# ELEMENTS OF GEOMETRY

PART I

*HAMBLIN SMITH*

1/6

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# ELEMENTS OF GEOMETRY

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# ELEMENTS OF GEOMETRY

## PART I

CONTAINING THE

## FIRST TWO BOOKS OF EUCLID

WITH

## EXERCISES AND NOTES

BY

J. HAMBLIN SMITH, M.A

GONVILLE AND CAIUS COLLEGE, AND LECTURER AT ST PETER'S COLLEGE, CAMBRIDGE



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# PREFACE.

To preserve Euclid's order, to supply omissions, to remove defects, to give brief notes of explanation and simpler methods of proof in cases of acknowledged difficulty—such are the main objects of this Edition of the First and Second Books of the Elements.

The work is based on the Greek text, as it is given in the Editions of August and Peyrard. To the suggestions of the late Professor De Morgan, published in the Companion to the British Almanack for 1849, I have paid constant deference.

A limited use of symbolic representation, wherein the symbols stand for words and not for operations, is generally regarded as desirable, and I have been assured, by the highest authorities on this point, that the symbols employed in this book are admissible in the Examinations at Oxford and Cambridge *.

I have generally followed Euclid's method of proof, but not to the exclusion of other methods recommended by their simplicity, such as the demonstrations by which I propose to replace (at least

* I regard this point as completely settled in Cambridge by the following notices prefixed to the papers on Euclid set in the Senate-House Examinations:

I. In the Previous Examination:

*In answers to these questions any intelligible symbols and abbreviations may be used.*

II. In the Mathematical Tripos:

*In answers to the questions on Euclid the symbol — must not be used. The only abbreviation admitted for the square on* AB *is "sq. on* AB," *and for the rectangle contained by* AB *and* CD, "*rect.* AB, CD."

for a first reading) the difficult Theorems 5 and 7 in the First Book. I have also attempted to render many of the proofs, as, for instance, Propositions 2, 13, and 35 in Book I, and Proposition 13 in Book II, less confusing to the learner.

In Propositions 4, 5, 6, 7, and 8 of the Second Book I have ventured to make an important change in Euclid's mode of exposition, by omitting the diagonals from the diagrams and the gnomons from the text.

In the Third Book, which I am now preparing, I intend to deviate with even greater boldness from the precise line of Euclid's method. For it is in treating of the properties of the circle that the importance of certain matters, to which reference is made in the Notes of the present volume, is fully brought out. I allude especially to the application of Superposition as a test of equality, to the conception of an Angle as a magnitude capable of unlimited increase, and to the development of the methods connected with Loci and Symmetry.

The Exercises have been selected with considerable care, chiefly from the Senate-House Examination Papers. They are intended to be progressive and easy, so that a learner may from the first be induced to work out something for himself.

I desire to express my thanks to the friends who have improved this work by their suggestions, and to beg for further help of the same kind.

J. HAMBLIN SMITH.

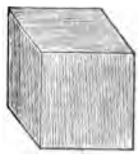
CAMBRIDGE, 1871.

# ELEMENTS OF GEOMETRY.

## *INTRODUCTORY REMARKS.*

WHEN a block of stone is hewn from the rock, we call it a Solid *Body.* The stone-cutter shapes it, and brings it into that which we call *regularity of form;* and then it becomes a Solid *Figure.*

Now suppose the figure to be such that the block has six flat sides, each the exact counterpart of the others; so that, to one who stands facing a corner of the block, the three sides which are visible present the appearance represented in this diagram.



Each side of the figure is called a *Surface;* and when smoothed and polished, it is called a *Plane* Surface.

The sharp and well-defined edges, in which each pair of sides meets, are called *Lines.*

The place, at which any three of the edges meet, is called a *Point.*

A *Magnitude* is any thing which is made up of parts in any way like itself. Thus, a line is a magnitude; because we may regard it as made up of parts which are themselves lines.

The properties Length, Breadth (or Width), and Thickness (or Depth or Height) of a body are called its *Dimensions.*

We make the following distinction between Solids, Surfaces, Lines, and Points:

A Solid has three dimensions, Length, Breadth, Thickness.

A Surface has two dimensions, Length, Breadth.

A Line has one dimension, Length.

A Point has no dimensions.

# EUCLID. BOOK I.

## *DEFINITIONS.*

I. A POINT is that which has no parts.

This is equivalent to saying that a Point has no magnitude, since we define it as that which cannot be divided into smaller parts.

II. A LINE is length without breadth.

We cannot conceive a visible line without breadth; but we can reason about lines as if they had no breadth, and this is what Euclid requires us to do.

III. The EXTREMITIES of finite LINES are points.

A Point marks *position*, as for instance, the place where a line begins or ends, or meets or crosses another line.

IV. A STRAIGHT LINE is one which lies in the same direction with regard to its points.

V. A SURFACE is that which has length and breadth only.

VI. The EXTREMITIES of a SURFACE are lines.

VII. A PLANE SURFACE is one in which, if any two points be taken, the straight line between them lies wholly in that surface.

Thus the ends of an uncut cedar-pencil are plane surfaces; but the rest of the surface of the pencil is not a plane surface, since two points may be taken in it such that the *straight* line joining them will not lie on the surface of the pencil.

In our introductory remarks we gave examples of a Surface, a Line, and a Point, as we know them through the evidence of the senses.

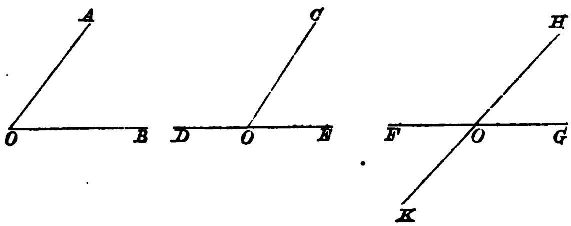
The Surfaces, Lines, and Points of Geometry may be regarded as mental pictures of the surfaces, lines, and points which we know from experience.

It is, however, to be observed that Geometry requires us to conceive the possibility of the existence

of a Surface apart from a Solid body,
of a Line apart from a Surface,
of a Point apart from a Line.

VIII. When two straight lines meet one another, but are not in the same straight line, the inclination of the lines to one another is called an ANGLE.

When *two* straight lines have one point common to both, they are said to *form* an angle (or angles) at that point. The point is called the *vertex* of the angle (or angles), and the lines are called the *arms* of the angle (or angles).

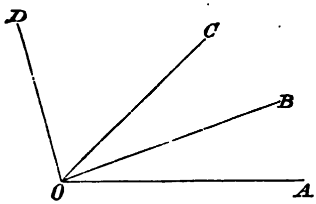



Thus, if the lines $OA$, $OB$ are terminated at the same point $O$, they form an angle, which is called *the angle at O*, or *the angle AOB*, or *the angle BOA*,—the letter which marks the vertex being put between those that mark the arms.

Again, if the line $CO$ meets the line $DE$ at a point in the line $DE$, so that $O$ is a point common to both lines, $CO$ is said to make with $DE$ the angles $COD$, $COE$; and these (as having one arm, $CO$, common to both) are called *adjacent* angles.

Lastly, if the lines $FG$, $HK$ cut each other in the point $O$, the lines make with each other four angles $FOH$, $HOG$, $GOK$, $KOF$; and of these $GOH$, $FOK$ are called *vertically opposite* angles, as also are $FOH$ and $GOK$.

When *three or more* straight lines as $OA$, $OB$, $OC$, $OD$ have a point $O$ common to all, the angle formed by one of



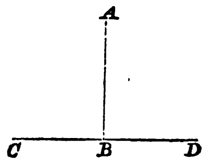

them, $OD$, with $OA$ may be regarded as being made up of the angles $AOB$, $BOC$, $COD$; that is, we may speak of the angle $AOD$ as a whole, of which the parts are the angles $AOB$, $BOC$ and $COD$.

Hence we may regard an angle as a *Magnitude*, inasmuch as any angle may be regarded as being made up of parts which are themselves angles.

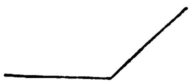
The size of an angle depends in no way on the length of the arms by which it is bounded.

We shall explain hereafter the restriction on the magnitude of angles enforced by Euclid's definition, and the important results that follow an extension of the definition.

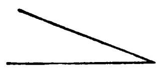
IX. When a straight line (as $AB$) meeting another straight line (as $CD$) makes the adjacent angles equal to one another, each of the angles is called a RIGHT ANGLE; and each line is said to be a PERPENDICULAR to the other.




X. An OBTUSE ANGLE is one which is greater than a right angle.



XI. An ACUTE ANGLE is one which is less than a right angle.

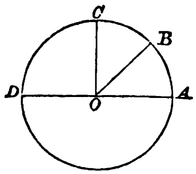


XII. A FIGURE is that which is enclosed by one or more boundaries.

XIII. A CIRCLE is a plane figure contained by one line, which is called the CIRCUMFERENCE, and is such, that all straight lines drawn to the circumference from a certain point (called the CENTRE) within the figure are equal to one another.

XIV. Any straight line drawn from the centre of a circle to the circumference is called a RADIUS.

XV. A DIAMETER of a circle is a straight line drawn through the centre and terminated both ways by the circumference.




Thus, in the diagram, $O$ is the centre of the circle $ABCD$, $OA$, $OB$, $OC$, $OD$ are Radii of the circle, and the straight line $AOD$ is a Diameter. Hence the radius of a circle is half the diameter.

XVI. A SEMICIRCLE is the figure contained by a diameter and the part of the circumference cut off by the diameter.

XVII. RECTILINEAR figures are those which are contained by straight lines.

The PERIMETER (or Periphery) of a rectilinear figure is the sum of its sides.

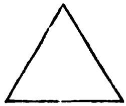
XVIII. A TRIANGLE is a plane figure contained by three straight lines.

XIX. A QUADRILATERAL is a plane figure contained by four straight lines.

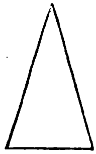
XX. A POLYGON is a plane figure contained by more than four straight lines.

When a polygon has all its sides equal and all its angles equal it is called a *regular* polygon.

XXI. An EQUILATERAL Triangle is one which has all its sides equal.



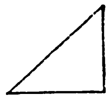
XXII. An ISOSCELES Triangle is one which has two sides equal.



The third side is often called the *base* of the triangle.

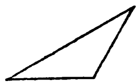
The term *base* is applied to any one of the sides of a triangle to distinguish it from the other two, especially when they have been previously mentioned.

XXIII. A RIGHT-ANGLED Triangle is one in which one of the angles is a right angle.



The side *subtending*, that is, *which is opposite* the right angle is called the *Hypotenuse.*

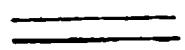
XXIV. An OBTUSE-ANGLED Triangle is one in which one of the angles is obtuse.



It will be shewn hereafter that a triangle can have only one of its angles either equal to, or greater than, a right angle.

XXV. An ACUTE-ANGLED Triangle is one in which ALL the angles are acute.

XXVI. PARALLEL STRAIGHT LINES are such as, being in the same plane, never meet when continually produced in both directions.



Euclid proceeds to put forward Six Postulates or Requests that he may be allowed to make certain assumptions on the construction of figures and the properties of geometrical magnitudes.

## POSTULATES.

Let it be granted

I. That a straight line may be drawn from any one point to any other point.

II. That a terminated straight line may be produced to any length in a straight line.

III. That a circle may be described from any centre at any distance from that centre.

IV. That all right angles are equal to one another.

V. That two straight lines cannot inclose a space.

VI. That if a straight line meet two other straight lines, so as to make the two interior angles on the same side of it, taken together, less than two right angles, these straight lines being continually produced shall at length meet upon that side, on which are the angles, which are together less than two right angles.

The word rendered "Postulates" is in the original *αἰτήματα*, "requests."

In the first three Postulates Euclid states the use, under certain restrictions, which he desires to make of certain instruments for the construction of lines and circles.

In Post. I. and II. he asks for the use of the straight ruler, wherewith to draw straight lines. The restriction is, that the ruler is not supposed to be marked with divisions so as to measure lines.

In Post. III. he asks for the use of a pair of compasses, wherewith to describe a circle whose centre is at one extremity of a given line and whose circumference passes through the other extremity of that line. The restriction is, that the compasses are not supposed to be capable of conveying distances.

Post. IV. and V. refer to simple geometrical facts, which Euclid desires to take for granted.

Post. VI. may, as we shall shew hereafter, be deduced as a Theorem from a more simple Postulate. The student must defer the consideration of this Postulate, till he has reached the 17th Proposition of Book I.

Euclid next enumerates, as statements of fact, nine Axioms, or, as he calls them, Common Notions, applicable (with the exception of the eighth) to all kinds of magnitudes, and not necessarily restricted, as are the Postulates, to *geometrical* magnitudes.

## *AXIOMS.*

I. Things which are equal to the same thing are equal to one another.

II. If equals be added to equals, the wholes are equal.

III. If equals be taken from equals, the remainders are equal.

IV. If equals and unequals be added together, the wholes are unequal.

V. If equals be taken from unequals, or unequals from equals, the remainders are unequal.

VI. Things which are double of the same thing are equal to one another.

VII. Things which are halves of the same thing are equal to one another.

VIII. Magnitudes which coincide with one another are equal to one another.

IX. The whole is greater than its part.

With his Common Notions Euclid takes the ground of authority, saying in effect, "To my Postulates I request, to my Common Notions I claim, your assent."

Euclid develops the science of Geometry in a series of Propositions, some of which are called Theorems and the rest Problems, though Euclid himself makes no such distinction.

By the name *Theorem* we understand a truth, capable of demonstration or proof by deduction from truths previously admitted or proved.

By the name *Problem* we understand a construction, capable of being effected by the employment of principles of construction previously admitted or proved.

A *Corollary* is a Theorem or Problem easily deduced from, or effected by means of, a Proposition to which it is attached.

We shall divide the First Book of the Elements into three sections. The reason for this division will appear in the course of the work.

## SYMBOLS AND ABBREVIATIONS USED IN BOOK I.

| | |
|---|---|
| $\because$ for because | $\odot$ for circle |
| $\therefore$ ............therefore | $\bigcirc$ce............circumference |
| = ............is (or are) equal to | $\parallel$ ...............parallel |
| $\angle$ ............angle | $\square$ ... ........parallelogram |
| $\triangle$ ............triangle | $\perp$ ..............perpendicular |

| | |
|---|---|
| equilat. .....equilateral | rt................right |
| ext$^r$..........exterior | sq. ............square |
| int$^r$..........interior | sqq. ............squares |
| pt............point | st................straight |
| rectil. ......rectilinear | |

It is well known that one of the chief difficulties with learners of Euclid is to distinguish between what is assumed, or given, and what has to be proved in some of the Propositions. To make the distinction clearer we shall put in italics the statements of what has to be done in a Problem, and what has to be proved in a Theorem. The last line in the proof of every Proposition states, that what had to be done or proved has been done or proved.

The letters Q. E. F. at the end of a Problem stand for *Quod erat faciendum*.

The letters Q. E. D. at the end of a Theorem stand for *Quod erat demonstrandum.*

In the marginal references:

Post. stands for Postulate.
Def. ............... Definition.
Ax. ............... Axiom.
I. 1. ............... Book I. Prop. 1.

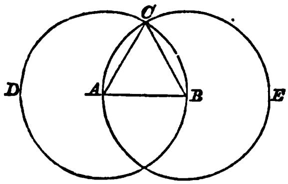
Hyp. stands for Hypothesis, *supposition*, and refers to something granted, or assumed to be true.

## SECTION I.

### *On the Properties of Triangles.*

PROPOSITION I. PROBLEM.

***To describe an equilateral triangle on a given straight line.***




Let $AB$ be the given st. line.

***It is required to describe an equilat.*** $\triangle$ ***on*** $AB$.

With centre $A$ and distance $AB$ describe $\odot BCD$. Post. 3.
With centre $B$ and distance $BA$ describe $\odot ACE$. Post. 3.

From $C$, the pt. in wh. the $\odot$s cut one another, draw the st. lines $CA$, $CB$. Post. 1.

Then will $ABC$ be an equilat. $\triangle$.

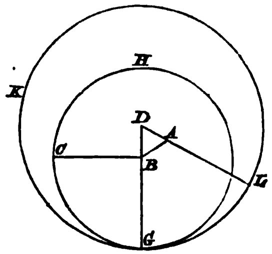
| | | |
|---|---|---|
| For | $\because$ $A$ is the centre of $\odot BCD$, | |
| | $\therefore$ $AC=AB$. | Def. 13. |
| And | $\because$ $B$ is the centre of $\odot ACE$, | |
| | $\therefore$ $BC=AB$. | Def. 13. |
| Now | $\because$ $AC$, $BC$ are each $=AB$, | |
| | $\therefore$ $AC=BC$. | Ax. 1. |

Thus $AC$, $AB$, $BC$ are all equal, and an equilat. $\triangle ABC$ has been described on $AB$.

Q. E. F.

## Proposition II. Problem.

***From a given point to draw a straight line equal to a given straight line.***




Let $A$ be the given pt., and $BC$ the given st. line.

*It is required to draw from $A$ a st. line equal to $BC$.*

From $A$ to $B$ draw the st. line $AB$. Post. 1.

On $AB$ describe the equilat. $\triangle ABD$. I. 1.

With centre $B$ and distance $BC$ describe $\odot CGH$. Post. 3.

Produce $DB$ to meet the $\bigcirc$ce $CGH$ in $G$.

With centre $D$ and distance $DG$ describe $\odot GKL$. Post. 3.

Produce $DA$ to meet the $\bigcirc$ce $GKL$ in $L$.

Then will $AL = BC$.

For $\because$ $B$ is the centre of $\odot CGH$,

$\therefore$ $BC = BG$. Def. 13.

And $\because$ $D$ is the centre of $\odot GKL$,

$\therefore$ $DL = DG$. Def. 13.

And parts of these, $DA$ and $DB$, are equal. Def. 21.

$\therefore$ remainder $AL$ = remainder $BG$. Ax. 3.

But $BC = BG$;

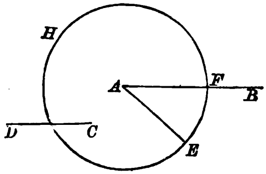
$\therefore$ $AL = BC$. Ax. 1.

Thus from pt. $A$ a st. line $AL$ has been drawn $= BC$.

Q. E. F.

PROPOSITION III. PROBLEM.

*From the greater of two given straight lines to cut off a part equal to the less.*




Let $AB$ be the greater of the two given st. lines $AB$, $CD$.

*It is required to cut off from $AB$ a part* $= CD$.

From $A$ draw the st. line $AE = CD$. I. 2.

With centre $A$ and distance $AE$ describe $\odot EFH$.

Then will $AF = CD$.

For $\because$ $A$ is the centre of $\odot EFH$,

$\therefore AF = AE$.

But $AE = CD$;

$\therefore AF = CD$. Ax. 1.

Thus from $AB$ a part $AF$ has been cut off $= CD$.

Q. E. F.

EXERCISES.

1. Shew that if straight lines be drawn from $A$ and $B$ in the diagram of Prop. I. to the other point in which the circles intersect, another equilateral triangle will be described on $AB$.

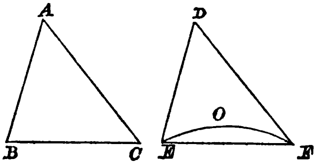
2. By a similar construction to that in Prop. I. describe on a given straight line an isosceles triangle, whose equal sides shall be each equal to another given straight line.

3. Draw a figure for the case in Prop. II., in which the given point coincides with $B$.

4. By a construction similar to that in Prop. III. produce the less of two given straight lines that it may be equal to the greater.

PROPOSITION IV. THEOREM.

***If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise the angles contained by those sides equal to one another, they must have their third sides equal; and the two triangles must be equal, and the other angles must be equal, each to each, viz. those to which the equal sides are opposite.***




In the $\triangle$s $ABC$, $DEF$,

let $AB=DE$, and $AC=DF$, and $\angle BAC = \angle EDF$.

*Then must $BC=EF$ and $\triangle ABC = \triangle DEF$, and the other $\angle$s, to which the equal sides are opposite, must be equal, that is, $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$.*

For, if $\triangle ABC$ be applied to $\triangle DEF$,

so that $A$ coincides with $D$, and $AB$ falls on $DE$,

then $\because AB=DE$, $\therefore B$ will coincide with $E$.

And $\because AB$ coincides with $DE$, and $\angle BAC = \angle EDF$, Hyp.

$\therefore AC$ will fall on $DF$.

Then $\because AC=DF$, $\therefore C$ will coincide with $F$.

And $\because B$ will coincide with $E$, and $C$ with $F$,

$\therefore BC$ will coincide with $EF$;

for if not, let it fall otherwise as $EOF$: then the two st. lines $BC$, $EF$ will enclose a space, which is impossible. Post. 5.

$\therefore BC$ will coincide with and $\therefore$ is equal to $EF$, Ax. 8.

and $\triangle ABC$ .......................................... $\triangle DEF$,

and $\angle ABC$ .......................................... $\angle DEF$,

and $\angle ACB$ .......................................... $\angle DFE$.

Q. E. D.

NOTE I. *On the Method of Superposition.*

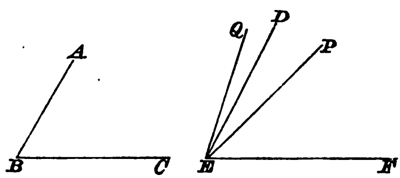
Two geometrical magnitudes are said, in accordance with Ax. VIII, to be *equal*, when they can be so placed, that the boundaries of the one coincide with the boundaries of the other.

Thus, two straight lines are equal, if they can be so placed that the points at their extremities coincide: and two angles are equal, if they can be so placed that their vertices coincide in position and their arms in direction: and two triangles are equal, if they can be so placed that their sides coincide in direction and magnitude.

In the application of the test of equality by this *Method of Superposition*, we assume that an angle or a triangle may be moved from one place, turned over, and put down in another place, without altering the relative positions of its boundaries.

We also assume that if one part of a straight line coincide with one part of another straight line, the other parts of the lines also coincide in direction; or, that straight lines, which coincide in two points, coincide when produced.

The method of Superposition enables us also to compare magnitudes of the same kind that are unequal. For example, suppose $ABC$ and $DEF$ to be two given angles.




Suppose the arm $BC$ to be placed on the arm $EF$, and the vertex $B$ on the vertex $E$.

