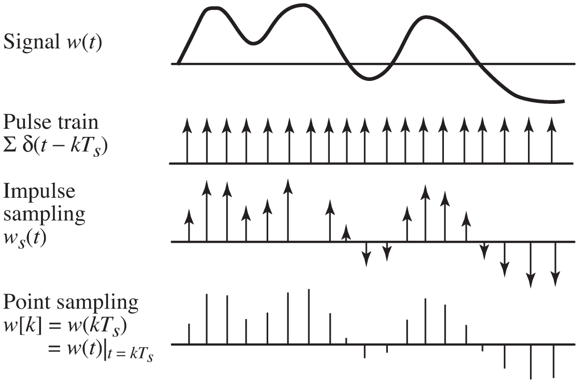
Discretization and reconstruction of signals

July 13, 2015

Outline

Discretizing signals

Sampling a continuous signal at discrete intervals



Discretizing signals

Is it possible to sample a continuous signal without loss of information?

Yes, as long as the signal has a limited bandwidth.
Bandwidth B = maximum frequency in a signal

How often should we sample the signal?

Nyquist theorem: if a signal has a bandwidth B then it can be fully reconstructed after being sampled with a frequency 2B

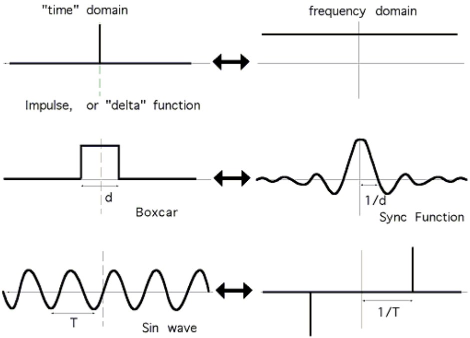
Outline

Fourier transform

$$f(t) \quad \Leftrightarrow \quad F(j\omega) \quad \in \mathbb{C}$$

$$\| \qquad\qquad \|$$

$$\int_{-\infty}^{\infty} e^{j\omega t} d\omega \qquad \int_{-\infty}^{\infty} f(\tau) e^{-j\omega\tau} d\tau = |F(j\omega) e^{j\phi(\omega)}|$$



Fourier transform: properties

- **Linearity**

$$\begin{cases} f_1(t) \leftrightarrow F_1(j\omega) \\ f_2(t) \leftrightarrow F_2(j\omega) \end{cases} \quad \Rightarrow \quad af_1(t) + bf_2(t) \leftrightarrow aF_1(j\omega) + bF_2(j\omega)$$

- **Time-scaling**

$$f(at) \leftrightarrow \left(\frac{1}{|a|}\right)F\left(\frac{j\omega}{a}\right)$$

- **Translation/Time-shifting**

$$f(t - t_0) \leftrightarrow e^{-j\omega t_0}F(j\omega)$$

- **Modulation/Frequency-shifting**

$$e^{j\omega_0 t}f(t) \leftrightarrow F(j(\omega - \omega_0))$$

Fourier transform: properties

- **Reciprocity**

$$F(-jt) \leftrightarrow 2\pi f(\omega)$$

- **Derivative in t**

$$\frac{df(t)}{dt} \leftrightarrow j\omega F(j\omega) \qquad \frac{d^n f(t)}{dt^n} \leftrightarrow (j\omega)^n F(j\omega)$$

- **Derivative in ω**

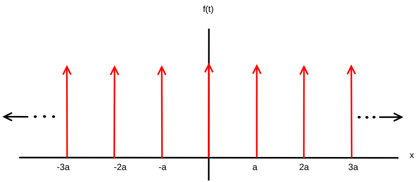
$$(-jt)^n f(t) \leftrightarrow \frac{d^n F(j\omega)}{d\omega^n} \qquad \frac{f(t)}{-jt} \leftrightarrow \int_{-\infty}^{\infty} F(j\Omega) d\Omega \text{ } if \text{ } f(0) = 0$$

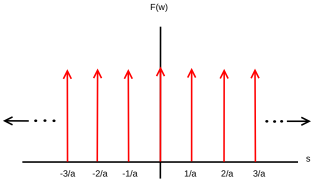
- **Convolution**

$$y(t) = h(t) * u(t) \leftrightarrow Y(j\omega) = H(j\omega)U(j\omega)$$
$$v(t) = h(t)u(t) \leftrightarrow V(j\omega) = \frac{1}{2\pi} H(j\omega) * U(j\omega)$$

