

Design in the frequency domain

Nyquist stability criterion

July 28, 2015

Outline

- 1 Introduction
- 2 Stability
- 3 Cauchy's argument principle
 - Cauchy's argument principle
 - Examples
- 4 Nyquist stability criterion
- 5 Nyquist plot
 - Nyquist plot
 - Poles on the imaginary axis
 - Discrete-time
- 6 Margins
 - Gain margin
 - Phase margin
 - Design example

Introduction

The root-locus method dealt with the s- and z-domain and the location of the poles and zeros formed the basis for that method.

For the **frequency domain**, we use the following substitution: $s = j\omega$, which means we will only regard perfect oscillations.

Sines, cosines and exponential signals are eigenfunctions of Linear Time Invariant systems (LTI).

Perfect oscillations form the natural decomposition of each signal when you are dealing with LTIs.

Introduction

We need to translate our **design criteria** to something that fits the discussed method. For the root-locus method, we had to express the criteria in positions of poles and zeros.

For the frequency domain, typical design criteria are:

- Phase and gain margin
- Bandwidth
- Zero-frequency magnitude (= DC gain)

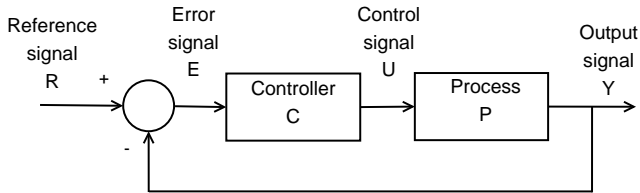
We will discuss two different **graphical representations** to design compensators in the frequency domain:

- Nyquist stability diagram and Nyquist plots
- Bode plots (for the design of lead, lag and lead-lag compensators)

Outline

- 1 Introduction
- 2 Stability**
- 3 Cauchy's argument principle
 - Cauchy's argument principle
 - Examples
- 4 Nyquist stability criterion
- 5 Nyquist plot
 - Nyquist plot
 - Poles on the imaginary axis
 - Discrete-time
- 6 Margins
 - Gain margin
 - Phase margin
 - Design example

Stability of the closed loop system



We write the output signal as a function of the input signal:

$$Y(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} R(s).$$

The closed loop system stability is determined by the poles of $\frac{P(s)C(s)}{1+P(s)C(s)}$, or the roots of $1 + P(s)C(s) = 0$.

Stability of the closed loop system

In the root-locus method, we determined the positions of the poles by plotting the roots of $1 + P(s)C(s) = 0$.

The system is stable if all the roots remain in the left half plane.

We are however not interested in the positions of the poles.

We only need to know whether there are poles in the right half plane.

There is a cheaper alternative: **the Nyquist stability criterion**.

Stability: Nyquist criterion

The Nyquist stability criterion avoids determining the roots of $1 + P(s)C(s)$ exactly (expensive). It only determines the number of roots in the right half plane.

It uses a theorem from complex calculus that finds the difference between the number of poles and the number of zeros within a contour (a closed curve).

We will apply this theorem to $1 + P(s)C(s)$ (which can be seen as a complex function) and the contour will encircle the entire right half plane (= the Nyquist contour).

Outline

- 1 Introduction
- 2 Stability
- 3 **Cauchy's argument principle**
 - Cauchy's argument principle
 - Examples
- 4 Nyquist stability criterion
- 5 Nyquist plot
 - Nyquist plot
 - Poles on the imaginary axis
 - Discrete-time
- 6 Margins
 - Gain margin
 - Phase margin
 - Design example

Outline

- 1 Introduction
- 2 Stability
- 3 Cauchy's argument principle
 - Cauchy's argument principle
 - Examples
- 4 Nyquist stability criterion
- 5 Nyquist plot
 - Nyquist plot
 - Poles on the imaginary axis
 - Discrete-time
- 6 Margins
 - Gain margin
 - Phase margin
 - Design example

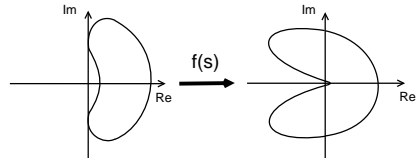
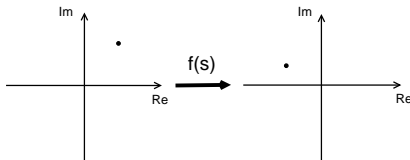
Complex function

Before we get to the theorem, we discuss the concept of a complex function.

A complex function $f(s) = u(x, y) + jv(x, y)$ maps the complex number $s = x + jy$ onto the complex number $w = u + jv$.