Then, if the arm $BA$ coincide in direction with the arm $ED$, the angle $ABC$ is equal to $DEF$.

If $BA$ fall between $ED$ and $EF$ in the direction $EP$, $ABC$ is less than $DEF$.

If $BA$ fall in the direction $EQ$ so that $ED$ is between $EQ$ and $EF$, $ABC$ is greater than $DEF$.

## NOTE II. *On the Conditions of Equality of two Triangles.*

A Triangle is composed of six parts, three sides and three angles.

There are four cases in which Euclid proves that two triangles are equal in all respects; viz. when the following parts are equal in the two triangles.

1. Two sides and the angle between them. I. 4.
2. Two angles and the side between them. I. 26.
3. The three sides of each. I. 8.
4. Two angles and the side opposite one of them. I. 26.

The Propositions, in which these cases are proved, are the most important in our First Section.

The first case we have proved in Prop. IV.

Availing ourselves of the method of superposition, we can prove Cases 2 and 3 by a process more simple than that employed by Euclid, and with the further advantage of bringing them into closer connection with Case 1. We shall therefore give three Propositions, which we designate A, B, and C, in the place of Euclid's Props. V, VI, VII, VIII.

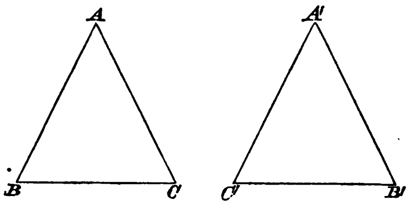
The displaced Propositions will be found at the end of this treatise, on pp. 108—112.

Our Proposition A corresponds with Euclid I. 5.
.................... B .............................. I. 26, first part.
.................... C .............................. I. 8.

## PROP. A. THEOREM.

*If two sides of a triangle be equal, the angles opposite those sides must also be equal.*

Fig. 1. Fig. 2.




In the isosceles triangle $ABC$, let $AC = AB$. (fig. 1.)

*Then must* $\angle ABC = \angle ACB$.

Imagine the $\triangle ABC$ to be taken up, turned round, and set down again in a reversed position as in fig. 2, and designate the angular points $A'$, $B'$, $C'$.

Then in $\triangle$s $ABC$, $A'C'B'$,

$\because AB = A'C'$, and $AC = A'B'$, and $\angle BAC = \angle C'A'B'$,

$\therefore \angle ABC = \angle A'C'B'$. I. 4.

But $\angle A'C'B' = \angle ACB$;

$\therefore \angle ABC = \angle ACB$. Ax. 1.

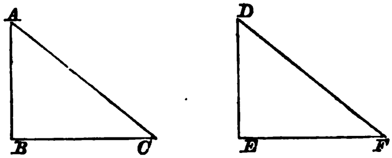
Q. E. D.

COR. Hence every equilateral triangle is also equiangular.

NOTE. When one side of a triangle is distinguished from the other sides by being called the *Base*, the angular point opposite to that side is called the *Vertex* of the triangle.

## Proposition B. Theorem.

*If two triangles have two angles of the one equal to two angles of the other, each to each, and the sides adjacent to the equal angles in each also equal; then must the triangles be equal in all respects.*




In $\triangle$s $ABC$, $DEF$,

let $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$, and $BC = EF$.

*Then must* $AB = DE$, *and* $AC = DF$, *and* $\angle BAC = \angle EDF$.

For if $\triangle DEF$ be applied to $\triangle ABC$, so that $E$ coincides with $B$, and $EF$ falls on $BC$;

then $\because EF = BC$, $\therefore F$ will coincide with $C$;

and $\because \angle DEF = \angle ABC$, $\therefore ED$ will fall on $BA$;

$\therefore D$ will fall on $BA$ or $BA$ produced.

Again, $\because \angle DFE = \angle ACB$, $\therefore FD$ will fall on $CA$;

$\therefore D$ will fall on $CA$ or $CA$ produced.

$\therefore D$ must coincide with $A$, the only pt. common to $BA$ and $CA$.

$\therefore DE$ will coincide with and $\therefore$ is equal to $AB$,

and $DF$ ............................................... $AC$,

and $\angle EDF$ ........................................... $\angle BAC$;

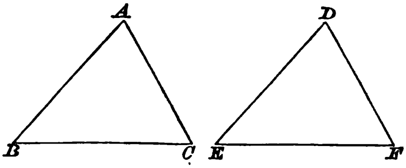
and $\therefore$ the triangles are equal in all respects. Q. E. D.

Cor. Hence, by a process like that in Prop. A, we can prove the following theorem:

*If two angles of a triangle be equal, the sides which subtend them are also equal.* (Eucl. I. 6.)

PROP. C. THEOREM.

*If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles must be equal in all respects.*

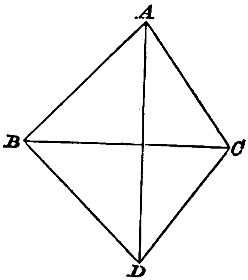



Let the three sides of the $\triangle$s $ABC$, $DEF$ be equal, each to each, that is, $AB=DE$, $AC=DF$, and $BC=EF$.

*Then must the triangles be equal in all respects.*

Imagine the $\triangle DEF$ to be turned over and applied to the $\triangle ABC$, in such a way that $EF$ coincides with $BC$, and the vertex $D$ falls on the side of $BC$ opposite to the side on which $A$ falls; and join $AD$.

CASE I. When $AD$ passes through $BC$.




Then in $\triangle ABD$, $\because BD=BA$, $\therefore \angle BAD = \angle BDA$. Prop. A.

And in $\triangle ACD$, $\because CD=CA$, $\therefore \angle CAD = \angle CDA$. Prop. A.

$\therefore$ sum of $\angle$s $BAD$, $CAD$ = sum of $\angle$s $BDA$, $CDA$, Ax. 2.

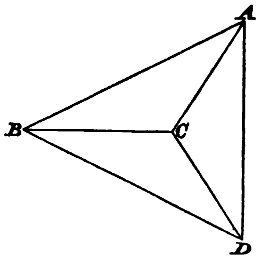
that is, $\angle BAC = \angle BDC$.

Hence we see, referring to the original triangles, that

$$\angle BAC = \angle EDF.$$

$\therefore$, by Prop. 4, the triangles are equal in all respects.

**Case II.** When the line joining the vertices does not pass through $BC$.




Then in $\triangle ABD$, $\because BD=BA$, $\therefore \angle BAD = \angle BDA$.

And in $\triangle ACD$, $\because CD=CA$, $\therefore \angle CAD = \angle CDA$.

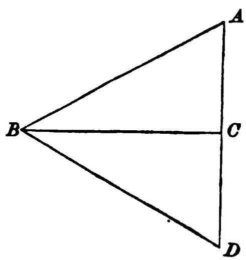
Hence since the whole angles $BAD$, $BDA$ are equal,

and parts of these $CAD$, $CDA$ are equal,

$\therefore$ the remainders $BAC$, $BDC$ are equal. Ax. 3.

Then, as in Case I., the equality of the original triangles may be proved.

**Case III.** When $AC$ and $CD$ are in the same straight line.



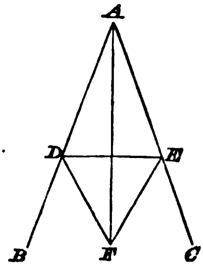

Then in $\triangle ABD$, $\because BD=BA$, $\therefore \angle BAD = \angle BDA$,

that is, $\angle BAC = \angle BDC$.

Then, as in Case I., the equality of the original triangles may be proved. Q. E. D.

## Proposition IX.

*To bisect a given angle.*



Let $BAC$ be the given angle.

*It is required to bisect* $\angle BAC$.

In $AB$ take any pt. $D$.

In $AC$ make $AE = AD$, and join $DE$.

On $DE$, on the side remote from $A$, describe an equilat. $\triangle DFE$.

Join $AF$. Then $AF$ will bisect $\angle BAC$.

For in $\triangle$s $AFD$, $AFE$,

$\because AD = AE$, and $AF$ is common, and $FD = FE$,

$\therefore \angle DAF = \angle EAF$, I. C.

that is, $\angle BAC$ is bisected by $AF$.

Q.E.F.

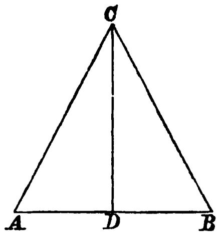
Ex. 1. Shew that we can prove this Proposition by means of Prop. IV. and Prop. A., without applying Prop. C.

Ex. 2. If the equilateral triangle, employed in the construction, be described with its vertex towards the given angle; shew that there is one case in which the construction will fail, and two in which it will hold good.

Note. The line dividing an angle into two equal parts is called the Bisector of the angle.

## Proposition X. Problem.

*To bisect a given finite straight line.*




Let $AB$ be the given st. line.

*It is required to bisect $AB$.*

On $AB$ describe an equilat. $\triangle ACB$.

Bisect $\angle ACB$ by the st. line $CD$ meeting $AB$ in $D$; I. 9.

then $AB$ shall be bisected in $D$.

For in $\triangle$s $ACD$, $BCD$,

$\because AC=BC$, and $CD$ is common, and $\angle ACD = \angle BCD$,

$\therefore AD=BD$; I. 4.

$\therefore AB$ is bisected in $D$.

Q. E. F.

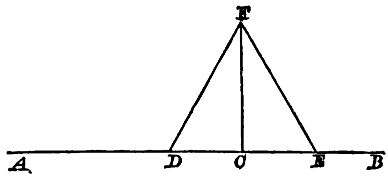
Ex. 1. The straight line, drawn to bisect the vertical angle of an isosceles triangle, also bisects the base.

Ex. 2. The straight line, drawn from the vertex of an isosceles triangle to bisect the base, also bisects the vertical angle.

Ex. 3. Produce a given finite straight line to a point, such that the part produced may be one-third of the line, which is made up of the whole and the part produced.

## Proposition XI. Problem.

*To draw a straight line at right angles to a given straight line from a given point in the same.*




Let $AB$ be the given st. line, and $C$ a given pt. in it.

*It is required to draw from C a st. line ⊥ to AB.*

Take any pt. $D$ in $AC$, and in $CB$ make $CE=CD$.

On $DE$ describe an equilat. △ $DFE$.

Join $FC$. $FC$ shall be ⊥ to $AB$.

For in △s $DCF$, $ECF$,

∵ $DC=CE$, and $CF$ is common, and $FD=FE$,

∴ $\angle DCF = \angle ECF$; I. c.

and these are adjacent angles,

∴ each of them is a rt. ∠; Def. 9.

and $FC$ is ⊥ to $AB$.

Q. E. F.

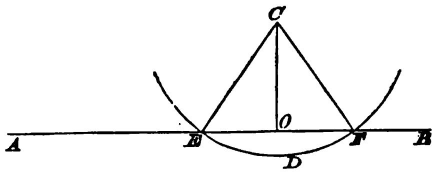
Ex. 1. Shew that in the diagram of Prop. IX. $AF$ and $ED$ intersect each other at right angles, and that $ED$ is bisected by $AF$.

Ex. 2. If $O$ be the point in which two lines, bisecting $AB$ and $AC$, two sides of an equilateral triangle, at right angles, meet; shew that $OA$, $OB$, $OC$ are all equal.

Ex. 3. Shew that Prop. XI. is a particular case of Prop. IX.

## Proposition XII. Problem.

*To draw a straight line perpendicular to a given straight line of an unlimited length from a given point without it.*




Let $AB$ be the given st. line of unlimited length; $C$ the given pt. without it.

*It is required to draw from C a st. line* $\perp$ *to AB.*

Take any pt. $D$ on the other side of $AB$.

With centre $C$ and distance $CD$ describe a $\odot$ cutting $AB$ in $E$ and $F$.

Bisect $EF$ in $O$, and join $CE$, $CO$, $CF$.

Then $CO$ shall be $\perp$ to $AB$.

For in $\triangle$s $COE$, $COF$,

$\because$ $EO = FO$, and $CO$ is common, and $CE = CF$,

$\therefore$ $\angle COE = \angle COF$; I. C.

$\therefore$ $CO$ is $\perp$ to $AB$. Def 9.

Q.E.F.

Ex. 1. If the straight line were not of unlimited length, how might the construction fail?

Ex. 2. If in a triangle the perpendicular from the vertex on the base bisect the base, the triangle is isosceles.

Ex. 3. The lines drawn from the angular points of an equilateral triangle to the middle points of the opposite sides are equal.

*Miscellaneous Exercises on Props. I. to XII.*

1. Draw a figure for Prop. II. for the case when the given point $A$ is

(α) below the line $BC$ and to the right of it,

(β) below the line $BC$ and to the left of it.

2. Divide a given angle into four equal parts.

3. The angles $B$, $C$, at the base of an isosceles triangle, are bisected by the straight lines $BD$, $CD$, meeting in $D$; shew that $BDC$ is an isosceles triangle.

4. $D$, $E$, $F$ are points taken in the sides $BC$, $CA$, $AB$, of an equilateral triangle, so that $BD=CE=AF$. Shew that the triangle $DEF$ is equilateral.

5. In a given straight line find a point equidistant from two given points; 1st, on the same side of it; 2nd, on opposite sides of it.

6. $ABC$ is any triangle. In $BA$, or $BA$ produced, find a point $D$ such that $BD=CD$.

7. The equal sides $AB$, $AC$, of an isosceles triangle $ABC$, are produced to points $F$ and $G$, so that $AF=AG$. $BG$ and $CF$ are joined, and $H$ is the point of their intersection. Prove that $BH=CH$, and also that the angle at $A$ is bisected by $AH$.

8. $BAC$, $BDC$ are isosceles triangles, standing on opposite sides of the same base $BC$. Prove that the straight line from $A$ to $D$ bisects $BC$ at right angles.

9. In how many directions may the line $AE$ be drawn in Prop. III.?

10. The two sides of a triangle being produced, if the angles on the other side of the base be equal, shew that the triangle is isosceles.

11. $ABC$, $ABD$ are two triangles on the same base $AB$ and on the same side of it, the vertex of each triangle being outside the other. If $AC=AD$, shew that $BC$ cannot $=BD$.

12. From $C$ any point in a straight line $AB$, $CD$ is drawn at right angles to $AB$, meeting a circle described with centre $A$ and distance $AB$ in $D$; and from $AD$, $AE$ is cut off $=AC$: shew that $AEB$ is a right angle.

## PROPOSITION XIII. THEOREM.

***The angles which one straight line makes with another upon one side of it are either two right angles, or together equal to two right angles.***

Fig. 1.

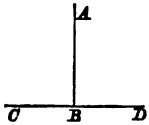
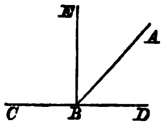



Fig. 2.




Let $AB$ make with $CD$ upon one side of it the $\angle$s $ABC$, $ABD$.

*Then must these be either two rt. $\angle$s, or together equal to two rt. $\angle$s.*

First, if $\angle ABC = \angle ABD$, as in fig. 1,
each of them is a rt. $\angle$. Def. 9.

Secondly, if $\angle ABC$ be not $= \angle ABD$, as in fig. 2,
from $B$ draw $BE \perp$ to $CD$. I. 11.

Then sum of $\angle$s $ABC$, $ABD$ = sum of $\angle$s $EBC$, $EBA$, $ABD$,
and sum of $\angle$s $EBC$, $EBD$ = sum of $\angle$s $EBC$, $EBA$, $ABD$;
$\therefore$ sum of $\angle$s $ABC$, $ABD$ = sum of $\angle$s $EBC$, $EBD$;
$\therefore$ sum of $\angle$s $ABC$, $ABD$ = sum of a rt. $\angle$ and a rt. $\angle$;
$\therefore$ $\angle$s $ABC$, $ABD$ are together = two rt. $\angle$s.

Q.E.D.

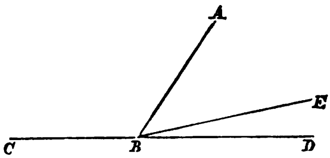
Ex. Straight lines drawn connecting the opposite angular points of a quadrilateral figure intersect each other in $O$. Shew that the angles at $O$ are together equal to four right angles.

NOTE. (i) If two angles together make up a right angle, each is called the COMPLEMENT of the other. Thus, in fig. 2, $\angle ABD$ is the complement of $\angle ABE$.

(ii) If two angles together make up two right angles, each is called the SUPPLEMENT of the other. Thus, in both figures, $\angle ABD$ is the supplement of $\angle ABC$.

### Proposition XIV. Theorem.

*If, at a point in a straight line, two other straight lines, upon the opposite sides of it, make the adjacent angles together equal to two right angles, these two straight lines must be in one and the same straight line.*




At the pt. $B$ in the st. line $AB$ let the st. lines $BC$, $BD$, on opposite sides of $AB$, make $\angle$ s $ABC$, $ABD$ together = two rt. angles.

*Then $BD$ must be in the same st. line with $BC$.*

For if not, let $BE$ be in the same st. line with $BC$.

Then $\angle$ s $ABC$, $ABE$ together = two rt. $\angle$ s. I. 13.

And $\angle$ s $ABC$, $ABD$ together = two rt. $\angle$ s. Hyp.

$\therefore$ sum of $\angle$ s $ABC$, $ABE$ = sum of $\angle$ s $ABC$, $ABD$.

Take away from each of these equals the $\angle$ $ABC$;

$\therefore$ $\angle$ $ABE$ = $\angle$ $ABD$, Ax. 3.

that is, the less = the greater; which is absurd,

$\therefore$ $BE$ is not in the same st. line with $BC$.

Similarly it may be shewn that no other line but $BD$ is in the same st. line with $BC$.

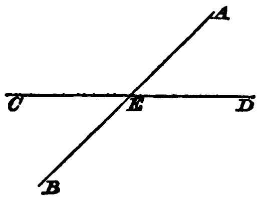
$\therefore$ $BD$ is in the same st. line with $BC$.

Q. E. D.

Ex. Shew the necessity of the words *the opposite sides* in the enunciation.

## PROPOSITION XV. THEOREM.

***If two straight lines cut one another, the vertically opposite angles must be equal.***




Let the st. lines $AB$, $CD$ cut one another in the pt. $E$.

*Then must* $\angle AEC = \angle BED$ *and* $\angle AED = \angle BEC$.

For $\because$ $AE$ meets $CD$,

$\therefore$ sum of $\angle$s $AEC$, $AED$ = two rt. $\angle$s. I. 13.

And $\because$ $DE$ meets $AB$,

$\therefore$ sum of $\angle$s $BED$, $AED$ = two rt. $\angle$s; I. 13.

$\therefore$ sum of $\angle$s $AEC$, $AED$ = sum of $\angle$s $BED$, $AED$;

$\therefore$ $\angle AEC = \angle BED$. Ax. 3.

Similarly it may be shewn that $\angle AED = \angle BEC$.

Q. E. D.

COROLLARY I. From this it is manifest, that if two straight lines cut one another, the four angles, which they make at the point of intersection, are together equal to four right angles.

COROLLARY II. All the angles, made by any number of straight lines meeting in one point, are together equal to four right angles.

Ex. 1. Shew that the bisectors of $AED$ and $BEC$ are in the same straight line.

Ex. 2. Prove that $\angle AED$ is equal to the angle between two straight lines drawn at right angles from $E$ to $AE$ and $EC$, if both lie above $CD$.

Ex. 3. If $AB$, $CD$ bisect each other in $E$; shew that the triangles $AED$, $BEC$ are equal in all respects.

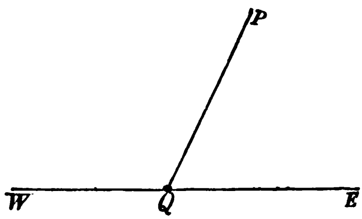
NOTE III. *On Euclid's definition of an Angle.*

Euclid directs us to regard an angle as the inclination of two straight lines to each other, which meet, *but are not in the same straight line.*

Thus he does not recognize the existence of a single angle equal in magnitude to two right angles.

Euclid's definition may be extended with advantage in the following terms:—

DEF. Let $WQE$ be a fixed straight line, and $QP$ a line which revolves about the fixed point $Q$, and which at first coincides with $QE$.




Then, when $QP$ has reached the position represented in the diagram, we say that it has described the angle $EQP$.

When $QP$ has revolved so far as to coincide with $QW$, we say that it has described an angle *equal to two right angles.*

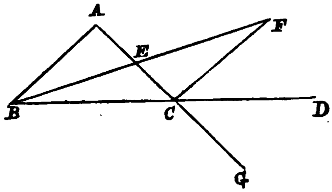
Hence we may obtain an easy proof of Prop. XIII.; for whatever the position of $PQ$ may be, the angles which it makes with $WE$ are together equal to two right angles.

Again, in Prop. XV. it is evident that $\angle AED = \angle BEC$, since each has the same supplementary $\angle AEC$.

We shall shew hereafter how this definition may be extended, so as to embrace angles *greater than two right angles.*

## PROPOSITION XVI. THEOREM.

*If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.*




Let the side $BC$ of $\triangle ABC$ be produced to $D$.

*Then must $\angle ACD$ be greater than either $\angle CAB$ or $\angle ABC$.*

Bisect $AC$ in $E$, and join $BE$.

Produce $BE$ to $F$, making $EF=BE$, and join $FC$.

Then in $\triangle$s $BEA$, $FEC$,

$\because BE=FE$, and $EA=EC$, and $\angle BEA=\angle FEC$, I. 15.

$\therefore \angle ECF=\angle EAB$. I. 4.

Now $\angle ACD$ is greater than $\angle ECF$;

$\therefore \angle ACD$ is greater than $\angle EAB$,

that is, $\angle ACD$ is greater than $\angle CAB$.

Similarly, if $AC$ be produced to $G$ it may be shewn that

$\angle BCG$ is greater than $\angle ABC$:

and $\angle BCG=\angle ACD$; I. 15.

$\therefore \angle ACD$ is greater than $\angle ABC$.

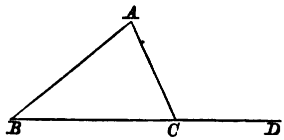
Q.E.D.

Ex. 1. From the same point there cannot be drawn more than two equal straight lines to meet a given straight line.

Ex. 2. If, from any point, a straight line be drawn to a given straight line making with it an acute and an obtuse angle, and if, from the same point, a perpendicular be drawn to the given line; the perpendicular will fall on the side of the acute angle.

