Numerical Linear Algebra - EXAM

- Submitted by Claire SUN (3630998)
- Submission Date: 2022.02.12

Declaration:

I have prepared the assignment myself and I have only used the sources declared in comments to the program

# #

[Programming Assignment Instructions (qr-method.pdf)]()

# #

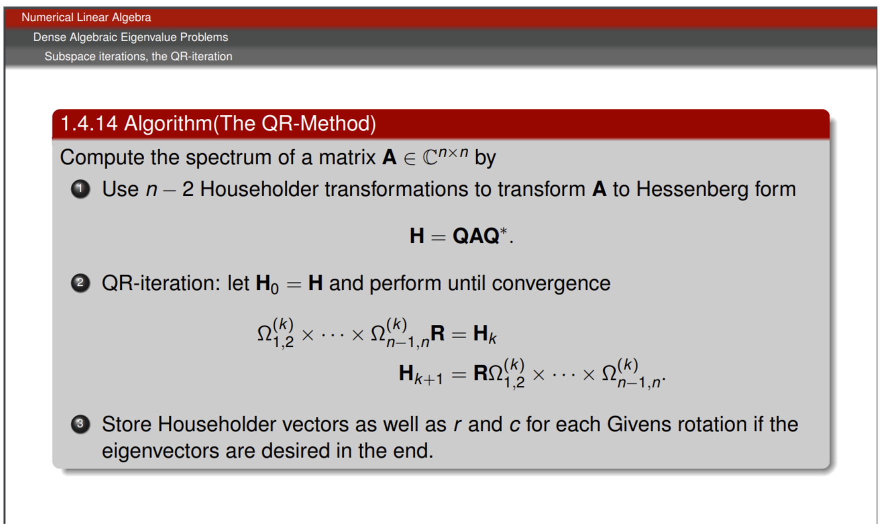


Image source: Lecture Notes

In [1]:
```
# Import Libraries

import numpy as np
import cmath
import matplotlib.pyplot as plt
from scipy.io import mmread
```

In [2]:
```
# Set Global Variables

np.set_printoptions(precision=3)
MAX_ITER = 1000 # maximum number of iterations
RTOL = 1e-6 # relative tolerance for convergence criterion, i.e. when subsequent iterates are smaller
 than RTOL (elementwise), stop iteration
ATOL = 1e-18

np.set_printoptions(precision=3, linewidth=300, suppress=True)
```

In [3]:

```python
# Generate Test Data

def generate_random_matrix(dtype, n):
    if dtype == 'float':
        A = np.random.rand(n,n)
    elif dtype == 'complex':
        A = np.random.rand(n,n) + np.random.rand(n,n)*1j
    return A

RANDOM_REAL_MATRICES = []
RANDOM_COMPLEX_MATRICES = []

for n in [3,5,10]:
    A = generate_random_matrix(dtype='float', n=n)
    RANDOM_REAL_MATRICES.append(A)

    A_ = generate_random_matrix(dtype='complex', n=n)
    RANDOM_COMPLEX_MATRICES.append(A_)


# symmetric real
A = np.random.randint(-3,3,(10,10))
A = (A.T + A).astype('float')

# diagonal dominant
B = A - np.triu(A,1)*0.99 - np.tril(A,-1)*0.99

# unbalanced
C = np.tril(np.ones((10,10),dtype='float'),0)

# Numerical example from https://people.inf.ethz.ch/arbenz/ewp/Lnotes/chapter4.pdf p.81 4.5.1
# Eigenvalues = {1 ± 2i, 3, 4, 5 ± 6i}

D = np.array([[7, 3, 4, -11, -9, -2],
              [-6, 4, -5, 7, 1, 12],
              [-1, -9, 2, 2, 9, 1],
              [-8, 0, -1, 5, 0, 8],
              [-4, 3, -5, 7, 2, 10],
              [6, 1, 4, -11, -7, -1]], dtype='float')

SPECIAL_REAL_MATRICES = [A, B, C, D]
```

Householder Transformation to Upper Hessenberg

Similarity Transformation

Two matrices A and B are similar if there exists an *invertible* matrix S such that $A = S^{-1}BS$.

Similar matrices have the same eigenvalues. Proof: $Av = \lambda v \Leftrightarrow B(S^{-1}v) = \lambda(S^{-1}v)$.

If S can be chosen to be a unitary matrix then A and B are *unitarily equivalent*.

Similarity transformation to Hessenberg form preserves eigenvalues.

The iterates $A_0, A_1, \ldots$ from the QR algorithm are also similar matrices since $A_{k+1} = R_k Q_k = Q_k^{-1}(Q_k R_k)Q_k = Q_k^{-1} A_k Q_k$. Therefore, $A_0, A1, \ldots,$ have the same eigenvalues.

#

Householder Reflector

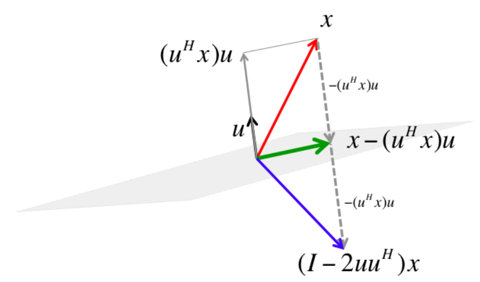
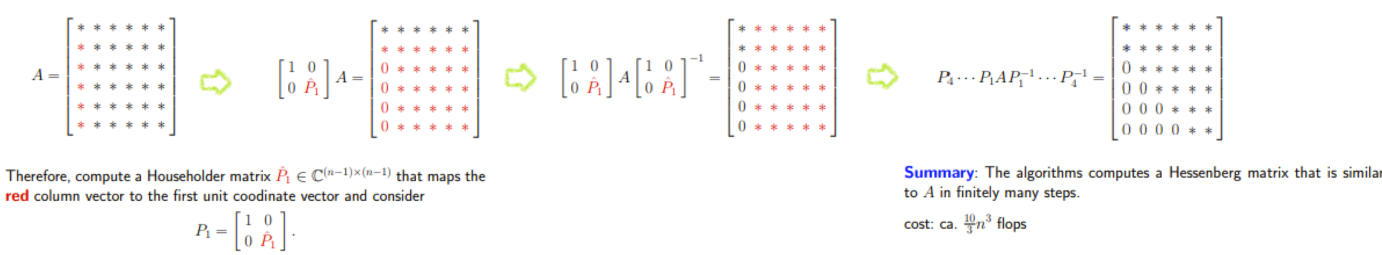


Image source: [https://www.cs.utexas.edu/users/flame/laff/alaff/chapter03-householder-transformation.html (https://www.cs.utexas.edu/users/flame/laff/alaff/chapter03-householder-transformation.html)](https://www.cs.utexas.edu/users/flame/laff/alaff/chapter03-householder-transformation.html)

#

Hessenberg Reduction



Modified based on image source: [https://www3.math.tu-berlin.de/Vorlesungen/SoSe11/NumMath2/Materials/hessenberg_eng.pdf (https://www3.math.tu-berlin.de/Vorlesungen/SoSe11/NumMath2/Materials/hessenberg_eng.pdf)](https://www3.math.tu-berlin.de/Vorlesungen/SoSe11/NumMath2/Materials/hessenberg_eng.pdf)

In [4]:
```
def Householder(a):
    """     Compute the Householder refelector H for a given vector a
    Input   ---------------------------------------------
            a: 1D np.array with real or complex entries
    Return  ---------------------------------------------
            H: 2D np.array with real or complex entries
    """

#    Initial try: naive implementation for real vectors

#    n = len(a)
#    e = np.zeros(n)
#    e[0] = np.sign(a[0])
#    v = a + np.linalg.norm(a,2) * e
#    H = np.eye(n) - 2* np.outer(v,v) / np.dot(v,v)

#   Improved version: for both real and complex vectors

    n = len(a)
    v = a.copy()

    if np.sign(a[0]) >= 0:
        sign = 1
    else:
        sign = -1

    # for real matrix: v @ v.T (transpose)
    if a.dtype != 'complex': #a.dtype =='float' or a.dtype == 'int':
        v[0] = a[0] + sign * np.linalg.norm(a,2)
        if np.linalg.norm(v,2)!= 0:
            u = v / np.linalg.norm(v,2)
        else:
            u = v
        H = np.eye(n) - 2 * np.outer(u,u)

    # for complex matrix: v @ v.conjugate.T, phase cancellation so that the first subdiagonal element
 is set to real
    else: # a.dtype == 'complex':
        # source: Lecture notes
        #theta = cmath.phase(a[0])
        #v[0] = a[0] + sign * np.linalg.norm(a,2) * np.exp(theta*1j)
        #H = np.exp(theta*-1j) * (np.eye(n) - 2 * np.outer(v,v.conjugate()) / np.dot(v,v.conjugate()))

        # Source: https://arxiv.org/pdf/math-ph/0609050.pdf p.19
        theta = cmath.phase(a[0])
        v[0] = a[0] + sign*np.linalg.norm(a,2)*np.exp(theta*1j)

        if np.linalg.norm(v,2)!= 0:
            u = v / np.linalg.norm(v,2)
        else:
            u = v

        H = np.exp(-theta*1j)*(np.eye(n) - 2* np.outer(u,u.conjugate()))

        #print(">>> check if H v = e1:", np.round(H.dot(a),3))
        #print('>>> check if H is Hermitian:',np.allclose(H@H.conjugate().T, np.eye(n))) # H @ H.conju
gate().T = I

    return H


def Hessenberg(A,eigvec=False, inplace=True):
    """ Reduce a general square matrix to an upper Hesseberg matrix by similarity transformation (with
same eigenvalues)
    Input   ---------------------------------------------
            A: 2D np.array with real or complex entries, shape is n x n (i.e. square matrix)
            eigvec : bool, indicating whether eigenvectors are required; default is False
            inplace: bool, indicating whether to update A in place; default is True
    Return  ---------------------------------------------
            H: 2D np.array with real or complex entries, same shape as A
            S: similarity transformation matrix such that H = S @ A @ S.T, if eigvec is True; otherwis
```

```
e S = identity matrix
    """

    # check if input is a square matrix
    if len(A.shape)!=2 or A.shape[0]!=A.shape[1]:
        raise ValueError('Input matrix is of shape {}. Expected square matrix.'.format(A.shape))

    n = A.shape[0]
    # if input matrix is 2x2 or smaller: already in Hessenberg
    if n <= 2:
        return A, np.eye(n)

    S = np.eye(n, dtype=A.dtype)

    if not inplace:
        A_ = A.copy()
    else:
        A_ = A

    for i in range(n-1): # (n-1) Householder transformations are needed
        a = A_[i+1:,i] # a = column vectors below diagonal, size (n-i-1), i = 0,...,n-2
        H_ = Householder(a)

        P = np.eye(n, dtype = A_.dtype) # Force P to have the same data type as input matrix
        P[i+1:,i+1:] = H_
        #print(">>> check if P is Hermitian and orthogonal:",np.allclose(P@P.conjugate().T, np.eye
(n)))

        if A_.dtype != 'complex':
            A_ = P @ A_ @ P.T
        else:
            A_ = P @ A_ @ P.conjugate().T

        #The first subdiagonal element should have only a real part; enforced here
        # A_[i+1,i] = A_[i+1,i].real
        # A_ = np.triu(A_,-1)

        if eigvec:
            S = P @ S

    return A_, S
```

In [6]:

```python
### Testing Hessenberg() on Real Matrices
from scipy.linalg import hessenberg

print('\n\n')
print("====================================================================")
print('Testing Hessenberg() function on randomly generated REAL matrices')
print("====================================================================")

#for n in [3,5,10,100]:
#    A = np.random.rand(n,n)

for A in RANDOM_REAL_MATRICES:
    print('\nReal >>> Input shape = {}'.format(A.shape))
    Hessenberg_A, S = Hessenberg(A,eigvec=True, inplace=False)
    scipy_hess_A = hessenberg(A)
    print('\nscipy.linalg.hessenberg and my Hessenberg close?',np.allclose(scipy_hess_A,Hessenberg_A))
    if not np.allclose(scipy_hess_A,Hessenberg_A):
        print('scipy.linalg.hessenberg and my Hessenberg close in abs() except signs?',np.allclose(np.abs(scipy_hess_A),np.abs(Hessenberg_A)))
    print('\nReconstructed matrix S.T@H@S close to input?',np.allclose(A, S.T@Hessenberg_A@S))

#    n = A.shape[0]
#    if n <=5:
#        print('\ninput matrix=\n', A)
#        print('\nreconstructed matrix=\n',S.T@Hessenberg_A@S)
#        print('\nscipy.linalg.hessenberg=\n',scipy_hess_A)
#        print('\nmy Hessenberg=\n',Hessenberg_A)


print('\n\n')
print("====================================================================")
print('Testing Hessenberg() function on special REAL matrices')
print("====================================================================")

for A in SPECIAL_REAL_MATRICES:
    print('\nReal >>> Input shape = {}'.format(A.shape))
    Hessenberg_A, S = Hessenberg(A,eigvec=True, inplace=False)
    scipy_hess_A = hessenberg(A)
    print('\nscipy.linalg.hessenberg and my Hessenberg close?',np.allclose(scipy_hess_A,Hessenberg_A))
    if not np.allclose(scipy_hess_A,Hessenberg_A):
        print('scipy.linalg.hessenberg and my Hessenberg close in abs() except signs?',np.allclose(np.abs(scipy_hess_A),np.abs(Hessenberg_A)))
    print('\nReconstructed matrix S.T@H@S close to input?',np.allclose(A, S.T@Hessenberg_A@S))

    print('\ninput matrix=\n', A)
    print('\nreconstructed matrix=\n',S.T@Hessenberg_A@S)
    print('\nscipy.linalg.hessenberg=\n',scipy_hess_A)
    print('\nmy Hessenberg=\n',Hessenberg_A)
```

```
===================================================================
Testing Hessenberg() function on randomly generated REAL matrices
===================================================================

Real >>> Input shape = (3, 3)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True

Reconstructed matrix S.T@H@S close to input? True

Real >>> Input shape = (5, 5)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True

Reconstructed matrix S.T@H@S close to input? True

Real >>> Input shape = (10, 10)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True

Reconstructed matrix S.T@H@S close to input? True



===================================================================
Testing Hessenberg() function on special REAL matrices
===================================================================

Real >>> Input shape = (10, 10)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True

Reconstructed matrix S.T@H@S close to input? True

input matrix=
 [[ 4. -2. -4.  3. -6.  2. -3. -2. -4.  2.]
 [-2.  4.  3. -3.  2. -3. -2. -3. -5. -6.]
 [-4.  3.  0.  3.  0.  3. -3. -1.  0.  2.]
 [ 3. -3.  3. -2. -5.  1.  3. -3.  2. -4.]
 [-6.  2.  0. -5. -6. -6.  1. -2. -5.  2.]
 [ 2. -3.  3.  1. -6.  4. -2. -1. -1. -1.]
 [-3. -2. -3.  3.  1. -2.  0. -3.  1. -3.]
 [-2. -3. -1. -3. -2. -1. -3. -6.  1. -4.]
 [-4. -5.  0.  2. -5. -1.  1.  1. -4.  1.]
 [ 2. -6.  2. -4.  2. -1. -3. -4.  1.  2.]]

reconstructed matrix=
 [[ 4. -2. -4.  3. -6.  2. -3. -2. -4.  2.]
 [-2.  4.  3. -3.  2. -3. -2. -3. -5. -6.]
 [-4.  3.  0.  3. -0.  3. -3. -1. -0.  2.]
 [ 3. -3.  3. -2. -5.  1.  3. -3.  2. -4.]
 [-6.  2. -0. -5. -6. -6.  1. -2. -5.  2.]
 [ 2. -3.  3.  1. -6.  4. -2. -1. -1. -1.]
 [-3. -2. -3.  3.  1. -2. -0. -3.  1. -3.]
 [-2. -3. -1. -3. -2. -1. -3. -6.  1. -4.]
 [-4. -5. -0.  2. -5. -1.  1.  1. -4.  1.]
 [ 2. -6.  2. -4.  2. -1. -3. -4.  1.  2.]]

scipy.linalg.hessenberg=
 [[ 4.     10.1   -0.     0.     0.     0.    -0.    -0.     0.     0.   ]
 [10.1   -4.196 -8.169 -0.    -0.    -0.    -0.     0.     0.    -0.   ]
 [ 0.    -8.169  5.375 -7.56   0.    -0.    -0.     0.    -0.    -0.   ]
 [ 0.     0.    -7.56  -7.257  5.807  0.    -0.    -0.     0.    -0.   ]
 [ 0.     0.     0.     5.807 -1.605  6.374  0.    -0.     0.     0.   ]
 [ 0.     0.     0.     0.     6.374 -3.799 -7.736 -0.    -0.     0.   ]
```

```
 [ 0.     0.     0.     0.     0.    -7.736 -0.966  2.819 -0.     0.   ]
 [ 0.     0.     0.     0.     0.     0.     2.819  0.78   4.333  0.   ]
 [ 0.     0.     0.     0.     0.     0.     0.     4.333  0.922 -5.58 ]
 [ 0.     0.     0.     0.     0.     0.     0.     0.    -5.58   2.745]]

my Hessenberg=
 [[ 4.    10.1   -0.     0.     0.     0.     0.    -0.     0.    -0.   ]
 [10.1   -4.196 -8.169  0.     0.    -0.     0.    -0.     0.    -0.   ]
 [-0.    -8.169  5.375 -7.56  -0.     0.    -0.     0.     0.     0.   ]
 [ 0.     0.    -7.56  -7.257  5.807 -0.     0.     0.     0.     0.   ]
 [ 0.     0.    -0.     5.807 -1.605  6.374  0.     0.    -0.     0.   ]
 [ 0.    -0.     0.    -0.     6.374 -3.799 -7.736  0.     0.     0.   ]
 [ 0.     0.    -0.     0.     0.    -7.736 -0.966  2.819  0.     0.   ]
 [-0.    -0.    -0.     0.     0.     0.     2.819  0.78   4.333  0.   ]
 [ 0.     0.    -0.    -0.    -0.     0.    -0.     4.333  0.922  5.58 ]
 [-0.    -0.     0.     0.    -0.    -0.     0.     0.     5.58   2.745]]

Real >>> Input shape = (10, 10)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True

Reconstructed matrix S.T@H@S close to input? True

input matrix=
 [[ 4.   -0.02 -0.04  0.03 -0.06  0.02 -0.03 -0.02 -0.04  0.02]
 [-0.02  4.    0.03 -0.03  0.02 -0.03 -0.02 -0.03 -0.05 -0.06]
 [-0.04  0.03  0.    0.03  0.    0.03 -0.03 -0.01  0.    0.02]
 [ 0.03 -0.03  0.03 -2.   -0.05  0.01  0.03 -0.03  0.02 -0.04]
 [-0.06  0.02  0.   -0.05 -6.   -0.06  0.01 -0.02 -0.05  0.02]
 [ 0.02 -0.03  0.03  0.01 -0.06  4.   -0.02 -0.01 -0.01 -0.01]
 [-0.03 -0.02 -0.03  0.03  0.01 -0.02  0.   -0.03  0.01 -0.03]
 [-0.02 -0.03 -0.01 -0.03 -0.02 -0.01 -0.03 -6.    0.01 -0.04]
 [-0.04 -0.05  0.    0.02 -0.05 -0.01  0.01  0.01 -4.    0.01]
 [ 0.02 -0.06  0.02 -0.04  0.02 -0.01 -0.03 -0.04  0.01  2.  ]]

reconstructed matrix=
 [[ 4.   -0.02 -0.04  0.03 -0.06  0.02 -0.03 -0.02 -0.04  0.02]
 [-0.02  4.    0.03 -0.03  0.02 -0.03 -0.02 -0.03 -0.05 -0.06]
 [-0.04  0.03  0.    0.03  0.    0.03 -0.03 -0.01  0.    0.02]
 [ 0.03 -0.03  0.03 -2.   -0.05  0.01  0.03 -0.03  0.02 -0.04]
 [-0.06  0.02  0.   -0.05 -6.   -0.06  0.01 -0.02 -0.05  0.02]
 [ 0.02 -0.03  0.03  0.01 -0.06  4.   -0.02 -0.01 -0.01 -0.01]
 [-0.03 -0.02 -0.03  0.03  0.01 -0.02  0.   -0.03  0.01 -0.03]
 [-0.02 -0.03 -0.01 -0.03 -0.02 -0.01 -0.03 -6.    0.01 -0.04]
 [-0.04 -0.05  0.    0.02 -0.05 -0.01  0.01  0.01 -4.    0.01]
 [ 0.02 -0.06  0.02 -0.04  0.02 -0.01 -0.03 -0.04  0.01  2.  ]]

scipy.linalg.hessenberg=
 [[ 4.     0.101  0.     0.    -0.    -0.     0.     0.     0.    -0.   ]
 [ 0.101 -2.779  3.296  0.    -0.    -0.    -0.     0.     0.    -0.   ]
 [ 0.     3.296 -0.83   2.933  0.     0.     0.     0.    -0.     0.   ]
 [ 0.     0.     2.933  0.258 -2.364 -0.    -0.    -0.     0.    -0.   ]
 [ 0.     0.     0.    -2.364 -1.808  1.79   0.     0.     0.    -0.   ]
 [ 0.     0.     0.     0.     1.79  -0.246 -1.975 -0.    -0.     0.   ]
 [ 0.     0.     0.     0.     0.    -1.975 -0.653  0.46  -0.     0.   ]
 [ 0.     0.     0.     0.     0.     0.     0.46  -5.93   0.272  0.   ]
 [ 0.     0.     0.     0.     0.     0.     0.     0.272  0.377 -1.138]
 [ 0.     0.     0.     0.     0.     0.     0.     0.    -1.138  3.611]]

my Hessenberg=
 [[ 4.     0.101 -0.    -0.     0.     0.    -0.    -0.    -0.    -0.   ]
 [ 0.101 -2.779  3.296  0.     0.    -0.    -0.    -0.     0.     0.   ]
 [-0.     3.296 -0.83   2.933  0.    -0.    -0.    -0.    -0.    -0.   ]
 [-0.    -0.     2.933  0.258 -2.364  0.    -0.     0.    -0.     0.   ]
 [ 0.     0.     0.    -2.364 -1.808  1.79   0.     0.     0.    -0.   ]
 [ 0.    -0.    -0.     0.     1.79  -0.246 -1.975  0.    -0.     0.   ]
 [-0.     0.    -0.    -0.    -0.    -1.975 -0.653  0.46  -0.    -0.   ]
 [-0.     0.    -0.     0.    -0.     0.     0.46  -5.93   0.272  0.   ]
 [-0.     0.     0.    -0.    -0.     0.    -0.     0.272  0.377  1.138]
 [-0.    -0.     0.     0.    -0.     0.    -0.    -0.     1.138  3.611]]
```

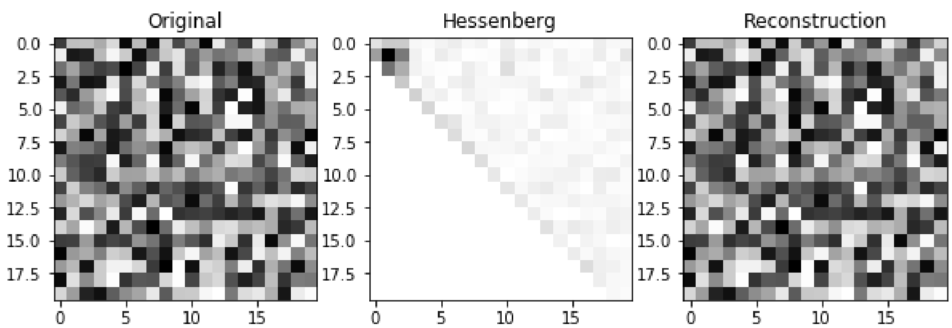
```
Real >>> Input shape = (10, 10)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True

Reconstructed matrix S.T@H@S close to input? True

input matrix=
 [[1. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [1. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [1. 1. 1. 0. 0. 0. 0. 0. 0. 0.]
 [1. 1. 1. 1. 0. 0. 0. 0. 0. 0.]
 [1. 1. 1. 1. 1. 0. 0. 0. 0. 0.]
 [1. 1. 1. 1. 1. 1. 0. 0. 0. 0.]
 [1. 1. 1. 1. 1. 1. 1. 0. 0. 0.]
 [1. 1. 1. 1. 1. 1. 1. 1. 0. 0.]
 [1. 1. 1. 1. 1. 1. 1. 1. 1. 0.]
 [1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]]

reconstructed matrix=
 [[ 1.  0.  0.  0.  0.  0.  0.  0.  0.  0.]
 [ 1.  1.  0.  0.  0.  0.  0.  0.  0. -0.]
 [ 1.  1.  1.  0.  0.  0.  0.  0.  0. -0.]
 [ 1.  1.  1.  1.  0.  0.  0.  0.  0.  0.]
 [ 1.  1.  1.  1.  1.  0.  0.  0.  0.  0.]
 [ 1.  1.  1.  1.  1.  1.  0.  0.  0.  0.]
 [ 1.  1.  1.  1.  1.  1.  1.  0.  0.  0.]
 [ 1.  1.  1.  1.  1.  1.  1.  1.  0.  0.]
 [ 1.  1.  1.  1.  1.  1.  1.  1.  1.  0.]
 [ 1.  1.  1.  1.  1.  1.  1.  1.  1.  1.]]

scipy.linalg.hessenberg=
 [[ 1.     0.     0.     0.     0.     0.     0.     0.     0.     0.   ]
 [-3.     5.     2.582  0.     0.    -0.     0.    -0.    -0.     0.   ]
 [ 0.    -2.582  0.5   -1.133 -0.     0.    -0.    -0.    -0.     0.   ]
 [ 0.     0.     1.133  0.5    0.717  0.    -0.     0.     0.     0.   ]
 [ 0.     0.     0.    -0.717  0.5   -0.508 -0.     0.     0.     0.   ]
 [ 0.     0.     0.     0.     0.508  0.5   -0.376 -0.    -0.    -0.   ]
 [ 0.     0.     0.     0.     0.     0.376  0.5   -0.28  -0.     0.   ]
 [ 0.     0.     0.     0.     0.     0.     0.28   0.5   -0.203  0.   ]
 [ 0.     0.     0.     0.     0.     0.     0.     0.203  0.5    0.129]
 [ 0.     0.     0.     0.     0.     0.     0.     0.    -0.129  0.5  ]]

my Hessenberg=
 [[ 1.     0.     0.     0.     0.     0.     0.     0.     0.     0.   ]
 [-3.     5.     2.582 -0.    -0.     0.    -0.     0.    -0.     0.   ]
 [-0.    -2.582  0.5   -1.133  0.    -0.    -0.    -0.    -0.    -0.   ]
 [ 0.     0.     1.133  0.5    0.717  0.     0.    -0.    -0.    -0.   ]
 [ 0.     0.    -0.    -0.717  0.5   -0.508  0.     0.     0.     0.   ]
 [-0.     0.     0.     0.     0.508  0.5   -0.376 -0.     0.    -0.   ]
 [ 0.    -0.     0.     0.    -0.     0.376  0.5   -0.28   0.     0.   ]
 [-0.    -0.    -0.    -0.    -0.     0.     0.28   0.5   -0.203  0.   ]
 [ 0.    -0.     0.     0.     0.     0.    -0.     0.203  0.5   -0.129]
 [ 0.    -0.    -0.    -0.     0.     0.    -0.    -0.     0.129  0.5  ]]

Real >>> Input shape = (6, 6)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True

Reconstructed matrix S.T@H@S close to input? True

input matrix=
 [[  7.   3.   4. -11.  -9.  -2.]
 [ -6.   4.  -5.   7.   1.  12.]
 [ -1.  -9.   2.   2.   9.   1.]
 [ -8.   0.  -1.   5.   0.   8.]
 [ -4.   3.  -5.   7.   2.  10.]
 [  6.   1.   4. -11.  -7.  -1.]]

reconstructed matrix=
 [[  7.   3.   4. -11.  -9.  -2.]
```

```
 [ -6.   4.  -5.   7.   1.  12.]
 [ -1.  -9.   2.   2.   9.   1.]
 [ -8.   0.  -1.   5.  -0.   8.]
 [ -4.   3.  -5.   7.   2.  10.]
 [  6.   1.   4. -11.  -7.  -1.]]

scipy.linalg.hessenberg=
 [[  7.      7.276   5.812  -0.14    9.015   7.936]
 [ 12.369   4.131  18.969  -1.207  10.683   2.416]
 [  0.     -7.16    2.448  -0.566  -4.181  -3.251]
 [  0.      0.     -8.599   2.915  -3.417   5.723]
 [  0.      0.      0.      1.046  -2.835 -10.979]
 [  0.      0.      0.      0.      1.414   5.342]]

my Hessenberg=
 [[ 7.     7.276  5.812 -0.14   9.015 -7.936]
 [12.369  4.131 18.969 -1.207 10.683 -2.416]
 [ 0.    -7.16   2.448 -0.566 -4.181  3.251]
 [-0.     0.    -8.599  2.915 -3.417 -5.723]
 [-0.    -0.    -0.     1.046 -2.835 10.979]
 [-0.    -0.     0.     0.    -1.414  5.342]]
```

In [7]:

```python
# Visualiation of Tranforming Matrix A to Hessenberg form via Householder Reflection

f, (ax0, ax1, ax2) = plt.subplots(1,3,figsize=(10,20))

n = 20
A = np.random.rand(n,n)
H, S = Hessenberg(A,eigvec=True, inplace=False)
recon = S.T @ H @ S

im = ax0.imshow(-abs(A),cmap='gray')
ax0.set_title('Original')
ax1.imshow(-abs(H),cmap='gray')
ax1.set_title('Hessenberg')
ax2.imshow(-abs(recon),cmap='gray')
ax2.set_title('Reconstruction')

print("Illustration of Transformation to Hessenberg Form ")
```

```
Illustration of Transformation to Hessenberg Form
```



#

Observations:

- Seems to be working for real matrices ##

In [8]:
```python
### Testing Hessenberg() on Random Complex Matrices

print('\n\n')
print("=====================================================================")
print('Testing Hessenberg() function on randomly generated COMPLEX matrices')
print("=====================================================================")

#for n in [3,5,10,100]:
#    A = np.random.rand(n,n)+np.random.rand(n,n)*1j

for A in RANDOM_COMPLEX_MATRICES:
    print('\n\nComplex >>> Input shape = {}'.format(A.shape))
    Hessenberg_A, S = Hessenberg(A,eigvec=True)
    scipy_hess_A = hessenberg(A)
    print('\nscipy.linalg.hessenberg and my Hessenberg close?',np.allclose(scipy_hess_A,Hessenberg_A))
    if not np.allclose(scipy_hess_A,Hessenberg_A):
        print('scipy.linalg.hessenberg and my Hessenberg close in abs() except signs?',np.allclose(np.
abs(scipy_hess_A),np.abs(Hessenberg_A)))
    print('\nReconstructed matrix S.T@H@S close to input?',np.allclose(A, S.conjugate().T@Hessenberg_A
@S))

#    n = A.shape[0]
#    if n <=5:
#        print('\ninput matrix=\n', A)
#        print('\nreconstructed matrix=\n',S.conjugate().T@Hessenberg_A@S)
#        print('\nscipy.linalg.hessenberg=\n',scipy_hess_A)
#        print('\nmy Hessenberg=\n',Hessenberg_A)
```

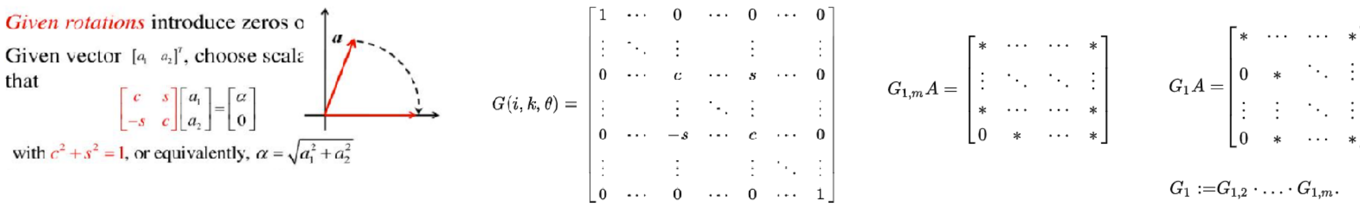
```
=====================================================================
Testing Hessenberg() function on randomly generated COMPLEX matrices
=====================================================================


Complex >>> Input shape = (3, 3)

scipy.linalg.hessenberg and my Hessenberg close? True

Reconstructed matrix S.T@H@S close to input? True


Complex >>> Input shape = (5, 5)

scipy.linalg.hessenberg and my Hessenberg close? True

Reconstructed matrix S.T@H@S close to input? True


Complex >>> Input shape = (10, 10)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True

Reconstructed matrix S.T@H@S close to input? True
```

# #

Observations:

- Works for both real and complex matrices
- Modified for the optional output of the similarity transformation matrix S ##

Givens Rotation



Modified from Image Source: [https://www.slideserve.com/pekelo/scientific-computing-chapter-3-linear-least-squares (https://www.slideserve.com/pekelo/scientific-computing-chapter-3-linear-least-squares)](https://www.slideserve.com/pekelo/scientific-computing-chapter-3-linear-least-squares), [https://de.wikipedia.org/wiki/Givens-Rotation (https://de.wikipedia.org/wiki/Givens-Rotation)](https://de.wikipedia.org/wiki/Givens-Rotation)

If x_j and x_k are real numbers:

- $\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} x_j \\ x_k \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$, where $r = \sqrt{x_j^2 + x_k^2}$, $c = x_j/r$ and $s = x_k/r$

If x_j and x_k are complex numbers:

- $\begin{bmatrix} c & s \\ -\bar{s} & c \end{bmatrix} \begin{bmatrix} x_j \\ x_k \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$, where
- $c = x_j^*/\sqrt{|x_j|^2 + |x_k|^2} = cos\theta_a e^{-i\theta_j}$
- $s = x_k^*/\sqrt{|x_j|^2 + |x_k|^2} = sin\theta_a e^{-i\theta_k}$
- $\theta_a = arctan(|x_k|/|x_j|) = arctan(Re(x_k)/cos\theta_k \times cos\theta_j/Re(x_j))$