For a complex number:

or for a contour:



The function $1 + P(s)C(s)$ can also be regarded as a complex function, so we can use it as a mapping.

Cauchy's argument principle

This is the engine behind the Nyquist stability criterion.

If a contour Γ in the s -plane circles Z zeros and P poles $f(s)$ in clockwise rotation, then the contour Γ' , which is the image of Γ as mapped by $f(s)$, circles the origin (in the ω -plane) $Z-P$ times in the clockwise direction.

So the only thing we are looking at is the encirclements of the origin.

On the next slides, we will prove this.

Cauchy's argument principle

Let's take a complex function $f(s) = \frac{(s-z_1)(s-z_2)(s-z_3)\dots}{(s-p_1)(s-p_2)(s-p_3)\dots}$.

If we would apply this function to a complex number c , this comes down to multiplying factors $c - z_i = A_{z_i} e^{j\theta_{z_i}}$ and $\frac{1}{c - p_i} = \frac{1}{A_{p_i}} e^{-j\theta_{p_i}}$.

So the modulus of $f(c)$ can be easily found by evaluating

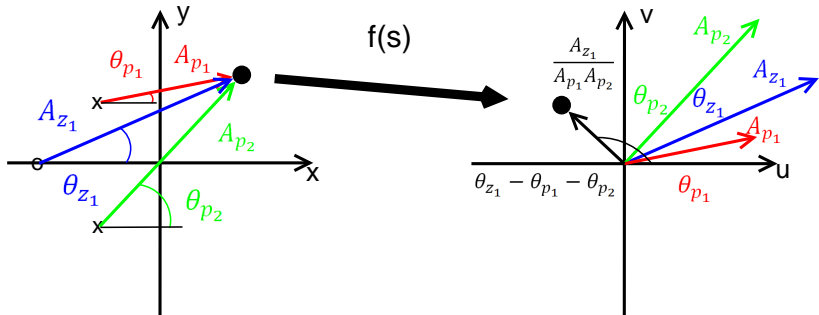
$$\frac{A_{z_1} A_{z_2} A_{z_3} \dots}{A_{p_1} A_{p_2} A_{p_3} \dots};$$

This might help if you want to map a point, but it is not important for us.

The evaluation of the argument of $f(c)$ is what will be interesting;

$$\angle f(c) = \theta_{z_1} + \theta_{z_2} + \theta_{z_3} + \dots - \theta_{p_1} - \theta_{p_2} - \theta_{p_3} - \dots$$

Cauchy's argument principle: graphically

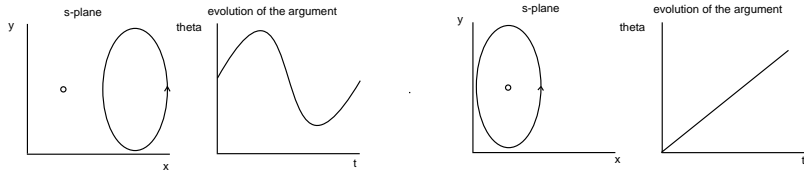


Cauchy's argument principle

When does the image of a contour in the ω -plane encircle the origin?

That happens when the argument of the image of the contour at the beginning and the end differ by 2π .

A pole or zero outside the contour will never have that effect. One inside the contour has the following effect:



Cauchy's argument principle

A pole results in -2π (counterclockwise rotation), if the contour is followed clockwise.

A zero results in $+2\pi$ (clockwise rotation).

This follows from the sign of their effect:

$$\angle f(c) = \theta_{z_1} + \theta_{z_2} + \theta_{z_3} + \dots - \theta_{p_1} - \theta_{p_2} - \theta_{p_3} - \dots$$

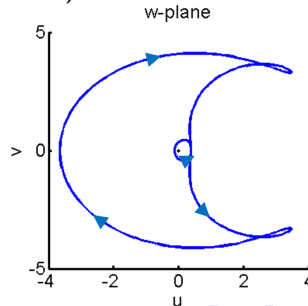
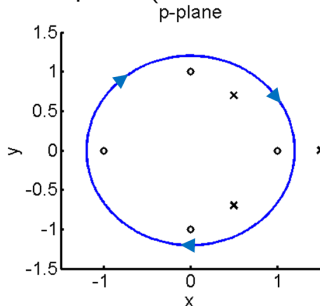
It is also possible that the origin is encircled when there are no poles or zeros in the contour (in the s-plane). But then the amount of clockwise encirclements equals the amount of counterclockwise encirclements hence no net encirclements.

