Location of the Largest Empty Rectangle among Arbitrary Obstacles

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Abstract : This paper outlines the following generalization of the classical maximal-empty-rectangle (MER) problem : given n arbitrarily-oriented non-intersecting line segments of finite length on a rectangular floor, locate an empty *isothetic* rectangle of maximum area. Thus, the earlier restriction on isotheticity of the obstacles is relaxed. Based on the well-known technique of matrix searching, a novel algorithm of time complexity $O(n\log^2 n)$ and space complexity $O(n)$, is proposed. Next, the technique is extended to handle the following two related open problems : locating the largest isothetic MER (i) inside an arbitrary simple polygon and (ii) amidst a set of arbitrary polygonal obstacles.

Keywords : Computational geometry, algorithms, matrix search, complexity.

1 Introduction

Geometry of rectangles plays significant role in VLSI layout design, design rule checking, routing and also in different problems of operations research. Recognition of all maximal-empty-rectangles (MER) is an important problem in computational geometry with manifold applications and was first introduced in [1]. The objective is to locate all isothetic MER's and/or the largest (area) MER among a given set of n points in a two-dimensional rectangular floor. The time complexity of the algorithm proposed in [1] is $O(min(n^2, R\log n))$, where R denoting the number of reported MER's, may be $O(n^2)$ in the worst case. The complexity was later improved to $O(R + n\log^2 n)$ in [2] and then to $O(R + n\log n)$ in [3, 4]. The algorithms in [5, 6] locate the largest empty rectangle among a point set without inspecting all MER's, in time $O(n\log^3 n)$ and $O(n\log^2 n)$ respectively, using divide-and-conquer and matrix-searching techniques. The MER problem is later generalized among a set of isothetic obstacles [7]. In three dimensional space, the largest maximal empty cuboid can be reported in $O(n^{8/3}\log^3 n)$ time and $O(n)$ space [8].

In this paper, we address a long-standing open problem of locating the largest isothetic MER amidst n arbitrarily-oriented non-intersecting line segments of finite length on a rectangular floor. The underlying difficulty stems from the fact that the number of MER's may be infinite in such a scenario. This motivates us to introduce the concept of prime-maximal-empty rectangles (PMER) among which the largest MER can be found. We also show that the number of PMER's is bounded above by $O(n^2)$ in the worst case. Using the well-known technique of matrix searching, a novel algorithm of time complexity $O(n\log^2 n)$ and space complexity $O(n)$ is then proposed for reporting the largest MER. Next, the technique is extended to handle the following two related open problems : locating the largest MER (i) inside an arbitrary polygon and (ii)

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amidst a set of arbitrary polygonal obstacles. Fig. 1 illustrates the various cases of the general MER problem. Location of the largest empty rectangle of arbitrary orientation among a set of polygonal obstacles, still remains open.

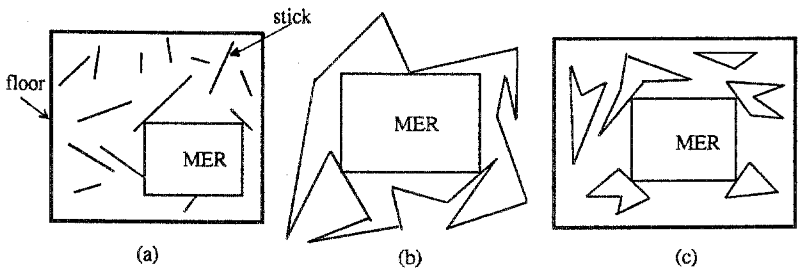


Fig. 1 : Largest (area) isothetic MER (a) amidst a set of sticks, (b) inside a polygon and (c) among polygonal obstacles

2 Basic Concepts

Consider a rectangular floor on which n arbitrarily-oriented non-overlapping straight line segments called sticks, are distributed. For our convenience, the set of sticks is assumed to include the four boundaries of the floor. Point obstacles, if any, may be thought of as infinitesimally small sticks. The bottom-left corner of the floor is considered to be the origin of the coordinate system. A *maximal empty rectangle* (MER) is an empty rectangle that is not properly subsumed in any other empty rectangle of larger size. Thus, every MER should touch several sticks so that its four arms are fixed. Throughout our discussion, we shall consider isothetic MER's only. We shall show shortly that the number of MER's may be infinity among a set of non-isothetic sticks.

