

Geometric algebra

August 28, 2015

Outline

- 1 Vectors
- 2 Matrices
- 3 Systems of linear equations
- 4 Eigenvalues and eigenvectors
- 5 Appendixes

Vectors and spatential interpretation

Properties of a vector

There are 3 properties of a vector \vec{x} :

- magnitude
- direction
- start point

with respect to a referention vector $\vec{0}$

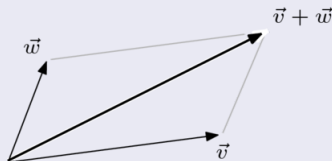
Multiplication between a scalar and a vector

$r \in \mathbb{R}$ ($r \in \mathbb{C}$ is possible, but hasn't a physical representation)

- $|r| < 1$: short the vector
- $|r| > 1$: increase the vector
- $r < 0$: reverse the direction

Addition of vectors

Parallelogram rule:



Vector spaces

First condition

A vector space V over a body L (set of operators) is a set of vectors that satisfy:

1. A vector sum is defined: $V \times V \rightarrow V : (\vec{x}, \vec{y}) \rightarrow \vec{x} + \vec{y}$

$$\vec{x}, \vec{y}, \vec{z} \in V$$

a) $\vec{x} + \vec{y} \in V$

b) $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$

c) $\exists! \vec{0} : \vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$

d) $\forall \vec{x}, \exists (-\vec{x}) : \vec{x} + (-\vec{x}) = (-\vec{x}) + \vec{x} = \vec{0}$

e) $\vec{x} + \vec{y} = \vec{y} + \vec{x}$

Vector spaces

Second condition

2. A outside law is defined: $L \times V \rightarrow V : (a, \vec{x}) \rightarrow a\vec{x}$

$$\vec{x}, \vec{y} \in V$$

$$a, b \in L$$

$$a) 1\vec{x} = \vec{x}$$

$$b) a(b\vec{x}) = (ab)\vec{x}$$

$$c) (a + b)\vec{x} = a\vec{x} + b\vec{x}$$

$$d) a(\vec{x} + \vec{y}) = a\vec{x} + a\vec{y}$$

Numberspaces of n-couples

Defenition

This is the set of all n-couples $\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$ with $x_i \in \mathbb{R}$ or $x_i \in \mathbb{C}$

($i=1,\dots,n$).

This set together with the operator set \mathbb{R} or \mathbb{C} is a vector space.

Subspaces

Defenition

V_1 is a subspace of vector space V if:

- 1 $V_1 \subset V$
- 2 With the same in- and outside law as V , is V_1 a vector space.

Properties

- 1 $\vec{0} \in$ every subspace
- 2 The intersection of two spaces is always a subspace.
- 3 Given: p vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_p \in V$.
The set vectors $a_1\vec{x}_1 + a_2\vec{x}_2 + \dots + a_p\vec{x}_p$ with $a_i \in \mathbb{R}$ is a subspace of V .

Linear independence, basis, dimensions

Defenition independence

Given: p vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_p \in V$.

Construct the null vector as a linear combination of those vectors (i.e. search the operators (numbers) a_1, a_2, \dots, a_p to form $a_1\vec{x}_1 + a_2\vec{x}_2 + \dots + a_n\vec{x}_n = \vec{0}$).

If the null vector only can be created by $a_1 = a_2 = \dots = a_p = 0$, then are the vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_p$ **linear independent**.

Linear independence, basis, dimensions

Properties linear independence

- 1 If the vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_p$ are linear independent, then can't none of them be written as a linear combination of the other $p-1$ vectors.
- 2 If the null vector is one of the p vectors, then is the set $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_p$ linear dependent (if $\vec{x}_1 = 0$ then is $a_1\vec{x}_1 + a_2\vec{x}_2 + \dots + a_n\vec{x}_n = 0$ with $a_1 \neq 0$ and $a_2, a_3, \dots, a_p = 0$).
- 3 Basis and dimension: p linear independent vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_p$ generate a vector space V^p . Every vector in V^p can be written **in only one way** as a linear combination of the p linear independent vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_p$ using operators a_1, a_2, \dots, a_p .

Linear independence, basis, dimensions

Basis, dimension

Given: $\vec{v} = a_1 \vec{x}_1 + a_2 \vec{x}_2 + \dots + a_p \vec{x}_p$.

The set operators $\{a_1, a_2, \dots, a_p\}$ are called the **coordinates** of the vector \vec{v} relative to the set vectors $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_p\}$. This set vectors is a **basis** of the vector space V^p , with **dimension** p .

Linear independence, basis, dimensions

Example

Given: $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The set $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is a linear independent combination. There doesn't exist numbers $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$ such that $a_1\vec{x}_1 + a_2\vec{x}_2 + a_3\vec{x}_3 = \vec{0}$. The set of all vectors $\vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2 + a_3\vec{x}_3$ is the three dimensional vector space V^3 .

If $\vec{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Then $3\vec{x}_1 + 2\vec{x}_2 + \vec{x}_3$ is the only way to write \vec{y} as a linear combination of $\vec{x}_1, \vec{x}_2, \vec{x}_3$.

Linear independence, basis, dimensions

Example

The set vectors $\vec{y} = a_1 \vec{x}_1 + a_2 \vec{x}_2$ is a **two dimensional** subspace V^2 .

The vectors in this subspace are:

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}.$$

Important difference

- All vectors of V^2 have **3** coordinates.
- The dimension of the subspace V^2 is **2**.

Linear independence, basis, dimensions

Convention of notation

Given: a n -dimensional vector space V^n .

The elements of this vector space are: \vec{x}, \vec{y}, \dots . If we choose $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ as a basis of V^n , then we can write every vector of V^n as a linear combination of those basis vectors in only one way: $\vec{x} = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \dots + x_n \vec{e}_n$. The numbers x_i are the coordinates of vector \vec{x} relative to the basis $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$. Between the vector

space of dimension n and the number space of dimension n exists a isomorphism.

Linear independence, basis, dimensions

Vector space V^p

Given: a p -dimensional vector space V^p where the vectors are n -couples (with $n \geq p$).

- ① In V^p exists a basis with p linear independent vectors.
- ② Every vector $\vec{x} \in V^p$ can be written in only one way as a linear combination of the p basis vectors using coordinates.

Example 1

Given: $n=5$, $p=2$, $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \\ 5 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

Linear independence, basis, dimensions

Example 1

The vectors \vec{x}_1 and \vec{x}_2 are linear independent, so they span a two dimensional subspace: $\vec{y} = a_1\vec{x}_1 + a_2\vec{x}_2$ with $a_1, a_2 \in \mathbb{R}$.

The coordinates of the vector $y_1 = \begin{bmatrix} 5 \\ -6 \\ 1 \\ 2 \\ 5 \end{bmatrix}$ are $a_1 = 1$ and $a_2 = 2$,

relative to the basis $\{\vec{x}_1, \vec{x}_2\}$.

Linear independence, basis, dimensions

Example 1

The vector $\vec{y}_2 = \begin{bmatrix} 5 \\ -7 \\ 1 \\ 2 \\ 5 \end{bmatrix}$ can't be written as a linear combination of

the vectors \vec{x}_1 and \vec{x}_2 . So y_2 doesn't belong to the subspace spanned by \vec{x}_1 and \vec{x}_2 .

This implies that \vec{y}_2 is linear independent with \vec{x}_1 and \vec{x}_2 .

Hence, the subspace spanned by \vec{y}_2 , \vec{x}_1 and \vec{x}_2 is a 3 dimensional subspace.

Linear independence, basis, dimensions

In general

When the set vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_p$ is linear independent, then is \vec{x}_i not totally in the subspace spanned by the vectors $\vec{x}_1, \dots, \vec{x}_{i-1}, \vec{x}_{i+1}, \dots, \vec{x}_p$.

The vector \vec{x}_i can be written as a sum of 2 components: $\vec{x}_{i\alpha}$ and $\vec{x}_{i\beta}$.

- ① $\vec{x}_{i\alpha} \in$ subspace spanned by $\vec{x}_1, \dots, \vec{x}_{i-1}, \vec{x}_{i+1}, \dots, \vec{x}_p$.
- ② $\vec{x}_{i\beta} \perp$ subspace spanned by $\vec{x}_1, \dots, \vec{x}_{i-1}, \vec{x}_{i+1}, \dots, \vec{x}_p$.

Inner product

Definition

The inner product of two vectors \vec{x} and $\vec{y} \in E^n$ (n-couples) is defined as the image: $E^n \times E^n \rightarrow \mathbb{R} : \{\vec{x}, \vec{y}\} \rightarrow \vec{x} \cdot \vec{y} \in \mathbb{R}$. This image is:

① Bilinear:

$$(\vec{x} + \vec{v}) \cdot \vec{y} = \vec{x} \cdot \vec{y} + \vec{v} \cdot \vec{y}$$

$$\vec{x}(\vec{v} + \vec{y}) = \vec{x} \cdot \vec{v} + \vec{x} \cdot \vec{y}$$

$$(a\vec{x}) \cdot \vec{y} = a(\vec{x} \cdot \vec{y})$$

$$\vec{x}(a\vec{y}) = a(\vec{x} \cdot \vec{y})$$

② Symmetric:

$$\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$$

③ Positive definite:

$$\forall \vec{x} \neq \vec{0} : \vec{x} \cdot \vec{x} > 0$$

Inner product

Matricial notation

The inner product is a **scalar**. If \vec{x} , \vec{y} and the basis $\in E^n$, then can the inner product be noted matricial:

$$\vec{x} \cdot \vec{y} = y^t A x = x^t A y = (x_1 \dots x_n) A \begin{pmatrix} y_1 \\ \dots \\ y_n \end{pmatrix}$$

with A positive definite and symmetric ($A = A^t$).

Inner product

Norm of a vector

$\|\vec{x}\|^2 = \vec{x} \cdot \vec{x}$ and because $\vec{x} \cdot \vec{x} > 0$:

$\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$ where $\|\vec{x}\|$ is called the norm of \vec{x} .