PROPOSITION XVII. THEOREM.

*Any two angles of a triangle are together less than two right angles.*




Let $ABC$ be any $\triangle$.

*Then must any two of its $\angle$s be together less than two rt. $\angle$s.*

Produce $BC$ to $D$.

Then $\angle ACD$ is greater than $\angle ABC$. I. 16.

$\therefore$ $\angle$s $ACD$, $ACB$ are together greater than $\angle$s $ABC$, $ACB$.

But $\angle$s $ACD$, $ACB$ together = two rt. $\angle$s. I. 13.

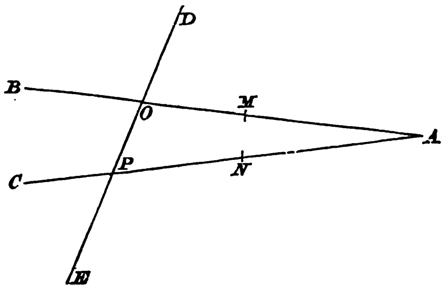
$\therefore$ $\angle$s $ABC$, $ACB$ are together less than two rt. $\angle$s.

Similarly it may be shewn that $\angle$s $ABC$, $BAC$ and also that $\angle$s $BAC$, $ACB$ are together less than two rt. $\angle$s.

Q. E. D.

NOTE IV. *On the Sixth Postulate.*

We learn from Prop. 17 that if two straight lines $BM$ and $CN$, which meet in $A$, are met by another straight line $DE$ in the points $O$, $P$,




the angles $MOP$ and $NPO$ are together less than two right angles.

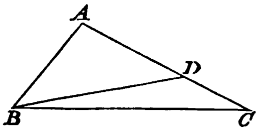
The sixth postulate asserts that if a line $DE$ meeting two other lines $BM$, $CN$ makes $MOP$, $NPO$, the two interior angles on the same side of it, together less than two right angles, $BM$ and $CN$ shall meet if produced on the same side of $DE$ on which are the angles $MOP$ and $NPO$.

Thus Postulate 6 is the converse of Proposition XVII.

We shall explain hereafter why the Postulate cannot be proved as readily as the Proposition.

## PROPOSITION XVIII. THEOREM.

*If one side of a triangle be greater than a second, the angle opposite the first must be greater than that opposite the second.*



In $\triangle ABC$, let side $AC$ be greater than $AB$.

*Then must $\angle ABC$ be greater than $\angle ACB$.*

From $AC$ cut off $AD = AB$, and join $BD$.

Then $\because AB = AD$,

$\therefore \angle ADB = \angle ABD$. I. A.

And $\because CD$, a side of $\triangle BDC$, is produced to $A$,

$\therefore \angle ADB$ is greater than $\angle ACB$; I. 16.

$\therefore$ also $\angle ABD$ is greater than $\angle ACB$.

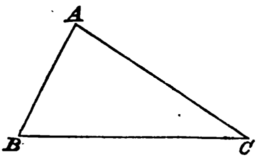
Much more is $\angle ABC$ greater than $\angle ACB$.

Q. E. D.

Ex. Shew that if two angles of a triangle be equal the sides which subtend them are equal also (Eucl. I. 6).

## PROPOSITION XIX. THEOREM.

*If one angle of a triangle be greater than a second, the side opposite the first must be greater than that opposite the second.*



In $\triangle ABC$, let $\angle ABC$ be greater than $\angle ACB$.

*Then must AC be greater than AB.*

For if $AC$ be not greater than $AB$,

$AC$ must either $= AB$, or be less than $AB$.

Now $AC$ cannot $= AB$; for then I. A.

$\angle ABC$ would $= \angle ACB$, which is not the case.

And $AC$ cannot be less than $AB$, for then I. 18.

$\angle ABC$ would be less than $\angle ACB$, which is not the case;

$\therefore$ $AC$ is greater than $AB$.

Q. E. D.

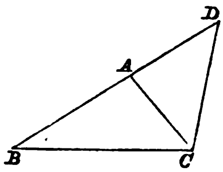
Ex. 1. In an obtuse-angled triangle, the greatest side is opposite the obtuse angle.

Ex. 2. $BC$, the base of an isosceles triangle $BAC$, is produced to any point $D$; shew that $AD$ is greater than $AB$.

Ex. 3. The perpendicular is the shortest straight line, which can be drawn from a given point to a given straight line; and of others, that which is nearer to the perpendicular is less than the more remote.

## PROPOSITION XX. THEOREM.

***Any two sides of a triangle are together greater than the third side.***



Let $ABC$ be a $\triangle$.

***Then any two of its sides must be together greater than the third side.***

Produce $BA$ to $D$, making $AD = AC$, and join $DC$.

Then $\because AD = AC$,

$\therefore \angle ACD = \angle ADC$, that is, $\angle BDC$. I. A.

Now $\angle BCD$ is greater than $\angle ACD$;

$\therefore \angle BCD$ is also greater than $\angle BDC$;

$\therefore BD$ is greater than $BC$. I. 19.

But $BD = BA$ and $AD$ together;

that is, $BD = BA$ and $AC$ together;

$\therefore BA$ and $AC$ together are greater than $BC$.

Similarly it may be shewn that

$AB$ and $BC$ together are greater than $AC$,

and $BC$ and $CA$ ................................ $AB$.

Q.E.D.

Ex. 1. Prove that any three sides of a quadrilateral figure are together greater than the fourth side.

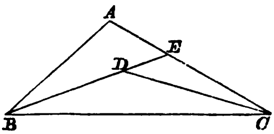
Ex. 2. Shew that any side of a triangle is greater than the difference between the other two sides.

Ex. 3. Prove that the sum of the distances of any point from the angular points of a quadrilateral is greater than half the perimeter of the quadrilateral.

Ex. 4. If one side of a triangle be bisected, the sum of the two other sides shall be more than double of the line joining the vertex and the point of bisection.

## Proposition XXI. Theorem.

*If, from the ends of the side of a triangle, there be drawn two straight lines to a point within the triangle; these will be less than the other sides of the triangle, but will contain a greater angle.*



Let $ABC$ be a $\triangle$, and from $D$, a pt. in the $\triangle$, draw st. lines to $B$ and $C$.

*Then will $BD$, $DC$ together be less than $BA$, $AC$, but $\angle BDC$ will be greater than $\angle BAC$.*

Produce $BD$ to meet $AC$ in $E$.

Then $\because$ $BA$, $AE$ are together greater than $BE$, I. 20.

add to each $EC$.

Then $BA$, $AC$ are together greater than $BE$, $EC$.

Again, $\because$ $DE$, $EC$ are together greater than $DC$,

add to each $BD$.

Then $BE$, $EC$ are together greater than $BD$, $DC$.

And it has been shewn that $BA$, $AC$ are together greater than $BE$, $EC$;

$\therefore$ $BA$, $AC$ are together greater than $BD$, $DC$.

Next, $\because$ $\angle BDC$ is greater than $\angle DEC$, I. 16.

and $\angle DEC$ is greater than $\angle BAC$, I. 16.

$\therefore$ $\angle BDC$ is greater than $\angle BAC$.

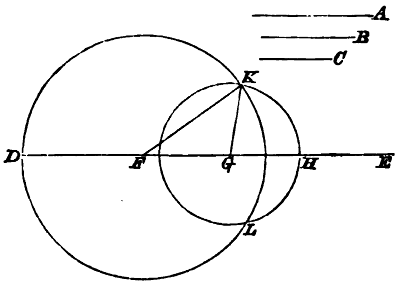
Q. E. D.

Ex. 1. Upon the base $AB$ of a triangle $ABC$ is described a quadrilateral figure $ADEB$, which is entirely within the triangle. Shew that the sides $AC$, $CB$ of the triangle are together greater than the sides $AD$, $DE$, $EB$ of the quadrilateral.

Ex. 2. Shew that the sum of the straight lines, joining the angles of a triangle with a point within the triangle, is less than the perimeter of the triangle, and greater than half the perimeter.

## Proposition XXII. Problem.

*To make a triangle, of which the sides shall be equal to three given straight lines, any two of which are greater than the third.*




Let $A$, $B$, $C$ be the three given lines, any two of which are greater than the third.

*It is reqd. to make a* $\triangle$ *having its sides* $= A$, $B$, $C$ *respectively.*

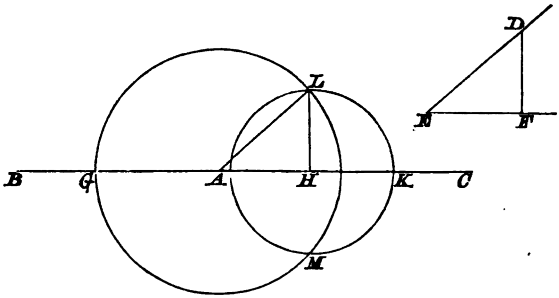
Take a st. line $DE$ of unlimited length.
In $DE$ make $DF = A$, $FG = B$, and $GH = C$.
With centre $F$ and distance $FD$, describe $\odot DKL$.
With centre $G$ and distance $GH$, describe $\odot HKL$.
Join $FK$ and $GK$.
Then $\triangle KFG$ has its sides $= A, B, C$ respectively.
For $FK = FD$; Def. 13.
$\therefore FK = A$;
and $GK = GH$; Def. 13.
$\therefore GK = C$;
and $FG = B$;
$\therefore$ a $\triangle KFG$ has been described as reqd.

Q. E. F.

Ex. Draw an isosceles triangle having each of the equal sides double of the base.

## PROPOSITION XXIII. PROBLEM.

*At a given point in a given straight line, to make an angle equal to a given angle.*




Let $A$ be the given pt., $BC$ the given line, $DEF$ the given $\angle$.

*It is reqd. to make at pt. $A$ an angle $= \angle DEF$.*

In $ED$, $EF$ take any pts. $D$, $F$; and join $DF$.

In $AB$, produced if necessary, make $AG = DE$.

In $AC$, produced if necessary, make $AH = EF$.

In $HC$, produced if necessary, make $HK = FD$.

With centre $A$, and distance $AG$, describe $\odot GLM$.

With centre $H$, and distance $HK$, describe $\odot LKM$.

Join $AL$ and $HL$.

Then $\because LA = AG$, $\therefore LA = DE$

and $\because HL = HK$, $\therefore HL = FD$.

Then in $\triangle$s $LAH$, $DEF$,

$\because LA = DE$, and $AH = EF$, and $HL = FD$;

$\therefore \angle LAH = \angle DEF$. I. C.

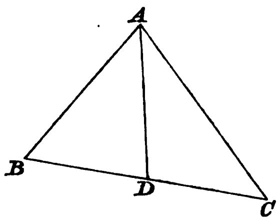
$\therefore$ an angle $LAH$ has been made at pt. $A$ as was reqd.

Q. E. F.

NOTE. We here give the proof of a theorem, necessary to the proof of Prop. XXIV. and applicable to several propositions in Book III.

### PROPOSITION D. THEOREM.

*Every straight line, drawn from the vertex of a triangle to the base, is less than the greater of the two sides, or than either, if they be equal.*




In the $\triangle ABC$, let the side $AC$ be not less than $AB$.

Take any pt. $D$ in $BC$, and join $AD$.

*Then must $AD$ be less than $AC$.*

For $\because$ $AC$ is not less than $AB$;

$\therefore$ $\angle ABD$ is not less than $\angle ACD$. I. A. and 18.

But $\angle ADC$ is greater than $\angle ABD$; I. 16.

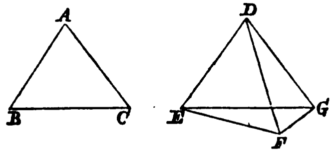
$\therefore$ $\angle ADC$ is greater than $\angle ACD$;

$\therefore$ $AC$ is greater than $AD$.

Q. E. D.

## Proposition XXIV. Theorem.

*If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle contained by the two sides of one of them greater than the angle contained by the two sides equal to them of the other; the base of that which has the greater angle must be greater than the base of the other.*




In the $\triangle$s $ABC, DEF$,
let $AB = DE$ and $AC = DF$,
and let $\angle BAC$ be greater than $\angle EDF$.
*Then must $BC$ be greater than $EF$.*

Of the two sides $DE, DF$ let $DE$ be not greater than $DF$*.
At pt. $D$ in st. line $ED$ make $\angle EDG = \angle BAC$, I. 23.
and make $DG = AC$ or $DF$, and join $EG, GF$.

Then $\because AB = DE$, and $AC = DG$, and $\angle BAC = \angle EDG$,
$\therefore BC = EG$. I. 4.

Again, $\because DG = DF$,
$\therefore \angle DFG = \angle DGF$; I. A.
$\therefore \angle EFG$ is greater than $\angle DGF$;
much more then $\angle EFG$ is greater than $\angle EGF$;
$\therefore EG$ is greater than $EF$. I. 19.

But $EG = BC$;
$\therefore BC$ is greater than $EF$.

Q. E. D.

* This line was added by Simson to obviate a defect in Euclid's proof. *Without* this condition, three distinct cases must be discussed. *With* the condition, we can prove that $F$ must lie below $EG$.

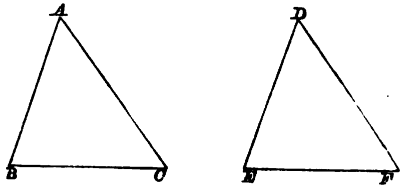
For since $DF$ is not less than $DE$, and $DG$ is drawn equal to $DF$, $DG$ is not less than $DE$.

Hence, by Prop. D, any line drawn from $D$ to meet $EG$ is less than $DG$, and therefore $DF$, being equal to $DG$, must extend beyond $EG$.

Another method of proving the Proposition is given at the end of this treatise, p. 113.

## PROPOSITION XXV. THEOREM.

***If two triangles have two sides of the one equal to two sides of the other, each to each, but the base of the one greater than the base of the other; the angle also, contained by the sides of that which has the greater base, must be greater than the angle contained by the sides equal to them of the other.***



In the $\triangle$s $ABC$, $DEF$,

let $AB = DE$ and $AC = DF$,

and let $BC$ be greater than $EF$.

*Then must* $\angle BAC$ *be greater than* $\angle EDF$.

For $\angle BAC$ is greater than, equal to, or less than $\angle EDF$.

Now $\angle BAC$ cannot $= \angle EDF$,

for then, by I. 4, $BC$ would $= EF$; which is not the case.

And $\angle BAC$ cannot be less than $\angle EDF$,

for then, by I. 24, $BC$ would be less than $EF$; which is not the case;

$\therefore$ $\angle BAC$ must be greater than $\angle EDF$.

Q. E. D.

NOTE. In Prop. XXVI. Euclid includes two cases, in which two triangles are equal in all respects; viz. when the following parts are equal in the two triangles:

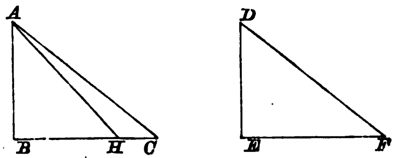
1. Two angles and the side between them.
2. Two angles and the side opposite one of them.

Of these we have already proved the first case, in Prop. B, so that we have only the second case left, to form the subject of our Prop. XXVI., which we shall prove by the method of superposition.

Euclid's proof of his 26th proposition is given at the end of this treatise, pp. 114, 115.

PROPOSITION XXVI. THEOREM.

*If two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, those sides being opposite to equal angles in each; then must the triangles be equal in all respects.*



In $\triangle$s $ABC$, $DEF$,

let $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$, and $AB = DE$.

*Then must* $BC = EF$, *and* $AC = DF$, *and* $\angle BAC = \angle EDF$.

Suppose $\triangle DEF$ to be applied to $\triangle ABC$,

so that $D$ coincides with $A$, and $DE$ falls on $AB$.

Then $\because DE = AB$, $\therefore E$ will coincide with $B$;

and $\because \angle DEF = \angle ABC$, $\therefore EF$ will fall on $BC$.

Then must $F$ coincide with $C$: for, if not,

let $F$ fall *between* $B$ and $C$, at the pt. $H$. Join $AH$.

Then $\because \angle AHB = \angle DFE$, I. 4.

$\therefore \angle AHB = \angle ACB$,

the ext$^{r}$. $\angle$ = the int$^{r}$. and opposite $\angle$, which is impossible.

$\therefore F$ does not fall between $B$ and $C$.

Similarly, it may be shewn that $F$ does not fall on $BC$ produced.

$\therefore F$ coincides with $C$, and $\therefore BC = EF$;

$\therefore AC = DF$, and $\angle BAC = \angle EDF$, I. 4.

and $\therefore$ the triangles are equal in all respects.

Q. E. D.

## MISCELLANEOUS EXERCISES ON PROPS. I. TO XXVI.

1. $M$ is the middle point of the base $BC$ of an isosceles triangle $ABC$, and $N$ is a point in $AC$. Shew that the difference between $MB$ and $MN$ is less than that between $AB$ and $AN$.

2. $ABC$ is a triangle, and the angle at $A$ is bisected by a straight line which meets $BC$ at $D$: shew that $BA$ is greater than $BD$, and $CA$ greater than $CD$.

3. $AB$, $AC$ are straight lines meeting in $A$, and $D$ is a given point. Draw through $D$ a straight line cutting off equal parts from $AB$, $AC$.

4. Draw a straight line through a given point, to make equal angles with two given straight lines which meet.

5. A given angle $BAC$ is bisected; if $CA$ be produced to $G$ and the angle $BAG$ bisected, the two bisecting lines are at right angles.

6. Two straight lines are drawn to the base of a triangle from the vertex, one bisecting the vertical angle, and the other bisecting the base. Prove that the latter is the greater of the two lines.

7. Shew that Prop. XVII. may be proved without producing a side of the triangle.

8. Shew that Prop. XVIII. may be proved by means of the following construction: cut off $AD = AB$, draw $AE$, bisecting $\angle BAC$ and meeting $BC$ in $E$, and join $DE$.

9. Shew that Prop. XX. can be proved, without producing one of the sides of the triangle, by bisecting one of the angles.

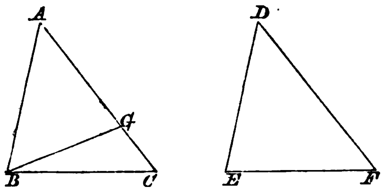
10. Given two angles of a triangle and the side adjacent to them, construct the triangle.

11. Shew that the perpendiculars, let fall on two sides of a triangle from any point in the straight line bisecting the angle contained by the two sides, are equal.

We conclude Section I. with the proof (omitted by Euclid) of another case in which two triangles are equal in all respects.

### Proposition E. Theorem.

*If two triangles have one angle of the one equal to one angle of the other, and the sides about a second angle in each equal; then, if the third angles in each be both acute, both obtuse, or if one of them be a right angle, the triangles are equal in all respects.*



In the $\triangle$s $ABC$, $DEF$, let $\angle BAC = \angle EDF$, $AB = DE$, $BC = EF$, and let $\angle$s $ACB$, $DFE$ be both acute, both obtuse, or let one of them be a right angle.

*Then must* $\triangle$*s* $ABC$, $DEF$ *be equal in all respects.*

For if $AC$ be not $= DF$, make $AG = DF$; and join $BG$.

Then in $\triangle$s $BAG$, $EDF$,

$\because BA = ED$, and $AG = DF$, and $\angle BAG = \angle EDF$,

$\therefore BG = EF$ and $\angle AGB = \angle DFE$.

But $BC = EF$, and $\therefore BG = BC$;

$\therefore \angle BCG = \angle BGC$.

First, let $\angle ACB$ and $\angle DFE$ be both acute,

then $\angle AGB$ is acute, and $\therefore \angle BGC$ is obtuse; I. 13.

$\therefore \angle BCG$ is obtuse, which is contrary to the hypothesis.

Next, let $\angle ACB$ and $\angle DFE$ be both obtuse,

then $\angle AGB$ is obtuse, and $\therefore \angle BGC$ is acute;

$\therefore \angle BCG$ is acute, which is contrary to the hypothesis.

Lastly, let one of the third angles $ACB$, $DFE$ be a right angle.

If $\angle ACB$ be a rt. $\angle$,

then $\angle BGC$ is also a rt. $\angle$;

$\therefore$ $\angle$s $BCG$, $BGC$ together = two rt. $\angle$s, which is impossible. I. 17.

Again, if $\angle DFE$ be a rt. $\angle$,

then $\angle AGB$ is a rt. $\angle$, and $\therefore$ $\angle BGC$ is a rt. $\angle$.

Hence $\angle BCG$ is also a rt. $\angle$,

$\therefore$ $\angle$s $BCG$, $BGC$ together = two rt. $\angle$s, which is impossible. I. 17.

Hence $AC$ is equal to $DF$,

and the $\triangle$s $ABC$, $DEF$ are equal in all respects.

Q. E. D.

Cor. From the first case of this proposition we deduce the following important theorem:

*If two right-angled triangles have the hypotenuse and one side of the one equal respectively to the hypotenuse and one side of the other, the triangles are equal in all respects.*

Note. In the enunciation of Prop. E, if, instead of the words *if one of them be a right angle*, we put the words *both right angles*, this case of the proposition would be identical with our I. 26.

# SECTION II.

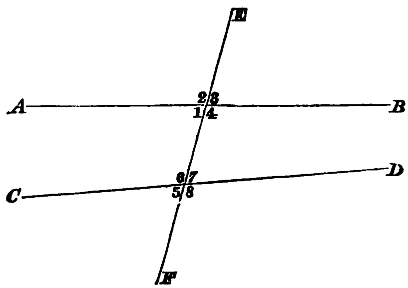
## *The Theory of Parallel Lines.*

### INTRODUCTION.

We have detached the Propositions, in which Euclid treats of Parallel Lines, from those which precede and follow them in the First Book; in order that the student may have a clearer notion of the difficulties attending this division of the subject, and of the way in which Euclid proposes to meet them.

We must first explain some technical terms used in this Section:

If a straight line $EF$ cut two other straight lines $AB$, $CD$, it makes with those lines eight angles, to which particular names are given.




The angles numbered 1, 4, 6, 7 are called *Interior* angles.

.......................... 2, 3, 5, 8 ............ *Exterior* .........

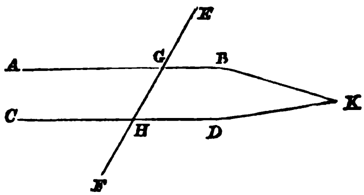
The angles marked 1 and 7 are called *alternate* angles.