In [25]:
```python
# Source: https://amir.sdsu.edu/Wen17A.pdf, page 2, equations (7) and (8)

def Givens(x_j,x_k):
    """ Find the Givens rotation matrix such that G @ [x_j,x_k] = [*, 0]
    Input   ----------------------------------------------
            x_j: jth row/column element from a np.array with real or complex entries
            x_k: kth row/column element from a np.array with real or complex entries

    Return  ---------------------------------------------
            G: 2x2 rotation matrix in np.array with real or complex entries
    """


    if x_k == 0.:# or x_k.real: #x_k== 0+0*1j : # including the case x_j=0 and x_k=0
        c = 1.
        s = 0.

    elif x_j == 0.:# or x_j.real == 0: #x_j == 0+0*1j : # i.e. x_k!=0
        c = 0.
        s = 1. # +1 or -1

    else: # f, g both non-zero
        theta_j = cmath.phase(x_j)
        theta_k = cmath.phase(x_k)
        theta_a = np.arctan(x_k.real/np.cos(theta_k)*np.cos(theta_j)/x_j.real) #div by zero excluded f
rom condition above
        c = np.cos(theta_a)*np.exp(-1j*theta_j)
        s = np.sin(theta_a)*np.exp(-1j*theta_k)

    if x_j.dtype == 'complex' or x_k.dtype == 'complex':
        G = np.array([[c,s],[-s.conjugate(),c.conjugate()]]).astype('complex')

    else:
        G = np.array([[c,s],[-s,c]]).astype('float')

    return G





# (Source: https://www.netlib.org/lapack/lawnspdf/lawn148.pdf, p.5) -> implementation does not work fo
r complex matrices...
#def __Givens__(f,g):
#    if g == 0:
#        c = 1
#        s = 0
#        r = f
#    elif f == 0: # g!=0
#        c = 0
#        s = np.sign(g.conjugate())
#        r = np.abs(g)
#    else: # f, g both non-zero
#        d = np.sqrt(np.abs(f)**2 +np.abs(g)**2 )
#        c = np.abs(f) / d
#        s = np.sign(f) * (g.conjugate()) / d
#        r = np.sign(f) * d
#    G = np.array([[c,s],[-s.conjugate(),c]])
#    return G


#def Givens_real(x_j, x_k):
#    """Avoiding squaring of small numbers
#    TODO: Need to adapt for complex numbers
#    """
#    # if the subdiagonal is already zero or close enough to zero, just return identity matrix
#    if abs(x_j) > abs(x_k) and x_j!=0:
#        tan = x_k / x_j
#        c = 1/np.sqrt(1+np.square(tan))
#        s = tan*c
#    else:
```

```python
#        cotan = x_j / x_k
#        s = 1/np.sqrt(1+np.square(cotan))
#        c = cotan*s
#    G = np.array([[c,s],[-s,c]])
#    return G


#def QR_Givens_naive(H):
#    '''Input: matrix H in Hessenberg form
#       Matrix multiplication directly
#    '''
#    n = H.shape[0]
#    G = np.eye(n)
#    for i in range(n-1): # apply (n-1) Givens rotations to zero-out the sub-diagonal elements in Hess
enberg matrix
#        G_ = np.eye(n)
#        G_[i:i+2, i:i+2] = Givens(H[i,i],H[i+1,i])
#        G = G_ @ G
#    R = G @ H
#    Q = G.conjugate().T
#    return Q, R

# [Gn ... G2 G1].T [Gn ... G2 G1] H = H;


#def QR_Givens_rows(H,eigvec=False):
#    '''Row-wise application of Givens rotation (2 rows at a time) rather than whole matrix multiplica
tion
#    Input: H = numpy array, Hessenberg matrix
#    eigvec = Bool, indicating whether eigenvectors is required; if True, store and output all Givens
 matrics
#    Return: Updated H_new overwritten on H_old // no explicit calculation of Q and R matrices
#    '''
#    n = H.shape[0]
#    G_store = []
#    G_iter = np.eye(n,dtype=H.dtype)
#
#    for i in range(n-1):
#        G = Givens(H[i,i],H[i+1,i])
#        new_rows = G @ H[i:i+2, :]
#        H[i:i+2, :] = new_rows
#        G_store.append(G)
#
#        # if eigenvectors are required
#        if eigvec:
#            G_ = np.eye(n,dtype=H.dtype)
#            G_[i:i+2,i:i+2] = G
#            G_iter = G_ @ G_iter
#
#    for i in range(n-1):
#        G = G_store.pop(0) #same order as above G1, G2, ...
#
#        if G.dtype != 'complex':
#            new_cols = H[:, i:i+2] @ G.T
#        else:
#            new_cols = H[:, i:i+2] @ G.conjugate().T
#
#        H[:, i:i+2] = new_cols
#
#    return H, G_iter



def select_idx(i,n,tridiagonal):
    """Output starting and ending indices for element application of Givens Rotation"""
    if tridiagonal:
        row_start = max(0,i-1) # for post multiplication of G.T
        col_end = i+3
    else:
        row_start = 0
        col_end = n
    return row_start, col_end
```

```python
def QR_Givens(H,eigvec=False, inplace=True, tridiagonal=False):

    """ Perform QR decomposition on a Hessenberg matrix by applying Givens Rotation (element-wise rath
er than whole matrix multiplication)
        No explicit computation of Q and R matrices; output updated Hessenberg matrix directly

    Input   ---------------------------------------------
            H: 2D np.array with real or complex entries, in the form of an upper Hessenberg matrix of
 shape n x n (i.e. square matrix)
            eigvec : bool, indicating whether eigenvectors are required; default is False
            inplace: bool, indicating whether to update H in place; default is True
            tridiagonal: bool, indicating whether H is tridiagonal; default is False
    Return  --------------------------------------------
            H_new: 2D np.array with real or complex entries, in the form of an *updated* Hessenberg ma
trix, effectively R @ Q
            G_iter: 2D np.array, the accumulated Givens rotation matrix, effectively Q.T; if eigvec is
False, not explicitly calculated and output identity matrix

            R = G @ H_old : applying Givens rotation G = Gn @ ... @ G2 @ G1 column by column to reduce
Hessenberg to upper triangular
            H_old = G.T @ R => Q = G.T
            H_new = R @ Q = G @ H_old @ G.T : update Hessenberg matrix (no explicite calculation )
            G_iter = G = Gn @ ... @ G2 @ G1 : this is effectively the Givens rotation per round of QR
 iteration
    """

    n = H.shape[0]
    G_store = []
    G_iter = np.eye(n,dtype=H.dtype)

    if inplace:
        H_old = H
    else:
        H_old = H.copy()

    for i in range(n-1):

        _, col_end = select_idx(i,n,tridiagonal)

        G = Givens(H_old[i,i],H_old[i+1,i])

        # Premultiplication by a rotation in the (i, i+1)-plane only involves rows i and i+1 and leave
s other rows unaffected
        # Furthermore, only consider non-zero entries in these two rows, to avoid unnecessary multipli
cation by zero
        H_old[i:i+2,i:col_end] = G @ H_old[i:i+2, i:col_end]

        G_store.append(G)

        # if eigenvectors are required
        if eigvec:
            G_ = np.eye(n,dtype=H.dtype)
            G_[i:i+2,i:i+2] = G
            G_iter = G_ @ G_iter

    # H_old is now effectively R; next is to obtain H_new = R @ Q where Q = G1.T G2.T ...Gn-1.T

    H_new = H_old

    for i in range(n-1):
        row_start, _ = select_idx(i,n,tridiagonal)

        G = G_store.pop(0) #same order as above G1, G2, ...

        if G.dtype != 'complex':
            G_T = G.T
        else:
            G_T = G.conjugate().T

        # Similar to above, postmultiplication by a rotation in the (i, i+1)-plane affects only column
```

```
s i and i+1.
        H_new[row_start:i+2, i:i+2] = H_new[row_start:i+2, i:i+2] @ G_T

    # H_new is the updated version, effectively R @ Q

    return H_new, G_iter
```

In [10]:

```python
### Testing QR_Givens() on Random Complex Matrices

print('\n\n')
print("=====================================================================")
print('Testing QR_Givens() function on randomly generated COMPLEX matrices')
print("=====================================================================")

for A in RANDOM_COMPLEX_MATRICES:
    print('\n\nComplex >>> Input shape = {}'.format(A.shape))
    H, S = Hessenberg(A,eigvec=True)
    print('A =\n',A)
    print('H =\n',H)

    H_new, G = QR_Givens(H,eigvec=True, tridiagonal=False)
    Q = G.T
    print('\nQR_Givens: H_new=\n',H_new)
    print('Q.conjugate().T @ Q = np.eye(n)? ',np.allclose(Q.conjugate().T@Q,np.eye(A.shape[0], dtype=
'complex')))


### Testing QR_Givens() on Random REAL Matrices

print('\n\n')
print("=====================================================================")
print('Testing QR_Givens() function on randomly generated REAL matrices')
print("=====================================================================")

for A in RANDOM_REAL_MATRICES:
    print('\n\nComplex >>> Input shape = {}'.format(A.shape))
    H, S = Hessenberg(A,eigvec=True)
    print('A =\n',A)
    print('H =\n',H)

    H_new, G = QR_Givens(H,eigvec=True, tridiagonal=False)
    Q = G.T
    print('\nQR_Givens: H_new=\n',H_new)
    print('Q.T @ Q = np.eye(n)? ',np.allclose(Q.T@Q,np.eye(A.shape[0])))



### Testing QR_Givens() on Random Complex Matrices

print('\n\n')
print("=====================================================================")
print('Testing QR_Givens() function on SPECIAL real matrices')
print("=====================================================================")

for A in SPECIAL_REAL_MATRICES:
    print('\n\nSpecial >>> Input shape = {}'.format(A.shape))
    H, S = Hessenberg(A,eigvec=True)
    print('A =\n',A)
    print('H =\n',H)

    H_new, G = QR_Givens(H,eigvec=True, tridiagonal=True)
    Q = G.T
    print('\nQR_Givens: H_new=\n',H_new)
    print('Q.T @ Q = np.eye(n)? ',np.allclose(Q.T@Q,np.eye(A.shape[0])))
```

```
====================================================================
Testing QR_Givens() function on randomly generated COMPLEX matrices
====================================================================


Complex >>> Input shape = (3, 3)
A =
 [[0.3  +0.511j 0.16 +0.142j 0.42 +0.515j]
 [0.052+0.216j 0.479+0.131j 0.906+0.751j]
 [0.002+0.02j  0.849+0.012j 0.64 +0.243j]]
H =
 [[ 0.3  +0.511j  0.142-0.23j   0.402-0.502j]
 [-0.223+0.j     0.626+0.2j    0.894+0.772j]
 [-0.   -0.j     0.85 +0.j     0.493+0.174j]]

QR_Givens: H_new=
 [[ 0.3  +0.511j  0.142-0.23j   0.402-0.502j]
 [-0.223+0.j     0.626+0.2j    0.894+0.772j]
 [-0.   -0.j     0.85 +0.j     0.493+0.174j]]
Q.conjugate().T @ Q = np.eye(n)?  True


Complex >>> Input shape = (5, 5)
A =
 [[0.344+0.535j 0.001+0.474j 0.596+0.834j 0.358+0.025j 0.886+0.639j]
 [0.168+0.708j 0.048+0.356j 0.473+0.686j 0.645+0.425j 0.847+0.436j]
 [0.984+0.403j 0.7  +0.528j 0.541+0.599j 0.4  +0.919j 0.642+0.75j ]
 [0.195+0.129j 0.416+0.407j 0.623+0.733j 0.855+0.817j 0.693+0.162j]
 [0.265+0.313j 0.444+0.583j 0.252+0.571j 0.822+0.535j 0.77 +0.606j]]
H =
 [[ 0.344+0.535j -0.012-1.194j -0.495+0.451j -0.132-0.43j  -0.595-0.407j]
 [-1.372+0.j     1.38 +1.613j -0.072-1.565j  0.15 +0.113j  0.039+0.004j]
 [ 0.   -0.j    -1.519+0.j     0.974+0.718j -0.257+0.147j -0.41 -0.47j ]
 [ 0.   +0.j    -0.   -0.j    -0.452-0.j     0.004+0.071j  0.151-0.1j  ]
 [ 0.   +0.j    -0.   +0.j     0.   +0.j    -0.486-0.j    -0.144-0.025j]]

QR_Givens: H_new=
 [[ 0.344+0.535j -0.012-1.194j -0.495+0.451j -0.132-0.43j  -0.595-0.407j]
 [-1.372+0.j     1.38 +1.613j -0.072-1.565j  0.15 +0.113j  0.039+0.004j]
 [ 0.   -0.j    -1.519+0.j     0.974+0.718j -0.257+0.147j -0.41 -0.47j ]
 [ 0.   +0.j    -0.   -0.j    -0.452-0.j     0.004+0.071j  0.151-0.1j  ]
 [ 0.   +0.j    -0.   +0.j     0.   +0.j    -0.486-0.j    -0.144-0.025j]]
Q.conjugate().T @ Q = np.eye(n)?  True


Complex >>> Input shape = (10, 10)
A =
 [[0.888+0.745j 0.354+0.961j 0.881+0.881j 0.768+0.572j 0.42 +0.319j 0.051+0.541j 0.98 +0.49j  0.911+
0.984j 0.232+0.29j  0.506+0.794j]
 [0.906+0.351j 0.999+0.772j 0.365+0.895j 0.701+0.347j 0.907+0.705j 0.275+0.875j 0.296+0.681j 0.371+0.
522j 0.502+0.777j 0.204+0.111j]
 [0.939+0.729j 0.143+0.564j 0.096+0.823j 0.579+0.807j 0.484+0.944j 0.63 +0.019j 0.467+0.138j 0.484+0.
097j 0.796+0.713j 0.03 +0.42j ]
 [0.41 +0.211j 0.466+0.692j 0.302+0.099j 0.301+0.497j 0.51 +0.994j 0.584+0.405j 0.827+0.609j 0.209+0.
734j 0.601+0.954j 0.407+0.745j]
 [0.003+0.718j 0.787+0.941j 0.588+0.316j 0.033+0.22j  0.914+0.965j 0.059+0.119j 0.948+0.148j 0.143+0.
672j 0.109+0.959j 0.271+0.934j]
 [0.013+0.269j 0.259+0.785j 0.916+0.258j 0.356+0.848j 0.844+0.826j 0.089+0.197j 0.756+0.115j 0.149+0.
99j  0.843+0.203j 0.737+0.539j]
 [0.177+0.352j 0.543+0.746j 0.467+0.721j 0.139+0.99j  0.895+0.289j 0.051+0.964j 0.671+0.427j 0.461+0.
426j 0.233+0.42j  0.833+0.438j]
 [0.646+0.11j  0.694+0.241j 0.063+0.279j 0.383+0.526j 0.039+0.058j 0.824+0.989j 0.45 +0.289j 0.345+0.
935j 0.814+0.749j 0.62 +0.746j]
 [0.423+0.813j 0.889+0.119j 0.558+0.666j 0.107+0.829j 0.258+0.74j  0.689+0.141j 0.542+0.975j 0.719+0.
243j 0.595+0.123j 0.744+0.268j]
 [0.888+0.031j 0.098+0.714j 0.57 +0.869j 0.332+0.875j 0.63 +0.344j 0.414+0.72j  0.589+0.091j 0.672+0.
514j 0.384+0.509j 0.104+0.513j]]
H =
 [[ 0.888+0.745j -0.327-2.322j -1.1  +0.486j  0.44 -0.207j  0.116+0.45j  -0.275+0.103j  0.099+0.192j
```

```
-0.103-0.358j  0.364-0.089j -0.228+0.501j]
 [-2.318+0.j     2.611+3.49j   0.032-2.875j -0.559+0.8j    0.418+0.029j -0.666-0.011j  0.135-0.213j
0.187-0.177j  0.035-0.275j -0.091-0.018j]
 [ 0.   -0.j    -3.712+0.j     1.006+1.43j   0.254-0.165j -0.565-0.098j  0.231+0.182j -0.1  +0.127j -
0.334+0.447j  0.05 -0.057j -0.066+0.027j]
 [ 0.   -0.j     0.   -0.j    -1.224-0.j    -0.032+0.403j -0.031+0.329j  0.096-0.201j  0.442+0.388j
0.29 +0.227j -0.41 -0.111j -0.068+0.012j]
 [ 0.   -0.j    -0.   -0.j     0.   -0.j     0.974+0.j     0.268+0.386j  0.149-0.117j  0.064-0.103j -
0.686+0.187j -0.078+0.719j -0.521-0.506j]
 [-0.   +0.j    -0.   -0.j    -0.   +0.j    -0.   +0.j     0.774+0.j    -0.164-0.23j   0.273-0.768j
0.396+0.095j -0.291+0.694j  0.202-0.305j]
 [ 0.   -0.j    -0.   -0.j     0.   +0.j     0.   +0.j    -0.   +0.j     0.755-0.j     0.33 +0.063j -
0.02 +0.04j   0.49 +0.021j  0.048-0.146j]
 [-0.   -0.j     0.   -0.j    -0.   +0.j     0.   +0.j    -0.   +0.j     0.   -0.j    -0.552+0.j    -
0.187+0.344j -0.487-0.216j -0.031-0.012j]
 [ 0.   +0.j    -0.   +0.j    -0.   +0.j    -0.   +0.j     0.   +0.j    -0.   -0.j    -0.   -0.j    -
0.392+0.j     0.783-0.277j  0.298+0.283j]
 [ 0.   -0.j    -0.   +0.j    -0.   -0.j     0.   +0.j    -0.   -0.j     0.   +0.j    -0.   -0.j
0.   +0.j    -0.369+0.j    -0.501-0.359j]]

QR_Givens: H_new=
 [[ 0.888+0.745j -0.327-2.322j -1.1  +0.486j  0.44 -0.207j  0.116+0.45j  -0.275+0.103j  0.099+0.192j
-0.103-0.358j  0.364-0.089j -0.228+0.501j]
 [-2.318+0.j     2.611+3.49j   0.032-2.875j -0.559+0.8j    0.418+0.029j -0.666-0.011j  0.135-0.213j
0.187-0.177j  0.035-0.275j -0.091-0.018j]
 [ 0.   -0.j    -3.712+0.j     1.006+1.43j   0.254-0.165j -0.565-0.098j  0.231+0.182j -0.1  +0.127j -
0.334+0.447j  0.05 -0.057j -0.066+0.027j]
 [ 0.   -0.j     0.   -0.j    -1.224-0.j    -0.032+0.403j -0.031+0.329j  0.096-0.201j  0.442+0.388j
0.29 +0.227j -0.41 -0.111j -0.068+0.012j]
 [ 0.   -0.j    -0.   -0.j     0.   -0.j     0.974+0.j     0.268+0.386j  0.149-0.117j  0.064-0.103j -
0.686+0.187j -0.078+0.719j -0.521-0.506j]
 [-0.   +0.j    -0.   -0.j    -0.   +0.j    -0.   +0.j     0.774+0.j    -0.164-0.23j   0.273-0.768j
0.396+0.095j -0.291+0.694j  0.202-0.305j]
 [ 0.   -0.j    -0.   -0.j     0.   +0.j     0.   +0.j    -0.   +0.j     0.755-0.j     0.33 +0.063j -
0.02 +0.04j   0.49 +0.021j  0.048-0.146j]
 [-0.   -0.j     0.   -0.j    -0.   +0.j     0.   +0.j    -0.   +0.j     0.   -0.j    -0.552+0.j    -
0.187+0.344j -0.487-0.216j -0.031-0.012j]
 [ 0.   +0.j    -0.   +0.j    -0.   +0.j    -0.   +0.j     0.   +0.j    -0.   -0.j    -0.   -0.j    -
0.392+0.j     0.783-0.277j  0.298+0.283j]
 [ 0.   -0.j    -0.   +0.j    -0.   -0.j     0.   +0.j    -0.   -0.j     0.   +0.j    -0.   -0.j
0.   +0.j    -0.369+0.j    -0.501-0.359j]]
Q.conjugate().T @ Q = np.eye(n)?  True



===================================================================
Testing QR_Givens() function on randomly generated REAL matrices
===================================================================


Complex >>> Input shape = (3, 3)
A =
 [[0.657 0.577 0.419]
 [0.152 0.689 0.853]
 [0.01  0.18  0.176]]
H =
 [[ 0.657 -0.603 -0.38 ]
 [-0.152  0.755  0.814]
 [ 0.     0.141  0.11 ]]

QR_Givens: H_new=
 [[ 0.657 -0.603 -0.38 ]
 [-0.152  0.755  0.814]
 [ 0.     0.141  0.11 ]]
Q.T @ Q = np.eye(n)?  True


Complex >>> Input shape = (5, 5)
A =
 [[0.115 0.312 0.619 0.011 0.972]
 [0.726 0.888 0.794 0.22  0.044]
 [0.69  0.526 0.155 0.21  0.547]
```

```
 [0.474 0.337 0.026 0.034 0.128]
 [0.441 0.446 0.038 0.211 0.282]]
H =
 [[ 0.115 -0.912  0.005  0.351 -0.686]
 [-1.193  1.364 -0.397  0.124  0.019]
 [-0.    -0.391  0.018  0.253  0.373]
 [ 0.    -0.    -0.09  -0.123 -0.332]
 [-0.    -0.     0.     0.191  0.1  ]]

QR_Givens: H_new=
 [[ 0.115 -0.912  0.005  0.351 -0.686]
 [-1.193  1.364 -0.397  0.124  0.019]
 [-0.    -0.391  0.018  0.253  0.373]
 [ 0.    -0.    -0.09  -0.123 -0.332]
 [-0.    -0.     0.     0.191  0.1  ]]
Q.T @ Q = np.eye(n)?  True


Complex >>> Input shape = (10, 10)
A =
 [[0.764 0.732 0.93  0.454 0.252 0.53  0.044 0.633 0.201 0.171]
 [0.732 0.73  0.879 0.219 0.097 0.742 0.046 0.856 0.831 0.116]
 [0.866 0.026 0.834 0.19  0.234 0.315 0.111 0.509 0.41  0.556]
 [0.315 0.807 0.503 0.722 0.81  0.    0.634 0.607 0.168 0.96 ]
 [0.102 0.928 0.512 0.62  0.615 0.512 0.639 0.886 0.767 0.606]
 [0.179 0.294 0.783 0.337 0.182 0.992 0.489 0.489 0.447 0.649]
 [0.371 0.617 0.411 0.851 0.656 0.611 0.507 0.355 0.057 0.261]
 [0.634 0.021 0.027 0.354 0.778 0.073 0.665 0.18  0.711 0.667]
 [0.717 0.534 0.535 0.625 0.346 0.592 0.401 0.365 0.751 0.444]
 [0.777 0.313 0.826 0.089 0.103 0.301 0.079 0.314 0.783 0.598]]
H =
 [[ 0.764 -1.309  0.321  0.134 -0.317 -0.07  -0.112  0.535 -0.073  0.423]
 [-1.757  3.462 -1.067  0.16  -0.012  0.534 -0.007 -0.228  0.175  0.139]
 [-0.    -2.36   1.108 -0.224 -0.079  0.103 -0.129  0.061  0.404  0.342]
 [ 0.    -0.    -1.009  0.22  -0.271 -0.525 -0.249  0.139 -0.373  0.123]
 [ 0.    -0.    -0.     0.358  0.148  0.158  0.035  0.42   0.25   0.174]
 [-0.     0.    -0.    -0.    -0.523  0.355  0.042  0.11  -0.074  0.463]
 [ 0.     0.    -0.     0.    -0.     0.618  0.501 -0.442  0.324 -0.533]
 [-0.    -0.    -0.    -0.     0.    -0.     0.236  0.236  0.451 -0.264]
 [ 0.     0.     0.     0.    -0.    -0.     0.    -0.087  0.019 -0.259]
 [-0.     0.    -0.    -0.     0.     0.     0.     0.     0.193 -0.12 ]]

QR_Givens: H_new=
 [[ 0.764 -1.309  0.321  0.134 -0.317 -0.07  -0.112  0.535 -0.073  0.423]
 [-1.757  3.462 -1.067  0.16  -0.012  0.534 -0.007 -0.228  0.175  0.139]
 [-0.    -2.36   1.108 -0.224 -0.079  0.103 -0.129  0.061  0.404  0.342]
 [ 0.    -0.    -1.009  0.22  -0.271 -0.525 -0.249  0.139 -0.373  0.123]
 [ 0.    -0.    -0.     0.358  0.148  0.158  0.035  0.42   0.25   0.174]
 [-0.     0.    -0.    -0.    -0.523  0.355  0.042  0.11  -0.074  0.463]
 [ 0.     0.    -0.     0.    -0.     0.618  0.501 -0.442  0.324 -0.533]
 [-0.    -0.    -0.    -0.     0.    -0.     0.236  0.236  0.451 -0.264]
 [ 0.     0.     0.     0.    -0.    -0.     0.    -0.087  0.019 -0.259]
 [-0.     0.    -0.    -0.     0.     0.     0.     0.     0.193 -0.12 ]]
Q.T @ Q = np.eye(n)?  True



===================================================================
Testing QR_Givens() function on SPECIAL real matrices
===================================================================


Special >>> Input shape = (10, 10)
A =
 [[ 4. -2. -4.  3. -6.  2. -3. -2. -4.  2.]
 [-2.  4.  3. -3.  2. -3. -2. -3. -5. -6.]
 [-4.  3.  0.  3.  0.  3. -3. -1.  0.  2.]
 [ 3. -3.  3. -2. -5.  1.  3. -3.  2. -4.]
 [-6.  2.  0. -5. -6. -6.  1. -2. -5.  2.]
 [ 2. -3.  3.  1. -6.  4. -2. -1. -1. -1.]
 [-3. -2. -3.  3.  1. -2.  0. -3.  1. -3.]
 [-2. -3. -1. -3. -2. -1. -3. -6.  1. -4.]
```

```
 [-4. -5.  0.  2. -5. -1.  1.  1. -4.  1.]
 [ 2. -6.  2. -4.  2. -1. -3. -4.  1.  2.]]
H =
 [[ 4.    10.1   -0.     0.     0.     0.     0.    -0.     0.    -0.   ]
 [10.1   -4.196 -8.169  0.     0.    -0.     0.    -0.     0.    -0.   ]
 [-0.    -8.169  5.375 -7.56  -0.     0.    -0.     0.     0.     0.   ]
 [ 0.     0.    -7.56  -7.257  5.807 -0.     0.     0.     0.     0.   ]
 [ 0.     0.    -0.     5.807 -1.605  6.374  0.     0.    -0.     0.   ]
 [ 0.    -0.     0.    -0.     6.374 -3.799 -7.736  0.     0.     0.   ]
 [ 0.     0.    -0.     0.     0.    -7.736 -0.966  2.819  0.     0.   ]
 [-0.    -0.    -0.     0.     0.     0.     2.819  0.78   4.333  0.   ]
 [ 0.     0.    -0.    -0.    -0.     0.    -0.     4.333  0.922  5.58 ]
 [-0.    -0.     0.     0.    -0.    -0.     0.     0.     5.58   2.745]]

QR_Givens: H_new=
 [[ 4.    10.1   -0.     0.     0.     0.     0.    -0.     0.    -0.   ]
 [10.1   -4.196 -8.169  0.     0.    -0.     0.    -0.     0.    -0.   ]
 [-0.    -8.169  5.375 -7.56  -0.     0.    -0.     0.     0.     0.   ]
 [ 0.     0.    -7.56  -7.257  5.807 -0.     0.     0.     0.     0.   ]
 [ 0.     0.    -0.     5.807 -1.605  6.374  0.     0.    -0.     0.   ]
 [ 0.    -0.     0.    -0.     6.374 -3.799 -7.736  0.     0.     0.   ]
 [ 0.     0.    -0.     0.     0.    -7.736 -0.966  2.819  0.     0.   ]
 [-0.    -0.    -0.     0.     0.     0.     2.819  0.78   4.333  0.   ]
 [ 0.     0.    -0.    -0.    -0.     0.    -0.     4.333  0.922  5.58 ]
 [-0.    -0.     0.     0.    -0.    -0.     0.     0.     5.58   2.745]]
Q.T @ Q = np.eye(n)?  True


Special >>> Input shape = (10, 10)
A =
 [[ 4.   -0.02 -0.04  0.03 -0.06  0.02 -0.03 -0.02 -0.04  0.02]
 [-0.02  4.    0.03 -0.03  0.02 -0.03 -0.02 -0.03 -0.05 -0.06]
 [-0.04  0.03  0.    0.03  0.    0.03 -0.03 -0.01  0.    0.02]
 [ 0.03 -0.03  0.03 -2.   -0.05  0.01  0.03 -0.03  0.02 -0.04]
 [-0.06  0.02  0.   -0.05 -6.   -0.06  0.01 -0.02 -0.05  0.02]
 [ 0.02 -0.03  0.03  0.01 -0.06  4.   -0.02 -0.01 -0.01 -0.01]
 [-0.03 -0.02 -0.03  0.03  0.01 -0.02  0.   -0.03  0.01 -0.03]
 [-0.02 -0.03 -0.01 -0.03 -0.02 -0.01 -0.03 -6.    0.01 -0.04]
 [-0.04 -0.05  0.    0.02 -0.05 -0.01  0.01  0.01 -4.    0.01]
 [ 0.02 -0.06  0.02 -0.04  0.02 -0.01 -0.03 -0.04  0.01  2.  ]]
H =
 [[ 4.     0.101 -0.    -0.     0.     0.    -0.    -0.    -0.    -0.   ]
 [ 0.101 -2.779  3.296  0.     0.    -0.    -0.    -0.     0.     0.   ]
 [-0.     3.296 -0.83   2.933  0.    -0.    -0.    -0.    -0.    -0.   ]
 [-0.    -0.     2.933  0.258 -2.364  0.    -0.     0.    -0.     0.   ]
 [ 0.     0.     0.    -2.364 -1.808  1.79   0.     0.     0.    -0.   ]
 [ 0.    -0.    -0.     0.     1.79  -0.246 -1.975  0.    -0.     0.   ]
 [-0.     0.    -0.    -0.    -0.    -1.975 -0.653  0.46  -0.    -0.   ]
 [-0.     0.    -0.     0.    -0.     0.     0.46  -5.93   0.272  0.   ]
 [-0.     0.     0.    -0.    -0.     0.    -0.     0.272  0.377  1.138]
 [-0.    -0.     0.     0.    -0.     0.    -0.    -0.     1.138  3.611]]

QR_Givens: H_new=
 [[ 4.     0.101 -0.    -0.     0.     0.    -0.    -0.    -0.    -0.   ]
 [ 0.101 -2.779  3.296  0.     0.    -0.    -0.    -0.     0.     0.   ]
 [-0.     3.296 -0.83   2.933  0.    -0.    -0.    -0.    -0.    -0.   ]
 [-0.    -0.     2.933  0.258 -2.364  0.    -0.     0.    -0.     0.   ]
 [ 0.     0.     0.    -2.364 -1.808  1.79   0.     0.     0.    -0.   ]
 [ 0.    -0.    -0.     0.     1.79  -0.246 -1.975  0.    -0.     0.   ]
 [-0.     0.    -0.    -0.    -0.    -1.975 -0.653  0.46  -0.    -0.   ]
 [-0.     0.    -0.     0.    -0.     0.     0.46  -5.93   0.272  0.   ]
 [-0.     0.     0.    -0.    -0.     0.    -0.     0.272  0.377  1.138]
 [-0.    -0.     0.     0.    -0.     0.    -0.    -0.     1.138  3.611]]
Q.T @ Q = np.eye(n)?  True


Special >>> Input shape = (10, 10)
A =
 [[1. 0. 0. 0. 0. 0. 0. 0. 0. 0.]
 [1. 1. 0. 0. 0. 0. 0. 0. 0. 0.]
 [1. 1. 1. 0. 0. 0. 0. 0. 0. 0.]
 [1. 1. 1. 1. 0. 0. 0. 0. 0. 0.]
```

```
 [1. 1. 1. 1. 1. 0. 0. 0. 0. 0.]
 [1. 1. 1. 1. 1. 1. 0. 0. 0. 0.]
 [1. 1. 1. 1. 1. 1. 1. 0. 0. 0.]
 [1. 1. 1. 1. 1. 1. 1. 1. 0. 0.]
 [1. 1. 1. 1. 1. 1. 1. 1. 1. 0.]
 [1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]]
H =
 [[ 1.     0.     0.     0.     0.     0.     0.     0.     0.     0.   ]
 [-3.     5.     2.582 -0.    -0.     0.    -0.     0.    -0.     0.   ]
 [-0.    -2.582  0.5   -1.133  0.    -0.    -0.    -0.    -0.    -0.   ]
 [ 0.     0.     1.133  0.5    0.717  0.     0.    -0.    -0.    -0.   ]
 [ 0.     0.    -0.    -0.717  0.5   -0.508  0.     0.     0.     0.   ]
 [-0.     0.     0.     0.     0.508  0.5   -0.376 -0.     0.    -0.   ]
 [ 0.    -0.     0.     0.    -0.     0.376  0.5   -0.28   0.     0.   ]
 [-0.    -0.    -0.    -0.    -0.     0.     0.28   0.5   -0.203  0.   ]
 [ 0.    -0.     0.     0.     0.     0.    -0.     0.203  0.5   -0.129]
 [ 0.    -0.    -0.    -0.     0.     0.    -0.    -0.     0.129  0.5  ]]

QR_Givens: H_new=
 [[ 1.     0.     0.     0.     0.     0.     0.     0.     0.     0.   ]
 [-3.     5.     2.582 -0.    -0.     0.    -0.     0.    -0.     0.   ]
 [-0.    -2.582  0.5   -1.133  0.    -0.    -0.    -0.    -0.    -0.   ]
 [ 0.     0.     1.133  0.5    0.717  0.     0.    -0.    -0.    -0.   ]
 [ 0.     0.    -0.    -0.717  0.5   -0.508  0.     0.     0.     0.   ]
 [-0.     0.     0.     0.     0.508  0.5   -0.376 -0.     0.    -0.   ]
 [ 0.    -0.     0.     0.    -0.     0.376  0.5   -0.28   0.     0.   ]
 [-0.    -0.    -0.    -0.    -0.     0.     0.28   0.5   -0.203  0.   ]
 [ 0.    -0.     0.     0.     0.     0.    -0.     0.203  0.5   -0.129]
 [ 0.    -0.    -0.    -0.     0.     0.    -0.    -0.     0.129  0.5  ]]
Q.T @ Q = np.eye(n)?  True


Special >>> Input shape = (6, 6)
A =
 [[  7.   3.   4. -11.  -9.  -2.]
 [ -6.   4.  -5.   7.   1.  12.]
 [ -1.  -9.   2.   2.   9.   1.]
 [ -8.   0.  -1.   5.   0.   8.]
 [ -4.   3.  -5.   7.   2.  10.]
 [  6.   1.   4. -11.  -7.  -1.]]
H =
 [[ 7.     7.276  5.812 -0.14   9.015 -7.936]
 [12.369  4.131 18.969 -1.207 10.683 -2.416]
 [ 0.    -7.16   2.448 -0.566 -4.181  3.251]
 [-0.     0.    -8.599  2.915 -3.417 -5.723]
 [-0.    -0.    -0.     1.046 -2.835 10.979]
 [-0.    -0.     0.     0.    -1.414  5.342]]

QR_Givens: H_new=
 [[ 7.     7.276  5.812 -0.14   9.015 -7.936]
 [12.369  4.131 18.969 -1.207 10.683 -2.416]
 [ 0.    -7.16   2.448 -0.566 -4.181  3.251]
 [-0.     0.    -8.599  2.915 -3.417 -5.723]
 [-0.    -0.    -0.     1.046 -2.835 10.979]
 [-0.    -0.     0.     0.    -1.414  5.342]]
Q.T @ Q = np.eye(n)?  True
```

Observation:

- Function QR_Givens() seems to be working for both real and complex matrices as expected
- **Where does the complex warning come from??**

#

QR Iteration

Some Background Information

Source: https://scicomp.stackexchange.com/questions/30407/how-does-the-qr-algorithm-applied-to-a-real-matrix-returns-complex-eigenvalues (https://scicomp.stackexchange.com/questions/30407/how-does-the-qr-algorithm-applied-to-a-real-matrix-returns-complex-eigenvalues)

- QR algorithm converges to the real Schur decomposition: a unitary matrix Q and a matrix R in block upper triangular form such that $A = QRQ^T$
- The key point is the *block upper triangular* form, which means here R_{ii} are real blocks of
- EITHER size 1x1 (R_{ii} is a (real) eigenvalue of A)
- OR size 2x2 (R_{ii} has a pair of complex conjugate eigenvalues). ##