Outline

- 1 Introduction
- 2 Stability
- 3 Cauchy's argument principle
 - Cauchy's argument principle
 - Examples
- 4 Nyquist stability criterion
- 5 Nyquist plot
 - Nyquist plot
 - Poles on the imaginary axis
 - Discrete-time
- 6 Margins
 - Gain margin
 - Phase margin
 - Design example

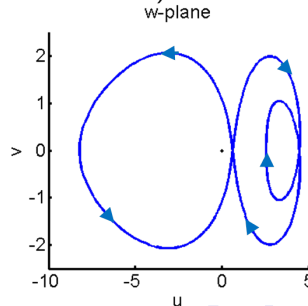
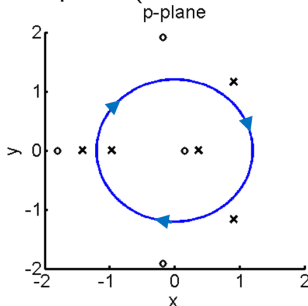
Example 1

- Two encircled poles (the x's): $P = 2$
- Four encircled zeros (the o's): $Z = 4$
- Hence: $N = Z - P = 2$
- Indeed, the image of the contour encircles the origin twice in the w -plane (in clockwise direction).



Example 2

- Two encircled poles: $P = 2$
- One encircled zero: $Z = 1$
- Hence: $N = Z - P = -1$
- Indeed, the image of the contour encircles the origin once in the ω -plane (in counterclockwise direction).

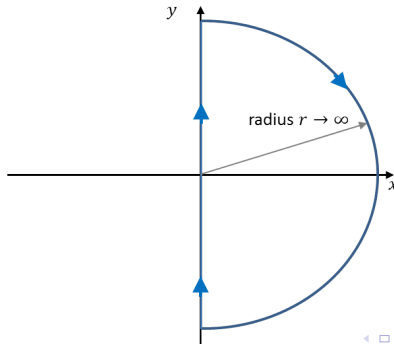


Outline

- 1 Introduction
- 2 Stability
- 3 Cauchy's argument principle
 - Cauchy's argument principle
 - Examples
- 4 **Nyquist stability criterion**
- 5 Nyquist plot
 - Nyquist plot
 - Poles on the imaginary axis
 - Discrete-time
- 6 Margins
 - Gain margin
 - Phase margin
 - Design example

Nyquist stability criterion

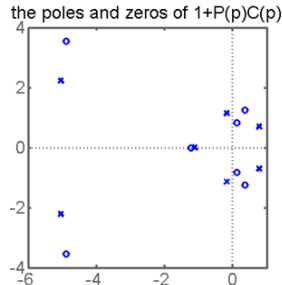
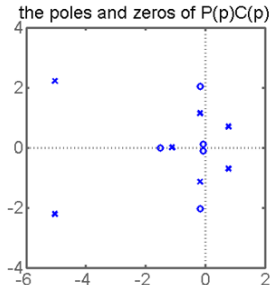
If we apply Cauchy's argument principle to the following contour (the Nyquist contour), the amount of clockwise encirclements around the origin of the image of this contour (called the Nyquist plot) in the ω -plane equals $Z - P$ of the right half plane (RHP).



Nyquist stability criterion

That way we will find the difference between the number of poles and zeros of $1 + P(s)C(s)$ in the RHP: $N = Z - P$.

So if we know P (the number of poles in the RHP of $1 + P(s)C(s)$), we know how many zeros of $1 + P(s)C(s)$ there are in the RHP.



Nyquist stability criterion

Luckily this last aspect is simple, since the poles of $1 + P(s)C(s)$ equal those of $P(s)C(s)$. Hence the amount of RHP poles is equal (the connection between the zeros is not clear).

So if we assume the number of RHP poles of $P(s)C(s)$ is known (which we'll assume), we know whether the system with unity (negative) feedback is stable.

Nyquist stability criterion

If we apply this to $1 + P(s)C(s)$, we need to count the number of encirclements of the origin.

However, the Nyquist stability criterion uses $P(s)C(s)$.

- The zeros of $1 + P(s)C(s)$ and the poles and zeros of $P(s)C(s)$ are hard to relate.
- This is in sharp contrast with how easily the Nyquist plots relate.
- The Nyquist plot of $P(s)C(s)$ equals the one of $(1 + P(s)C(s))$, after it has been moved to the right over a distance 1.
- To find $Z - P$, one has to count the number of clockwise encirclements of the image of $P(s)C(s)$ around the point $(-1, 0)$, since this equals the number of clockwise encirclements of the image of $1 + P(s)C(s)$ around the origin.