Normalizing is dividing a vector by its norm. The result is a vector with norm = 1.

$$\left\| \frac{\vec{x}}{\|\vec{x}\|} \right\| = \sqrt{\frac{\vec{x} \cdot \vec{x}}{\|\vec{x}\| \|\vec{x}\|}} = \sqrt{\frac{\|\vec{x}\|^2}{\|\vec{x}\| \|\vec{x}\|}} = 1.$$

Inner product

Cauchy Schwarz inequality

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \|\vec{y}\| \text{ or}$$

$$-\|\vec{x}\| \|\vec{y}\| \leq \vec{x} \cdot \vec{y} \leq \|\vec{x}\| \|\vec{y}\| \Rightarrow$$

$$-1 \leq \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|} \leq 1$$

By definition is:

$$\cos(\theta) = \cos(\angle(\vec{x}, \vec{y})) = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|} \text{ (with } \theta \text{ the angle between the two vectors)}$$

Hence: the angle between the vectors \vec{x} and $\vec{y} = \theta = \arccos\left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \|\vec{y}\|}\right)$ (in product of $\frac{\vec{x}}{\|\vec{x}\|}$ and $\frac{\vec{y}}{\|\vec{y}\|}$).

Inner product

Orthogonality

$$\begin{aligned}\vec{x} \text{ and } \vec{y} \text{ are orthogonal} &\Leftrightarrow \\ \theta = \angle(\vec{x}, \vec{y}) = 90^\circ = \frac{\pi}{2} \text{ rad} &\Leftrightarrow \\ \cos(\theta) = 0 &\Leftrightarrow \\ \vec{x} \cdot \vec{y} = 0\end{aligned}$$

Hence, if $\vec{x}, \vec{y} \neq 0$:

$$\vec{x} \perp \vec{y} \Leftrightarrow \vec{x} \cdot \vec{y} = 0.$$

Parallellism

$$\vec{x} \parallel \vec{y} \Leftrightarrow \theta = 0^\circ \text{ or } 180^\circ \Leftrightarrow \cos(\theta) = \pm 1 \Leftrightarrow |\vec{x} \cdot \vec{y}| = \|\vec{x}\| \|\vec{y}\|$$

Inner product

Distance between two vectors

Distance = $\|\vec{x} - \vec{y}\| = \|\vec{z}\|$ with $\vec{z} = \vec{x} - \vec{y}$.

$$\|\vec{x} - \vec{y}\|^2 = (\vec{x} - \vec{y})(\vec{x} - \vec{y})$$

$$= \vec{x}\vec{x} - \vec{x}\vec{y} - \vec{y}\vec{x} + \vec{y}\vec{y}$$

$$= \vec{x}\vec{x} + \vec{y}\vec{y} - 2\vec{x}\vec{y}$$

$$= \|\vec{x}\|^2 + \|\vec{y}\|^2 - 2\|\vec{x}\|\|\vec{y}\|\cos(\theta) \text{ with } \theta \text{ the angle between } \vec{x} \text{ and } \vec{y}.$$

Pythagorean theorem

If $\vec{x} \perp \vec{y}$ then $\cos(\theta) = 0$ and thus:

$$\|\vec{x} - \vec{y}\|^2 = \|\vec{x}\|^2 + \|\vec{y}\|^2.$$

Inner product

The 'simple' inner product

If in the definition $\vec{x} \cdot \vec{y} = y^t A x = x^t A y$ (with A positive definite and symmetric) $A=I$, then the inner product becomes the simple

$$\text{inner product: } \vec{x} \cdot \vec{y} = y^t I x = x^t I y = (x_1 \dots x_n) \begin{pmatrix} y_1 \\ \dots \\ y_n \end{pmatrix} = \sum_{i=1}^n x_i y_i.$$

This simple inner product can always be found in a basis transformation: $x=Rx'$ and $y=Ry'$, then $\vec{x} \cdot \vec{y} = y'^t (R^t A R) x'$. Now, R must be taken such that $R^t A R = I$. This can be done by converting A to its normal form by a congruent transformation (e.g. the method of quadratic forms).

In what follows, we mean by 'inner product' always 'the simple inner product'.

Gram Schmidt orthogonalization

Making two independent vectors orthogonal

Geometric derivation:

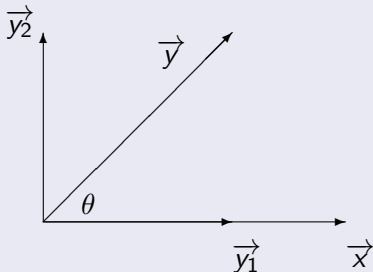


Figure 1: decomposition of vector \vec{y} in a component parallel (\vec{y}_1) and a component orthogonal (\vec{y}_2) to \vec{x} .

Gram Schmidt orthogonalization

Making two independent vectors orthogonal

- 1 Project \vec{y} orthogonal on \vec{x} , this generates the vector \vec{y}_1 , the component parallel with \vec{x} .
- 2 Subtract \vec{y} by \vec{y}_1 , the result is \vec{y}_2 which is orthogonal to \vec{x} .

\vec{y}_1 is a multiple of the normalized vector \vec{x} : $\vec{y}_1 = \alpha \frac{\vec{x}}{\|\vec{x}\|}$.

$\vec{y}_1 \parallel \vec{x}$: $\vec{y}_1 \vec{x} = \pm \|\vec{y}_1\| \|\vec{x}\|$ (+ if $\theta \leq 90^\circ$ and - if $\theta > 90^\circ$).

From fig. 1: $\|\vec{y}_1\| = \cos(\theta) \|\vec{y}\|$.

From the inner product: $\cos(\theta) = \frac{\vec{x} \vec{y}}{\|\vec{x}\| \|\vec{y}\|}$. So

$$\vec{y}_1 \vec{x} = \alpha \frac{\vec{x}}{\|\vec{x}\|} \vec{x} = \alpha \|\vec{x}\| = \|\vec{y}_1\| \|\vec{x}\| = \cos(\theta) \|\vec{y}\| \|\vec{x}\| = \vec{x} \vec{y}.$$

So we get: $\alpha = \frac{\vec{x} \vec{y}}{\|\vec{x}\|^2}$.

Gram Schmidt orthogonalization

Conclusion

$$\vec{y}_1 = \alpha \frac{\vec{x}}{\|\vec{x}\|} = \left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2} \right) \vec{x} \text{ with } \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2} \text{ a scalar.}$$

$$\text{And } \vec{y}_2 = \vec{y} - \vec{y}_1 = \vec{y} - \left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2} \right) \vec{x}.$$

Hence:

The vector \vec{y} gets orthogonalized on the vector \vec{x} by subtract \vec{y} by the component of \vec{y} parallel with \vec{x} .

Verify of $\vec{y}_2 \perp \vec{x}$:

$$\vec{y}_2 \cdot \vec{x} = \left(\vec{y} - \left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2} \right) \vec{x} \right) \cdot \vec{x} = \vec{y} \cdot \vec{x} - \left(\frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2} \right) \|\vec{x}\|^2 = 0.$$

Gram Schmidt orthogonalization

Generalization to multiple vectors

Given: 3 vectors: 2 orthogonal unit vectors \vec{x}_1 and \vec{x}_2
($\|\vec{x}_1\| = 1 = \|\vec{x}_2\|$, $\vec{x}_1 \vec{x}_2 = 0$) and a vector \vec{y} .

Asked: orthogonalize \vec{y} on \vec{x}_1 and \vec{x}_2 .

Solution:

We first search the component of \vec{y} parallel with \vec{x}_1 and subtract \vec{y} by this component. This gives \vec{y}_1 .

$$\vec{y}_1 = \vec{y} - \left(\frac{\vec{x}_1 \vec{y}}{\|\vec{x}_1\|^2} \right) \vec{x}_1 = \vec{y} - (\vec{x}_1 \vec{y}) \vec{x}_1 \quad (\|\vec{x}_1\|^2 = 1)$$

$$\vec{y}_1 \perp \vec{x}_1.$$

Gram Schmidt orthogonalization

Generalization to multiple vectors

Next, we subtract \vec{y}_1 by the component of \vec{y}_1 that is parallel with \vec{x}_2 , to get \vec{z} (which is perpendicular to both \vec{x}_1 and \vec{x}_2).

$\vec{z} = \vec{y}_1 - \vec{x}_2(\vec{y}_1 \vec{x}_2)$. We can write \vec{z} in another way:

$$\vec{z} = \vec{y}_1 - \vec{x}_2(\vec{y}_1 \vec{x}_2) = \vec{y} - (\vec{x}_1 \vec{y}) \vec{x}_1 - \vec{x}_2([\vec{y} - (\vec{x}_1 \vec{y}) \vec{x}_1] \vec{x}_2)$$

$$\vec{z} = \vec{y} - \vec{x}_1(\vec{x}_1 \vec{y}) - \vec{x}_2(\vec{x}_2 \vec{y})$$

Conclusion

The vector \vec{y} becomes orthogonalized on two orthogonal unit vectors \vec{x}_1 and \vec{x}_2 by subtracting \vec{y} by the components of \vec{y} parallel with \vec{x}_1 and \vec{x}_2 .

Complementary subspace

Definition

Given: a n dimensional vector space V^n , with $p < n$ linear independent vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_p$. These vectors create a p dimensional subspace V^p and can be orthogonalized via the Gram Schmidt method to an orthonormal basis $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_p$ with $\vec{e}_i \vec{e}_j = \delta_{ij}$ (with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$).

This p vectors can be complemented by $n - p$ linear independent vectors $\vec{f}_1, \vec{f}_2, \dots, \vec{f}_{n-p}$ that are linear independent with $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_p$ and orthonormal.

These vectors $\vec{f}_1, \vec{f}_2, \dots, \vec{f}_{n-p}$ generate the orthogonal complement of the subspace created by $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_p$.

The orthogonal complement of the p -dimensional subspace V^p of V^n ($p < n$), has dimension $n - p$.

