Numerical Linear Algebra - EXAM

- Submitted by Claire SUN (3630998)
- Submission Date: 2022.02.12

Declaration:

I have prepared the assignment myself and I have only used the sources declared in comments to the program

[Programming Assignment Instructions (qr-method.pdf)](qr-method.pdf)

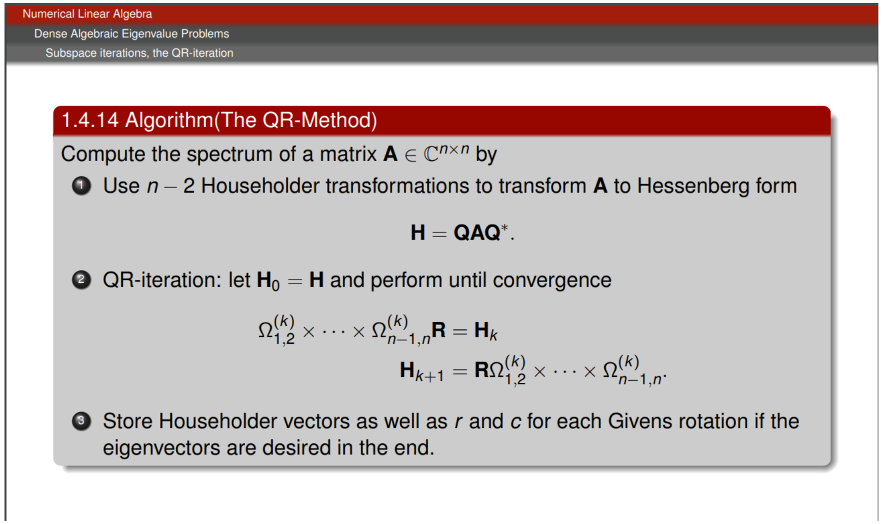


Image source: Lecture Notes

In [1]:
```python
# Import Libraries

import numpy as np
import cmath
import matplotlib.pyplot as plt
from scipy.linalg import hessenberg
from scipy.io import mmread
```

In [2]:
```python
# Set Global Variables

MAX_ITER = 1000 # maximum number of iterations for testing
RTOL = 1e-6 # relative tolerance for convergence, i.e., when subsequent iterates are smaller than RTOL, stop iteration
ATOL = 1e-18 # same as numpy: absolute(a - b) <= (atol + rtol * absolute(b))
np.set_printoptions(precision=3, linewidth=300, suppress=True)
```

In [3]:
```python
# Generate Test Data

def generate_random_matrix(dtype, n):
    if dtype == 'float':
        A = np.random.rand(n,n)
    elif dtype == 'complex':
        A = np.random.rand(n,n) + np.random.rand(n,n)*1j
    return A

RANDOM_REAL_MATRICES = []
RANDOM_COMPLEX_MATRICES = []

for n in [3,5,10]:
    A = generate_random_matrix(dtype='float', n=n)
    RANDOM_REAL_MATRICES.append(A)

    A_ = generate_random_matrix(dtype='complex', n=n)
    RANDOM_COMPLEX_MATRICES.append(A_)

# symmetric real
A = np.random.randint(-3,3,(10,10))
A = (A.T + A).astype('float')

# diagonal dominant
B = A - np.triu(A,1)*0.99 - np.tril(A,-1)*0.99

# unbalanced
C = np.tril(np.ones((10,10),dtype='float'),0)

# Numerical example from https://people.inf.ethz.ch/arbenz/ewp/Lnotes/chapter4.pdf p.81 4.5.1
# Eigenvalues = {1 ± 2i, 3, 4, 5 ± 6i}

D = np.array([[7, 3, 4, -11, -9, -2],
              [-6, 4, -5, 7, 1, 12],
              [-1, -9, 2, 2, 9, 1],
              [-8, 0, -1, 5, 0, 8],
              [-4, 3, -5, 7, 2, 10],
              [6, 1, 4, -11, -7, -1]], dtype='float')

SPECIAL_REAL_MATRICES = [A, B, C, D]
```

Householder Transformation to Upper Hessenberg

Similarity Transformation

Two matrices A and B are similar if there exists an *invertible* matrix S such that $A = S^{-1}BS$.

Similar matrices have the same eigenvalues. Proof: $Av = \lambda v \Leftrightarrow B(S^{-1}v) = \lambda(S^{-1}v)$.

If S can be chosen to be a unitary matrix then A and B are *unitarily equivalent*.

Similarity transformation to Hessenberg form preserves eigenvalues.

The iterates $A_0, A_1, \ldots$ from the QR algorithm are also similar matrices since $A_{k+1} = R_k Q_k = Q_k^{-1}(Q_k R_k)Q_k = Q_k^{-1}A_k Q_k$. Therefore, $A_0, A1, \ldots,$ have the same eigenvalues.

Householder Reflector

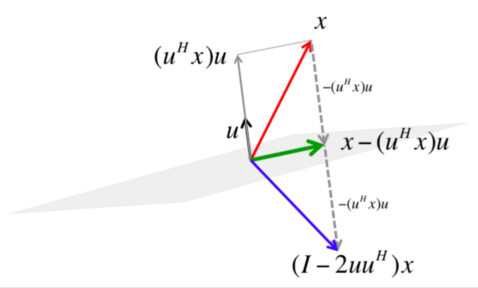


Image source: [https://www.cs.utexas.edu/users/flame/laff/alaff/chapter03-householder-transformation.html (https://www.cs.utexas.edu/users/flame/laff/alaff/chapter03-householder-transformation.html)](https://www.cs.utexas.edu/users/flame/laff/alaff/chapter03-householder-transformation.html)

Hessenberg Reduction

$$A = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & \hat{P}_1 \end{bmatrix} A = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & \hat{P}_1 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & \hat{P}_1 \end{bmatrix}^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{bmatrix} \Rightarrow P_4 \cdots P_1 A P_1^{-1} \cdots P_4^{-1} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * & * \end{bmatrix}$$

Therefore, compute a Householder matrix $\hat{P}_1 \in \mathbb{C}^{(n-1)\times(n-1)}$ that maps the **red** column vector to the first unit coodinate vector and consider

$$P_1 = \begin{bmatrix} 1 & 0 \\ 0 & \hat{P}_1 \end{bmatrix}.$$

Summary: The algorithms computes a Hessenberg matrix that is similar to A in finitely many steps.

cost: ca. $\frac{10}{3}n^3$ flops

Modified based on image source: [https://www3.math.tu-berlin.de/Vorlesungen/SoSe11/NumMath2/Materials/hessenberg_eng.pdf (https://www3.math.tu-berlin.de/Vorlesungen/SoSe11/NumMath2/Materials/hessenberg_eng.pdf)](https://www3.math.tu-berlin.de/Vorlesungen/SoSe11/NumMath2/Materials/hessenberg_eng.pdf)

In [42]:
```
def Householder(a):

    """     Compute the Householder refelector H for a given vector a
            H = I - 2 u @ u.T ;  a' = H @ a
    Input   ---------------------------------------------------------
            a: 1D np.array with real or complex entries, size n
    Return  ---------------------------------------------------------
            H: 2D np.array with real or complex entries, size nxn
    """

#    Initial try: naive implementation for real vectors

#    n = len(a)
#    e = np.zeros(n)
#    e[0] = np.sign(a[0])
#    v = a + np.linalg.norm(a,2) * e
#    H = np.eye(n) - 2* np.outer(v,v) / np.dot(v,v)


#   Improved version: for both real and complex vectors; element update rather than entire vector

    n = len(a)
    v = a.copy()

    if np.sign(a[0]) >= 0: #sign of the first element to avoid cancellation
        sign = 1
    else:
        sign = -1

    # for real matrix: v @ v.T (transpose)

    if a.dtype != 'complex':
        v[0] = a[0] + sign * np.linalg.norm(a,2)
        if np.linalg.norm(v,2)!= 0:
            u = v / np.linalg.norm(v,2)
        else:
            u = v
        H = np.eye(n) - 2 * np.outer(u,u)


    # for complex matrix: v @ v.conjugate.T, phase cancellation so that the first subdiagonal element should be real
    # (Source: https://arxiv.org/pdf/math-ph/0609050.pdf p.19)

    else: # a.dtype == 'complex':

        theta = cmath.phase(a[0])
        v[0] = a[0] + sign * np.linalg.norm(a,2) * np.exp(theta*1j)

        if np.linalg.norm(v,2)!= 0:
            u = v / np.linalg.norm(v,2)
        else:
            u = v
```

```

        H = np.exp(-theta*1j)*(np.eye(n) - 2* np.outer(u,u.conjugate()))

        #print(">>> check if H v = e1:", np.round(H.dot(a),3))
        #print('>>> check if H is Hermitian:',np.allclose(H@H.conjugate().T, np.eye(n))) # H @ H.conjugate().T = I

    return H



def Hessenberg(A,eigvec=False, inplace=True):

    """ Reduce a general square matrix to an upper Hesseberg matrix by similarity transformation
    Input   ----------------------------------------------
            A: 2D np.array with real or complex entries, shape is n x n (i.e. square matrix)
            eigvec : bool, indicating whether eigenvectors are required; default is False
            inplace: bool, indicating whether to update A in place; default is True
    Return  ---------------------------------------------
            H: 2D np.array with real or complex entries, same shape as A
            S: similarity transformation matrix such that H = S @ A @ S.T, if eigvec is True; otherwise S = identity matrix
    """

    # check if input is a square matrix
    if len(A.shape)!=2 or A.shape[0]!=A.shape[1]:
        raise ValueError('Input matrix is of shape {}. Expected square matrix.'.format(A.shape))

    n = A.shape[0]

    # if input matrix is 2x2 or smaller: already in Hessenberg
    if n <= 2:
        return A, np.eye(n)

    S = np.eye(n, dtype=A.dtype)

    if not inplace:
        A_ = A.copy()
    else:
        A_ = A

    for i in range(n-1): # (n-2) Householder transformations are needed
        a = A_[i+1:,i] # a = column vectors below diagonal, size (n-i-1), i = 0,...,n-2
        H_ = Householder(a)

        P = np.eye(n, dtype = A_.dtype) # Force P to have the same data type as input matrix
        P[i+1:,i+1:] = H_
        #print(">>> check if P is Hermitian and orthogonal:",np.allclose(P@P.conjugate().T, np.eye(n)))

        if A_.dtype != 'complex':
            A_ = P @ A_ @ P.T
        else:
            A_ = P @ A_ @ P.conjugate().T