Nyquist stability criterion

Nyquist stability criterion

If the open loop system $P(s)C(s)$ has ℓ poles in the open right half plane (of the s -plane),
then the system with unity feedback is stable if and only if the
Nyquist plot encircles the point $(-1,0)$ ℓ times in the counter
clockwise direction.

Outline

- 1 Introduction
- 2 Stability
- 3 Cauchy's argument principle
 - Cauchy's argument principle
 - Examples
- 4 Nyquist stability criterion
- 5 Nyquist plot**
 - Nyquist plot
 - Poles on the imaginary axis
 - Discrete-time
- 6 Margins
 - Gain margin
 - Phase margin
 - Design example

Outline

- 1 Introduction
- 2 Stability
- 3 Cauchy's argument principle
 - Cauchy's argument principle
 - Examples
- 4 Nyquist stability criterion
- 5 Nyquist plot
 - Nyquist plot
 - Poles on the imaginary axis
 - Discrete-time
- 6 Margins
 - Gain margin
 - Phase margin
 - Design example

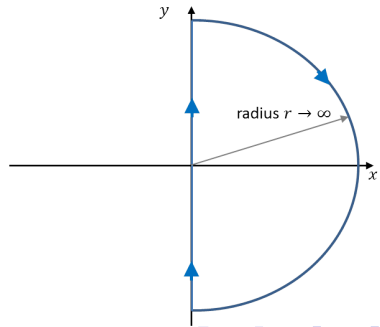
Nyquist plot

We know how to deduce stability of the closed loop system from the Nyquist plot.

Now we will discuss how these plots can be found.

First some simplifications:

- For physically realizable systems (i.e. relevant systems to engineers) the circle bow will be mapped on a point.
- The image of the positive imaginary axis is the mirror image of the negative imaginary axis



Nyquist plot: physically realizable systems

Where does the very convenient property of physically realizable systems come from?

- 1 Every physically realizable system is causal
- 2 Every causal system has a transfer function with an order of the denominator that is larger than or equal to the order of the numerator
- 3 From this shape of the transfer function, we can easily show that the circle bow maps onto a point

Nyquist plot: physically realizable systems

1 Every physically realizable system is causal

This is logical: you cannot build a system that knows the future

2 Every causal system has a transfer function with an order of the denominator that is larger than or equal to the order of the numerator

This can be easily understood in the discrete time case:

$$a_0 y_n + a_1 y_{n-1} + a_2 y_{n-2} + \dots = b_0 u_n + b_1 u_{n-1} + b_2 u_{n-2} + \dots$$

Apply the z-transformation:

$$\frac{Y(z)}{U(z)} = H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots}$$

Nyquist plot: physically realizable systems

3 From this shape of the transfer function, we can easily show that the circle bow maps onto a point, in the case of physically realizable systems

- If the order of the denominator is strictly higher than the order of the numerator:

If $s \rightarrow \infty$, then $P(s)C(s) \rightarrow 0$

- If the order of the denominator is equal to the order of the numerator:

If $s \rightarrow \infty$, then $P(s)C(s) \rightarrow c$, a real number

Nyquist plot: symmetry

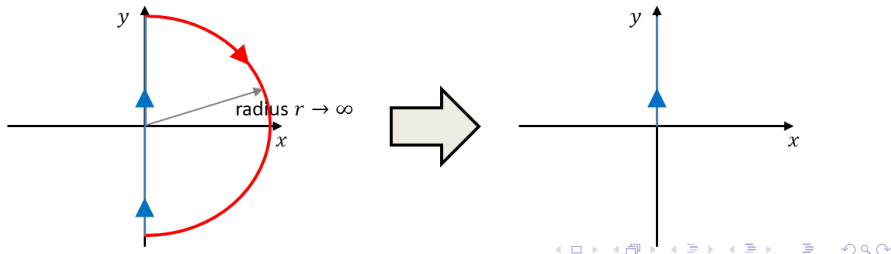
The symmetry follows directly from the way $f(c)$ can be evaluated.

Since the position of $f(c)$ only depends on the location of the poles and zeros, and those poles and zeros only occur symmetrically round the real axis, the Nyquist plot will be symmetrical round the real axis.

Nyquist plot

So we will only have to study the positive (or the negative) imaginary axis!