Definition : A *stick* is a straight line segment $L_i[(a_i, b_i), (c_i, d_i)]$, where (a_i, b_i) and (c_i, d_i) denote the end points of the stick, such that $b_i \geq d_i$. We shall assume that the equation of the stick $L_i[(a_i, b_i), (c_i, d_i)]$ is : $y_i = m_i x + k_i$, where m_i is the slope of the stick, and k_i is its intercept on the y-axis.

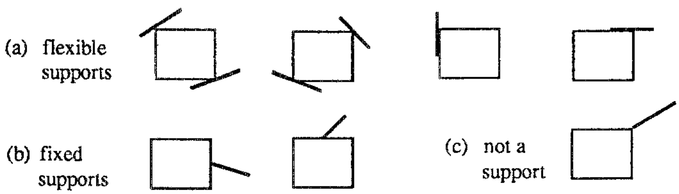


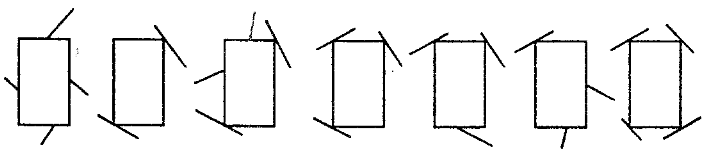
Fig. 2 : Fixed and flexible supports of a PMER

Definition : A stick is said to be a *support* of an empty rectangle if it suitably touches the rectangle at some point on its boundary as described below (Fig. 2). A support may be of two types :

• If a stick touches the top-left or bottom-right (resp. top-right or bottom-left) corner of the rectangle, and its slope $m \geq 0$ ($m \leq 0$), then it is called a *flexible support*; it is so called because the corner of the rectangle can lie anywhere on the stick.

• A stick touching any point on the boundary of the rectangle excepting the corners, called a *fixed support*; in this case, an arm of the rectangle is blocked by one end of the stick and hence fixed.

To form an MER, we need various combinations of fixed and flexible supports touching the rectangle. Few examples are shown in Figs.3a-3g.



(a) (b) (c) (d) (e) (f) (g)

Fig. 3 : MER's formed by various combinations of supports

It is easy to verify that in some cases (Figs. 3a, 3c), the MER becomes unique. In many other cases (Figs. 3b, 3d-g), the number of possible MER's formed by the same set of sticks becomes infinite (as a corner may slide continuously on a flexible stick), and hence inspection of all of them is not feasible. However, the largest one among them can readily be identified from simple geometric observations. This motivates to define a *prime MER* as described in the next section.

3 Prime Maximal Empty Rectangles (PMER)

Definition : An MER supported by a given set of sticks with their specified designations (fixed or flexible), is said to be a *prime* MER (PMER) if there exists (i) no MER of larger size formed by the same set of supporting sticks with same designations, and (ii) no MER formed by a different set of supporting sticks, that properly encloses it.

As mentioned earlier, the presence of a flexible support often makes the number of MER's infinite (see Fig. 3b). The PMER, being the largest among them, is of interest to us. In this context one should remember that the corner(s) of a PMER may be constrained to slide on a portion of a flexible support because of the presence of another stick(s) as shown in Fig. 4.

3.1 Properties of PMER's

To locate the PMER in the presence of a flexible support L, a variable point (α, β) is chosen on L and an MER (R) is drawn with one corner at (α, β) and touching the other supporting sticks. The PMER is the one for which the area $A(R)$ is maximum. It is also easy to observe that the expression of $A(R)$ is a quadratic function of α, and has a unique maxima. In order to maximize $A(R)$ with respect to α, one has to consider the presence of other sticks, if any, that constrain the search space. In the next section we shall introduce the concept of a *window*, that defines the search space for the given set of supporting sticks satisfying the constraints imposed by the neighboring sticks. The largest-area rectangle inside the window will then be the corresponding PMER. The various configurations of supports are described in the following cases:

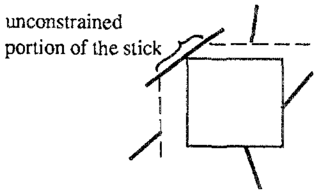


Fig. 4 : A flexible support constrained by one / two fixed support(s)

