Design in the frequency domain
Nyquist stability criterion

July 29, 2015

Outline

Introduction

The root-locus method dealt with the s- and z-domains and the location of the poles and zeros formed the basis for that method.

For the **frequency domain**, we use the following substitution: $s = j\omega$, which means we will only regard perfect oscillations.

Sines, cosines and exponential signals are eigenfunctions of Linear Time Invariant systems (LTI).
Perfect oscillations form the natural decomposition of each signal when you are dealing with LTIs.

Introduction

We need to translate our **design criteria** to something that fits the discussed method. For the root-locus method, we had to express the criteria in positions of poles and zeros.
For the frequency domain, typical design criteria are:

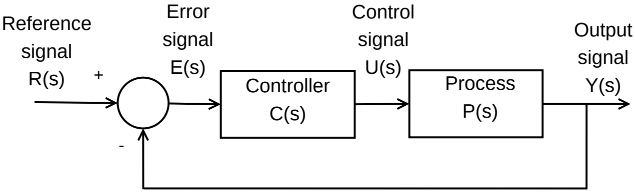
- Phase and gain margin
- Bandwidth
- Zero-frequency magnitude (= DC gain)

We will discuss two different **graphical representations** to design compensators in the frequency domain:

- Nyquist plots
- Bode plots (for the design of lead, lag and lead-lag compensators: next lecture)

Outline

Stability of the closed loop system



We write the output signal as a function of the input signal:

$$Y(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} R(s).$$

The closed loop system stability is determined by the poles of $\frac{P(s)C(s)}{1+P(s)C(s)}$, or the roots of $1 + P(s)C(s) = 0$.

Stability of the closed loop system

In the root-locus method, we determined the positions of the poles by plotting the roots of $1 + P(s)C(s) = 0$.

The system is stable if all the roots remain in the left half plane.

We are however not interested in the positions of the poles. We only need to know whether there *are* poles in the right half plane.

There is a cheaper alternative: **the Nyquist stability criterion**.

Stability: Nyquist criterion

The Nyquist stability criterion avoids determining the roots of $1 + P(s)C(s)$ exactly. It only determines the *number* of roots in the right half plane.

It uses a theorem from complex calculus that finds the difference between the number of poles and the number of zeros within a **contour** (a closed curve).

We will apply this theorem to $1 + P(s)C(s)$ (which can be seen as a complex function) and the contour will encircle the entire right half plane (= **the Nyquist contour**).

Outline

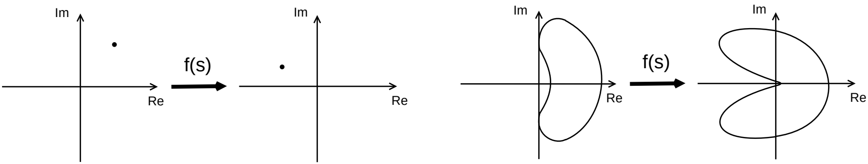
Outline

Complex function

Before we get to the theorem, we discuss the concept of a complex function.

A complex function $f(s) = u(x, y) + jv(x, y)$ maps the complex number $s = x + jy$ onto the complex number $w = u + jv$.

For a complex number: or for a contour:



The function $1 + P(s)C(s)$ can also be regarded as a complex function, so we can use it as a mapping.

Cauchy's argument principle

This is the engine behind the Nyquist stability criterion.

If a contour Γ in the s-plane encircles Z zeros and P poles of $f(s)$ in clockwise direction, the contour Γ′, which is the image of Γ as mapped by $f(s)$, encircles the origin (in the w-plane) $Z - P$ times in the clockwise direction.

So the only thing we are looking at is the **number of encirclements** of the origin.
On the next slides, we will prove this.