In [11]:
```python
#def QR_Iteration_Eigenvalues_naive(A, max_iter, tol):
#    # step 1: tranform A to Hessenberg
#    H, _ = Hessenberg(A)
#    # step 2: QR iteration
#    H_old = H
#    for i in range(max_iter):
#        Q, R = QR_Givens_naive(H_old)
#        H_new = R @ Q
#        # TODO: test for convergence
#        if np.linalg.norm(H_new - H) < tol:
#            print('break at i=',i,np.linalg.norm(H_new - H),H_new)
#            break
#        H_old = H_new
#    R = H_new
#    return np.diag(R)

def diagonal_block(R):

    """     Read eigenvalues from the diagonal block matrix R, either in 1x1 block (i.e. real eigenval
ue) or in 2x2 block (i.e. complex conjugate pair)
    Input   ---------------------------------------------
            R: 2D np.array with real or complex entries, in the form of a block upper triangular matri
x
    Return  ---------------------------------------------
            eigenvalues: 1D np.array with real or complex entries
    """

    eigenvalues = []

    for i in range(len(R)):
        if i < len(eigenvalues): # if current diagonal values has already been used in previous step,
 skip
            continue

        if i ==len(R)-1 or np.abs(R[i+1,i]) < ATOL: # if subdiagonal element is sufficiently small, tr
eat as zero OR if last row
            eigenvalues.append(R[i,i])

        else:
            # solving eigenvalues from characteristic polynomial lam^2 - (R[i,i]+R[i+1,i+1])*lam + (R
[i,i]*R[i+1,i+1] - R[i,i+1]*R[i+1,i]) = 0
            b = - (R[i,i]+R[i+1,i+1])
            c = R[i,i]*R[i+1,i+1] - R[i,i+1]*R[i+1,i]

            lam_1 = -b/2
            sign = np.sign(lam_1)

            if b**2 - 4*c >= 0:
                lam_2 = np.sqrt(b**2-4*c)/2
                eigenvalues.append(lam_1 + sign*lam_2)
                eigenvalues.append(lam_1 - sign*lam_2)
            else:
                lam_2 = np.sqrt(-(b**2-4*c))/2
                eigenvalues.append(lam_1 + sign*lam_2*1j)
                eigenvalues.append(lam_1 - sign*lam_2*1j)

    return np.array(eigenvalues)


def QR_Iteration_Eigenvalues(A, max_iter, tol):

    """     Perform QR iteration to obtain eigenvalues
    Input   ---------------------------------------------
            A: 2D np.array with real or complex entries
            max_iter: int, maximum number of iterations to be performed
            tol: float, relative tolerance between eigenvalues by successive iterations
    Return  ---------------------------------------------
            eigvals_new: 1D np.array with real or complex entries
    """
```

```
    # step 1: tranform A to Hessenberg
    H, _ = Hessenberg(A)

    # step 2: QR iteration

    H_old = H.copy()

    eigvals_old = diagonal_block(H)

    for i in range(max_iter):
        H_new,_ = QR_Givens(H_old) # updating H_old in-place # default QR_Givens(H,eigvec=False, inpla
ce=True, tridiagonal=False)


        eigvals_new = diagonal_block(H_new)
        #print('>>>\n', H_new, '\n\n',np.diag(H_new), '\n')

        # Test for convergence: relative tol = 1- w / w' for each eigenvalue
        if np.allclose(eigvals_new, eigvals_old, rtol=tol):
            print('Iteration terminates at i={} with tol={} reached'.format(i,tol))
            break

        if i == max_iter-1:
            err = np.max(abs((eigvals_new - eigvals_old)) / abs(eigvals_old))
            idx = np.argwhere(np.isclose(eigvals_new, eigvals_old, rtol=tol)==False).tolist()
            print('max_iter={} reached, max error={} vs tol={}; index of non-convergence={}'.format(ma
x_iter, err, tol, idx))

        eigvals_old = eigvals_new.copy() # need to save a copy for comparison in the next iteration
        # H_old = H_new # this is not needed as H_old is updated inplace

    return eigvals_new
```

In [12]:
```python
# Testing

def sort_(eigenvalues):

    """     Sort eigenvalues in descending order by magitude; Numpy extended sort order: Real: [R, na
n]; Complex: [R + Rj, R + nanj, nan + Rj, nan + nanj]
    Input   ----------------------------------------------
            eigenvalues: 1D np.array with real or complex entries
    Return  ----------------------------------------------
            eigenvaluese_sorted: 1D np.array with real or complex entries, sorted in descending order
    """
    eigenvalues_sorted = np.flip(np.sort(eigenvalues))

    return eigenvalues_sorted

#    if eigenvalues.dtype != 'complex':
#        eigenvalues_sorted = np.sort(eigenvalues) #np.flip(sorted(eigenvalues,key=abs))
#    else:
#        eigenvalues_sorted = np.sort_complex(eigenvalues)
#    return eigenvalues_sorted



def test_eigenvalues_function(A,function):

    """     Perform eigenvalue testing on selected function; eigenvalues are compared with results fro
m np.linalg.eig()
    Input   ----------------------------------------------
            A: input matrix, 2D np.array with real or complex entries
            function: name of the function being tested
    Return  ----------------------------------------------
            None; test results will be printed
    """

    print("------ Testing for function: {}() ------".format(function.__name__))
    w,v = np.linalg.eig(A)
    w_sorted = sort_(w)
    eigenvalues = function(A,max_iter=MAX_ITER, tol=RTOL)
    eigenvalues_sorted = sort_(eigenvalues)
    print()
    print('my eigenvalues: {}'.format(eigenvalues_sorted))
    print('np.linalg.eig:  {}'.format(w_sorted)) #sort in descending order of magnitutes

    if np.allclose(w_sorted,eigenvalues_sorted):
        print('\nnp.linalg.eig and my eigenvalues close? True \n')
    else:
        print('\nnp.linalg.eig and my eigenvalues close? False \n')
        print('\nmy eigenvalues - np.linalg.eig =\n', eigenvalues_sorted - w_sorted,'\n')

    print('\n')


def test_eigenvalues(eigenvalues,w, verbose=True):

    """     Comparing two sets of eigenvalues
    Input   ----------------------------------------------
            eigenvalues: 1D np.array with real or complex entries, eigenvalues from my function
            w: eigenvalues from np.linalg.eig(), basis for comparison
            verbose: bool, indicating whether to print the difference between the two sets of eigenval
ues
    Return  ----------------------------------------------
            None; test results will be printed
    """

    eigenvalues_sorted = sort_(eigenvalues)
    w_sorted = sort_(w)

    if verbose:
        print()
        print('my eigenvalues: {}'.format(eigenvalues_sorted))
        print('np.linalg.eig:  {}'.format(w_sorted)) #sort in descending order of magnitutes
```

```
    if np.allclose(w_sorted,eigenvalues_sorted):
        print('\nnp.linalg.eig and my eigenvalues close? True \n')
    else:
        print('\nnp.linalg.eig and my eigenvalues close? False \n')
        if verbose:
            print('my eigenvalues - np.linalg.eig =\n', eigenvalues_sorted - w_sorted,'\n')
            print('max abs diff = ',np.abs(eigenvalues_sorted - w_sorted).max(),'\n')
        print('\n')
```

In [26]:

```python
### Testing QR_Iteration_Eigenvalues() on Random Complex Matrices

print('\n\n')
print("====================================================================================")
print('Testing QR_Iteration_Eigenvalues() function on randomly generated COMPLEX matrices')
print("====================================================================================")

for A in RANDOM_COMPLEX_MATRICES:
    print('\nComplex >>> Input shape = {}\n'.format(A.shape))
    test_eigenvalues_function(A,QR_Iteration_Eigenvalues)


### Testing QR_Iteration_Eigenvalues() on Random Real Matrices

print('\n\n')
print("====================================================================================")
print('Testing QR_Iteration_Eigenvalues() function on randomly generated REAL matrices')
print("====================================================================================")

for A in RANDOM_REAL_MATRICES:
    print('\nReal >>> Input shape = {}\n'.format(A.shape))
    test_eigenvalues_function(A,QR_Iteration_Eigenvalues)



### Testing QR_Iteration_Eigenvalues() on Special Matrices

print('\n\n')
print("====================================================================================")
print('Testing QR_Iteration_Eigenvalues() function on SPECIAL real matrices')
print("====================================================================================")

for A in SPECIAL_REAL_MATRICES:
    print('\nSpecial >>> Input shape = {}\n'.format(A.shape))
    test_eigenvalues_function(A,QR_Iteration_Eigenvalues)
```

```
===================================================================================
Testing QR_Iteration_Eigenvalues() function on randomly generated COMPLEX matrices
===================================================================================

Complex >>> Input shape = (3, 3)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=38 with tol=1e-06 reached

my eigenvalues: [ 1.478+0.611j  0.301+0.405j -0.36 -0.131j]
np.linalg.eig:  [ 1.478+0.611j  0.301+0.405j -0.36 -0.131j]

np.linalg.eig and my eigenvalues close? True



Complex >>> Input shape = (5, 5)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=52 with tol=1e-06 reached

my eigenvalues: [ 2.622+2.657j  0.509-0.054j  0.157+0.706j -0.072-0.338j -0.657-0.058j]
np.linalg.eig:  [ 2.622+2.657j  0.509-0.054j  0.157+0.706j -0.072-0.338j -0.657-0.058j]

np.linalg.eig and my eigenvalues close? True



Complex >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=400 with tol=1e-06 reached

my eigenvalues: [ 4.936+5.572j  1.213+0.249j  0.879-0.054j  0.797-0.64j  -0.049+0.528j -0.157+1.27j
-0.221-0.654j -0.39 -0.341j -0.676-0.138j -1.329+0.203j]
np.linalg.eig:  [ 4.936+5.572j  1.213+0.249j  0.879-0.054j  0.797-0.64j  -0.049+0.528j -0.157+1.27j
-0.221-0.654j -0.39 -0.341j -0.676-0.138j -1.329+0.203j]

np.linalg.eig and my eigenvalues close? True




===================================================================================
Testing QR_Iteration_Eigenvalues() function on randomly generated REAL matrices
===================================================================================

Real >>> Input shape = (3, 3)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=6 with tol=1e-06 reached

my eigenvalues: [ 1.098  0.468 -0.044]
np.linalg.eig:  [ 1.098  0.468 -0.044]

np.linalg.eig and my eigenvalues close? True



Real >>> Input shape = (5, 5)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=24 with tol=1e-06 reached
```

```
my eigenvalues: [ 2.014+0.j    0.052+0.25j  0.052-0.25j -0.139+0.j   -0.506+0.j  ]
np.linalg.eig:  [ 2.014+0.j    0.052+0.25j  0.052-0.25j -0.139+0.j   -0.506+0.j  ]

np.linalg.eig and my eigenvalues close? True




Real >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------

/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:33: ComplexWarning: Casting complex valu
es to real discards the imaginary part
```

```
Iteration terminates at i=551 with tol=1e-06 reached

my eigenvalues: [ 4.821+0.j     0.958+0.j     0.532+0.j     0.358+0.342j  0.358-0.342j -0.011+0.2j
-0.011-0.2j   -0.056+0.j    -0.128+0.597j -0.128-0.597j]
np.linalg.eig:  [ 4.821+0.j     0.958+0.j     0.532+0.j     0.358+0.342j  0.358-0.342j -0.011+0.2j
-0.011-0.2j   -0.056+0.j    -0.128+0.597j -0.128-0.597j]

np.linalg.eig and my eigenvalues close? True




==================================================================================
Testing QR_Iteration_Eigenvalues() function on SPECIAL real matrices
==================================================================================

Special >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=140 with tol=1e-06 reached

my eigenvalues: [ 14.993   9.198   8.242   6.585   2.133  -0.717  -5.727  -9.631 -13.122 -15.953]
np.linalg.eig:  [ 14.993   9.198   8.242   6.585   2.133  -0.717  -5.727  -9.631 -13.122 -15.953]

np.linalg.eig and my eigenvalues close? True



Special >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
max_iter=1000 reached, max error=0.010554175755873824 vs tol=1e-06; index of non-convergence=[[3],
[5]]

my eigenvalues: [ 4.049  3.972  1.999  1.328  0.029 -0.03  -1.344 -2.001 -5.982 -6.022]
np.linalg.eig:  [ 4.049  3.984  3.972  1.999  0.029 -0.03  -2.001 -3.999 -5.982 -6.022]

np.linalg.eig and my eigenvalues close? False


my eigenvalues - np.linalg.eig =
 [-0.    -0.012 -1.973 -0.671 -0.     0.     0.657  1.999  0.    -0.   ]



Special >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=795 with tol=1e-06 reached

my eigenvalues: [1.026+0.009j 1.026-0.009j 1.015+0.022j 1.015-0.022j 0.999+0.027j 0.999-0.027j 0.984+
0.021j 0.984-0.021j 0.975+0.008j 0.975-0.008j]
np.linalg.eig:  [1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]

np.linalg.eig and my eigenvalues close? False


my eigenvalues - np.linalg.eig =
 [ 0.026+0.009j  0.026-0.009j  0.015+0.022j  0.015-0.022j -0.001+0.027j -0.001-0.027j -0.016+0.021j -
0.016-0.021j -0.025+0.008j -0.025-0.008j]



Special >>> Input shape = (6, 6)

------ Testing for function: QR_Iteration_Eigenvalues() ------
```

```
Iteration terminates at i=52 with tol=1e-06 reached

my eigenvalues: [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]
np.linalg.eig:  [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]

np.linalg.eig and my eigenvalues close? True
```

Observations:

- Function QR_Iteration_Eigenvalues() seems to be working as expected with both real and complex matrices, with real eigenvalues or complex conjugate pairs
- Underperformance in special matrix -> diagonal dominant with ones in lower triangular **#TODO**

In [27]:
```
### Testing QR_Iteration_Eigenvalues() on COMPLEX Matrices with only pure imaginary parts

print('\n\n')
print("====================================================================================")
print('Testing QR_Iteration_Eigenvalues() function on SPECIAL complex matrices (Im Only)')
print("====================================================================================")

A = np.random.rand(10,10) * 1j
print('\nSpecial >>> Input shape = {}\n'.format(A.shape))
test_eigenvalues_function(A,QR_Iteration_Eigenvalues)
```

```
====================================================================================
Testing QR_Iteration_Eigenvalues() function on SPECIAL complex matrices (Im Only)
====================================================================================

Special >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------

/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:25: RuntimeWarning: divide by zero encou
ntered in double_scalars

max_iter=1000 reached, max error=1.0054667935413541e-05 vs tol=1e-06; index of non-convergence=[[2],
[3], [4], [5]]

my eigenvalues: [ 1.56 -2.242j  0.652+0.434j  0.552-0.337j  0.151+0.37j   0.   -0.158j  0.   -0.081j
-0.151+0.37j  -0.552-0.337j -0.652+0.434j -1.56 -2.242j]
np.linalg.eig:  [ 0.665+0.467j  0.462-0.132j  0.318+0.454j  0.   +5.225j  0.   -0.23j   0.   -0.695j
-0.   -0.313j -0.318+0.454j -0.462-0.132j -0.665+0.467j]

np.linalg.eig and my eigenvalues close? False


my eigenvalues - np.linalg.eig =
 [ 0.895-2.709j  0.19 +0.566j  0.234-0.791j  0.151-4.854j -0.   +0.072j -0.   +0.614j -0.151+0.683j -
0.234-0.791j -0.19 +0.566j -0.895-2.709j]
```

Observations:

- Does not seem to work for matrices with pure imaginary parts WHY??? **#TODO**
- *Suspect Givens rotation not working as expected for pure imaginary numbers...*

#

Wilkinson Shift

1.4.22 Definition(Wilkinson shift)

Let

$$\mathbf{M} = \begin{pmatrix} h^{(k)}_{n-1,n-1} & h^{(k)}_{n-1,n} \\ h^{(k)}_{n,n-1} & h^{(k)}_{nn} \end{pmatrix}.$$

Then, for σ_k use the eigenvalue of $\mathbf{M}$ which is closer to $h^{(k)}_{nn}$.

1.4.23 Remark

The Wilkinson shift is reliable and the $h_{n,n-1}$ and h_{nn} converge to zero and the smallest eigenvalue by magnitude, respectively. They converge at least quadratically and cubically in the symmetric case [GvL83, Section 8.2].

In [87]:
```
def Wilkinson_shift(H):

    """     Using the bottom right 2x2 block of Hessenberg matrix to determine if the eigenvalues are
 real or complex (b^2-4ac)
            Calculate the Wilkinson's shift according to Lecture Notes 1.4.22 Definition (Wilkinson sh
ift)
    Input   ----------------------------------------------
            H: 2D np.array with real or complex entries, in the form of an upper Hessenberg matrix
    Return  ---------------------------------------------
            datatype: 'complex' or 'float'
            lam1 or lam2: shift to be performed
    """

    M = H[-2:,-2:]

    # characteristic polynomial x^2 + bx + c = 0
    b = -(M[0,0] + M[1,1])
    c = (M[0,0]*M[1,1] - M[1,0]*M[0,1])

    # if real eigenvalues - return the one closer to A[n,n]'
    if b**2 >= 4*c:
        lam1 = (-b + np.sqrt(b**2-4*c))/2
        lam2 = (-b - np.sqrt(b**2-4*c))/2
        datatype = 'float'

    # if complex eigenvalues - return Re(x) and Im(x)
    else:
        re_ = -b/2
        im_ = np.sqrt(4*c-b**2)/2
        lam1 = re_+1j*im_
        lam2 = re_+1j*im_
        datatype = 'complex'

    if abs(lam1-M[1,1]) < abs(lam2-M[1,1]):
        return (datatype, lam1)
    else:
        return (datatype, lam2)



def QR_Iteration_WilkinsonShift(A, max_iter, tol):

    """     Perform QR iteration with Wilkinson's shift to obtain eigenvalues
    Input   ----------------------------------------------
            A: 2D np.array with real or complex entries
            max_iter: int, maximum number of iterations to be performed
            tol: float, relative tolerance between eigenvalues by successive iterations
    Return  ---------------------------------------------
            eigvals_new: 1D np.array with real or complex entries
    """

    n = A.shape[0]
    # step 1: tranform A to Hessenberg
    H, _ = Hessenberg(A)

    # step 2: QR iteration
    H_old = H.copy()
    eigvals_old = diagonal_block(H)

    for i in range(max_iter):

        datatype, lam = Wilkinson_shift(H_old)
        #print(">>> Wilkinson_shift = {}".format(lam))

        # H_old = H_old - lam * np.eye(n)
        for j in range(n):
            H_old[j,j] -= lam

        #H_new, _ = QR_Givens(H_old) +  lam * np.eye(n) # updating H_old in-place  # default QR_Givens
(H,eigvec=False, inplace=True, tridiagonal=False
        H_new, _ = QR_Givens(H_old)
```

```
        for j in range(n):
            H_new[j,j] += lam

        eigvals_new = diagonal_block(H_new)
        H_old = H_new

        # Test for convergence: relative tol = 1- w / w' for each eigenvalue
        if np.allclose(eigvals_new, eigvals_old, rtol=tol):
            print('Iteration terminates at i={} with tol={} reached'.format(i,tol))
            break

        if i == max_iter-1:
            err = np.max(abs((eigvals_new - eigvals_old)) / abs(eigvals_old))
            idx = np.argwhere(np.isclose(eigvals_new, eigvals_old, rtol=tol)==False).tolist()
            print('max_iter={} reached, max error={} vs tol={}; index of non-convergence={}'.format(ma
x_iter, err, tol, idx))

        eigvals_old = eigvals_new

    return eigvals_new
```

```
In [88]: ### Testing QR_Iteration_WilkinsonShift() on Random Complex Matrices

print('\n\n')
print("====================================================================================")
print('Testing QR_Iteration_WilkinsonShift() function on randomly generated COMPLEX matrices')
print("====================================================================================")

for A in RANDOM_COMPLEX_MATRICES:
    print('\nComplex >>> Input shape = {}\n'.format(A.shape))
    test_eigenvalues_function(A,QR_Iteration_Eigenvalues)
    test_eigenvalues_function(A,QR_Iteration_WilkinsonShift)


### Testing QR_Iteration_WilkinsonShift() on Random Real Matrices

print('\n\n')
print("====================================================================================")
print('Testing QR_Iteration_WilkinsonShift() function on randomly generated REAL matrices')
print("====================================================================================")

for A in RANDOM_REAL_MATRICES:
    print('\nReal >>> Input shape = {}\n'.format(A.shape))
    test_eigenvalues_function(A,QR_Iteration_Eigenvalues)
    test_eigenvalues_function(A,QR_Iteration_WilkinsonShift)


### Testing QR_Iteration_WilkinsonShift() on Special Matrices

print('\n\n')
print("====================================================================================")
print('Testing QR_Iteration_WilkinsonShift() function on SPECIAL real matrices')
print("====================================================================================")

for A in SPECIAL_REAL_MATRICES:
    print('\nSpecial >>> Input shape = {}\n'.format(A.shape))
    test_eigenvalues_function(A,QR_Iteration_Eigenvalues)
    test_eigenvalues_function(A,QR_Iteration_WilkinsonShift)
```

```
==================================================================================
Testing QR_Iteration_WilkinsonShift() function on randomly generated COMPLEX matrices
==================================================================================

Complex >>> Input shape = (3, 3)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=38 with tol=1e-06 reached

my eigenvalues: [ 1.478+0.611j  0.301+0.405j -0.36 -0.131j]
np.linalg.eig:  [ 1.478+0.611j  0.301+0.405j -0.36 -0.131j]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=3 with tol=1e-06 reached

my eigenvalues: [ 1.478+0.611j  0.301+0.405j -0.36 -0.131j]
np.linalg.eig:  [ 1.478+0.611j  0.301+0.405j -0.36 -0.131j]

np.linalg.eig and my eigenvalues close? True




Complex >>> Input shape = (5, 5)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=52 with tol=1e-06 reached

my eigenvalues: [ 2.622+2.657j  0.509-0.054j  0.157+0.706j -0.072-0.338j -0.657-0.058j]
np.linalg.eig:  [ 2.622+2.657j  0.509-0.054j  0.157+0.706j -0.072-0.338j -0.657-0.058j]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=54 with tol=1e-06 reached

my eigenvalues: [ 2.622+2.657j  0.509-0.054j  0.157+0.706j -0.072-0.338j -0.657-0.058j]
np.linalg.eig:  [ 2.622+2.657j  0.509-0.054j  0.157+0.706j -0.072-0.338j -0.657-0.058j]

np.linalg.eig and my eigenvalues close? True




Complex >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=400 with tol=1e-06 reached

my eigenvalues: [ 4.936+5.572j  1.213+0.249j  0.879-0.054j  0.797-0.64j  -0.049+0.528j -0.157+1.27j
-0.221-0.654j -0.39 -0.341j -0.676-0.138j -1.329+0.203j]
np.linalg.eig:  [ 4.936+5.572j  1.213+0.249j  0.879-0.054j  0.797-0.64j  -0.049+0.528j -0.157+1.27j
-0.221-0.654j -0.39 -0.341j -0.676-0.138j -1.329+0.203j]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=129 with tol=1e-06 reached

my eigenvalues: [ 4.936+5.572j  1.213+0.249j  0.879-0.054j  0.797-0.64j  -0.049+0.528j -0.157+1.27j
-0.221-0.654j -0.39 -0.341j -0.676-0.138j -1.329+0.203j]
```

```
np.linalg.eig:  [ 4.936+5.572j  1.213+0.249j  0.879-0.054j  0.797-0.64j  -0.049+0.528j -0.157+1.27j
-0.221-0.654j -0.39 -0.341j -0.676-0.138j -1.329+0.203j]

np.linalg.eig and my eigenvalues close? True





=====================================================================================
Testing QR_Iteration_WilkinsonShift() function on randomly generated REAL matrices
=====================================================================================

Real >>> Input shape = (3, 3)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=6 with tol=1e-06 reached

my eigenvalues: [ 1.098  0.468 -0.044]
np.linalg.eig:  [ 1.098  0.468 -0.044]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=3 with tol=1e-06 reached

my eigenvalues: [ 1.098  0.468 -0.044]
np.linalg.eig:  [ 1.098  0.468 -0.044]

np.linalg.eig and my eigenvalues close? True




Real >>> Input shape = (5, 5)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=24 with tol=1e-06 reached

my eigenvalues: [ 2.014+0.j    0.052+0.25j  0.052-0.25j -0.139+0.j   -0.506+0.j  ]
np.linalg.eig:  [ 2.014+0.j    0.052+0.25j  0.052-0.25j -0.139+0.j   -0.506+0.j  ]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------

/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:33: ComplexWarning: Casting complex valu
es to real discards the imaginary part
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:65: ComplexWarning: Casting complex valu
es to real discards the imaginary part
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:70: ComplexWarning: Casting complex valu
es to real discards the imaginary part
```

```
Iteration terminates at i=255 with tol=1e-06 reached

my eigenvalues: [ 2.014+0.j    0.052+0.25j  0.052-0.25j -0.139+0.j   -0.506+0.j  ]
np.linalg.eig:  [ 2.014+0.j    0.052+0.25j  0.052-0.25j -0.139+0.j   -0.506+0.j  ]

np.linalg.eig and my eigenvalues close? True




Real >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=551 with tol=1e-06 reached

my eigenvalues: [ 4.821+0.j     0.958+0.j     0.532+0.j     0.358+0.342j  0.358-0.342j -0.011+0.2j
-0.011-0.2j   -0.056+0.j    -0.128+0.597j -0.128-0.597j]
np.linalg.eig:  [ 4.821+0.j     0.958+0.j     0.532+0.j     0.358+0.342j  0.358-0.342j -0.011+0.2j
-0.011-0.2j   -0.056+0.j    -0.128+0.597j -0.128-0.597j]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=603 with tol=1e-06 reached

my eigenvalues: [ 4.821+0.j     0.958+0.j     0.532+0.j     0.358+0.342j  0.358-0.342j -0.011+0.2j
-0.011-0.2j   -0.056+0.j    -0.128+0.597j -0.128-0.597j]
np.linalg.eig:  [ 4.821+0.j     0.958+0.j     0.532+0.j     0.358+0.342j  0.358-0.342j -0.011+0.2j
-0.011-0.2j   -0.056+0.j    -0.128+0.597j -0.128-0.597j]

np.linalg.eig and my eigenvalues close? True






====================================================================================
Testing QR_Iteration_WilkinsonShift() function on SPECIAL real matrices
====================================================================================

Special >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=140 with tol=1e-06 reached

my eigenvalues: [ 14.993   9.198   8.242   6.585   2.133  -0.717  -5.727  -9.631 -13.122 -15.953]
np.linalg.eig:  [ 14.993   9.198   8.242   6.585   2.133  -0.717  -5.727  -9.631 -13.122 -15.953]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=37 with tol=1e-06 reached

my eigenvalues: [ 14.993   9.198   8.242   6.585   2.133  -0.717  -5.727  -9.631 -13.122 -15.953]
np.linalg.eig:  [ 14.993   9.198   8.242   6.585   2.133  -0.717  -5.727  -9.631 -13.122 -15.953]

np.linalg.eig and my eigenvalues close? True




Special >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=2146 with tol=1e-06 reached
```

```
my eigenvalues: [ 4.049  3.983  3.972  1.999  0.029 -0.03  -2.001 -3.999 -5.982 -6.022]
np.linalg.eig:  [ 4.049  3.984  3.972  1.999  0.029 -0.03  -2.001 -3.999 -5.982 -6.022]

np.linalg.eig and my eigenvalues close? False


my eigenvalues - np.linalg.eig =
 [-0.    -0.001 -0.    -0.    -0.     0.    -0.     0.001 -0.     0.   ]



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=48 with tol=1e-06 reached

my eigenvalues: [ 4.049  3.984  3.972  1.999  0.029 -0.03  -2.001 -3.999 -5.982 -6.022]
np.linalg.eig:  [ 4.049  3.984  3.972  1.999  0.029 -0.03  -2.001 -3.999 -5.982 -6.022]

np.linalg.eig and my eigenvalues close? True




Special >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=795 with tol=1e-06 reached

my eigenvalues: [1.026+0.009j 1.026-0.009j 1.015+0.022j 1.015-0.022j 0.999+0.027j 0.999-0.027j 0.984+
0.021j 0.984-0.021j 0.975+0.008j 0.975-0.008j]
np.linalg.eig:  [1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]

np.linalg.eig and my eigenvalues close? False


my eigenvalues - np.linalg.eig =
 [ 0.026+0.009j  0.026-0.009j  0.015+0.022j  0.015-0.022j -0.001+0.027j -0.001-0.027j -0.016+0.021j -
0.016-0.021j -0.025+0.008j -0.025-0.008j]



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=103 with tol=1e-06 reached

my eigenvalues: [1.024+0.008j 1.024-0.008j 1.014+0.021j 1.014-0.021j 0.999+0.025j 0.999-0.025j 0.985+
0.019j 0.985-0.019j 0.977+0.007j 0.977-0.007j]
np.linalg.eig:  [1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]

np.linalg.eig and my eigenvalues close? False


my eigenvalues - np.linalg.eig =
 [ 0.024+0.008j  0.024-0.008j  0.014+0.021j  0.014-0.021j -0.001+0.025j -0.001-0.025j -0.015+0.019j -
0.015-0.019j -0.023+0.007j -0.023-0.007j]




Special >>> Input shape = (6, 6)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=52 with tol=1e-06 reached

my eigenvalues: [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]
np.linalg.eig:  [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=18 with tol=1e-06 reached
```

```
my eigenvalues: [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]
np.linalg.eig:  [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]

np.linalg.eig and my eigenvalues close? True
```

Observations:

- Function QR_Iteration_WilkinsonShift() seems to be working as expected with both real and complex matrices, with real eigenvalues or complex conjugate pairs
- Underperformance in special matrix -> diagonal dominant with ones in lower triangular **#TODO**
- Convergence comparison in visualization **#TODO**

In [ ]:
```
# Some visualization of comparision of convergence - TODO!
```

#

Eigenvectors

Two similar matrices, A and B, with similarity transformation $B = S^{-1}AS$, have the same set of eigenvalues and their eigenvectors are related $v_B = P^{-1}v_A$.

Reference sources:

- Implementing the QR algorithm for efficiently computing matrix eigenvalues and eigenvectors - Section 4.5 (p.51) https://addi.ehu.es/bitstream/handle/10810/26427/TFG_Erana_Robles_Gorka.pdf?sequence=1 (https://addi.ehu.es/bitstream/handle/10810/26427/TFG_Erana_Robles_Gorka.pdf?sequence=1)
- Find eigenvectors of an upper triangular matrix

(1) https://math.stackexchange.com/questions/2632460/general-form-of-left-and-right-eigenvectors-of-upper-triangular-matrices (https://math.stackexchange.com/questions/2632460/general-form-of-left-and-right-eigenvectors-of-upper-triangular-matrices)

(2) https://math.stackexchange.com/questions/3947108/how-to-get-eigenvectors-using-qr-algorithm (https://math.stackexchange.com/questions/3947108/how-to-get-eigenvectors-using-qr-algorithm)

In [17]:
```
# Find Eigenvector of an upper triangular matrix given eigenvectors

def eigenvectors_upperTriangularMatrix(U,eigenvalues):

    """     Compute the eigenvectors of an upper triangular matrix, based on given eigenvalues
            Source: https://addi.ehu.es/bitstream/handle/10810/26427/TFG_Erana_Robles_Gorka.pdf?sequen
ce=1
    Input   ---------------------------------------------
            U: 2D np.array in the form of an upper triangular matrix
            eigenvalues: 1D np.array, real eigenvalues associated with matrix U
    Return  --------------------------------------------
            V: 2D np.array with each column represents an eigenvector, in the same order as the associ
ated eigenvalue
    """

    n = U.shape[0]

    V = np.eye(n, dtype=U.dtype) # in case of complex matrix, enforce dtype

    for i in range(1,n,1):
        for j in range(i-1,-1,-1):
            V[j,i]= - np.dot(U[j,:],V[:,i]) / (U[j,j]-eigenvalues[i])

        V[:,i] = V[:,i]/np.linalg.norm(V[:,i]) # vector normalization

    return V
```

In [18]:
```
# Test for eigenvectors

def test_eigenvectors(eigenvectors, v, verbose=True):

    """     Compare two sets of eigenvectors
    Input   ----------------------------------------------
            eigenvectors: 2D np.array with real or complex entries, eigenvectors from my function
            v: 2D np.array, eigenvectors from np.linalg.eig(), basis for comparison
            verbose: bool, indicating whether to print the two sets of eigenvectors and ratios of eige
nvectors/v
    Return  ---------------------------------------------
            None; test results will be printed
    """


    if np.allclose(eigenvectors,v,rtol=RTOL):
        print('np.linalg.eig and my function return the same eigenvectors? True \n')

    else:
        print('np.linalg.eig and my function return the same eigenvectors? False \n')
        d = np.divide(eigenvectors,v, where=(v!=0))
        if verbose:
            print('my eigenvectors =\n',eigenvectors,'\nnp.linalg.eig =\n',v, '\nmy eigenvectors / np.
linalg.eig =\n', d,'\n')
        d_ = np.divide(d,d[0], where=(d[0]!=0))
        print('Eigenvectors all close except sign and/or scaling?', np.allclose(d_,np.ones(d.shape,d.d
type),rtol=RTOL))

        # special case of upper triangular matrix
        if np.allclose(np.tril(v,-1),np.zeros(d.shape,d.dtype)):
            print('[Upper Triagular Matrix] Eigenvectors all close except signs?', np.allclose(np.abs(
eigenvectors),np.abs(v),rtol=RTOL),'\n')
        print('\n\n')


def test_eigenvectors_function(A,eigenvalues,function):

    """     Perform eigenvector testing on selected function; eigenvectors are compared with results f
rom np.linalg.eig()
    Input   ----------------------------------------------
            A: input matrix, 2D np.array with real or complex entries
            function: name of the function being tested
    Return  ---------------------------------------------
            None; test results will be printed
    """

    print("------ Testing for function: {}() ------".format(function.__name__))
    eigenvectors = function(A, eigenvalues)
    w,v = np.linalg.eig(A)

    test_eigenvectors(eigenvectors, v)
```