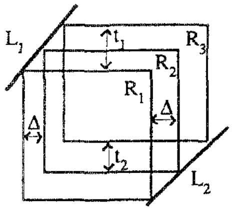


Fig. 5 : Monotonicity property of the MERs' with only two flexible supports at its two diagonally opposite corners

Case 1 : Two flexible supports at diagonally opposite corners of an MER

Consider the set of MER's formed by two flexible supports, L_1 and L_2, at two opposite corners (Fig. 3b). Let the absolute values of the gradients of L_1 and L_2 be m_1 and m_2 respectively (m_1 and m_2 are of same sign). Let $\ell_x(R)(\ell_y(R))$ denote the length of the horizontal(vertical) arm of a rectangle R, touching L_1 and L_2. Consider now three rectangles R_1, R_2 and R_3 each touching L_1 and L_2, as shown in Fig. 5. The rectangle R_2 is obtained by shifting R_1 horizontally by an amount Δ. Similarly, R_3 is obtained from R_2 by shifting it through the same distance Δ. It is easy to see that $A(R_1) - A(R_2) = \Delta\ell_x(R_1) \times (m_1 - m_2) = A(R_2) - A(R_3)$.
Thus, $A(R_1)$ is $>$ $(=,<)A(R_2)$ implies $A(R_2)$ $>$ $(=,<)A(R_3)$. This leads to the following lemma describing the monotonicity property of MER's.

Lemma 1 *Consider the set of MER's whose horizontal arms are of equal length and touch two flexible supports at two opposite corners. If $m_1 > m_2 > 0$ (resp. $m_2 > m_1 > 0$), then the area of an MER monotonically increases as its top-left corner slides upward (resp. downward) along L_1. Otherwise, if $m_1 = m_2$, then the area of all such MER's will be same.* □

In other words, the largest empty rectangle formed by only two flexible supports at two diagonally opposite corners will always touch an end point of each of L_1 and L_2. Furthermore, it will always be subsumed by another MER with L_1 and/or L_2 as fixed supports. It is easy to see that no other configuration of exactly two sticks can support the arms of a PMER. Therefore, no PMER is possible with only two supporting sticks. However, in the presence of some other additional sticks, a PMER may exist. This motivates us to define the *minimal set of supporting sticks* as follows.

Definition : A set of supporting sticks of a PMER is said to be *minimal* if the removal of any one of them cannot define the PMER.

If two flexible supports touch the opposite corners of an MER, the minimal set of supporting sticks must include either (i) an additional flexible support or (ii) two more fixed supports touching an appropriate pair of adjacent arms of the MER.

Case 1.1 : with one more flexible support

Let $L_1[(a_1,b_1),(c_1,d_1)]$ and $L_2[(a_2,b_2),(c_2,d_2)]$ be two flexible supports touching the top-left and bottom-right corners and assume $m_1 > m_2$. Let $L_3[(a_3,b_3),(c_3,d_3)]$ appear as another support at the bottom-left or the top-right corner (see Figs. 6a, 6b).

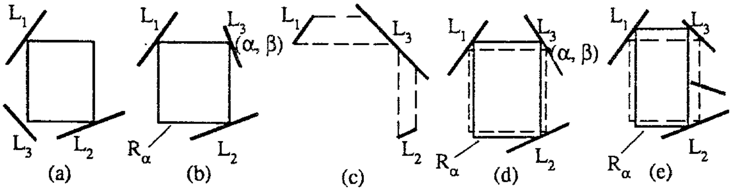


Fig. 6 : PMER with three flexible supports

Since $m_1 > m_2$, the area of an MER increases as the top-left or the bottom-right corner slides upward. So, if L_3 appears at the bottom-left corner (Fig. 6a), the largest-area empty rectangle touching L_1 and L_2 will not touch L_3. Hence, these three flexible supports do not serve as a minimal set of supporting sticks. But if L_3 appears at the top-right corner (Fig. 6b), the PMER will be the one whose three corners touch L_1, L_2 and L_3. In order to locate the PMER, let us choose a point (α,β) on L_3 as the top-right corner, and consider the corresponding MER (R_α). The top-left and bottom-right corners will be at $(\frac{\beta-k_1}{m_1},\beta)$ and $(\alpha, m_2\alpha+k_2)$ respectively. The area of the MER is $A(R_\alpha) = (\frac{\beta-k_1}{m_1}-\alpha)(\beta-(m_2\alpha+k_2))$.

