = b_1c_1 + b_2c_2 \text{ and } B_m = b_2c_1 - b_1c_2$$

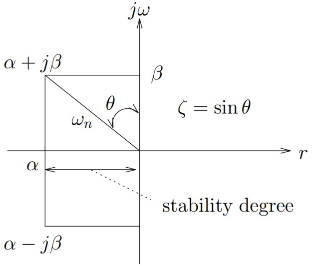
$$\Rightarrow C_m = \sqrt{(b_1^2 + b_2^2)(c_1^2 + c_2^2)}$$

$$\Rightarrow \gamma = \arctan\left(\frac{-B_m}{A_m}\right)$$

Continuous time

Define the damping ratio ζ and natural frequency ω_n :

$$\alpha = -\zeta\omega_n, \ \beta = \omega_n\sqrt{1-\zeta^2}$$

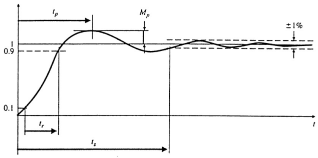


Continuous time

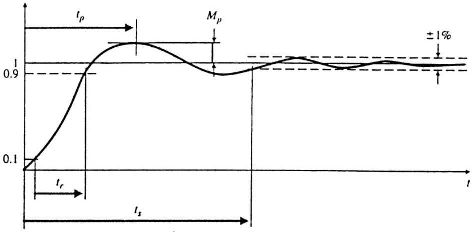
For a second order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

the time response looks like :



Continuous time



$$\text{rise time } t_r \simeq \frac{1.8}{\omega_n}$$

$$\text{settling time } t_s = \frac{4.6}{\zeta\omega_n}$$

$$\text{peak time } t_p = \frac{\pi}{\omega_d}, \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\text{overshoot } M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Continuous time

In general : if a continuous-time system $(A, B, C, 0)$ has poles at $\alpha \pm j\beta \Rightarrow$ the impulse response will have time modes of the form

$$A_m e^{\alpha t} cos(\beta t + \gamma)$$

$$\left.\begin{array}{l} \text{A}_m \text{ :amplitude} \\ \gamma \text{ : phase} \end{array}\right\} \text{determined by B and C.}$$

Discrete time

In discrete time the impulse response matrix $G_d(k)$ takes on the form :

$$G_d(k) = C_d A_d^{k-1} B_d, \; k \geq 1$$

which can be proven to be a sum of terms of the form :

$$C_m b^{k-1} \cos(\omega(k-1)+\gamma)$$

each of which satisfies a second order linear system

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cc|c} \alpha & \beta & b_1 \\ -\beta & \alpha & b_2 \\ \hline c_1 & c_2 & 0 \end{array}\right]$$

Discrete time

Parameterize A as

$$A = \begin{bmatrix} b\cos\omega & b\sin\omega \\ -b\sin\omega & b\cos\omega \end{bmatrix}$$

then

$$A^k = b^k \begin{bmatrix} b\cos\omega & b\sin\omega \\ -b\sin\omega & b\cos\omega \end{bmatrix}$$

Discrete time

Now,

$$
\begin{aligned}
CA_kB &= (A_m \cos\omega_k + B_m \sin\omega_k) b^k \\
&= C_m b^k \cos(\omega_k + \gamma) \\
\text{with } A_m &= C_m \cos\gamma = b_1 c_1 + b_2 c_2 \\
\text{and } B_m &= -C_m \sin\gamma = b_2 c_1 + b_1 c_2
\end{aligned}
$$