In [19]:
```python
### Testing eigenvectors_upperTriangularMatrix() on Random Real Upper Triangular Matrices

print('\n\n')
print("====================================================================================")
print('Testing eigenvectors_upperTriangularMatrix() on Random REAL Upper Triangular Matrices')
print("====================================================================================")

for A in RANDOM_REAL_MATRICES:
    U = np.triu(A)
    print('\nREAL >>> Input shape = {}\n'.format(U.shape))
    print('U = \n',U)
    eigenvalues = np.diag(U)
    test_eigenvectors_function(U,eigenvalues,eigenvectors_upperTriangularMatrix)


### Testing eigenvectors_upperTriangularMatrix() on Random COMPLEX Upper Triangular Matrices

print('\n\n')
print("====================================================================================")
print('Testing eigenvectors_upperTriangularMatrix() on Random COMPLEX Upper Triangular Matrices')
print("====================================================================================")

for A in RANDOM_COMPLEX_MATRICES:
    U = np.triu(A)
    print('\nCOMPLEX >>> Input shape = {}\n'.format(U.shape))
    print('U = \n',U)
    eigenvalues = np.diag(U)
    test_eigenvectors_function(U,eigenvalues,eigenvectors_upperTriangularMatrix)
```

```
==================================================================================
Testing eigenvectors_upperTriangularMatrix() on Random REAL Upper Triangular Matrices
==================================================================================

REAL >>> Input shape = (3, 3)

U =
 [[0.657 0.577 0.419]
 [0.    0.689 0.853]
 [0.    0.    0.176]]
------ Testing for function: eigenvectors_upperTriangularMatrix() ------
np.linalg.eig and my function return the same eigenvectors? True


REAL >>> Input shape = (5, 5)

U =
 [[0.115 0.312 0.619 0.011 0.972]
 [0.    0.888 0.794 0.22  0.044]
 [0.    0.    0.155 0.21  0.547]
 [0.    0.    0.    0.034 0.128]
 [0.    0.    0.    0.    0.282]]
------ Testing for function: eigenvectors_upperTriangularMatrix() ------
np.linalg.eig and my function return the same eigenvectors? True


REAL >>> Input shape = (10, 10)

U =
 [[0.764 0.732 0.93  0.454 0.252 0.53  0.044 0.633 0.201 0.171]
 [0.    0.73  0.879 0.219 0.097 0.742 0.046 0.856 0.831 0.116]
 [0.    0.    0.834 0.19  0.234 0.315 0.111 0.509 0.41  0.556]
 [0.    0.    0.    0.722 0.81  0.    0.634 0.607 0.168 0.96 ]
 [0.    0.    0.    0.    0.615 0.512 0.639 0.886 0.767 0.606]
 [0.    0.    0.    0.    0.    0.992 0.489 0.489 0.447 0.649]
 [0.    0.    0.    0.    0.    0.    0.507 0.355 0.057 0.261]
 [0.    0.    0.    0.    0.    0.    0.    0.18  0.711 0.667]
 [0.    0.    0.    0.    0.    0.    0.    0.    0.751 0.444]
 [0.    0.    0.    0.    0.    0.    0.    0.    0.    0.598]]
------ Testing for function: eigenvectors_upperTriangularMatrix() ------
np.linalg.eig and my function return the same eigenvectors? True




==================================================================================
Testing eigenvectors_upperTriangularMatrix() on Random COMPLEX Upper Triangular Matrices
==================================================================================

COMPLEX >>> Input shape = (3, 3)

U =
 [[0.3  +0.511j 0.16 +0.142j 0.42 +0.515j]
 [0.   +0.j    0.479+0.131j 0.906+0.751j]
 [0.   +0.j    0.   +0.j    0.64 +0.243j]]
------ Testing for function: eigenvectors_upperTriangularMatrix() ------
np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 1.   +0.j    -0.127+0.436j  0.041+0.593j]
 [ 0.   +0.j     0.891+0.j     0.79 +0.067j]
 [ 0.   +0.j     0.   +0.j     0.132+0.j   ]]
np.linalg.eig =
 [[ 1.   +0.j    -0.127+0.436j  0.091+0.588j]
 [ 0.   +0.j     0.891+0.j     0.793+0.j   ]
 [ 0.   +0.j     0.   +0.j     0.132-0.011j]]
my eigenvectors / np.linalg.eig =
 [[1.   +0.j    1.   +0.j    0.996+0.084j]
 [0.   +0.j    1.   +0.j    0.996+0.084j]
```

```
 [0.   +0.j    0.   +0.j    0.996+0.084j]]

Eigenvectors all close except sign and/or scaling? False
[Upper Triagular Matrix] Eigenvectors all close except signs? True




COMPLEX >>> Input shape = (5, 5)

U =
 [[0.344+0.535j 0.001+0.474j 0.596+0.834j 0.358+0.025j 0.886+0.639j]
 [0.   +0.j    0.048+0.356j 0.473+0.686j 0.645+0.425j 0.847+0.436j]
 [0.   +0.j    0.   +0.j    0.541+0.599j 0.4  +0.919j 0.642+0.75j ]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.855+0.817j 0.693+0.162j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.77 +0.606j]]
------ Testing for function: eigenvectors_upperTriangularMatrix() ------
np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 1.   +0.j    -0.419-0.69j   0.449+0.859j  0.321+0.781j -0.47 -0.747j]
 [ 0.   +0.j     0.59 +0.j     0.179+0.1j    0.277+0.269j -0.321-0.113j]
 [ 0.   +0.j     0.   +0.j     0.135+0.j     0.295+0.182j -0.309+0.025j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.132+0.j    -0.053+0.074j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.029+0.j   ]]
np.linalg.eig =
 [[ 1.   +0.j     0.808+0.j     0.969+0.j     0.845+0.j     0.883+0.j   ]
 [ 0.   +0.j    -0.306+0.504j  0.171-0.112j  0.354-0.154j  0.266-0.212j]
 [ 0.   +0.j    -0.   +0.j     0.062-0.12j   0.281-0.203j  0.143-0.274j]
 [ 0.   +0.j    -0.   +0.j     0.   +0.j     0.05 -0.122j -0.035-0.084j]
 [ 0.   +0.j    -0.   +0.j     0.   +0.j     0.   +0.j    -0.015+0.025j]]
my eigenvectors / np.linalg.eig =
 [[ 1.   +0.j    -0.519-0.855j  0.463+0.887j  0.381+0.925j -0.532-0.847j]
 [ 0.   +0.j    -0.519-0.855j  0.463+0.887j  0.381+0.925j -0.532-0.847j]
 [ 0.   +0.j     0.   +0.j     0.463+0.887j  0.381+0.925j -0.532-0.847j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.381+0.925j -0.532-0.847j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j    -0.532-0.847j]]

Eigenvectors all close except sign and/or scaling? False
[Upper Triagular Matrix] Eigenvectors all close except signs? True




COMPLEX >>> Input shape = (10, 10)

U =
 [[0.888+0.745j 0.354+0.961j 0.881+0.881j 0.768+0.572j 0.42 +0.319j 0.051+0.541j 0.98 +0.49j  0.911+
0.984j 0.232+0.29j  0.506+0.794j]
 [0.   +0.j    0.999+0.772j 0.365+0.895j 0.701+0.347j 0.907+0.705j 0.275+0.875j 0.296+0.681j 0.371+0.
522j 0.502+0.777j 0.204+0.111j]
 [0.   +0.j    0.   +0.j    0.096+0.823j 0.579+0.807j 0.484+0.944j 0.63 +0.019j 0.467+0.138j 0.484+0.
097j 0.796+0.713j 0.03 +0.42j ]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.301+0.497j 0.51 +0.994j 0.584+0.405j 0.827+0.609j 0.209+0.
734j 0.601+0.954j 0.407+0.745j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.914+0.965j 0.059+0.119j 0.948+0.148j 0.143+0.
672j 0.109+0.959j 0.271+0.934j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.089+0.197j 0.756+0.115j 0.149+0.
99j  0.843+0.203j 0.737+0.539j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.671+0.427j 0.461+0.
426j 0.233+0.42j  0.833+0.438j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.345+0.
935j 0.814+0.749j 0.62 +0.746j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.
j    0.595+0.123j 0.744+0.268j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.
j    0.   +0.j    0.104+0.513j]]
------ Testing for function: eigenvectors_upperTriangularMatrix() ------
np.linalg.eig and my function return the same eigenvectors? False
```

```
my eigenvectors =
 [[ 1.   +0.j     0.552+0.827j -0.813-0.172j -0.268-0.773j  0.895-0.409j -0.069+0.503j  0.385+0.435j
-0.052+0.908j -0.15 -0.588j  0.366-0.664j]
 [ 0.   +0.j     0.111+0.j    -0.132-0.384j  0.321-0.203j  0.164-0.059j -0.341-0.226j  0.289-0.091j -
0.289+0.185j -0.23 +0.164j  0.402+0.097j]
 [ 0.   +0.j     0.   +0.j     0.38 +0.j    -0.153+0.373j  0.013+0.026j  0.363-0.395j -0.354-0.456j -
0.079-0.215j -0.081+0.494j -0.355+0.325j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.156+0.j     0.013+0.006j -0.404+0.225j -0.204+0.19j  -
0.03 -0.024j  0.428-0.021j  0.137-0.043j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.01 +0.j    -0.03 -0.011j -0.19 +0.236j -
0.018+0.009j  0.149-0.051j  0.011+0.004j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.268+0.j     0.196-0.045j
0.009-0.008j -0.121-0.223j -0.028+0.032j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.165+0.j
0.004-0.02j  -0.093-0.111j -0.001-0.011j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j
0.02 +0.j    -0.028+0.06j   0.001+0.014j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j
0.   +0.j     0.051+0.j    -0.004-0.007j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j
0.   +0.j     0.   +0.j     0.006+0.j   ]]
np.linalg.eig =
 [[ 1.   +0.j     0.994+0.j     0.831+0.j     0.818+0.j     0.984+0.j    -0.417+0.289j  0.581+0.j
0.91 +0.j     0.607+0.j     0.758+0.j   ]
 [ 0.   +0.j     0.062-0.092j  0.208+0.349j  0.087+0.37j   0.173+0.015j -0.064-0.404j  0.124-0.277j
0.201+0.278j -0.102-0.264j  0.109+0.399j]
 [ 0.   +0.j     0.   +0.j    -0.371+0.079j -0.302-0.266j  0.001+0.029j  0.537+0.j    -0.576-0.037j -
0.21 +0.091j -0.459-0.201j -0.456-0.154j]
 [ 0.   +0.j     0.   +0.j    -0.   +0.j    -0.051+0.148j  0.009+0.011j -0.439-0.146j  0.007+0.278j -
0.022+0.031j -0.085+0.42j   0.104+0.099j]
 [ 0.   +0.j     0.   +0.j    -0.   +0.j    -0.   +0.j     0.009+0.004j -0.012-0.029j  0.051+0.299j
0.01 +0.018j  0.013+0.157j  0.001+0.011j]
 [ 0.   +0.j     0.   +0.j    -0.   +0.j    -0.   +0.j     0.   +0.j     0.181+0.198j  0.096-0.177j -
0.008-0.008j  0.246-0.062j -0.042-0.009j]
 [ 0.   +0.j     0.   +0.j    -0.   +0.j    -0.   +0.j     0.   +0.j     0.   +0.j     0.109-0.124j -
0.02 -0.002j  0.131-0.062j  0.009-0.007j]
 [ 0.   +0.j     0.   +0.j    -0.   +0.j    -0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j    -
0.001-0.02j  -0.051-0.042j -0.011+0.008j]
 [ 0.   +0.j     0.   +0.j    -0.   +0.j    -0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j
0.   -0.j    -0.013+0.049j  0.004-0.007j]
 [ 0.   +0.j     0.   +0.j    -0.   +0.j    -0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j
0.   -0.j    -0.   +0.j     0.003+0.005j]]
my eigenvectors / np.linalg.eig =
 [[ 1.   +0.j     0.555+0.832j -0.978-0.207j -0.327-0.945j  0.91 -0.415j  0.676-0.737j  0.662+0.749j
-0.058+0.998j -0.247-0.969j  0.483-0.876j]
 [ 0.   +0.j     0.555+0.832j -0.978-0.207j -0.327-0.945j  0.91 -0.415j  0.676-0.737j  0.662+0.749j -
0.058+0.998j -0.247-0.969j  0.483-0.876j]
 [ 0.   +0.j     0.   +0.j    -0.978-0.207j -0.327-0.945j  0.91 -0.415j  0.676-0.737j  0.662+0.749j -
0.058+0.998j -0.247-0.969j  0.483-0.876j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j    -0.327-0.945j  0.91 -0.415j  0.676-0.737j  0.662+0.749j -
0.058+0.998j -0.247-0.969j  0.483-0.876j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.91 -0.415j  0.676-0.737j  0.662+0.749j -
0.058+0.998j -0.247-0.969j  0.483-0.876j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.676-0.737j  0.662+0.749j -
0.058+0.998j -0.247-0.969j  0.483-0.876j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.662+0.749j -
0.058+0.998j -0.247-0.969j  0.483-0.876j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j    -
0.058+0.998j -0.247-0.969j  0.483-0.876j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j
0.   +0.j    -0.247-0.969j  0.483-0.876j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j
0.   +0.j     0.   +0.j     0.483-0.876j]]

Eigenvectors all close except sign and/or scaling? False
[Upper Triagular Matrix] Eigenvectors all close except signs? True
```

Observations:

- Seems to be working except for the signs of eigenvectors

In [20]:
```
# General method for obtaining eigenvectors

def inverse_power_method_with_shift(A, s, max_iter=MAX_ITER, rtol=RTOL):

    """     Apply the inverse power method with shift to compute the eigenvectors associated with an e
igenvalue which is close to the shift
    Input   ----------------------------------------------
            A: input matrix, 2D np.array wiht real or complex entries
            s: shift to be performed, which is closed to an eigenvalue
            max_iter: int, maximum number of iterations to be performed
            rtol: float, relative tolerance between eigenvalues by successive iterations
    Return  ----------------------------------------------
            x_new: 1D np.array, the eigenvector associated with the eigenvalue which is closed to the
 given shift
    """

    n = A.shape[0]
    A_ = A.astype(s.dtype) - s*np.eye(n,dtype=s.dtype)
    x_old = np.ones(n, dtype=s.dtype) #initialization as [1,1,1,...]

    for i in range(max_iter):
        y = np.linalg.solve(A_,x_old) # assuming this can be used? Potentially Singular Matrix Erro
r...
        x_new = y/np.linalg.norm(y)

        if np.allclose(x_new,x_old,rtol=rtol):
            print('... num of iterations with inverse power method =',i)
            break
        x_old = x_new

    return x_new



def eigenvectors_inversePowerMethod(U,eigenvalues):

    """     Compute the eigenvectors of a block triangular matrix, based on given eigenvalues
    Input   ----------------------------------------------
            U: 2D np.array in the form of a block triangular matrix with complex eigenvalues
            eigenvalues: 1D np.array, complex eigenvalues associated with matrix U
    Return  ----------------------------------------------
            V: 2D np.array with each column represents an eigenvector, in the same order as the associ
ated eigenvalues
    """

    n = U.shape[0]
    V = np.eye(n, dtype='complex') # only used in case of complex matrix, need to enforce dtype

    for i in range(n):
        V[:,i] = inverse_power_method_with_shift(U, eigenvalues[i])

    # print('\n ----\n',V.dtype, 'eigenvectors_inversePowerMethod=\n',V,)

    return V
```

In [21]:
```python
# Testing

print('\n\n')
print("====================================================================================")
print('Testing eigenvectors_inversePowerMethod() on Random REAL Matrices')
print("====================================================================================")

for A in RANDOM_REAL_MATRICES[-1:]:
    print('\nREAL >>> Input shape = {}\n'.format(A.shape))
    w,v = np.linalg.eig(A)
    eigenvalues = w
    test_eigenvectors_function(A,eigenvalues,eigenvectors_inversePowerMethod)


print('\n\n')
print("====================================================================================")
print('Testing eigenvectors_inversePowerMethod() on Random COMPLEX Matrices')
print("====================================================================================")

for A in RANDOM_COMPLEX_MATRICES[-1:]:
    print('\nCOMPLEX >>> Input shape = {}\n'.format(A.shape))
    w,v = np.linalg.eig(A)
    eigenvalues = w
    test_eigenvectors_function(A,eigenvalues,eigenvectors_inversePowerMethod)


print('\n\n')
print("====================================================================================")
print('Testing eigenvectors_inversePowerMethod() on Random COMPLEX Upper *BLOCK* Triangular Matrices')
print("====================================================================================")

for A in RANDOM_COMPLEX_MATRICES[-1:]:
    print('\nCOMPLEX >>> Input shape = {}\n'.format(A.shape))
    U = np.triu(A,-1)
    print('U = \n',U)
    w,v = np.linalg.eig(U)
    eigenvalues = w
    test_eigenvectors_function(U,eigenvalues,eigenvectors_inversePowerMethod)
```

```
================================================================================
Testing eigenvectors_inversePowerMethod() on Random REAL Matrices
================================================================================

REAL >>> Input shape = (10, 10)

------ Testing for function: eigenvectors_inversePowerMethod() ------
... num of iterations with inverse power method = 1
... num of iterations with inverse power method = 1
np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 0.295+0.j     0.127+0.j    -0.097+0.326j -0.097-0.326j  0.057+0.j     0.106+0.087j  0.106-0.087j
0.053+0.j    -0.176+0.209j -0.176-0.209j]
 [ 0.328+0.j     0.163+0.j    -0.345-0.074j -0.345+0.074j -0.336+0.j     0.412+0.279j  0.412-0.279j
0.043+0.j    -0.382+0.033j -0.382-0.033j]
 [ 0.246+0.j     0.258+0.j    -0.032-0.152j -0.032+0.152j  0.169+0.j    -0.197-0.06j  -0.197+0.06j
0.287+0.j     0.105-0.331j  0.105+0.331j]
 [ 0.361+0.j    -0.327+0.j     0.026+0.176j  0.026-0.176j  0.497+0.j    -0.271-0.484j -0.271+0.484j -
0.11 +0.j     0.512+0.114j  0.512-0.114j]
 [ 0.405+0.j    -0.418+0.j    -0.307-0.141j -0.307+0.141j  0.041+0.j    -0.016+0.002j -0.016-0.002j -
0.36 +0.j    -0.175-0.296j -0.175+0.296j]
 [ 0.3  +0.j     0.206+0.j    -0.187-0.063j -0.187+0.063j -0.576+0.j     0.011+0.328j  0.011-0.328j -
0.29 +0.j    -0.188-0.002j -0.188+0.002j]
 [ 0.314+0.j    -0.416+0.j     0.266+0.375j  0.266-0.375j  0.26 +0.j    -0.284+0.14j  -0.284-0.14j
0.775+0.j    -0.065+0.022j -0.065-0.022j]
 [ 0.278+0.j    -0.394+0.j     0.446-0.265j  0.446+0.265j  0.318+0.j    -0.187-0.132j -0.187+0.132j -
0.043+0.j     0.277+0.146j  0.277-0.146j]
 [ 0.345+0.j    -0.028+0.j     0.256-0.01j   0.256+0.01j  -0.068+0.j     0.045+0.017j  0.045-0.017j -
0.017+0.j     0.163-0.137j  0.163+0.137j]
 [ 0.256+0.j     0.487+0.j    -0.037-0.065j -0.037+0.065j -0.318+0.j     0.309-0.159j  0.309+0.159j -
0.292+0.j    -0.103+0.264j -0.103-0.264j]]
np.linalg.eig =
 [[-0.295+0.j     0.127+0.j    -0.25 +0.231j -0.25 -0.231j -0.057+0.j     0.128-0.05j   0.128+0.05j
0.053+0.j    -0.127+0.242j -0.127-0.242j]
 [-0.328+0.j     0.163+0.j    -0.259-0.24j  -0.259+0.24j   0.336+0.j     0.445-0.223j  0.445+0.223j
0.043+0.j    -0.366+0.115j -0.366-0.115j]
 [-0.246+0.j     0.258+0.j     0.05 -0.147j  0.05 +0.147j -0.169+0.j    -0.149+0.143j -0.149-0.143j
0.287+0.j     0.031-0.346j  0.031+0.346j]
 [-0.361+0.j    -0.327+0.j    -0.068+0.165j -0.068-0.165j -0.497+0.j    -0.555+0.j    -0.555-0.j    -
0.11 +0.j     0.525+0.j     0.525-0.j   ]
 [-0.405+0.j    -0.418+0.j    -0.191-0.278j -0.191+0.278j -0.041+0.j    -0.006+0.015j -0.006-0.015j -
0.36 +0.j    -0.235-0.251j -0.235+0.251j]
 [-0.3  +0.j     0.206+0.j    -0.129-0.149j -0.129+0.149j  0.576+0.j     0.291+0.151j  0.291-0.151j -
0.29 +0.j    -0.184+0.038j -0.184-0.038j]
 [-0.314+0.j    -0.416+0.j     0.036+0.458j  0.036-0.458j -0.26 +0.j    -0.017+0.317j -0.017-0.317j
0.775+0.j    -0.058+0.035j -0.058-0.035j]
 [-0.278+0.j    -0.394+0.j     0.519+0.j     0.519-0.j    -0.318+0.j    -0.207+0.098j -0.207-0.098j -
0.043+0.j     0.302+0.082j  0.302-0.082j]
 [-0.345+0.j    -0.028+0.j     0.225+0.122j  0.225-0.122j  0.068+0.j     0.037-0.031j  0.037+0.031j -
0.017+0.j     0.129-0.169j  0.129+0.169j]
 [-0.256+0.j     0.487+0.j     0.001-0.075j  0.001+0.075j  0.318+0.j     0.013-0.347j  0.013+0.347j -
0.292+0.j    -0.044+0.281j -0.044-0.281j]]
my eigenvectors / np.linalg.eig =
 [[-1.   -0.j     1.   +0.j     0.859-0.512j  0.859+0.512j -1.   -0.j     0.489+0.872j  0.489-0.872j
1.   +0.j     0.976+0.216j  0.976-0.216j]
 [-1.   -0.j     1.   +0.j     0.859-0.512j  0.859+0.512j -1.   +0.j     0.489+0.872j  0.489-0.872j
1.   +0.j     0.976+0.216j  0.976-0.216j]
 [-1.   -0.j     1.   +0.j     0.859-0.512j  0.859+0.512j -1.   -0.j     0.489+0.872j  0.489-0.872j
1.   +0.j     0.976+0.216j  0.976-0.216j]
 [-1.   -0.j     1.   -0.j     0.859-0.512j  0.859+0.512j -1.   -0.j     0.489+0.872j  0.489-0.872j
1.   -0.j     0.976+0.216j  0.976-0.216j]
 [-1.   -0.j     1.   -0.j     0.859-0.512j  0.859+0.512j -1.   -0.j     0.489+0.872j  0.489-0.872j
1.   -0.j     0.976+0.216j  0.976-0.216j]
 [-1.   -0.j     1.   +0.j     0.859-0.512j  0.859+0.512j -1.   +0.j     0.489+0.872j  0.489-0.872j
1.   -0.j     0.976+0.216j  0.976-0.216j]
 [-1.   -0.j     1.   -0.j     0.859-0.512j  0.859+0.512j -1.   -0.j     0.489+0.872j  0.489-0.872j
1.   +0.j     0.976+0.216j  0.976-0.216j]
 [-1.   -0.j     1.   -0.j     0.859-0.512j  0.859+0.512j -1.   -0.j     0.489+0.872j  0.489-0.872j
```

```
1.   -0.j     0.976+0.216j  0.976-0.216j]
 [-1.   -0.j     1.   -0.j     0.859-0.512j  0.859+0.512j -1.   +0.j     0.489+0.872j  0.489-0.872j
1.   -0.j     0.976+0.216j  0.976-0.216j]
 [-1.   -0.j     1.   +0.j     0.859-0.512j  0.859+0.512j -1.   +0.j     0.489+0.872j  0.489-0.872j
1.   -0.j     0.976+0.216j  0.976-0.216j]]

Eigenvectors all close except sign and/or scaling? True






=================================================================================
Testing eigenvectors_inversePowerMethod() on Random COMPLEX Matrices
=================================================================================

COMPLEX >>> Input shape = (10, 10)

------ Testing for function: eigenvectors_inversePowerMethod() ------
np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[-0.357-0.135j  0.08 +0.291j -0.359-0.223j  0.026-0.544j -0.225+0.22j   0.285-0.359j  0.113+0.155j
-0.149-0.049j -0.26 -0.177j  0.267-0.097j]
 [-0.33 -0.12j   0.146-0.051j  0.106-0.058j -0.009+0.178j -0.087-0.079j -0.162-0.284j -0.069+0.181j -
0.335+0.376j -0.164+0.337j -0.341-0.127j]
 [-0.276-0.113j  0.166+0.533j  0.508-0.059j  0.028+0.042j -0.002-0.4j    0.153+0.231j -0.026+0.094j
0.202+0.079j  0.267+0.026j -0.052+0.019j]
 [-0.285-0.135j -0.206-0.222j  0.173+0.267j -0.053+0.17j  -0.24 +0.005j -0.045-0.04j  -0.118-0.095j
0.423+0.245j  0.329+0.134j -0.218+0.359j]
 [-0.265-0.161j  0.124+0.014j -0.158+0.141j  0.464+0.33j   0.042+0.235j  0.253+0.23j   0.312-0.096j
0.207-0.113j  0.072-0.271j  0.239+0.116j]
 [-0.277-0.102j -0.005-0.342j -0.069-0.214j  0.097+0.26j   0.042-0.149j  0.245+0.406j  0.285+0.154j -
0.025-0.328j -0.358-0.439j  0.497+0.04j ]
 [-0.277-0.137j -0.009-0.199j -0.186-0.13j  -0.132+0.032j  0.247+0.448j -0.272+0.092j -0.5  +0.023j -
0.236-0.178j -0.291+0.064j  0.01 -0.259j]
 [-0.28 -0.077j -0.482-0.211j  0.064+0.091j -0.141-0.328j  0.063+0.003j -0.21 -0.305j  0.266-0.198j
0.19 +0.065j  0.232+0.066j -0.291-0.041j]
 [-0.293-0.083j  0.028+0.144j -0.22 +0.27j  -0.066-0.005j -0.262-0.383j -0.188+0.07j  -0.444-0.044j -
0.089-0.357j  0.035+0.013j -0.122-0.078j]
 [-0.276-0.102j  0.057+0.083j  0.237-0.328j -0.195-0.218j  0.254+0.207j -0.002-0.025j  0.162-0.311j -
0.092+0.021j  0.115-0.02j   0.291+0.169j]]
np.linalg.eig =
 [[ 0.381+0.j     0.302+0.01j  -0.331-0.263j -0.294-0.458j  0.084+0.304j -0.16 -0.429j -0.105-0.16j
0.062+0.144j  0.302-0.089j  0.259-0.119j]
 [ 0.351-0.004j -0.005-0.155j  0.112-0.045j  0.096+0.15j  -0.112+0.038j -0.327-0.007j  0.077-0.178j
0.503+0.j    -0.158-0.34j  -0.351-0.099j]
 [ 0.298+0.008j  0.558+0.j     0.511+0.j     0.047+0.018j -0.351-0.191j  0.277-0.011j  0.03 -0.093j -
0.075-0.203j -0.189+0.191j -0.051+0.023j]
 [ 0.315+0.025j -0.273+0.131j  0.141+0.285j  0.055+0.169j -0.112+0.213j -0.058+0.018j  0.114+0.1j   -
0.099-0.479j -0.312+0.17j  -0.188+0.376j]
 [ 0.305+0.057j  0.05 -0.115j -0.173+0.122j  0.569+0.j     0.226+0.076j  0.327-0.098j -0.316+0.081j -
0.222-0.08j   0.164+0.227j  0.247+0.096j]
 [ 0.296-0.003j -0.328-0.096j -0.044-0.22j   0.23 +0.156j -0.111-0.109j  0.474+0.j    -0.277-0.167j -
0.228+0.237j  0.567+0.j     0.498+0.j   ]
 [ 0.308+0.03j  -0.193-0.051j -0.17 -0.15j  -0.089+0.103j  0.511+0.j    -0.062+0.281j  0.5  +0.j
0.024+0.295j  0.134-0.266j -0.011-0.259j]
 [ 0.289-0.027j -0.345+0.398j  0.053+0.098j -0.306-0.186j  0.033-0.054j -0.37 +0.022j -0.275+0.186j -
0.078-0.185j -0.198+0.138j -0.294-0.018j]
 [ 0.303-0.026j  0.146+0.015j -0.25 +0.243j -0.057+0.034j -0.462+0.045j -0.037+0.198j  0.441+0.065j -
0.208+0.304j -0.032+0.019j -0.128-0.068j]
 [ 0.295-0.002j  0.097-0.03j   0.273-0.299j -0.285-0.065j  0.304-0.122j -0.022-0.011j -0.177+0.303j
0.077+0.055j -0.057+0.102j  0.304+0.145j]]
my eigenvectors / np.linalg.eig =
 [[-0.935-0.354j  0.296+0.955j  0.993-0.115j  0.815+0.58j   0.483+0.876j  0.516+0.857j -0.999+0.046j
-0.665+0.747j -0.632-0.775j  0.997+0.081j]
 [-0.935-0.354j  0.296+0.955j  0.993-0.115j  0.815+0.58j   0.483+0.876j  0.516+0.857j -0.999+0.046j -
0.665+0.747j -0.632-0.775j  0.997+0.081j]
 [-0.935-0.354j  0.296+0.955j  0.993-0.115j  0.815+0.58j   0.483+0.876j  0.516+0.857j -0.999+0.046j -
0.665+0.747j -0.632-0.775j  0.997+0.081j]
 [-0.935-0.354j  0.296+0.955j  0.993-0.115j  0.815+0.58j   0.483+0.876j  0.516+0.857j -0.999+0.046j -
```

```
0.665+0.747j -0.632-0.775j  0.997+0.081j]
 [-0.935-0.354j  0.296+0.955j  0.993-0.115j  0.815+0.58j   0.483+0.876j  0.516+0.857j -0.999+0.046j -
0.665+0.747j -0.632-0.775j  0.997+0.081j]
 [-0.935-0.354j  0.296+0.955j  0.993-0.115j  0.815+0.58j   0.483+0.876j  0.516+0.857j -0.999+0.046j -
0.665+0.747j -0.632-0.775j  0.997+0.081j]
 [-0.935-0.354j  0.296+0.955j  0.993-0.115j  0.815+0.58j   0.483+0.876j  0.516+0.857j -0.999+0.046j -
0.665+0.747j -0.632-0.775j  0.997+0.081j]
 [-0.935-0.354j  0.296+0.955j  0.993-0.115j  0.815+0.58j   0.483+0.876j  0.516+0.857j -0.999+0.046j -
0.665+0.747j -0.632-0.775j  0.997+0.081j]
 [-0.935-0.354j  0.296+0.955j  0.993-0.115j  0.815+0.58j   0.483+0.876j  0.516+0.857j -0.999+0.046j -
0.665+0.747j -0.632-0.775j  0.997+0.081j]
 [-0.935-0.354j  0.296+0.955j  0.993-0.115j  0.815+0.58j   0.483+0.876j  0.516+0.857j -0.999+0.046j -
0.665+0.747j -0.632-0.775j  0.997+0.081j]]

Eigenvectors all close except sign and/or scaling? True





=====================================================================================
Testing eigenvectors_inversePowerMethod() on Random COMPLEX Upper *BLOCK* Triangular Matrices
=====================================================================================

COMPLEX >>> Input shape = (10, 10)

U =
 [[0.888+0.745j 0.354+0.961j 0.881+0.881j 0.768+0.572j 0.42 +0.319j 0.051+0.541j 0.98 +0.49j  0.911+
0.984j 0.232+0.29j  0.506+0.794j]
 [0.906+0.351j 0.999+0.772j 0.365+0.895j 0.701+0.347j 0.907+0.705j 0.275+0.875j 0.296+0.681j 0.371+0.
522j 0.502+0.777j 0.204+0.111j]
 [0.   +0.j    0.143+0.564j 0.096+0.823j 0.579+0.807j 0.484+0.944j 0.63 +0.019j 0.467+0.138j 0.484+0.
097j 0.796+0.713j 0.03 +0.42j ]
 [0.   +0.j    0.   +0.j    0.302+0.099j 0.301+0.497j 0.51 +0.994j 0.584+0.405j 0.827+0.609j 0.209+0.
734j 0.601+0.954j 0.407+0.745j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.033+0.22j  0.914+0.965j 0.059+0.119j 0.948+0.148j 0.143+0.
672j 0.109+0.959j 0.271+0.934j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.844+0.826j 0.089+0.197j 0.756+0.115j 0.149+0.
99j  0.843+0.203j 0.737+0.539j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.051+0.964j 0.671+0.427j 0.461+0.
426j 0.233+0.42j  0.833+0.438j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.45 +0.289j 0.345+0.
935j 0.814+0.749j 0.62 +0.746j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.719+0.
243j 0.595+0.123j 0.744+0.268j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.
j    0.384+0.509j 0.104+0.513j]]
------ Testing for function: eigenvectors_inversePowerMethod() ------
np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[-0.051+0.717j  0.171+0.7j   -0.679-0.096j  0.198-0.577j  0.129+0.553j  0.235-0.434j -0.374+0.073j
0.051-0.614j -0.264+0.153j -0.046-0.178j]
 [ 0.178+0.625j  0.393+0.533j -0.578+0.306j  0.233+0.507j -0.608-0.423j -0.453-0.202j  0.149-0.169j
0.06 +0.3j    0.108+0.26j  -0.008-0.349j]
 [-0.109+0.214j -0.033+0.202j -0.   +0.001j -0.319+0.009j  0.312-0.08j  -0.155+0.406j  0.363+0.252j
0.286+0.509j  0.048-0.452j  0.34 +0.514j]
 [-0.015+0.056j  0.01 +0.011j  0.216+0.028j -0.009+0.126j -0.05 +0.111j  0.409+0.339j -0.104-0.365j
0.103-0.352j  0.285+0.1j   -0.457-0.11j ]
 [-0.012+0.018j -0.002-0.014j  0.17 -0.026j  0.161-0.1j   -0.004+0.029j -0.009+0.04j   0.213+0.24j  -
0.116+0.045j -0.081-0.173j  0.13 +0.078j]
 [-0.007+0.013j -0.002-0.014j  0.124-0.024j  0.175+0.158j -0.043+0.019j  0.044-0.036j  0.112-0.135j
0.005+0.069j -0.364-0.154j  0.08 -0.194j]
 [-0.007+0.006j  0.002-0.011j  0.042+0.036j -0.186-0.234j  0.073+0.005j  0.11 -0.113j -0.215-0.284j
0.059-0.026j -0.074+0.418j -0.031-0.107j]
 [-0.002+0.003j -0.   -0.006j -0.038-0.031j  0.077-0.056j -0.002+0.027j  0.033-0.045j -0.334-0.101j
0.008-0.124j -0.   -0.266j  0.059+0.076j]
 [ 0.   +0.002j -0.002-0.002j -0.049-0.003j  0.048+0.067j -0.026-0.001j -0.01 +0.034j  0.022+0.056j -
0.031-0.005j -0.129+0.204j  0.252-0.002j]
 [-0.   +0.j    -0.   -0.001j -0.02 -0.013j -0.055+0.004j  0.013-0.017j -0.053+0.038j  0.267+0.087j
0.018+0.096j  0.129-0.106j -0.303+0.006j]]
```

```
np.linalg.eig =
 [[ 0.718+0.j     0.721+0.j     0.686+0.j     0.609+0.j    -0.422-0.38j  -0.096-0.484j -0.266+0.273j
0.616+0.j    -0.18 -0.246j -0.173-0.06j ]
 [ 0.61 -0.222j  0.611-0.255j  0.53 -0.384j -0.404+0.385j  0.741+0.j    -0.478+0.134j  0.026-0.224j -
0.294+0.085j -0.247+0.134j -0.295-0.186j]
 [ 0.221+0.094j  0.188+0.08j  -0.   -0.001j -0.112-0.299j -0.211+0.244j  0.14 +0.412j  0.442+0.j    -
0.484+0.328j  0.455+0.j     0.616+0.j   ]
 [ 0.057+0.011j  0.013-0.007j -0.218+0.002j -0.122+0.032j -0.022-0.12j   0.531+0.j    -0.293-0.241j
0.36 +0.074j -0.07 +0.294j -0.344+0.32j ]
 [ 0.019+0.01j  -0.014-0.001j -0.165+0.049j  0.147+0.12j  -0.013-0.026j  0.019+0.037j  0.311+0.076j -
0.054-0.112j  0.164-0.098j  0.137-0.066j]
 [ 0.014+0.006j -0.014-0.001j -0.12 +0.041j -0.092+0.217j  0.025-0.041j  0.011-0.056j  0.015-0.174j -
0.068+0.01j   0.115-0.378j -0.118-0.174j]
 [ 0.006+0.007j -0.01 -0.005j -0.047-0.03j   0.161-0.252j -0.063+0.037j  0.012-0.158j -0.339-0.11j
0.03 +0.057j -0.423-0.03j  -0.107-0.033j]
 [ 0.003+0.002j -0.006-0.001j  0.041+0.026j  0.078+0.055j -0.014-0.023j -0.003-0.055j -0.332+0.107j
0.124-0.002j  0.265-0.028j  0.096-0.008j]
 [ 0.001-0.j    -0.002+0.001j  0.048-0.004j -0.048+0.067j  0.022-0.015j  0.014+0.032j  0.05 +0.033j
0.002-0.032j -0.216-0.107j  0.137-0.211j]
 [ 0.   -0.j    -0.001+0.j     0.022+0.01j  -0.022-0.05j  -0.001+0.022j -0.017+0.062j  0.269-0.08j  -
0.095+0.026j  0.119+0.117j -0.162+0.256j]]
my eigenvectors / np.linalg.eig =
 [[-0.072+0.997j  0.237+0.971j -0.99 -0.14j   0.324-0.946j -0.821-0.571j  0.77 +0.639j  0.821+0.57j
0.083-0.997j  0.105-0.995j  0.552+0.834j]
 [-0.072+0.997j  0.237+0.971j -0.99 -0.14j   0.324-0.946j -0.821-0.571j  0.77 +0.639j  0.821+0.57j
0.083-0.997j  0.105-0.995j  0.552+0.834j]
 [-0.072+0.997j  0.237+0.971j -0.99 -0.14j   0.324-0.946j -0.821-0.571j  0.77 +0.639j  0.821+0.57j
0.083-0.997j  0.105-0.995j  0.552+0.834j]
 [-0.072+0.997j  0.237+0.971j -0.99 -0.14j   0.324-0.946j -0.821-0.571j  0.77 +0.639j  0.821+0.57j
0.083-0.997j  0.105-0.995j  0.552+0.834j]
 [-0.072+0.997j  0.237+0.971j -0.99 -0.14j   0.324-0.946j -0.821-0.571j  0.77 +0.639j  0.821+0.57j
0.083-0.997j  0.105-0.995j  0.552+0.834j]
 [-0.072+0.997j  0.237+0.971j -0.99 -0.14j   0.324-0.946j -0.821-0.571j  0.77 +0.639j  0.821+0.57j
0.083-0.997j  0.105-0.995j  0.552+0.834j]
 [-0.072+0.997j  0.237+0.971j -0.99 -0.14j   0.324-0.946j -0.821-0.571j  0.77 +0.639j  0.821+0.57j
0.083-0.997j  0.105-0.995j  0.552+0.834j]
 [-0.072+0.997j  0.237+0.971j -0.99 -0.14j   0.324-0.946j -0.821-0.571j  0.77 +0.639j  0.821+0.57j
0.083-0.997j  0.105-0.995j  0.552+0.834j]
 [-0.072+0.997j  0.237+0.971j -0.99 -0.14j   0.324-0.946j -0.821-0.571j  0.77 +0.639j  0.821+0.57j
0.083-0.997j  0.105-0.995j  0.552+0.834j]
 [-0.072+0.997j  0.237+0.971j -0.99 -0.14j   0.324-0.946j -0.821-0.571j  0.77 +0.639j  0.821+0.57j
0.083-0.997j  0.105-0.995j  0.552+0.834j]]

Eigenvectors all close except sign and/or scaling? True
```