Cauchy's argument principle

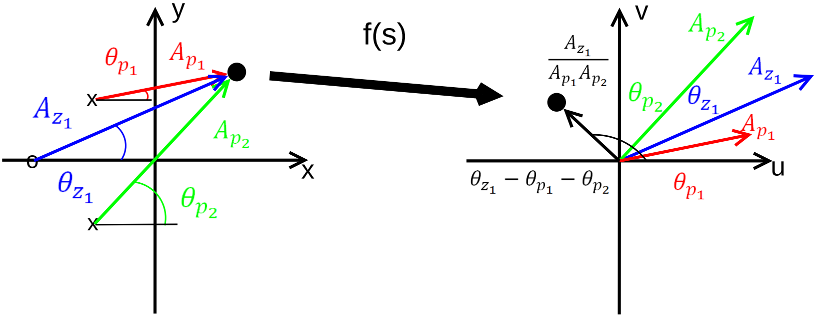
Let's take a complex function $f(s) = \frac{(s-z_1)(s-z_2)(s-z_3)\dots}{(s-p_1)(s-p_2)(s-p_3)\dots}$.

If we would apply this function to a complex number c, this comes down to multiplying factors $c - z_i = A_{z_i} e^{j\theta_{z_i}}$ and $\frac{1}{c-p_i} = \frac{1}{A_{p_i}} e^{-j\theta_{p_i}}$.

So the **modulus** of $f(c)$ can be easily found by evaluating $\frac{A_{z_1} A_{z_2} A_{z_3} \dots}{A_{p_1} A_{p_2} A_{p_3} \dots}$. This might help if you want to map a point, but it is not important for us.

The evaluation of the **argument** of $f(c)$ is what will be interesting: $\angle f(c) = \theta_{z_1} + \theta_{z_2} + \theta_{z_3} + \dots - \theta_{p_1} - \theta_{p_2} - \theta_{p_3} - \dots$

Cauchy's argument principle: graphically

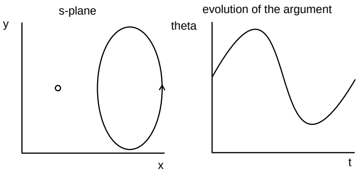


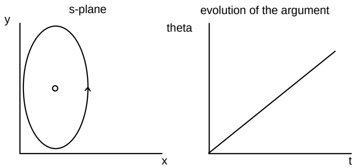
Cauchy's argument principle

When does the image of a contour in the w-plane encircle the origin?

That happens when the argument of the image of the contour at the beginning and the end differ by 2π.

A pole or zero outside the contour will never have that effect. One inside the contour has the following effect:





Cauchy's argument principle

A pole results in -2π (counterclockwise rotation), if the contour is followed clockwise.

A zero results in $+2\pi$ (clockwise rotation).

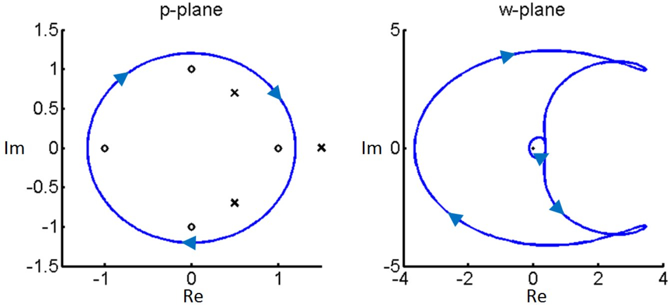
This follows from the sign of their effect:
$\angle f(c) = \theta_{z_1} + \theta_{z_2} + \theta_{z_3} + \ldots - \theta_{p_1} - \theta_{p_2} - \theta_{p_3} - \ldots$

It is also possible that the origin is encircled when there are no poles or zeros in the contour (in the s-plane). But then the amount of clockwise encirclements equals the amount of counterclockwise encirclements, hence no net encirclements.