- The circular part maps onto one point, which is the same as where $j\omega$ maps onto
- The other half of the imaginary axis will give the mirror image of the studied half



Nyquist plot

Extracting the number of times $(-1, 0)$ is circled isn't that difficult anymore:

- You search for the image of $j0^+$
- You search for the image of $j\infty$
- You search for the positive real y for which $f(jy)$'s imaginary part changes sign

This information allows you to determine if you encircle $(-1, 0)$
Let's visualize this with a simple example, but of course you can use software to do this (e.g. nyquist in Matlab)

Nyquist plot: a simple example

Let's take the following open loop system:

$$P(s)C(s) = \frac{1}{s^2 - 2s + 2}$$

Substitute s with $j\omega$

$$\frac{1}{-\omega^2 - 2j\omega + 2} = \frac{1}{-\omega^2 + 2 - 2j\omega} \frac{-\omega^2 + 2 + 2j\omega}{-\omega^2 + 2 + 2j\omega} = \frac{-\omega^2 + 2 + 2j\omega}{(-\omega^2 + 2)^2 + 4\omega^2}$$

- $f(j0^+) = \frac{1}{2}$
- $f(j\infty) = 0$
- Imaginary part: $\frac{2\omega}{(-\omega^2+2)^2+4\omega^2} = 0 \Rightarrow \omega = 0$

The real axis gets crossed 2 times: first at $\frac{1}{2}(\omega = 0^+)$ and then at $0(\omega = \infty) \rightarrow (-1, 0)$ is not encircled

Nyquist plot: a simple example

Remember our open loop system:

$$P(s)C(s) = \frac{1}{s^2 - 2s + 2}.$$

Z and P are respectively the number of zeros and poles of $1 + P(s)C(s)$.

The poles of $P(s)C(s)$ and $1 + P(s)C(s)$ are the same ($P = 2$ in our example).

Since $(-1, 0)$ is not encircled: $Z - P = 0$, hence there are 2 zeros in the right half plane.

Remember that the zeros of $1 + P(s)C(s)$ are the poles of the closed loop system and thus, the unity feedback controller is unstable.

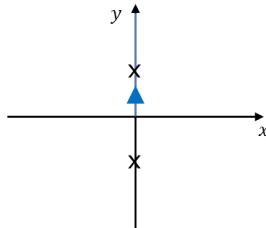
Outline

- 1 Introduction
- 2 Stability
- 3 Cauchy's argument principle
 - Cauchy's argument principle
 - Examples
- 4 Nyquist stability criterion
- 5 Nyquist plot
 - Nyquist plot
 - Poles on the imaginary axis
 - Discrete-time
- 6 Margins
 - Gain margin
 - Phase margin
 - Design example

Nyquist plot: poles on the imaginary axis

Poles on the imaginary axis: why are they a problem?

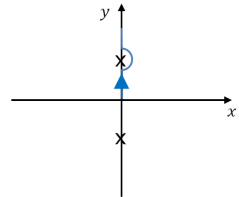
- Take for instance the case with one pair of imaginary poles at $j\omega_c$ and $-j\omega_c$
- When coming close to $j\omega_c$, the argument will remain 0 and the gain will increase to infinity
- At $j\omega_c$ itself, the gain will be infinite, but the argument is undetermined, hence we cannot map this point



Nyquist plot: poles on the imaginary axis

How do we solve this?

- Instead of going through the poles, we will evade them by an infinitesimally small amount (see figure)
- That way we do not have the problem of an undetermined mapping at the pole
- Since we avoid them by an infinitesimally small amount, we also know we will not wrongly avoid a pole the lies in the right half plane



Now the Nyquist plot will go to infinity as the pole is approached, then the argument will change from 0 to π as the semi-circle is traversed and then the Nyquist plot will return from infinity.

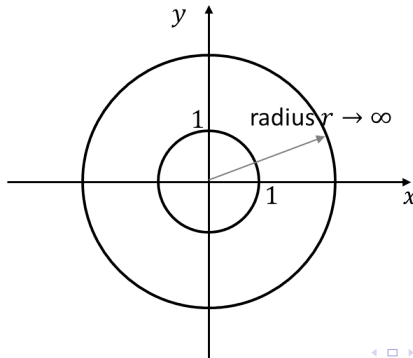
Outline

- 1 Introduction
- 2 Stability
- 3 Cauchy's argument principle
 - Cauchy's argument principle
 - Examples
- 4 Nyquist stability criterion
- 5 Nyquist plot
 - Nyquist plot
 - Poles on the imaginary axis
 - Discrete-time
- 6 Margins
 - Gain margin
 - Phase margin
 - Design example

Discrete-time case

The Cauchy argument principle still applies, since $P(z)C(z)$ also has the shape of a rational polynomial.