Complementary subspace

Example

$$\text{Given: } n = 5, p = 3, \vec{e}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Asked: the orthogonal complement

Solution: The orthogonal complement has dimension $n - p = 5 - 3 = 2$ and consists of the set vectors that are perpendicular to the vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_p$.

$$\begin{cases} \vec{e}_1 \cdot \vec{x} = 0 \\ \vec{e}_2 \cdot \vec{x} = 0 \\ \vec{e}_3 \cdot \vec{x} = 0 \end{cases}$$

Complementary subspace

Example

$$\begin{cases} \begin{bmatrix} 2 & 0 & -1 & 0 & 0 \end{bmatrix} \vec{x} = 0 \\ \begin{bmatrix} 1 & 0 & 0 & 4 & 0 \end{bmatrix} \vec{x} = 0 \\ \begin{bmatrix} 1 & 3 & 1 & 0 & 0 \end{bmatrix} \vec{x} = 0 \end{cases}$$

$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 1 & 3 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

This is a homogeneous system of equations. The solution of this system is the orthogonal complement.

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Row- and column vectors

Example

The rows of a $m \times n$ matrix A can be considered as m row vectors with n components.

$$A^{m \times n} = \begin{bmatrix} \vec{r_1} \\ \vec{r_2} \\ \dots \\ \vec{r_m} \end{bmatrix}$$

The columns of a $m \times n$ matrix A can be considered as n column vectors with m components.

$$A^{m \times n} = [\vec{r_1} \quad \vec{r_2} \quad \dots \quad \vec{r_m}]$$

Row- and column space, rank

Column space

We consider the columns of $A^{m \times n}$ as vectors with m components and define the vectors \vec{x} as every possible linear combination of the column vectors \vec{k}_i :

$$\vec{x} = a_1 \vec{k}_1 + a_2 \vec{k}_2 + \dots + a_n \vec{k}_n \text{ with } a_i \in \mathbb{R} \text{ and } i \text{ the } i^{\text{th}} \text{ column of } A.$$

The set of all the vectors \vec{x} is called the column space of A .
The column space = all possible linear combinations that can be created of the columns of A .

Row- and column space, rank

Column space

If only r of the n vectors are linear independent, that means

- None of this r vectors can be written as a linear combination of the other $r - 1$ vectors
- all others $n - r$ column vectors can be written as linear combinations of the r linear independent vectors

Then r is called:

- 1 the (column) rank of matrix A
- 2 the dimension of the column space of matrix A

Row- and column space, rank

Row space

The concept row space and row rank can be derived in the same way as the column space is derived.

Rank

It is a fundamental matrix property that:

$$\text{row rank } A = \text{column rank } A$$

That means: the number linear independent columns in a matrix is equal to the number linear independent rows.

Hence:

$$\begin{aligned} \text{rank } A &= \text{row rank } A = \text{column rank } A \\ &= \text{dimension row space } A = \text{dimension column space } A \end{aligned}$$

Row- and column space, rank

Example

$$A^{4 \times 3} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \\ 2 & 4 & -2 \end{bmatrix} = [k_1 \quad k_2 \quad k_3] = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix}$$

We determine the rank of A by converting it to his echelon form by elementary row operations (explained in the appendix).

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 1 & 3 & -1 \\ 2 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Canonical form: the first element of every row is 1. Above and below this ones is every number 0.

Row- and column space, rank

Conclusion

The rank = the number of rows that differs from zero. So the rank is 2. In other words: there are only 2 linear independent rows and columns in A .

Hence:

The column space of A is a 2 dimensional subspace of the 4 dimensional vector space.

The row space is a 2 dimensional subspace of the 3 dimensional space.

Link between determinant and rank by square matrices

Determinant-rank

If all the columns of a square matrix are linear dependent, then the determinant is 0.

If $A^{m \times m}$:

$\det A = 0 \Leftrightarrow \text{rank } A < m \Leftrightarrow \text{columns dependent} \Leftrightarrow \text{rows dependent}.$

If $\det A \neq 0$ then $\text{rank } A = m$ and A is of **full rank**. A matrix can be inverted if its determinant differs from 0 (when its of full rank).

Link between determinant and rank by square matrices

Determinant-rank

If $\text{rank}(B^{n \times n}) = r$ with $r \leq n$ and $\text{rank}(A^{n \times n}) = n$ (A is of full rank) then is:

$$\text{rank}(AB) = \text{rank}(BA) = \text{rank}(B).$$

Elementary row- and column operations are always of full rank (appendix). When there is a multiplication between a matrix and a elementary row- or column matrix, then has the product always the rank of the matrix.

Column space test

Belongs the vector to the column space?

Given: n vectors with m components relative to a basis in V^m :

$$\begin{bmatrix} x_{11} \\ x_{21} \\ \dots \\ x_{m1} \end{bmatrix}, \dots, \begin{bmatrix} x_{1n} \\ x_{2n} \\ \dots \\ x_{mn} \end{bmatrix}.$$

If r of this n vectors are linear independent then has the matrix

$$X^{m \times n} = [\vec{x}_1, \dots, \vec{x}_n] = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \dots & & \dots \\ x_{m1} & \dots & x_{mn} \end{bmatrix}$$

rank r . The column space of X has dimension r .

Column space test

Belongs the vector to the column space?

How can you determine if a given vector \vec{y} with m components belongs to the column space of X ?

If \vec{y} is a element of the column space of X , then can \vec{y} be written as a linear combination of the vectors $\vec{x}_1, \dots, \vec{x}_n$. By adding \vec{y} to this vectors, the spanned space will be the same and the rank will still be r .

In other words, if \vec{y} belongs to the column space of X , then will the rank of the expanded matrix $[\vec{x}_1, \dots, \vec{x}_n, \vec{y}]$ be the same as the rank of the normal matrix X .

$$\vec{y} \in \text{column space} \Leftrightarrow \text{rank} [X \quad \vec{y}] = \text{rank} [X] = r$$

Column space test

Vector \vec{y} not in column space

If \vec{y} doesn't belong to the column space of X , then can't \vec{y} be written as a linear combination of the vectors $\vec{x}_1, \dots, \vec{x}_n$, in other words: \vec{y} can be splitted in two vectors $\vec{y} = \vec{z}_1 + \vec{z}_2$ with \vec{z}_1 in the column space of X and \vec{z}_2 not in the column space of X . By adding \vec{y} to the vectors $\vec{x}_1, \dots, \vec{x}_n$ increases the dimension of the spanned space.

$$\vec{y} \notin \text{column space} \Leftrightarrow \text{rank} [X \quad \vec{y}] = r + 1$$

Remark: $\text{rank} [X \quad \vec{z}_1] = r$ and $\text{rank} [X \quad \vec{z}_2] = r + 1$.

Column space test

Resume

- ① n vectors with m components $\vec{x}_1, \dots, \vec{x}_n$
- ② Belongs \vec{y} to the space spanned by $\vec{x}_1, \dots, \vec{x}_n$?
- ③ Solution:
 - a) Determine $r_1 = \text{rank} [X] = \text{rank} [\vec{x}_1 \ \dots \ \vec{x}_n]$
 - b) Determine $r_2 = \text{rank} [X \ \vec{y}] = \text{rank} [\vec{x}_1 \ \dots \ \vec{x}_n \ \vec{y}]$
 - c) Is $r_1 = r_2 \Rightarrow y \in \text{column space}$ or $r_1 + 1 = r_2 \Rightarrow y \notin \text{column space}$

Remark: analogous for the row space test.

Column space test

Example

$$\text{Given: } \vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 5 \\ 7 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 0 \\ 4 \\ 3 \\ 9 \\ 12 \end{bmatrix}, \vec{y} = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

Asked: Belongs \vec{y} to the space spanned by $\vec{x}_1, \vec{x}_2, \vec{x}_3$?

Solution: 1) find the rang of X:

$$X = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 4 \\ 3 & 0 & 3 \\ 4 & 5 & 9 \\ 5 & 7 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 0 \\ 3 & 0 & 0 \\ 4 & 5 & 0 \\ 5 & 7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{3}{2} & \frac{3}{4} & 0 \\ -\frac{1}{2} & \frac{9}{4} & 0 \\ -1 & \frac{12}{4} & 0 \end{bmatrix}$$

Hence, $\text{rank}(X) = 2$.

Column space test

Example

2) Determine the rank $[X \quad \vec{y}]$

$$[X \quad \vec{y}] = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 2 & 2 & 4 & -1 \\ 3 & 0 & 3 & 2 \\ 4 & 5 & 9 & 1 \\ 5 & 7 & 12 & 0 \end{bmatrix} \quad \text{rank}[X \quad \vec{y}] = 3.$$

Hence, $\text{rank}[X \quad \vec{y}] = \text{rank}[X] + 1$.

That means: \vec{y} can **not** be written as a linear combination of the vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$ and doesn't belong to the space spanned by this vectors. Because $\text{rank}(X) = 2$ is the dimension of the space 2.

Hence \vec{x}_3 can be written as a linear combination of \vec{x}_1, \vec{x}_2 .

$\text{Rank}(X \vec{y}) = 3$, so the space spanned by $\vec{x}_1, \vec{x}_2, \vec{x}_3$ and \vec{y} has dimension 3.

Orthonormal matrices

Definition

The set vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ is orthonormal if:

- $\|\vec{e}_i\| = 1$ with $i = 1, \dots, n$
- $\vec{e}_i \perp \vec{e}_j$ or $\vec{e}_i \cdot \vec{e}_j = 0$ with $i \neq j$

This is noted as:

$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$ with $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$ (δ_{ij} is called 'Kronecker delta').

If \vec{e}_i has the components $\begin{bmatrix} \vec{e}_{1i} \\ \dots \\ \vec{e}_{ni} \end{bmatrix}$ relative to some basis, then:

$$\vec{e}_i \cdot \vec{e}_j = \begin{bmatrix} e_{1i} & \dots & e_{ni} \end{bmatrix} \begin{bmatrix} e_{1j} \\ \dots \\ e_{nj} \end{bmatrix} = e_{1i}e_{1j} + \dots + e_{ni}e_{nj} = \delta_{ij}$$

Orthonormal matrices

Defenition

If we create from $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ the matrix

$$E = [\vec{e}_1, \dots, \vec{e}_n] = \begin{bmatrix} e_{11} & \dots & e_{1n} \\ \dots & & \dots \\ e_{n1} & \dots & e_{nn} \end{bmatrix}$$

then applies

$$E^t E = \begin{bmatrix} e_{11} & \dots & e_{n1} \\ \dots & & \dots \\ e_{1n} & \dots & e_{nn} \end{bmatrix} \begin{bmatrix} e_{11} & \dots & e_{1n} \\ \dots & & \dots \\ e_{n1} & \dots & e_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{bmatrix} =$$

$I^{n \times n}$

Hence, if the columns and rows of a matrix form a orthonormal system, then is the matrix orthonormal: $E^t E = I = E E^t$.