Observations:

- Seems to be working except for the signs of eigenvectors

In [83]:

```
# Modified function to take Hessenberg matrix as an input and added optionality to output Q matrix for
eigenvector calculation

def eigen_solver(H, S, max_iter, tol, eigvec, shift, tridiagonal):

    """     Compute the eigenvalues and eigenvectors of a Hessenberg matrix
    Input   ---------------------------------------------
            H: 2D np.array in the form of a Hessenberg matrix with real or complex entries
            S: similarity transformation associated with the Hessenberg reduction, H = S @ A @ S.T whe
re A is the input matrix
            max_iter: int, maximum number of iterations to be performed
            tol: float, relative tolerance between eigenvalues by successive iterations
            eigenvec: bool, indicating whether eigenvectors are to be calculated
            shift: default None; option 'Wilkinson'
            tridiagonal: bool, indicating whether the input H is a triadiagonal matrix (i.e. original
 input A is symmetric)
    Return  ---------------------------------------------
            eigvals_new: 1D np.array, eigenvalues
            eigenvectors: 2D np.array with each column represents an eigenvector, in the same order as
the associated eigenvalues;
                          if eigvec is False, then this default to a zero matrix
    """

    n = H.shape[0]

    H_old = H.copy() # H will NOT be updated inplace
    eigvals_old = diagonal_block(H)

    if eigvec:
        T = np.eye(n,dtype=eigvals_old.dtype)

    if not shift:
        print('>>> QR Iteration WITHOUT shift')
    else:
        print('>>> QR Iteration with {} shift'.format(shift))

    for i in range(max_iter):

        if shift == 'Wilkinson':
            datatype, lam = Wilkinson_shift(H_old)
            #print(">>> QR Iteration with Wilkinson's Shift: lambda = {}, {}".format(lam, datatype))

        if shift:
            # H_old = H_old - lam * np.eye(n,dtype=datatype) #naive implementation O(2n^2)
            for j in range(n):
                H_old[j,j] -= lam #elementwise only O(n)

        H_new, G_iter = QR_Givens(H_old,eigvec,True, tridiagonal) # updating H_old in-place # default
 QR_Givens(H,eigvec=False, inplace=True, tridiagonal=False)

        if shift:
            # H_new = H_new +  lam * np.eye(n,dtype=datatype) #naive implementation O(2n^2)
            for j in range(n):
                H_new[j,j] += lam #elementwise only O(n)

        eigvals_new = diagonal_block(H_new)
        H_old = H_new # necessary to reset H_old = H_new


        # if eigenvectors are required
        if eigvec:
            T = G_iter @ T   # Q = I.Q1.Q2.Q3...Qk; Q1 = G1*; T*= Q = G1*...Gk* ==> T = Gk...G1.I

        # Test for convergence: relative tol = 1- w / w' for each eigenvalue
        if np.allclose(eigvals_new, eigvals_old, rtol=tol):
            print('Iteration terminates at i={} with tol={} reached'.format(i,tol))
            break

        if i == max_iter-1:
            err = np.max(abs((eigvals_new - eigvals_old)) / abs(eigvals_old))
            idx = np.argwhere(np.isclose(eigvals_new, eigvals_old, rtol=tol)==False).tolist()
```

```python
            print('max_iter={} reached, max error={} vs tol={}; index of non-convergence={}'.format(ma
x_iter, err, tol, idx))

        eigvals_old = eigvals_new

    # sort eigenvalues in descending order
    eigvals_new = sort_(eigvals_new)

    # step 3: Compute eigenvectors if required
    eigenvectors = np.zeros((n,n),dtype=eigvals_new.dtype)

    if eigvec:
        #print(H_new.dtype, 'H = \n',H_new)
        if eigvals_new.dtype == 'complex':
            #print('complex eigenvalues -',eigvals_new)
            eig_U = eigenvectors_inversePowerMethod(H_new,eigvals_new)
        else:
            #print('real eigenvalues - ',eigvals_new)
            eig_U = eigenvectors_upperTriangularMatrix(H_new,eigvals_new)

        for i in range(n):
            eigenvectors[:,i] = S.conjugate().T @ T.conjugate().T @ eig_U[:,i]

    return eigvals_new, eigenvectors

def sort_eigen(w, v):

    """     Sort eigenvalues and eigenvectors
    Input   ----------------------------------------------
            w: 1D np.array containing eigenvalues
            v: 2D np.array containing eigenvectors in each column
    Return  ---------------------------------------------
            w_sorted: 1D np.array, eigenvalues sorted in descending order
            v_sorted: 2D np.array, eigenvectors sorted in the same order as its correspondong eigenval
ue in w_sorted
    """

    w_sorted = np.flip(np.sort(w)) #same as sort_(w) # sort eigenvectors in descending order
    w_sorted_ix = np.flip(np.argsort(w))
    # w_sorted_ix = [np.argwhere(w==x).item() for x in w_sorted] # this does not deal with repeating e
igenvalues ...

    v_sorted = np.empty_like(v)

    for i,j in enumerate(w_sorted_ix):
        v_sorted[:,i] = v[:,j]

    return w_sorted, v_sorted
```

In [29]:

```python
print('\n\n')
print("====================================================================================")
print('Testing eigen_solver() on Random REAL Matrices')
print("====================================================================================")

for A in RANDOM_REAL_MATRICES[-1:]:
    print('\nREAL >>> Input shape = {}\n'.format(A.shape))

    w,v = np.linalg.eig(A)
    w_sorted, v_sorted = sort_eigen(w, v)

    H, S = Hessenberg(A,eigvec=True)

    eigenvalues, eigenvectors = eigen_solver(H, S, max_iter=MAX_ITER, tol=RTOL, eigvec=True, shift=None, tridiagonal=False)
    test_eigenvalues(eigenvalues,w_sorted)
    test_eigenvectors(eigenvectors,v_sorted)


    eigenvalues, eigenvectors = eigen_solver(H, S, max_iter=MAX_ITER, tol=RTOL, eigvec=True, shift='Wilkinson',tridiagonal=False)
    test_eigenvalues(eigenvalues,w_sorted)
    test_eigenvectors(eigenvectors,v_sorted)
```

```
=====================================================================================
Testing eigen_solver() on Random REAL Matrices
=====================================================================================

REAL >>> Input shape = (10, 10)

>>> QR Iteration without shift
Iteration terminates at i=551 with tol=1e-06 reached
... num of iterations with inverse power method = 1
... num of iterations with inverse power method = 1

my eigenvalues: [ 4.821-0.j     0.958-0.j     0.532+0.j     0.358+0.342j  0.358-0.342j -0.011+0.2j
-0.011-0.2j   -0.056-0.j    -0.128-0.597j -0.128+0.597j]
np.linalg.eig:  [ 4.821+0.j     0.958+0.j     0.532+0.j     0.358+0.342j  0.358-0.342j -0.011+0.2j
-0.011-0.2j   -0.056+0.j    -0.128+0.597j -0.128-0.597j]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [-0.-0.j     0.-0.j     0.+0.j    -0.-0.j    -0.+0.j     0.-0.j     0.+0.j    -0.-0.j     0.-1.195j
0.+1.195j]

max abs diff =  1.1946621537224646



np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 0.295+0.j     0.127-0.j    -0.057+0.j     0.042-0.131j  0.102-0.091j  0.183-0.203j  0.196+0.19j
-0.053+0.j     0.104+0.324j  0.34 +0.012j]
 [ 0.328+0.j     0.163-0.j     0.336-0.j     0.11 -0.485j  0.399-0.297j  0.383-0.021j  0.384-0.005j -
0.043+0.j     0.343-0.081j  0.015+0.352j]
 [ 0.246+0.j     0.258-0.j    -0.169+0.j     0.016+0.206j -0.195+0.069j -0.115+0.328j -0.137-0.319j -
0.287+0.j     0.029-0.153j -0.139+0.069j]
 [ 0.361+0.j    -0.327+0.j    -0.497+0.j    -0.352+0.429j -0.249+0.495j -0.509-0.129j -0.499+0.164j
0.11 -0.j    -0.022+0.177j  0.164-0.069j]
 [ 0.405+0.j    -0.418+0.j    -0.041+0.j     0.008+0.014j -0.016-0.001j  0.166+0.301j  0.145-0.311j
0.36 -0.j     0.303-0.148j -0.06 +0.332j]
 [ 0.3  +0.j     0.206-0.j     0.576-0.j     0.301-0.129j -0.004-0.328j  0.187+0.008j  0.186-0.021j
0.29 -0.j     0.186-0.067j -0.014+0.197j]
 [ 0.314+0.j    -0.416+0.j    -0.26 +0.j     0.234+0.214j -0.29 -0.127j  0.065-0.02j   0.067+0.015j -
0.775+0.j    -0.257+0.381j  0.297-0.351j]
 [ 0.278+0.j    -0.394+0.j    -0.318+0.j    -0.055+0.222j -0.181+0.14j  -0.273-0.154j -0.262+0.172j
0.043-0.j    -0.451-0.255j -0.368-0.365j]
 [ 0.345+0.j    -0.028+0.j     0.068-0.j    -0.   -0.048j  0.044-0.019j -0.167+0.132j -0.175-0.12j
0.017-0.j    -0.256-0.005j -0.074-0.245j]
 [ 0.256+0.j     0.487-0.j     0.318-0.j    -0.26 -0.23j   0.316+0.145j  0.112-0.261j  0.129+0.253j
0.292-0.j     0.036-0.066j -0.054+0.052j]]
np.linalg.eig =
 [[-0.295+0.j     0.127+0.j    -0.057+0.j     0.128-0.05j   0.128+0.05j  -0.127+0.242j -0.127-0.242j
0.053+0.j    -0.25 +0.231j -0.25 -0.231j]
 [-0.328+0.j     0.163+0.j     0.336+0.j     0.445-0.223j  0.445+0.223j -0.366+0.115j -0.366-0.115j
0.043+0.j    -0.259-0.24j  -0.259+0.24j ]
 [-0.246+0.j     0.258+0.j    -0.169+0.j    -0.149+0.143j -0.149-0.143j  0.031-0.346j  0.031+0.346j
0.287+0.j     0.05 -0.147j  0.05 +0.147j]
 [-0.361+0.j    -0.327+0.j    -0.497+0.j    -0.555+0.j    -0.555-0.j     0.525+0.j     0.525-0.j    -
0.11 +0.j    -0.068+0.165j -0.068-0.165j]
 [-0.405+0.j    -0.418+0.j    -0.041+0.j    -0.006+0.015j -0.006-0.015j -0.235-0.251j -0.235+0.251j -
0.36 +0.j    -0.191-0.278j -0.191+0.278j]
 [-0.3  +0.j     0.206+0.j     0.576+0.j     0.291+0.151j  0.291-0.151j -0.184+0.038j -0.184-0.038j -
0.29 +0.j    -0.129-0.149j -0.129+0.149j]
 [-0.314+0.j    -0.416+0.j    -0.26 +0.j    -0.017+0.317j -0.017-0.317j -0.058+0.035j -0.058-0.035j
0.775+0.j     0.036+0.458j  0.036-0.458j]
 [-0.278+0.j    -0.394+0.j    -0.318+0.j    -0.207+0.098j -0.207-0.098j  0.302+0.082j  0.302-0.082j -
0.043+0.j     0.519+0.j     0.519-0.j   ]
 [-0.345+0.j    -0.028+0.j     0.068+0.j     0.037-0.031j  0.037+0.031j  0.129-0.169j  0.129+0.169j -
0.017+0.j     0.225+0.122j  0.225-0.122j]
 [-0.256+0.j     0.487+0.j     0.318+0.j     0.013-0.347j  0.013+0.347j -0.044+0.281j -0.044-0.281j -
```

```
0.292+0.j     0.001-0.075j  0.001+0.075j]]
my eigenvectors / np.linalg.eig =
 [[-1.   -0.j     1.   -0.j     1.   -0.j     0.634-0.774j  0.45 -0.893j -0.969-0.246j -0.95 +0.312j
-1.   +0.j     0.422-0.907j -0.759+0.651j]
 [-1.   -0.j     1.   -0.j     1.   -0.j     0.634-0.774j  0.45 -0.893j -0.969-0.246j -0.95 +0.312j -
1.   +0.j    -0.557+0.831j  0.649-0.761j]
 [-1.   -0.j     1.   -0.j     1.   -0.j     0.634-0.774j  0.45 -0.893j -0.969-0.246j -0.95 +0.312j -
1.   +0.j     0.99 -0.141j  0.132+0.991j]
 [-1.   -0.j     1.   -0.j     1.   -0.j     0.634-0.774j  0.45 -0.893j -0.969-0.246j -0.95 +0.312j -
1.   +0.j     0.964-0.267j  0.004+1.j   ]
 [-1.   -0.j     1.   -0.j     1.   -0.j     0.634-0.774j  0.45 -0.893j -0.969-0.246j -0.95 +0.312j -
1.   +0.j    -0.15 +0.989j  0.911-0.412j]
 [-1.   -0.j     1.   -0.j     1.   -0.j     0.634-0.774j  0.45 -0.893j -0.969-0.246j -0.95 +0.312j -
1.   +0.j    -0.359+0.933j  0.802-0.598j]
 [-1.   -0.j     1.   -0.j     1.   -0.j     0.634-0.774j  0.45 -0.893j -0.969-0.246j -0.95 +0.312j -
1.   +0.j     0.782+0.623j  0.812+0.584j]
 [-1.   -0.j     1.   -0.j     1.   -0.j     0.634-0.774j  0.45 -0.893j -0.969-0.246j -0.95 +0.312j -
1.   +0.j    -0.87 -0.493j -0.709-0.705j]
 [-1.   -0.j     1.   -0.j     1.   -0.j     0.634-0.774j  0.45 -0.893j -0.969-0.246j -0.95 +0.312j -
1.   +0.j    -0.888+0.459j  0.202-0.979j]
 [-1.   -0.j     1.   -0.j     1.   -0.j     0.634-0.774j  0.45 -0.893j -0.969-0.246j -0.95 +0.312j -
1.   +0.j     0.886+0.463j  0.685+0.728j]]

Eigenvectors all close except sign and/or scaling? False



>>> QR Iteration with Wilkinson's Shift: lambda = (-0.05017623850927704+0.2124584982248223j), complex
Iteration terminates at i=344 with tol=1e-06 reached

my eigenvalues: [ 4.821-0.j     0.958+0.j     0.532-0.j     0.358+0.342j  0.358-0.342j -0.011+0.2j
-0.011-0.2j   -0.056-0.j    -0.128+0.597j -0.128-0.597j]
np.linalg.eig:  [ 4.821+0.j     0.958+0.j     0.532+0.j     0.358+0.342j  0.358-0.342j -0.011+0.2j
-0.011-0.2j   -0.056+0.j    -0.128+0.597j -0.128-0.597j]

np.linalg.eig and my eigenvalues close? True

np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 0.266-0.128j -0.062-0.111j  0.014+0.055j -0.053+0.126j -0.124+0.058j  0.273+0.005j -0.264+0.07j
0.018+0.05j   0.291+0.175j -0.329+0.087j]
 [ 0.295-0.142j -0.079-0.143j -0.084-0.325j -0.151+0.474j -0.467+0.17j   0.267+0.276j -0.193+0.332j
0.014+0.04j  -0.158+0.315j  0.063+0.347j]
 [ 0.222-0.107j -0.125-0.225j  0.042+0.164j  0.002-0.206j  0.206-0.01j  -0.323+0.127j  0.344+0.046j
0.095+0.27j  -0.155-0.007j  0.151+0.037j]
 [ 0.326-0.157j  0.159+0.286j  0.124+0.481j  0.387-0.397j  0.381-0.403j -0.235-0.469j  0.116-0.512j -
0.037-0.104j  0.177+0.019j -0.175-0.031j]
 [ 0.365-0.176j  0.203+0.365j  0.01 +0.039j -0.007-0.015j  0.015+0.006j -0.119+0.322j  0.193+0.284j -
0.12 -0.34j  -0.213+0.261j  0.132+0.311j]
 [ 0.27 -0.13j  -0.1  -0.18j  -0.144-0.558j -0.311+0.103j -0.09 +0.315j  0.116+0.147j -0.078+0.171j -
0.096-0.274j -0.108+0.165j  0.057+0.189j]
 [ 0.283-0.136j  0.202+0.364j  0.065+0.252j -0.215-0.233j  0.242+0.205j  0.058+0.036j -0.047+0.049j
0.257+0.731j  0.43 -0.163j -0.367-0.277j]
 [ 0.251-0.121j  0.192+0.345j  0.08 +0.308j  0.074-0.217j  0.213-0.083j -0.062-0.307j -0.013-0.313j -
0.014-0.041j -0.145-0.498j  0.278-0.437j]
 [ 0.311-0.15j   0.014+0.025j -0.017-0.065j -0.004+0.048j -0.047+0.006j -0.209-0.04j   0.193-0.089j -
0.006-0.016j  0.054-0.25j   0.018-0.256j]
 [ 0.231-0.111j -0.237-0.425j -0.08 -0.308j  0.24 +0.252j -0.261-0.229j  0.27 -0.086j -0.283-0.019j -
0.097-0.275j -0.072+0.02j   0.064+0.039j]]
np.linalg.eig =
 [[-0.295+0.j     0.127+0.j    -0.057+0.j     0.128-0.05j   0.128+0.05j  -0.127+0.242j -0.127-0.242j
0.053+0.j    -0.25 +0.231j -0.25 -0.231j]
 [-0.328+0.j     0.163+0.j     0.336+0.j     0.445-0.223j  0.445+0.223j -0.366+0.115j -0.366-0.115j
0.043+0.j    -0.259-0.24j  -0.259+0.24j ]
 [-0.246+0.j     0.258+0.j    -0.169+0.j    -0.149+0.143j -0.149-0.143j  0.031-0.346j  0.031+0.346j
0.287+0.j     0.05 -0.147j  0.05 +0.147j]
 [-0.361+0.j    -0.327+0.j    -0.497+0.j    -0.555+0.j    -0.555-0.j     0.525+0.j     0.525-0.j    -
0.11 +0.j    -0.068+0.165j -0.068-0.165j]
 [-0.405+0.j    -0.418+0.j    -0.041+0.j    -0.006+0.015j -0.006-0.015j -0.235-0.251j -0.235+0.251j -
0.36 +0.j    -0.191-0.278j -0.191+0.278j]
 [-0.3  +0.j     0.206+0.j     0.576+0.j     0.291+0.151j  0.291-0.151j -0.184+0.038j -0.184-0.038j -
```

```
0.29 +0.j    -0.129-0.149j -0.129+0.149j]
 [-0.314+0.j    -0.416+0.j    -0.26 +0.j    -0.017+0.317j -0.017-0.317j -0.058+0.035j -0.058-0.035j
0.775+0.j     0.036+0.458j  0.036-0.458j]
 [-0.278+0.j    -0.394+0.j    -0.318+0.j    -0.207+0.098j -0.207-0.098j  0.302+0.082j  0.302-0.082j -
0.043+0.j     0.519+0.j     0.519-0.j   ]
 [-0.345+0.j    -0.028+0.j     0.068+0.j     0.037-0.031j  0.037+0.031j  0.129-0.169j  0.129+0.169j -
0.017+0.j     0.225+0.122j  0.225-0.122j]
 [-0.256+0.j     0.487+0.j     0.318+0.j     0.013-0.347j  0.013+0.347j -0.044+0.281j -0.044-0.281j -
0.292+0.j     0.001-0.075j  0.001+0.075j]]
my eigenvectors / np.linalg.eig =
 [[-0.901+0.434j -0.486-0.874j -0.25 -0.968j -0.698+0.716j -0.687+0.727j -0.448-0.894j  0.221-0.975j
0.332+0.943j -0.279-0.96j   0.537-0.844j]
 [-0.901+0.434j -0.486-0.874j -0.25 -0.968j -0.698+0.716j -0.687+0.727j -0.448-0.894j  0.221-0.975j
0.332+0.943j -0.279-0.96j   0.537-0.844j]
 [-0.901+0.434j -0.486-0.874j -0.25 -0.968j -0.698+0.716j -0.687+0.727j -0.448-0.894j  0.221-0.975j
0.332+0.943j -0.279-0.96j   0.537-0.844j]
 [-0.901+0.434j -0.486-0.874j -0.25 -0.968j -0.698+0.716j -0.687+0.727j -0.448-0.894j  0.221-0.975j
0.332+0.943j -0.279-0.96j   0.537-0.844j]
 [-0.901+0.434j -0.486-0.874j -0.25 -0.968j -0.698+0.716j -0.687+0.727j -0.448-0.894j  0.221-0.975j
0.332+0.943j -0.279-0.96j   0.537-0.844j]
 [-0.901+0.434j -0.486-0.874j -0.25 -0.968j -0.698+0.716j -0.687+0.727j -0.448-0.894j  0.221-0.975j
0.332+0.943j -0.279-0.96j   0.537-0.844j]
 [-0.901+0.434j -0.486-0.874j -0.25 -0.968j -0.698+0.716j -0.687+0.727j -0.448-0.894j  0.221-0.975j
0.332+0.943j -0.279-0.96j   0.537-0.844j]
 [-0.901+0.434j -0.486-0.874j -0.25 -0.968j -0.698+0.716j -0.687+0.727j -0.448-0.894j  0.221-0.975j
0.332+0.943j -0.279-0.96j   0.537-0.844j]
 [-0.901+0.434j -0.486-0.874j -0.25 -0.968j -0.698+0.716j -0.687+0.727j -0.448-0.894j  0.221-0.975j
0.332+0.943j -0.279-0.96j   0.537-0.844j]
 [-0.901+0.434j -0.486-0.874j -0.25 -0.968j -0.698+0.716j -0.687+0.727j -0.448-0.894j  0.221-0.975j
0.332+0.943j -0.279-0.96j   0.537-0.844j]]

Eigenvectors all close except sign and/or scaling? True
```

In [31]:
```python
print('\n\n')
print("====================================================================================")
print('Testing eigen_solver() on Random COMPLEX Matrices')
print("====================================================================================")

for A in RANDOM_COMPLEX_MATRICES[-1:]:
    print('\nCOMPLEX >>> Input shape = {}\n'.format(A.shape))

    w,v = np.linalg.eig(A)
    w_sorted, v_sorted = sort_eigen(w, v)

    H, S = Hessenberg(A,eigvec=True)

    eigenvalues, eigenvectors = eigen_solver(H, S, max_iter=MAX_ITER, tol=RTOL, eigvec=True, shift=None,tridiagonal=False)
    test_eigenvalues(eigenvalues,w_sorted)
    test_eigenvectors(eigenvectors,v_sorted)


    eigenvalues, eigenvectors = eigen_solver(H, S, max_iter=MAX_ITER, tol=RTOL, eigvec=True, shift='Wilkinson',tridiagonal=False)
    test_eigenvalues(eigenvalues,w_sorted)
    test_eigenvectors(eigenvectors,v_sorted)
```

```
================================================================================
Testing eigen_solver() on Random COMPLEX Matrices
================================================================================

COMPLEX >>> Input shape = (10, 10)

>>> QR Iteration without shift
Iteration terminates at i=400 with tol=1e-06 reached

my eigenvalues: [ 4.936+5.572j  1.213+0.249j  0.879-0.054j  0.797-0.64j  -0.049+0.528j -0.157+1.27j
-0.221-0.654j -0.39 -0.341j -0.676-0.138j -1.329+0.203j]
np.linalg.eig:  [ 4.936+5.572j  1.213+0.249j  0.879-0.054j  0.797-0.64j  -0.049+0.528j -0.157+1.27j
-0.221-0.654j -0.39 -0.341j -0.676-0.138j -1.329+0.203j]

np.linalg.eig and my eigenvalues close? True

np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 0.027-0.38j  -0.109-0.534j  0.458+0.007j  0.197-0.246j  0.192+0.006j  0.185+0.238j  0.113-0.108j
-0.268-0.094j -0.039+0.312j -0.415-0.08j ]
 [ 0.02 -0.351j  0.035+0.175j  0.126-0.302j  0.096+0.068j  0.101+0.165j  0.116-0.103j -0.175-0.472j
0.183+0.315j  0.375-0.008j  0.079-0.092j]
 [ 0.029-0.297j  0.037+0.034j -0.09 +0.262j  0.05 +0.397j  0.058+0.078j  0.357+0.429j -0.164+0.141j
0.052+0.018j -0.099-0.25j   0.453-0.236j]
 [ 0.047-0.312j -0.01 +0.178j  0.004-0.06j   0.238-0.034j -0.147+0.035j -0.275-0.126j -0.414+0.259j
0.396-0.14j  -0.031-0.354j  0.257+0.187j]
 [ 0.078-0.3j    0.53 +0.206j -0.028+0.341j -0.07 -0.228j  0.116-0.305j  0.12 -0.035j  0.002+0.236j -
0.111-0.241j -0.274+0.059j -0.097+0.188j]
 [ 0.018-0.295j  0.158+0.228j -0.173+0.442j -0.023+0.153j  0.296-0.131j -0.136-0.314j  0.301+0.131j -
0.359-0.346j -0.228+0.519j -0.141-0.175j]
 [ 0.051-0.305j -0.12 +0.064j -0.239-0.16j  -0.299-0.414j -0.288+0.409j -0.085-0.181j  0.268-0.125j -
0.172+0.194j  0.19 +0.23j  -0.22 -0.055j]
 [-0.006-0.29j  -0.218-0.284j  0.114-0.353j -0.063+0.004j  0.007-0.332j -0.526-0.011j -0.146+0.138j
0.199+0.217j -0.047-0.237j  0.092+0.062j]
 [-0.005-0.304j -0.065+0.011j -0.171-0.106j  0.307+0.348j -0.307+0.324j  0.081+0.122j  0.357+0.089j
0.045+0.137j -0.004-0.037j -0.109+0.331j]
 [ 0.019-0.294j -0.243-0.164j  0.018-0.017j -0.277-0.175j -0.146-0.319j  0.085+0.055j  0.024-0.091j -
0.118-0.315j -0.07 -0.093j  0.104-0.391j]]
np.linalg.eig =
 [[ 0.381+0.j    -0.294-0.458j -0.16 -0.429j  0.084+0.304j -0.105-0.16j   0.302+0.01j   0.062+0.144j
0.259-0.119j  0.302-0.089j -0.331-0.263j]
 [ 0.351-0.004j  0.096+0.15j  -0.327-0.007j -0.112+0.038j  0.077-0.178j -0.005-0.155j  0.503+0.j    -
0.351-0.099j -0.158-0.34j   0.112-0.045j]
 [ 0.298+0.008j  0.047+0.018j  0.277-0.011j -0.351-0.191j  0.03 -0.093j  0.558+0.j    -0.075-0.203j -
0.051+0.023j -0.189+0.191j  0.511+0.j   ]
 [ 0.315+0.025j  0.055+0.169j -0.058+0.018j -0.112+0.213j  0.114+0.1j   -0.273+0.131j -0.099-0.479j -
0.188+0.376j -0.312+0.17j   0.141+0.285j]
 [ 0.305+0.057j  0.569+0.j     0.327-0.098j  0.226+0.076j -0.316+0.081j  0.05 -0.115j -0.222-0.08j
0.247+0.096j  0.164+0.227j -0.173+0.122j]
 [ 0.296-0.003j  0.23 +0.156j  0.474+0.j    -0.111-0.109j -0.277-0.167j -0.328-0.096j -0.228+0.237j
0.498+0.j     0.567+0.j    -0.044-0.22j ]
 [ 0.308+0.03j  -0.089+0.103j -0.062+0.281j  0.511+0.j     0.5  +0.j    -0.193-0.051j  0.024+0.295j -
0.011-0.259j  0.134-0.266j -0.17 -0.15j ]
 [ 0.289-0.027j -0.306-0.186j -0.37 +0.022j  0.033-0.054j -0.275+0.186j -0.345+0.398j -0.078-0.185j -
0.294-0.018j -0.198+0.138j  0.053+0.098j]
 [ 0.303-0.026j -0.057+0.034j -0.037+0.198j -0.462+0.045j  0.441+0.065j  0.146+0.015j -0.208+0.304j -
0.128-0.068j -0.032+0.019j -0.25 +0.243j]
 [ 0.295-0.002j -0.285-0.065j -0.022-0.011j  0.304-0.122j -0.177+0.303j  0.097-0.03j   0.077+0.055j
0.304+0.145j -0.057+0.102j  0.273-0.299j]]
my eigenvectors / np.linalg.eig =
 [[ 0.071-0.998j  0.932+0.362j -0.364+0.932j -0.585-0.811j -0.576+0.818j  0.639+0.769j -0.348-0.937j
-0.72 -0.694j -0.402+0.916j  0.887-0.462j]
 [ 0.071-0.998j  0.932+0.362j -0.364+0.932j -0.585-0.811j -0.576+0.818j  0.639+0.769j -0.348-0.937j -
0.72 -0.694j -0.402+0.916j  0.887-0.462j]
 [ 0.071-0.998j  0.932+0.362j -0.364+0.932j -0.585-0.811j -0.576+0.818j  0.639+0.769j -0.348-0.937j -
0.72 -0.694j -0.402+0.916j  0.887-0.462j]
 [ 0.071-0.998j  0.932+0.362j -0.364+0.932j -0.585-0.811j -0.576+0.818j  0.639+0.769j -0.348-0.937j -
0.72 -0.694j -0.402+0.916j  0.887-0.462j]
```

```
 [ 0.071-0.998j  0.932+0.362j -0.364+0.932j -0.585-0.811j -0.576+0.818j  0.639+0.769j -0.348-0.937j -
0.72 -0.694j -0.402+0.916j  0.887-0.462j]
 [ 0.071-0.998j  0.932+0.362j -0.364+0.932j -0.585-0.811j -0.576+0.818j  0.639+0.769j -0.348-0.937j -
0.72 -0.694j -0.402+0.916j  0.887-0.462j]
 [ 0.071-0.998j  0.932+0.362j -0.364+0.932j -0.585-0.811j -0.576+0.818j  0.639+0.769j -0.348-0.937j -
0.72 -0.694j -0.402+0.916j  0.887-0.462j]
 [ 0.071-0.998j  0.932+0.362j -0.364+0.932j -0.585-0.811j -0.576+0.818j  0.639+0.769j -0.348-0.937j -
0.72 -0.694j -0.402+0.916j  0.887-0.462j]
 [ 0.071-0.998j  0.932+0.362j -0.364+0.932j -0.585-0.811j -0.576+0.818j  0.639+0.769j -0.348-0.937j -
0.72 -0.694j -0.402+0.916j  0.887-0.462j]
 [ 0.071-0.998j  0.932+0.362j -0.364+0.932j -0.585-0.811j -0.576+0.818j  0.639+0.769j -0.348-0.937j -
0.72 -0.694j -0.402+0.916j  0.887-0.462j]]

Eigenvectors all close except sign and/or scaling? True



>>> QR Iteration with Wilkinson's Shift: lambda = (-0.4089463713521236-0.2711453428077542j), float
Iteration terminates at i=83 with tol=1e-06 reached

my eigenvalues: [ 4.936+5.572j  1.213+0.249j  0.879-0.054j  0.797-0.64j  -0.049+0.528j -0.157+1.27j
-0.221-0.654j -0.39 -0.341j -0.676-0.138j -1.329+0.203j]
np.linalg.eig:  [ 4.936+5.572j  1.213+0.249j  0.879-0.054j  0.797-0.64j  -0.049+0.528j -0.157+1.27j
-0.221-0.654j -0.39 -0.341j -0.676-0.138j -1.329+0.203j]

np.linalg.eig and my eigenvalues close? True

np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[-0.365+0.111j -0.542+0.059j  0.433+0.151j  0.08 -0.305j  0.096+0.166j  0.216+0.211j  0.099+0.121j
0.012+0.284j  0.068-0.307j -0.119-0.406j]
 [-0.335+0.106j  0.177-0.019j  0.214-0.247j  0.116+0.023j -0.087+0.173j  0.101-0.118j  0.484-0.137j
0.225-0.287j -0.374-0.027j  0.118+0.027j]
 [-0.287+0.079j  0.037-0.034j -0.168+0.22j   0.208+0.342j -0.036+0.091j  0.412+0.377j -0.128-0.175j -
0.002-0.056j  0.075+0.258j  0.418+0.294j]
 [-0.308+0.067j  0.176+0.026j  0.023-0.056j  0.204-0.128j -0.107-0.107j -0.29 -0.088j -0.225-0.434j -
0.276-0.317j -0.002+0.355j -0.049+0.314j]
 [-0.308+0.034j  0.254-0.509j -0.133+0.315j -0.157-0.18j   0.32 -0.062j  0.115-0.051j -0.235-0.016j -
0.183+0.192j  0.279-0.033j -0.212+0.j   ]
 [-0.282+0.089j  0.242-0.136j -0.303+0.365j  0.041+0.15j   0.266+0.183j -0.177-0.293j -0.154+0.29j  -
0.189+0.461j  0.275-0.496j  0.091-0.206j]
 [-0.303+0.061j  0.052+0.126j -0.177-0.227j -0.442-0.257j -0.499-0.03j  -0.108-0.168j  0.104+0.277j
0.244+0.088j -0.168-0.246j -0.052-0.221j]
 [-0.269+0.109j -0.303+0.19j   0.219-0.299j -0.056+0.03j   0.286-0.169j -0.523+0.06j  -0.126-0.157j
0.128-0.265j  0.025+0.24j  -0.012+0.11j ]
 [-0.283+0.113j  0.005+0.066j -0.129-0.155j  0.422+0.193j -0.437-0.091j  0.097+0.11j  -0.117+0.349j
0.111-0.092j  0.001+0.037j -0.344+0.055j]
 [-0.282+0.087j -0.186+0.226j  0.023-0.01j  -0.324-0.047j  0.195-0.292j  0.092+0.043j  0.089+0.031j -
0.249+0.226j  0.061+0.099j  0.395-0.087j]]
np.linalg.eig =
 [[ 0.381+0.j    -0.294-0.458j -0.16 -0.429j  0.084+0.304j -0.105-0.16j   0.302+0.01j   0.062+0.144j
0.259-0.119j  0.302-0.089j -0.331-0.263j]
 [ 0.351-0.004j  0.096+0.15j  -0.327-0.007j -0.112+0.038j  0.077-0.178j -0.005-0.155j  0.503+0.j    -
0.351-0.099j -0.158-0.34j   0.112-0.045j]
 [ 0.298+0.008j  0.047+0.018j  0.277-0.011j -0.351-0.191j  0.03 -0.093j  0.558+0.j    -0.075-0.203j -
0.051+0.023j -0.189+0.191j  0.511+0.j   ]
 [ 0.315+0.025j  0.055+0.169j -0.058+0.018j -0.112+0.213j  0.114+0.1j   -0.273+0.131j -0.099-0.479j -
0.188+0.376j -0.312+0.17j   0.141+0.285j]
 [ 0.305+0.057j  0.569+0.j     0.327-0.098j  0.226+0.076j -0.316+0.081j  0.05 -0.115j -0.222-0.08j
0.247+0.096j  0.164+0.227j -0.173+0.122j]
 [ 0.296-0.003j  0.23 +0.156j  0.474+0.j    -0.111-0.109j -0.277-0.167j -0.328-0.096j -0.228+0.237j
0.498+0.j     0.567+0.j    -0.044-0.22j ]
 [ 0.308+0.03j  -0.089+0.103j -0.062+0.281j  0.511+0.j     0.5  +0.j    -0.193-0.051j  0.024+0.295j -
0.011-0.259j  0.134-0.266j -0.17 -0.15j ]
 [ 0.289-0.027j -0.306-0.186j -0.37 +0.022j  0.033-0.054j -0.275+0.186j -0.345+0.398j -0.078-0.185j -
0.294-0.018j -0.198+0.138j  0.053+0.098j]
 [ 0.303-0.026j -0.057+0.034j -0.037+0.198j -0.462+0.045j  0.441+0.065j  0.146+0.015j -0.208+0.304j -
0.128-0.068j -0.032+0.019j -0.25 +0.243j]
 [ 0.295-0.002j -0.285-0.065j -0.022-0.011j  0.304-0.122j -0.177+0.303j  0.097-0.03j   0.077+0.055j
0.304+0.145j -0.057+0.102j  0.273-0.299j]]
my eigenvectors / np.linalg.eig =
```

```
 [[-0.957+0.29j   0.447-0.895j -0.638+0.77j  -0.865-0.502j -0.998-0.061j  0.737+0.676j  0.962-0.273j
-0.38 +0.925j  0.485-0.874j  0.818+0.576j]
 [-0.957+0.29j   0.447-0.895j -0.638+0.77j  -0.865-0.502j -0.998-0.061j  0.737+0.676j  0.962-0.273j -
0.38 +0.925j  0.485-0.874j  0.818+0.576j]
 [-0.957+0.29j   0.447-0.895j -0.638+0.77j  -0.865-0.502j -0.998-0.061j  0.737+0.676j  0.962-0.273j -
0.38 +0.925j  0.485-0.874j  0.818+0.576j]
 [-0.957+0.29j   0.447-0.895j -0.638+0.77j  -0.865-0.502j -0.998-0.061j  0.737+0.676j  0.962-0.273j -
0.38 +0.925j  0.485-0.874j  0.818+0.576j]
 [-0.957+0.29j   0.447-0.895j -0.638+0.77j  -0.865-0.502j -0.998-0.061j  0.737+0.676j  0.962-0.273j -
0.38 +0.925j  0.485-0.874j  0.818+0.576j]
 [-0.957+0.29j   0.447-0.895j -0.638+0.77j  -0.865-0.502j -0.998-0.061j  0.737+0.676j  0.962-0.273j -
0.38 +0.925j  0.485-0.874j  0.818+0.576j]
 [-0.957+0.29j   0.447-0.895j -0.638+0.77j  -0.865-0.502j -0.998-0.061j  0.737+0.676j  0.962-0.273j -
0.38 +0.925j  0.485-0.874j  0.818+0.576j]
 [-0.957+0.29j   0.447-0.895j -0.638+0.77j  -0.865-0.502j -0.998-0.061j  0.737+0.676j  0.962-0.273j -
0.38 +0.925j  0.485-0.874j  0.818+0.576j]
 [-0.957+0.29j   0.447-0.895j -0.638+0.77j  -0.865-0.502j -0.998-0.061j  0.737+0.676j  0.962-0.273j -
0.38 +0.925j  0.485-0.874j  0.818+0.576j]
 [-0.957+0.29j   0.447-0.895j -0.638+0.77j  -0.865-0.502j -0.998-0.061j  0.737+0.676j  0.962-0.273j -
0.38 +0.925j  0.485-0.874j  0.818+0.576j]]

Eigenvectors all close except sign and/or scaling? True
```