The PMER can be obtained by maximizing the expression $A(R_\alpha)$ with respect to α subject to the constraints $max(d_1,d_3) \le \beta \le \min(b_1,b_3)$, and $max(c_2,a_3) \le \alpha \le min(c_3,a_2)$.

Now, $A'(R_\alpha) = 0 \Longrightarrow \alpha^* = \frac{(m_3-m_1)(k_2-k_3)+(m_3-m_2)(k_1-k_3)}{2(m_3-m_1)(m_3-m_2)}$

and the specified constraints are $\frac{max(d_1,d_3)-k_3}{m_3} \le \alpha \le \frac{min(b_1,b_3)-k_3}{m_3}$, and $max(c_2,a_3) \le \alpha \le min(c_3,a_2)$.

If there is no point on L_3 satisfying the above two constraints on α, as shown in Fig. 6c, no PMER will exist touching L_1, L_2 and L_3. Otherwise, let $\alpha_1 < \alpha < \alpha_2$ be the region satisfying both the constraints.
Now, if $\alpha_1 < \alpha^* < \alpha_2$ (Fig. 6b), then R_{α^*} is the PMER;
if $\alpha^* \le \alpha_1$ ($\alpha^* \ge \alpha_2$) then R_{α_1} (R_{α_2}) will be the PMER (see Fig. 6d), since $A(R_\alpha)$ is a quadratic function of α.

Now, if the rectangle is empty, it is the desired PMER. If the rectangle is non-empty because of the presence of some other neighboring sticks (Fig. 6e), an appropriate additional constraint will be imposed on the search space of α.

Case 1.2 : with two more fixed supports

Consider now two flexible supports $L_1[(a_1,b_1)(c_1,d_1)]$ and $L_2[(a_2,b_2)(c_2,d_2)]$ ($m_1 > m_2$) at diagonally opposite corners, and two other fixed supports $L_3[(a_3,b_3)(c_3,d_3)]$ and $L_4[(a_4,b_4)(c_4,d_4)]$ touching two adjacent arms of the MER. Since $m_1 > m_2$, the PMER will exist only if L_3 and L_4 provide constraints to the top-right corner as shown in

Fig. 7. The unique MER touching all the four sticks will then be the resulting PMER (R) whose top-left and bottom-right corners are at $(\frac{d_3-k_1}{m_1}, d_3)$ and $(a_4, m_2a_4 + k_2)$ respectively, and the area is given by : $A(R) = (\frac{d_3-k_1}{m_1} - a_4)(d_3 - m_2a_4 + k_2)$.

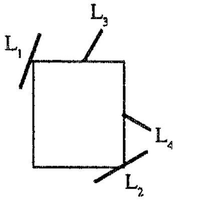


Fig. 7 : PMER with two fixed supports and two flexible supports at diagonally opposite corners

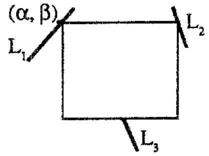


Fig. 8 : PMER with two flexible supports at its two adjacent corners

Case 2 : MER's with two flexible supports at adjacent corners

Two flexible supports $L_1[(a_1, b_1), (c_1, d_1)]$ and $L_2[(a_2, b_2), (c_2, d_2)]$ touching two adjacent corners, and a fixed support $L_3[(a_3, b_3), (c_3, d_3)]$ touching the opposite arm (Fig. 8), also define a minimal set of supporting sticks.

In order to locate the PMER in this scenario, choose an MER (R_α) whose top-left corner touches L_1 at the point (α, β), where $\beta = m_1\alpha + k_1$. The top-right corner on L_2 is at $(m_2\beta + k_2, \beta)$, and the bottom-right corner is at $(m_2\beta + k_2, b_3)$. Hence, the area of the rectangle R_α will be $A(R_\alpha) = (m_2\beta + k_2 - \alpha)(\beta - b_3)$.

Maximizing the expression $A(R_\alpha)$ with respect to α satisfying the constraints $\alpha \leq min(a_1, a_3)$, and $max(d_1, d_2, b_3) \leq \beta \leq min(b_1, b_2)$ as in the Case 1, one can derive a similar rule for locating the PMER. If the solution rectangle is not empty, because of the presence of other neighboring sticks, additional constraints need to be considered.