The angles marked 4 and 6 are also called alternate angles.

The pairs of angles 1 and 5, 2 and 6, 4 and 8, 3 and 7 are called *corresponding* angles.

PROPOSITION XXVII. THEOREM.

*If a straight line, falling upon two other straight lines, make the alternate angles equal to one another; these two straight lines must be parallel.*




Let the st. line $EF$, falling on the st. lines $AB$, $CD$,

make the alternate $\angle$s $AGH$, $GHD$ equal.

*Then must $AB$ be $\parallel$ to $CD$.*

For if not, $AB$ and $CD$ will meet, if produced, either towards $B$, $D$, or towards $A$, $C$.

Let them be produced and meet towards $B$, $D$ in $K$.

Then $GHK$ is a $\triangle$;

and $\therefore \angle AGH$ is greater than $\angle GHD$. I. 16.

But $\angle AGH = \angle GHD$, Hyp.

which is impossible.

$\therefore AB$, $CD$ do not meet when produced towards $B$, $D$.

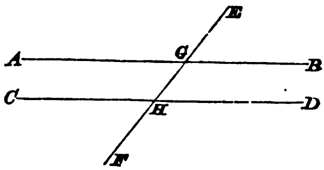
In like manner it may be shewn that they do not meet when produced towards $A$, $C$.

$\therefore AB$ and $CD$ are parallel. Def. 26.

Q. E. D.

## PROPOSITION XXVIII. THEOREM.

*If a straight line, falling upon two other straight lines, make the exterior angle equal to the interior and opposite upon the same side of the line, or make the interior angles upon the same side together equal to two right angles; the two straight lines are parallel to one another.*



Let the st. line $EF$, falling on st. lines $AB$, $CD$, make

I. $\angle\, EGB =$ corresponding $\angle\, GHD$; or

II. $\angle$ s $BGH$, $GHD$ together = two rt. $\angle$ s.

*Then, in either case, $AB$ must be $\parallel$ to $CD$.*

I. $\because\ \angle\, EGB$ is given $= \angle\, GHD$, Hyp.

and $\angle\, EGB$ is known to be $= \angle\, AGH$, I. 15.

$\therefore\ \angle\, AGH = \angle\, GHD$;

and these are alternate $\angle$ s;

$\therefore\ AB$ is $\parallel$ to $CD$. I. 27.

II. $\because\ \angle$ s $BGH$, $GHD$ together = two rt. $\angle$ s, Hyp.

and $\angle$ s $BGH$, $AGH$ together = two rt. $\angle$ s, I. 13.

$\therefore\ \angle$ s $BGH$, $AGH$ together $= \angle$ s $BGH$, $GHD$ together;

$\therefore\ \angle\, AGH = \angle\, GHD$;

$\therefore\ AB$ is $\parallel$ to $CD$. I. 27.

Q.E.D.

## NOTE V. *On the Sixth Postulate.*

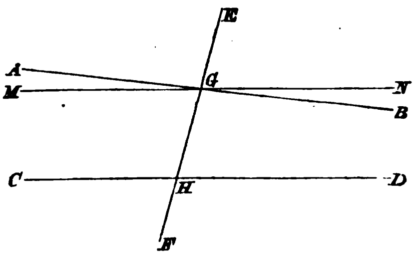
We explained in Note IV., page 32, that Euclid's Sixth Postulate is the converse of the 17th Proposition.

In the place of this Postulate many modern writers on Geometry propose, as more evident to the senses, the following Postulate:

"*Two straight lines which cut one another cannot* BOTH *be parallel to the same straight line.*"

If this be assumed, we can prove Post. 6, as a Theorem, thus:

Let the line *EF* falling on the lines *AB*, *CD* make the ∠s *BGH*, *GHD* together less than two rt. ∠s. Then must *AB*, *CD* meet when produced towards *B*, *D*.




For if not, suppose *AB* and *CD* to be parallel.

Then ∵ ∠s *AGH*, *BGH* together = two rt. ∠s, I. 13.

and ∠s *GHD*, *BGH* are together less than two rt. ∠s,

∴ ∠ *AGH* is greater than ∠ *GHD*.

Make ∠ *MGH* = ∠ *GHD*, and produce *MG* to *N*.

Then ∵ the alternate ∠s *MGH*, *GHD* are equal,

∴ *MN* is ∥ to *CD*. I. 27.

Thus two lines *MN*, *AB* which cut one another are both parallel to *CD*, which is impossible.

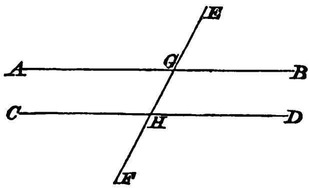
∴ *AB* and *CD* are not parallel.

It is also cle ır that they meet towards *B*, *D*, because *GB* lies between *GN* and *HD*.

Q. F. D.

### PROPOSITION XXIX. THEOREM.

*If a straight line fall upon two parallel straight lines, it makes the alternate angles equal to one another, and the exterior angle equal to the interior and opposite upon the same side; and likewise the two interior angles upon the same side together equal to two right angles.*




Let the st. line $EF$ fall on the parallel st. lines $AB$, $CD$.

*Then must*

*I.* $\angle AGH$ = alternate $\angle GHD$.

*II.* $\angle EGB$ = corresponding $\angle GHD$.

*III.* $\angle$*s* $BGH$, $GHD$ *together* = *two rt.* $\angle$*s*.

I. If $\angle AGH$ be not $= \angle GHD$, let $\angle AGH$ be greater than $\angle GHD$.

Add to each $\angle BGH$.

Then $\angle$s $AGH$, $BGH$ are together greater than $\angle$s $GHD$, $BGH$ together.

Now $\angle$s $AGH$, $BGH$ together = two rt. $\angle$s; I. 13.

$\therefore$ $\angle$s $GHD$, $BGH$ are together less than two rt. $\angle$s;

$\therefore$ $AB$ and $CD$ will meet if produced towards $B$, $D$. Post. 6.

But they cannot meet, $\because$ they are parallel;

$\therefore$ $\angle AGH$ is not greater than $\angle GHD$.

Similarly it may be shewn that

$\angle AGH$ is not less than $\angle GHD$;

$\therefore$ $\angle AGH = \angle GHD$.

II. $\because \angle EGB = \angle AGH$, I. 15.

and $\angle GHD$ has been proved $= \angle AGH$;

$\therefore \angle EGB = \angle GHD$.

III. $\because \angle GHD$ has been proved $= \angle EGB$,

$\therefore$ adding to each $\angle BGH$,

$\angle$s $BGH$, $GHD$ together $=$ $\angle$s $BGH$, $EGB$ together.

But $\angle$s $BGH$, $EGB$ together $=$ two rt. $\angle$s; I. 13.

$\therefore$ $\angle$s $BGH$, $GHD$ together $=$ two rt. $\angle$s.

Q. E. D.

## Exercises.

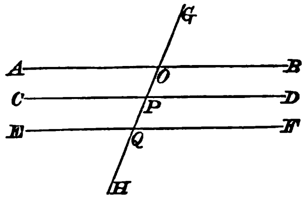
1. If through a point, equidistant from two parallel straight lines, two straight lines be drawn cutting the parallel straight lines; they will intercept equal portions of those lines.

2. If a straight line be drawn, bisecting one of the angles of a triangle, to meet the opposite side; the straight lines drawn from the point of section, parallel to the other sides and terminated by those sides, will be equal.

3. If any straight line joining two parallel straight lines be bisected, any other straight line, drawn through the point of bisection to meet the two lines, will be bisected in that point.

PROPOSITION XXX. THEOREM.

*Straight lines which are parallel to the same straight line are parallel to one another.*



Let the st. lines $AB$, $CD$ be each $\parallel$ to $EF$.

*Then must $AB$ be $\parallel$ to $CD$.*

Draw the st. line $GH$, cutting $AB$, $CD$, $EF$ in the pts. $O$, $P$, $Q$.

Then $\because$ $GH$ cuts the $\parallel$ lines $AB$, $EF$,

$\therefore$ $\angle AOP$ = alternate $\angle PQF$. I. 29.

And $\because$ $GH$ cuts the $\parallel$ lines $CD$, $EF$,

$\therefore$ extr. $\angle OPD$ = intr. $\angle PQF$; I. 29.

$\therefore$ $\angle AOP = \angle OPD$;

and these are alternate angles;

$\therefore$ $AB$ is $\parallel$ to $CD$. I. 27.

Q. E. D.

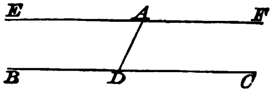
The following Theorems are important. They admit of easy proof, and are therefore left as Exercises for the student.

1. If two straight lines be parallel to two other straight lines, each to each, the first pair make the same angles with one another as the second.

2. If two straight lines be perpendicular to two other straight lines, each to each, the first pair make the same angles with one another as the second.

## PROPOSITION XXXI. PROBLEM.

***To draw a straight line through a given point parallel to a given straight line.***



Let $A$ be the given pt. and $BC$ the given st. line.

*It is required to draw through $A$ a st. line $\parallel$ to $BC$.*

In $BC$ take any pt. $D$, and join $AD$.

Make $\angle DAE = \angle ADC$. I. 23.

Produce $EA$ to $F$. Then $EF$ shall be $\parallel$ to $BC$.

For $\because$ $AD$, meeting $EF$ and $BC$, makes the alternate angles equal, that is, $\angle EAD = \angle ADC$,

$\therefore$ $EF$ is $\parallel$ to $BC$. I. 27.

$\therefore$ a st. line has been drawn through $A$ $\parallel$ to $BC$.

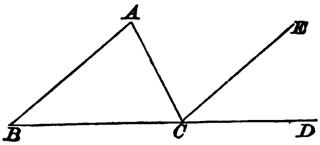
Q. E. D.

Ex. 1. From a given point draw a straight line, to make an angle with a given straight line that shall be equal to a given angle.

Ex. 2. Through a given point $A$ draw a straight line $ABC$, meeting two parallel straight lines in $B$ and $C$, so that $BC$ may be equal to a given straight line.

## PROPOSITION XXXII. THEOREM.

*If a side of any triangle be produced, the exterior angle is equal to the two interior and opposite angles, and the three interior angles of every triangle are together equal to two right angles.*



Let $ABC$ be a $\triangle$, and let one of its sides, $BC$, be produced to $D$.

*Then will*

I. $\angle ACD = \angle$s $ABC, BAC$ *together.*

II. $\angle$s $ABC, BAC, ACB$ *together* = *two rt.* $\angle$*s.*

From $C$ draw $CE \parallel$ to $AB$. I. 31.

Then I. $\because$ $BD$ meets the $\parallel$s $EC, AB$,

$\therefore$ extr. $\angle ECD$ = intr. $\angle ABC$. I. 29.

And $\because$ $AC$ meets the $\parallel$s $EC, AB$,

$\therefore$ $\angle ACE$ = alternate $\angle BAC$. I. 29.

$\therefore$ $\angle$s $ECD, ACE$ together = $\angle$s $ABC, BAC$ together;

$\therefore$ $\angle ACD$ = $\angle$s $ABC, BAC$ together.

And II. $\because$ $\angle$s $ABC, BAC$ together = $\angle ACD$,

to each of these equals add $\angle ACB$;

then $\angle$s $ABC, BAC, ACB$ together = $\angle$s $ACD, ACB$ together,

$\therefore$ $\angle$s $ABC, BAC, ACB$ together = two rt. $\angle$s. I. 13.

Q. E. D.

Ex. 1. In an acute-angled triangle, any two angles are greater than the third.

Ex. 2. The straight line, which bisects the external vertical angle of an isosceles triangle, is parallel to the base.

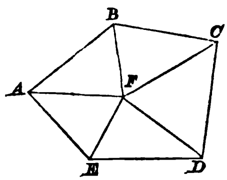
Ex. 3. If the side $BC$ of the triangle $ABC$ be produced to $D$, and $AE$ be drawn bisecting the angle $BAC$ and meeting $BC$ in $E$; shew that the angles $ABD$, $ACD$ are together double of the angle $AED$.

Ex. 4. If the straight lines bisecting the angles at the base of an isosceles triangle be produced to meet; shew that they will contain an angle equal to an exterior angle at the base of the triangle.

Ex. 5. If the straight line bisecting the external angle of a triangle be parallel to the base; prove that the triangle is isosceles.

The following Corollaries to Prop. 32 were first given in Simson's Edition of Euclid.

Cor. 1. *The sum of the interior angles of any rectilinear figure together with four right angles is equal to twice as many right angles as the figure has sides.*




Let $ABCDE$ be any rectilinear figure.

Take any pt. $F$ within the figure, and from $F$ draw the st. lines $FA$, $FB$, $FC$, $FD$, $FE$ to the angular pts. of the figure.

Then there are formed as many $\triangle$s as the figure has sides.

The three $\angle$s in *each* of these $\triangle$s together = two rt. $\angle$s.

$\therefore$ *all* the $\angle$s in these $\triangle$s together = twice as many right $\angle$s as there are $\triangle$s, that is, twice as many right $\angle$s as the figure has sides.

Now angles of all the $\triangle$s = $\angle$s at $A$, $B$, $C$, $D$, $E$ and $\angle$s at $F$,

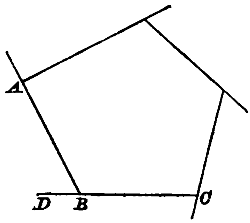
that is, = $\angle$s of the figure and $\angle$s at $F$,

and $\therefore$ = $\angle$s of the figure and four rt. $\angle$s. I. 15. Cor. 2.

$\therefore$ $\angle$s of the figure and four rt. $\angle$s = twice as many rt. $\angle$s as the figure has sides.

COR. 2. *The exterior angles of any convex rectilinear figure, made by producing each of its sides in succession, are together equal to four right angles.*

Every interior angle, as $ABC$, and its adjacent exterior angle, as $ABD$, together are = two rt. $\angle$ s.



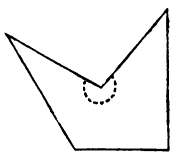

∴ all the intr. $\angle$ s together with all the extr. $\angle$ s = twice as many rt. $\angle$ s as the figure has sides.

But all the intr. $\angle$ s together with four rt. $\angle$ s = twice as many rt. $\angle$ s as the figure has sides.

∴ all the intr. $\angle$ s together with all the extr. $\angle$ s = all the intr. $\angle$ s together with four rt. $\angle$ s.

∴ all the extr. $\angle$ s = four rt. $\angle$ s.

NOTE. The latter of these corollaries refers only to *convex* figures, that is, figures in which every interior angle is less than two right angles. When a figure contains an angle greater



than two right angles, as the angle marked by the dotted line in the diagram, this is called a *re-entering angle.*

EX. 1. The exterior angles of a quadrilateral made by producing the sides successively are together equal to the interior angles.

Ex. 2. Prove that the interior angles of a hexagon are equal to eight right angles.

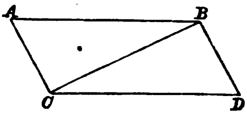
Ex. 3. Shew that the angle of an equiangular pentagon is $\frac{6}{5}$ of a right angle.

Ex. 4. How many sides has the rectilinear figure, the sum of whose interior angles is double that of its exterior angles?

Ex. 5. How many sides has an equiangular polygon, four of whose angles are together equal to seven right angles?

## Proposition XXXIII. Theorem.

*The straight lines which join the extremities of two equal and parallel straight lines, towards the same parts, are also themselves equal and parallel.*




Let the equal and ∥ st. lines $AB$, $CD$ be joined towards the same parts by the st. lines $AC$, $BD$.

*Then must $AC$ and $BD$ be equal and ∥.*

Join $BC$.

Then ∵ $AB$ is ∥ to $CD$,

∴ $\angle ABC$ = alternate $\angle DCB$. I. 29.

Then in $\triangle$s $ABC$, $BCD$,

∵ $AB = CD$, and $BC$ is common, and $\angle ABC = \angle DCB$,

∴ $AC = BD$, and $\angle ACB = \angle DBC$. I. 4.

Then ∵ $BC$, meeting $AC$ and $BD$, makes the alternate $\angle$s $ACB$, $DBC$ equal,

∴ $AC$ is ∥ to $BD$, and it has been shewn that $AC = BD$.

Q. E. D.

*Miscellaneous Exercises on Sections I and II.*

1. If two exterior angles of a triangle be bisected by straight lines which meet in $O$; prove that the perpendiculars from $O$ on the sides, or the sides produced, of the triangle are equal.

2. Trisect a right angle.

3. The bisectors of the three angles of a triangle meet in one point.

4. The perpendiculars to the three sides of a triangle drawn from the middle points of the sides meet in one point.

5. The angle between the bisector of the angle $BAC$ of the triangle $ABC$ and the perpendicular from $A$ on $BC$, is equal to half the difference between the angles at $B$ and $C$.

6. If the straight line $AD$ bisect the angle at $A$ of the triangle $ABC$, and $BDE$ be drawn perpendicular to $AD$, and meeting $AC$, or $AC$ produced, in $E$; shew that $BD$ is equal to $DE$.

7. Divide a right-angled triangle into two isosceles triangles.

8. $AB$, $CD$ are two given straight lines. Through a point $E$ between them draw a straight line $GEH$, such that the intercepted portion $GH$ shall be bisected in $E$.

9. The vertical angle $O$ of a triangle $OPQ$ is a right, acute, or obtuse angle, according as $OR$, the line bisecting $PQ$, is equal to, greater or less than the half of $PQ$.

10. Shew by means of Ex. 9 how to draw a perpendicular to a given straight line from its extremity without producing it.

# SECTION III.

## *On the Equality of Rectilineal Figures in respect of Area.*

THE amount of space enclosed by a Figure is called the Area of that figure.

Euclid calls two figures *equal* when they enclose the same amount of space. They may be dissimilar in shape, but if the areas contained within the boundaries of the figures be the same, then he calls the figures *equal*. He regards a triangle, for example, as a figure having sides and angles and area, and he proves in this section that two triangles may have equality of area, though the sides and angles of each may be unequal.

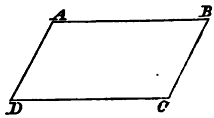
Coincidence of their boundaries is a test of the equality of all geometrical magnitudes, as we explained in Note 1, page 14.

In the case of lines and angles it is the only test: in the case of *figures* it is *a test, but not the only test;* as we shall shew in this Section.

The sign =, standing between the symbols denoting two *figures*, must be read *is equal in area to.*

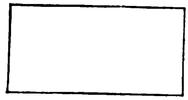
Before we proceed to prove the Propositions included in this Section, we must complete the list of Definitions required in Book I, continuing the numbers prefixed to the definitions in page 6.

DEFINITIONS.

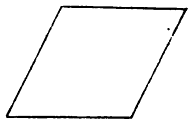



XXVII. A PARALLELOGRAM is a four-sided figure whose opposite sides are parallel.

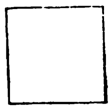
For brevity we often designate a parallelogram by two letters only, which mark opposite angles. Thus we call the figure in the margin the parallelogram $AC$.



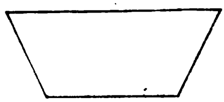
XXVIII. A RECTANGLE is a parallelogram, having one of its angles a right angle.



XXIX. A RHOMBUS is a parallelogram, having its sides equal.



XXX. A SQUARE is a parallelogram, having its sides equal and one of its angles a right angle.



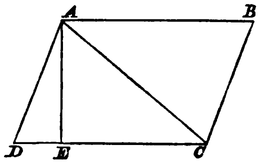
XXXI. A TRAPEZIUM is a four-sided figure of which two sides only are parallel.

XXXII. A DIAGONAL of a four-sided figure is the straight line joining two of the opposite angular points.

XXXIII. The ALTITUDE of a Parallelogram is the perpendicular distance of one of its sides from the side opposite, regarded as the Base.

The altitude of a triangle is the perpendicular distance of one of its angular points from the side opposite, regarded as the base.

Thus if *ABCD* be a parallelogram, and *AE* a perpendicular let fall from *A* to *CD*, *AE* is the *altitude* of the parallelogram, and also of the triangle *ACD*.




If a perpendicular be let fall from *B* to *DC* produced, meeting *DC* in *F*, *BF* is the altitude of the parallelogram.

### Exercises.

Prove from the definitions just given the following theorems :—

1. All the angles of a Square are right angles.

2. All the angles of a Rectangle are right angles.

3. The diagonals of a square make with each of the sides an angle equal to half a right angle.

4. If two straight lines bisect each other, the lines joining their extremities will form a parallelogram.

5. Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles.

6. If the straight lines joining two opposite angular points of a parallelogram bisect the angles, the parallelogram is a rhombus.

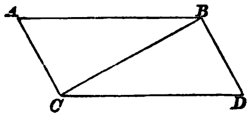
7. If the opposite angles of a quadrilateral be equal, the quadrilateral is a parallelogram.

8. If two opposite sides of a quadrilateral figure be equal to one another, and the two remaining sides be also equal to one another, the figure is a parallelogram.

9. If one angle of a rhombus be equal to two-thirds of two right angles, the diagonal drawn from that angular point divides the rhombus into two equilateral triangles.

### PROPOSITION XXXIV. THEOREM.

*The opposite sides and angles of a parallelogram are equal to one another, and the diagonal bisects it.*



Let $ABDC$ be a $\square$, and $BC$ a diagonal of the $\square$.

*Then must* $AB = DC$ and $AC = DB$,

*and* $\angle BAC = \angle CDB$, and $\angle ABD = \angle ACD$

*and* $\triangle ABC = \triangle DCB$.

For $\because$ $AB$ is $\parallel$ to $CD$, and $BC$ meets them,

$\therefore$ $\angle ABC =$ alternate $\angle DCB$; I. 29.

and $\because$ $AC$ is $\parallel$ to $BD$, and $BC$ meets them,

$\therefore$ $\angle ACB =$ alternate $\angle DBC$. I. 29.

Then in $\triangle$s $ABC$, $DCB$,

$\because$ $\angle ABC = \angle DCB$, and $\angle ACB = \angle DBC$,

and $BC$ is common, a side adjacent to the equal $\angle$s in each;

$\therefore$ $AB = DC$, and $AC = DB$, and $\angle BAC = \angle CDB$,

and $\triangle ABC = \triangle DCB$. I. B.