Outline

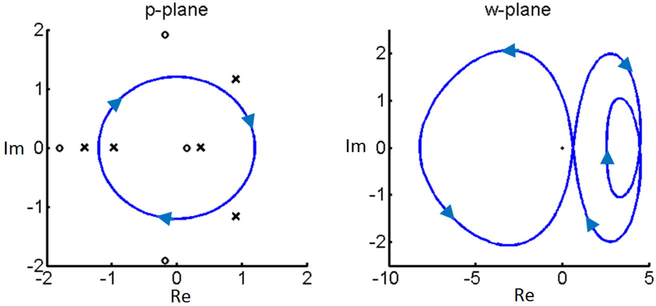
Example 1

- Two encircled poles (the x's): $P = 2$
- Four encircled zeros (the o's): $Z = 4$
- Hence: $N = Z - P = 2$
- Indeed, the image of the contour encircles the origin twice in the w-plane (in clockwise direction).



Example 2

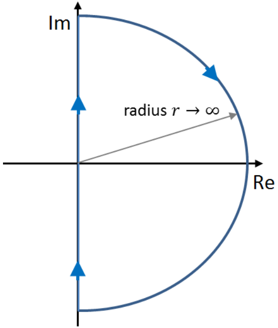
- Two encircled poles: $P = 2$
- One encircled zero: $Z = 1$
- Hence: $N = Z - P = -1$
- Indeed, the image of the contour encircles the origin once in the w-plane (in counterclockwise direction).



Outline

Nyquist stability criterion

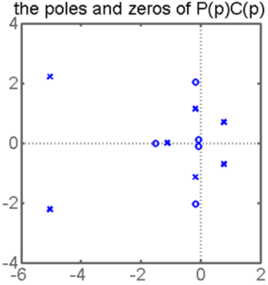
If we apply Cauchy's argument principle to the following contour (the **Nyquist contour**), the amount of clockwise encirclements around the origin of the mapping of this contour (the **Nyquist plot**) in the w-plane equals $Z - P$ of the right half plane (RHP).

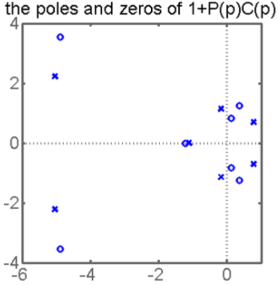


Nyquist stability criterion

That way, we will find the difference between the number of poles and zeros of $1 + P(s)C(s)$ in the RHP: $N = Z - P$.

So if we know P (the number of poles in the RHP of $1 + P(s)C(s)$), we know how many roots of $1 + P(s)C(s)$ there are in the RHP.





Nyquist stability criterion

Luckily this last aspect is simple, since the roots of $1 + P(s)C(s)$ equal those of $P(s)C(s)$. Hence the amount of RHP poles is equal (the connection between the zeros is not as clear).

So if the number of RHP poles of $P(s)C(s)$ is known (which is assumed), we know whether the system with unity (negative) feedback is stable.

Nyquist stability criterion

If we apply this to $1 + P(s)C(s)$, we need to count the number of encirclements of the origin.
However, the Nyquist stability criterion uses $P(s)C(s)$.

- The zeros of $1 + P(s)C(s)$ and the poles and zeros of $P(s)C(s)$ are hard to relate.
- This is in sharp contrast with how easily the Nyquist plots relate: the Nyquist plot of $P(s)C(s)$ equals the one of $(1 + P(s)C(s)$, after it has been moved to the right over a distance 1.
- To find $Z - P$, one has to count the number of clockwise encirclements of the image of $P(s)C(s)$ around the point $(-1, 0)$, since this equals the number of clockwise encirclements of the image of $1 + P(s)C(s)$ around the origin.

Nyquist stability criterion

Nyquist stability criterion

If the open loop system $P(s)C(s)$ has ℓ poles in the right half plane (of the s-plane), then the system with unity feedback is stable if and only if the Nyquist plot encircles the point (-1,0) ℓ times in the counter clockwise direction.