Case 3 : MER's formed by one flexible and two fixed supports

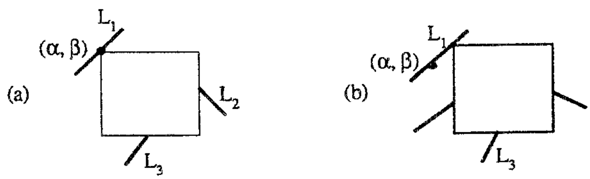


Fig. 9 : PMER with a single flexible support and a few fixed supports

A flexible support $L_1[(a_1, b_1), (c_1, d_1)]$ touching one corner of the MER and two fixed supports $L_2[(a_2, b_2), (c_2, d_2)]$ and $L_3[(a_3, b_3), (c_3, d_3)]$ touching two adjacent arms as in Fig 9a, also define an instance of minimal set of supporting sticks.

The area of the rectangle (R_α) whose top-left corner point (α, β) lies on L_1 is $A(R_\alpha) = (a_2 - \alpha)(\beta - b_3)$.

To locate the PMER, one has to maximize $A(R_\alpha)$ over α satisfying the constraints $c_1 \leq \alpha \leq min(a_1, a_3)$, and $max(b_2, d_1) \leq \beta \leq b_1$ as in the Case 1. If the corresponding rectangle is empty, then we get the desired PMER. Otherwise, the desired position of the top-left corner of the PMER on L_1 will further be constrained by the intruding sticks (see Fig. 9b) and can be determined easily.

It is now easy to follow that the minimal set of supporting sticks defining a PMER must be either of the following configurations :

(i) four fixed supports to the four sides of the MER;
(ii) two flexible supports at diagonally opposite corners and fixed supports to two adjacent sides of an appropriate third corner of the MER;
(iii) three flexible supports at three corners of the MER;
(iv) two flexible supports at two adjacent corners of the MER and a fixed support to its opposite side;
(v) one flexible support at a corner and fixed supports to two adjacent sides of its diagonally opposite corner.

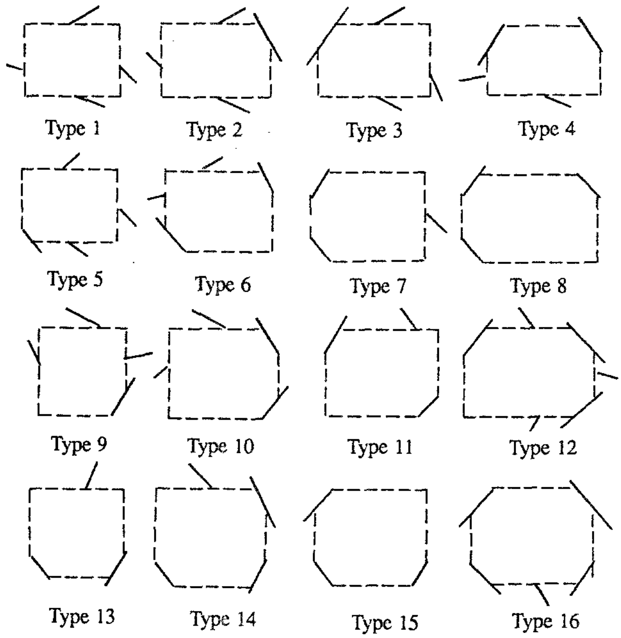


Fig. 10 : 16 possible types of windows

3.2 Enumeration of PMER's

To locate the PMER's amidst a set of sticks we now introduce the concept of a window that defines the search space as follows.

Definition : A *window* is a maximal empty convex polygon having at most eight arms among which four are isothetic and at most four are non-isothetic. The non-isothetic arms are portions of sticks. The isothetic arms are determined by the neighboring sticks ensuring maximality and emptiness.

Observation : There are 16 possible types of windows depending on the presence or absence of flexible supports at four corners. This list is exhaustive (Fig. 10).

Lemma 2 *Corresponding to every window there exists at most one PMER; and for each PMER there exists a unique window enclosing it.* □

For each of the 16 types of windows, the PMER can be obtained from the results outlined in Case 1 to Case 3 of section 3.1, as shown in Table 1.