Also $\because$ $\angle ABC = \angle BCD$, and $\angle CBD = \angle ACB$,

$\therefore$ $\angle$s $ABC$, $CBD$ together $= \angle$s $BCD$, $ACB$ together,

that is, $\angle ABD = \angle ACD$. I. 4.

Q. E. D.

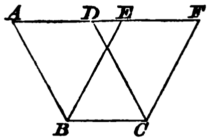
Ex. 1. Shew that the diagonals of a parallelogram bisect each other.

Ex. 2. Shew that the diagonals of a rectangle are equal.

Ex. 3. Prove that the four triangles, into which a parallelogram is divided by its diagonals, are equal to each other.

## PROPOSITION XXXV. THEOREM.

*Parallelograms on the same base and between the same parallels are equal.*




Let the ▱s $ABCD$, $EBCF$ be on the same base $BC$, and between the same ∥s $AF$, $BC$.

*Then must* ▱ $ABCD$ = ▱ $EBCF$.

CASE I. If there be a space between the sides $AD$, $EF$.

Join $DE$.

Then in the △s $FDC$, $EAB$,

∵ extr. $\angle FDC$ = intr. $\angle EAB$, I. 29.

and intr. $\angle DFC$ = extr. $\angle AEB$, I. 29.

and $DC = AB$, I. 34.

∴ △ $FDC$ = △ $EAB$. I. 26.

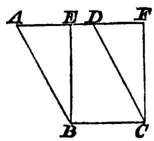
Now ▱ $ABCD$ with △ $FDC$ = figure $ABCF$;

and ▱ $EBCF$ with △ $EAB$ = figure $ABCF$;

∴ ▱ $ABCD$ with △ $FDC$ = ▱ $EBCF$ with △ $EAB$;

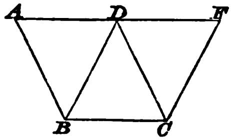
∴ ▱ $ABCD$ = ▱ $EBCF$.

CASE II. If the sides $AD$, $EF$ overlap one another, thus:




the same method of proof applies.

CASE III. If the sides opposite to $BC$ be terminated in the same point $D$, thus:




the same method of proof is applicable,
but it is easier to reason thus:

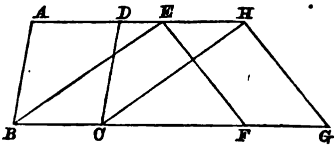
Each of the $\square$s is double of $\triangle BDC$; I. 34.

$\therefore \square ABCD = \square DBCF$.

Q. E. D.

PROPOSITION XXXVI. THEOREM.

*Parallelograms on equal bases, and between the same parallels, are equal to one another.*




Let the $\square$s $ABCD$, $EFGH$ be on equal bases $BC$, $FG$, and between the same $\parallel$s $AH$, $BG$.

*Then must* $\square ABCD = \square EFGH$.

Join $BE$, $CH$.

Then $\because BC = FG$, Hyp.

and $EH = FG$; I. 34.

$\therefore BC = EH$,

and $BC$ is $\parallel$ to $EH$. Hyp.

$\therefore EB$ is $=$ and $\parallel$ to $CH$; I. 33.

$\therefore EBCH$ is a parallelogram.

Now $\square EBCH = \square ABCD$, I. 35.

$\because$ they are on the same base $BC$ and between the same $\parallel$s;

and $\square EBCH = \square EFGH$, I. 35.

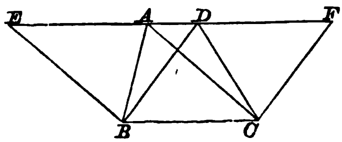
$\because$ they are on the same base $EH$ and between the same $\parallel$s;

$\therefore \square ABCD = \square EFGH$.

Q. E. D.

## PROPOSITION XXXVII. THEOREM.

*Triangles upon the same base, and between the same parallels, are equal to one another.*



Let $\triangle$s $ABC$, $DBC$ be on same base $BC$ and between same $\parallel$s $AD$, $BC$.

*Then must* $\triangle ABC = \triangle DBC$.

From $B$ draw $BE \parallel$ to $CA$ to meet $DA$ produced in $E$.

From $C$ draw $CF \parallel$ to $BD$ to meet $AD$ produced in $F$.

Then $EBCA$ and $FCBD$ are parallelograms

and $\square EBCA = \square FCBD$, I. 35.

$\because$ they are on the same base and between the same $\parallel$s.

Now $\triangle ABC$ is half of $\square EBCA$, I. 34.

and $\triangle DBC$ is half of $\square FCBD$; I. 34.

$\therefore$ $\triangle ABC = \triangle DBC$. Ax. 7.

Q. E. D.

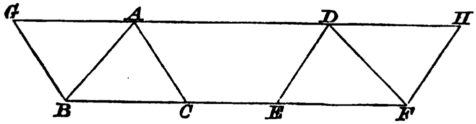
Ex. 1. If $P$ be a point in a side $AB$ of a parallelogram $ABCD$, and $PC$, $PD$ be joined, the triangles $PAD$, $PBC$ are together equal to the triangle $PDC$.

Ex. 2. Two straight lines $AB$, $CD$ intersect in $E$, and the triangle $AEC$ is equal to the triangle $BED$. Shew that $BC$ is parallel to $AD$.

Ex. 3. If $A$, $B$ be points in one, and $C$, $D$ points in another of two parallel straight lines, and the lines $AD$, $BC$ intersect in $E$, then the triangles $AEC$, $BED$ are equal.

## Proposition XXXVIII. Theorem.

*Triangles upon equal bases, and between the same parallels, are equal to one another.*



Let $\triangle$s $ABC$, $DEF$ be on equal bases, $BC$, $EF$ and between the same $\parallel$s $BF$, $AD$.

*Then must* $\triangle ABC = \triangle DEF$.

From $B$ draw $BG \parallel$ to $CA$ to meet $DA$ produced in $G$.

From $F$ draw $FH \parallel$ to $ED$ to meet $AD$ produced in $H$.

Then $CG$ and $EH$ are parallelograms, and they are equal,

$\because$ they are on equal bases $BC$, $EF$ and between the same $\parallel$s $BF$, $GH$. I. 36.

Now $\triangle ABC$ is half of $\square CG$,

and $\triangle DEF$ is half of $\square EH$;

$\therefore \triangle ABC = \triangle DEF$. Ax. 7.

Q. E. D.

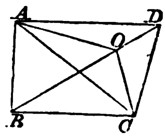
Ex. 1. Shew that a straight line, drawn from the vertex of a triangle to bisect the base, divides the triangle into two equal parts.

Ex. 2. If the triangles in the Proposition are not towards the same parts, shew that the straight line, joining the vertices of the triangles, is bisected by the line containing the bases.

Ex. 3. In the equal sides $AB$, $AC$ of an isosceles triangle $ABC$ points $D$, $E$ are taken such that $BD = AE$. Shew that the triangles $CBD$, $ABE$ are equal.

## PROPOSITION XXXIX. THEOREM.

*Equal triangles upon the same base, and upon the same side of it, are between the same parallels.*




Let the equal $\triangle$s $ABC$, $DBC$ be on the same base $BC$, and on the same side of it.

Join $AD$.

*Then must $AD$ be $\parallel$ to $BC$.*

For if not, through $A$ draw $AO \parallel$ to $BC$, so as to meet $BD$, or $BD$ produced, in $O$, and join $OC$.

Then $\because$ $\triangle$s $ABC$, $OBC$ are on the same base and between the same $\parallel$s,

$\therefore \triangle ABC = \triangle OBC$. I. 37.

But $\triangle ABC = \triangle DBC$; Hyp.

$\therefore \triangle OBC = \triangle DBC$,

the less = the greater, which is impossible;

$\therefore AO$ is not $\parallel$ to $BC$.

In the same way it may be shewn that no other line but $AD$ is $\parallel$ to $BC$;

$\therefore AD$ is $\parallel$ to $BC$.

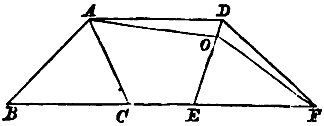
Q. E. D.

Ex. 1. $AD$ is parallel to $BC$; $AC$, $BD$ meet in $E$; $BC$ is produced to $P$ so that the triangle $PEB$ is equal to the triangle $ABC$: shew that $PD$ is parallel to $AC$.

Ex. 2. If of the four triangles into which the diagonals divide a quadrilateral, two opposite ones are equal, the quadrilateral has two opposite sides parallel.

PROPOSITION XL. THEOREM.

*Equal triangles upon equal bases, in the same straight line, and towards the same parts, are between the same parallels.*




Let the equal $\triangle$s $ABC, DEF$ be on equal bases $BC, EF$ in the same st. line $BF$ and towards the same parts.

Join $AD$.

*Then must $AD$ be $\parallel$ to $BF$.*

For if not, through $A$ draw $AO \parallel$ to $BF$, so as to meet $ED$, or $ED$ produced, in $O$, and join $OF$.

Then $\triangle ABC = \triangle OEF$, $\because$ they are on equal bases and between the same $\parallel$s. I. 38.

But $\triangle ABC = \triangle DEF$; Hyp.

$\therefore \triangle OEF = \triangle DEF$,

the less = the greater, which is impossible.

$\therefore AO$ is not $\parallel$ to $BF$.

In the same way it may be shewn that no other line but $AD$ is $\parallel$ to $BF$,

$\therefore AD$ is $\parallel$ to $BF$.

Q. E. D.

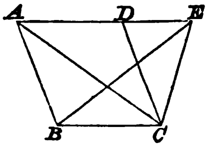
Ex. 1. If the triangles be not towards the same parts, shew that the straight line joining the vertices of the triangles is bisected by the line containing the bases.

Ex. 2. The straight line, joining the points of bisection of two sides of a triangle, is parallel to the base.

Ex. 3. The straight lines, joining the middle points of the sides of a triangle, divide it into four equal triangles.

PROPOSITION XLI. THEOREM.

***If a parallelogram and a triangle be upon the same base, and between the same parallels, the parallelogram is double of the triangle.***



Let the ▱$ABCD$ and the △$EBC$ be on the same base $BC$ and between the same ||s $AE$, $BC$.

*Then must* ▱$ABCD$ *be double of* △$EBC$.

Join $AC$.

Then △$ABC$ = △$EBC$, ∵ they are on the same base and between the same ||s; I. 37.

and ▱$ABCD$ is double of △$ABC$, ∵ $AC$ is a diagonal of $ABCD$; I. 34.

∴ ▱$ABCD$ is double of △$EBC$.

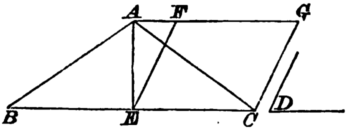
Q. E. D.

Ex. 1. If from a point, without a parallelogram, there be drawn two straight lines to the extremities of the two opposite sides, between which, when produced, the point does not lie, the difference of the triangles thus formed is equal to half the parallelogram.

Ex. 2. The two triangles, formed by drawing straight lines from any point within a parallelogram to the extremities of its opposite sides, are together half of the parallelogram.

## Proposition XLII. Problem.

*To describe a parallelogram that shall be equal to a given triangle, and have one of its angles equal to a given angle.*



Let $ABC$ be the given $\triangle$, and $D$ the given $\angle$.

*It is required to describe a* $\square$ *equal to* $\triangle$ *ABC, having one of its* $\angle$*s* $=\angle$ *D.*

Bisect $BC$ in $E$ and join $AE$.

At $E$ make $\angle\, CEF = \angle\, D$.

Draw $AFG \parallel$ to $BC$, and from $C$ draw $CG \parallel$ to $EF$.

Then $FECG$ is a parallelogram.

Now $\triangle\ AEB = \triangle\ AEC$,

$\because$ they are on equal bases and between the same $\parallel$s; I. 38.

$\therefore\ \triangle\ ABC$ is double of $\triangle\ AEC$.

But $\square\ FECG$ is double of $\triangle\ AEC$,

$\because$ they are on same base and between same $\parallel$s; I. 41.

$\therefore\ \square\ FECG = \triangle\ ABC$, Ax. 6.

and $\square\ FECG$ has one of its $\angle$s, $CEF = \angle\, D$.

$\therefore\ \square\ FECG$ has been described as was reqd.

Q. E. F.

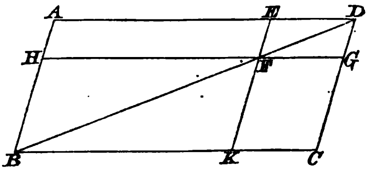
Ex. 1. Describe a triangle, which shall be equal to a given parallelogram, and have one of its angles equal to a given rectilineal angle.

Ex. 2. Construct a parallelogram, equal to a given triangle, and such that the sum of its sides shall be equal to the sum of the sides of the triangle.

Ex. 3. The perimeter of an isosceles triangle is greater than the perimeter of a rectangle, which is of the same altitude with, and equal to, the given triangle.

## Proposition XLIII. Theorem.

*The complements of the parallelograms, which are about the diameter of any parallelogram, are equal to one another.*



Let $ABCD$ be a ▱, of which $BD$ is a diagonal, and $EG$, $HK$ the ▱s about $BD$, that is, through which $BD$ passes,

and $AF$, $FC$ the other ▱s, which make up the whole figure $ABCD$,

and which are ∴ called the Complements.

*Then must complement $AF$ = complement $FC$.*

For ∵ $BD$ is a diagonal of ▱ $AC$,
∴ $\triangle ABD = \triangle CDB$; I. 34.
and ∵ $BF$ is a diagonal of ▱ $HK$,
∴ $\triangle HBF = \triangle KFB$;
and ∵ $FD$ is a diagonal of ▱ $EG$,
∴ $\triangle EFD = \triangle GDF$.

Hence sum of $\triangle$s $HBF$, $EFD$ = sum of $\triangle$s $KFB$, $GDF$.

Take these equals from $\triangle$s $ABD$, $CDB$ respectively,
then remaining ▱ $AF$ = remaining ▱ $FC$. Ax. 3.

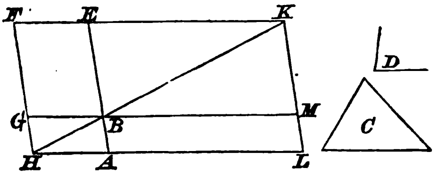
Q. E. D.

Ex. 1. If through a point $O$, within a parallelogram $ABCD$, two straight lines are drawn parallel to the sides, and the parallelograms $OB$, $OD$ are equal; the point $O$ is in the diagonal $AC$.

Ex. 2. $ABCD$ is a parallelogram, $AMN$ a straight line meeting the sides $BC$, $CD$ (one of them being produced) in $M$, $N$. Shew that the triangle $MBN$ is equal to the triangle $MDC$.

### PROPOSITION XLIV. PROBLEM.

*To a given straight line to apply a parallelogram, which shall be equal to a given triangle, and have one of its angles equal to a given angle.*




Let $AB$ be the given st. line, $C$ the given $\triangle$, $D$ the given $\angle$.

*It is required to apply to $AB$ a $\square = \triangle C$ and having one of its $\angle$s $= \angle D$.*

Make a $\square = \triangle C$, and having one of its angles $= \angle D$, I. 42.

and suppose it to be removed to such a position that one of the sides containing this angle is in the same st. line with $AB$, and let the $\square$ be denoted by $BEFG$.

Produce $FG$ to $H$, draw $AH \parallel$ to $BG$ or $EF$, and join $BH$.

Then $\because$ $FH$ meets the $\parallel$s $AH$, $EF$,

$\therefore$ sum of $\angle$s $AHF$, $HFE =$ two rt. $\angle$s;

$\therefore$ sum of $\angle$s $BHG$, $HFE$ is less than two rt. $\angle$s;

$\therefore$ $HB$, $FE$ will meet if produced towards $B$, $E$. Post. 6.

Let them meet in $K$.

Through $K$ draw $KL \parallel$ to $EA$ or $FH$,

and produce $HA$, $GB$ to meet $KL$ in the pts. $L$, $M$.

Then $HFKL$ is a $\square$, and $HK$ is its diagonal;

and $AG$, $ME$ are $\square$s about $HK$,

$\therefore$ complement $BL =$ complement $BF$,

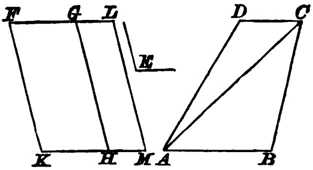
$\therefore$ $\square BL = \triangle C$.

Also the $\square$ $BL$ has one of its $\angle$s, $ABM = \angle EBG$, and $\therefore$ equal to $\angle D$.

Q.E.F.

## PROPOSITION XLV. PROBLEM.

***To describe a parallelogram, which shall be equal to a given rectilinear figure, and have one of its angles equal to a given angle.***




Let $ABCD$ be the given rectil. figure, and $E$ the given $\angle$.

Join $AC$.

*It is required to describe a* ▱ *= to ABCD, having one of its* $\angle$*s* $= \angle E$.

Describe a ▱ $FGHK = \triangle\ ABC$, having $\angle FKH = \angle E$. I. 42.

To $GH$ apply a ▱ $GHML = \triangle\ CDA$, having $\angle GHM = \angle E$. I. 44.

Then $FKML$ is the ▱ reqd.

For ∵ $\angle GHM$ and $\angle FKH$ are each $= \angle E$;

∴ $\angle GHM = \angle FKH$,

∴ sum of $\angle$s $GHM, GHK$ = sum of $\angle$s $FKH, GHK$

= two rt. $\angle$s; I. 29.

∴ $KHM$ is a st. line. I. 14.

Again, ∵ $HG$ meets the ‖s $FG, KM$,

$\angle FGH = \angle GHM$,

∴ sum of $\angle$s $FGH, LGH$ = sum of $\angle$s $GHM, LGH$

= two rt. $\angle$s; I. 29.

∴ $FGL$ is a st. line. I. 14.

Then ∵ $KF$ is ‖ to $HG$, and $HG$ is ‖ to $LM$,

∴ $KF$ is ‖ to $LM$; I. 30.

and $KM$ has been shewn to be ‖ to $FL$,

∴ $FKML$ is a parallelogram,

and ∵ $FH = \triangle\ ABC$, and $GM = \triangle\ CDA$,

∴ ▱ $FM$ = whole rectil. fig. $ABCD$,

and ▱ $FM$ has one of its $\angle$s, $FKM = \angle E$.

In the same way a ▱ may be constructed equal to a given rectil. fig. of any number of sides, and having one of its angles equal to a given angle. Q. E. F.

## *Miscellaneous Exercises.*

1. If one diagonal of a quadrilateral bisect the other, it divides the quadrilateral into two equal triangles.

2. If from any point in the diagonal, or the diagonal produced, of a parallelogram, straight lines be drawn to the opposite angles, they will cut off equal triangles.

3. In a trapezium the straight line, joining the middle points of the parallel sides, bisects the trapezium.

4. The diagonals $AC$, $BD$ of a parallelogram intersect in $O$, and $P$ is a point within the triangle $AOB$; prove that the difference of the triangles $APB$, $CPD$ is equal to the sum of the triangles $APC$, $BPD$.

5. If either diameter of a parallelogram be equal to a side of the figure, the other diameter shall be greater than any side of the figure.

6. If through the angles of a parallelogram four straight lines be drawn parallel to its diagonals, another parallelogram will be formed, the area of which will be double that of the original parallelogram.

7. If two triangles have two sides respectively equal and the included angles supplemental, the triangles are equal.

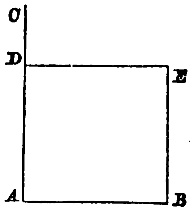
8. Bisect a given triangle by a straight line drawn from a given point in one of the sides.

9. If the base of a triangle $ABC$ be produced to a point $D$ such that $BD$ is equal to $AB$, and if straight lines be drawn from $A$ and $D$ to $E$, the middle point of $BC$; prove that the triangle $ADE$ is equal to the triangle $ABC$.

10. Prove that a pair of the diagonals of the parallelograms, which are about the diameter of any parallelogram, are parallel to each other.

## Proposition XLVI. Problem.

*To describe a square upon a given straight line.*



Let $AB$ be the given st. line.

*It is required to describe a square on AB.*

From $A$ draw $AC \perp$ to $AB$.

In $AC$ make $AD = AB$.

Through $D$ draw $DE \parallel$ to $AB$.

Through $B$ draw $BE \parallel$ to $AD$.

Then $AE$ is a parallelogram,

and $\therefore$ $AB = ED$, and $AD = BE$.

But $AB = AD$,

$\therefore$ $AB$, $BE$, $ED$, $DA$ are all equal;

$\therefore$ $AE$ is equilateral.

And $\angle BAD$ is a right angle,

$\therefore$ $AE$ is a square, Def. XXX.

and it is described on $AB$.

Q. E. F.

Ex. 1. Shew how to construct a rectangle whose sides are equal to two given straight lines.

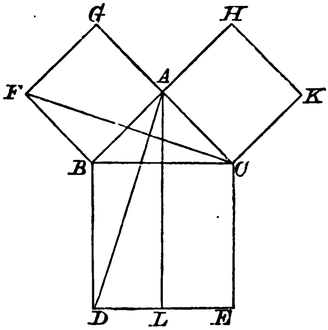
Ex. 2. Shew that the squares on equal straight lines are equal.

Ex. 3. Shew that equal squares must be on equal straight lines.

Note. The theorems in Ex. 2 and 3 are assumed by Euclid in the proof of Prop. XLVIII. See also Prop. A. of Book II., p. 77.

PROPOSITION XLVII. THEOREM.

*In any right-angled triangle the square which is described on the side subtending the right angle is equal to the squares described on the sides which contain the right angle.*




Let $ABC$ be a right-angled $\triangle$, having the rt. $\angle\, BAC$.

*Then must sq. on $BC$ = sum of sqq. on $BA$, $AC$.*

On $BC$, $CA$, $AB$ descr. the sqq. $BDEC$, $CKHA$, $AGFB$.

Through $A$ draw $AL \parallel$ to $BD$ or $CE$, and join $AD$, $FC$.

Then $\because \angle\, BAC$ and $\angle\, BAG$ are both rt. $\angle$ s,

$\therefore CAG$ is a st. line; I. 14.

and $\because \angle\, BAC$ and $\angle\, CAH$ are both rt. $\angle$ s;

$\therefore BAH$ is a st. line. I. 14.