Orthonormal matrices

Properties

- ① The inverse of a orthonormal matrix is his transpose:

$$E^t E = I = E E^t \Rightarrow E^{-1} E E^t = E^{-1} I \Rightarrow I E^t = E^{-1} \Rightarrow E^t = E^{-1}$$
- ② Maintaining the norm of a vector: \vec{x} is a vector with norm $\|\vec{x}\|$ and E an orthonormal matrix $\|E \vec{x}\| = \|\vec{x}\|$.
 When there is a 'fault' Δx on x , then is

$$\|E(\vec{x} + \Delta \vec{x})\| = \|\vec{x} + \Delta \vec{x}\|.$$
 Hence, an orthonormal matrix doesn't change the magnitude of a fault. This is important in numerical applications.
- ③ The determinant of an orthonormal matrix is ± 1 , when it is 1 the matrix represent a rotation matrix.

Change of basis and matrix representation

Change of basis

In a n dimensional vector space V^n are two different bases given:

'old basis': $\vec{e}_1, \dots, \vec{e}_n$

'new basis': $\vec{f}_1, \dots, \vec{f}_n$

A vector x has coordinates $\begin{bmatrix} a_1 \\ \dots \\ a_n \end{bmatrix}$ relative to the old basis and

coordinates $\begin{bmatrix} b_1 \\ \dots \\ b_n \end{bmatrix}$ relative to the new basis.

Change of basis and matrix representation

Change of basis

Every basis vector \vec{f}_i can be expressed as a linear combination of the old basis vectors: $\vec{f}_i = f_{1i}\vec{e}_1 + \dots + f_{ni}\vec{e}_n$. So \vec{f}_i has coordinates

$\begin{bmatrix} f_{1i} \\ \dots \\ f_{ni} \end{bmatrix}$ relative to the old basis.

If we group the old basis vectors in a matrix: $E = [\vec{e}_1 \quad \dots \quad \vec{e}_n]$

then you can write: $\vec{x} = [\vec{e}_1 \quad \dots \quad \vec{e}_n] \begin{bmatrix} a_1 \\ \dots \\ a_n \end{bmatrix} = E \vec{x}_{old}$ and

$$\vec{f}_i = [\vec{e}_1 \quad \dots \quad \vec{e}_n] \begin{bmatrix} f_{1i} \\ \dots \\ f_{ni} \end{bmatrix} = E \begin{bmatrix} f_{1i} \\ \dots \\ f_{ni} \end{bmatrix}.$$

Change of basis and matrix representation

Change of basis

If we now consider \vec{x} as a linear combination of the 'new' basis:

$$\vec{x} = b_1 \vec{f}_1 + \dots + b_n \vec{f}_n = \begin{bmatrix} \vec{f}_1 & \dots & \vec{f}_n \end{bmatrix} \begin{bmatrix} b_1 \\ \dots \\ b_n \end{bmatrix} =$$

$$\begin{bmatrix} \vec{f}_1 & \dots & \vec{f}_n \end{bmatrix} \vec{x}_{new} = \begin{bmatrix} E \begin{bmatrix} f_{11} \\ \dots \\ f_{n1} \end{bmatrix} & \dots & E \begin{bmatrix} f_{1n} \\ \dots \\ f_{nn} \end{bmatrix} \end{bmatrix} \vec{x}_{new} = EF \vec{x}_{new}$$

with $F = \begin{bmatrix} f_{11} & \dots & f_{1n} \\ \dots & \dots & \dots \\ f_{n1} & \dots & f_{nn} \end{bmatrix}$. The columns of F contains the

coordinates of the new basis vectors relative to the old basis vectors $\vec{e}_1, \dots, \vec{e}_n$.

Change of basis and matrix representation

Change of basis

So we get: $\vec{x} = E \vec{x}_{old} = EF \vec{x}_{new}$.

From which follows:

$$\vec{x}_{old} = \begin{bmatrix} a_1 \\ \dots \\ a_n \end{bmatrix} = F \vec{x}_{new} = F \begin{bmatrix} b_1 \\ \dots \\ b_n \end{bmatrix} \text{ with the columns of } F \text{ the coordinates of the new basis vectors relative to the old basis.}$$

So, $\vec{x}_{old} = F \vec{x}_{new}$

or:

$$\vec{x}_{new} = F^{-1} \vec{x}_{old}$$

Change of basis and matrix representation

Example

Given: the 3 dimensional vector space V^3 with basis vectors:

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The 'new' basis vectors are defined as:

$$\vec{f}_1 = \vec{e}_1 - \vec{e}_2 + \vec{e}_3$$

$$\vec{f}_2 = 2\vec{e}_1 + 3\vec{e}_2 + 0\vec{e}_3$$

$$\vec{f}_3 = -\vec{e}_1 + \vec{e}_2 + 2\vec{e}_3.$$

And there is a vector \vec{x} :

$$\vec{x} = 2\vec{e}_1 + \vec{e}_2 - \vec{e}_3.$$

Asked: The coordinates of \vec{x} relative to the basis $\vec{f}_1, \vec{f}_2, \vec{f}_3$.

Change of basis and matrix representation

Example

Solution: The components of \vec{f}_1 are: $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

The components of \vec{f}_2 are: $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$.

The components of \vec{f}_3 are: $\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$.

The components of \vec{x} are: $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

Change of basis and matrix representation

Example

So, $\vec{x}_{old} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. We know that $\vec{x}_{old} = F\vec{x}_{new}$ with

$$F = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

$$\text{Now is } \vec{x}_{new} = F^{-1}\vec{x}_{old} = \frac{1}{15} \begin{bmatrix} 6 & -4 & 5 \\ 3 & 3 & 0 \\ -3 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

$$\text{So } \vec{x}_{new} = \frac{1}{15} \begin{bmatrix} 3 \\ 9 \\ -9 \end{bmatrix} = \frac{1}{5}\vec{f}_1 + \frac{3}{5}\vec{f}_2 - \frac{3}{5}\vec{f}_3.$$

Change of basis and matrix representation

Remarks

- 1 If the old basis is orthonormal and the new basis too, then is the transition matrix orthonormal as well ($F^{-1} = F^T$).
- 2 If the old basis is from dimension n , then the new basis has the same dimension n too, because they span the same space. So F has to be of full rank: $\text{rank}(F)=n$.

Outline

- 1 Vectors
- 2 Matrices
- 3 Systems of linear equations
- 4 Eigenvalues and eigenvectors
- 5 Appendixes

Systems of linear equations

Definitions

In general, a system of linear equations is described in a matrix-vector identity: $A^{m \times n} x^{n \times 1} = y^{m \times 1}$. If $y \neq 0$ then the system is called not homogeneous, if $y=0$ then the system is called homogeneous. It will turn out that in all the possible cases, the rank r of the matrix A is important for the existence of a solution and the number of solutions.

A good geometric view in the properties and solutions of homogeneous and not homogeneous systems is essential for an understanding of linear images, eigenvalues, etc.

Not homogeneous equations

Existence of a solution

In general, a system of m equations in n variables x_1, x_2, \dots, x_n has the form: $A^{m \times n} x^{n \times 1} = y^{m \times 1}$. The problem is: search the variables x_1, x_2, \dots, x_n which satisfy this relationship (A and y are known).

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \dots \\ y_m \end{bmatrix} \text{ or}$$

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = y_1 \\ \dots \\ a_{m1}x_1 + \dots + a_{mn}x_n = y_m \end{cases}$$

Not homogeneous equations

Existence of a solution

If we think of A as column vectors with m components

$A = [\vec{a}_1 \quad \dots \quad \vec{a}_n]$, then we can write the system in another way:

$$x_1 \begin{bmatrix} a_{11} \\ \dots \\ a_{m1} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ \dots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} y_1 \\ \dots \\ y_m \end{bmatrix} \text{ or } x_1 \vec{a}_1 + \dots + x_n \vec{a}_n = \vec{y}.$$

The geometric interpretation:

Given a vector \vec{y} of m components and n vectors \vec{a}_i of m components, search **all** numbers x_i , $i=1,\dots,n$, so that vector \vec{y} can be written as a linear combination of the vectors \vec{a}_i .

All the linear combinations of the vectors \vec{a}_i can be written as: $x_1 \vec{a}_1 + \dots + x_n \vec{a}_n = \vec{y}$ and is called the column space of A .

Not homogeneous equations

Existence of a solution

If $\text{rank}(A)=r$, then has the columns space dimension r . In other words: there are r linear independent columns in A and the other $n-r$ columns in A can be written as a linear combination of the r linear independent columns.

Since \vec{y} has to be a linear combination of the columns of A , \vec{y} has to be in the columns space of A .

If \vec{y} isn't in the column space of A , then \vec{y} can't be written as a linear combination of the columns of A , then the system **doesn't have** a solution.

Not homogeneous equations

Existence of a solution

The condition for the existence of a solution is that \vec{y} belongs to the column space of A . Now, the column space test becomes the solution test.

The system $Ax=y$ with $\text{rank}(A)=r$, has a solution only if $\text{rank}(A)=\text{rank}(Ay)=r$ with Ay a $m \times (n+1)$ matrix.

Not homogeneous equations

Example 1

$$\begin{cases} -x_1 + 2x_2 = 0 \\ x_1 + 3x_2 = 1 \\ 0x_1 + x_2 = 2 \end{cases} \quad \text{or} \quad \underbrace{\begin{bmatrix} -1 & 2 \\ 1 & 3 \\ 0 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_x = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}}_y$$

$$\text{Rank}(A)=2, \text{rank}(Ay)=3$$

$\text{Rank}(A) \neq \text{rank}(Ay)$, so there **isn't** a solution.

Not homogeneous equations

Example 2

$$\begin{cases} x_1 + x_2 + x_3 = 2 \\ -x_1 + 2x_2 + 3x_3 = -1 \end{cases} \quad \text{or} \quad \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_x = \underbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}_y$$

$\text{Rank}(A)=2=\text{rank}(Ay)=2$, so this system can be solved.