Table - 1

Window type	*Result to be used*
1	obvious
8,12,14,15,16	Case 1.1
6,11	Case 1.2
4,7,10,13	Case 2
2,3,5,9	Case 3

From the maximality and emptiness of a window, it follows that, whenever two horizontal arms of a window are fixed, the remaining arms are also determined uniquely by the neighboring sticks (see Fig. 10).

Now, let us consider the top horizontal arm of a window. The y-coordinate of the arm is determined either by the top end-point of a stick (e.g. type 4, 7 of Fig. 10), or by the bottom end-point of a stick (e.g. type 1, 6, 10 of Fig. 10). Similarly, the bottom (horizontal) arm of the window is determined by some end-point of a stick. Thus for every window, the two horizontal arms are determined by a pair of end-points which in turn define the window uniquely. The following result now follows from the above discussion and Lemma 2.

Theorem 1 *The total number of possible PMER's is $O(n^2)$ in the worst case.* □

4 Location of the Largest MER

4.1 Basic concepts

The largest MER is actually the largest PMER, and can be located without inspecting all windows by using divide-and-conquer and the technique of *matrix searching* [6].

Divide-and-conquer scheme

Let S be the set of sticks present on the floor. Our algorithm is based on two-level divide-and-conquer approach. First, the endpoints of sticks are equally subdivided into two halves by a vertical line, say VP. The largest empty rectangle on the floor is the largest one among the following cases (see Fig. 11) : (i) the largest MER passing through VP, (ii) the largest MER to the left of VP, and (iii) the largest empty rectangle to the right of VP.

The sub-problem of case (i) can be solved using a second level of divide-and-conquer, i.e., slicing the floor into two halves with horizontal cuts recursively. The last two cases may yield a number of first-level sub-problems, each of which can be solved recursively, as the original one.

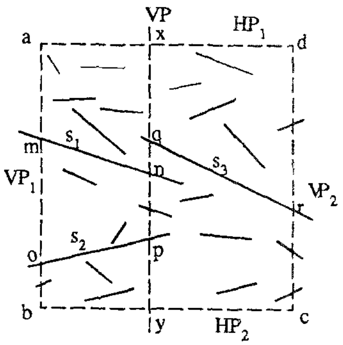


Fig. 11 : Vertical partition of a plane

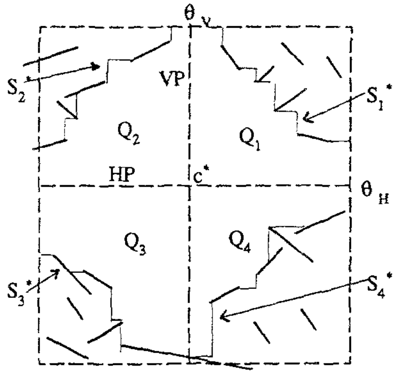


Fig. 12 : Four maximal stairs defining the orthoconvex polygon around c*

4.2 Formulation using matrix-searching

To solve the sub-problem of case (i), we slice the floor with a horizontal line HP that partitions the end-points of sticks into two equal halves. Let HP and VP intersect at c^* (see Fig. 12). We now have to consider three sub-cases : (i) the MER's enclosing the point c^*, (ii) the MER's above the line HP, (iii) the MER's below the line HP. The last two sub-cases can be solved recursively by further horizontal partitioning. Note that if a stick s hits both the horizontal boundaries of a sub-floor, then the portion of the floor separated by s from c^*, need not be considered. Before describing the algorithm for solving sub-case (i), we recall few results of matrix-searching.

Matrix searching

Definition [9] : A matrix $A(\mu \times \nu) = [A_{ij}]$ is said to be *totally monotone* if for every i, j, k, ℓ such that $i < k$, $j < \ell$ and $A_{ij} \leq A_{i\ell}$, we have $A_{kj} \leq A_{k\ell}$.

The largest entry in a totally monotone matrix can be found in $O(\mu + \nu)$ time [6], provided an entry of the matrix is computable in constant time. We now show that the location of the largest PMER enclosing the point c^*, can be transformed into the problem of finding the largest element of a totally monotone matrix.

Largest PMER enclosing c^*

The two orthogonal lines VP and HP partition the concerned part of the floor into four quadrants, say Q_1, Q_2, Q_3 and Q_4 (see Fig. 12). Let θ_V and θ_H denote the points where VP and HP meet the top and right boundaries of the floor respectively.