Now $\because \angle\, DBC = \angle\, FBA$, each being a rt. $\angle$,

adding to each $\angle\, ABC$, we have

$\angle\, ABD = \angle\, FBC$.

Then in $\triangle$ s $ABD$, $FBC$,

$\because AB = FB$, and $BD = BC$, and $\angle\, ABD = \angle\, FBC$,

$\therefore \triangle\, ABD = \triangle\, FBC$. I. 4.

Now $\square\, BL$ is double of $\triangle\, ABD$, on same base $BD$ and between same $\parallel$ s $AL$, $BD$, I. 41.

and sq. $BG$ is double of $\triangle\, FBC$, on same base $FB$ and between same $\parallel$ s $FB$, $GC$; I. 41.

$\therefore \square\, BL =$ sq. $BG$.

Similarly, by joining $AE$, $BK$ it may be shewn that

$$\square\ CL = \text{sq. } AK.$$

Now sq. on $BC$ = sum of $\square\ BL$ and $\square\ CL$,
= sum of sq. $BG$ and sq. $AK$,
= sum of sqq. on $BA$ and $AC$.

Q. E. D.

Ex. 1. Prove that the square, described upon the diagonal of any given square, is equal to twice the given square.

Ex. 2. Find a line, the square on which shall be equal to the sum of the squares on three given straight lines.

Ex. 3. If one angle of a triangle be equal to the sum of the other two, and one of the sides containing this angle being divided into four equal parts, the other contains three of those parts; the remaining side of the triangle contains five such parts.

Ex. 4. The triangles $ABC$, $DEF$, having the angles $ACB$, $DFE$ right angles, have also the sides $AB$, $AC$ equal to $DE$, $DF$, each to each; shew that the triangles are equal in every respect.

NOTE. This Theorem has been already deduced as a Corollary from Prop. E, page 43.

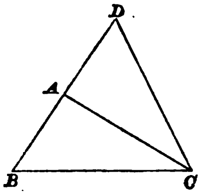
Ex. 5. Divide a given straight line into two parts, so that the square on one part shall be double of the square on the other.

Ex. 6. If from one of the acute angles of a right-angled triangle a line be drawn to the opposite side, the squares on that side and on the line so drawn are together equal to the sum of the squares on the segment adjacent to the right angle and on the hypothenuse.

Ex. 7. In any triangle, if a line be drawn from the vertex at right angles to the base, the difference between the squares on the sides is equal to the difference between the squares on the segments of the base.

## Proposition XLVIII. Theorem.

***If the square described upon one of the sides of a triangle be equal to the squares described upon the other two sides of it, the angle contained by those sides is a right angle.***




Let the sq. on $BC$, a side of $\triangle$ $ABC$, be equal to the sum of the sqq. on $AB$, $AC$.

*Then must* $\angle BAC$ *be a rt. angle.*

From pt. $A$ draw $AD$ $\perp$ to $AC$.

Make $AD=AB$, and join $DC$.

Then $\because AD=AB$,

$\therefore$ sq. on $AD$=sq. on $AB$; I. 46, Ex. 2.

add to each sq. on $AC$:

then sum of sqq. on $AD$, $AC$=sum of sqq. on $AB$, $AC$.

But sq. on $DC$=sum of sqq. on $AD$, $AC$, I. 47.

and sq. on $BC$=sum of sqq. on $AB$, $AC$; Hyp.

$\therefore$ sq. on $DC$=sq. on $BC$,

$\therefore DC=BC$. I. 46, Ex. 3.

Then in $\triangle$s $ABC$, $ADC$,

$\because AB=AD$, and $AC$ is common, and $BC=DC$,

$\therefore \angle BAC = \angle DAC$; I. c.

and $\angle DAC$ is a rt. angle, by construction,

$\therefore \angle BAC$ is a rt. angle.

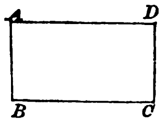
Q. E. D.

# BOOK II.

## *INTRODUCTORY REMARKS.*

THE geometrical figure with which we are chiefly concerned in this book is the RECTANGLE. A rectangle is said to be *contained by* any two of its adjacent sides.

Thus if $ABCD$ be a rectangle, it is said to be contained by $AB$, $AD$, or by any other pair of adjacent sides.



We shall use the abbreviation *rect.* $AB, AD$ to express the words "the rectangle contained by $AB, AD$."

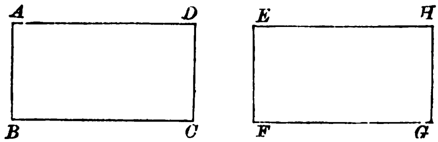
We shall make frequent use of a Theorem (employed, but not demonstrated, by Euclid) which may be thus stated and proved:

### PROPOSITION A. THEOREM.

*If the adjacent sides of one rectangle be equal to the adjacent sides of another rectangle, each to each, the rectangles are equal in area.*

Let $ABCD$, $EFGH$ be two rectangles:

and let $AB = EF$ and $BC = FG$.



*Then must rect.* $ABCD =$ *rect.* $EFGH$.

For if the rect. $EFGH$ be applied to the rect. $ABCD$, so that $EF$ coincides with $AB$,

then $FG$ will fall on $BC$, $\because \angle EFG = \angle ABC$,

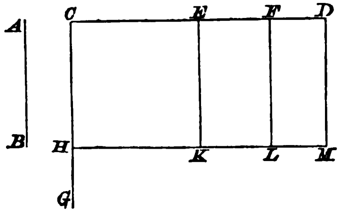
and $G$ will coincide with $C$, $\because BC = FG$.

Similarly it may be shewn that $H$ will coincide with $D$,

$\therefore$ rect. $EFGH$ coincides with and is $\therefore$ equal to rect. $ABCD$. Q. E. D.

PROPOSITION I. THEOREM.

*If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the rectangles contained by the undivided line and the several parts of the divided line.*




Let $AB$ and $CD$ be two given st. lines,

and let $CD$ be divided into any parts in $E$, $F$.

*Then must rect.* $AB$, $CD$ = *sum of rect.* $AB$, $CE$ *and rect.* $AB$, $EF$ *and rect.* $AB$, $FD$.

From $C$ draw $CG \perp$ to $CD$, and in $CG$ make $CH = AB$.

Through $H$ draw $HM \parallel$ to $CD$. I. 31.

Through $E$, $F$, and $D$ draw $EK$, $FL$, $DM \parallel$ to $CH$.

Then $CM$ = sum of $CK$ and $EL$ and $FM$.

Now $CM$ = rect. $AB$, $CD$, $\because CH = AB$,
$CK$ = rect. $AB$, $CE$, $\because CH = AB$,
$EL$ = rect. $AB$, $EF$, $\because EK = AB$,
$FM$ = rect. $AB$, $FD$, $\because FL = AB$;

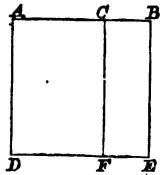
$\therefore$ rect. $AB$, $CD$ = sum of rect. $AB$, $CE$ and rect. $AB$, $EF$ and rect. $AB$, $FD$.

Q. E. D.

Ex. If two straight lines be each divided into any number of parts, the rectangle contained by the two lines is equal to the rectangles contained by all the parts of the one taken separately with all the parts of the other.

PROPOSITION II. THEOREM.

*If a straight line be divided into any two parts, the rectangles contained by the whole and each of the parts are together equal to the square on the whole line.*



Let the st. line $AB$ be divided into any two parts in $C$.

*Then must*

*sq. on $AB$ = sum of rect. $AB, AC$ and rect. $AB, CB$.*

On $AB$ describe the sq. $ADEB$. I. 46.

Through $C$ draw $CF \parallel$ to $AD$. I. 31.

Then $AE$ = sum of $AF$ and $CE$.

Now $AE$ is the sq. on $AB$,

$AF$ = rect. $AB, AC$, $\because AD = AB$,

$CE$ = rect. $AB, CB$, $\because BE = AB$..

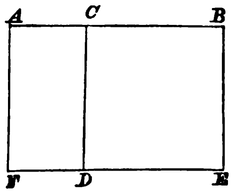
$\therefore$ sq. on $AB$ = sum of rect. $AB, AC$ and rect. $AB, CB$.

Q. E. D.

Ex. The square on a straight line is equal to four times the square on half the line.

PROPOSITION III. THEOREM.

*If a straight line be divided into any two parts, the rectangle contained by the whole and one of the parts is equal to the rectangle contained by the two parts together with the square on the aforesaid part.*




Let the st. line $AB$ be divided into any two parts in $C$.

*Then must*
*rect.* $AB$, $CB$ = *sum of rect.* $AC$, $CB$ *and sq. on* $CB$.

On $CB$ descr. the sq. $CDEB$. I. 46.

From $A$ draw $AF \parallel$ to $CD$, meeting $ED$ produced in $F$.

Then $AE$ = sum of $AD$ and $CE$.

Now $AE$ = rect. $AB$, $CB$, $\because BE = CB$,

$AD$ = rect. $AC$, $CB$, $\because CD = CB$,

$CE$ = sq. on $CB$.

$\therefore$ rect. $AB$, $CB$ = sum of rect. $AC$, $CB$ and sq. on $CB$.

Q. E. D.

NOTE. When a straight line is cut in a point, the distances of the point of section from the ends of the line are called the *segments* of the line.

If a line $AB$ be divided in $C$,

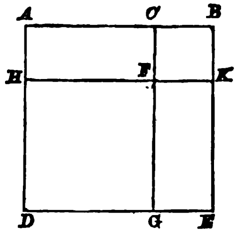
$AC$ and $CB$ are called the *internal* segments of $AB$.

If a line $AC$ be produced to $B$,

$AB$ and $CB$ are called the *external* segments of $AC$.

PROPOSITION IV. THEOREM.

*If a straight line be divided into any two parts, the square on the whole line is equal to the squares on the two parts together with twice the rectangle contained by the parts.*



Let the st. line $AB$ be divided into any two parts in $C$.

*Then must*

*sq. on $AB$ = sum of sqq. on $AC$, $CB$ and twice rect. $AC$, $CB$.*

On $AB$ describe the sq. $ADEB$.

From $AD$ cut off $AH = CB$. Then $HD = AC$.

Draw $CG \parallel$ to $AD$, and $HK \parallel$ to $AB$, meeting $CG$ in $F$.

Then $\because BK = AH$, $\therefore BK = CB$,

$\therefore BK$, $KF$, $FC$, $CB$ are all equal; and $KBC$ is a rt. $\angle$;

$\therefore CK$ is the sq. on $CB$. Def. xxx.

Also $HG$ = sq. on $AC$, $\because HF$ and $HD$ each $= AC$.

Now $AE$ = sum of $HG$, $CK$, $AF$, $FE$,

and $AE$ = sq. on $AB$,

$HG$ = sq. on $AC$,

$CK$ = sq. on $CB$,

$AF$ = rect. $AC$, $CB$, $\because CF = CB$,

$FE$ = rect. $AC$, $CB$, $\because FG = AC$ and $FK = CB$.

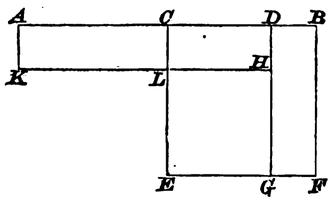
$\therefore$ sq. on $AB$ = sum of sqq. on $AC$, $CB$ and twice rect. $AC$, $CB$.

Q. E. D.

Ex. In a triangle, whose vertical angle is a right angle, a straight line is drawn from the vertex perpendicular to the base. Shew that the rectangle, contained by the segments of the base, is equal to the square on the perpendicular.

PROPOSITION V. THEOREM.

***If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.***




Let the st. line $AB$ be divided equally in $C$ and unequally in $D$.

*Then must*

*rect.* $AD$, $DB$ *together with sq. on* $CD$ = *sq. on* $CB$.

On $CB$ describe the sq. $CEFB$:

Draw $DG \parallel$ to $CE$, and from it cut off $DH = DB$.

Draw $HLK \parallel$ to $AD$, and $AK \parallel$ to $DH$.

Then rect. $DF$ = rect. $AL$, $\because BF = AC$, and $BD = CL$.

Also $LG$ = sq. on $CD$, $\because LH = CD$, and $HG = CD$.

Then rect. $AD$, $DB$ together with sq. on $CD$

= $AH$ together with $LG$

= sum of $AL$ and $CH$ and $LG$

= sum of $DF$ and $CH$ and $LG$

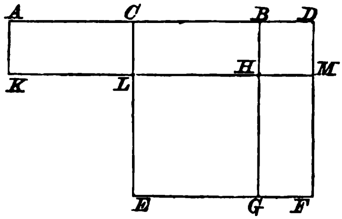
= $CF$

= sq. on $CB$.

Q. E. D.

## PROPOSITION VI. THEOREM.

*If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, is equal to the square on the straight line which is made up of the half and the part produced.*




Let the st. line $AB$ be bisected in $C$ and produced to $D$.

*Then must*

*rect. $AD$, $DB$ together with sq. on $CB$ = sq. on $CD$.*

On $CD$ describe the sq. $CEFD$.

Draw $BG \parallel$ to $CE$, and cut off $BH = BD$.

Through $H$ draw $KLM \parallel$ to $AD$.

Through $A$ draw $AK \parallel$ to $CE$.

Now $\because BG = CD$ and $BH = BD$;

$\therefore HG = CB$; Ax. 3.

$\therefore$ rect. $MG$ = rect. $AL$. II. A.

Then rect. $AD$, $DB$ together with sq. on $CB$

= sum of $AM$ and $LG$

= sum of $AL$, $CM$, and $LG$

= sum of $MG$, $CM$, and $LG$

= $CF$

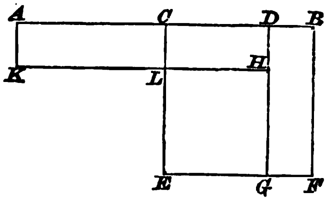
= sq. on $CD$.

Q. E. D.

NOTE. We here give the proof of an important theorem, which is usually placed as a corollary to Prop. 5.

PROPOSITION B. THEOREM.

*The difference between the squares on any two straight lines is equal to the rectangle contained by the sum and difference of those lines.*




Let $AC$, $CD$ be two st. lines, of which $AC$ is the greater, and let them be placed so as to form one st. line $AD$.

Produce $AD$ to $B$, making $CB=AC$.

Then $AD=$ the sum of the lines $AC$, $CD$,

and $DB=$ the difference of the lines $AC$, $CD$.

*Then must difference between sqq. on $AC$, $CD=$ rect. $AD$, $DB$.*

On $CB$ describe the sq. $CEFB$.

Draw $DG \parallel$ to $CE$, and from it cut off $DH=DB$.

Draw $HLK \parallel$ to $AD$, and $AK \parallel$ to $DH$.

Then rect. $DF=$ rect. $AL$, $\because BF=AC$, and $BD=CL$.

Also $LG=$ sq. on $CD$, $\because LH=CD$, and $HG=CD$.

Then difference between sqq. on $AC$, $CD$

$=$ difference between sqq. on $CB$, $CD$

$=$ sum of $CH$ and $DF$

$=$ sum of $CH$ and $AL$

$=AH$

$=$ rect. $AD$, $DH$

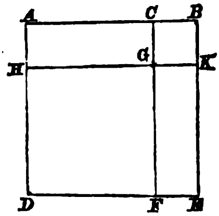
$=$ rect. $AD$, $DB$.

Q. E. D.

Ex. Shew that Propositions V. and VI. might be deduced from this Proposition.

## PROPOSITION VII. THEOREM.

*If a straight line be divided into any two parts, the squares on the whole line and on one of the parts are equal to twice the rectangle contained by the whole and that part together with the square on the other part.*




Let $AB$ be divided into any two parts in $C$.

*Then must*

*sqq. on $AB$, $BC$ = twice rect. $AB$, $BC$ together with sq. on $AC$.*

On $AB$ describe the sq. $ADEB$.

From $AD$ cut off $AH = CB$.

Draw $CF \parallel$ to $AD$ and $HGK \parallel$ to $AB$.

Then $HF$ = sq. on $AC$, and $CK$ = sq. on $CB$.

Then sqq. on $AB$, $BC$ = sum of $AE$ and $CK$

= sum of $AK$, $HF$, $GE$ and $CK$

= sum of $AK$, $HF$ and $CE$.

Now $AK$ = rect. $AB$, $BC$,

$CE$ = rect. $AB$, $BC$, $\because BE = AB$;

$HF$ = sq. on $AC$.

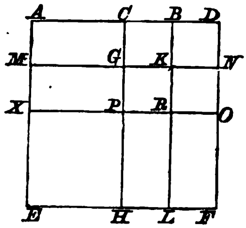
$\therefore$ sqq. on $AB$, $BC$ = twice rect. $AB$, $BC$ together with sq. on $AC$.

Q. E. D.

Ex. If straight lines be drawn from $G$ to $B$ and from $G$ to $D$, shew that $BGD$ is a straight line.

### Proposition VIII. Theorem.

*If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and the first part.*




Let the st. line $AB$ be divided into any two parts in $C$.

Produce $AB$ to $D$, so that $BD = BC$.

*Then must four times rect. $AB$, $BC$ together with sq. on $AC$ = sq. on $AD$.*

On $AD$ describe the sq. $AEFD$.

From $AE$ cut off $AM$ and $MX$ each $= CB$.

Through $C$, $B$ draw $CH$, $BL \parallel$ to $AE$.

Through $M$, $X$ draw $MGKN$, $XPRO \parallel$ to $AD$.

Now $\because XE = AC$, and $XP = AC$, $\therefore XH =$ sq. on $AC$.

Also $AG = MP = PL = RF$, II. A.

and $CK = GR = BN = KO$; II. A.

$\therefore$ sum of these eight rectangles

= four times the sum of $AG$, $CK$

= four times $AK$

= four times rect. $AB$, $BC$.

Then four times rect. $AB$, $BC$ and sq. on $AC$

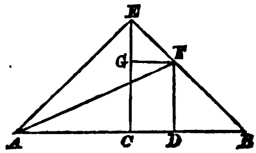
= sum of the eight rectangles and $XH$

$= AEFD$

= sq. on $AD$. Q. E. D.

## PROPOSITION IX. THEOREM.

*If a straight line be divided into two equal, and also into two unequal parts, the squares on the two unequal parts are together double of the square on half the line and of the square on the line between the points of section.*




Let $AB$ be divided equally in $C$ and unequally in $D$.

*Then must*

*sum of sqq. on $AD$, $DB$ = twice sum of sqq. on $AC$, $CD$.*

Draw $CE = AC$ at rt. $\angle$s to $AB$, and join $EA$, $EB$.
Draw $DF$ at rt. $\angle$s to $AB$, meeting $EB$ in $F$.
Draw $FG$ at rt. $\angle$s to $EC$, and join $AF$.

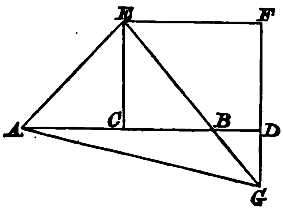
Then $\because \angle ACE$ is a rt. $\angle$,
$\therefore$ sum of $\angle$s $AEC$, $EAC$ = a rt. $\angle$; I. 32.
and $\because \angle AEC = \angle EAC$, I. A.
$\therefore \angle AEC$ = half a rt. $\angle$.
So also $\angle BEC$ and $\angle EBC$ are each = half a rt. $\angle$.
Hence $\angle AEF$ is a rt. $\angle$.
Also, $\because \angle GEF$ is half a rt. $\angle$, and $\angle EGF$ is a rt. $\angle$;
$\therefore \angle EFG$ is half a rt. $\angle$; $\therefore EG = GF$. I. B. Cor.
So also $\angle BFD$ is half a rt. $\angle$, and $BD = DF$.
Now sum of sqq. on $AD$, $DB$
= sq. on $AD$ together with sq. on $DF$
= sq. on $AF$ I. 47.
= sq. on $AE$ together with sq. on $EF$
= sqq. on $AC$, $EC$ together with sqq. on $EG$, $GF$ I. 47.
= twice sq. on $AC$ together with twice sq. on $GF$
= twice sq. on $AC$ together with twice sq. on $CD$.

Q. E. D.

Ex. If in any triangle $BAC$ a line $AD$ be drawn bisecting $BC$ in $D$, shew that the sum of the squares on $AB$, $AC$ is equal to twice the sum of the squares on $AD$, $BD$.

### Proposition X. Theorem.

*If a straight line be bisected and produced to any point, the square on the whole line thus produced and the square on the part of it produced are together double of the square on half the line bisected and of the square on the line made up of the half and the part produced.*




Let the st. line $AB$ be bisected in $C$ and produced to $D$.

*Then must*

*sum of sqq. on $AD$, $BD$ = twice sum of sqq. on $AC$, $CD$.*

Draw $CE \perp$ to $AB$, and make $CE = AC$.

Join $EA$, $EB$ and draw $EF \parallel$ to $AD$ and $DF \parallel$ to $CE$.

Then $\because \angle$s $FEB$, $EFD$ are together less than two rt. $\angle$s,

$\therefore EB$ and $FD$ will meet if produced towards $B$, $D$ in some pt. $G$.

Join $AG$.

Then $\because \angle ACE$ is a rt. $\angle$,

$\therefore \angle$s $EAC$, $AEC$ together = a rt. $\angle$,

and $\because \angle EAC = \angle AEC$, I. A.

$\therefore \angle AEC$ = half a rt. $\angle$.

So also $\angle BEC$ = half a rt. $\angle$.

Hence $\angle DBG$, which = $\angle EBC$, is half a rt. $\angle$,

and $\therefore \angle BGD$ is half a rt. $\angle$;

$\therefore BD = DG$.

Again, $\because \angle FGE$ = half a rt. $\angle$, and $\angle EFG$ is a rt. $\angle$,

$\therefore \angle FEG$ = half a rt. $\angle$, and $EF = FG$.

Then sum of sqq. on $AD$, $DB$

= sum of sqq. on $AD$, $DG$

= sq. on $AG$ I. 47.

= sq. on $AE$ together with sq. on $EG$

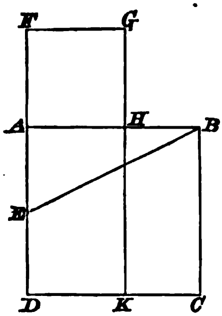
= sqq. on $AC$, $EC$ together with sqq. on $EF$, $FG$ I. 47.