In [32]:
```python
print('\n\n')
print("====================================================================================")
print('Testing eigen_solver() on Real SPECIAL Matrices')
print("====================================================================================")

for A in SPECIAL_REAL_MATRICES[-1:]:
    print('\nSPECIAL >>> Input shape = {}\n'.format(A.shape))

    w,v = np.linalg.eig(A)
    w_sorted, v_sorted = sort_eigen(w, v)

    H, S = Hessenberg(A,eigvec=True)

    eigenvalues, eigenvectors = eigen_solver(H, S, max_iter=MAX_ITER, tol=RTOL, eigvec=True, shift=None,tridiagonal=False)
    test_eigenvalues(eigenvalues,w_sorted)
    test_eigenvectors(eigenvectors,v_sorted)


    eigenvalues, eigenvectors = eigen_solver(H, S, max_iter=MAX_ITER, tol=RTOL, eigvec=True, shift='Wilkinson',tridiagonal=False)
    test_eigenvalues(eigenvalues,w_sorted)
    test_eigenvectors(eigenvectors,v_sorted)
```

```
=====================================================================================
Testing eigen_solver() on Real SPECIAL Matrices
=====================================================================================

SPECIAL >>> Input shape = (6, 6)

>>> QR Iteration without shift
Iteration terminates at i=52 with tol=1e-06 reached
... num of iterations with inverse power method = 1

my eigenvalues: [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]
np.linalg.eig:  [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]

np.linalg.eig and my eigenvalues close? True

np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 0.443-0.232j  0.443+0.232j  0.061+0.j     0.247+0.j    -0.077+0.151j -0.077-0.151j]
 [-0.232-0.443j -0.232+0.443j  0.539+0.j    -0.124+0.j     0.538-0.057j  0.538+0.057j]
 [-0.   +0.j    -0.   -0.j    -0.453+0.j     0.823+0.j     0.509-0.223j  0.509+0.223j]
 [-0.   +0.j    -0.   -0.j    -0.453+0.j     0.412+0.j    -0.144-0.114j -0.144+0.114j]
 [-0.232-0.443j -0.232+0.443j  0.539+0.j    -0.124+0.j     0.544+0.107j  0.544-0.107j]
 [ 0.443-0.232j  0.443+0.232j  0.061+0.j     0.247+0.j     0.088+0.144j  0.088-0.144j]]
np.linalg.eig =
 [[ 0.   +0.5j    0.   -0.5j    0.061+0.j    -0.247+0.j    -0.131+0.107j -0.131-0.107j]
 [ 0.5  +0.j     0.5  -0.j     0.539+0.j     0.124+0.j     0.515+0.163j  0.515-0.163j]
 [ 0.   +0.j     0.   -0.j    -0.453+0.j    -0.823+0.j     0.556+0.j     0.556-0.j   ]
 [ 0.   +0.j     0.   -0.j    -0.453+0.j    -0.412+0.j    -0.086-0.162j -0.086+0.162j]
 [ 0.5  +0.j     0.5  -0.j     0.539+0.j     0.124+0.j     0.456+0.316j  0.456-0.316j]
 [ 0.   +0.5j    0.   -0.5j    0.061+0.j    -0.247+0.j     0.022+0.167j  0.022-0.167j]]
my eigenvectors / np.linalg.eig =
 [[-0.464-0.886j -0.464+0.886j  1.   +0.j    -1.   -0.j     0.916-0.401j  0.916+0.401j]
 [-0.464-0.886j -0.464+0.886j  1.   +0.j    -1.   +0.j     0.916-0.401j  0.916+0.401j]
 [-0.41 +0.863j -0.41 -0.863j  1.   -0.j    -1.   -0.j     0.916-0.401j  0.916+0.401j]
 [ 1.768+1.398j  1.768-1.398j  1.   -0.j    -1.   -0.j     0.916-0.401j  0.916+0.401j]
 [-0.464-0.886j -0.464+0.886j  1.   +0.j    -1.   +0.j     0.916-0.401j  0.916+0.401j]
 [-0.464-0.886j -0.464+0.886j  1.   +0.j    -1.   -0.j     0.916-0.401j  0.916+0.401j]]

Eigenvectors all close except sign and/or scaling? False



>>> QR Iteration with Wilkinson's Shift: lambda = 2.3424691828527715, float
Iteration terminates at i=42 with tol=1e-06 reached


/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:33: ComplexWarning: Casting complex valu
es to real discards the imaginary part
```

```
my eigenvalues: [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]
np.linalg.eig:  [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]

np.linalg.eig and my eigenvalues close? True

np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 0.498-0.045j  0.498+0.045j  0.061+0.j     0.247+0.j    -0.168+0.014j -0.168-0.014j]
 [-0.045-0.498j -0.045+0.498j  0.539+0.j    -0.124+0.j     0.331+0.428j  0.331-0.428j]
 [-0.   -0.j    -0.   +0.j    -0.453+0.j     0.823+0.j     0.457+0.317j  0.457-0.317j]
 [-0.   +0.j    -0.   -0.j    -0.453+0.j     0.412+0.j     0.022-0.182j  0.022+0.182j]
 [-0.045-0.498j -0.045+0.498j  0.539+0.j    -0.124+0.j     0.194+0.52j   0.194-0.52j ]
 [ 0.498-0.045j  0.498+0.045j  0.061+0.j     0.247+0.j    -0.077+0.15j  -0.077-0.15j ]]
np.linalg.eig =
 [[ 0.   +0.5j    0.   -0.5j    0.061+0.j    -0.247+0.j    -0.131+0.107j -0.131-0.107j]
 [ 0.5  +0.j     0.5  -0.j     0.539+0.j     0.124+0.j     0.515+0.163j  0.515-0.163j]
 [ 0.   +0.j     0.   -0.j    -0.453+0.j    -0.823+0.j     0.556+0.j     0.556-0.j   ]
 [ 0.   +0.j     0.   -0.j    -0.453+0.j    -0.412+0.j    -0.086-0.162j -0.086+0.162j]
 [ 0.5  +0.j     0.5  -0.j     0.539+0.j     0.124+0.j     0.456+0.316j  0.456-0.316j]
 [ 0.   +0.5j    0.   -0.5j    0.061+0.j    -0.247+0.j     0.022+0.167j  0.022-0.167j]]
my eigenvectors / np.linalg.eig =
 [[-0.089-0.996j -0.089+0.996j  1.   +0.j    -1.   -0.j     0.822+0.57j   0.822-0.57j ]
 [-0.089-0.996j -0.089+0.996j  1.   +0.j    -1.   +0.j     0.822+0.57j   0.822-0.57j ]
 [-0.975+0.271j -0.975-0.271j  1.   -0.j    -1.   -0.j     0.822+0.57j   0.822-0.57j ]
 [-0.34 +0.59j  -0.34 -0.59j   1.   -0.j    -1.   -0.j     0.822+0.57j   0.822-0.57j ]
 [-0.089-0.996j -0.089+0.996j  1.   +0.j    -1.   +0.j     0.822+0.57j   0.822-0.57j ]
 [-0.089-0.996j -0.089+0.996j  1.   +0.j    -1.   -0.j     0.822+0.57j   0.822-0.57j ]]

Eigenvectors all close except sign and/or scaling? False
```

Observations

- Seems to be working for real and complex matrices
- Does not seem to be working for special real matrices #TODO

In [ ]:

In [ ]:

References for Francis Double-Shift QR Algorithm

- https://www.cs.cornell.edu/~bindel/class/cs6210-f12/notes/lec28.pdf (https://www.cs.cornell.edu/~bindel/class/cs6210-f12/notes/lec28.pdf)
- https://people.inf.ethz.ch/arbenz/ewp/Lnotes/chapter4.pdf (https://people.inf.ethz.ch/arbenz/ewp/Lnotes/chapter4.pdf)
- https://people.eecs.berkeley.edu/~wkahan/Math128/Parlett.pdf (https://people.eecs.berkeley.edu/~wkahan/Math128/Parlett.pdf)
- https://www.math.usm.edu/lambers/mat610/class0331.pdf (https://www.math.usm.edu/lambers/mat610/class0331.pdf)
- http://math.ntnu.edu.tw/~min/matrix_computation/QR_alg_ch2_1.pdf (http://math.ntnu.edu.tw/~min/matrix_computation/QR_alg_ch2_1.pdf)
- https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.215.3553&rep=rep1&type=pdf (https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.215.3553&rep=rep1&type=pdf)
- https://www.youtube.com/watch?v=RSm_Mqi0aSA (https://www.youtube.com/watch?v=RSm_Mqi0aSA)

Background

Implicit double shifts

Apply 1st shift s_1 on H_0:

- $H_0 - s_1 I = Q_1 R_1$ ......(1)
- $H_1 = R_1 Q_1 + s_1 I$ ......(2)

Apply 2nd shift s_2 on H_1:

- $H_1 - s_2 I = Q_2 R_2$ ......(3)
- $H_2 = R_2 Q_2 + s_2 I$ ......(4)

Now combining the two shifts (without explicitly forming H_1), from (2) and (3):

- $R_1 Q_1 + s_1 I = Q_2 R_2 + s_2 I$

Apply Q1 from left and R1 from right on both sides and expand terms:

- $Q_1(R_1 Q_1)R_1 + Q_1(s_1 I)R_1 = Q_1(Q_2 R_2)R_1 + Q_1(s_2 I)R_1$

Substitude $Q_1 R_1$ from (1) and re-arrange:

- $(Q_1 Q_2)(R_2 R_1) = (H_0 - s_1 I)^2 + (H_0 - s_1 I)(s_1 - s_2) = (H_0 - s_1 I)(H_0 - s_2 I)$

This is effectively the QR decomposition of matrix $M = (H_0 - s_1 I)(H_0 - s_2 I)$, i.e. performing two shifts on H_0 to obtain H_2:

- $H_2 = (Q_1 Q_2)^* H_0 (Q_1 Q_2)$

When s_1 and s_2 are a conjugate pair of a complex eigenvalue λ, matrix M is real, thus avoiding QR decomposition in complex arithmatics

- $M = H_0^2 - 2Re(\lambda)H + |\lambda|^2 I$

A few observations:

- Naive implementation involves forming M explicitly by matrix multiplication, which is computational expensive ~$O(n^3)$
- Note also the matrix M is *not* in Hessenberg form; it has two lower sub-diagonals due to the multiplication of two Hessenberg matrices.
- Implicit Q-Theream ...
- Need to restore M to Hessenberg by Givens rotation

Implicit Q thereom

- "The gist of the implicit Q theorem is that if Q.T AQ = H and Z.T AZ = G are both unreduced Hessenberg matrices and Q and Z have the same first column, the G and H are “essentially equal” in the sense that G = DHD with D = diag(±1, . . . , ±1)." -- Golub and van Loan [5, p.347]

Bulge chasing not implemented

- Instead of forming M in its entirety, we only form its first column, which being a second degree polynomial of a Hessenberg matrix, has only three nonzero entries.
- Compute a Householder reflection P0 that makes Me1 a multiple of e1.
- Then, we compute P0.T H P0, which is no longer Hessenberg, because P0 operates on the first three rows and columns of H.

- Finally, we apply a series of Householder reflections $P1, P2, \ldots, Pn-2$ that restore Hessenberg form
- Because the reflections $P1, P2, \ldots, Pn-2$ do not affect the first row (when applied on the left) or column (when applied on the right),
- it follows that if we define $\tilde{Z} = P0\ P1\ P2 \cdots Pn-2$, then Z and $\tilde{Z}$ have the same first column
- Since both matrices implement similarity transformations that preserve the Hessenberg form of H, it follows from the Implicit Q Theorem that
- Z and $\tilde{Z}$ are essentially equal, and that they essentially produce the same updated matrix H2
- This variation of a Hessenberg QR step is called a Francis QR Step.
- (Source: [https://www.math.usm.edu/lambers/mat610/class0331.pdf (https://www.math.usm.edu/lambers/mat610/class0331.pdf)](https://www.math.usm.edu/lambers/mat610/class0331.pdf))

In [ ]:
```
# Naive implementation of Francis Double-shift QR Algorithm

def construct_M_full(H,lam_re,lam_im):

    """     Naive implementation of the full matrix M = (H-s1.I)(H-s2.I) where s1 and s2 are complex c
onjugate pair

    Input   ---------------------------------------------
            H: 2D np.array in the form of a Hessenberg matrix with real entries
            lam_re: real part of a complex eigenvalue
            lam_im: imaginary part of a complex eigenvalue
    Return  --------------------------------------------
            M: 2D np.array with only real entries
    """

    I = np.eye(H.shape[0])

    M = H@H - 2*lam_re*H + (lam_re**2+lam_im**2)*I  #Note: H@H complexity = O(n^3) expensive!
    # RuntimeWarning: overflow encountered in matmul
    # RuntimeWarning: invalid value encountered in matmul
    #print(M)

    return M



# for eigenvalues only

def QR_Iteration_DoubleShift(A, max_iter, tol):

    """     Perform QR iteration with double shift to obtain eigenvalues
    Input   ---------------------------------------------
            A: 2D np.array with real or complex entries
            max_iter: int, maximum number of iterations to be performed
            tol: float, relative tolerance between eigenvalues by successive iterations
    Return  --------------------------------------------
            eigvals_new: 1D np.array with real or complex entries
    """

    n = A.shape[0]

    # step 1: tranform A to Hessenberg
    H, _ = Hessenberg(A)

    # step 2: QR iteration
    H_old = H.copy()
    eigvals_old = diagonal_block(H)

    for i in range(max_iter):

        # calculate Wilkinson shift
        datatype, lam = Wilkinson_shift(H_old)
        # print(">>> Wilkinson_shift = {},{}".format(lam,datatype))

        if datatype == 'complex':
            M = construct_M_full(H_old,lam.real,lam.imag)
            H_old, _ = Hessenberg(M) # Hessenberg(A,eigvec=False, inplace=True)
            H_new, _ = QR_Givens(H_old) # updating H_old in-place ## default QR_Givens(H,eigvec=False,
inplace=True, tridiagonal=False)

        else:
            H_old = H_old - lam * np.eye(n)
            H_new, _ = QR_Givens(H_old) +  lam * np.eye(n) # updating H_old in-place

        eigvals_new = diagonal_block(H_new)

        H_old = H_new

        # Test for convergence: relative tol = 1- w / w' for each eigenvalue
        if np.allclose(eigvals_new, eigvals_old, rtol=tol):
            print('Iteration terminates at i={} with tol={} reached'.format(i,tol))
```

```
            break

        if i == max_iter-1:
            err = np.max(abs((eigvals_new - eigvals_old)) / abs(eigvals_old))
            idx = np.argwhere(np.isclose(eigvals_new, eigvals_old, rtol=tol)==False).tolist()
            print('max_iter={} reached, max error={} vs tol={}; index of non-convergence={}'.format(ma
x_iter, err, tol, idx))

        eigvals_old = eigvals_new

    return eigvals_new
```

In [ ]:
```
print('\n\n')
print("======================================================================================")
print('Testing QR_Iteration_DoubleShift on Random REAL Matrices')
print("======================================================================================")

for A in RANDOM_REAL_MATRICES[1:-1]:
    print('\nREAL >>> Input shape = {}\n'.format(A.shape))

    test_eigenvalues_function(A,QR_Iteration_Eigenvalues)
    test_eigenvalues_function(A,QR_Iteration_WilkinsonShift)
    test_eigenvalues_function(A,QR_Iteration_DoubleShift)
```

Observations:

- Double-shift does not work **#TODO**
- Overflow issues with repeated matrix multiplication H@H in order to get M in the iteration
- Need implement element-wise Buldg-chasing (based on implicit Q)

In [44]:
```
##############
### IGNORE ###
##############

def eigen_solver_real(H, S, max_iter, tol, eigvec):

    """     Compute the eigenvalues and eigenvectors of a Hessenberg matrix with real entries
            with Wilkinson' shift (if real) or double shift (if complex)
    Input   ----------------------------------------------
            H: 2D np.array in the form of a Hessenberg matrix with real entries
            S: similarity transformation associated with the Hessenberg reduction, H = S @ A @ S.T whe
re A is the input real matrix
            max_iter: int, maximum number of iterations to be performed
            tol: float, relative tolerance between eigenvalues by successive iterations
            eigenvec: bool, indicating whether eigenvectors are to be calculated
    Return  ---------------------------------------------
            eigvals_new: 1D np.array, eigenvalues
            eigenvectors: 2D np.array with each column represents an eigenvector, in the same order as
the associated eigenvalues;
                          if eigvec is False, then this default to a zero matrix
    """

    datatype, lam = Wilkinson_shift(H_old)

    if datatype == 'complex':
        shift = 'Double-'
    else:
        shift = "Wilkinson's "
    print(">>> QR Iteration with Real Shift: lambda = {}, {} -> Applying {}Shift".format(lam, datatype
, shift))

    pass # TODO


def eigen_solver_realsymmetric(H, S, max_iter, tol, eigvec):
    pass # TODO
```

In [ ]:

In [ ]:

In [ ]:

In [48]:

```
# Visualization

def plot_eigenvalues_real(w_sorted,eigenvalues):
    plt.plot(w_sorted, color='red', marker='o', linestyle='dashed', label = 'np.linalg.eig()')
    plt.plot(eigenvalues, color='green', marker='x', linestyle='dotted', label='my eigenvalues')
    plt.title('Real Eigenvalues')
    plt.legend()
    plt.show()


def plot_eigenvalues_complex(w_sorted,eigenvalues):
    f, ax = plt.subplots(1,2,figsize=(10,5))
    ax[0].plot(w_sorted.real, color='red', marker='o', linestyle='dashed', label = 'np.linalg.eig()')
    ax[0].plot(eigenvalues.real, color='green', marker='x', linestyle='dotted', label='my eigenvalues'
)
    ax[0].set_title('Complex Eigenvalues - Real')
    ax[0].legend()


    x = np.arange(len(eigenvalues))  # the label locations
    width = 0.02  # the width of the bars
    ax[1].bar(x - width/2, w_sorted.imag, width, color='red',label = 'np.linalg.eig()')
    ax[1].bar(x + width/2, eigenvalues.imag, width,color='green', label='my eigenvalues')
    ax[1].set_title('Complex Eigenvalues - Imaginary')
    ax[1].legend()

    plt.show()
```

In [79]:
```python
# Object-Oriented Programming: implement algorithm 1.4.14 in a way, that H is either Hessenberg or tridiagonal, either real or complex, but encapsulate the differences inside the QR-decompositions applied in every step.

class Matrix:

    # General matrix class for both complex and real matrices

    def __init__(self, A, calc_eigvec=False):
        self.calc_eigvec = calc_eigvec
        self.A = A
        self.n = A.shape[0]
        self.dtype = A.dtype

        self.is_symmetric = False
        if np.allclose(A,A.T) or np.allclose(A, A.conjugate().T):
            self.is_symmetric = True

        self.is_hessenberg = False
        zeros = np.zeros([self.n,self.n],dtype=self.dtype)
        if np.allclose(np.tril(self.A, -2),zeros):
            self.is_hessenberg = True
            self.H = self.A
            self.S = None
        else:
            self.H,self.S = Hessenberg(self.A, self.calc_eigvec)

        self.is_tridiagonal = False
        tridiag = self.A - np.tril(self.A,-2) - np.triu(self.A,2)
        if self.is_hessenberg and np.allclose(np.triu(self.A,2),zeros) and not np.allclose(tridiag,zeros):
            self.is_tridiagonal = True

    def __eq__(self, other):
        return np.allclose(self.A,other.A)

    def __repr__(self):
        return self.A

    def __str__(self):
        return 'Matrix dtype={}, shape={}x{}, is_symmetric={}, is_hessenberg={}, is_tridiagonal={} \n{}'.format(self.dtype,self.n,self.n,self.is_symmetric,self.is_hessenberg,self.is_tridiagonal, np.array2string(self.A, precision=3, separator=', \t', suppress_small=True))


    # Wilkson's shift in complex arithmatics
    def eigen_solver(self, shift=None):
        # shift = None or 'Wilkinson'
        tridiagonal = self.is_symmetric
        # tridiagonal = False
        print('activiating tridigonal:', tridiagonal)

        if self.calc_eigvec:
            print('\n ------------ Processing both eigenvalues and eigenvectors ------------ ')
            self.eigenvalues, self.eigenvectors = eigen_solver(self.H, self.S, MAX_ITER, RTOL, self.calc_eigvec,shift,tridiagonal)
            #eigen_solver(H, S, max_iter, tol, eigvec, shift, tridiagonal)
        else:
            print('\n ------------ Processing eigenvalues only ------------ ')
            self.eigenvalues, _ = eigen_solver(self.H, self.S, MAX_ITER, RTOL, self.calc_eigvec,shift,tridiagonal)

    def eigen_test(self, verbose):
        w,v = np.linalg.eig(self.A)
        w_sorted, v_sorted = sort_eigen(w, v)

        print('\n ------------ Checking Eigenvalues ------------')
        test_eigenvalues(self.eigenvalues,w_sorted,verbose)
        if self.calc_eigvec:
            print('\n ------------ Checking Eigenvectors ------------')
            test_eigenvectors(self.eigenvectors,v_sorted,verbose)
```

```
        if verbose:
            print(self.eigenvalues.dtype)

            if self.eigenvalues.dtype == 'complex128':
                plot_eigenvalues_complex(w_sorted,self.eigenvalues)
            else:
                plot_eigenvalues_real(w_sorted,self.eigenvalues)
```

In [ ]:

```
##############
### IGNORE ###
##############


class RealMatrix(Matrix):

    def __init__(self, A, calc_eigvec=False):
        super().__init__(A, calc_eigvec)


    # double-shifts for complex eigenvalues
    def eigen_solver(self):

        if self.calc_eigvec:
            print('\n ------------ Processing both eigenvalues and eigenvectors ------------ ')
            self.eigenvalues, self.eigenvectors = eigen_solver_real(self.H, self.S, max_iter=MAX_ITER,
tol=RTOL, eigvec=self.calc_eigvec)

        else:
            print('\n ------------ Processing eigenvalues only ------------ ')
            self.eigenvalues, _ = eigen_solver_real(self.H, self.S, max_iter=MAX_ITER, tol=RTOL, eigve
c=self.calc_eigvec)


class SymmetricRealMatrix(RealMatrix):

    def __init__(self, A, calc_eigvec=False):
        super().__init__(A, calc_eigvec)


    # transformation to tridiagonal form with optimized storage
    def eigen_solver(self):

        if self.calc_eigvec:
            print('\n ------------ Processing both eigenvalues and eigenvectors ------------ ')
            self.eigenvalues, self.eigenvectors = eigen_solver_realsymmetric(self.H, self.S, max_iter=
MAX_ITER, tol=RTOL, eigvec=self.calc_eigvec)

        else:
            print('\n ------------ Processing eigenvalues only ------------ ')
            self.eigenvalues, _ = eigen_solver_realsymmetric(self.H, self.S, max_iter=MAX_ITER, tol=RT
OL, eigvec=self.calc_eigvec)
```

In [ ]:

Matric Market

from scipy.io import mmread

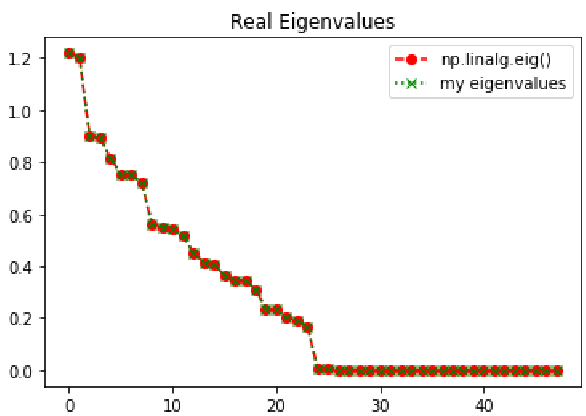
- BCSSTK01 (real symmetric positive definite, 48 by 48, 224 entries), Small test problem; source: https://math.nist.gov/MatrixMarket/data/Harwell-Boeing/bcsstruc1/bcsstruc1.html (https://math.nist.gov/MatrixMarket/data/Harwell-Boeing/bcsstruc1/bcsstruc1.html)
- CK104 (real unsymmetric, 104 by 104, 992 entries) The matrices have several multiple eigenvalues and clustered eigenvalues. source: https://math.nist.gov/MatrixMarket/data/NEP/chuck/ck104.html (https://math.nist.gov/MatrixMarket/data/NEP/chuck/ck104.html)
- QC324 (complex symmetric indefinite, 324 x 324, 26730 entries) H2PLUS: Model of H2+ in an Electromagnetic Field source: https://math.nist.gov/MatrixMarket/data/NEP/h2plus/h2plus.html (https://math.nist.gov/MatrixMarket/data/NEP/h2plus/h2plus.html)
- MHD1280 (complex unsymmetric, 1280 by 1280, 47906 entries) MHD: Alfven Spectra in Magnetohydrodynamicssource: https://math.nist.gov/MatrixMarket/data/NEP/mhd/mhd.html (https://math.nist.gov/MatrixMarket/data/NEP/mhd/mhd.html)

- RBS480A (real unsymmetric, 480 by 480, 17088 entries) The computational task is to compute the real eigenvalues and the corresponding eigenvectors (most of the eigenvalues are complex). source: https://math.nist.gov/MatrixMarket/data/NEP/robotics/robotics.html (https://math.nist.gov/MatrixMarket/data/NEP/robotics/robotics.html)
- QH1484 (real unsymmetric, 1484 by 1484, 6110 entries) This set of matrices if from the application of the Hydro-Quebec power systems' small-signal model. In the application, one wants to compute all eigenvalues a + b i in a box of the complex plane. https://math.nist.gov/MatrixMarket/data/NEP/quebec/quebec.html (https://math.nist.gov/MatrixMarket/data/NEP/quebec/quebec.html)
- ORANI678 (real unsymmetric, 2529 by 2529, 90158 entries) Australian Economic Models Economic model of Australia, 1968-69 data source: https://math.nist.gov/MatrixMarket/data/Harwell-Boeing/econaus/orani678.html (https://math.nist.gov/MatrixMarket/data/Harwell-Boeing/econaus/orani678.html)

In [71]:
```
def normalize(A):
    """ normalize a matrix to [-1,1]
    """
    scaler = max(abs(A.max()),abs(A.min()))
    return A/scaler, scaler

def runtest_matrixmarket(source, matrix_class,shift):
    """source: str, filename in the form of 'xxxn.mtx.gz'
    matrix_class: options = [Matrix,RealMatrix, SymmetricRealMatrix]
    """
    A = mmread(source)
    print("From Matrix Market: \nmatrix source={}, matrix type={}, shape={}, data type={}, max={}, min
={}"
          .format(source, type(A), A.shape,A.dtype, A.max(),A.min()))
    A = np.asarray(A.todense())
    A, s = normalize(A)
    a = matrix_class(A, calc_eigvec=False)
    print('is_symmetric={}', a.is_symmetric)
    a.eigen_solver(shift)
    a.eigen_test(verbose=True)
```

In [72]:
```
# small real symmetric 48x48
source = 'matricmarket/bcsstk01.mtx.gz'
matrix_class = Matrix
runtest_matrixmarket(source, matrix_class,shift=None)
```

```
From Matrix Market:
matrix source=matricmarket/bcsstk01.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=
(48, 48), data type=float64, max=2472387301.98, min=-109779731.332
is_symmetric={} True
activiating tridigonal: True

 ------------ Processing eigenvalues only ------------
>>> QR Iteration without shift

/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:33: ComplexWarning: Casting complex valu
es to real discards the imaginary part

Iteration terminates at i=181 with tol=1e-06 reached

 ------------ Checking Eigenvalues ------------

my eigenvalues: [1.22  1.201 0.898 0.893 0.816 0.752 0.75  0.722 0.561 0.551 0.544 0.516 0.452 0.415
0.407 0.362 0.346 0.346 0.31  0.236 0.234 0.2   0.193 0.167 0.003 0.003 0.002 0.002 0.002 0.002 0.002
0.002 0.002 0.002 0.001 0.001 0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
]
np.linalg.eig:  [1.22  1.201 0.898 0.893 0.816 0.752 0.75  0.722 0.561 0.551 0.544 0.516 0.452 0.415
0.407 0.362 0.346 0.346 0.31  0.236 0.234 0.2   0.193 0.167 0.003 0.003 0.002 0.002 0.002 0.002 0.002
0.002 0.002 0.002 0.001 0.001 0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.
]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [ 0. -0.  0. -0. -0. -0.  0.  0. -0.  0.  0. -0.  0. -0.  0. -0. -0.  0.  0.  0. -0.  0. -0. -0. -0.
-0. -0.  0. -0. -0.  0. -0.  0.  0. -0. -0. -0. -0.  0.  0. -0. -0.  0. -0. -0.  0. -0. -0.]

max abs diff =  6.407067545399769e-06



float64
```



#

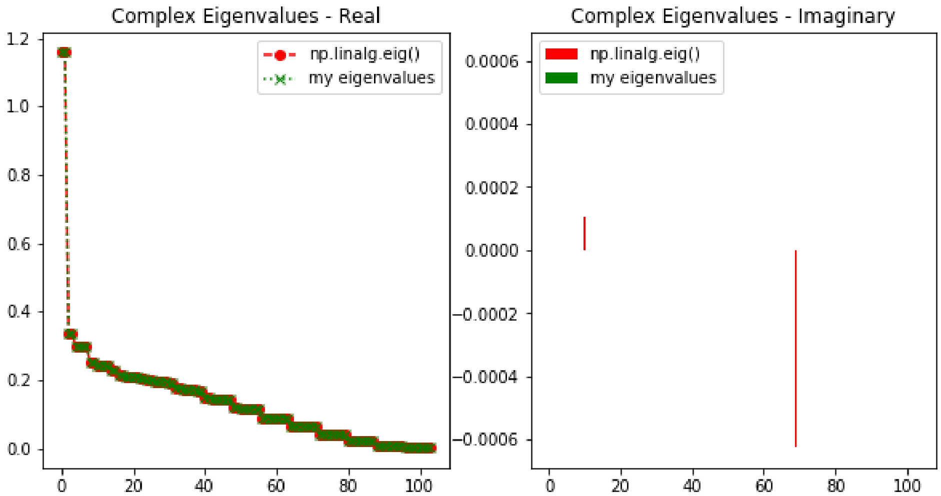
Observation

- General QR iteration without shift seems to be working for small real symmetric matrix