```

```
        #The first subdiagonal element should have only a real part; enforced here
        A_[i+1,i] = A_[i+1,i].real
        #A_ = np.triu(A_,-1)

        if eigvec:
            S = P @ S

    return A_, S

```

In [5]:

```python
### Testing Hessenberg() on Real Matrices
#from scipy.linalg import hessenberg

print('\n\n')
print("====================================================================")
print('Testing Hessenberg() function on randomly generated REAL matrices')
print("====================================================================")

#for n in [3,5,10,100]:
#    A = np.random.rand(n,n)

for A in RANDOM_REAL_MATRICES:
    print('\nReal >>> Input shape = {}'.format(A.shape))
    Hessenberg_A, S = Hessenberg(A,eigvec=True, inplace=False)
    scipy_hess_A = hessenberg(A)
    print('\nscipy.linalg.hessenberg and my Hessenberg close?',np.allclose(scipy_hess_A,Hessenberg_A))
    if not np.allclose(scipy_hess_A,Hessenberg_A):
        print('scipy.linalg.hessenberg and my Hessenberg close in abs() except signs?',
              np.allclose(np.abs(scipy_hess_A),np.abs(Hessenberg_A)))
    print('\nReconstructed matrix S.T@H@S close to input?',np.allclose(A, S.T@Hessenberg_A@S))

#    n = A.shape[0]
#    if n <=5:
#        print('\ninput matrix=\n', A)
#        print('\nreconstructed matrix=\n',S.T@Hessenberg_A@S)
#        print('\nscipy.linalg.hessenberg=\n',scipy_hess_A)
#        print('\nmy Hessenberg=\n',Hessenberg_A)


print('\n\n')
print("====================================================================")
print('Testing Hessenberg() function on special REAL matrices')
print("====================================================================")

for A in SPECIAL_REAL_MATRICES:
    print('\nReal >>> Input shape = {}'.format(A.shape))
    Hessenberg_A, S = Hessenberg(A,eigvec=True, inplace=False)
    scipy_hess_A = hessenberg(A)
    print('\nscipy.linalg.hessenberg and my Hessenberg close?',np.allclose(scipy_hess_A,Hessenberg_A))
    if not np.allclose(scipy_hess_A,Hessenberg_A):
        print('scipy.linalg.hessenberg and my Hessenberg close in abs() except signs?',
              np.allclose(np.abs(scipy_hess_A),np.abs(Hessenberg_A)))
    print('\nReconstructed matrix S.T@H@S close to input?',np.allclose(A, S.T@Hessenberg_A@S))

print('\ninput matrix=\n', A)
print('\nreconstructed matrix=\n',S.T@Hessenberg_A@S)
print('\nscipy.linalg.hessenberg=\n',scipy_hess_A)
print('\nmy Hessenberg=\n',Hessenberg_A)

```

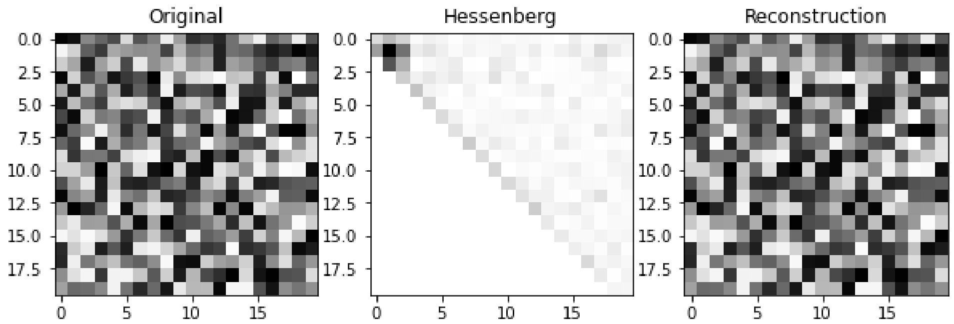
```
====================================================================
Testing Hessenberg() function on randomly generated REAL matrices
====================================================================

Real >>> Input shape = (3, 3)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True

Reconstructed matrix S.T@H@S close to input? True

Real >>> Input shape = (5, 5)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True

Reconstructed matrix S.T@H@S close to input? True

Real >>> Input shape = (10, 10)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True

Reconstructed matrix S.T@H@S close to input? True



====================================================================
Testing Hessenberg() function on special REAL matrices
====================================================================

Real >>> Input shape = (10, 10)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True

Reconstructed matrix S.T@H@S close to input? True

Real >>> Input shape = (10, 10)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True

Reconstructed matrix S.T@H@S close to input? True

Real >>> Input shape = (10, 10)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True
```

```
Reconstructed matrix S.T@H@S close to input? True

Real >>> Input shape = (6, 6)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in abs() except signs? True

Reconstructed matrix S.T@H@S close to input? True

input matrix=
 [[  7.   3.   4. -11.  -9.  -2.]
 [ -6.   4.  -5.   7.   1.  12.]
 [ -1.  -9.   2.   2.   9.   1.]
 [ -8.   0.  -1.   5.   0.   8.]
 [ -4.   3.  -5.   7.   2.  10.]
 [  6.   1.   4. -11.  -7.  -1.]]

reconstructed matrix=
 [[  7.   3.   4. -11.  -9.  -2.]
 [ -6.   4.  -5.   7.   1.  12.]
 [ -1.  -9.   2.   2.   9.   1.]
 [ -8.  -0.  -1.   5.  -0.   8.]
 [ -4.   3.  -5.   7.   2.  10.]
 [  6.   1.   4. -11.  -7.  -1.]]

scipy.linalg.hessenberg=
 [[  7.      7.276   5.812  -0.14    9.015   7.936]
 [ 12.369   4.131  18.969  -1.207  10.683   2.416]
 [  0.     -7.16    2.448  -0.566  -4.181  -3.251]
 [  0.      0.     -8.599   2.915  -3.417   5.723]
 [  0.      0.      0.      1.046  -2.835 -10.979]
 [  0.      0.      0.      0.      1.414   5.342]]

my Hessenberg=
 [[ 7.     7.276  5.812 -0.14   9.015 -7.936]
 [12.369  4.131 18.969 -1.207 10.683 -2.416]
 [ 0.    -7.16   2.448 -0.566 -4.181  3.251]
 [-0.     0.    -8.599  2.915 -3.417 -5.723]
 [-0.    -0.    -0.     1.046 -2.835 10.979]
 [-0.    -0.     0.    -0.    -1.414  5.342]]
```

In [6]:

```python
# Visualiation of Tranforming Matrix A to Hessenberg form via Householder Reflection

f, (ax0, ax1, ax2) = plt.subplots(1,3,figsize=(10,20))

n = 20
A = np.random.rand(n,n)
H, S = Hessenberg(A,eigvec=True, inplace=False)
recon = S.T @ H @ S

im = ax0.imshow(-abs(A),cmap='gray')
ax0.set_title('Original')
ax1.imshow(-abs(H),cmap='gray')
ax1.set_title('Hessenberg')
ax2.imshow(-abs(recon),cmap='gray')
ax2.set_title('Reconstruction')

print("Illustration of Transformation to Hessenberg Form ")

```

```
Illustration of Transformation to Hessenberg Form
```



Observations:

- Hessenberg() function seems to be working for real matrices

In [7]:

```python
### Testing Hessenberg() on Random Complex Matrices

print('\n\n')
print("====================================================================")
print('Testing Hessenberg() function on randomly generated COMPLEX matrices')
print("====================================================================")

#for n in [3,5,10,100]:
#    A = np.random.rand(n,n)+np.random.rand(n,n)*1j

for A in RANDOM_COMPLEX_MATRICES:
    print('\n\nComplex >>> Input shape = {}'.format(A.shape))
    Hessenberg_A, S = Hessenberg(A,eigvec=True,inplace=False)
    scipy_hess_A = hessenberg(A)
    print('\nscipy.linalg.hessenberg and my Hessenberg close?',np.allclose(scipy_hess_A,Hessenberg_A))
    if not np.allclose(scipy_hess_A,Hessenberg_A):
        #print('scipy.linalg.hessenberg and my Hessenberg close in abs() except signs?',np.allclose(np.abs(scipy_hess_A),np.abs(Hessenberg_A)))
        print('scipy.linalg.hessenberg and my Hessenberg close in absolute values of re() and im()?',
              np.allclose(np.abs(np.real(scipy_hess_A)),np.abs(np.real(Hessenberg_A))),
              np.allclose(np.abs(np.imag(scipy_hess_A)),np.abs(np.imag(Hessenberg_A))))
        #print(np.real(scipy_hess_A),'\n',np.real(Hessenberg_A),'\n\n',np.imag(scipy_hess_A),'\n',np.imag(Hessenberg_A))
    print('\nReconstructed matrix S.T@H@S close to input?',np.allclose(A, S.conjugate().T@Hessenberg_A@S))

#    n = A.shape[0]
#    if n <=5:
#        print('\ninput matrix=\n', A)
#        print('\nreconstructed matrix=\n',S.conjugate().T@Hessenberg_A@S)
#        print('\nscipy.linalg.hessenberg=\n',scipy_hess_A)
#        print('\nmy Hessenberg=\n',Hessenberg_A)