Shannon-Nyquist sampling theorem

Sampling = multiplication by a train of Dirac-impulses
Fourier transform of an impulse train with period T is an impulse train with period T^{-1}:



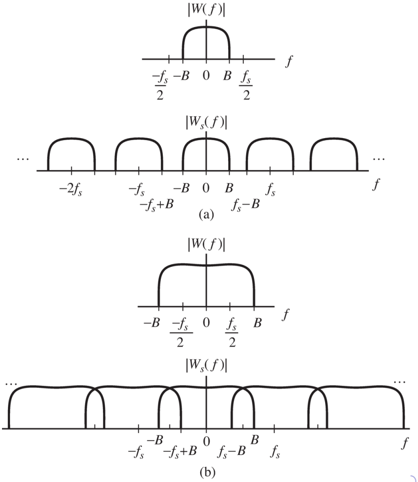


Sampling in the frequency domain = convolution of signal spectrum and impuls

Shannon-Nyquist sampling theorem

The signal can be fully reconstructed if there are no overlaps in the frequency domain. If the sampling frequency is too low then information will be lost (overlap).
If the sampling frequency is at least twice the bandwidth B, then the signal can be reconstructed without a problem (no overlap).

Sampling frequence $f_s \geq 2B$



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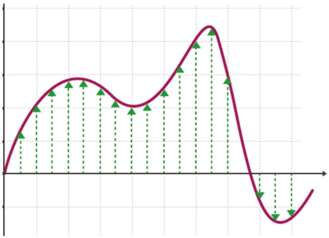
Time sampling

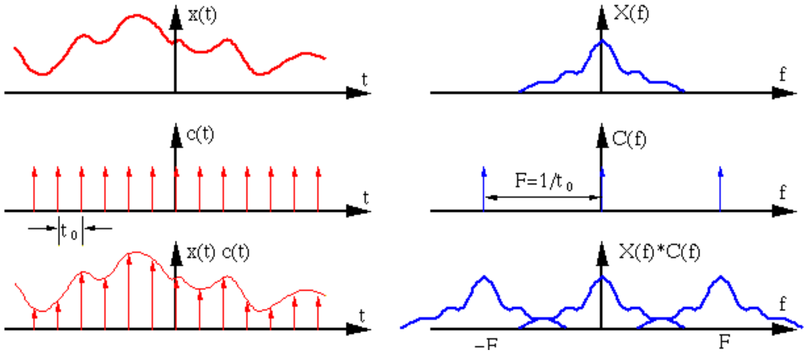
Time sampling is the operation that turns the signal f(t) into a pulse train where the magnitude of the impulse at kT equals the value f(kT).

The choice of the sampling-interval T is important. If T is small enough, the values f(kT) and f((k+1)T) will differ very little. This way we can reconstruct the intermediary values with full accuracy.



Time sampling

$$y(t) = x(t)c(t) \leftrightarrow Y(f) = \mathcal{F}[y(t)] = \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}dt$$
$$Y(f) = X(f) * C(f)$$



Truncation

To get a finite amount of data, we need to truncate the signal by a windowing function of finite duration T, e.g.:

A rectangular window

$$w(t) = \begin{cases} 1 & 0 \leq t \leq T \\ 0 & otherwise \end{cases}$$

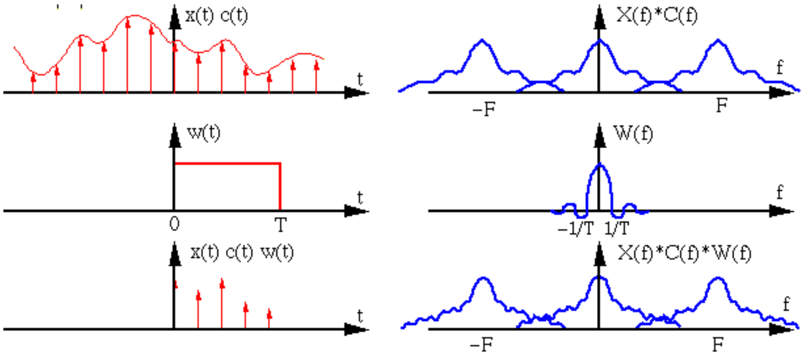
with spectrum

$$W(f) = \mathcal{F}[w(t)] = \frac{sin(\pi f T)}{\pi f} e^{-j\pi f T}$$

Truncation

$$z(t) = y(t)w(t) = \begin{cases} y(t) & 0 \leq t < T \\ 0 & otherwise \end{cases}$$