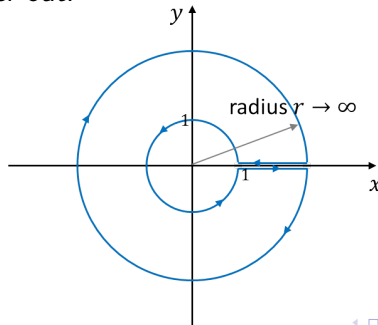
The contour will have to encircle the entire complex plane except for the unity circle:



Discrete-time case

However, this is not a contour, but it can be solved with the following trick:

The two horizontal pieces are both infinitely close to the real axis, that way they are identical but with opposite signs. They will cancel each other out.



Outline

- 1 Introduction
- 2 Stability
- 3 Cauchy's argument principle
 - Cauchy's argument principle
 - Examples
- 4 Nyquist stability criterion
- 5 Nyquist plot
 - Nyquist plot
 - Poles on the imaginary axis
 - Discrete-time
- 6 Margins
 - Gain margin
 - Phase margin
 - Design example

Margins

The Nyquist plot allows us to determine the stability of the system
We know the stability changes when $1 + P(s)C(s)$ has an imaginary zero (then the system is marginally stable).

We can see such a zero in the Nyquist plot of $P(s)C(s)$.
After all, the Nyquist plot is the image of the imaginary axis, so if there is a zero on the imaginary axis, the Nyquist plot of $1 + P(s)C(s)$ would go through 0 and the Nyquist plot of $P(s)C(s)$ would go through -1.

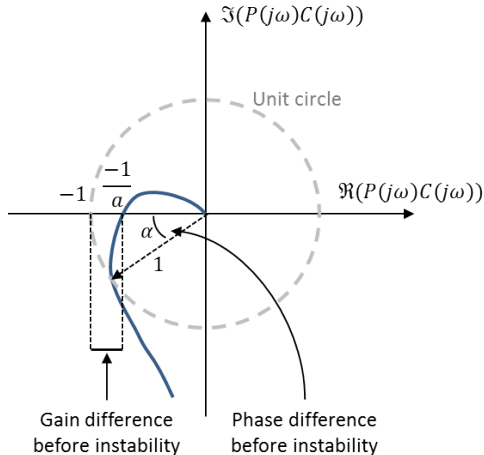
Margins

This explains why we can use the 'distance to -1' as a measure of stability.

We will see two different stability margins, which can be easily read off the Nyquist diagram:

- **The gain margin:** the amount of extra gain you can allow before instability occurs (in dB)
- **The phase margin:** the amount of phase you can add before instability occurs

Margins



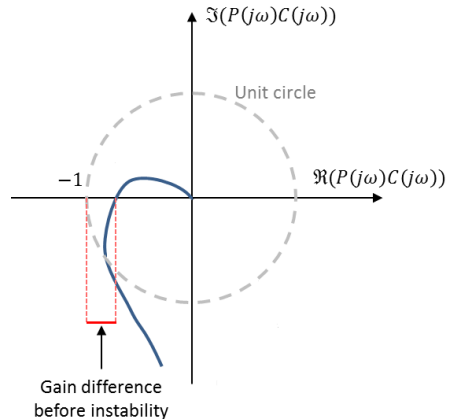
Outline

- 1 Introduction
- 2 Stability
- 3 Cauchy's argument principle
 - Cauchy's argument principle
 - Examples
- 4 Nyquist stability criterion
- 5 Nyquist plot
 - Nyquist plot
 - Poles on the imaginary axis
 - Discrete-time
- 6 Margins
 - Gain margin
 - Phase margin
 - Design example

Gain margin

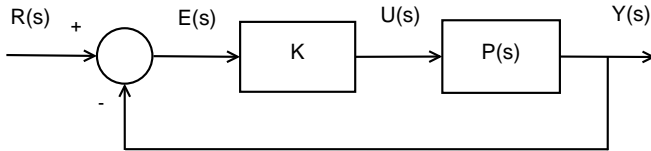
The gain margin is the amount of extra gain allowed before the system becomes unstable (or how much larger the gain has to be, before the system becomes unstable).

The gain margin is **multiplicative**, so it is the factor you have to multiply the gain with, so the Nyquist plot goes through -1 in the ω -plane. It will be expressed in dB ($20\log_{10}K$).



Gain margin

Let's look at it the following way:



The stability margin of $P(s)$ with unity feedback is the K for which the system above is marginally stable.

Gain margin

So $KP(s)$ should equal -1 for an imaginary $s = j\omega$.

This requires $\angle(KP(j\omega)) = \angle P(j\omega)$ to equal -180° . This ω_π is called the **Gain Crossover Frequency (GCF)**.