Discrete time

$$\Rightarrow C_m = \sqrt{(b_1^2 + b_2^2)(c_1^2 + c_2^2)}$$

$$\Rightarrow b = \sqrt{\alpha^2 + \beta^2}$$

$$\Rightarrow \omega = \arctan\left(\frac{\beta}{\alpha}\right)$$

$$\Rightarrow \gamma = \arctan\left(\frac{-B_m}{A_m}\right)$$

Outline

Definition

Definition: The zeros of a LTI system are defined as those values $\zeta \in C$ for which the rank of the transfer matrix $G(\xi)$ is lower than its normal rank (=rank of $G(\xi)$ for almost all values of ξ):

$$\text{rank} G(\zeta) < \text{normal rank}$$

Property

Property: Let ζ be a zero of $G(\xi)$ $(p \times m)$, then

$$\begin{aligned}
\text{rank} G(\zeta) &> \text{normal rank} \\
&\Downarrow \text{ if } \zeta \text{is not a pole of } G(\xi) \\
\exists v \neq \text{ s.t. } [D + C(\zeta I - A)^{-1}B]v &= 0, \\
&\Downarrow \text{ define } = \Delta C(\zeta I - A)^{-1} \\
Dv + Cw = 0, C(\zeta I - A)w - Bv &= 0 \\
&\Downarrow
\end{aligned}$$

$$\begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} = 0$$

Transmission zero's

How to find zeros for square MIMO systems ($p = m$) with invertible D ?

$$(\zeta I - A)w - Bv = 0, \; v = D^{-1}Cw$$

$$\Downarrow$$

$$Aw + Bv = w\zeta, \; v = D^{-1}Cw$$

$$\Downarrow$$

$$(A - BD^{-1}C)w = w\zeta$$

$$\begin{aligned} \zeta &= \text{ eigenvalue of } A - BD^{-1}C, \\ w &= \text{ corresponding eigenvector !} \end{aligned}$$

Other case

For other cases :

- Generalized eigenvalue problem.
- Use Matlab function tzero.

Minimum and non-minimum phase systems: If a system has an unstable zero (in the right halfplane (RHP) or outside the unit circle), then it is a non-minimum phase (NMP) system, otherwise it is a minimum phase (MP) system.

Physical explanation (continuous time)

Let ζ be a real zero, then there will exist vectors x_0 and u_0 such that:

$$\begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

This means if we have an input $u_0 e^{\zeta t}$, there exists an initial state x_0 such that the response is

$$y(t) = 0.$$

Physical explanation (continuous time)

For complex zeros :
if ζ is a complex zero, then also its complex conjugateζ^* is a zero.
Try to prove that if

$$\begin{bmatrix} \zeta I - A & -B \\ C & D \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

the output $y(t)$ will be exactly zero if the input is

$$u(t)|u_0|e^{\Re e(\zeta)t}.\cos(\Im_m\{\zeta\}t + \phi_{u0})$$

and the initial state is chosen to be $\Re_e\{x_0\}$.

Physical explanation (continuous time)

is the elementwise multiplication, both cos() and e () are assumed to be elementwise operators,

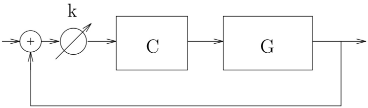
$\Re_e\{x_0\}$ is the real part of x_0,
$\Im_m\{\zeta\}$ is the imaginary part of ζ and $u_0 = |u_0|e^{j\phi u_0}$.

Try to derive equivalent formulas for the discrete-time case.

Transmission zero's

Why zeros are important :

- Limited control system performance



If G is NMP, k can not go to ∞, since unstable zeros (which become unstable poles) put a limit on high gains.

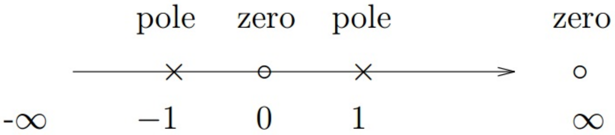
Transmission zero's

- Stability of a stabilizing controller, Parity Interlacing Property (PIP) :
  Let G be unstable. Then, G can be stabilized by a controller C which itself is stable $\Leftrightarrow$ between every pair of real RHP zeros of G (including at ∞), there is an even number of poles.

Example

Example:

$$G(s) = \frac{s}{s^2 - 1}$$



$G(s)$ can not be stabilized by a stable controller.

Transmission zero's

RHP zeros can cause undershoot !