Definition : A continuous curve from θ_H to θ_V in the quadrant Q_1 is said to form a *stair* around c^* if for any two points (x_i, y_i) and (x_j, y_j), $x_i \leq x_j$ on the curve, we have $y_i \geq y_j$.
The *maximal-empty-stair* (S_1^*) in Q_1 in the presence of sticks, is the stair such that the polygon bounded by the segments $c^*\theta_H$, $c^*\theta_V$, and the stair, is empty and has maximum area (see Fig. 12).

The maximal-empty-stairs in other quadrants (Q_2, Q_3 and Q_4) can be defined similarly and are denoted as S_2^*, S_3^* and S_4^* respectively. Clearly, the polygon bounded by the four maximal-empty-stairs is *orthoconvex*, and is the union of the windows enclosing c^*. The largest PMER enclosing c^* in the sub-floor is the largest PMER inside the orthoconvex polygon.

Maximal-empty-stairs in all the four quadrants can be obtained in $O(k)$ time from a sorted list (w.r.t y-coordinates) of k end-points of the sticks present on the sub-floor. The portions of sticks and the convex corners of the orthoconvex polygon will be referred to as objects later on.

Let $P = \{p_1, \ldots, p_m\}$ be the set of end-points of all the objects in S_2^*. From each point $p_i \in P$, draw a horizontal and a vertical line hitting the maximal-empty-stairs of Q_1 and Q_3 at the points q_1 and q_3 respectively. If $q_1(q_3)$ falls on a non-isothetic stick of S_1^* (S_3^*), then the stick is split into two portions replacing the original one. On the other hand, if the line hits an isothetic line, then a new object $q_1(q_3)$ is created and inserted in S_1^* (S_3^*). Now, from each end-point of the updated S_1^* (S_3^*), a vertical (horizontal) line is drawn hitting an object of S_4^*. As before, the set S_4^* is enhanced.

Thus, if k_i be the number of objects in the initial set S_i^*, ($i = 1,2,3,4$), then the above process increases the number of objects in S_1^* and S_3^* by at most $2k_2$, and in S_4^* by at most $2(k_1 + k_2 + k_3)$. The process can easily be accomplished by merging the above sets in an appropriate sequence.

Now consider a matrix $A(\mu \times \nu)$, whose columns (resp. rows) correspond to the objects and the convex corners not attached to any stick of S_2^* (resp. enhanced S_4^*). The set of points in S_2^* (S_4^*) may serve as the top-left (bottom-right) corners of the PMER around c^*. Thus $\mu \leq (k_1 + k_2 + k_3 + 2k_4)$, and $\nu \leq 2k_2$. Henceforth, we will denote by $\widehat{S}_i^*$, the enhanced set of objects as well as the convex corners not attached to any stick of S_i^*, i = 1, 2, 3, 4. The elements in $\widehat{S}_2^*$ are sorted in increasing order of their x-coordinates, and denoted by $\{g_1, g_2, \ldots, g_\nu\}$; those in $\widehat{S}_4^*$ are sorted in decreasing order of x-coordinates and denoted by $\{h_1, h_2, \ldots, h_\mu\}$. The entries of the matrix are as follows :

$A_{k\ell}$ = area of the largest PMER with top-left and bottom-right corners on $g_k \in \widehat{S}_2^*$ and $h_\ell \in \widehat{S}_4^*$ respectively, provided such a PMER exists;
= blank, otherwise.

The non-blank entries of a row can be found from the following result :

Lemma 3 *The horizontal (vertical) projections of both the end-points of any object $h_\ell \in \widehat{S}_4^*$ fall either on the same vertical (horizontal) arm or on the same stick of the stair $\widehat{S}_3^*$ ($\widehat{S}_1^*$). Similarly, the vertical (horizontal) projections of each object of $\widehat{S}_3^*$ ($\widehat{S}_1^*$) fall either on the same horizontal (vertical) line or on the same stick of $\widehat{S}_2^*$.*

From the bottom (top) end-point of h_ℓ draw a horizontal (vertical) line that meets the stair $\widehat{S}_1^*$ ($\widehat{S}_3^*$) at a point say, q_1 (q_2). From q_1 (q_2), draw a vertical (horizontal) line meeting the stair $\widehat{S}_2^*$ at a point $\gamma_1(x_1, y_1)$ ($\gamma_2(x_2, y_2)$). Now, a PMER, whose top-left (bottom-right) corner lies on $g_k \in \widehat{S}_2^*$ ($h_\ell \in \widehat{S}_4^*$), exists only if at least one end-point of g_k lies in the interval $[x_1, x_2]$. Thus the objects on which γ_1 and γ_2 lie, yield the non-zero entries of the ℓ-th row of the matrix A. It can be readily seen that the matrix $A(\mu \times \nu)$ forms a special class of *totally monotone partial matrix* [9].