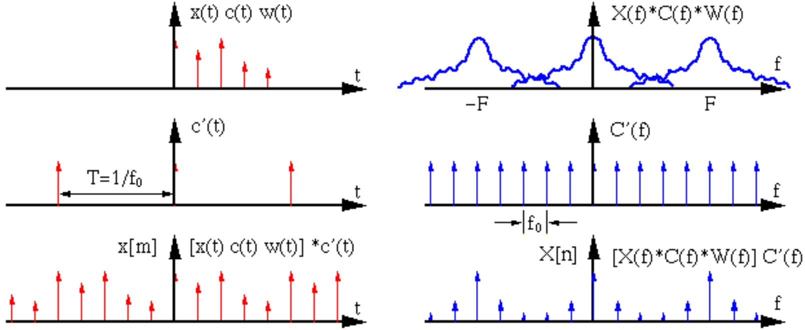
$$Z(f) = Y(f) * W(f)$$



Frequency sampling

$$P(j2\pi f) = \sum_{n=-\infty}^{\infty} \delta(f - nf_0) \qquad \leftrightarrow \qquad p(t) = \mathcal{F}^{-1}[P(j2\pi f]$$

$$V(j2\pi f) = Z(j2\pi f)P(j2\pi f) \qquad \leftrightarrow \qquad v(t) = z(t)p(t)$$



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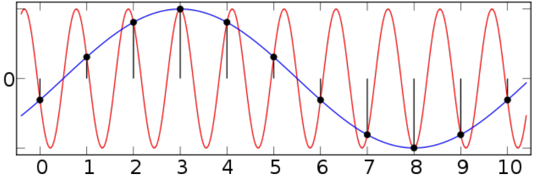
Aliasing

Aliasing is an effect that causes different signals to become indistinguishable when sampled. Frequencies that are too high to be sampled are folded onto lower frequencies.
A too low sample rate doesn't just lose information in the higher frequencies. It also causes faulty values for the lower frequencies.



The red sine wave is being sampled at just over it's bandwidth, however the blue sine wave will be recreated as it also fit's all data points and is within the expected bandwidth.

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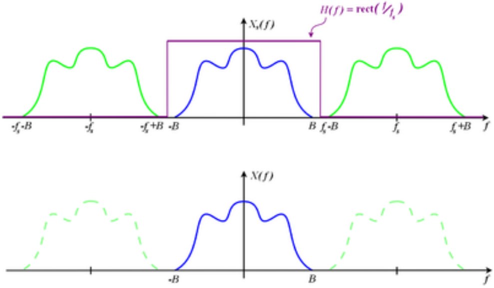
1. Discretization
   - Fourier transform
   - Discretization
2. Aliasing
3. Reconstruction
4. Methods
5. Example

Reconstruction

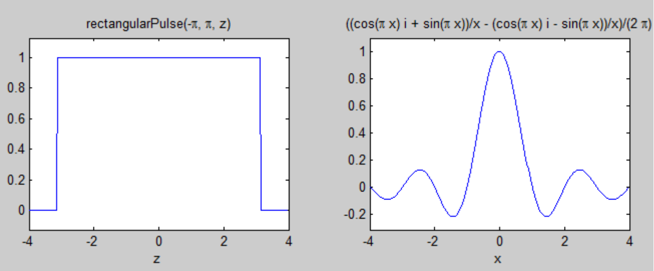
To retrieve the original spectrum of a sampled signal we have to multiply this signal with a block function $H(f) = rect(\frac{t}{t_0})$



This is the same as convoluting the signal with the inverse Fourier transform of a block function.

Because convoluting with a shifted impulse shifts a signal, this can be rewritten as:

$$\sum_{n=-\infty}^{\infty} f(nT)\frac{sin(\frac{\omega_0(t-nT)}{2})}{\frac{\omega_0(t-nT)}{2}}$$



Outline

Methods

Forward Euler
Backward Euler
Trapezium
Bilinear
Prewarping
Zero pole matching
Step invariant (ZOH)
Impuls invariant

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