In the next we suppose that there is always a solution, thus:
 $\text{rank}(A)=\text{rank}(Ay)$.

Number of solutions and solution methods

Defenitions

$$A^{m \times n} x^{n \times 1} = y^{m \times 1}$$

There are **three** cases:

- ① $m > n$: overdetermined system
- ② $m = n$: square system
- ③ $m < n$: under determined system

Number of solutions and solution methods

$m > n$: overdetermined system

There is a solution if $\text{rang}(A) = \text{rang}(Ay) = r$. If this condition is satisfied, we can write: $x_1 \vec{a}_1 + \dots + x_n \vec{a}_n = \vec{y}$ with a_i the i^{th} column of A . As $\text{rank}(A) = r$, there are r linear independent columns $\vec{a}_1, \dots, \vec{a}_r$. Now we can write:

$$x_1 \vec{a}_1 + \dots + x_r \vec{a}_r = \vec{y} - x_{r+1} \vec{a}_{r+1} - \dots - x_n \vec{a}_n$$

$$\text{If: } \vec{y} = \vec{y} - x_{r+1} \vec{a}_{r+1} - \dots - x_n \vec{a}_n \text{ then } \begin{bmatrix} \vec{a}_1 & \dots & \vec{a}_r \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_r \end{bmatrix} = \vec{y}.$$

From the column test follows that \vec{y} belongs to the column space of A . The vectors $\vec{a}_{r+1}, \dots, \vec{a}_n$ belong to the same space. The vectors $\vec{a}_1, \dots, \vec{a}_r$ create the **basis** of this space.

Number of solutions and solution methods

$m > n$: overdetermined system

For every choice of the vector \vec{y} , there exist a **unique** set (only one solution) of numbers x_1, \dots, x_r : this are the coordinates of the vector \vec{y} in the basis $\vec{a}_1, \dots, \vec{a}_r$. There is only one solution because every vector in only one way can be described as a linear combination of basis vectors.

Now is: $\vec{y} = \vec{y} - x_{r+1}\vec{a}_{r+1} - \dots - x_n\vec{a}_n$.

For each set $[x_{r+1}, \dots, x_n]$ there is a vector \vec{y} , and for every vector \vec{y} exist just one solution $[x_1, \dots, x_r]$. For the choice of the set $[x_{r+1}, \dots, x_n]$, there are ∞^{n-r} possibilities.

Number of solutions and solution methods

$m > n$: overdetermined system

If $m > n$ and $\text{rank}(Ay) = \text{rank}(A) = r$, then has the system $Ax = y$ ∞^{n-r} solutions.

The variables x_1, x_2, \dots, x_r are the main variables and $x_{r+1}, x_{r+2}, \dots, x_n$ are the help variables. For each set help variables exist only one set main variables.

If $\text{rank}(A) = \text{rank}(Ay) = r$ and if $n = r$, then has the system $Ax = y$ exact **one** solution ($\infty^0 = 1$).

Number of solutions and solution methods

$m > n$: overdetermined system

How do you determine the solution for a set help variables $[x_{r+1}, x_{r+2}, \dots, x_n]$.

$$\text{If } [\vec{a}_1 \quad \dots \quad \vec{a}_r] \begin{bmatrix} x_1 \\ \dots \\ x_r \end{bmatrix} = \vec{y} - x_{r+1} \vec{a}_{r+1} - \dots - x_n \vec{a}_n \Rightarrow A' x' = y'$$

with $A' = [\vec{a}_1 \quad \dots \quad \vec{a}_r]$ and

$$\vec{y}' = \vec{y} - x_{r+1} \vec{a}_{r+1} - \dots - x_n \vec{a}_n = \begin{bmatrix} y'_1 \\ \dots \\ y'_m \end{bmatrix}.$$

We know that $\text{rang}(A') = r$. This means that r rows of A' are linear independent. The other rows of A' can be created by linear combinations of the linear independent rows.

Number of solutions and solution methods

$m > n$: overdetermined system

The same is true for the matrix Ay , since $\text{rank}(Ay) = r$ has matrix Ay r linear independent rows. The other rows can be created by linear combinations of the linear independent rows. This means that there are only r rows necessary to solve the system. The other $m-r$ equations are linear combinations of the r independent equations.

Thus: select from A' r linear independent equations with the components of y' according to this equations:

$$A'' = \begin{bmatrix} \vec{a}_1' \\ \dots \\ \vec{a}_r' \end{bmatrix}. \text{ And we get: } A''x' = \begin{bmatrix} y_1' \\ \dots \\ y_r' \end{bmatrix} = y''.$$

Number of solutions and solution methods

$m > n$: overdetermined system

A'' is a square matrix and has full rank, so it is an invertible matrix.
 $A''x' = y'' \Rightarrow (A'')^{-1}A''x' = (A'')^{-1}y''$ and we get:

$$\underbrace{x'}_{r \times 1} = \underbrace{(A'')^{-1}}_{r \times r} \underbrace{y'}_{r \times 1}$$

This is the general solution of the overdetermined system. The matrix A'' is the main matrix of the system. A'' is created by the choice of help variables and independent rows. There are $m-r$ help variables, and for each set of them there is a vector y'' . So there exist ∞^{n-r} solutions in x .

Number of solutions and solution methods

$m > n$: overdetermined system

Example:

$$\begin{cases} x_1 - 2x_2 - x_3 = 3 \\ 2x_1 + 0x_2 + 2x_3 = -2 \\ 3x_1 + 3x_2 + 6x_3 = -9 \\ 4x_1 + 6x_2 + 10x_3 = -16 \\ 5x_2 + 7x_2 + 12x_3 = -19 \end{cases} \quad m=5, n=3$$

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 2 \\ 3 & 3 & 6 \\ 4 & 6 & 10 \\ 5 & 7 & 12 \end{bmatrix}, y = \begin{bmatrix} 3 \\ -2 \\ -9 \\ -16 \\ -19 \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Number of solutions and solution methods

$m > n$: overdetermined system

① Solution test:

$$\text{rank}(A) = 2 = \text{rank}(Ay) = 2 \Rightarrow \text{solvable}$$

② Solution:

We choose the first two columns of A as linear independent columns. \rightarrow So we fix the help and main variables.

$$\begin{bmatrix} 1 & -2 \\ 2 & 0 \\ 3 & 3 \\ 4 & 6 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -9 \\ -16 \\ -19 \end{bmatrix} - x_3 \begin{bmatrix} -1 \\ 2 \\ 6 \\ 10 \\ 12 \end{bmatrix}$$

Number of solutions and solution methods

$m > n$: overdetermined system

$\text{Rank}(A_y) = 2$, so there are 2 linear independent rows. The other rows are linear combinations of the independent rows. We choose the first two rows to be linear independent:

$$\begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} - x_3 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ this is } A''x' = y''.$$

With $A''^{-1} = \frac{1}{4} \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix}$, we get:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} - \frac{x_3}{4} \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 - x_3 \\ -2 - x_3 \end{bmatrix}.$$

This is the general solution of the system (verify by doing substitution). For each choice of x_3 you get a different solution. In this case there are $\infty^{n-r} = \infty^{3-2} = \infty$ solutions.

Number of solutions and solution methods

m=n: square system

We suppose there is a solution: $\text{rank}(Ay) = \text{rank}(A) = r$. The same rules can be applied here: if $\vec{a}_1, \dots, \vec{a}_r$ are the linear independent columns of A, we can write:

$$\begin{bmatrix} \vec{a}_1 & \dots & \vec{a}_r \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_r \end{bmatrix} = y - r_{r+1} \vec{a}_{r+1} - \dots - x_n \vec{a}_n \text{ and again there}$$

are ∞^{n-r} solutions (one solution for each set $[x_{r+1}, \dots, x_n]$).

Remark: if $m=n=r$, there is exactly one solution, the system is called **Cramer's system**.

Conclusion: A square system can be solved on the same way as a overdetermined system.

Number of solutions and solution methods

$m < n$: underdetermined system

Again there has to be satisfied to: $\text{rank}(A) = \text{rank}(Ay) = r$. If this condition is true, then we can find a solution on the same way as described above.

Example 1:

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 1 & -4 & -6 \\ 2 & 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -9 \\ 2 \end{bmatrix}$$

$$\text{Rank}(A) \Rightarrow \begin{bmatrix} 1 & 1 & 3 & 1 \\ 0 & 0 & -7 & -7 \\ 0 & 0 & -7 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank}(A) = 2$$

Number of solutions and solution methods

$m < n$: underdetermined system

Rank(A_y)

$$\Rightarrow \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 1 & 1 & -4 & -6 & -9 \\ 2 & 2 & -1 & -5 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & 0 & -7 & -7 & -15 \\ 0 & 0 & -7 & -7 & -14 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 0 & 0 & 1 & 1 & \frac{15}{7} \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -2 & 6 - \frac{45}{7} \\ 0 & 0 & 1 & 1 & \frac{15}{7} \\ 0 & 0 & 0 & 0 & \frac{-1}{7} \end{bmatrix} \Rightarrow \text{rank}(A_y) = 3$$

Rank(A) \neq rank(A_y) so the system is **unsolvable**.