```

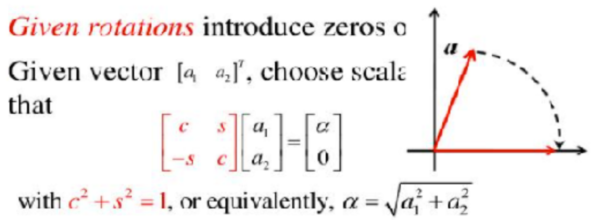
```
====================================================================
Testing Hessenberg() function on randomly generated COMPLEX matrices
====================================================================


Complex >>> Input shape = (3, 3)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in absolute values of re() and im()? True True

Reconstructed matrix S.T@H@S close to input? True


Complex >>> Input shape = (5, 5)

scipy.linalg.hessenberg and my Hessenberg close? False
```

```
scipy.linalg.hessenberg and my Hessenberg close in absolute values of re() and im()? True True

Reconstructed matrix S.T@H@S close to input? True


Complex >>> Input shape = (10, 10)

scipy.linalg.hessenberg and my Hessenberg close? False
scipy.linalg.hessenberg and my Hessenberg close in absolute values of re() and im()? True True

Reconstructed matrix S.T@H@S close to input? True
```

Observations:

- Hessenberg() function works for complex matrices as well
- Modified for the optional output of the similarity transformation matrix S

Givens Rotation



$$G(i,k,\theta) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & -s & \cdots & c & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

$$G_{1,m}A = \begin{bmatrix} * & \cdots & \cdots & * \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & \cdots & * \\ 0 & * & \cdots & * \end{bmatrix}$$

$$G_1 A = \begin{bmatrix} * & \cdots & \cdots & * \\ 0 & * & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{bmatrix}$$

$$G_1 := G_{1,2} \cdot \ldots \cdot G_{1,m}.$$

Modified from Image Source: *https://www.slideserve.com/pekelo/scientific-computing-chapter-3-linear-least-squares (https://www.slideserve.com/pekelo/scientific-computing-chapter-3-linear-least-squares)*, *https://de.wikipedia.org/wiki/Givens-Rotation (https://de.wikipedia.org/wiki/Givens-Rotation)*

If x_j and x_k are real numbers:

- $\begin{bmatrix} c & s \\ -s & c \end{bmatrix}\begin{bmatrix} x_j \\ x_k \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$, where $r = \sqrt{x_j^2 + x_k^2}$, $c = x_j/r$ and $s = x_k/r$

If x_j and x_k are complex numbers:

- $\begin{bmatrix} c & s \\ -\bar{s} & c \end{bmatrix}\begin{bmatrix} x_j \\ x_k \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}$, where

- $c = x_j^* / \sqrt{|x_j|^2 + |x_k|^2} = cos\theta_a e^{-i\theta_j}$
- $s = x_k^* / \sqrt{|x_j|^2 + |x_k|^2} = sin\theta_a e^{-i\theta_k}$
- $\theta_a = arctan(|x_k|/|x_j|) = arctan(Re(x_k)/cos\theta_k \times cos\theta_j / Re(x_j))$

Source: [https://amir.sdsu.edu/Wen17A.pdf (https://amir.sdsu.edu/Wen17A.pdf)](https://amir.sdsu.edu/Wen17A.pdf), page 2, equations (7) and (8)

In [9]:
```python
def Givens(x_j,x_k):

    """ Find the Givens rotation matrix such that G @ [x_j,x_k] = [*, 0]
    Input   ---------------------------------------------
            x_j: jth row/column element from a np.array with real or complex entries
            x_k: kth row/column element from a np.array with real or complex entries

    Return  ---------------------------------------------
            G: 2x2 rotation matrix in np.array with real or complex entries
    """


    if x_k == 0. or x_k.real == 0: #x_k== 0+0*1j : # including the case x_j=0 and x_k=0
        c = 1.
        s = 0.

    elif x_j == 0. or x_j.real == 0: #x_j == 0+0*1j : # i.e. x_k!=0
        c = 0.
        s = 1.

    else: # f, g both non-zero
        theta_j = cmath.phase(x_j)
        theta_k = cmath.phase(x_k)
        theta_a = np.arctan(x_k.real/np.cos(theta_k)*np.cos(theta_j)/x_j.real) #div by zero excluded from condition above
        c = np.cos(theta_a)*np.exp(-1j*theta_j)
        s = np.sin(theta_a)*np.exp(-1j*theta_k)

    if x_j.dtype == 'complex' or x_k.dtype == 'complex':
        G = np.array([[c,s],[-s.conjugate(),c.conjugate()]]).astype('complex')

    else:
        G = np.array([[c,s],[-s,c]]).astype('float')

    return G



# Previous unsuccessful attempts
# (Source: https://www.netlib.org/lapack/lawnspdf/lawn148.pdf, p.5) -> implementation does not work for complex matrices.

#def __Givens__(f,g):
#    if g == 0:
#        c = 1
#        s = 0
#        r = f
#    elif f == 0: # g!=0
#        c = 0
#        s = np.sign(g.conjugate())
#        r = np.abs(g)
#    else: # f, g both non-zero
#        d = np.sqrt(np.abs(f)**2 +np.abs(g)**2 )
#        c = np.abs(f) / d
```

```
#        s = np.sign(f) * (g.conjugate()) / d
#        r = np.sign(f) * d
#    G = np.array([[c,s],[-s.conjugate(),c]])
#    return G


#def Givens_real(x_j, x_k):
#    """Avoiding squaring of small numbers
#    TODO: Need to adapt for complex numbers
#    """
#    # if the subdiagonal is already zero or close enough to zero, just return identity matrix
#    if abs(x_j) > abs(x_k) and x_j!=0:
#        tan = x_k / x_j
#        c = 1/np.sqrt(1+np.square(tan))
#        s = tan*c
#    else:
#        cotan = x_j / x_k
#        s = 1/np.sqrt(1+np.square(cotan))
#        c = cotan*s
#    G = np.array([[c,s],[-s,c]])
#    return G


#def QR_Givens_naive(H):
#    '''Input: matrix H in Hessenberg form
#       Matrix multiplication directly
#    '''
#    n = H.shape[0]
#    G = np.eye(n)
#    for i in range(n-1): # apply (n-1) Givens rotations to zero-out the sub-diagonal elements in Hessenberg matrix
#        G_ = np.eye(n)
#        G_[i:i+2, i:i+2] = Givens(H[i,i],H[i+1,i])
#        G = G_ @ G
#    R = G @ H
#    Q = G.conjugate().T
#    return Q, R

# [Gn ... G2 G1].T [Gn ... G2 G1] H = H;


#def QR_Givens_rows(H,eigvec=False):
#    '''Row-wise application of Givens rotation (2 rows at a time) rather than whole matrix multiplication
#    Input: H = numpy array, Hessenberg matrix
#    eigvec = Bool, indicating whether eigenvectors is required; if True, store and output all Givens matrics
#    Return: Updated H_new overwritten on H_old // no explicit calculation of Q and R matrices
#    '''
#    n = H.shape[0]
#    G_store = []
#    G_iter = np.eye(n,dtype=H.dtype)
#
#    for i in range(n-1):
#        G = Givens(H[i,i],H[i+1,i])
```

```
#        new_rows = G @ H[i:i+2, :]
#        H[i:i+2, :] = new_rows
#        G_store.append(G)
#
#        # if eigenvectors are required
#        if eigvec:
#            G_ = np.eye(n,dtype=H.dtype)
#            G_[i:i+2,i:i+2] = G
#            G_iter = G_ @ G_iter
#
#    for i in range(n-1):
#        G = G_store.pop(0) #same order as above G1, G2, ...
#
#        if G.dtype != 'complex':
#            new_cols = H[:, i:i+2] @ G.T
#        else:
#            new_cols = H[:, i:i+2] @ G.conjugate().T
#
#        H[:, i:i+2] = new_cols
#
#    return H, G_iter



def select_idx(i,n,tridiagonal):

    """ Find the start and end indices for element-wise application of Givens Rotation
    Input  ----------------------------------------------
            i: ith row/column of current application
            n: total number of rows / columns
            tridiagonal: bool, indicating whether the matrix is tridiagonal
    Return ---------------------------------------------
            row_start: start index for row operations
            col_end: end index for column operations
    """

    if tridiagonal: # only operates on 3 elements per row / column
        row_start = max(0,i-1) # for post multiplication of G.T
        col_end = i+3

    else:
        row_start = 0
        col_end = n

    return row_start, col_end



def QR_Givens(H, eigvec=False, inplace=True, tridiagonal=False):

    """ Perform QR decomposition on a Hessenberg matrix by applying Givens Rotation
        (element-wise rather than whole matrix multiplication)
```

```
        No explicit computation of Q and R matrices; output updated Hessenberg matrix directly

    Input   ---------------------------------------------
            H: 2D np.array with real or complex entries, in the form of an upper Hessenberg matrix of shape n x n (square matrix)
            eigvec : bool, indicating whether eigenvectors are required; default is False
            inplace: bool, indicating whether to update H in place; default is True
            tridiagonal: bool, indicating whether H is tridiagonal; default is False

    Return  ---------------------------------------------
            H_new: 2D np.array with real or complex entries, in the form of an *updated* Hessenberg matrix, effectively R@Q

            G_iter: 2D np.array, the accumulated Givens rotation matrix, effectively Q.T;
                    if eigvec is False, not explicitly calculated and output identity matrix

            R = G @ H_old : applying Givens rotation G = Gn @ ... @ G2 @ G1 column by column to reduce Hessenberg to upper triangular
            H_old = G.T @ R => Q = G.T
            H_new = R @ Q = G @ H_old @ G.T : update Hessenberg matrix (no explicite calculation )
            G_iter = G = Gn @ ... @ G2 @ G1 : this is effectively the Givens rotation per round of QR iteration
    """

    n = H.shape[0]
    G_store = []
    G_iter = np.eye(n,dtype=H.dtype)

    if inplace:
        H_old = H
    else:
        H_old = H.copy()

    for i in range(n-1):

        _, col_end = select_idx(i,n,tridiagonal)

        G = Givens(H_old[i,i],H_old[i+1,i])

        # Premultiplication by a rotation in the (i, i+1)-plane only involves rows i and i+1 and leaves other rows unaffected
        # Furthermore, only consider non-zero entries in these two rows, to avoid unnecessary multiplication by zero
        H_old[i:i+2,i:col_end] = G @ H_old[i:i+2, i:col_end]

        G_store.append(G)

        # if eigenvectors are required
        if eigvec:
            G_ = np.eye(n,dtype=H.dtype)
            G_[i:i+2,i:i+2] = G
            G_iter = G_ @ G_iter

    # H_old is now effectively R; next is to obtain H_new = R @ Q where Q = G1.T G2.T ...Gn-1.T

    H_new = H_old

    for i in range(n-1):
```

```
        row_start, _ = select_idx(i,n,tridiagonal)

        G = G_store.pop(0) #same order as above G1, G2, ...

        if G.dtype != 'complex':
            G_T = G.T
        else:
            G_T = G.conjugate().T

        # Similar to above, postmultiplication by a rotation in the (i, i+1)-plane affects only columns i and i+1.
        H_new[row_start:i+2, i:i+2] = H_new[row_start:i+2, i:i+2] @ G_T

    # H_new is the updated version, effectively R @ Q

    return H_new, G_iter
```

