Machine Learning - Exercise 3 - LDA

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3.1

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\begin{aligned}
0 &= \frac{\partial}{\partial b} \sum_{i=1}^{N} (w^T x_i + b - y_i)^2 && (1)\\
&= \sum_{i=1}^{N} \frac{\partial}{\partial b} (w^T x_i + b - y_i)^2 && (2)\\
&= \sum_{i=1}^{N} \frac{\partial}{\partial b} (b^2 + 2(w^T x_i)b - 2by_i) && (3)\\
&= 2Nb + \sum_{i=1}^{N} 2(w^T x_i) - \sum_{i=1}^{N} 2y_i && (4)\\
&= 2Nb + \sum_{i=1}^{N} 2(w^T x_i) && (5)\\
\iff \hat{b} &= -\frac{1}{N} \sum_{i=1}^{N} (w^T x_i) && (6)\\
&= -w^T \frac{1}{N} \sum_{i=1}^{N} x_i && (7)\\
&= -w^T \mu && (8)
\end{aligned}
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3.2

$$\begin{aligned}
0 &= \frac{\partial}{\partial w}\sum_{i=1}^{N}(w^T x_i + \hat{b} - y_i)^2 && (9)\\
&= \sum_{i=1}^{N}\frac{\partial}{\partial w}((w^T x_i)^2 + 2(w^T x_i)\hat{b} - 2(w^T x_i)y_i) && (10)\\
&= \sum_{i=1}^{N}(2(w^T x_i)x_i^T + 2x_i^T\hat{b} - 2x_i^T y_i) && (11)\\
\iff 0 &= \sum_{i=1}^{N} x_i^T w^T (x_i - \mu) - \sum_{i=1}^{N} x_i^T y_i && (12)\\
\iff 0 &= \sum_{i=1}^{N} x_i w^T (x_i - \mu) - \sum_{i=1}^{N} x_i y_i && (13)\\
\iff 0 &= \sum_{i=1}^{N} x_i (x_i - \mu)^T w - \sum_{i=1}^{N} x_i y_i && (14)\\
\iff 0 &= (\sum_{i=1}^{N} x_i x_i^T - N\mu\mu^T)\hat{w} - \sum_{i=1}^{N} x_i y_i && (15)\\
& && (16)
\end{aligned}$$

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\begin{align}
S_W &= \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu_{y_i})(x_i - \mu_{y_i})^T \tag{17}\\
&= \frac{1}{N}\sum_{i:y_i=1}(x_i - \mu_1)(x_i - \mu_1)^T + \frac{1}{N}\sum_{i:y_i=-1}(x_i - \mu_{-1})(x_i - \mu_{-1})^T \tag{18}\\
&= \frac{1}{N}\sum_{i=1}^{N} x_i x_i^T - (\frac{1}{N}\sum_{i:y_i=1}^{N} x_i)\mu_1^T - \mu_1(\frac{1}{N}\sum_{i:y_i=1} x_i^T) + \frac{1}{N}\sum_{i:y_i=1}^{N}\mu_1\mu_1^T \tag{19}\\
&- (\frac{1}{N}\sum_{i:y_i=-1}^{N} x_i)\mu_{-1}^T - \mu_{-1}(\frac{1}{N}\sum_{i:y_i=1} x_i^T) + \frac{1}{N}\sum_{i:y_i=1}^{N}\mu_{-1}\mu_{-1}^T \tag{20}\\
&= \frac{1}{N}\sum_{i=1}^{N} x_i x_i^T - \frac{1}{2}\mu_1\mu_1^T - \frac{1}{2}\mu_1\mu_1^T + \frac{1}{2}\mu_1\mu_1^T \tag{21}\\
&- \frac{1}{2}\mu_{-1}\mu_{-1}^T - \frac{1}{2}\mu_{-1}\mu_{-1}^T + \frac{1}{2}\mu_{-1}\mu_{-1}^T \tag{22}\\
&= \frac{1}{N}\sum_{i=1}^{N} x_i x_i^T - \frac{1}{2}\mu_1\mu_1^T - \frac{1}{2}\mu_{-1}\mu_{-1}^T \tag{23}
\end{align}
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\begin{align}
S_B &= (\mu_1 - \mu_{-1})(\mu_1 - \mu_{-1})^T \tag{24}\\
&= \mu_1\mu_1^T - \mu_1\mu_{-1}^T - \mu_{-1}\mu_1^T + \mu_{-1}\mu_{-1}^T \tag{25}
\end{align}
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\begin{align}
0 &= (S_W + \frac{1}{4}S_B)\hat{w} - \frac{\mu_1 - \mu_{-1}}{2} \tag{26}\\
&= (\frac{1}{N}\sum_{i=1}^{N} x_i x_i^T - \frac{1}{4}(\mu_1\mu_1^T + \mu_1\mu_{-1}^T + \mu_{-1}\mu_1^T + \mu_{-1}\mu_{-1}^T))\hat{w} \tag{27}\\
&- \frac{1}{N}(\sum_{i:y_i=1} x_i y_i + \sum_{i:y_i=-1} x_i y_i) \tag{28}\\
&= (\frac{1}{N}\sum_{i=1}^{N} x_i x_i^T - \frac{1}{4}(\mu_1 + \mu_{-1})(\mu_1 + \mu_{-1})^T)\hat{w} - \frac{1}{N}\sum_{i=1}^{N} x_i y_i \tag{29}\\
&= (\frac{1}{N}\sum_{i=1}^{N} x_i x_i^T - \mu\mu^T)\hat{w} - \frac{1}{N}\sum_{i=1}^{N} x_i y_i \tag{30}\\
\iff 0 &= (\sum_{i=1}^{N} x_i x_i^T - N\mu\mu^T)\hat{w} - \sum_{i=1}^{N} x_i y_i \tag{31}\\
&QED. \tag{32}
\end{align}
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3.3

See lecture notes 6.1.

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\begin{aligned}
\frac{\mu_1 - \mu_{-1}}{2} &= (S_W + \frac{1}{4} S_B)\hat{w} && (33)\\
&= S_W \hat{w} + \frac{1}{4}(\mu_1 - \mu_{-1})((\mu_1 - \mu_{-1})^T \hat{w}) && (34)\\
\iff S_W \hat{w} &= (\mu_1 - \mu_{-1})(\frac{1}{2} - \frac{1}{4}(\mu_1 - \mu_{-1})^T \hat{w}) && (35)\\
&= \tau(\mu_1 - \mu_{-1}) && (36)\\
\iff \hat{w} &= \tau S_W^{-1}(\mu_1 - \mu_{-1}) && (37)
\end{aligned}
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