K then has to be set such that $|KP(j\omega_\pi)| = 1$.

So a large gain can lead to instability and this risk only exists when there exists a ω_π for which $\angle P(j\omega) = -180^\circ$.

We will illustrate this with an example.

Gain margin: example

Consider the process

$$P(s) = \frac{1}{s(s+2)}.$$

We derive the argument as a function of ω :

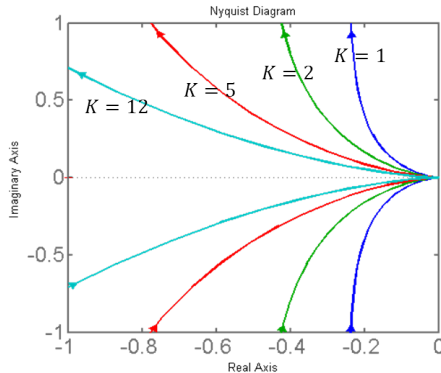
$$\angle P(j\omega) = -\tan^{-1}\left(\frac{\omega}{0}\right) - \tan^{-1}\left(\frac{\omega}{2}\right) = -90^\circ - \tan^{-1}\left(\frac{\omega}{2}\right).$$

For this to equal to -180° , it requires $\omega \rightarrow \infty$.

This makes $P(j\omega) = 0$ and this means the gain margin is infinite.

Gain margin: example

This solution is shown in the Nyquist plot below:



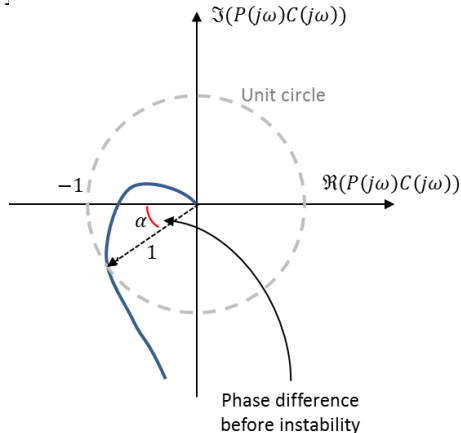
The only crossover with the real axis occurs at $\omega = 0$ and that does not change with increasing K .

Outline

- 1 Introduction
- 2 Stability
- 3 Cauchy's argument principle
 - Cauchy's argument principle
 - Examples
- 4 Nyquist stability criterion
- 5 Nyquist plot
 - Nyquist plot
 - Poles on the imaginary axis
 - Discrete-time
- 6 Margins
 - Gain margin
 - **Phase margin**
 - Design example

Phase margin

The phase margin is how much you can rotate the Nyquist diagram before -1 is crossed



Phase margin

This can be interpreted as multiplying $P(s)$ with $e^{j\theta}$ until -1 is crossed, so $P(s)e^{j\theta} = -1$ for an imaginary $s = j\omega$.

This requires $|P(j\omega)e^{j\theta}| = |P(j\omega)|$ to equal 1. This ω_0 is called the **Phase Crossover Frequency (PCF)**.

$|\theta|$ then has to be set such that

$$\angle(P(s)e^{j\theta}) = \angle P(s) + \theta = -180^\circ.$$

The gain margin is defined as positive, but that doesn't matter, because of the symmetry with respect to the real axis.

If a rotation of θ degrees results in a crossing of -1 , then a rotation of $-\theta$ does the same.

Phase margin: example

Consider the process

$$P(s) = \frac{1}{s(s+2)}.$$

We derive the modulus as a function of ω :

$$|P(j\omega)| = \frac{1}{\omega} \frac{1}{\sqrt{\omega^2 + 4}}.$$

For this to equal to 1, we find $\sqrt{\omega^4 + 4\omega^2} = 1$ or
 $\omega = \sqrt{\sqrt{5} - 2} = 0.486$.

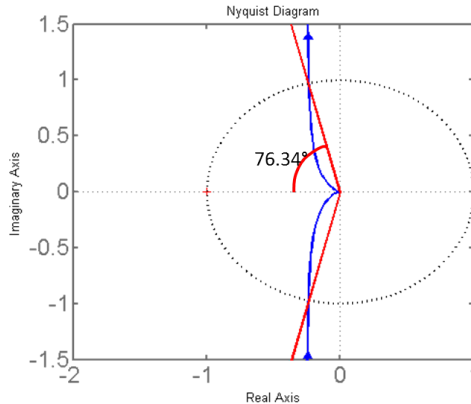
Now we need to find θ so $\angle(P(j\omega)e^{j\theta}) = -180^\circ$.