In [10]:
```
### Testing QR_Givens() on Random Complex Matrices

print('\n\n')
print("====================================================================")
print('Testing QR_Givens() function on randomly generated COMPLEX matrices')
print("====================================================================")

for A in RANDOM_COMPLEX_MATRICES:
    print('\n\nComplex >>> Input shape = {}'.format(A.shape))
    H, S = Hessenberg(A,eigvec=True)
    #print('A =\n',A)
    #print('H =\n',H)

    H_new, G = QR_Givens(H,eigvec=True, tridiagonal=False)
    Q = G.T
    print('\nQR_Givens: H_new=\n',H_new)
    print('Q.conjugate().T @ Q = np.eye(n)? ',np.allclose(Q.conjugate().T@Q,np.eye(A.shape[0], dtype='complex')))




### Testing QR_Givens() on Random REAL Matrices

print('\n\n')
print("====================================================================")
print('Testing QR_Givens() function on randomly generated REAL matrices')
print("====================================================================")

for A in RANDOM_REAL_MATRICES:
    print('\n\nComplex >>> Input shape = {}'.format(A.shape))
    H, S = Hessenberg(A,eigvec=True)
    #print('A =\n',A)
    #print('H =\n',H)

    H_new, G = QR_Givens(H,eigvec=True, tridiagonal=False)
    Q = G.T
    print('\nQR_Givens: H_new=\n',H_new)
    print('Q.T @ Q = np.eye(n)? ',np.allclose(Q.T@Q,np.eye(A.shape[0])))




### Testing QR_Givens() on Random Complex Matrices

print('\n\n')
print("====================================================================")
print('Testing QR_Givens() function on SPECIAL real matrices')
print("====================================================================")

for A in SPECIAL_REAL_MATRICES:
    print('\n\nSpecial >>> Input shape = {}'.format(A.shape))
    H, S = Hessenberg(A,eigvec=True)
```

```
    #print('A =\n',A)
    #print('H =\n',H)

    H_new, G = QR_Givens(H,eigvec=True, tridiagonal=True)
    Q = G.T
    print('\nQR_Givens: H_new=\n',H_new)
    print('Q.T @ Q = np.eye(n)? ',np.allclose(Q.T@Q,np.eye(A.shape[0])))