Number of solutions and solution methods

$m < n$: underdetermined system

Example 2:

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 1 & -4 & -6 \\ 2 & 2 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \end{bmatrix}$$

$\text{Rank}(A) = \text{rank}(Ay) = r$, so there is a solution. Column one and three are linear independent. Now we can write:

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \end{bmatrix} \text{ or}$$

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \end{bmatrix} - x_2 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - x_4 \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix}.$$

Number of solutions and solution methods

$m < n$: underdetermined system

We choose the first two rows:

$$\begin{bmatrix} 1 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \end{bmatrix} - x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} - x_4 \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$$A''^{-1} = \frac{1}{7} \begin{bmatrix} 4 & 3 \\ 1 & -1 \end{bmatrix} \text{ so we get:}$$

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 4 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ -8 \end{bmatrix} - \frac{-x_2}{7} \begin{bmatrix} 4 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{x_4}{7} \begin{bmatrix} 4 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -7x_2 + 14x_3 \\ 14 + 0x_2 - 7x_1 \end{bmatrix} = \begin{bmatrix} -x_2 + 2x_4 \\ 2 - x_4 \end{bmatrix}$$

There are $\infty^{n-r} = \infty^2$ solutions. Verify via substitution:

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 1 & -4 & -6 \\ 2 & 2 & -2 & -5 \end{bmatrix} \begin{bmatrix} -x_2 + 2x_4 \\ x_2 \\ 2 - x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \end{bmatrix}.$$

Number of solutions and solution methods

$m < n$: underdetermined system

There are $\infty^{n-r} = \infty^2$ solutions. Verify via substitution:

$$\begin{bmatrix} 1 & 1 & 3 & 1 \\ 1 & 1 & -4 & -6 \\ 2 & 2 & -2 & -5 \end{bmatrix} \begin{bmatrix} -x_2 + 2x_4 \\ x_2 \\ 2 - x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \end{bmatrix}.$$

Not homogeneous systems

Resume

$$A^{m \times n} x^{n \times 1} = y^{m \times 1}$$

- ① Determine rank(A):
rank(A)=r
- ② Determine rank(Ay):
rank(Ay)=r+1 → unsolvable
rank(Ay)=r → solvable
- ③ If solvable: choose r independent columns of A and the components $[x_1, \dots, x_r]$ according to it.

$$[\vec{a}_1 \quad \dots \quad \vec{a}_r] \begin{bmatrix} x_1 \\ \dots \\ x_r \end{bmatrix} = \vec{y} - x_{r+1} \vec{a}_{r+1} - \dots - x_r \vec{a}_r = \vec{y}'$$

Not homogeneous systems

Resume

For every choice of $[x_{r+1}, \dots, x_n]$ exist one solution. Hence, there are ∞^{n-r} solutions.

Because $\text{rank}(A_y)=r$, there are r rows of A_y linear independent and the other $m-r$ rows are linear combinations of it. Choose r independent rows of A_y . Then is

$$\underbrace{A''}_{r \times r} \underbrace{x'}_{r \times 1} = \underbrace{y''}_{m \times 1} \Rightarrow x' = A''^{-1} y''$$

the general solution of the system. Verify by substitution.

Not homogeneous systems

Resume

For every choice of $[x_{r+1}, \dots, x_n]$ exist one solution. Hence, there are ∞^{n-r} solutions.

Because $\text{rank}(A_y)=r$, there are r rows of A_y linear independent and the other $m-r$ rows are linear combinations of it. Choose r independent rows of A_y . Then is

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the general solution of the system. Verify by substitution.

Not homogeneous systems

Remark

The method above can be described in a **matricial** way:

$$A_{m \times n} = \begin{bmatrix} A_{rxr} & A_{rx(n-r)} \\ A_{(m-r)xr} & A_{(m-r)x(n-r)} \end{bmatrix}$$

We get: $\begin{bmatrix} A_{rxr} & A_{rx(n-r)} \\ A_{(m-r)xr} & A_{(m-r)x(n-r)} \end{bmatrix} \begin{bmatrix} x_r \\ x_{n-r} \end{bmatrix} = \begin{bmatrix} y_r \\ y_{m-r} \end{bmatrix}$ with x_r the main variables and x_{n-r} the help variables.

Now we can write:

$$\begin{cases} A_{rxr}x_r + A_{rx(n-r)}x_{n-r} = y_r \\ A_{(m-r)xr}x_r + A_{(m-r)x(n-r)}x_{n-r} = y_{m-r} \end{cases} \quad . \text{ Thus:}$$

$$\begin{cases} x_r = A_{rxr}^{-1}y_r - A_{rxr}^{-1}A_{rx(n-r)}x_{n-r} \\ x_{n-r} = x_{n-r} \end{cases} \quad \text{is the general } \infty^{n-r} \text{ fold}$$

solution.

Homogeneous systems

General

$$\underbrace{A}_{m \times n} \underbrace{x}_{n \times 1} = 0 \text{ and } \text{rank}(A)=r$$

$x=0$ is always a solution.

Existence of a solution

Now we consider the **row** vectors of A.

$$\begin{bmatrix} \vec{a_1} \\ \dots \\ \vec{a_m} \end{bmatrix} \vec{x} = 0 \text{ or } \begin{bmatrix} \vec{a_1} \vec{x} \\ \dots \\ \vec{a_m} \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \dots \\ 0 \end{bmatrix}.$$

The homogeneous system is nothing more than m simple inner products that has to be 0.

Homogeneous systems

Existence of a solution

The inner product of two vectors is zero if the two vectors are orthogonal. In other words: $\vec{x} \perp$ on every row of A .

Hence, the solution of the homogeneous system consists of all the vectors \vec{x} which are perpendicular to the rows of A .

Because $\text{rank}(A)=r$ is the dimension of the row space r . In A are r linear independent rows, the other rows are a linear combination of the independent rows.

Homogeneous systems

Existence of a solution

The rows of A are vectors with n components. The row space is a r dimensional subspace of the n dimensional vector space. This global n dimensional space consists of two subspaces.

- 1 The subspace spanned by the rows of A : subspace with dimension r .
- 2 The complementary subspace: the vectors in this space are linear independently from the rows of A . The complementary subspace has dimension $n-r$.

Homogeneous systems

Resume

The solution of $\underbrace{A}_{m \times n} \underbrace{x}_{n \times 1} = 0$ with $\text{rank}(A)=r$ is a $n-r$ dimension space: the complementary space of the row space of A .

When $n=r$, the **only** solution of the homogeneous system is 0. There isn't a vector which is perpendicular to the the rows of A . Since $n=r$, the rows of A span the total n dimensional space and the complementary space has dimension 0.

$\underbrace{A}_{m \times n} \underbrace{x}_{n \times 1} = 0$ with $\text{rank}(A)=r$ has a solution different from 0 if $\text{rank}(A) < n$

Homogeneous systems

Number of solutions

The solutions of $Ax=0$ lay in the orthogonal complement of the row space of A .

The number of linear independent solutions of the homogeneous system is equal to the number of linear independent vectors in the orthogonal complement. The dimension of this space is $n-r$, so there are $n-r$ basis vectors in the complement. All the solutions of $Ax=0$ are a linear combination of these $n-r$ basis vectors.

If x_1, \dots, x_{n-r} are the basis vectors, then are the linear combinations: $\vec{x} = a_1\vec{x}_1 + \dots + a_{n-r}\vec{x}_{n-r}$ and we get:

$$Ax = A(a_1x_1 + \dots + a_{n-r}x_{n-r}) = 0.$$

Homogeneous systems

Number of solutions

If $a_1 \neq 0$, then is $x_1 + \frac{a_2}{a_1} \dots + \frac{a_{n-r}}{a_1} x_{n-r}$ a solution.

The solutions $\vec{x} = a_1 \vec{x}_1 + \dots + a_{n-r} \vec{x}_{n-r}$ and

$x_1 + \frac{a_2}{a_1} \dots + \frac{a_{n-r}}{a_1} x_{n-r}$ are actually the same: they differs from a factor a_1 . We get **different** solutions for each choice of numbers $b_1 = \frac{a_2}{a_1}, \dots, b_{n-r-1} \frac{a_{n-r}}{a_1}$.

Since there are ∞^{n-r-1} possible choices, the solutions of the homogeneous system $Ax=0$ with $\text{rank}(A)=r$ are ∞^{n-r-1} fold under determined.

Homogeneous systems

Solution method

$$\underbrace{A}_{m \times n} \underbrace{x}_{n \times 1} = 0 \Rightarrow \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = 0$$

Convert A by elementary row operations to its echelon form:

$$A \sim \begin{bmatrix} \overbrace{1 \ 0 \ \dots \ 0}^r & b_{1 \times (r+1)} & \dots & b_{1 \times n} \\ 0 \ 1 \ \dots \ 0 & \dots & & \dots \\ \dots & \dots & \dots & \dots \\ 0 \ 0 \ \dots \ 1 & b_{r \times (r+1)} & \dots & b_{r \times n} \\ 0 \ \dots \ \dots \ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 \ \dots \ \dots \ 0 & 0 & \dots & 0 \end{bmatrix}$$

Homogeneous systems

Solution method

Now you have the rank of A (number of rows different from 0) and the solution method:

$$\begin{bmatrix} 1 & 0 & \dots & 0 & b_{1 \times (r+1)} & \dots & b_{1 \times n} \\ 0 & 1 & \dots & 0 & \dots & \dots & \dots \\ \dots & & & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & b_{r \times (r+1)} & \dots & b_{r \times n} \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 \\ \dots & & & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_r \\ x_{r+1} \\ \dots \\ x_n \end{bmatrix} = 0$$

Homogeneous systems

Solution method

From which follows:

$$\begin{cases} x_1 = -b_{1x(r+1)}x_{r+1} - \dots - b_{1xn}x_n \\ \dots = \dots \\ x_r = -b_{rx(r+1)}x_{r+1} - \dots - b_{rxn}x_n \end{cases}$$

Again, x_{r+1}, \dots, x_n are called the help variables. We can write:

$$\begin{bmatrix} x_1 \\ \dots \\ x_r \\ x_{r+1} \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_{1x(r+1)} & \dots & \dots & b_{1n} \\ \dots & & & \dots \\ b_{rx(r+1)} & \dots & \dots & b_{rxn} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & & & \dots \\ 0 & \dots & \dots & 1 \end{bmatrix} \begin{bmatrix} x_{r+1} \\ \dots \\ x_n \end{bmatrix}$$

Homogeneous systems

Solution method

Which gives:

$$\begin{bmatrix} x_1 \\ \dots \\ x_r \\ x_{r+1} \\ \dots \\ x_n \end{bmatrix} = x_{r+1} \begin{bmatrix} b_{1 \times (r+1)} \\ \dots \\ b_{r \times (r+1)} \\ 1 \\ 0 \\ \dots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} b_{1 \times n} \\ \dots \\ b_{r \times n} \\ 0 \\ 0 \\ \dots \\ 1 \end{bmatrix} .$$

Homogeneous systems

Solution method

The $n-r$ vectors $\begin{bmatrix} b_{1i} \\ \dots \\ b_{ri} \\ 0 \\ \dots \\ 1 \\ \dots \\ 0 \end{bmatrix}$ with $i=1, n-r$ form a basis for the solution

space (the orthogonal complement of the row space of A) of the homogeneous system. All the linear combinations of this bases are solutions of the homogeneous system.