Let

$$G(s) = \frac{\prod_{i=1}^{n} 1 - \frac{s}{\zeta_i}}{\prod_{i=1}^{n+r} 1 - \frac{s}{\lambda_i}},$$

where r = relative degree of $G(s)$. Let $y(t)$ be the step response.

$$\text{Fact: } \begin{cases} y(0) \text{ and its first } r-1 \text{ derivative are } 0; \\ y^{(r)} \text{ is the first non-zero derivative;} \\ y(\infty) = G(0). \end{cases}$$

Transmission zero's

The system has undershoot $\Leftrightarrow$ the steady state value $y(\infty)$ has a sign opposite to $y^{(r)}(0)$, i.e.

$$y^{(r)}(0)y(\infty) < 0.$$

It can be proven that the system has undershoot $\Leftrightarrow$ there is an odd number of RHP zeros.

PBH test:

There are 6 (independent) left eigenvectors associated with the 6 eigenvalues of A :

$$\begin{bmatrix} -0.1480 \pm 0.0826i \\ 0.0703 \pm 0.5380i \\ 0.1480 \pm 0.0826i \\ -0.0703 \pm 0.5380i \\ 0.2994 \pm 0.2954i \\ -0.2994 \pm 0.2954i \end{bmatrix}, \begin{bmatrix} 0.0766 \\ 0.5624 \\ 0.0766 \\ 0.5624 \\ 0.4218 \\ 0.4218 \end{bmatrix}$$

PBH test:

There are 6 (independent) left eigenvectors associated with the 6 eigenvalues of A :

$$\begin{bmatrix} 0 \\ 0.4892 \\ 0.0000 \\ 0.4892 \\ 0.5105 \\ 0.5105 \end{bmatrix}, \begin{bmatrix} -0.0046 \\ -0.0227 \\ 0.0046 \\ 0.0227 \\ -0.7067 \\ 0.7067 \end{bmatrix}, \begin{bmatrix} 0.0000 \\ -0.0295 \\ 0.0000 \\ -0.0295 \\ -0.7065 \\ -0.7065 \end{bmatrix}$$

It can be easily checked that none of them is orthogonal to B. The result is the same as the rank test above. The same can be done to investigate the observability.

Poles and zeros:

Poles: the eigenvalues of A are

$$-0.2370 \pm 0.5947i, 0, -0.2813, 0.9760, -0.9687$$

The system is not stable since there is a pole at zero.

The first two oscillatory poles are from the spring-mass system consisting of the tape and the motor/capstan inertias. The zero pole reflects the pure integrative effect of the tape on the capstans.

Poles and zeros:

Zeros: it can be checked in Matlab using function tzero that there is only one zero at -2. The eigenvector of the matrix

$$\begin{bmatrix} -2I - A & -B \\ C & 0 \end{bmatrix}$$

associated with the eigenvalue 0 is

$$\begin{bmatrix} x_0 \\ u_0 \end{bmatrix} \; with \; x_0 = \begin{bmatrix} -0.1959 \\ 0.1959 \\ 0.1959 \\ -0.1959 \\ -0.4571 \\ 0.4571 \end{bmatrix}, u_0 = \begin{bmatrix} 0.4630 \\ -0.4630 \end{bmatrix}$$

Outline

Physical explanation of the zero:

Since the initial values $i_1(0) = -i_2(0)$ and the inputs $e_1(t) = -e_2(t)$, and the motor drive systems in both sides are symmetric, the two capstans will rotate at the same speed, but in opposite directions. So the tape position over the read/write head will remain at 0 if the initial value is at 0.
The tension in the tape is the sum of the tensions in the tape spring and the tape damping. The input voltages vary such a way that the tensions in the tape spring and the tape damping cancel each other out.

Outline

Matlab Functions

c2d
c2dm
canon
expm
tf2ss
ss2tf

(d)impulse
(d)step
minreal
tzero
ctrb
obsv

ctrbf
obsvf
(d)lsim
ss
ssdata