```

```
====================================================================
Testing QR_Givens() function on randomly generated COMPLEX matrices
====================================================================


Complex >>> Input shape = (3, 3)

QR_Givens: H_new=
 [[ 1.547+1.438j  0.288-0.896j  0.856-0.45j ]
 [-0.677-0.j     0.402-0.333j -0.096-0.059j]
 [ 0.   -0.j     0.033+0.081j  0.035-0.13j ]]
Q.conjugate().T @ Q = np.eye(n)?  True


Complex >>> Input shape = (5, 5)

QR_Givens: H_new=
 [[ 2.081+2.091j -0.111-0.981j -0.224+0.181j  0.197-0.359j  0.353+0.104j]
 [-0.733+0.j     0.077+0.777j -0.204-0.177j  0.066-0.478j  0.475-0.164j]
 [ 0.   +0.j     0.443+0.j    -0.144-0.247j  0.003-0.088j -0.344-0.207j]
 [ 0.   -0.j     0.   +0.j    -0.531+0.j    -0.36 +0.057j  0.07 -0.j   ]
 [ 0.   +0.j    -0.   -0.j    -0.   +0.j    -0.028-0.168j -0.084+0.72j ]]
Q.conjugate().T @ Q = np.eye(n)?  True


Complex >>> Input shape = (10, 10)

QR_Givens: H_new=
 [[ 3.65 +4.147j  0.258-2.361j  0.102-0.964j -0.403+0.485j -0.269+0.468j  0.414-0.241j -0.411+0.212j -0.569+0.084j -0.183+0.146j  0.93 -0.711j]
 [-3.069+0.j     0.917+1.403j -0.566+0.44j  -0.148-0.674j -0.009-0.237j  0.248-0.012j  0.636+0.266j -0.573-0.708j -0.088-0.47j   0.143-0.05j ]
 [ 0.   -0.j     1.104+0.j     0.308+0.162j  0.013+0.298j -0.18 -0.073j  0.28 -0.22j   0.25 +0.09j  -0.267+0.112j -0.58 +0.029j -0.013+0.73j ]
 [ 0.   -0.j    -0.   +0.j    -0.919+0.j    -0.82 +0.018j -0.095+0.488j -0.094-0.097j -0.3  -0.09j   0.142+0.034j -0.155+0.269j -0.201+0.37j ]
 [-0.   +0.j    -0.   -0.j    -0.   +0.j    -1.11 +0.j     0.597-0.219j -0.144-0.066j -0.284+0.373j  0.225+0.256j  0.543-0.28j  -0.295-0.441j]
 [ 0.   +0.j     0.   -0.j     0.   +0.j     0.   +0.j     0.944+0.j    -0.159+0.048j -0.151+0.444j -0.093+0.103j -0.303-0.007j -0.208-0.099j]
 [-0.   -0.j     0.   +0.j    -0.   -0.j    -0.   +0.j     0.   +0.j    -0.902+0.j     0.404-0.188j  0.246+0.019j  0.21 +0.029j  0.01 +0.087j]
 [-0.   +0.j     0.   +0.j    -0.   +0.j     0.   +0.j    -0.   +0.j    -0.   -0.j     0.374+0.j    -0.331-0.035j  0.31 +0.259j -0.207+0.306j]
 [ 0.   -0.j    -0.   +0.j    -0.   +0.j    -0.   +0.j     0.   -0.j     0.   +0.j    -0.   +0.j     0.192+0.j    -0.513+0.064j  0.271+0.081j]
 [-0.   -0.j    -0.   +0.j    -0.   -0.j     0.   -0.j    -0.   -0.j    -0.   +0.j    -0.   +0.j    -0.   +0.j    -0.358-0.263j  0.055+0.222j]]
```

```
Q.conjugate().T @ Q = np.eye(n)?  True



====================================================================
Testing QR_Givens() function on randomly generated REAL matrices
====================================================================


Complex >>> Input shape = (3, 3)

QR_Givens: H_new=
 [[ 0.737 -0.473  0.06 ]
 [-0.434  0.349 -0.381]
 [-0.     0.354 -0.068]]
Q.T @ Q = np.eye(n)?  True


Complex >>> Input shape = (5, 5)

QR_Givens: H_new=
 [[ 1.076 -0.96  -0.065  0.577  0.089]
 [-1.481  0.692 -0.924  0.368 -0.17 ]
 [-0.    -0.127  0.072 -0.04   0.321]
 [-0.     0.     0.397  0.182  0.204]
 [-0.     0.    -0.     0.007  0.067]]
Q.T @ Q = np.eye(n)?  True


Complex >>> Input shape = (10, 10)

QR_Givens: H_new=
 [[ 4.074 -1.492 -0.118  0.082 -0.496  0.335 -0.538 -0.027  0.089 -0.36 ]
 [-2.145  0.537 -0.244 -0.342  0.093  0.94   0.018 -0.45   0.525  0.518]
 [ 0.    -0.744  0.267 -0.053  0.355 -0.016 -0.17   0.18   0.29   0.151]
 [ 0.    -0.     0.833 -0.195 -0.594  0.055 -0.081  0.06   0.106 -0.032]
 [ 0.    -0.    -0.     0.819 -0.141  0.032  0.097 -0.36  -0.42   0.214]
 [ 0.     0.     0.     0.     0.436  0.083  0.155  0.411  0.196 -0.435]
 [ 0.     0.    -0.    -0.     0.    -0.222 -0.5   -0.007 -0.165  0.215]
 [-0.     0.    -0.     0.    -0.    -0.    -0.158  0.406 -0.208 -0.423]
 [ 0.    -0.     0.    -0.     0.     0.     0.     0.045  0.247  0.277]
 [ 0.    -0.    -0.    -0.    -0.     0.     0.     0.    -0.193 -0.362]]
Q.T @ Q = np.eye(n)?  True



====================================================================
Testing QR_Givens() function on SPECIAL real matrices
====================================================================


Special >>> Input shape = (10, 10)
```

```
QR_Givens: H_new=
 [[-2.853  9.49   0.     0.    -0.     0.     0.    -0.     0.     0.   ]
 [ 9.49  -1.212 -3.894 -0.     0.     0.     0.     0.    -0.     0.   ]
 [-0.    -3.894 -9.049  6.26   0.    -0.    -0.    -0.    -0.     0.   ]
 [ 0.    -0.     6.26   4.223  5.993 -0.     0.    -0.    -0.    -0.   ]
 [-0.     0.     0.     5.993 -7.009  4.185  0.    -0.    -0.    -0.   ]
 [ 0.     0.    -0.    -0.     4.185 -3.363 -5.347 -0.     0.    -0.   ]
 [ 0.     0.    -0.     0.     0.    -5.347  5.205 -3.516 -0.    -0.   ]
 [-0.     0.    -0.    -0.    -0.     0.    -3.516 -2.659  3.083  0.   ]
 [ 0.    -0.    -0.    -0.    -0.     0.     0.     3.083 -0.084  0.12 ]
 [ 0.     0.     0.    -0.    -0.    -0.    -0.     0.     0.12  -1.199]]
Q.T @ Q = np.eye(n)?  True


Special >>> Input shape = (10, 10)

QR_Givens: H_new=
 [[ 0.077  2.979 -0.     0.     0.     0.    -0.    -0.    -0.     0.   ]
 [ 2.979 -1.097 -3.511  0.     0.     0.    -0.     0.    -0.     0.   ]
 [ 0.    -3.511 -0.763 -1.662  0.     0.     0.     0.    -0.    -0.   ]
 [ 0.    -0.    -1.662 -4.181  0.311  0.     0.     0.    -0.     0.   ]
 [ 0.     0.     0.     0.311  1.002 -2.148 -0.     0.     0.    -0.   ]
 [ 0.     0.     0.    -0.    -2.148  0.85   3.502  0.    -0.     0.   ]
 [-0.    -0.     0.     0.    -0.     3.502 -3.55   0.921  0.     0.   ]
 [-0.     0.     0.    -0.    -0.     0.     0.921 -4.319 -0.008  0.   ]
 [-0.    -0.    -0.    -0.     0.     0.    -0.    -0.008 -6.019 -0.   ]
 [ 0.     0.    -0.     0.     0.     0.     0.     0.    -0.     0.   ]]
Q.T @ Q = np.eye(n)?  True


Special >>> Input shape = (10, 10)

QR_Givens: H_new=
 [[ 5.5    2.872 -0.     0.     0.     0.     0.     0.     0.     0.   ]
 [-2.872  0.5    1.265 -0.    -0.     0.    -0.     0.    -0.     0.   ]
 [-0.    -1.265  0.5   -0.806 -0.    -0.    -0.     0.     0.    -0.   ]
 [ 0.     0.     0.806  0.5    0.577 -0.     0.    -0.     0.    -0.   ]
 [ 0.    -0.    -0.    -0.577  0.5   -0.435  0.     0.     0.    -0.   ]
 [-0.     0.     0.    -0.     0.435  0.5   -0.334 -0.     0.    -0.   ]
 [ 0.    -0.     0.     0.    -0.     0.334  0.5   -0.256 -0.    -0.   ]
 [-0.     0.    -0.     0.     0.    -0.     0.256  0.5   -0.188  0.   ]
 [ 0.    -0.     0.    -0.    -0.    -0.     0.     0.188  0.5   -0.121]
 [ 0.     0.    -0.     0.     0.    -0.     0.     0.     0.121  0.5  ]]
Q.T @ Q = np.eye(n)?  True


Special >>> Input shape = (6, 6)

QR_Givens: H_new=
 [[ 13.248 -12.062 -17.543  -0.14    9.015  -7.936]
 [  7.268   1.573   0.487   5.827  10.683  -2.416]
```

```
 [  0.     -7.658   1.654   1.797   5.373   3.251]
 [ -0.      0.     -1.01   -3.139  -3.496   9.98 ]
 [ -0.     -0.     -0.      4.939  -1.009   4.566]
 [ -0.     -0.      0.     -0.     -1.99    6.674]]
Q.T @ Q = np.eye(n)?  True
```

```
C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:32: ComplexWarning: Casting complex values to real discards the imaginary part
```

Observation:

- Function QR_Givens() seems to be working for both real and complex matrices as expected
- **Where does the complex warning come from??**

QR Iteration

Some Background Information

Source: https://scicomp.stackexchange.com/questions/30407/how-does-the-qr-algorithm-applied-to-a-real-matrix-returns-complex-eigenvalues (https://scicomp.stackexchange.com/questions/30407/how-does-the-qr-algorithm-applied-to-a-real-matrix-returns-complex-eigenvalues)

- QR algorithm converges to the real Schur decomposition: a unitary matrix Q and a matrix R in block upper triangular form such that $A = QRQ^T$
- The key point is the *block upper triangular* form, which means here R_{ii} are real blocks of
  - EITHER **size 1x1** (R_{ii} is a (real) eigenvalue of A)
  - OR **size 2x2** (R_{ii} has a pair of complex conjugate eigenvalues).

In [11]:

```
#def QR_Iteration_Eigenvalues_naive(A, max_iter, tol):
#    # step 1: tranform A to Hessenberg
#    H, _ = Hessenberg(A)
#    # step 2: QR iteration
#    H_old = H
#    for i in range(max_iter):
#        Q, R = QR_Givens_naive(H_old)
#        H_new = R @ Q
#        # TODO: test for convergence
#        if np.linalg.norm(H_new - H) < tol:
#            print('break at i=',i,np.linalg.norm(H_new - H),H_new)
#            break
#        H_old = H_new
#    R = H_new
#    return np.diag(R) # this is wrong


def diagonal_block(R):

    """     Read eigenvalues from the diagonal block matrix R, either in 1x1 block (i.e. real eigenvalue)
            or in 2x2 block (i.e. complex conjugate pair)
    Input   ---------------------------------------------
            R: 2D np.array with real or complex entries, in the form of a block upper triangular matrix
    Return  ---------------------------------------------
            eigenvalues: 1D np.array with real or complex entries
    """

    eigenvalues = []

    for i in range(len(R)):
        if i < len(eigenvalues): # if current diagonal values has already been used in previous step, skip
            continue

        if i ==len(R)-1 or np.abs(R[i+1,i]) < ATOL: # if subdiagonal element is sufficiently small, treat as zero OR if last row
            eigenvalues.append(R[i,i])

        else:
            # solving eigenvalues from characteristic polynomial lam^2 - (R[i,i]+R[i+1,i+1])*lam + (R[i,i]*R[i+1,i+1] - R[i,i+1]*R[i+1,i]) = 0
            b = - (R[i,i]+R[i+1,i+1])
            c = R[i,i]*R[i+1,i+1] - R[i,i+1]*R[i+1,i]

            lam_1 = -b/2
            sign = np.sign(lam_1)

            if b**2 - 4*c >= 0:
                lam_2 = np.sqrt(b**2-4*c)/2
                eigenvalues.append(lam_1 + sign*lam_2)
                eigenvalues.append(lam_1 - sign*lam_2)
            else:
                lam_2 = np.sqrt(-(b**2-4*c))/2
                eigenvalues.append(lam_1 + sign*lam_2*1j)
                eigenvalues.append(lam_1 - sign*lam_2*1j)
```

```python

    return np.array(eigenvalues)



def QR_Iteration_Eigenvalues(A, max_iter, tol):

    """     Perform QR iteration to obtain eigenvalues
    Input   ---------------------------------------------
            A: 2D np.array with real or complex entries
            max_iter: int, maximum number of iterations to be performed
            tol: float, relative tolerance between eigenvalues by successive iterations
    Return  ---------------------------------------------
            eigvals_new: 1D np.array with real or complex entries
            i: total number of iterations
    """


    # step 1: tranform A to Hessenberg
    H, _ = Hessenberg(A) # default: eigvec=False, inplace=True

    # step 2: QR iteration
    H_old = H #.copy()
    eigvals_old = diagonal_block(H)

    for i in range(max_iter):
        H_new,_ = QR_Givens(H_old) # updating H_old in-place # default QR_Givens(H,eigvec=False, inplace=True, tridiagonal=False)

        eigvals_new = diagonal_block(H_new)
        #print('>>>\n', H_new, '\n\n',np.diag(H_new), '\n')

        # Test for convergence: relative tol = 1- w / w' for each eigenvalue
        if np.allclose(eigvals_new, eigvals_old, rtol=tol):
            print('Iteration terminates at i={} with tol={} reached'.format(i,tol))
            break

        if i == max_iter-1:
            err = np.max(abs((eigvals_new - eigvals_old)) / abs(eigvals_old))
            idx = np.argwhere(np.isclose(eigvals_new, eigvals_old, rtol=tol)==False).tolist()
            print('max_iter={} reached, max error={} vs tol={}; index of non-convergence={}'.format(max_iter, err, tol, idx))

        eigvals_old = eigvals_new.copy() # need to save a copy for comparison in the next iteration
        # H_old = H_new # this is not needed as H_old is updated inplace

    return eigvals_new, i