This results in $\theta = -180^\circ + \tan^{-1}(\frac{\omega}{0}) + \tan^{-1}(\frac{\omega}{2}) = -76.34^\circ$.

This means the phase margin is 76.34° .

Phase margin: example

This solution is shown graphically in the Nyquist plot below:



What should the margins be?

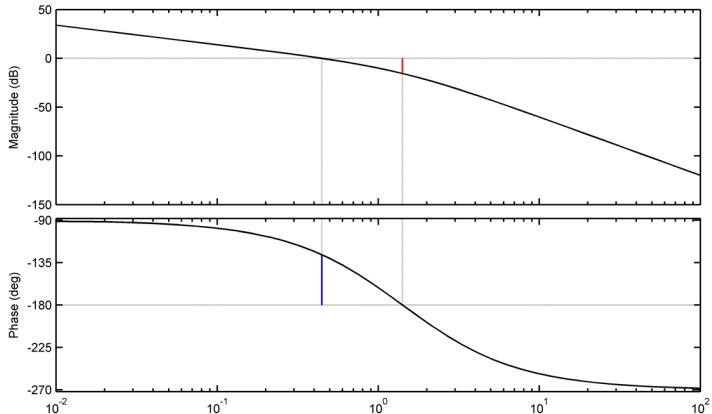
Phase margin:

- This is more subtle than the gain margin
- If it is too small, instability might occur due to practicalities
- If it is too small, we get large overshoots and large oscillations that fade away very slowly
- Sometimes a good value is 60° , but it is highly case-dependent

A good margin does not offer certainty about the stability, whereas a bad phase margin (very large or very small) does give certainty about instability.

Margins using Bode plots

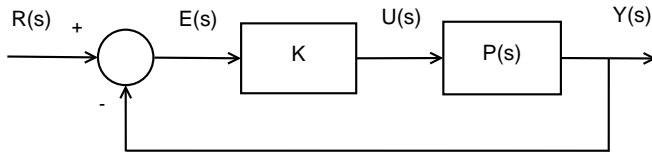
We can also easily derive the **gain margin** and **phase margin** from the Bode plot of $P(s)C(s)$:



Outline

- 1 Introduction
- 2 Stability
- 3 Cauchy's argument principle
 - Cauchy's argument principle
 - Examples
- 4 Nyquist stability criterion
- 5 Nyquist plot
 - Nyquist plot
 - Poles on the imaginary axis
 - Discrete-time
- 6 Margins
 - Gain margin
 - Phase margin
 - Design example

Margins: design example



Consider the following process

$$P(s) = \frac{1 - s}{(1 + s)^2}.$$

Design a simple proportional controller K with a phase margin of 45° .

Margins: design example

We find ω by demanding

$$\begin{aligned}\angle(P(j\omega)e^{\frac{j\pi}{4}}) &= -180^\circ \\ 45^\circ + \tan^{-1}(-\omega) - \tan^{-1}(\omega) - \tan^{-1}(\omega) &= -180^\circ \\ -3\tan^{-1}(\omega) &= -225^\circ \\ \omega &= \tan(75^\circ) = 3.73\end{aligned}$$

Now we can find K (which doesn't influence the argument) by setting $|KP(j\omega)| = 1$:

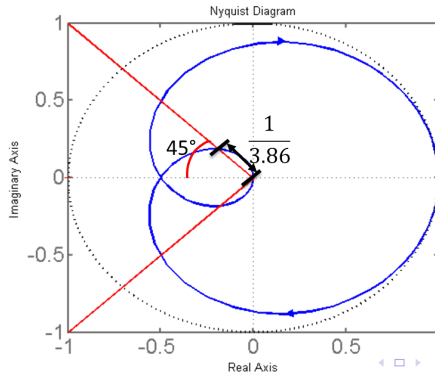
$$K \frac{\sqrt{\omega^2 + 1}}{\sqrt{\omega^2 + 1}^2} = \frac{K}{\sqrt{\omega^2 + 1}} = \frac{K}{3.86} = 1 \rightarrow K = 3.86 = 5.87 \text{ dB}$$

Margins: design example

We can also do this graphically with the Nyquist plot.

First determine the point that corresponds to $\theta = 45^\circ$.

Then determine the modulus, K is the inverse of that modulus.



Recapitulation

- The Nyquist stability criterion came to existence as a cheap alternative to determine stability of a closed loop system with unity feedback
- It also shows the phase margin and the gain margin, which are used to measure the stability of the system
- It is relevant as a design tool, as shown in the last example
- Bode plots make up a similar tool, yet the Nyquist stability criterion is (slightly) more generally applicable