Homogeneous systems

Solution method

Example 1:

Find the solution of:
$$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 \\ 1 & 3 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

$$A \sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 4 & 0 \\ 0 & 3 & \frac{3}{2} & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 8 & 0 \\ 0 & 1 & \frac{1}{2} & 0 & 0 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 8 & 0 \\ 0 & 1 & 0 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & 8 & 0 \end{bmatrix}$$

Homogeneous systems

Solution method

We get:

$$\begin{bmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & 8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0 \text{ from which follows:}$$

$$\begin{cases} x_1 = -4x_4 + 0x_5 \\ x_2 = 4x_4 + 0x_5 \\ x_3 = -8x_4 + 0x_5 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -4 \\ 4 \\ -8 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \text{ The last two}$$

vectors are the basis vectors of the solution space.

Homogeneous systems

Solution method

Example 2:

$$\begin{cases} x_1 + 0x_2 + 0x_3 = 0 \\ 0x_1 + x_2 + 0x_3 = 0 \\ 0x_1 + 0x_2 + x_3 = 0 \\ x_1 + 0x_2 + 0x_3 = 0 \\ 0x_1 + x_2 - x_3 = 0 \end{cases} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad m=5, n=3.$$

Rank(A)=3, so there isn't a solution different from 0. The rows of A span the full three dimensional space.

Remark: a square homogeneous system is solvable if $\det(A)=0$ ($\text{rank}(A)<n$).

Systems with complex coefficients

Notation

Complex systems emerge for example by eigenvalue problems.

If $A = A_r + jA_i$ with $j = \sqrt{-1}$, $x = x_r + jx_i$ and $y = y_r + jy_i$, then applies: $[A_r + jA_i] [x_r + jx_i] = [y_r + jy_i]$.

We split the real and the complex part, and we get:

$$\begin{cases} A_r x_r - A_i x_i = y_r \\ A_i x_r + A_r x_i = y_i \end{cases} \quad . \text{ This is equal to: } \underbrace{\begin{bmatrix} A_r & -A_i \\ A_i & A_r \end{bmatrix}}_{2m \times 2n} \underbrace{\begin{bmatrix} x_r \\ x_i \end{bmatrix}}_{2n \times 1} = \underbrace{\begin{bmatrix} y_r \\ y_i \end{bmatrix}}_{2m \times 1}.$$

This is a real system with double dimension. Every complex system can be written as a real system with double dimension.

Systems with complex coefficients

Example

Given the homogeneous system:
$$\begin{cases} (1 + 2j)t_1 + (1 - j)t_2 = 0 \\ (-2 + j)t_1 + (1 + j)t_2 = 0 \end{cases}$$

with $t_1, t_2 \in \mathbb{C}$.

We can write:
$$\begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{real part}} + j \underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\text{imaginary part}}$$

The system $At=0$ becomes: $[A_r + jA_i] [x_r + jy] = 0$ or

$$\begin{cases} A_r x_r - A_i y = 0 \\ A_i x_r + A_r y = 0 \end{cases} \Rightarrow \begin{bmatrix} A_r & -A_i \\ A_i & A_r \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0 \text{ with}$$

$$A_r = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}, A_i = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}.$$

Systems with complex coefficients

Example

Wich gives:
$$\begin{bmatrix} 1 & 1 & -2 & 1 \\ -2 & 1 & -1 & -1 \\ 2 & -1 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0.$$

The solution is:

$$\begin{cases} x_1 = \frac{1}{3}y_1 - \frac{2}{3}y_2 \\ x_2 = \frac{5}{3}y_1 - \frac{1}{3}y_2 \\ y_1 = y_1 \\ y_2 = y_2 \end{cases} \quad \text{or} \quad \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} + j & \frac{-2}{3} \\ \frac{5}{3} & \frac{-1}{3} + j \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Verify by substitution.

Linear images

General

Applications and exercises of linear images can always be reduced to the theory and geometry of linear equations.

Definition

Given: a n dimensional space V^n and a m dimensional space V^m .
A linear image is a image $\theta : V^n \rightarrow V^m : \vec{x} \in V^n \rightarrow \vec{y} \in V^m$ for
which:
$$\begin{cases} \theta(\vec{x} + \vec{y}) = \theta(\vec{x}) + \theta(\vec{y}) \\ \theta(k\vec{x}) = k\theta(\vec{x}) \end{cases}$$

Linear images

Matrix representation

\vec{x} is written relative to a basis $\vec{e}_1, \dots, \vec{e}_n$ in V^n which gives:

$$\vec{x} = \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} \text{ or } \vec{x} = x_1 \vec{e}_1 + \dots + x_n \vec{e}_n.$$

\vec{y} is written relative to a basis $\vec{f}_1, \dots, \vec{f}_m$ in V^m which gives:

$$\vec{y} = \begin{bmatrix} y_1 \\ \dots \\ y_m \end{bmatrix} \text{ or } \vec{y} = y_1 \vec{f}_1 + \dots + y_m \vec{f}_m.$$

There is a linear image θ that \vec{x} maps on

$$\vec{y} : \vec{y} = \theta(\vec{x}) = \theta(x_1 \vec{e}_1 + \dots + x_n \vec{e}_n) = x_1 \theta(\vec{e}_1) + \dots + x_n \theta(\vec{e}_n)$$

with $\theta(\vec{e}_i) \in V^m$.

Linear images

Matrix representation

If we group all this basis vectors, we get:

$$y = \begin{bmatrix} y_1 \\ \dots \\ y_m \end{bmatrix} = [\theta(e_1) \quad \dots \quad \theta(e_n)] \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = A \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix}.$$

Hence, $y=Ax$ with the columns of A the images of the basis vectors in V^n

Linear images

Image of a linear mapping

We consider the columns of A :

$$y = Ax = [\vec{a}_1 \quad \dots \quad \vec{a}_n] \begin{bmatrix} x_1 \\ \dots \\ x_n \end{bmatrix} = x_1 \vec{a}_1 + \dots + x_n \vec{a}_n.$$

The image of a linear mapping is the set of all possible vectors that can be created by the columns of A and is a subspace of V^m . The vectors \vec{y} are all the possible linear combinations of the columns of A , and so is the image of the linear mapping the column space of A .

If $\text{rank}(A)=r$, then has this space dimension r .

The image of the linear mapping = column space of A
Dimension image= $\text{rank}(A)=r$

Linear images

Kernell

The kernell is the set of all vectors \vec{x} that are projected on the null vector. The kernel is a subspace of V^n . It is the solution space of $Ax=0$ with $\text{rank}(A)=r$. The dimension of this solution space is $n-r$.

Kernel= solution space of $Ax=0$ with dimension $n-r$.

Outline

- 1 Vectors
- 2 Matrices
- 3 Systems of linear equations
- 4 Eigenvalues and eigenvectors
- 5 Appendixes

Defenition

Defenition

Every vector \vec{x} and each number λ that satisfy $Ax = \lambda x$, are called eigenvectors and eigenvalues of the square matrix A .

The matrix A represent a linear mapping: $V^n \rightarrow V^n$.

$Ax = \lambda x$ is the set of all vectors \vec{x} which are projected on a complex multiple of itself.

General

$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$. This is a square homogeneous system and has only a solution if: $\det(A - \lambda I) = 0$.

This is the condition for the existence of a solution and so of the existence of a eigenvector.

Solutions

General

$Ax = \lambda x \Rightarrow (A - \lambda I)x = 0$. This is a square homogeneous system and has only a solution if: $\det(A - \lambda I) = 0$.

This is the condition for the existence of a solution and so of the existence of a eigenvector.

Solutions

General - polynomials

$\det(A - \lambda I)$ is a **polynomial of degree n**. A fundamental theorem of algebra:

Every polynomial of degree n has n (complex) zeros.

So $\det(A - \lambda I) = a_n \lambda^n + \dots + a_1 \lambda + a_0 = 0$ has zeros $\lambda_1, \dots, \lambda_n$. Some zeros can be multiple. The number of occurrence of a zero λ_i is called the multiplicity of the zero λ_i .

If λ_i is a complex zero $\lambda_i = a_i + jb_i$ then its complex conjugate $\lambda_i^* = a_i - jb_i$ is also a zero.

The eigenvalues of A: $\lambda_1, \dots, \lambda_n$ are the zeros of the characteristic polynomial $\det(A - \lambda I) = 0$.

Solutions

General - solutions

For every eigenvalue λ_i , we can find the related eigenvectors as a solution of the homogeneous system: $(A - \lambda_i I)x = 0$. The number of eigenvectors that belong to a eigenvalue λ_i is the dimension of the solution space $(A - \lambda_i I)x = 0$.

If $\text{rank}(A - \lambda_i I) = r_i$, then $n - r_i$ eigenvectors belong to λ_i .

If the multiplicity of λ_i is η_i , then 1 or 2 or ... or η_i eigenvectors belong to λ_i . The exact number of eigenvectors that belong to λ_i is $n - r_i$ with $r_i = \text{rank}(A - \lambda_i I)$.

Solutions

General

The solution of the eigenvalue problem:

- 1 Search the zeros of $\det(A - \lambda I) = 0$.
These are $\lambda_1, \dots, \lambda_n$.
- 2 Determine the solution space of the homogeneous system (with different λ_i) $(A - \lambda_i I)x = 0$, all these solutions are eigenvectors.

Remark: the solutions of the homogeneous systems are determined by a factor. We take the norm of the vector 1 and the factor is the eigenvalue λ_i . So the eigenvectors have always norm = 1.

Solutions

Properties

- ① Independence between eigenvectors: The eigenvectors of **different** eigenvalues are linear independent.
- ② If the eigenvalue λ_i has a multiplicity ≥ 1 , then different eigenvectors can be related to this eigenvalue. If $\text{rank}(A - \lambda_i I) = r_i$, then λ_i has $n - r_i$ independent eigenvectors. These $n - r_i$ eigenvectors span the eigenspace of eigenvalue λ_i . According to property 1, the different eigenspaces are independent. That means: all eigenvectors span the full n dimensional space only if:

$$(n - r_1) + (n - r_2) + \dots = n \text{ or if: } \forall i (n - r_i) = \eta_i$$

Remark: this property is not always true: the eigenvalue 0 of

the matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has multiplicity 2, while $n - r = 2 - 1 = 1$.