```

In [12]:
```python
# Testing

def sort_(eigenvalues):

    """     Sort eigenvalues in descending order by magitude;
            following Numpy's extended sort order: Real: [R, nan]; Complex: [R + Rj, R + nanj, nan + Rj, nan + nanj]
    Input   ----------------------------------------------
            eigenvalues: 1D np.array with real or complex entries
    Return  ---------------------------------------------
            eigenvaluese_sorted: 1D np.array with real or complex entries, sorted in descending order
    """
    eigenvalues_sorted = np.flip(np.sort(eigenvalues))

    return eigenvalues_sorted

#    if eigenvalues.dtype != 'complex':
#        eigenvalues_sorted = np.sort(eigenvalues) #np.flip(sorted(eigenvalues,key=abs))
#    else:
#        eigenvalues_sorted = np.sort_complex(eigenvalues)
#    return eigenvalues_sorted



def test_eigenvalues_function(A,function):

    """     Perform eigenvalue testing on selected function; eigenvalues are compared with results from np.linalg.eig()
    Input   ----------------------------------------------
            A: input matrix, 2D np.array with real or complex entries
            function: name of the function being tested
    Return  ---------------------------------------------
            None; test results will be printed
    """

    print("------ Testing for function: {}() ------".format(function.__name__))
    w,v = np.linalg.eig(A)
    w_sorted = sort_(w)
    eigenvalues,i = function(A,max_iter=MAX_ITER, tol=RTOL)
    eigenvalues_sorted = sort_(eigenvalues)
    print()
    print('my eigenvalues: {}'.format(eigenvalues_sorted))
    print('np.linalg.eig:  {}'.format(w_sorted)) #sort in descending order of magnitutes

    if np.allclose(w_sorted,eigenvalues_sorted):
        print('\nnp.linalg.eig and my eigenvalues close? True \n')
    else:
        print('\nnp.linalg.eig and my eigenvalues close? False \n')
        print('\nmy eigenvalues - np.linalg.eig =\n', eigenvalues_sorted - w_sorted,'\n')

    print('\n')


def test_eigenvalues(eigenvalues,w, verbose=True):
```

```python

    """     Comparing two sets of eigenvalues
    Input   ----------------------------------------------
            eigenvalues: 1D np.array with real or complex entries, eigenvalues from my function
            w: eigenvalues from np.linalg.eig(), basis for comparison
            verbose: bool, indicating whether to print the difference between the two sets of eigenvalues
    Return  ---------------------------------------------
            None; test results will be printed
    """

    eigenvalues_sorted = sort_(eigenvalues)
    w_sorted = sort_(w)

    if verbose:
        print()
        print('my eigenvalues: {}'.format(eigenvalues_sorted))
        print('np.linalg.eig:  {}'.format(w_sorted)) #sort in descending order of magnitutes

    if np.allclose(w_sorted,eigenvalues_sorted):
        print('\nnp.linalg.eig and my eigenvalues close? True \n')
    else:
        print('\nnp.linalg.eig and my eigenvalues close? False \n')
        if verbose:
            print('my eigenvalues - np.linalg.eig =\n', eigenvalues_sorted - w_sorted,'\n')
            print('max abs diff = ',np.abs(eigenvalues_sorted - w_sorted).max(),'\n')
        print('\n')

```

In [43]:

```
### Testing QR_Iteration_Eigenvalues() on Random Complex Matrices

print('\n\n')
print("==================================================================================")
print('Testing QR_Iteration_Eigenvalues() function on randomly generated COMPLEX matrices')
print("==================================================================================")

for A in RANDOM_COMPLEX_MATRICES:
    print('\nComplex >>> Input shape = {}\n'.format(A.shape))
    test_eigenvalues_function(A,QR_Iteration_Eigenvalues)



### Testing QR_Iteration_Eigenvalues() on Random Real Matrices

print('\n\n')
print("==================================================================================")
print('Testing QR_Iteration_Eigenvalues() function on randomly generated REAL matrices')
print("==================================================================================")

for A in RANDOM_REAL_MATRICES:
    print('\nReal >>> Input shape = {}\n'.format(A.shape))
    test_eigenvalues_function(A,QR_Iteration_Eigenvalues)



### Testing QR_Iteration_Eigenvalues() on Special Matrices

print('\n\n')
print("==================================================================================")
print('Testing QR_Iteration_Eigenvalues() function on SPECIAL real matrices')
print("==================================================================================")

for A in SPECIAL_REAL_MATRICES:
    print('\nSpecial >>> Input shape = {}\n'.format(A.shape))
    test_eigenvalues_function(A,QR_Iteration_Eigenvalues)
```

```
==================================================================================
Testing QR_Iteration_Eigenvalues() function on randomly generated COMPLEX matrices
==================================================================================

Complex >>> Input shape = (3, 3)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=11 with tol=1e-06 reached
```

```
my eigenvalues: [ 1.71 +1.655j  0.288-0.519j -0.015-0.161j]
np.linalg.eig:  [ 1.71 +1.655j  0.288-0.519j -0.015-0.161j]

np.linalg.eig and my eigenvalues close? True




Complex >>> Input shape = (5, 5)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=935 with tol=1e-06 reached

my eigenvalues: [ 2.251+2.287j  0.132+0.508j -0.182+0.806j -0.189-0.474j -0.442+0.271j]
np.linalg.eig:  [ 2.251+2.287j  0.132+0.508j -0.182+0.806j -0.189-0.474j -0.442+0.271j]

np.linalg.eig and my eigenvalues close? True




Complex >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=74 with tol=1e-06 reached

my eigenvalues: [ 4.624+5.231j  1.242-0.661j  0.321-0.067j  0.194+0.373j  0.193-1.047j  0.058+1.069j -0.285+0.567j -0.419-0.316j -0.705-0.182j -1.116+0.6
55j]
np.linalg.eig:  [ 4.624+5.231j  1.242-0.661j  0.321-0.067j  0.194+0.373j  0.193-1.047j  0.058+1.069j -0.285+0.567j -0.419-0.316j -0.705-0.182j -1.116+0.6
55j]

np.linalg.eig and my eigenvalues close? True






=====================================================================================
Testing QR_Iteration_Eigenvalues() function on randomly generated REAL matrices
=====================================================================================

Real >>> Input shape = (3, 3)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=35 with tol=1e-06 reached

my eigenvalues: [0.992+0.j    0.013+0.295j 0.013-0.295j]
np.linalg.eig:  [0.992+0.j    0.013+0.295j 0.013-0.295j]

np.linalg.eig and my eigenvalues close? True
```

```
Real >>> Input shape = (5, 5)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=28 with tol=1e-06 reached

my eigenvalues: [ 2.116+0.j     0.207+0.229j  0.207-0.229j  0.073+0.j    -0.515+0.j   ]
np.linalg.eig:  [ 2.116+0.j     0.207+0.229j  0.207-0.229j  0.073+0.j    -0.515+0.j   ]

np.linalg.eig and my eigenvalues close? True



Real >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------

C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:32: ComplexWarning: Casting complex values to real discards the imaginary part

Iteration terminates at i=558 with tol=1e-06 reached

my eigenvalues: [ 4.82 +0.j     0.579+0.149j  0.579-0.149j  0.377+0.j     0.146+0.j    -0.24 +0.j    -0.352+0.944j -0.352-0.944j -0.571+0.117j -0.571-0.1
17j]
np.linalg.eig:  [ 4.82 +0.j     0.579+0.149j  0.579-0.149j  0.377+0.j     0.146+0.j    -0.24 +0.j    -0.352+0.944j -0.352-0.944j -0.571+0.117j -0.571-0.1
17j]

np.linalg.eig and my eigenvalues close? True



=====================================================================================
Testing QR_Iteration_Eigenvalues() function on SPECIAL real matrices
=====================================================================================

Special >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=194 with tol=1e-06 reached

my eigenvalues: [  9.684   8.685   7.302   1.666  -1.201  -3.008  -5.856  -8.817 -11.764 -14.69 ]
np.linalg.eig:  [  9.684   8.685   7.302   1.666  -1.201  -3.008  -5.856  -8.817 -11.764 -14.69 ]

np.linalg.eig and my eigenvalues close? True
```

```
Special >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=644 with tol=1e-06 reached

my eigenvalues: [ 4.03   3.972  0.064  0.    -0.064 -3.97  -4.03  -5.954 -6.018 -6.03 ]
np.linalg.eig:  [ 4.03   3.972  0.064  0.    -0.064 -3.97  -4.03  -5.953 -6.018 -6.03 ]

np.linalg.eig and my eigenvalues close? False


my eigenvalues - np.linalg.eig =
 [-0. -0. -0. -0.  0. -0.  0. -0.  0.  0.]




Special >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=780 with tol=1e-06 reached

my eigenvalues: [1.026+0.009j 1.026-0.009j 1.016+0.022j 1.016-0.022j 0.999+0.027j 0.999-0.027j 0.984+0.021j 0.984-0.021j 0.975+0.008j 0.975-0.008j]
np.linalg.eig:  [1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]

np.linalg.eig and my eigenvalues close? False


my eigenvalues - np.linalg.eig =
 [ 0.026+0.009j  0.026-0.009j  0.016+0.022j  0.016-0.022j -0.001+0.027j -0.001-0.027j -0.016+0.021j -0.016-0.021j -0.025+0.008j -0.025-0.008j]




Special >>> Input shape = (6, 6)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=52 with tol=1e-06 reached

my eigenvalues: [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]
np.linalg.eig:  [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]

np.linalg.eig and my eigenvalues close? True
```