= twice sq. on $AC$ together with twice sq. on $EF$

= twice sq. on $AC$ together with twice sq. on $CD$. Q.E.D.

## PROPOSITION XI. PROBLEM.

***To divide a given straight line into two parts, so that the rectangle contained by the whole and one of the parts shall be equal to the square on the other part.***




Let $AB$ be the given st. line.
On $AB$ descr. the sq. $ADCB$.
Bisect $AD$ in $E$ and join $EB$.
Produce $DA$ to $F$, making $EF = EB$.
On $AF$ descr. the sq. $AFGH$.

*Then $AB$ is divided in $H$ so that rect. $AB, BH =$ sq. on $AH$.*

Produce $GH$ to $K$.

Then $\because$ $DA$ is bisected in $E$ and produced to $F$,
$\therefore$ rect. $DF, FA$ together with sq. on $AE$
$=$ sq. on $EF$ II. 6.
$=$ sq. on $EB$, $\because$ $EB = EF$,
$=$ sum of sqq. on $AB$, $AE$. I. 47.

Take from each the square on $AE$.
Then rect. $DF, FA =$ sq. on $AB$.
Now $FK =$ rect. $DF, FA$, $\because$ $FG = FA$,
$\therefore$ $FK = AC$.

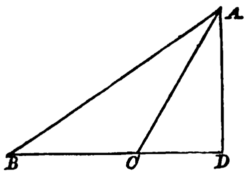
Take from each the common part $AK$.
Then $FH = HC$;
that is, sq. on $AH =$ rect. $AB, BH$, $\because$ $BC = AB$.
Thus $AB$ is divided in $H$ as was reqd.

Q. E. F.

Ex. Shew that the squares on the whole line and one of the parts are equal to three times the square on the other part.

## PROPOSITION XII. THEOREM.

*In obtuse-angled triangles, if a perpendicular be arawn rom either of the acute angles to the opposite side produced, the square on the side subtending the obtuse angle is greater than the squares on the sides containing the obtuse angle, by twice the rectangle contained by the side, upon which, when produced, the perpendicular falls, and the straight line intercepted without the triangle between the perpendicular and the obtuse angle.*




Let $ABC$ be an obtuse-angled $\triangle$, having $\angle ACB$ obtuse.

From $A$ draw $AD \perp$ to $BC$ produced.

*Then must sq. on $AB$ be greater than sum of sqq. on $BC$, $CA$ by twice rect. $BC$, $CD$.*

For since $BD$ is divided into two parts in $C$,

sq. on $BD$ = sum of sqq. on $BC$, $CD$ and twice rect. $BC$, $CD$. II. 4.

Add to each sq. on $DA$: then

sum of sqq. on $BD$, $DA$ = sum of sqq. on $BC$, $CD$, $DA$ and twice rect. $BC$, $CD$.

Now sqq. on $BD$, $DA$ = sq. on $AB$, I. 47.

and sqq. on $CD$, $DA$ = sq. on $CA$; I. 47.

$\therefore$ sq. on $AB$ = sum of sqq. on $BC$, $CA$ and twice rect. $BC$, $CD$.

$\therefore$ sq. on $AB$ is greater than sum of sqq. on $BC$, $CA$ by twice rect. $BC$, $CD$.

Q.E.D.

Ex. 1. The squares on the diagonals of a trapezium are together equal to the squares on its two sides, which are not parallel, and twice the rectangle contained by the sides, which are parallel.

Ex. 2. If $ABC$ be an equilateral triangle, and $AD$, $BE$ be perpendiculars to the opposite sides intersecting in $F$; shew that the square on $AB$ is equal to three times the square on $AF$.

## Proposition XIII. Theorem.

*In every triangle, the square on the side subtending any of the acute angles is less than the squares on the sides containing that angle, by twice the rectangle contained by either of these sides and the straight line intercepted between the perpendicular, let fall upon it from the opposite angle, and the acute angle.*

Fig. 1.

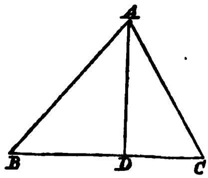
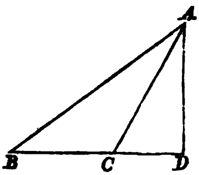



Fig. 2.




Let $ABC$ be any $\triangle$, having the $\angle ABC$ acute.

From $A$ draw $AD \perp$ to $BC$ or $BC$ produced.

*Then must sq. on $AC$ be less than the sum of sqq. on $AB$, $BC$ by twice rect. $BC, BD$.*

For in fig. 1 $BC$ is divided into two parts in $D$,

and in fig. 2 $BD$ is divided into two parts in $C$;

$\therefore$ in both cases

sum of sqq. on $BC, BD$ = sum of twice rect. $BC, BD$ and sq. on $CD$. II. 7.

Add to each the sq. on $DA$, then

sum of sqq. on $BC, BD, DA$ = sum of twice rect. $BC, BD$ and sqq. on $CD, DA$;

$\therefore$ sum of sqq. on $BC, AB$ = sum of twice rect. $BC, BD$ and sq. on $AC$; I. 47.

$\therefore$ sq. on $AC$ is less than sum of sqq. on $AB, BC$ by twice rect. $BC, BD$.

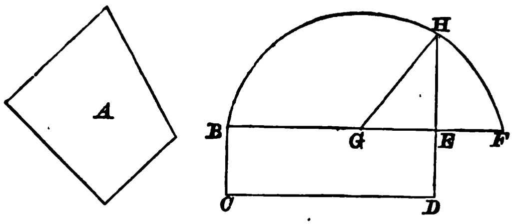
The case, in which the $\perp$ $AD$ coincides with $AC$, needs no proof.

Q.E.D.

Ex. 1. Prove that the sum of the squares on any two sides of a triangle is equal to twice the sum of the squares on half the base and on the line joining the vertical angle with the middle point of the base.

## Proposition XIV. Problem.

*To describe a square that shall be equal to a given rectilinear figure.*




Let $A$ be the given rectil. figure.

*It is reqd. to describe a square that shall* $=A$.

Describe the rectangular ▱ $BCDE=A$. I. 45.

Then if $BE=ED$, the ▱ $BCDE$ is a square, and what was reqd. is done.

But if $BE$ be not $=ED$, produce $BE$ to $F$, so that $EF=ED$.

Bisect $BF$ in $G$; and with centre $G$ and distance $GB$, describe the semicircle $BHF$.

Produce $DE$ to $H$ and join $GH$.

Then, $\because$ $BF$ is divided equally in $G$ and unequally in $E$,

$\therefore$ rect. $BE$, $EF$ together with sq. on $GE$

$=$ sq. on $GF$ II. 5.

$=$ sq. on $GH$

$=$ sum of sqq. on $EH$, $GE$. I. 47.

Take from each the square on $GE$.

Then rect. $BE$, $EF=$ sq. on $EH$.

But rect. $BE$, $EF=BD$, $\because$ $EF=ED$;

$\therefore$ sq. on $EH=BD$;

$\therefore$ sq. on $EH=$ rectil. figure $A$.

Q. E. F.

Ex. 1. Shew how to describe a rectangle equal to a given square, and having one of its sides equal to a given straight line.

Ex. 2. Divide a given straight line into two parts, so that the rectangle contained by them shall be equal to the square described upon a straight line, which is less than half the line divided.

## *Miscellaneous Exercises on Book II.*

1. In a triangle whose vertical angle is a right angle a straight line is drawn from the vertex perpendicular to the base; shew that the square on either of the sides adjacent to the right angle is equal to the rectangle contained by the base and the segment of it adjacent to that side.

2. The squares on the diagonals of a parallelogram are together equal to the squares on the four sides.

3. If $ABCD$ be any rectangle, and $O$ any point either within or without the rectangle, shew that the sum of the squares on $OA$, $OC$ is equal to the sum of the squares on $OB$, $OD$.

4. If either diagonal of a parallelogram be equal to one of the sides about the opposite angle of the figure, the square on it shall be less than the square on the other diameter by twice the square on the other side about that opposite angle.

5. Produce a given straight line $AB$ to $C$, so that the rectangle contained by the sum and difference of $AB$ and $AC$ may be equal to a given square.

6. Shew that the sum of the squares on the diagonals of any quadrilateral is less than the sum of the squares on the four sides by four times the square on the line joining the middle points of the diagonals.

7. If the square on one perpendicular from the vertex of a triangle is equal to the rectangle contained by the segments of the base, the vertical angle is a right angle.

8. Produce a given straight line so that the rectangle contained by the whole line thus produced and another given straight line may be equal to the square on the produced part.

9. $ABC$ is a triangle right-angled at $A$; in the hypothenuse two points $D$, $E$ are taken, such that $BD=BA$ and $CE=CA$; shew that the square on $DE$ is equal to twice the rectangle contained by $BE$, $CD$.

10. In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.

11. If straight lines be drawn from each angle of a triangle to bisect the opposite sides, four times the sum of the squares on these lines is equal to three times the sum of the squares on the sides of the triangle.

12. $CD$ is drawn perpendicular to $AB$, a side of the triangle $ABC$, in which $AC = AB$. Shew that the square on $CD$ is equal to the square on $BD$ together with twice the rectangle $AD$, $DB$.

## NOTE VI. *On the Measurement of Areas.*

To *measure* a Magnitude, we fix upon some magnitude of the same kind to serve as a standard or unit; and then any magnitude of that kind is measured by the number of times it contains this unit, and this number is called the MEASURE of the quantity.

Suppose, for instance, we wish to measure a straight line $AB$. We take another straight line $EF$ for our standard,


and then we say

if $AB$ contain $EF$ three times, the measure of $AB$ is 3,
if .................... four ........................................ 4,
if .................... $x$ ........................................ $x$.

Next, suppose we wish to measure two straight lines $AB$, $CD$ by the same standard $EF$.

If $AB$ contain $EF$ $m$ times
and $CD$ .............. $n$ times,

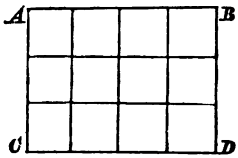
where $m$ and $n$ stand for numbers, whole or fractional, we say that $AB$ and $CD$ are *commensurable*.

But it may happen that we may be able to find a standard line $EF$, such that it is contained an exact number of times in $AB$; and yet there is no number, whole or fractional, which will express the number of times $EF$ is contained in $CD$.

In such a case, where no unit-line can be found, such that it is contained an exact number of times in *each* of two lines $AB$, $CD$; these two lines are called *incommensurable*.

In the processes of Geometry we constantly meet with incommensurable magnitudes. Thus the side and diagonal of a square are incommensurables; and so are the diameter and circumference of a circle.

Next, suppose two lines $AB$, $AC$ to be at right angles to each other and to be commensurable, so that $AB$ contains four times a certain unit of linear measurement, which is contained by $AC$ three times.



Divide $AB$, $AC$ into four and three equal parts respectively, and draw lines through the points of division parallel to $AC$, $AB$ respectively; then the rectangle $ACDB$ is divided into a number of equal squares, each constructed on a line equal to the unit of linear measurement.

If one of these squares be taken as the unit of area, the *measure* of the area of the rectangle $ACDB$ will be the number of these squares.

Now this number will evidently be the same as that obtained by multiplying the measure of $AB$ by the measure of $AC$; that is, the measure of $AB$ being 4 and the measure of $AC$ 3, the measure of $ACDB$ is $4 \times 3$ or 12. (Algebra, Art. 38.)

And *generally*, if the measures of two adjacent sides of a rectangle, supposed to be commensurable, be $a$ and $b$, then the measure of the rectangle will be $ab$. (Algebra, Art. 39.)

If all lines were commensurable, then, whatever might be the length of two adjacent sides of a rectangle, we might select the unit of length, so that the measures of the two sides should be whole numbers; and then we might apply the processes of Algebra to establish many Propositions in Geometry by simpler methods than those adopted by Euclid.

Take, for example, the theorem in Book II. Prop. IV.

If all lines were commensurable we might proceed thus:

Let the measure of $AC$ be $x$,
........................ of $CB$ ... $y$,
Then the measure of $AB$ is $x+y$.

Now $\qquad (x+y)^2=x^2+y^2+2xy,$

which proves the theorem.

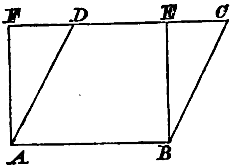
But, inasmuch as all lines are not commensurable, we have in Geometry to treat of *magnitudes* and not of *measures*: that is, when we use the symbol $A$ to represent a line, (as in I. 22), $A$ stands for the line itself and not, as in Algebra, for the number of units of length contained by the line.

The method, adopted by Euclid in Book II. to explain the relations between the rectangles contained by certain lines, is more exact than any method founded upon Algebraical principles can be; because his method applies not merely to the case in which the sides of a rectangle are commensurable, but also to the case in which they are incommensurable.

The student is now in a position to understand the practical application of the theory of Equivalence of Areas, of which the foundation is the 35th Proposition of Book I. We shall give a few examples of the use made of this theory in Mensuration.

*Area of a Parallelogram.*

The area of a parallelogram $ABCD$ is equal to the area of the rectangle $ABEF$ on the same base $AB$ and between the same parallels $AB$, $FC$.



Now $BE$ is the altitude of the parallelogram $ABCD$, if $AB$ be taken as the base.

Hence area of $\square ABCD$ = rect. $AB$, $BE$.

If then the measure of the base be denoted by $b$,
and .......................... altitude ............... $h$,
the measure of the area of the $\square$ will be denoted by $bh$.
That is, when the base and altitude are commensurable,
measure of area = measure of base into measure of altitude.

*Area of a Triangle.*

If from one of the angular points $A$ of a triangle $ABC$, a perpendicular $AD$ be drawn to $BC$, fig. 1, or to $BC$ produced, fig. 2,

Fig. 1.

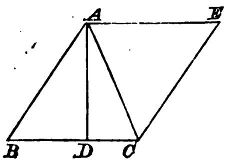
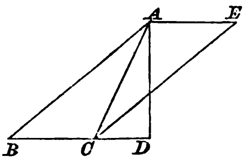


Fig. 2.



and if, in both cases, a parallelogram $ABCE$ be completed of which $AB$, $BC$ are adjacent sides,

area of $\triangle ABC$ = half of area of $\square ABCE$.

Now if the measure of $BC$ be $b$,
and ........................$AD$ ... $h$,

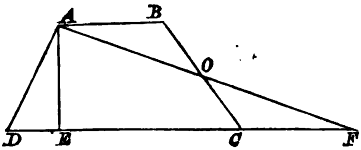
measure of area of $\square ABCE$ is $bh$;

$\therefore$ measure of area of $\triangle ABC$ is $\frac{bh}{2}$.

### *Area of a Trapezium.*

Let $ABCD$ be the given trapezium, having the sides $AB$, $CD$ parallel.

Draw $AE$ at right angles to $CD$.




Produce $DC$ to $F$, making $CF = AB$.

Join $AF$, cutting $BC$ in $O$.

Then in $\triangle$s $AOB$, $COF$,

$\because \angle BAO = \angle CFO$, and $\angle AOB = \angle FOC$, and $AB = CF$;

$$\therefore \triangle COF = \triangle AOB. \qquad \text{I. 26.}$$

Hence trapezium $ABCD = \triangle ADF$.

Now suppose the measures of $AB$, $CD$, $AE$ to be $m$, $n$, $p$ respectively;

$$\therefore \text{ measure of } DF = m + n, \because CF = AB.$$

Then measure of area of trapezium

$$= \tfrac{1}{2} (\text{measure of } DF \times \text{measure of } AE)$$

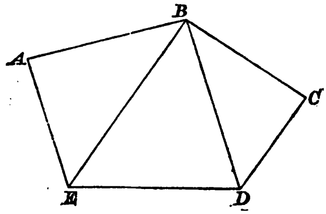
$$= \tfrac{1}{2} (m + n) \times p.$$

That is, the measure of the area of trapezium is found by multiplying half the measure of the sum of the parallel sides by the measure of the perpendicular distance between the parallel sides.

### *Area of an Irregular Polygon.*

There are three methods of finding the area of an irregular polygon, which we shall here briefly notice.

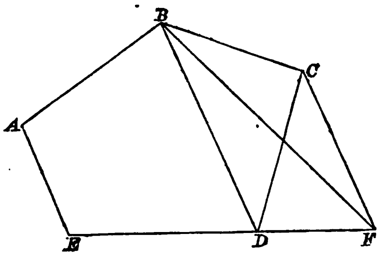
I. *The polygon may be divided into triangles*, and the area of each of these triangles be found separately.




Thus the area of the irregular polygon $ABCDE$ is equal to the sum of the areas of the triangles $ABE$, $EBD$, $DBC$.

II. *The polygon may be converted into a single triangle of equal area.*

If $ABCDE$ be a pentagon we can convert it into an equivalent quadrilateral by the following process:




Join $BD$ and draw $CF$ parallel to $BD$, meeting $ED$ produced in $F$, and join $BF$.

Then will quadrilateral $ABFE$ = pentagon $ABCDE$.

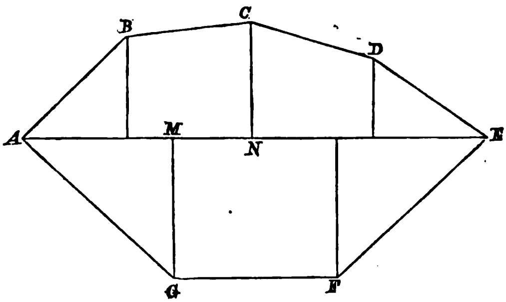
For $\triangle BDF = \triangle BCD$, on same base $BD$ and between same parallels.

If, then, from the pentagon we remove $\triangle BCD$, and add $\triangle BDF$ to the remainder, we obtain a quadrilateral $ABFE$ equivalent to the pentagon $ABCDE$.

The quadrilateral may then by a similar process be converted into an equivalent triangle, and thus a polygon of any number of sides may be gradually converted into an equivalent triangle.

The area of this triangle may then be found.

III. The third method is chiefly employed in practice by Surveyors.




Let *ABCDEFG* be an irregular polygon.

Draw *AE*, the longest diagonal, and drop perpendiculars on *AE* from the other angular points of the polygon.

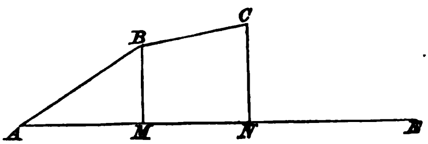
The polygon is thus divided into figures which are either right-angled triangles, rectangles, or trapeziums; and the areas of each of these figures may be readily calculated.

## NOTE VII. *On Projections.*

The projection of a *point* $B$, on a straight line of unlimited length $AE$, is the point $M$ at the foot of the perpendicular dropped from $B$ on $AE$.

The projection of a *straight line* $BC$, on a straight line of unlimited length $AE$, is $MN$,—the part of $AE$ intercepted between perpendiculars drawn from $B$ and $C$.

When two lines, as $AB$ and $AE$, form an angle, the projection of $AB$ on $AE$ is $AM$.




We might employ the term projection with advantage to shorten and make clearer the enunciations of Props. XII. and XIII. of Book II.

Thus the enunciation of Prop. XII. might be:—

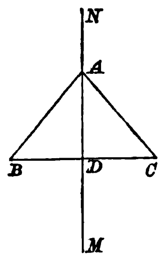
"In oblique-angled triangles, the square on the side subtending the obtuse angle is greater than the squares on the sides containing that angle, by twice the rectangle contained by one of these sides and the projection of the other on it."

The enunciation of Prop. XIII. might be altered in a similar manner.

## Note VIII. *On Loci.*

Suppose we have to determine the position of a point, which is equidistant from the extremities of a given straight line *BC*.

There is an infinite number of points satisfying this condition, for the vertex of any isosceles triangle, described on *BC* as its base, is equidistant from *B* and *C*.




Let *ABC* be *one* of the isosceles triangles described on *BC*.

If *BC* be bisected in *D*, *MN* a perpendicular to *BC* drawn through *D* will pass through *A*.

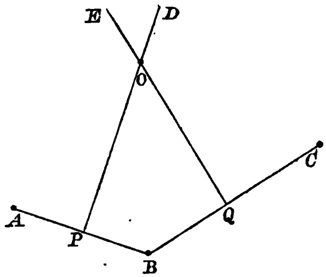
It is easy to shew that any point in *MN*, or *MN* produced in either direction, is equidistant from *B* and *C*.

It may also be proved that no point out of *MN* is equidistant from *B* and *C*.

The line *MN* is called the Locus of all the points, infinite in number, which are equidistant from *B* and *C*.

Def. In plane Geometry *Locus* is the name given to a line, straight or curved, all of whose points satisfy a certain geometrical condition (or have a common property), to the exclusion of all other points.

Next, suppose we have to determine the position of a point which is equidistant from three given points *A*, *B*, *C*, not in the same straight line.




If we join *A* and *B* we know that all points equidistant from *A* and *B* lie in the line *PD*, which bisects *AB* at right angles.

If we join *B* and *C* we know that all points equidistant from *B* and *C* lie in the line *QE*, which bisects *BC* at right angles.

Hence *O* the point of intersection of *PD* and *QE* is the only point equidistant from *A*, *B* and *C*.

*PD* is the Locus of points equidistant from *A* and *B*,

*QE* ............................................... *B* and *C*,

and the Intersection of these Loci determines the point which is equidistant from *A*, *B* and *C*.

### *Examples of Loci.*

Find the loci of

(1) Points at a given distance from a given point.

(2) Points at a given distance from a given straight line.

(3) The middle points of straight lines drawn from a given point to a given straight line.

(4) Points equidistant from the arms of an angle.

(5) Points equidistant from a given circle.

(6) Points equally distant from two straight lines which intersect.

## Note IX. *On the Methods employed in the solution of Problems.*

In the solution of Geometrical Exercises, certain methods may be applied with success to particular classes of questions.

We propose to make a few remarks on these methods, so far as they are applicable to the first two books of Euclid's Elements.

### I. *The Method of Synthesis.*

In the Exercises, attached to the Propositions in the preceding pages, the construction of the diagram, necessary for the solution of each question, has usually been fully described, or sufficiently suggested.

The student has in most cases been required simply to apply the geometrical fact, proved in the Proposition preceding the exercise, in order to arrive at the conclusion demanded in the question.