Lemma 4 *Consider two distinct objects g_i and g_j from $\widehat{S}_2^*$ and two distinct objects h_k and h_ℓ from $\widehat{S}_4^*$ as in Fig. 13. Then (a) if $A_{ik} \leq A_{jk}$ then $A_{i\ell} \leq A_{j\ell}$ and (b) if $A_{j\ell} \leq A_{jk}$ then $A_{i\ell} \leq A_{ik}$.*

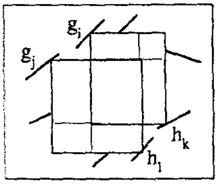


Fig. 13 : Proof of monotonicity of the matrix A(μ x ν)

Proof : Follows from geometry of rectangles. □

The above lemma leads to the fact that if the quadrants Q_1 and Q_3 are empty, all entries in the matrix A are non-blank, and hence it is a totally monotone matrix. If Q_1 and/or Q_3 are non-empty, A becomes a totally monotone partial matrix. Similarly, if the objects of $\widehat{S}_1^*$ and $\widehat{S}_3^*$ designate the rows and columns of a matrix, one obtains a totally monotone matrix. Details of the algorithm appear in [10].

Theorem 2 *The largest entry of the matrix $A(\mu \times \nu)$ can be found in $O(\mu + \nu)$ time.*

Proof : The blank entries (A_{ik}) of the matrix correspond to non-empty rectangles and do not produce valid PMER's. The matrix because of its special attributes, becomes totally monotone if we put 0 in all blank entries. While doing matrix searching, the blank entries are not actually filled with zeros, but can be found by manipulating the starting and finishing boundary sequences [9]. Hence, the largest element in the matrix can be obtained in linear time [6]. □

4.3 Time complexity

To estimate the time complexity of the algorithm, we consider two levels of divide-and-conquer separately. In both the first and second level of divide-and-conquer, each stick is considered at most $O(logn)$ times [10].

Sorting x-coordinates of end-points of the sticks requires $O(n\text{log}n)$ time. The time complexity of solving a first-level subproblem depends on the number of second-level subproblems generated by horizontal splitting. In a second-level subproblem, the orthoconvex polygon around c^* is obtained, and then matrix searching is performed to locate the largest PMER around c^*.

The orthoconvex polygon can be constructed in time linear in the number of sticks in the corresponding subfloor. The updated list of objects $\widehat{S}_i^*$ can also be obtained by merging the stairs, in linear time.

Corresponding to an entry of the matrix, if a valid PMER exists, then its location and area can be found from Table 1 by determining the window type in constant time. Therefore each call to matrix-search takes linear time in the number of objects present inside the concerned orthoconvex polygon. Thus, the total time complexity of the algorithm is $O(nlog^2n)$. The space complexity depends on the space requirement for all leaf-level subproblems of divide-and-conquer steps, which is $O(n)$.

5 Other Applications

We now highlight a minor, but important modification in our algorithm for locating the largest MER among a set of arbitrary polygonal obstacles. The arms of all the polygons are considered as sticks. Since our algorithm is based on the assumption of non-intersecting sticks, we assume that there is an infinitesimal gap at the tip of each pair of sticks forming a corner of a polygon, so that the area of the rectangle passing through the gap is very close to zero. In addition, each stick is assumed to have two *sides* indicating the exterior and interior of the concerned polygon. In our second level of divide-and-conquer step, if the point c^* falls inside a polygon, no empty rectangle exists enclosing c^*. The decision about whether or not c^* is inside a polygon can be obtained by observing the sticks forming the orthoconvex polygon around c^*. If the interior side of any stick is facing c^* then matrix searching is skipped. The complexity of the algorithm will be $O(N\log^2 N)$, where N is the total number of vertices of all the polygons. The algorithm with similar modification, can be used to locate the largest MER inside an arbitrary simple N-gon in $O(N\log^2 N)$ time.

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