Observations:

- Function QR_Iteration_Eigenvalues() seems to be working as expected with both real and complex matrices, with real eigenvalues or complex conjugate pairs
- Underperformance in special matrices: diagonal dominant and unbalanced
  - *real symmetric* A = np.random.randint(-3,3,(10,10)) A = (A.T + A).astype('float')
  - *diagonal dominant* B = A - np.triu(A,1)*0.99 - np.tril(A,-1)*0.99
  - *unbalanced* C = np.tril(np.ones((10,10),dtype='float'),0)

In [14]:

```python
### Testing QR_Iteration_Eigenvalues() on COMPLEX Matrices with only pure imaginary parts

print('\n\n')
print("==================================================================================")
print('Testing QR_Iteration_Eigenvalues() function on SPECIAL complex matrices (Im Only)')
print("==================================================================================")

A = np.random.rand(5,5) * 1j
print('\nSpecial >>> Input shape = {}\n'.format(A.shape))
test_eigenvalues_function(A,QR_Iteration_Eigenvalues)

```

```
==================================================================================
Testing QR_Iteration_Eigenvalues() function on SPECIAL complex matrices (Im Only)
==================================================================================

Special >>> Input shape = (5, 5)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=23 with tol=1e-06 reached

my eigenvalues: [ 0.561+0.013j  0.   -0.296j  0.   +0.978j  0.   -2.01j  -0.561+0.013j]
np.linalg.eig:  [ 0.752+0.172j  0.   +2.607j  0.   -0.248j  0.   -0.023j -0.752+0.172j]

np.linalg.eig and my eigenvalues close? False


my eigenvalues - np.linalg.eig =
 [-0.192-0.159j  0.   -2.903j  0.   +1.225j  0.   -1.986j  0.192-0.159j]
```

Observations:

- Does not seem to work for matrices with pure imaginary parts ... **#TODO**

Wilkinson Shift

1.4.22 Definition(Wilkinson shift)

Let

$$\mathbf{M} = \begin{pmatrix} h_{n-1,n-1}^{(k)} & h_{n-1,n}^{(k)} \\ h_{n,n-1}^{(k)} & h_{nn}^{(k)} \end{pmatrix}.$$

Then, for σ_k use the eigenvalue of $\mathbf{M}$ which is closer to $h_{nn}^{(k)}$.

1.4.23 Remark

The Wilkinson shift is reliable and the $h_{n,n-1}$ and h_{nn} converge to zero and the smallest eigenvalue by magnitude, respectively. They converge at least quadratically and cubically in the symmetric case [GvL83, Section 8.2].

Source: NLA Lecture Notes

In [15]:

```python

def Wilkinson_shift(H):

    """     Using the bottom right 2x2 block of Hessenberg matrix to determine if eigenvalues are real or complex (b^2-4ac)
            Calculate the Wilkinson's shift according to Lecture Notes 1.4.22 Definition (Wilkinson shift)
    Input   ----------------------------------------------
            H: 2D np.array with real or complex entries, in the form of an upper Hessenberg matrix
    Return  ---------------------------------------------
            datatype: 'complex' or 'float'
            lam1 or lam2: shift to be performed
    """

    n = H.shape[0]

    # find the current index where the subdiagonal is non-zero
    # (if the subdiagonal is zero, it means the shift has worked and converged to an eigenvalue)
    while abs(H[n-1,n-2]) < ATOL: #global variable ATOL = 1e-18, proxy for zero
        n -= 1

    # M is the bottom right 2x2 block with nonzero subdiagonal (equivalent to the idea of deflation)
    M = H[n-2:n,n-2:n]

    # characteristic polynomial x^2 + bx + c = 0
    b = -(M[0,0] + M[1,1])
    c = (M[0,0]*M[1,1] - M[1,0]*M[0,1])

    # if real eigenvalues - return the one closer to A[n,n]'
    if b**2 >= 4*c:
        lam1 = (-b + np.sqrt(b**2-4*c))/2
        lam2 = (-b - np.sqrt(b**2-4*c))/2
        datatype = 'float'

    # if complex eigenvalues - return Re(x) and Im(x)
    else:
        re_ = -b/2
        im_ = np.sqrt(4*c-b**2)/2
        lam1 = re_+1j*im_
        lam2 = re_+1j*im_
        datatype = 'complex'

    if abs(lam1-M[1,1]) < abs(lam2-M[1,1]):
        return (datatype, lam1)

    else:
        return (datatype, lam2)



def QR_Iteration_WilkinsonShift(A, max_iter, tol):

    """     Perform QR iteration with Wilkinson's shift to obtain eigenvalues
    Input   ----------------------------------------------
```

```
            A: 2D np.array with real or complex entries
            max_iter: int, maximum number of iterations to be performed
            tol: float, relative tolerance between eigenvalues by successive iterations
    Return  ---------------------------------------------
            eigvals_new: 1D np.array with real or complex entries
            i: total number of iterations
    """

    n = A.shape[0]

    # step 1: tranform A to Hessenberg
    H, _ = Hessenberg(A)

    # step 2: QR iteration
    H_old = H.copy()
    eigvals_old = diagonal_block(H)

    for i in range(max_iter):

        datatype, lam = Wilkinson_shift(H_old)
        # print(">>> Wilkinson_shift = {}".format(lam))

        # H_old = H_old - lam * np.eye(n)
        for j in range(n):
            H_old[j,j] -= lam

        #H_new, _ = QR_Givens(H_old) +  lam * np.eye(n) # updating H_old in-place  # default QR_Givens(H,eigvec=False, inplace=True, tridiagonal=False
        H_new, _ = QR_Givens(H_old)
        for j in range(n):
            H_new[j,j] += lam

        eigvals_new = diagonal_block(H_new)
        H_old = H_new

        # Test for convergence: relative tol = 1- w / w' for each eigenvalue
        if np.allclose(eigvals_new, eigvals_old, rtol=tol):
            print('Iteration terminates at i={} with tol={} reached'.format(i,tol))
            break

        if i == max_iter-1:
            err = np.max(abs((eigvals_new - eigvals_old)) / abs(eigvals_old))
            idx = np.argwhere(np.isclose(eigvals_new, eigvals_old, rtol=tol)==False).tolist()
            print('max_iter={} reached, max error={} vs tol={}; index of non-convergence={}'.format(max_iter, err, tol, idx))

        eigvals_old = eigvals_new

    return eigvals_new, i
```