This way of proceeding is called Synthesis (σύνθεσις = composition), because in it we proceed by a regular chain of reasoning from what is *given* to what is *sought*. This being the method employed by Euclid throughout the Elements, we have no need to exemplify it here.

### II. *The Method of Analysis.*

The solution of many Problems is rendered more easy *by supposing the problem solved and the diagram constructed.* It is then often possible to observe relations between lines, angles and figures in the diagram, which are suggestive of the steps by which the necessary construction might have been effected.

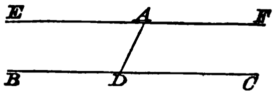
This is called the Method of Analysis (ἀνάλυσις = resolution). It is a method of discovering truth by reasoning concerning things unknown or propositions merely supposed, as if the one were given or the other were really true. The process can best be explained by the following examples.

Our first example of the Analytical process shall be the 31st Proposition of Euclid's First Book.

Ex. 1. *To draw a straight line through a given point parallel to a given straight line.*

Let $A$ be the given point, and $BC$ be the given straight line.

Suppose the problem to be effected, and $EF$ to be the straight line required.




Now we know that any straight line $AD$ drawn from $A$ to meet $BC$ makes equal angles with $EF$ and $BC$. (I. 29.)

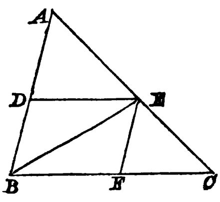
This is a fact from which we can work backward, and arrive at the steps necessary for the solution of the problem; thus:

Take any point $D$ in $BC$, join $AD$, make $\angle EAD = \angle ADC$ and produce $EA$ to $F$: then $EF$ must be parallel to $BC$.

Ex. 2. *To inscribe in a triangle a rhombus having one of its angles coincident with an angle of the triangle.*

Let $ABC$ be the given triangle.

Suppose the problem to be effected, and $DBFE$ to be the rhombus.




Then if $EB$ be joined, $\angle DBE = \angle FBE$.

This is a fact from which we can work backward, and deduce the necessary construction; thus:

Bisect $\angle ABC$ by the straight line $BE$, meeting $AC$ in $E$.

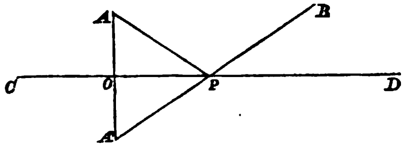
Draw $ED$ and $EF$ parallel to $BC$ and $AB$ respectively.

Then $DBFE$ is the rhombus required. (See Ex. 6, p. 59.)

Ex. 3. *To determine the point in a given straight line, at which straight lines, drawn from two given points on the same side of the given line, make equal angles with it.*

Let $CD$ be the given line, and $A$ and $B$ the given points.

Suppose the problem to be effected, and $P$ to be the point required.




We then reason thus:

If $BP$ were produced to some point $A'$,

$\angle CPA'$, being $= \angle BPD$, will be $= \angle APC$.

Again, if $PA'$ be made equal to $PA$,

$AA'$ will be bisected by $CP$ at right angles.

This is a fact from which we can work backward, and find the steps necessary for the solution of the problem; thus:

From $A$ draw $AO \perp$ to $CD$.

Produce $AO$ to $A'$, making $OA' = OA$.

Join $BA'$, cutting $CD$ in $P$.

Then $P$ is the point required.

## Note X. *On Symmetry.*

The problem, which we have just been considering, suggests the following remarks:

If two points, $A$ and $A'$, be so situated with respect to a straight line $CD$, that $CD$ bisects at right angles the straight line joining $A$ and $A'$, then $A$ and $A'$ are said to be *symmetrical* with regard to $CD$.

The importance of symmetrical relations, as suggestive of methods for the solution of problems, cannot be fully shewn to a learner, who is unacquainted with the properties of the circle. The following example, however, will illustrate this part of the subject sufficiently for our purpose at present.

*Find a point in a given straight line, such that the sum of its distances from two fixed points on the same side of the line is a minimum, that is, less than the sum of the distances of any other point in the line from the fixed points.*

Taking the diagram of the last example, suppose $CD$ to be the given line, and $A$, $B$ the given points.

Now if $A$ and $A'$ be symmetrical with respect to $CD$, we know that *every* point in $CD$ is equally distant from $A$ and $A'$. (See Note VIII. p. 103.)

Hence the sum of the distances of any point in $CD$ from $A$ and $B$ is equal to the sum of the distances of that point from $A'$ and $B$.

But the sum of the distances of a point in $CD$ from $A'$ and $B$ is the least possible when it lies in the straight line joining $A'$ and $B$.

Hence the point $P$, *determined as in the last example*, is the point required.

NOTE. Propositions IX., X., XI., XII. of Book I. give good examples of symmetrical constructions.

## NOTE XI. *Euclid's Prop. V. of Book I.*

*The angles at the base of an isosceles triangle are equal to one another; and if the equal sides be produced, the angles upon the other side of the base shall be equal.*

Let $ABC$ be an isosceles $\triangle$, having $AB = AC$.

Produce $AB$, $AC$ to $D$ and $E$.

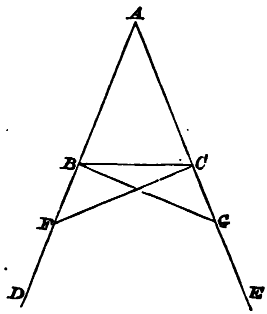
*Then must* $\angle ABC = \angle ACB$,

*and* $\angle DBC = \angle ECB$.

In $BD$ take any pt. $F$.

From $AE$ cut off $AG = AF$.

Join $FC$ and $GB$.




Then in $\triangle$s $AFC$, $AGB$,

$\because FA = GA$, and $AC = AB$, and $\angle FAC = \angle GAB$,

$\therefore FC = GB$, and $\angle AFC = \angle AGB$, and $\angle ACF = \angle ABG$. I. 4.

Again, $\because AF = AG$,

of which the parts $AB$, $AC$ are equal,

$\therefore$ remainder $BF$ = remainder $CG$. Ax. 3.

Then in $\triangle$s $BFC$, $CGB$,

$\because BF = CG$, and $FC = GB$, and $\angle BFC = \angle CGB$,

$\therefore \angle FBC = \angle GCB$, and $\angle BCF = \angle CBG$. I. 4.

Now it has been proved that $\angle ACF = \angle ABG$,
of which the parts $\angle BCF$ and $\angle CBG$ are equal;

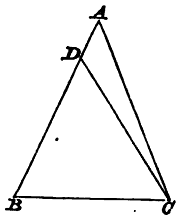
$\therefore$ remaining $\angle ACB$ = remaining $\angle ABC$. Ax. 3.

Also it has been proved that $\angle FBC = \angle GCB$,
that is, $\angle DBC = \angle ECB$.

Q. E. D.

### NOTE XII. *Euclid's Prop. VI. of Book I.*

*If two angles of a triangle be equal to one another, the sides also which subtend the equal angles shall be equal to one another.*



In $\triangle ABC$ let $\angle ACB = \angle ABC$.

*Then must* $AB = AC$.

For if not, $AB$ is either greater or less than $AC$.

Suppose $AB$ to be greater than $AC$.

From $AB$ cut off $BD = AC$.

Then in $\triangle$s $DBC$, $ACB$,

$\because$ $DB = AC$, and $BC$ is common, and $\angle DBC = \angle ACB$,

$\therefore$ $\triangle DBC = \triangle ACB$; I. 4.

that is, the less = the greater; which is absurd.

$\therefore$ $AB$ is not greater than $AC$.

Similarly it may be shewn that $AB$ is not less than $AC$;

$\therefore$ $AB = AC$.

Q. E. D.

### NOTE XIII. *Euclid's Prop. VII. of Book I.*

*Upon the same base and on the same side of it, there cannot be two triangles that have their sides which are terminated in one extremity of the base equal to one another, and their sides which are terminated in the other extremity of the base equal also.*

If it be possible, on the same base $AB$, and on the same side of it, let there be two $\triangle$s $ACB$, $ADB$, such that $AC = AD$, and also $BC = BD$.

Join $CD$.

First, when the vertex of each of the $\triangle$s is *outside* the other $\triangle$ (fig. 1);

Fig. 1.

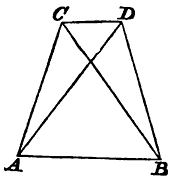
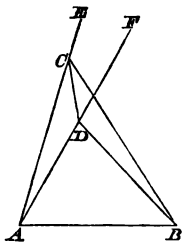


Fig. 2.



$\because AD = AC$,

$\therefore \angle ACD = \angle ADC$. I. 5.

But $\angle ACD$ is greater than $\angle BCD$;

$\therefore \angle ADC$ is greater than $\angle BCD$;

much more is $\angle BDC$ greater than $\angle BCD$.

Again, $\because BC = BD$,

$\therefore \angle BDC = \angle BCD$,

that is, $\angle BDC$ is both equal to and greater than $\angle BCD$; which is absurd.

Secondly, when the vertex $D$ of one of the $\triangle$s falls *within* the other $\triangle$ (fig. 2);

Produce $AC$ and $AD$ to $E$ and $F$.

Then $\because AC = AD$,

$\therefore \angle ECD = \angle FDC$. I. 5.

But $\angle ECD$ is greater than $\angle BCD$;

$\therefore \angle FDC$ is greater than $\angle BCD$;

much more is $\angle BDC$ greater than $\angle BCD$.

Again, $\because BC = BD$,

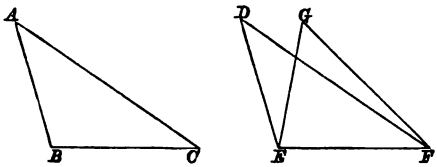
$\therefore \angle BDC = \angle BCD$;

that is, $\angle BDC$ is both equal to and greater than $\angle BCD$; which is absurd.

Lastly, when the vertex $D$ of one of the $\triangle$s falls on a side $BC$ of the other, it is plain that $BC$ and $BD$ cannot be equal. Q. E. D.

NOTE XIV. *Euclid's Prop. VIII. of Book I.*

*If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle which is contained by the two sides of the one must be equal to the angle contained by the two sides of the other.*



Let the sides of the $\triangle$s $ABC$, $DEF$ be equal, each to each, that is, $AB = DE$, $AC = DF$ and $BC = EF$.

*Then must* $\angle BAC = \angle EDF$.

Apply the $\triangle$ $ABC$ to the $\triangle$ $DEF$,
so that pt. $B$ is on pt. $E$, and $BC$ on $EF$.

Then $\because BC = EF$,
$\therefore$ $C$ will coincide with $F$,
and $BC$ will coincide with $EF$.

Then $AB$ and $AC$ must coincide with $DE$ and $DF$.

For if $AB$ and $AC$ have a different position, as $GE$, $GF$, then upon the same base and upon the same side of it there can be two $\triangle$s which have their sides which are terminated in one extremity of the base equal, and their sides which are terminated in the other extremity of the base also equal: which is impossible. I. 7.

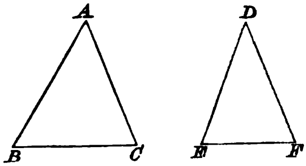
$\therefore$ since base $BC$ coincides with base $EF$,
$AB$ must coincide with $DE$, and $AC$ with $DF$;
$\therefore$ $\angle BAC$ coincides with and is equal to $\angle EDF$.

Q.E.D.

NOTE XV. *Another proof of I. 24.*

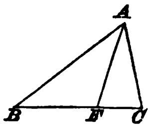
In the $\triangle$s $ABC, DEF$, let $AB=DE$ and $AC=DF$, and let $\angle BAC$ be greater than $\angle EDF$.

*Then must BC be greater than EF.*




Apply the $\triangle DEF$ to the $\triangle ABC$
so that $DE$ coincides with $AB$.

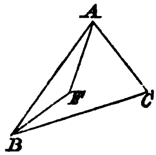
Then $\because \angle EDF$ is less than $\angle BAC$,
$DF$ will fall between $BA$ and $AC$,
and $F$ will fall *on*, or *above*, or *below*, $BC$.




I. If $F$ fall *on* $BC$,

$BF$ is less than $BC$;

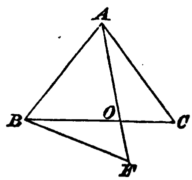
$\therefore$ $EF$ is less than $BC$.




II. If $F$ fall *above* $BC$,

$BF, FA$ together are less than $BC, CA$,
and $FA=CA$;

$\therefore$ $BF$ is less than $BC$;
$\therefore$ $EF$ is less than $BC$.

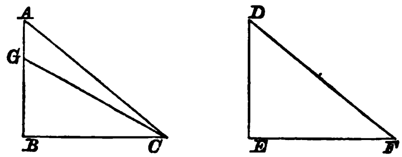



III. If $F$ fall *below* $BC$,
let $AF$ cut $BC$ in $O$.

Then $BO, OF$ together are greater than $BF$, I. 20.
and $OC, AO$..................................$AC$; I. 20.
$\therefore$ $BC, AF$..................................$BF, AC$ together,
and $AF=AC$,
$\therefore$ $BC$ is greater than $BF$;
and $\therefore$ $EF$ is less than $BC$. Q. E. D.

NOTE XVI. *Euclid's Proof of Prop. XXVI. of Book I.*

*If two triangles have two angles of the one equal to two angles of the other, each to each; and one side equal to one side, viz. either the sides adjacent to the equal angles, or the sides opposite to equal angles in each; then shall the other sides be equal, each to each; and also the third angle of the one to the third angle of the other.*




In $\triangle$s $ABC$, $DEF$,

Let $\angle ABC = \angle DEF$, and $\angle ACB = \angle DFE$;

and *first*,

Let the sides adjacent to the equal $\angle$s in each be equal, that is, let $BC = EF$.

Then must $AB = DE$, and $AC = DF$, and $\angle BAC = \angle EDF$.

For if $AB$ be not $= DE$, one of them must be the greater.

Let $AB$ be the greater, and make $GB = DE$, and join $GC$.

Then in $\triangle$s $GBC$, $DEF$,

$\because$ $GB = DE$, and $BC = EF$, and $\angle GBC = \angle DEF$,

$\therefore$ $\angle GCB = \angle DFE$. I. 4.

But $\angle ACB = \angle DFE$, by hypothesis;

$\therefore$ $\angle GCB = \angle ACB$;

that is, the less $=$ the greater, which is impossible.

$\therefore$ $AB$ is not greater than $DE$.

In the same way it may be shewn that $AB$ is not less than $DE$;

$\therefore$ $AB = DE$.

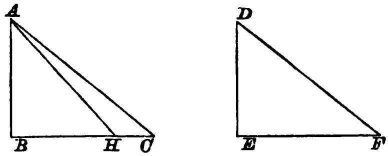
Then in $\triangle$s $ABC$, $DEF$,

$\because$ $AB = DE$, and $BC = EF$, and $\angle ABC = \angle DEF$,

$\therefore$ $AC = DF$, and $\angle BAC = \angle EDF$. I. 4.

*Next*, let the sides which are opposite to equal angles in each triangle be equal, viz. $AB=DE$.

*Then must* $AC=DF$, *and* $BC=EF$, *and* $\angle BAC=\angle EDF$.




For if $BC$ be not $=EF$, let $BC$ be the greater, and make $BH=EF$, and join $AH$.

Then in $\triangle$s $ABH$, $DEF$,

$\because AB=DE$, and $BH=EF$, and $\angle ABH=\angle DEF$,

$\therefore \angle AHB=\angle DFE$. I. 4.

But $\angle ACB=\angle DFE$, by hypothesis,

$\therefore \angle AHB=\angle ACB$;

that is, the exterior $\angle$ of $\triangle AHC$ is equal to the interior and opposite $\angle ACB$, which is impossible.

$\therefore BC$ is not greater than $EF$.

In the same way it may be shewn that $BC$ is not less than $EF$;

$\therefore BC=EF$.

Then in $\triangle$s $ABC$, $DEF$,

$\because AB=DE$, and $BC=EF$, and $\angle ABC=\angle DEF$,

$\therefore AC=DF$, and $\angle BAC=\angle EDF$.

Q. E. D.

## *Miscellaneous Examples on Books I. and II.*

1. *AB* and *CD* are equal straight lines bisecting one another at right angles. Shew that *ACBD* is a square.

2. From a point in the side of a parallelogram draw a line dividing the parallelogram into two equal parts.

3. Draw through a point lying between two lines that intersect a line terminated by the given lines and bisected in the given point.

4. The square on the hypotenuse of an isosceles right-angled triangle is equal to four times the square on the perpendicular from the right angle on the hypotenuse.

5. Describe a rhombus which shall be equal to a given triangle and have each of its sides equal to one side of the triangle.

6. Shew how to describe a square when the difference between the lengths of a diagonal and a side is given.

7. Two rings slide on two straight lines which intersect at right angles in a point *O*, and are connected by an inextensible string passing round a peg fixed at that point. Shew that the rings will be nearest to each other when they are equidistant from *O*.

8. *ABCD* is a parallelogram, whose diagonals *AC*, *BD* intersect in *O*; shew that if the parallelograms *AOBP*, *DOCQ* be completed, the straight line joining *P* and *Q* passes through *O*.

9. *ABCD*, *EBCF* are two parallelograms on the same base *BC* and so situated that *CF* passes through *A*. Join *DF*, and produce it to meet *BE* produced in *K*; join *FB*, and prove that the triangle *FAB* equals the triangle *FEK*.

10. The alternate sides of a polygon are produced to meet; shew that all the angles at their points of intersection together with four right angles are equal to all the interior angles of the polygon.

11. Shew that the perimeter of a rectangle is always greater than that of the square equal to it.

12. Shew that the opposite sides of an equiangular hexagon are parallel, though they be not equal; and that any two sides that are adjacent are together equal to the two which are parallel.

13. If two equal straight lines intersect each other anywhere at right angles, shew that the area of the quadrilateral formed by joining their extremities is invariable and equal to one-half the square on either line.

14. Two triangles $ACB$, $ADB$ are constructed on the same side of the same base $AB$. Shew that if $AC=BD$ and $AD=BC$, then $CD$ is parallel to $AB$; but if $AC=BC$ and $AD=BD$, then $CD$ is perpendicular to $AB$.

15. $AB$ is the hypotenuse of a right-angled triangle $ABC$: find a point $D$ in $AB$ such that $DB$ may be equal to the perpendicular from $D$ on $AC$.

16. Find the locus of the vertices of triangles of equal area on the same base.

17. Shew that the perimeter of an isosceles triangle is less than that of any triangle of equal area on the same base.

18. If each of the equal angles of an isosceles triangle be equal to one-fourth the vertical angle, and from one of them a perpendicular be drawn to the base, meeting the opposite side produced, then will the part produced, the perpendicular, and the remaining side, form an equilateral triangle.

19. If a straight line terminated by the sides of a triangle be bisected, shew that no other line terminated by the same two sides can be bisected in the same point.

20. From a given point draw to two parallel straight lines two equal straight lines at right angles to each other.

21. Given the lengths of the two diagonals of a rhombus, construct it.

22. $ABCD$ is a quadrilateral figure: construct a triangle whose base shall be in the line $AB$, such that its altitude shall be equal to a given line, and its area equal to that of the quadrilateral.

23. If $ABC$ be a triangle in which $C$ is a right angle, shew how to draw a straight line parallel to a given straight line, so as to be terminated by $CA$ and $CB$ and bisected by $AB$.

24. If $ABC$ be a triangle in which $C$ is a right angle, and $DE$ be drawn from a point $D$ in $AC$ at right angles to $AB$, prove that the rectangles $AB$, $AE$ and $AC$, $AD$ are equal.

25. A line is drawn bisecting a parallelogram $ABCD$ and meeting $AD$, $BC$ in $E$ and $F$; shew that the triangles $EBF$, $CED$ are equal.

26. Upon the hypotenuse $BC$ and the sides $CA$, $AB$ of a right-angled triangle $ABC$, squares $BDEC$, $AF$ and $AG$ are described: shew that the squares on $DG$ and $EF$ are together equal to five times the square on $BC$.

27. If from the vertical angle of a triangle three straight lines be drawn, one bisecting the angle, the second bisecting the base, and the third perpendicular to the base, shew that the first lies, both in position and magnitude, between the other two.

28. Shew that the area of a rhombus is equal to half the rectangle contained by the diagonals.

29. Let $ACB$, $ADB$ be two right-angled triangles having a common hypotenuse $AB$. Join $CD$ and on $CD$ produced both ways draw perpendiculars $AE$, $BF$. Shew that the sum of the squares on $CE$ and $CF$ is equal to the sum of the squares on $DE$ and $DF$.

30. In the base $AC$ of a triangle take any point $D$: bisect $AD$, $DC$, $AB$, $BC$ at the points $E$, $F$, $G$, $H$ respectively. Shew that $EG$ is equal and parallel to $FH$.

31. If $AD$ be drawn from the vertex of an isosceles triangle $ABC$ to a point $D$ in the base, shew that the rectangle $BD$, $DC$ is equal to the difference between the squares on $AB$ and $AD$.

32. If in the sides of a square four points be taken at equal distances from the four angular points taken in order, the figure contained by the straight lines which join them shall also be a square.

33. If perpendiculars $AP$, $BQ$, $CR$ be drawn from the angular points of a triangle $ABC$ upon the sides, shew that they will bisect the angles of the triangle $PQR$.

34. If of the four triangles into which the diagonals divide a quadrilateral, any two opposite ones are equal, the quadrilateral is a trapezium.

35. $ABCD$, $AECF$ are two parallelograms, $EA$, $AD$ being in a straight line. Let $FG$ drawn parallel to $AC$ meet $BA$ produced in $G$. Then the triangle $ABE$ equals the triangle $ADG$.

36. From $AC$, the diagonal of a square $ABCD$, cut off $AE$ equal to one-fourth of $AC$, and join $BE$, $DE$. Shew that the figure $BADE$ is equal to twice the square on $AE$.

37. If $ABC$ be a triangle with the angles at $B$ and $C$ each double of the angle at $A$, prove that the square on $AB$ is equal to the square on $BC$ together with the rectangle $AB$, $BC$.

38. If two sides of a quadrilateral be parallel, the triangle contained by either of the other sides and the two straight lines drawn from its extremities to the middle point of the opposite side is half the quadrilateral.

39. If two opposite angles of a quadrilateral be right angles, the angles subtended by either side at the two opposite angular points will be equal.

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