#

In [75]:
```
MAX_ITER = 10000

# real unsymmetric matrix 104x104
source = 'matricmarket/ck104.mtx.gz'
matrix_class = Matrix
runtest_matrixmarket(source, matrix_class,shift=None)
```

```
From Matrix Market:
matrix source=matricmarket/ck104.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=(10
4, 104), data type=float64, max=4.7481454523109, min=-2.6168329561089
is_symmetric={} False
activiating tridigonal: False

 ------------ Processing eigenvalues only ------------
>>> QR Iteration without shift

/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:33: ComplexWarning: Casting complex valu
es to real discards the imaginary part

Iteration terminates at i=5183 with tol=1e-06 reached

 ------------ Checking Eigenvalues ------------

my eigenvalues: [1.159+0.j    1.159+0.j    0.336+0.j    0.336+0.j    0.299+0.j    0.299+0.j    0.297+
0.j    0.297+0.j    0.252+0.j    0.252+0.j    0.24 +0.j    0.24 -0.j    0.24 +0.j    0.24 -0.j    0.2
28+0.j    0.228+0.j    0.216+0.j    0.216-0.j    0.211+0.j    0.211-0.j    0.211+0.j    0.211-0.j
0.204+0.j
 0.204-0.j    0.198+0.j    0.198+0.j    0.194+0.j    0.194-0.j    0.194+0.j    0.194-0.j    0.189+0.j
0.189+0.j    0.175+0.j    0.175+0.j    0.172+0.j    0.172+0.j    0.172+0.j    0.172+0.j    0.167+0.j
0.167-0.j    0.148+0.j    0.148+0.j    0.145+0.j    0.145+0.j    0.145+0.j    0.145+0.j
 0.142+0.j    0.142+0.j    0.119+0.j    0.119+0.j    0.117+0.j    0.117+0.j    0.117+0.j    0.117+0.j
0.116+0.j    0.116+0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089+0.j
0.089-0.j    0.088+0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.062+0.001j
 0.062-0.001j 0.062+0.001j 0.062-0.001j 0.04 +0.j    0.04 +0.j    0.039+0.j    0.039+0.j    0.039+0.j
0.039-0.j    0.039+0.j    0.039-0.j    0.021+0.j    0.021+0.j    0.021+0.j    0.021+0.j    0.02 +0.j
0.02 +0.j    0.02 +0.j    0.02 -0.j    0.008+0.j    0.008+0.j    0.008+0.j    0.008+0.j
 0.008+0.j    0.008+0.j    0.007+0.j    0.007+0.j    0.001+0.j    0.001-0.j    0.001+0.j    0.001+0.j
0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j   ]
np.linalg.eig:  [1.159+0.j    1.159+0.j    0.336+0.j    0.336+0.j    0.299+0.j    0.299+0.j    0.297+
0.j    0.297+0.j    0.252+0.j    0.252+0.j    0.24 +0.j    0.24 -0.j    0.24 +0.j    0.24 -0.j    0.2
28+0.j    0.228+0.j    0.216+0.j    0.216+0.j    0.211+0.j    0.211-0.j    0.211+0.j    0.211-0.j
0.204+0.j
 0.204+0.j    0.198+0.j    0.198+0.j    0.194+0.j    0.194-0.j    0.194+0.j    0.194-0.j    0.189+0.j
0.189+0.j    0.175+0.j    0.175+0.j    0.172+0.j    0.172+0.j    0.172+0.j    0.172-0.j    0.167+0.j
0.167+0.j    0.148+0.j    0.148+0.j    0.145+0.j    0.145+0.j    0.145+0.j    0.145+0.j
 0.142+0.j    0.142+0.j    0.119+0.j    0.119-0.j    0.117+0.j    0.117-0.j    0.117+0.j    0.117+0.j
0.116+0.j    0.116+0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089-0.j
0.089+0.j    0.089-0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.062+0.001j
 0.062-0.001j 0.062+0.001j 0.062-0.001j 0.04 +0.j    0.04 +0.j    0.039+0.j    0.039+0.j    0.039+0.j
0.039-0.j    0.039+0.j    0.039-0.j    0.021+0.j    0.021+0.j    0.021+0.j    0.021+0.j    0.02 +0.j
0.02 +0.j    0.02 +0.j    0.02 +0.j    0.008+0.j    0.008+0.j    0.008+0.j    0.008+0.j
 0.008+0.j    0.008+0.j    0.007+0.j    0.007+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j
0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j   ]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [-0.+0.j     0.+0.j     0.+0.j     0.+0.j     0.+0.j    -0.+0.j     0.+0.j    -0.+0.j    -0.+0.j
-0.+0.j     0.+0.j     0.-0.j     0.-0.j     0.+0.j    -0.+0.j    -0.+0.j    -0.+0.j    -0.-0.j    -
0.+0.j    -0.-0.j    -0.-0.j    -0.+0.j     0.+0.j     0.-0.j    -0.+0.j     0.+0.j     0.+0.j
  0.-0.j     0.-0.j     0.+0.j    -0.+0.j    -0.+0.j    -0.+0.j     0.+0.j     0.+0.j    -0.+0.j
0.-0.j    -0.+0.j    -0.+0.j     0.-0.j    -0.+0.j    -0.+0.j     0.+0.j    -0.+0.j     0.+0.j    -0.
+0.j    -0.+0.j    -0.+0.j    -0.-0.j    -0.+0.j     0.-0.j     0.+0.j    -0.+0.j    -0.+0.j
 -0.+0.j    -0.+0.j    -0.+0.j    -0.+0.j     0.+0.j     0.+0.j     0.-0.j    -0.+0.001j -0.-0.001j -
0.+0.j     0.+0.j     0.+0.j     0.+0.j    -0.+0.j     0.+0.j     0.-0.j    -0.-0.j    -0.+0.j    -0.
+0.j    -0.+0.j     0.+0.j    -0.+0.j     0.-0.j     0.+0.j    -0.+0.j    -0.-0.j    -0.+0.j
  0.+0.j     0.+0.j    -0.+0.j    -0.+0.j    -0.+0.j    -0.+0.j     0.-0.j    -0.+0.j     0.+0.j    -
0.+0.j    -0.+0.j     0.+0.j     0.+0.j    -0.+0.j     0.+0.j    -0.+0.j     0.-0.j    -0.+0.j     0.
+0.j    -0.+0.j    -0.+0.j    -0.+0.j     0.+0.j   ]

max abs diff =  0.0007620477194806169


complex128
```

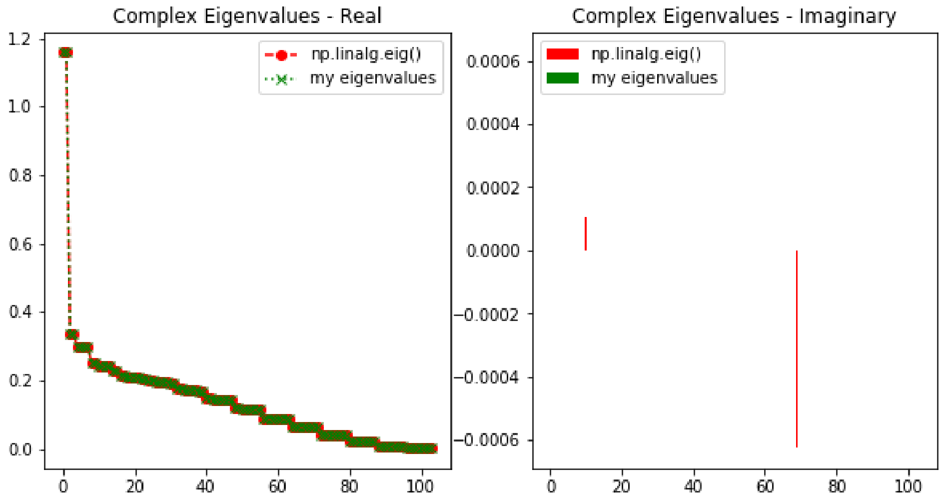


Complex Eigenvalues - Real
np.linalg.eig()
my eigenvalues
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
20
40
60
80
100
Complex Eigenvalues - Imaginary
np.linalg.eig()
my eigenvalues
0.0006
0.0004
0.0002
0.0000
−0.0002
−0.0004
−0.0006
0
20
40
60
80
100

In [76]:

```
MAX_ITER = 10000

# real unsymmetric matrix 104x104
source = 'matricmarket/ck104.mtx.gz'
matrix_class = Matrix
runtest_matrixmarket(source, matrix_class,shift='Wilkinson')
```

```
From Matrix Market:
matrix source=matricmarket/ck104.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=(10
4, 104), data type=float64, max=4.7481454523109, min=-2.6168329561089
is_symmetric={} False
activiating tridigonal: False

 ------------ Processing eigenvalues only ------------
>>> QR Iteration with Wilkinson's Shift: lambda = 0.06258762000408616, float

/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:33: ComplexWarning: Casting complex valu
es to real discards the imaginary part

Iteration terminates at i=6468 with tol=1e-06 reached

 ------------ Checking Eigenvalues ------------

my eigenvalues: [1.159+0.j    1.159+0.j    0.336+0.j    0.336+0.j    0.299+0.j    0.299+0.j    0.297+
0.j    0.297+0.j    0.252+0.j    0.252+0.j    0.24 +0.j    0.24 -0.j    0.24 +0.j    0.24 -0.j    0.2
28+0.j    0.228+0.j    0.216+0.j    0.216-0.j    0.211+0.j    0.211-0.j    0.211+0.j    0.211-0.j
0.204+0.j
 0.204+0.j    0.198+0.j    0.198+0.j    0.194+0.j    0.194-0.j    0.194+0.j    0.194-0.j    0.189+0.j
0.189+0.j    0.175+0.j    0.175+0.j    0.172+0.j    0.172+0.j    0.172+0.j    0.172+0.j    0.167+0.j
0.167+0.j    0.148+0.j    0.148+0.j    0.145+0.j    0.145+0.j    0.145+0.j    0.145+0.j
 0.142+0.j    0.142+0.j    0.119+0.j    0.119+0.j    0.117+0.j    0.117+0.j    0.117+0.j    0.117+0.j
0.116+0.j    0.116+0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089-0.j
0.089+0.j    0.089-0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.062+0.001j
 0.062-0.001j 0.062+0.001j 0.062-0.001j 0.04 +0.j    0.04 +0.j    0.039+0.j    0.039+0.j    0.039+0.j
0.039-0.j    0.039+0.j    0.039-0.j    0.021+0.j    0.021+0.j    0.021+0.j    0.021+0.j    0.02 +0.j
0.02 +0.j    0.02 +0.j    0.02 +0.j    0.008+0.j    0.008+0.j    0.008+0.j    0.008+0.j
 0.008+0.j    0.008+0.j    0.007+0.j    0.007+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j
0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j   ]
np.linalg.eig:  [1.159+0.j    1.159+0.j    0.336+0.j    0.336+0.j    0.299+0.j    0.299+0.j    0.297+
0.j    0.297+0.j    0.252+0.j    0.252+0.j    0.24 +0.j    0.24 -0.j    0.24 +0.j    0.24 -0.j    0.2
28+0.j    0.228+0.j    0.216+0.j    0.216+0.j    0.211+0.j    0.211-0.j    0.211+0.j    0.211-0.j
0.204+0.j
 0.204+0.j    0.198+0.j    0.198+0.j    0.194+0.j    0.194-0.j    0.194+0.j    0.194-0.j    0.189+0.j
0.189+0.j    0.175+0.j    0.175+0.j    0.172+0.j    0.172+0.j    0.172+0.j    0.172-0.j    0.167+0.j
0.167+0.j    0.148+0.j    0.148+0.j    0.145+0.j    0.145+0.j    0.145+0.j    0.145+0.j
 0.142+0.j    0.142+0.j    0.119+0.j    0.119-0.j    0.117+0.j    0.117-0.j    0.117+0.j    0.117+0.j
0.116+0.j    0.116+0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089-0.j
0.089+0.j    0.089-0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.062+0.001j
 0.062-0.001j 0.062+0.001j 0.062-0.001j 0.04 +0.j    0.04 +0.j    0.039+0.j    0.039+0.j    0.039+0.j
0.039-0.j    0.039+0.j    0.039-0.j    0.021+0.j    0.021+0.j    0.021+0.j    0.021+0.j    0.02 +0.j
0.02 +0.j    0.02 +0.j    0.02 +0.j    0.008+0.j    0.008+0.j    0.008+0.j    0.008+0.j
 0.008+0.j    0.008+0.j    0.007+0.j    0.007+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j
0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j   ]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [-0.+0.j  0.+0.j  0.+0.j  0.+0.j  0.+0.j -0.+0.j  0.+0.j -0.+0.j -0.+0.j -0.+0.j  0.+0.j  0.-0.j -0.
-0.j -0.+0.j -0.+0.j -0.+0.j -0.+0.j -0.-0.j -0.-0.j -0.+0.j -0.+0.j -0.-0.j -0.+0.j -0.+0.j  0.+0.j
-0.+0.j  0.+0.j  0.-0.j -0.-0.j -0.+0.j -0.+0.j -0.+0.j -0.+0.j  0.+0.j -0.+0.j -0.+0.j  0.-0.j
  0.+0.j  0.+0.j  0.+0.j -0.+0.j -0.+0.j  0.+0.j  0.+0.j -0.+0.j -0.+0.j -0.+0.j -0.+0.j -0.-0.j -0.+
0.j  0.-0.j -0.+0.j  0.+0.j  0.+0.j -0.+0.j -0.+0.j -0.+0.j  0.+0.j -0.+0.j  0.+0.j -0.-0.j -0.+0.j -
0.+0.j -0.-0.j -0.+0.j  0.+0.j -0.+0.j -0.+0.j -0.+0.j -0.-0.j -0.-0.j -0.+0.j -0.+0.j  0.+0.j
 -0.+0.j -0.+0.j  0.+0.j  0.-0.j -0.-0.j -0.+0.j -0.+0.j  0.+0.j  0.+0.j -0.+0.j  0.+0.j -0.+0.j  0.+
0.j  0.+0.j -0.+0.j -0.+0.j  0.+0.j -0.+0.j -0.+0.j -0.+0.j  0.+0.j  0.+0.j  0.+0.j -0.+0.j -0.+0.j -
0.+0.j  0.+0.j  0.+0.j  0.+0.j  0.+0.j]

max abs diff =  2.607548914700976e-06



complex128
```



Observation

- General QR iteration without shift seems to be working for real unsymmetric matrix
- However, applying Wilkinson's shift seems to converge slower than without shift...

#

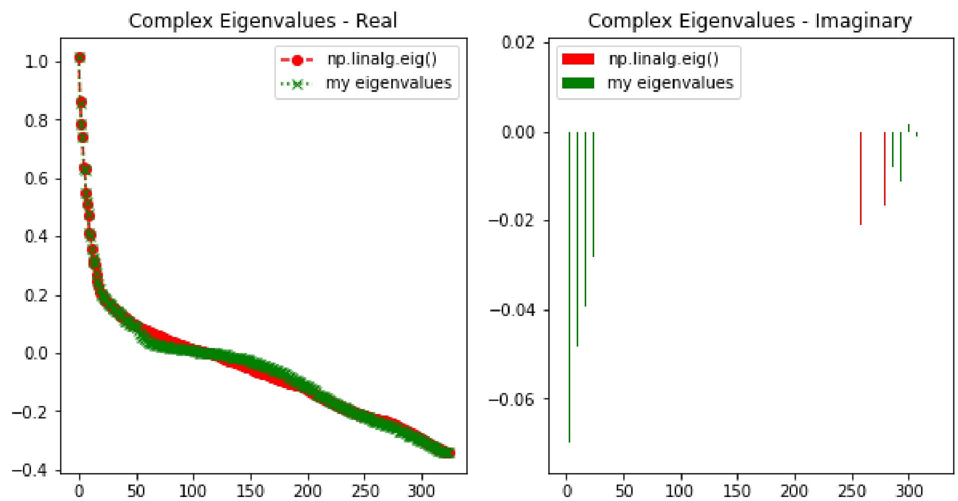
In [77]:

```
MAX_ITER = 10000

# complex symmetric matrix
source = 'matricmarket/qc324.mtx.gz'
matrix_class = Matrix
runtest_matrixmarket(source, matrix_class,shift=None)
```

```
From Matrix Market:
matrix source=matricmarket/qc324.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=(32
4, 324), data type=complex128, max=(1.4957127687965-0.087585823232133j), min=(-0.40837438550439-0.017
204228197979j)
is_symmetric={} True
activiating tridigonal: True

 ------------ Processing eigenvalues only ------------
>>> QR Iteration without shift
max_iter=10000 reached, max error=0.003883810927479336 vs tol=1e-06; index of non-convergence=[[52],
[54], [163], [164], [270], [272], [273]]

 ------------ Checking Eigenvalues ------------

my eigenvalues: [ 1.014-0.063j  0.859-0.064j  0.786-0.072j  0.741-0.07j   0.632-0.069j  0.628-0.06j
0.548-0.058j  0.519-0.048j  0.48 -0.056j  0.41 -0.042j  0.401-0.048j  0.354-0.054j  0.324-0.039j  0.3
12-0.042j  0.305-0.041j  0.275-0.039j  0.248-0.034j  0.235-0.039j  0.218-0.038j  0.203-0.033j  0.2  -
0.038j
  0.193-0.039j  0.189-0.036j  0.184-0.026j  0.182-0.028j  0.17 -0.027j  0.168-0.03j   0.167-0.031j
0.164-0.028j  0.155-0.027j  0.152-0.023j  0.148-0.032j  0.144-0.03j   0.141-0.031j  0.141-0.03j   0.1
37-0.032j  0.133-0.028j  0.131-0.033j  0.124-0.037j  0.115-0.033j  0.112-0.03j   0.112-0.037j
  0.111-0.043j  0.108-0.027j  0.101-0.025j  0.096-0.017j  0.094-0.023j  0.092-0.022j  0.091-0.017j
0.091-0.024j  0.088-0.031j  0.085-0.032j  0.079-0.042j  0.066-0.042j  0.064-0.048j  0.056-0.043j  0.0
56-0.049j  0.051-0.04j   0.046-0.045j  0.041-0.039j  0.039-0.036j  0.036-0.036j  0.033-0.044j
  0.032-0.035j  0.031-0.037j  0.03 -0.038j  0.028-0.03j   0.028-0.027j  0.027-0.031j  0.027-0.034j
0.025-0.024j  0.025-0.036j  0.024-0.022j  0.023-0.024j  0.022-0.024j  0.021-0.032j  0.021-0.018j  0.0
2 -0.022j  0.02 -0.034j  0.018-0.019j  0.017-0.015j  0.017-0.022j  0.016-0.019j  0.016-0.017j
  0.015-0.019j  0.014-0.02j   0.014-0.015j  0.014-0.016j  0.014-0.018j  0.013-0.025j  0.013-0.016j
0.012-0.019j  0.012-0.01j   0.012-0.017j  0.011-0.019j  0.011-0.019j  0.01 -0.008j  0.009-0.015j  0.0
09-0.014j  0.008-0.007j  0.008-0.014j  0.007-0.013j  0.006-0.01j   0.005-0.006j  0.003-0.003j
  0.002-0.009j  0.002-0.008j  0.002-0.011j  0.002-0.008j  0.002-0.025j  0.001-0.014j  0.    -0.016j -
0.   -0.004j -0.   -0.01j  -0.   -0.006j -0.001-0.013j -0.001-0.002j -0.002-0.002j -0.002-0.004j -0.0
02-0.005j -0.003-0.003j -0.004-0.009j -0.005-0.006j -0.005-0.007j -0.006-0.006j -0.006-0.001j
 -0.007-0.004j -0.007-0.007j -0.008-0.006j -0.009-0.007j -0.009-0.008j -0.012-0.008j -0.013-0.012j -
0.015-0.007j -0.015-0.007j -0.016-0.008j -0.017-0.01j  -0.017-0.015j -0.018-0.013j -0.018-0.009j -0.0
18-0.011j -0.019-0.011j -0.019-0.01j  -0.02 -0.014j -0.02 -0.008j -0.02 -0.01j  -0.021-0.016j
 -0.022-0.014j -0.025-0.014j -0.028-0.014j -0.029-0.012j -0.031-0.012j -0.034-0.012j -0.034-0.012j -
0.035-0.013j -0.035-0.011j -0.038-0.01j  -0.038-0.014j -0.04 -0.013j -0.041-0.015j -0.044-0.009j -0.0
44-0.015j -0.044-0.008j -0.047-0.011j -0.049-0.016j -0.049-0.01j  -0.051-0.013j -0.052-0.017j
 -0.053-0.017j -0.055-0.011j -0.056-0.022j -0.059-0.012j -0.06 -0.022j -0.06 -0.017j -0.061-0.011j -
0.064-0.016j -0.064-0.009j -0.067-0.017j -0.069-0.012j -0.071-0.018j -0.072-0.024j -0.074-0.011j -0.0
77-0.018j -0.08 -0.017j -0.082-0.019j -0.083-0.013j -0.089-0.021j -0.089-0.017j -0.093-0.02j
 -0.093-0.017j -0.094-0.018j -0.098-0.02j  -0.099-0.021j -0.099-0.014j -0.107-0.027j -0.109-0.01j  -
0.111-0.024j -0.113-0.017j -0.114-0.014j -0.114-0.009j -0.115-0.01j  -0.117-0.023j -0.119-0.015j -0.1
24-0.019j -0.126-0.007j -0.128-0.013j -0.128-0.021j -0.131-0.02j  -0.141-0.012j -0.141-0.015j
 -0.144-0.014j -0.148-0.012j -0.148-0.016j -0.149-0.009j -0.15 -0.007j -0.152-0.02j  -0.156-0.007j -
0.156-0.014j -0.159-0.019j -0.165-0.014j -0.167-0.013j -0.168-0.015j -0.169-0.005j -0.171-0.001j -0.1
73-0.014j -0.178-0.011j -0.178-0.017j -0.181-0.008j -0.182-0.008j -0.183-0.023j -0.186-0.02j
 -0.186-0.012j -0.187-0.011j -0.189-0.017j -0.193-0.009j -0.193-0.009j -0.196-0.015j -0.196-0.015j -
0.201-0.016j -0.204-0.012j -0.205-0.014j -0.207-0.008j -0.207-0.005j -0.209-0.013j -0.209-0.012j -0.2
11-0.013j -0.213-0.007j -0.213-0.017j -0.217-0.009j -0.218-0.012j -0.22 -0.015j -0.22 -0.018j
 -0.221-0.015j -0.222-0.014j -0.227-0.015j -0.227-0.01j  -0.229-0.005j -0.23 -0.008j -0.234-0.01j  -
0.235-0.014j -0.235-0.01j  -0.237-0.009j -0.239-0.006j -0.239-0.004j -0.24 -0.012j -0.243-0.013j -0.2
44-0.009j -0.245-0.014j -0.246-0.006j -0.246-0.012j -0.249-0.008j -0.25 -0.012j -0.251-0.015j
 -0.253-0.016j -0.254-0.008j -0.256-0.013j -0.256-0.012j -0.259-0.013j -0.26 -0.011j -0.261-0.014j -
0.267-0.011j -0.267-0.015j -0.268-0.01j  -0.269-0.009j -0.272-0.014j -0.272-0.006j -0.274-0.008j -0.2
75-0.007j -0.278-0.011j -0.284-0.003j -0.285-0.013j -0.287-0.014j -0.288-0.008j -0.288-0.011j
 -0.289-0.009j -0.292+0.006j -0.295-0.004j -0.297-0.004j -0.299-0.004j -0.302-0.005j -0.302+0.002j -
0.302-0.002j -0.306+0.007j -0.308-0.004j -0.309-0.006j -0.315-0.003j -0.316-0.013j -0.317-0.001j -0.3
18+0.006j -0.319-0.005j -0.323-0.005j -0.323-0.009j -0.325+0.004j -0.332-0.004j -0.333+0.004j
 -0.335+0.005j -0.336-0.002j -0.337+0.007j -0.338+0.004j -0.34 +0.009j -0.341-0.002j -0.341+0.008j -
0.342+0.017j -0.343+0.003j]
np.linalg.eig:  [ 1.014-0.064j  0.859-0.063j  0.785-0.064j  0.739-0.064j  0.637-0.064j  0.631-0.063j
0.549-0.063j  0.511-0.064j  0.474-0.062j  0.41 -0.06j   0.408-0.064j  0.355-0.058j  0.32 -0.063j  0.3
07-0.055j  0.301-0.045j  0.267-0.053j  0.245-0.062j  0.232-0.05j   0.208-0.038j  0.202-0.048j  0.201-
0.032j
  0.193-0.03j   0.184-0.029j  0.181-0.06j   0.177-0.028j  0.176-0.046j  0.17 -0.027j  0.164-0.026j
0.159-0.035j  0.158-0.025j  0.154-0.044j  0.152-0.024j  0.146-0.023j  0.14 -0.022j  0.136-0.042j  0.1
35-0.022j  0.13 -0.021j  0.127-0.031j  0.126-0.058j  0.125-0.021j  0.12 -0.04j   0.12 -0.02j
  0.115-0.019j  0.111-0.019j  0.108-0.038j  0.106-0.019j  0.103-0.02j   0.1  -0.018j  0.098-0.037j
0.096-0.017j  0.092-0.017j  0.092-0.035j  0.088-0.016j  0.087-0.034j  0.085-0.015j  0.082-0.033j  0.0
```

```
81-0.015j  0.079-0.055j  0.078-0.014j  0.077-0.033j  0.074-0.014j  0.073-0.045j  0.072-0.032j
  0.071-0.013j  0.068-0.013j  0.067-0.031j  0.064-0.012j  0.062-0.03j   0.061-0.012j  0.058-0.011j
0.057-0.03j   0.055-0.011j  0.053-0.029j  0.052-0.01j   0.049-0.01j   0.048-0.028j  0.046-0.009j  0.0
44-0.009j  0.043-0.027j  0.041-0.008j  0.039-0.027j  0.038-0.008j  0.038-0.053j  0.036-0.007j
  0.034-0.026j  0.033-0.007j  0.031-0.006j  0.03 -0.025j  0.028-0.006j  0.026-0.006j  0.025-0.024j
0.024-0.005j  0.021-0.005j  0.021-0.024j  0.019-0.004j  0.017-0.004j  0.016-0.023j  0.015-0.004j  0.0
14-0.003j  0.012-0.022j  0.012-0.003j  0.01 -0.003j  0.008-0.002j  0.008-0.022j  0.007-0.002j
  0.006-0.002j  0.004-0.001j  0.004-0.021j  0.003-0.05j   0.003-0.001j  0.002-0.001j  0.001-0.j
0.   -0.j    -0.   -0.02j  -0.   -0.j    -0.001-0.j    -0.002-0.j    -0.004-0.j    -0.004-0.02j  -0.0
06-0.j    -0.008-0.j    -0.008-0.019j -0.011-0.j    -0.012-0.018j -0.014-0.j    -0.016-0.018j
 -0.017-0.j    -0.02 -0.017j -0.021-0.j    -0.021-0.038j -0.024-0.017j -0.025-0.j    -0.027-0.048j -
0.027-0.016j -0.027-0.032j -0.03 -0.j    -0.031-0.015j -0.034-0.015j -0.035-0.j    -0.036-0.03j  -0.0
38-0.014j -0.04 -0.j    -0.041-0.014j -0.044-0.029j -0.045-0.013j -0.046-0.j    -0.048-0.013j
 -0.051-0.012j -0.051-0.028j -0.052-0.j    -0.052-0.046j -0.054-0.011j -0.058-0.011j -0.058-0.j    -
0.058-0.027j -0.061-0.01j  -0.064-0.01j  -0.065-0.026j -0.065-0.j    -0.066-0.009j -0.069-0.009j -0.0
7 -0.035j -0.071-0.025j -0.072-0.008j -0.074-0.044j -0.075-0.008j -0.077-0.024j -0.078-0.008j
 -0.08 -0.007j -0.083-0.007j -0.083-0.023j -0.085-0.006j -0.087-0.006j -0.088-0.022j -0.09 -0.005j -
0.092-0.005j -0.093-0.042j -0.094-0.022j -0.094-0.004j -0.096-0.004j -0.098-0.004j -0.099-0.021j -0.1
-0.003j -0.102-0.031j -0.102-0.003j -0.104-0.003j -0.104-0.021j -0.105-0.002j -0.107-0.002j
 -0.108-0.04j  -0.109-0.002j -0.109-0.02j  -0.11 -0.001j -0.111-0.001j -0.112-0.001j -0.113-0.j    -
0.114-0.019j -0.118-0.019j -0.121-0.038j -0.122-0.019j -0.125-0.02j  -0.129-0.018j -0.13 -0.037j -0.1
33-0.017j -0.137-0.017j -0.137-0.035j -0.14 -0.016j -0.142-0.034j -0.144-0.015j -0.147-0.033j
 -0.147-0.015j -0.151-0.014j -0.152-0.033j -0.154-0.014j -0.157-0.032j -0.158-0.013j -0.161-0.013j -
0.162-0.031j -0.164-0.012j -0.166-0.03j  -0.167-0.012j -0.171-0.011j -0.171-0.03j  -0.174-0.011j -0.1
76-0.029j -0.177-0.01j  -0.179-0.01j  -0.181-0.028j -0.182-0.009j -0.185-0.009j -0.185-0.027j
 -0.188-0.008j -0.19 -0.027j -0.191-0.008j -0.193-0.007j -0.195-0.026j -0.196-0.007j -0.198-0.006j -
0.199-0.025j -0.2  -0.006j -0.203-0.006j -0.203-0.024j -0.205-0.005j -0.207-0.005j -0.208-0.024j -0.2
09-0.004j -0.211-0.004j -0.212-0.023j -0.213-0.004j -0.215-0.003j -0.216-0.022j -0.217-0.003j
 -0.219-0.003j -0.22 -0.002j -0.221-0.022j -0.222-0.002j -0.223-0.002j -0.224-0.001j -0.225-0.021j -
0.226-0.001j -0.227-0.001j -0.228-0.j    -0.228-0.j    -0.229-0.02j  -0.229-0.j    -0.23 -0.j    -0.2
31-0.j    -0.233-0.j    -0.233-0.02j  -0.234-0.j    -0.237-0.j    -0.237-0.019j -0.239-0.j
 -0.241-0.018j -0.242-0.j    -0.245-0.018j -0.246-0.j    -0.248-0.017j -0.25 -0.j    -0.252-0.017j -
0.254-0.j    -0.256-0.016j -0.258-0.j    -0.259-0.015j -0.263-0.015j -0.263-0.j    -0.266-0.014j -0.2
69-0.j    -0.27 -0.014j -0.273-0.013j -0.274-0.j    -0.277-0.013j -0.28 -0.012j -0.28 -0.j
 -0.283-0.011j -0.286-0.011j -0.287-0.j    -0.289-0.01j  -0.292-0.01j  -0.294-0.j    -0.295-0.009j -
0.298-0.009j -0.301-0.008j -0.304-0.008j -0.306-0.008j -0.309-0.007j -0.311-0.007j -0.314-0.006j -0.3
16-0.006j -0.318-0.005j -0.321-0.005j -0.323-0.004j -0.325-0.004j -0.327-0.004j -0.329-0.003j
 -0.331-0.003j -0.332-0.003j -0.334-0.002j -0.336-0.002j -0.337-0.002j -0.339-0.001j -0.34 -0.001j -
0.341-0.001j -0.342-0.j   ]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [-0.   +0.001j -0.001-0.002j  0.001-0.008j  0.002-0.006j -0.005-0.005j -0.003+0.003j -0.001+0.005j
0.008+0.016j  0.005+0.006j  0.   +0.018j -0.007+0.016j -0.001+0.004j  0.003+0.024j  0.005+0.013j  0.0
04+0.003j  0.008+0.014j  0.003+0.028j  0.003+0.011j  0.011+0.j     0.002+0.015j -0.001-0.006j
  0.001-0.008j  0.004-0.007j  0.003+0.033j  0.005-0.001j -0.006+0.019j -0.003-0.004j  0.003-0.005j
0.006+0.007j -0.003-0.002j -0.003+0.02j  -0.004-0.008j -0.002-0.007j  0.001-0.009j  0.005+0.012j  0.0
02-0.01j   0.003-0.007j  0.004-0.002j -0.002+0.021j -0.01 -0.013j -0.008+0.01j  -0.008-0.017j
 -0.004-0.024j -0.003-0.007j -0.007+0.013j -0.01 +0.002j -0.009-0.003j -0.008-0.003j -0.007+0.019j -
0.006-0.007j -0.004-0.014j -0.007+0.003j -0.01 -0.026j -0.021-0.008j -0.021-0.032j -0.026-0.01j  -0.0
26-0.034j -0.028+0.016j -0.031-0.031j -0.036-0.006j -0.036-0.022j -0.036+0.008j -0.039-0.012j
 -0.039-0.022j -0.037-0.024j -0.037-0.007j -0.036-0.018j -0.034+0.003j -0.034-0.02j  -0.031-0.023j -
0.032+0.006j -0.03 -0.025j -0.029+0.007j -0.029-0.014j -0.027-0.014j -0.026-0.004j -0.026-0.009j -0.0
23-0.014j -0.023-0.007j -0.023-0.01j  -0.021+0.012j -0.021-0.015j -0.022+0.033j -0.019-0.01j
 -0.019+0.007j -0.019-0.013j -0.017-0.009j -0.016+0.009j -0.014-0.012j -0.013-0.02j  -0.013+0.008j -
0.011-0.014j -0.009-0.005j -0.009+0.007j -0.008-0.015j -0.006-0.015j -0.006+0.015j -0.006-0.012j -0.0
05-0.011j -0.004+0.015j -0.004-0.012j -0.003-0.011j -0.003-0.008j -0.003+0.016j -0.004-0.001j
 -0.003-0.007j -0.002-0.007j -0.002+0.01j  -0.001+0.042j -0.001-0.024j -0.001-0.013j -0.001-0.016j -
0.   -0.004j -0.   +0.01j   0.   -0.006j  0.   -0.013j  0.001-0.002j  0.002-0.002j  0.002+0.016j  0.0
03-0.005j  0.005-0.003j  0.004+0.01j   0.006-0.006j  0.007+0.011j  0.008-0.006j  0.01 +0.017j
  0.01 -0.004j  0.012+0.01j   0.013-0.006j  0.013+0.031j  0.014+0.009j  0.013-0.008j  0.014+0.036j
0.013+0.009j  0.012+0.025j  0.013-0.008j  0.014+0.006j  0.017-0.j     0.017-0.013j  0.018+0.021j  0.0
2 +0.003j  0.021-0.011j  0.023+0.004j  0.024+0.015j  0.025+0.006j  0.025-0.01j   0.027-0.003j
  0.029-0.002j  0.026+0.014j  0.024-0.014j  0.024+0.034j  0.023-0.j     0.024-0.001j  0.025-0.012j
0.023+0.013j  0.025-0.001j  0.026+0.j     0.026+0.012j  0.025-0.013j  0.025-0.006j  0.026-0.j     0.0
26+0.02j   0.027+0.016j  0.025-0.003j  0.026+0.028j  0.026-0.002j  0.026+0.011j  0.026-0.009j
  0.027-0.01j   0.028-0.004j  0.027+0.001j  0.026-0.006j  0.028-0.016j  0.028+0.005j  0.029-0.006j
0.028-0.011j  0.028+0.033j  0.026+0.005j  0.025-0.008j  0.026-0.014j  0.026-0.021j  0.024+0.01j   0.0
24-0.014j  0.022+0.014j  0.02 -0.016j  0.021-0.01j   0.016-0.j     0.016-0.015j  0.014-0.018j
  0.015+0.023j  0.014-0.016j  0.011-0.j     0.011-0.02j   0.012-0.013j  0.005-0.026j  0.005-0.01j
0.003-0.005j  0.005+0.002j  0.007+0.024j  0.008+0.01j   0.011+0.01j   0.012-0.004j  0.011+0.022j  0.0
```

```
09-0.002j  0.011+0.01j   0.009+0.022j  0.012-0.005j  0.01 +0.015j  0.003+0.003j  0.005+0.019j
  0.004+0.001j  0.003+0.002j  0.003+0.017j  0.005+0.005j  0.007+0.024j  0.005-0.007j  0.005+0.005j
0.005+0.018j  0.005-0.007j  0.002+0.016j  0.    -0.001j  0.003-0.004j  0.003+0.024j  0.002+0.009j  0.0
03+0.015j -0.001-0.001j  0.001-0.008j  0.    +0.02j   0.    +0.001j  0.002-0.015j -0.    +0.007j
  0.002-0.004j  0.003+0.015j  0.001-0.01j   0.    -0.002j  0.002+0.017j -0.    -0.009j  0.002-0.008j -
0.002+0.009j -0.003-0.006j -0.003-0.008j -0.004+0.017j -0.002+0.j    -0.001-0.008j -0.001+0.012j -0.0
02-0.009j -0.001-0.003j -0.001+0.007j -0.004-0.006j -0.003-0.009j -0.003+0.007j -0.003-0.015j
 -0.003-0.012j -0.002-0.012j -0.007+0.006j -0.005-0.008j -0.006-0.003j -0.006-0.007j -0.01 +0.011j -
0.009-0.013j -0.008-0.01j  -0.009-0.009j -0.011-0.006j -0.011+0.017j -0.011-0.012j -0.013-0.013j -0.0
13-0.009j -0.013-0.014j -0.013+0.014j -0.012-0.012j -0.012-0.008j -0.013+0.007j -0.012-0.015j
 -0.012+0.003j -0.012-0.008j -0.011+0.005j -0.01  -0.012j -0.011+0.004j -0.011-0.011j -0.009+0.003j -
0.013-0.011j -0.011+0.001j -0.01  -0.01j  -0.009+0.007j -0.009+0.j    -0.008-0.006j -0.007+0.006j -0.0
06-0.007j -0.009+0.003j -0.011+0.01j  -0.011-0.013j -0.01  -0.002j -0.008+0.004j -0.007-0.011j
 -0.006+0.002j -0.006+0.017j -0.008-0.004j -0.008+0.007j -0.006+0.006j -0.008-0.005j -0.006+0.011j -
0.004+0.007j -0.006+0.016j -0.005+0.004j -0.002+0.001j -0.006+0.004j -0.004-0.006j -0.003+0.005j -0.0
02+0.012j -0.    +0.j    -0.002-0.j    -0.    -0.004j  0.    +0.008j -0.005-0.j    -0.004+0.007j
 -0.004+0.008j -0.004+0.j    -0.002+0.009j -0.003+0.006j -0.003+0.011j -0.002-0.001j -0.001+0.009j -
0.001+0.017j -0.001+0.004j]

max abs diff =  0.04490683528993241



complex128
```