Solutions

Properties

- ③ If A has n linear independent eigenvectors (property 2):

$$\begin{cases} Ax_1 = x_1 \lambda_1 \\ \dots \\ Ax_n = x_n \lambda_n \end{cases} \quad \text{or} \quad A \begin{bmatrix} x_1 & \dots & \dots & x_n \end{bmatrix} =$$

$$\begin{bmatrix} x_1 & \dots & \dots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \lambda_n \end{bmatrix} \Rightarrow A^{n \times n} X^{n \times n} = X^{n \times n} \Lambda^{n \times n}.$$

Since, $\vec{x}_1, \dots, \vec{x}_n$ are linear independent, applies: $\det(X) \neq 0$, so X^{-1} exists and we can write:

$$AX = X\Lambda \Rightarrow X^{-1}AX = X^{-1}X\Lambda \Rightarrow X^{-1}AX = \Lambda.$$

Solutions

Properties

- 3 The transformation $A \rightarrow x^{-1}Ax$ is called the similarity transformation.

Similar matrices have the same eigenvalues.

- ④ If $\det(x) \neq 0$ (n linear independent eigenvectors):

$$Ax = x\Lambda \Rightarrow \det(Ax) = \det(x\Lambda)$$

$$\Rightarrow \det(A)\det(x) = \det(x)\det(\Lambda)$$

$$\Rightarrow \det(A) = \det(\Lambda) = \lambda_1 \dots \lambda_n.$$

So the determinant of a matrix is the product of his eigenvalues.

- 5 If $\text{rank}(A)=r$, then there are r eigenvalues different from 0.

Solutions

Eigenvalues of a symmetric matrix

Properties of a symmetric matrix $A=A^t$:

- ① The eigenvalues of a real symmetric matrix are real.
- ② The eigenvectors of a real symmetric matrix of full rank form a orthonormal set.
 - Eigenvectors of different eigenvalues are orthogonal.
 - A eigenvalue with multiplicity η_i has $n - r_i$ eigenvectors. Via Gram-Schmidt can these vectors be orthogonalized.
- ③ If there are n eigenvectors: $X = [\vec{x}_1 \quad \dots \quad \vec{x}_n]$, then we can write: $X^t X = I_n \Rightarrow X^{-1} = X^t$.

Solutions

Eigenvalues of a symmetric matrix

Example:

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad A \text{ is symmetric, so all the eigenvalues are real.}$$

The characteristic polynomial:

$$(3 - \lambda)((1 - \lambda)^2 - 1) = (3 - \lambda)(\lambda^2 - 2\lambda) = 0 \Rightarrow \lambda_1 = 3, \lambda_2 = 2, \lambda_3 = 0$$

Eigenvectors of λ_1 : homogeneous system:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solutions

Eigenvalues of a symmetric matrix

Eigenvectors of λ_2 : homogeneous system:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow$$

$$\begin{cases} x_1 = 0 \\ x_2 = x_3 \\ x_3 = x_3 \end{cases} \quad \vec{x}_2 = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Remark: mostly we scale the eigenvector so that its norm is 1.

$$\text{And we get: } \vec{x}_2 = x_3 \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

Solutions

Eigenvalues of a symmetric matrix

Eigenvectors of $\lambda_3 = 0$: homogeneous system:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \sim \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \Rightarrow$$

$$\begin{cases} x_1 = 0 \\ x_2 = -x_3 \\ x_3 = x_3 \end{cases} \quad \vec{x}_2 = x_3 \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}. \text{ Hence, } AX = X\Lambda \text{ becomes:}$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Outline

- 1 Vectors
- 2 Matrices
- 3 Systems of linear equations
- 4 Eigenvalues and eigenvectors
- 5 Appendixes

Appendix 1: inverting a square matrix A

A necessary condition: $\det(A) \neq 0$.

- ① Change each element by its cofactor
- ② Transpose
- ③ Divide by $\det(A)$

Cofactor of element a_{ij} (a scalar):

- Remove the row and column which contains the element a_{ij} (row i and column j). Result: a $(n-1) \times (n-1)$ matrix M_{ij} .
- $\text{cofactor}(a_{ij}) = (-1)^{i+j} \det(M_{ij})$

Example: $A = \begin{bmatrix} 1 & 3 & 4 \\ 5 & 7 & 8 \\ 2 & 0 & 0 \end{bmatrix}$ and $M_{22} = \begin{bmatrix} 1 & 4 \\ 2 & 0 \end{bmatrix}$ with $\det(M_{22}) = -8$.

$\text{Cofactor}(a_{22}) = (-1)^{2+2}(-8) = 8$.

Appendix 2: elementary row operations

When you multiply a matrix with a elementary matrix you can:

- 1 change rows/columns
- 2 scale rows/columns with a factor
- 3 add rows/columns with each other

This operations can be done by:

- Multiplying the matrix left with a elementary matrix for **row operations**
- Multiplying the matrix right with a elementary matrix for **column operations**

Elementary matrices:

- are square
- have full rank
- When a matrix is multiplied by a elementary matrix, his rank don't change

Appendix 2: elementary row operations

The row operations:

① Changing two rows: If $A^{m \times n} = \begin{bmatrix} a_1 \\ \dots \\ a_i \\ \dots \\ a_j \\ \dots \\ a_n \end{bmatrix}$ with \vec{a}_i the rows of A.

We want to change row a_i and a_j by multiplying A left by a matrix $R^{m \times m}$.

Appendix 2: elementary row operations

If $RA = \begin{bmatrix} a_1 \\ \dots \\ a_j \\ \dots \\ a_i \\ \dots \\ a_n \end{bmatrix}$, then has R the form: $R = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \\ \dots & & & \\ 0 & .. & 0 & 1 & .. & 0 \\ \dots & & & & & \\ 0 & .. & 1 & 0 & .. & 0 \\ \dots & & & & & \\ 0 & & & & & 1 \end{bmatrix}$

This is the unit matrix with changing elements: ii, ij, ji, jj .

Appendix 2: elementary row operations

- 2 Scale with a factor α :

If we want to scale the i^{th} row of A with α , then we have to

multiply $A^{m \times n}$ with $R^{m \times m} =$
$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \\ \dots & & & \\ 0 & \dots & \alpha & \dots & 0 \\ \dots & & & & \\ 0 & & & & 1 \end{bmatrix}.$$

This is the unit matrix with α on element ii .

Appendix 2: elementary row operations

- ③ Add two rows and one of the two rows change by this sum:

$$\text{Row } i + \text{row } j \rightarrow \text{row } i. \quad R = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \\ \dots & & & \\ 0 & \dots & 1 & 1 & \dots & 0 \\ \dots & & & & & \\ 0 & \dots & 0 & 1 & \dots & 0 \\ \dots & & & & & \\ 0 & & & & & 1 \end{bmatrix}$$

This is the unit matrix with a 1 on element ij .

By using elementar row operations we can determine $\text{rank}(A)$ by reducing A to its canonical form.

Appendix 2: elementary row operations

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 4 & 4 & 4 \\ 10 & 12 & 14 & 16 \end{bmatrix}$$

$$\begin{cases} \text{row 3} + \text{row 1} \rightarrow \text{row 3} \\ \text{row 4} - \text{row 2} \rightarrow \text{row 4} \end{cases}$$

$$R_1 A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

Appendix 2: elementary row operations

$$\begin{cases} \text{row 3} - \text{row 2} \rightarrow \text{row 3} \\ \text{row 4} - \text{row 2} \rightarrow \text{row 4} \end{cases}$$

$$R_2[R_1 A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} R_1 A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} \text{row 2} - 4 \times \text{row 1} \rightarrow \text{row 2} \end{cases}$$

$$R_3[R_2 R_1 A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_2 R_1 A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -2 & -5 & -16 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Appendix 2: elementary row operations

{ row 2 - row 1 \rightarrow row 2

$$R_4[R_3 R_2 R_1 A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_3 R_2 R_1 A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -8 & -20 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

{ (row 2)/(-4) \rightarrow row 2

$$R_5[R_4 R_3 R_2 R_1 A] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_4 R_3 R_2 R_1 A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Appendix 2: elementary row operations

$$\left\{ \begin{array}{l} \text{row 1} - 2 \times \text{row 2} \rightarrow \text{row 1} \end{array} \right.$$

$$R_6[R_5 R_4 R_3 R_2 R_1 A] = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_5 R_4 R_3 R_2 R_1 A =$$

$$\begin{bmatrix} 1 & 0 & -1 & -6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, $\text{rank}(A)=2$ because the canonical form has two rows different from 0.

Remark: column operations can be derived on the same way.

Appendix 3: canonical form

The canonical form of a matrix $A^{m \times n}$ with $\text{rank}(A)=r$ can be derived by row/column operations.

With **row operations**:

$$A \sim \begin{bmatrix} 1 & 0 & \dots & 0 & x & x & \dots & x \\ 0 & 1 & \dots & 0 & x & x & \dots & x \\ \dots & & & & & & & \\ 0 & \dots & & 1 & x & x & \dots & x \\ \dots & & & & & & & \\ 0 & 0 & \dots & 0 & 0 & \dots & \dots & 0 \\ \dots & & & & & & & \\ 0 & \dots & \dots & 0 & 0 & \dots & \dots & 0 \end{bmatrix} \quad (x \text{ is a number } \neq 0)$$

- m-r rows will be zero
- Above and below every 1 are zeros

Appendix 3: canonical form

With **column operations**:

$$A \sim \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & & 0 \\ \dots & & & & \dots & & & \dots \\ 0 & \dots & & 1 & 0 & \dots & & 0 \\ \dots & & & & & & & \\ x & x & \dots & x & 0 & \dots & & 0 \\ \dots & & & & & & & \\ x & \dots & \dots & x & 0 & & & 0 \end{bmatrix} \quad (x \text{ is a number } \neq 0)$$

- n-r rows will be zero
- Left and right of every 1 are zeros