Observation

- General QR iteration without shift seems to be working for complex symmetric matrix

#

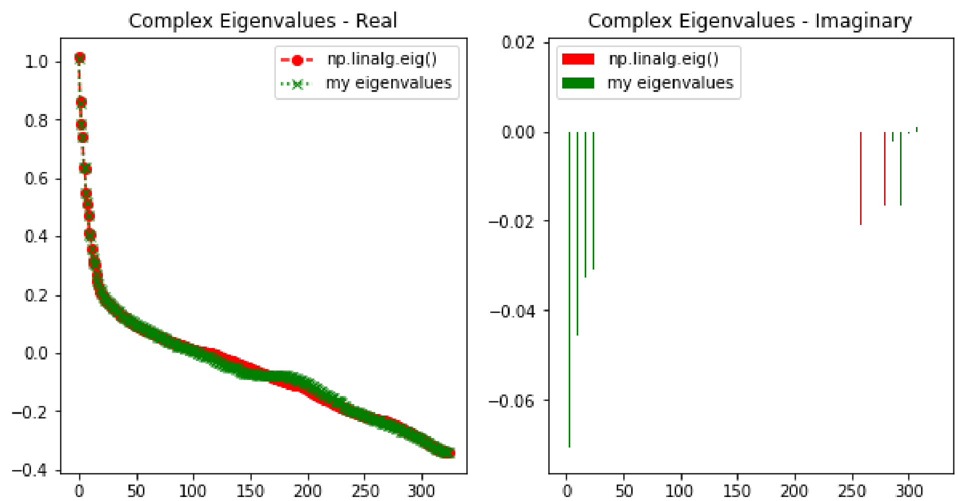
In [78]:

```
MAX_ITER = 10000

# complex symmetric matrix
source = 'matricmarket/qc324.mtx.gz'
matrix_class = Matrix
runtest_matrixmarket(source, matrix_class,shift='Wilkinson')
```

```
From Matrix Market:
matrix source=matricmarket/qc324.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=(32
4, 324), data type=complex128, max=(1.4957127687965-0.087585823232133j), min=(-0.40837438550439-0.017
204228197979j)
is_symmetric={} True
activiating tridigonal: True

 ------------ Processing eigenvalues only ------------
>>> QR Iteration with Wilkinson's Shift: lambda = (-0.07521622424344354-0.00721409631161954j), float
max_iter=10000 reached, max error=0.01796299554513952 vs tol=1e-06; index of non-convergence=[[40],
[42], [64], [65], [123], [124], [208], [210]]

 ------------ Checking Eigenvalues ------------

my eigenvalues: [ 1.013-0.063j  0.859-0.064j  0.788-0.069j  0.741-0.071j  0.638-0.063j  0.634-0.072j
0.547-0.058j  0.515-0.06j   0.471-0.052j  0.409-0.056j  0.402-0.046j  0.357-0.037j  0.327-0.039j  0.3
07-0.041j  0.304-0.044j  0.268-0.038j  0.247-0.034j  0.235-0.033j  0.218-0.031j  0.212-0.029j  0.206-
0.03j
  0.191-0.032j  0.191-0.034j  0.183-0.039j  0.181-0.031j  0.176-0.035j  0.172-0.033j  0.168-0.033j
0.164-0.032j  0.16 -0.033j  0.156-0.026j  0.152-0.022j  0.147-0.035j  0.144-0.026j  0.143-0.026j  0.1
43-0.026j  0.133-0.018j  0.129-0.019j  0.127-0.026j  0.123-0.028j  0.119-0.03j   0.118-0.023j
  0.116-0.041j  0.116-0.029j  0.111-0.032j  0.11 -0.034j  0.109-0.029j  0.105-0.03j   0.104-0.026j
0.101-0.035j  0.097-0.034j  0.096-0.027j  0.094-0.013j  0.092-0.012j  0.09 -0.015j  0.087-0.025j  0.0
87-0.016j  0.085-0.027j  0.083-0.022j  0.08 -0.025j  0.079-0.023j  0.076-0.025j  0.074-0.027j
  0.073-0.023j  0.069-0.027j  0.066-0.028j  0.065-0.028j  0.063-0.033j  0.06 -0.04j   0.057-0.035j
0.056-0.044j  0.056-0.036j  0.054-0.045j  0.054-0.041j  0.051-0.035j  0.049-0.031j  0.044-0.028j  0.0
37-0.022j  0.037-0.002j  0.036-0.034j  0.036-0.038j  0.033-0.008j  0.033-0.014j  0.031-0.014j
  0.03 -0.062j  0.03 -0.016j  0.024-0.021j  0.024-0.027j  0.023-0.024j  0.022-0.011j  0.022-0.021j
0.019-0.02j   0.018-0.029j  0.016-0.027j  0.015-0.025j  0.015-0.023j  0.013-0.016j  0.013-0.03j   0.0
12-0.032j  0.01 -0.013j  0.005-0.019j  0.005-0.024j  0.003-0.022j -0.001-0.016j -0.002-0.02j
 -0.003-0.026j -0.003-0.027j -0.005-0.033j -0.007-0.008j -0.008-0.027j -0.009-0.019j -0.011-0.011j -
0.011-0.011j -0.013-0.009j -0.013-0.015j -0.016-0.006j -0.017-0.008j -0.02 -0.011j -0.024-0.007j -0.0
28-0.015j -0.029-0.009j -0.029-0.016j -0.032-0.008j -0.033-0.009j -0.033-0.014j -0.036-0.012j
 -0.039-0.01j  -0.043-0.011j -0.044-0.007j -0.044-0.01j  -0.046-0.013j -0.047-0.016j -0.048-0.004j -
0.05 -0.011j -0.05 -0.008j -0.051-0.009j -0.051-0.011j -0.052-0.006j -0.053-0.009j -0.055-0.01j  -0.0
59-0.005j -0.063-0.01j  -0.063-0.003j -0.064-0.007j -0.068-0.01j  -0.069-0.006j -0.069-0.012j
 -0.071-0.009j -0.072-0.014j -0.072-0.01j  -0.073-0.005j -0.073-0.008j -0.073-0.007j -0.074-0.006j -
0.074-0.004j -0.074-0.007j -0.074-0.008j -0.074-0.006j -0.074-0.008j -0.075-0.007j -0.075-0.015j -0.0
75-0.015j -0.075-0.007j -0.075-0.011j -0.075-0.007j -0.075-0.008j -0.075-0.007j -0.075-0.007j
 -0.075-0.007j -0.075-0.007j -0.075-0.007j -0.076-0.009j -0.076-0.005j -0.076-0.007j -0.076-0.006j -
0.076-0.005j -0.076-0.01j  -0.076-0.008j -0.077-0.005j -0.077-0.008j -0.077-0.007j -0.078-0.018j -0.0
78-0.007j -0.08 -0.012j -0.081+0.j    -0.083-0.01j  -0.083-0.014j -0.084-0.009j -0.086-0.012j
 -0.087-0.01j  -0.089-0.017j -0.092-0.014j -0.093-0.015j -0.093-0.011j -0.095-0.018j -0.096-0.017j -
0.099-0.012j -0.101-0.014j -0.103-0.012j -0.104-0.013j -0.105-0.015j -0.106-0.018j -0.11 -0.019j -0.1
13-0.02j  -0.115-0.015j -0.117-0.009j -0.118-0.018j -0.121-0.014j -0.122-0.011j -0.125-0.007j
 -0.13 -0.019j -0.132-0.014j -0.132-0.007j -0.135-0.018j -0.142-0.015j -0.142-0.018j -0.144-0.009j -
0.145-0.015j -0.146-0.021j -0.151-0.015j -0.151-0.012j -0.154-0.009j -0.156-0.006j -0.156-0.017j -0.1
62-0.022j -0.162-0.014j -0.163-0.012j -0.165-0.011j -0.167-0.017j -0.17 -0.006j -0.17 -0.004j
 -0.18 -0.009j -0.181-0.011j -0.184-0.023j -0.19 -0.008j -0.191-0.017j -0.195-0.02j  -0.196-0.008j -
0.196-0.018j -0.197-0.029j -0.197-0.02j  -0.201-0.017j -0.204-0.027j -0.205-0.017j -0.205-0.009j -0.2
06-0.005j -0.208-0.015j -0.211-0.018j -0.213-0.02j  -0.213-0.014j -0.214-0.02j  -0.216-0.021j
 -0.219-0.017j -0.221-0.014j -0.222-0.014j -0.224-0.017j -0.227-0.021j -0.227-0.011j -0.228-0.022j -
0.228-0.022j -0.231-0.013j -0.231-0.019j -0.232-0.001j -0.232-0.01j  -0.237-0.016j -0.24 -0.012j -0.2
4 -0.021j -0.24 -0.019j -0.242-0.016j -0.243-0.025j -0.246-0.024j -0.246-0.015j -0.246+0.001j
 -0.247-0.01j  -0.252-0.004j -0.253-0.025j -0.254-0.015j -0.257-0.01j  -0.259-0.01j  -0.26 -0.003j -
0.26 -0.01j  -0.261-0.008j -0.267-0.013j -0.269-0.017j -0.269-0.014j -0.27 +0.001j -0.27 -0.002j -0.2
71-0.015j -0.274-0.012j -0.275-0.009j -0.277-0.008j -0.28 -0.01j  -0.281-0.021j -0.283-0.017j
 -0.284-0.005j -0.285-0.007j -0.289-0.002j -0.291-0.005j -0.291+0.j    -0.296+0.001j -0.299-0.001j -
0.301-0.005j -0.302-0.005j -0.304-0.003j -0.307-0.001j -0.308-0.003j -0.31 -0.01j  -0.313+0.001j -0.3
17-0.001j -0.32 +0.004j -0.32 +0.006j -0.325+0.004j -0.326-0.j    -0.327-0.002j -0.327-0.004j
 -0.331-0.002j -0.333+0.002j -0.334+0.006j -0.334+0.009j -0.337+0.002j -0.338+0.002j -0.34 +0.009j -
0.341+0.017j -0.343+0.012j]
np.linalg.eig:  [ 1.014-0.064j  0.859-0.063j  0.785-0.064j  0.739-0.064j  0.637-0.064j  0.631-0.063j
0.549-0.063j  0.511-0.064j  0.474-0.062j  0.41 -0.06j   0.408-0.064j  0.355-0.058j  0.32 -0.063j  0.3
07-0.055j  0.301-0.045j  0.267-0.053j  0.245-0.062j  0.232-0.05j   0.208-0.038j  0.202-0.048j  0.201-
0.032j
  0.193-0.03j   0.184-0.029j  0.181-0.06j   0.177-0.028j  0.176-0.046j  0.17 -0.027j  0.164-0.026j
0.159-0.035j  0.158-0.025j  0.154-0.044j  0.152-0.024j  0.146-0.023j  0.14 -0.022j  0.136-0.042j  0.1
35-0.022j  0.13 -0.021j  0.127-0.031j  0.126-0.058j  0.125-0.021j  0.12 -0.04j   0.12 -0.02j
  0.115-0.019j  0.111-0.019j  0.108-0.038j  0.106-0.019j  0.103-0.02j   0.1  -0.018j  0.098-0.037j
0.096-0.017j  0.092-0.017j  0.092-0.035j  0.088-0.016j  0.087-0.034j  0.085-0.015j  0.082-0.033j  0.0
```

```
81-0.015j  0.079-0.055j  0.078-0.014j  0.077-0.033j  0.074-0.014j  0.073-0.045j  0.072-0.032j
  0.071-0.013j  0.068-0.013j  0.067-0.031j  0.064-0.012j  0.062-0.03j   0.061-0.012j  0.058-0.011j
0.057-0.03j   0.055-0.011j  0.053-0.029j  0.052-0.01j   0.049-0.01j   0.048-0.028j  0.046-0.009j  0.0
44-0.009j  0.043-0.027j  0.041-0.008j  0.039-0.027j  0.038-0.008j  0.038-0.053j  0.036-0.007j
  0.034-0.026j  0.033-0.007j  0.031-0.006j  0.03 -0.025j  0.028-0.006j  0.026-0.006j  0.025-0.024j
0.024-0.005j  0.021-0.005j  0.021-0.024j  0.019-0.004j  0.017-0.004j  0.016-0.023j  0.015-0.004j  0.0
14-0.003j  0.012-0.022j  0.012-0.003j  0.01 -0.003j  0.008-0.002j  0.008-0.022j  0.007-0.002j
  0.006-0.002j  0.004-0.001j  0.004-0.021j  0.003-0.05j   0.003-0.001j  0.002-0.001j  0.001-0.j
0.   -0.j    -0.   -0.02j  -0.   -0.j    -0.001-0.j    -0.002-0.j    -0.004-0.j    -0.004-0.02j  -0.0
06-0.j    -0.008-0.j    -0.008-0.019j -0.011-0.j    -0.012-0.018j -0.014-0.j    -0.016-0.018j
 -0.017-0.j    -0.02 -0.017j -0.021-0.j    -0.021-0.038j -0.024-0.017j -0.025-0.j    -0.027-0.048j -
0.027-0.016j -0.027-0.032j -0.03 -0.j    -0.031-0.015j -0.034-0.015j -0.035-0.j    -0.036-0.03j  -0.0
38-0.014j -0.04 -0.j    -0.041-0.014j -0.044-0.029j -0.045-0.013j -0.046-0.j    -0.048-0.013j
 -0.051-0.012j -0.051-0.028j -0.052-0.j    -0.052-0.046j -0.054-0.011j -0.058-0.011j -0.058-0.j    -
0.058-0.027j -0.061-0.01j  -0.064-0.01j  -0.065-0.026j -0.065-0.j    -0.066-0.009j -0.069-0.009j -0.0
7 -0.035j -0.071-0.025j -0.072-0.008j -0.074-0.044j -0.075-0.008j -0.077-0.024j -0.078-0.008j
 -0.08 -0.007j -0.083-0.007j -0.083-0.023j -0.085-0.006j -0.087-0.006j -0.088-0.022j -0.09 -0.005j -
0.092-0.005j -0.093-0.042j -0.094-0.022j -0.094-0.004j -0.096-0.004j -0.098-0.004j -0.099-0.021j -0.1
-0.003j -0.102-0.031j -0.102-0.003j -0.104-0.003j -0.104-0.021j -0.105-0.002j -0.107-0.002j
 -0.108-0.04j  -0.109-0.002j -0.109-0.02j  -0.11 -0.001j -0.111-0.001j -0.112-0.001j -0.113-0.j    -
0.114-0.019j -0.118-0.019j -0.121-0.038j -0.122-0.019j -0.125-0.02j  -0.129-0.018j -0.13 -0.037j -0.1
33-0.017j -0.137-0.017j -0.137-0.035j -0.14 -0.016j -0.142-0.034j -0.144-0.015j -0.147-0.033j
 -0.147-0.015j -0.151-0.014j -0.152-0.033j -0.154-0.014j -0.157-0.032j -0.158-0.013j -0.161-0.013j -
0.162-0.031j -0.164-0.012j -0.166-0.03j  -0.167-0.012j -0.171-0.011j -0.171-0.03j  -0.174-0.011j -0.1
76-0.029j -0.177-0.01j  -0.179-0.01j  -0.181-0.028j -0.182-0.009j -0.185-0.009j -0.185-0.027j
 -0.188-0.008j -0.19 -0.027j -0.191-0.008j -0.193-0.007j -0.195-0.026j -0.196-0.007j -0.198-0.006j -
0.199-0.025j -0.2  -0.006j -0.203-0.006j -0.203-0.024j -0.205-0.005j -0.207-0.005j -0.208-0.024j -0.2
09-0.004j -0.211-0.004j -0.212-0.023j -0.213-0.004j -0.215-0.003j -0.216-0.022j -0.217-0.003j
 -0.219-0.003j -0.22 -0.002j -0.221-0.022j -0.222-0.002j -0.223-0.002j -0.224-0.001j -0.225-0.021j -
0.226-0.001j -0.227-0.001j -0.228-0.j    -0.228-0.j    -0.229-0.02j  -0.229-0.j    -0.23 -0.j    -0.2
31-0.j    -0.233-0.j    -0.233-0.02j  -0.234-0.j    -0.237-0.j    -0.237-0.019j -0.239-0.j
 -0.241-0.018j -0.242-0.j    -0.245-0.018j -0.246-0.j    -0.248-0.017j -0.25 -0.j    -0.252-0.017j -
0.254-0.j    -0.256-0.016j -0.258-0.j    -0.259-0.015j -0.263-0.015j -0.263-0.j    -0.266-0.014j -0.2
69-0.j    -0.27 -0.014j -0.273-0.013j -0.274-0.j    -0.277-0.013j -0.28 -0.012j -0.28 -0.j
 -0.283-0.011j -0.286-0.011j -0.287-0.j    -0.289-0.01j  -0.292-0.01j  -0.294-0.j    -0.295-0.009j -
0.298-0.009j -0.301-0.008j -0.304-0.008j -0.306-0.008j -0.309-0.007j -0.311-0.007j -0.314-0.006j -0.3
16-0.006j -0.318-0.005j -0.321-0.005j -0.323-0.004j -0.325-0.004j -0.327-0.004j -0.329-0.003j
 -0.331-0.003j -0.332-0.003j -0.334-0.002j -0.336-0.002j -0.337-0.002j -0.339-0.001j -0.34 -0.001j -
0.341-0.001j -0.342-0.j   ]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [-0.001+0.001j -0.001-0.002j  0.003-0.005j  0.002-0.006j  0.001+0.001j  0.004-0.009j -0.002+0.006j
0.004+0.004j -0.003+0.01j  -0.001+0.004j -0.006+0.019j  0.003+0.021j  0.006+0.024j -0.   +0.014j  0.0
02+0.j     0.002+0.015j  0.002+0.027j  0.003+0.017j  0.011+0.006j  0.01 +0.019j  0.004+0.002j
 -0.001-0.002j  0.006-0.005j  0.001+0.021j  0.003-0.003j -0.   +0.011j  0.001-0.006j  0.005-0.007j
0.006+0.003j  0.003-0.008j  0.002+0.017j  0.001+0.002j  0.001-0.012j  0.004-0.003j  0.008+0.015j  0.0
08-0.004j  0.004+0.003j  0.003+0.012j  0.001+0.031j -0.002-0.008j -0.001+0.01j  -0.002-0.003j
  0.001-0.022j  0.005-0.01j   0.003+0.006j  0.004-0.015j  0.006-0.01j   0.005-0.012j  0.006+0.01j
0.004-0.018j  0.005-0.018j  0.004+0.007j  0.005+0.003j  0.005+0.022j  0.006+0.j     0.005+0.009j  0.0
06-0.001j  0.006+0.028j  0.005-0.007j  0.003+0.008j  0.004-0.01j   0.003+0.019j  0.002+0.005j
  0.002-0.01j   0.002-0.014j -0.001+0.003j  0.   -0.016j  0.001-0.003j -0.001-0.028j -0.001-0.024j -
0.002-0.015j  0.001-0.026j  0.002-0.017j  0.002-0.031j  0.001-0.025j  0.001-0.003j -0.003-0.019j -0.0
06-0.013j -0.006+0.026j -0.005-0.026j -0.003-0.012j -0.005-0.j    -0.005+0.039j -0.005-0.007j
 -0.004-0.036j -0.003-0.009j -0.006-0.015j -0.006-0.001j -0.005-0.019j -0.004-0.006j -0.004+0.004j -
0.004-0.015j -0.004-0.024j -0.004-0.003j -0.004-0.02j  -0.002-0.019j -0.003+0.007j -0.002-0.026j -0.0
02-0.029j -0.002+0.01j  -0.007-0.016j -0.006-0.022j -0.005-0.02j  -0.009+0.005j -0.009-0.018j
 -0.008-0.025j -0.007-0.026j -0.009-0.012j -0.01 +0.042j -0.011-0.026j -0.011-0.018j -0.012-0.01j  -
0.012-0.011j -0.012+0.011j -0.012-0.015j -0.015-0.006j -0.015-0.008j -0.016-0.011j -0.02 +0.013j -0.0
22-0.015j -0.02 -0.009j -0.021+0.003j -0.022-0.008j -0.02 +0.009j -0.019-0.014j -0.02 +0.006j
 -0.022-0.01j  -0.023+0.006j -0.023-0.007j -0.023+0.028j -0.022+0.003j -0.021-0.016j -0.022+0.044j -
0.023+0.005j -0.023+0.024j -0.021-0.009j -0.021+0.005j -0.018+0.009j -0.018-0.009j -0.019+0.02j  -0.0
22+0.01j  -0.023-0.01j  -0.022+0.011j -0.02 +0.022j -0.024+0.003j -0.023-0.006j -0.021+0.001j
 -0.02 +0.003j -0.021+0.014j -0.021-0.01j  -0.02 +0.04j  -0.018+0.003j -0.016+0.003j -0.016-0.006j -
0.015+0.023j -0.013+0.003j -0.011+0.002j -0.009+0.02j  -0.009-0.008j -0.008+0.003j -0.006-0.006j -0.0
05+0.02j  -0.004+0.018j -0.003-0.002j -0.001+0.037j -0.   +0.j     0.002+0.017j  0.002+0.j
  0.005-0.j     0.007-0.001j  0.007+0.016j  0.01 -0.003j  0.012+0.j     0.012+0.015j  0.014-0.j
0.016+0.j     0.016+0.032j  0.017+0.014j  0.017-0.001j  0.019-0.004j  0.021-0.003j  0.021+0.004j  0.0
22-0.003j  0.022+0.019j  0.021+0.003j  0.021-0.008j  0.021+0.007j  0.022-0.006j  0.021-0.01j
  0.021+0.03j   0.019-0.016j  0.017+0.006j  0.017-0.013j  0.018-0.01j   0.017-0.017j  0.018-0.016j
0.015+0.008j  0.017+0.005j  0.018+0.026j  0.018+0.006j  0.021+0.005j  0.022+0.001j  0.021+0.018j  0.0
```

```
19-0.003j  0.022+0.002j  0.02 +0.026j  0.022-0.002j  0.021+0.02j   0.022+0.005j  0.021+0.026j
  0.017-0.004j  0.019+0.j     0.02 +0.025j  0.019-0.004j  0.015+0.017j  0.016-0.004j  0.017+0.004j
0.017+0.016j  0.019-0.008j  0.016+0.016j  0.016-0.001j  0.016+0.002j  0.015+0.024j  0.017-0.006j  0.0
14+0.006j  0.014-0.004j  0.017-0.002j  0.015+0.017j  0.015-0.008j  0.016+0.003j  0.015+0.023j
  0.007-0.001j  0.009+0.015j  0.006-0.015j  0.003-0.001j  0.004+0.009j  0.001-0.013j  0.002-0.002j
0.003+0.007j  0.004-0.024j  0.006-0.014j  0.002+0.007j  0.001-0.021j  0.003-0.012j  0.003+0.015j  0.0
04-0.001j  0.004-0.011j  0.001+0.005j -0.    -0.017j  0.002-0.011j  0.003+0.002j  0.001-0.018j
 -0.001-0.014j -0.001-0.012j -0.001+0.008j -0.002-0.015j -0.004-0.019j -0.003-0.01j  -0.003-0.001j -
0.002-0.021j -0.004-0.012j -0.003-0.018j -0.003-0.001j -0.003+0.011j -0.008-0.016j -0.01 -0.012j -0.0
09-0.021j -0.007-0.019j -0.009+0.004j -0.008-0.025j -0.009-0.024j -0.009+0.004j -0.007+0.001j
 -0.007+0.009j -0.009-0.004j -0.008-0.007j -0.008-0.015j -0.008+0.007j -0.009-0.01j  -0.008+0.013j -
0.006-0.01j  -0.005+0.008j -0.008-0.013j -0.009-0.002j -0.006+0.001j -0.007+0.001j -0.004+0.012j -0.0
02-0.015j -0.004+0.002j -0.002+0.004j -0.003-0.008j -0.003+0.002j -0.001-0.009j -0.003-0.017j
 -0.001+0.007j  0.001+0.004j -0.003-0.002j -0.002+0.006j  0.001+0.01j  -0.003+0.001j -0.004+0.009j -
0.003+0.004j -0.001+0.003j -0.001+0.005j -0.001+0.007j  0.001+0.004j  0.001-0.003j  0.001+0.007j -0.0
01+0.005j -0.001+0.009j  0.001+0.011j -0.002+0.009j -0.001+0.004j  0.    +0.002j  0.002-0.001j
 -0.001+0.001j -0.001+0.005j  0.    +0.008j  0.001+0.011j  0.001+0.004j  0.    +0.003j -0.    +0.01j  -
0.   +0.017j -0.001+0.012j]

max abs diff =  0.04890143704273701


complex128
```

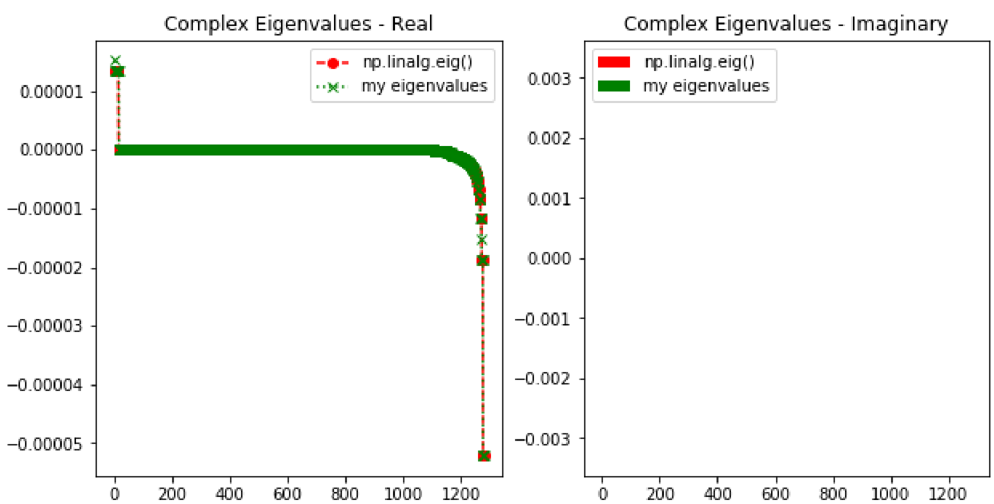


Observation

- QR iteration with Wilkinson's shift seems to be working for complex symmetric matrix
- However, not clear if convergence is faster than general method without shift
- Max iteration already increased to 1e4; execution takes a long time... #

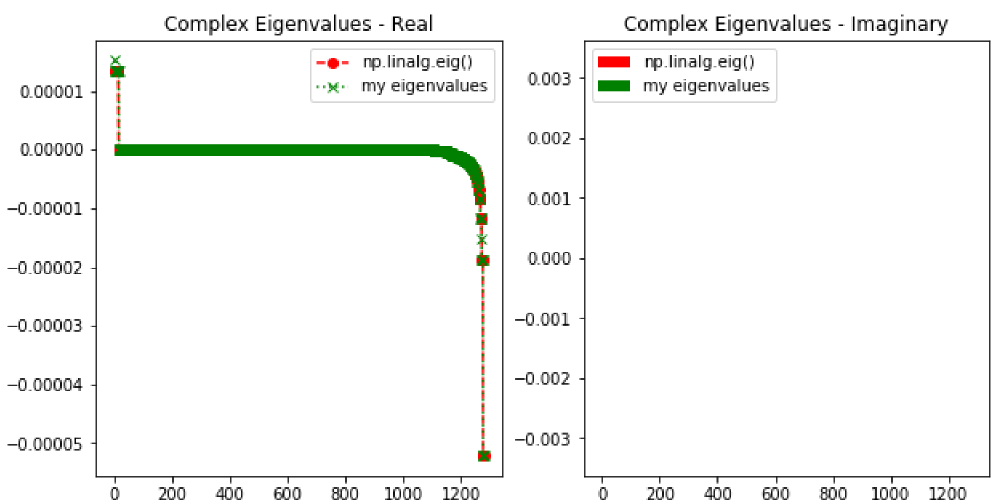
In [84]:
```python
MAX_ITER = 10000

# Complex unsymmetric
source = 'matricmarket/mhd1280a.mtx.gz'
matrix_class = Matrix
runtest_matrixmarket(source, matrix_class,shift=None)
```

```
From Matrix Market:
matrix source=matricmarket/mhd1280a.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=
(1280, 1280), data type=complex128, max=(74619.3808+3.84011238e-11j), min=(-15906.7974-2.73633283e-07
j)
is_symmetric={} False
activiating tridigonal: False

 ------------ Processing eigenvalues only ------------
>>> QR Iteration WITHOUT shift
max_iter=10000 reached, max error=34.74374895817617 vs tol=1e-06; index of non-convergence=[[105], [1
06], [107], [108], [154], [155], [156], [157], [245], [246], [323], [324], [401], [402]]

 ------------ Checking Eigenvalues ------------

my eigenvalues: [ 0.-0.j     0.+0.j     0.+0.j    ... -0.-0.003j -0.-0.003j -0.+0.003j]
np.linalg.eig:  [ 0.+0.j     0.+0.j     0.+0.j    ... -0.-0.003j -0.-0.003j -0.+0.003j]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [ 0.-0.j  0.+0.j  0.+0.j ... -0.-0.j  0.+0.j  0.-0.j]

max abs diff =  0.0015719045645541168



complex128
```



In [89]:
```python
MAX_ITER = 10000

# Complex unsymmetric
source = 'matricmarket/mhd1280a.mtx.gz'
matrix_class = Matrix
runtest_matrixmarket(source, matrix_class,shift='Wilkinson')
```

```
From Matrix Market:
matrix source=matricmarket/mhd1280a.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=
(1280, 1280), data type=complex128, max=(74619.3808+3.84011238e-11j), min=(-15906.7974-2.73633283e-07
j)
is_symmetric={} False
activiating tridigonal: False

 ------------ Processing eigenvalues only ------------
>>> QR Iteration with Wilkinson shift
max_iter=10000 reached, max error=34.7437488841539 vs tol=1e-06; index of non-convergence=[[105], [10
6], [107], [108], [154], [155], [156], [157], [245], [246], [323], [324], [401], [402]]

 ------------ Checking Eigenvalues ------------

my eigenvalues: [ 0.-0.j     0.+0.j     0.+0.j    ... -0.-0.003j -0.-0.003j -0.+0.003j]
np.linalg.eig:  [ 0.+0.j     0.+0.j     0.+0.j    ... -0.-0.003j -0.-0.003j -0.+0.003j]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [ 0.-0.j  0.+0.j  0.+0.j ... -0.-0.j  0.+0.j  0.-0.j]

max abs diff =  0.0015719045645541184



complex128
```



Observation

- Increased max iteration number to 1e4
- General QR iteration without shift seems to be working for complex unsymmetric matrix
- Not clear whether QR iteration with Wilkinson shift converges faster...

#

In [ ]:

In [ ]:

In [ ]:

```
In [ ]:
```

Checklist / To-Dos:

Option 1

General complex matrices with transformation to Hessenberg form

- Hessenberg form via **Housholder** Tranformation
  $$\color{blue}{Done}$$
- QR Iteration with **Givens** Rotation
  $$\color{blue}{Done}$$
- Test for **convergence**
  $$\color{blue}{Done}$$
- Modify Householder to accommodate for complex matrices
  $$\color{blue}{Done}$$
- Modify Givens to accommodate for complex matrices
  $$\color{blue}{Done}$$
- **Diagonal blocks** for real matrix with complex conjugate eigenvalues
  $$\color{blue}{Done}$$
- Use **Wilkinson shifts** to accelerate convergence
  $$\color{blue}{Done}$$
- Compute not only eigenvalues but also **eigenvectors**
  $$\color{blue}{Done}$$
- Apply your program to several suitable matrices from the **Matrix Market**. Use suitable library functions for reading the matrices.
  $$\color{blue}{Done}$$

Option 2

General real matrices with transformation to Hessenberg form and double shifts for complex eigenvalues

- Use **double shifts** to avoid complex arithmetic _ Matrix multiplication
  $$\color{blue}{Done}$$
- **"Bulge-chasing"** for element-wise computation **#TODO**

Option 3

Symmetric real matrices with transformation to tridiagonal form with optimized storage

- Use **deflation** when the subdiagonal element in the last row is sufficiently small (implementation may be hard if not option 3)
- Use **deflation**, if any other subdiagonal element is small (seems very hard)
- Add **SVD** capability to option 3

Additional Considerations

- Well-chosen tests for correctness
- Investigation into convergence
- Write very well structured code
- Well-prepared jupyter notebooks with code, accompanying text, and results
- **Object-oriented programming**: is it possible to implement algorithm 1.4.14 in a way, that H is either Hessenberg or tridiagonal, either real or complex, but encapsulate the differences inside the QR-decompositions applied in every step **#TODO**