In [16]:

```python
### Testing QR_Iteration_WilkinsonShift() on Random Complex Matrices

print('\n\n')
print("====================================================================================")
print('Testing QR_Iteration_WilkinsonShift() function on randomly generated COMPLEX matrices')
print("====================================================================================")

for A in RANDOM_COMPLEX_MATRICES:
    print('\nComplex >>> Input shape = {}\n'.format(A.shape))
    test_eigenvalues_function(A,QR_Iteration_Eigenvalues)
    test_eigenvalues_function(A,QR_Iteration_WilkinsonShift)


### Testing QR_Iteration_WilkinsonShift() on Random Real Matrices

print('\n\n')
print("====================================================================================")
print('Testing QR_Iteration_WilkinsonShift() function on randomly generated REAL matrices')
print("====================================================================================")

for A in RANDOM_REAL_MATRICES:
    print('\nReal >>> Input shape = {}\n'.format(A.shape))
    test_eigenvalues_function(A,QR_Iteration_Eigenvalues)
    test_eigenvalues_function(A,QR_Iteration_WilkinsonShift)


### Testing QR_Iteration_WilkinsonShift() on Special Matrices

print('\n\n')
print("====================================================================================")
print('Testing QR_Iteration_WilkinsonShift() function on SPECIAL real matrices')
print("====================================================================================")

for A in SPECIAL_REAL_MATRICES:
    print('\nSpecial >>> Input shape = {}\n'.format(A.shape))
    test_eigenvalues_function(A,QR_Iteration_Eigenvalues)
    test_eigenvalues_function(A,QR_Iteration_WilkinsonShift)
```

```
====================================================================================
Testing QR_Iteration_WilkinsonShift() function on randomly generated COMPLEX matrices
====================================================================================

Complex >>> Input shape = (3, 3)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=11 with tol=1e-06 reached
```

```
my eigenvalues: [ 1.71 +1.655j  0.288-0.519j -0.015-0.161j]
np.linalg.eig:  [ 1.71 +1.655j  0.288-0.519j -0.015-0.161j]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=5 with tol=1e-06 reached

my eigenvalues: [ 1.71 +1.655j  0.288-0.519j -0.015-0.161j]
np.linalg.eig:  [ 1.71 +1.655j  0.288-0.519j -0.015-0.161j]

np.linalg.eig and my eigenvalues close? True




Complex >>> Input shape = (5, 5)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=935 with tol=1e-06 reached

my eigenvalues: [ 2.251+2.287j  0.132+0.508j -0.182+0.806j -0.189-0.474j -0.442+0.271j]
np.linalg.eig:  [ 2.251+2.287j  0.132+0.508j -0.182+0.806j -0.189-0.474j -0.442+0.271j]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=12 with tol=1e-06 reached

my eigenvalues: [ 2.251+2.287j  0.132+0.508j -0.182+0.806j -0.189-0.474j -0.442+0.271j]
np.linalg.eig:  [ 2.251+2.287j  0.132+0.508j -0.182+0.806j -0.189-0.474j -0.442+0.271j]

np.linalg.eig and my eigenvalues close? True




Complex >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=74 with tol=1e-06 reached

my eigenvalues: [ 4.624+5.231j  1.242-0.661j  0.321-0.067j  0.194+0.373j  0.193-1.047j  0.058+1.069j -0.285+0.567j -0.419-0.316j -0.705-0.182j -1.116+0.6
55j]
np.linalg.eig:  [ 4.624+5.231j  1.242-0.661j  0.321-0.067j  0.194+0.373j  0.193-1.047j  0.058+1.069j -0.285+0.567j -0.419-0.316j -0.705-0.182j -1.116+0.6
55j]
```

```
np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=40 with tol=1e-06 reached

my eigenvalues: [ 4.624+5.231j  1.242-0.661j  0.321-0.067j  0.194+0.373j  0.193-1.047j  0.058+1.069j -0.285+0.567j -0.419-0.316j -0.705-0.182j -1.116+0.6
55j]
np.linalg.eig:  [ 4.624+5.231j  1.242-0.661j  0.321-0.067j  0.194+0.373j  0.193-1.047j  0.058+1.069j -0.285+0.567j -0.419-0.316j -0.705-0.182j -1.116+0.6
55j]

np.linalg.eig and my eigenvalues close? True






====================================================================================
Testing QR_Iteration_WilkinsonShift() function on randomly generated REAL matrices
====================================================================================

Real >>> Input shape = (3, 3)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=35 with tol=1e-06 reached

my eigenvalues: [0.992+0.j    0.013+0.295j 0.013-0.295j]
np.linalg.eig:  [0.992+0.j    0.013+0.295j 0.013-0.295j]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=36 with tol=1e-06 reached

my eigenvalues: [0.992+0.j    0.013+0.295j 0.013-0.295j]
np.linalg.eig:  [0.992+0.j    0.013+0.295j 0.013-0.295j]

np.linalg.eig and my eigenvalues close? True




Real >>> Input shape = (5, 5)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=28 with tol=1e-06 reached

my eigenvalues: [ 2.116+0.j     0.207+0.229j  0.207-0.229j  0.073+0.j    -0.515+0.j   ]
```

```
np.linalg.eig:  [ 2.116+0.j     0.207+0.229j  0.207-0.229j  0.073+0.j    -0.515+0.j   ]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=14 with tol=1e-06 reached

my eigenvalues: [ 2.116+0.j     0.207+0.229j  0.207-0.229j  0.073+0.j    -0.515+0.j   ]
np.linalg.eig:  [ 2.116+0.j     0.207+0.229j  0.207-0.229j  0.073+0.j    -0.515+0.j   ]

np.linalg.eig and my eigenvalues close? True




Real >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------

C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:32: ComplexWarning: Casting complex values to real discards the imaginary part
C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:75: ComplexWarning: Casting complex values to real discards the imaginary part
C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:80: ComplexWarning: Casting complex values to real discards the imaginary part

Iteration terminates at i=558 with tol=1e-06 reached

my eigenvalues: [ 4.82 +0.j     0.579+0.149j  0.579-0.149j  0.377+0.j     0.146+0.j    -0.24 +0.j    -0.352+0.944j -0.352-0.944j -0.571+0.117j -0.571-
0.117j]
np.linalg.eig:  [ 4.82 +0.j     0.579+0.149j  0.579-0.149j  0.377+0.j     0.146+0.j    -0.24 +0.j    -0.352+0.944j -0.352-0.944j -0.571+0.117j -0.571-
0.117j]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=465 with tol=1e-06 reached

my eigenvalues: [ 4.82 +0.j     0.579+0.149j  0.579-0.149j  0.377+0.j     0.146+0.j    -0.24 +0.j    -0.352+0.944j -0.352-0.944j -0.571+0.117j -0.571-
0.117j]
np.linalg.eig:  [ 4.82 +0.j     0.579+0.149j  0.579-0.149j  0.377+0.j     0.146+0.j    -0.24 +0.j    -0.352+0.944j -0.352-0.944j -0.571+0.117j -0.571-
0.117j]

np.linalg.eig and my eigenvalues close? True





=====================================================================================
Testing QR_Iteration_WilkinsonShift() function on SPECIAL real matrices
```

```
=====================================================================================

Special >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=194 with tol=1e-06 reached

my eigenvalues: [  9.684   8.685   7.302   1.666  -1.201  -3.008  -5.856  -8.817 -11.764 -14.69 ]
np.linalg.eig:  [  9.684   8.685   7.302   1.666  -1.201  -3.008  -5.856  -8.817 -11.764 -14.69 ]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=52 with tol=1e-06 reached

my eigenvalues: [  9.684   8.685   7.302   1.666  -1.201  -3.008  -5.856  -8.817 -11.764 -14.69 ]
np.linalg.eig:  [  9.684   8.685   7.302   1.666  -1.201  -3.008  -5.856  -8.817 -11.764 -14.69 ]

np.linalg.eig and my eigenvalues close? True




Special >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=644 with tol=1e-06 reached

my eigenvalues: [ 4.03   3.972  0.064  0.    -0.064 -3.97  -4.03  -5.954 -6.018 -6.03 ]
np.linalg.eig:  [ 4.03   3.972  0.064  0.    -0.064 -3.97  -4.03  -5.953 -6.018 -6.03 ]

np.linalg.eig and my eigenvalues close? False


my eigenvalues - np.linalg.eig =
 [-0. -0. -0. -0.  0. -0.  0. -0.  0.  0.]



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=30 with tol=1e-06 reached

my eigenvalues: [ 4.03   3.972  0.064  0.    -0.064 -3.97  -4.03  -5.953 -6.018 -6.03 ]
np.linalg.eig:  [ 4.03   3.972  0.064  0.    -0.064 -3.97  -4.03  -5.953 -6.018 -6.03 ]

np.linalg.eig and my eigenvalues close? True
```

```
Special >>> Input shape = (10, 10)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=780 with tol=1e-06 reached

my eigenvalues: [1.026+0.009j 1.026-0.009j 1.016+0.022j 1.016-0.022j 0.999+0.027j 0.999-0.027j 0.984+0.021j 0.984-0.021j 0.975+0.008j 0.975-0.008j]
np.linalg.eig:  [1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]

np.linalg.eig and my eigenvalues close? False


my eigenvalues - np.linalg.eig =
 [ 0.026+0.009j  0.026-0.009j  0.016+0.022j  0.016-0.022j -0.001+0.027j -0.001-0.027j -0.016+0.021j -0.016-0.021j -0.025+0.008j -0.025-0.008j]



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=605 with tol=1e-06 reached

my eigenvalues: [1.026+0.j    1.02 +0.015j 1.02 -0.015j 1.007+0.024j 1.007-0.024j 0.992+0.023j 0.992-0.023j 0.98 +0.014j 0.98 -0.014j 0.976+0.j   ]
np.linalg.eig:  [1. 1. 1. 1. 1. 1. 1. 1. 1. 1.]

np.linalg.eig and my eigenvalues close? False


my eigenvalues - np.linalg.eig =
 [ 0.026+0.j     0.02 +0.015j  0.02 -0.015j  0.007+0.024j  0.007-0.024j -0.008+0.023j -0.008-0.023j -0.02 +0.014j -0.02 -0.014j -0.024+0.j   ]




Special >>> Input shape = (6, 6)

------ Testing for function: QR_Iteration_Eigenvalues() ------
Iteration terminates at i=52 with tol=1e-06 reached

my eigenvalues: [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]
np.linalg.eig:  [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]

np.linalg.eig and my eigenvalues close? True



------ Testing for function: QR_Iteration_WilkinsonShift() ------
Iteration terminates at i=16 with tol=1e-06 reached

my eigenvalues: [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]
np.linalg.eig:  [5.+6.j 5.-6.j 4.+0.j 3.+0.j 1.+2.j 1.-2.j]

np.linalg.eig and my eigenvalues close? True
```

Observations:

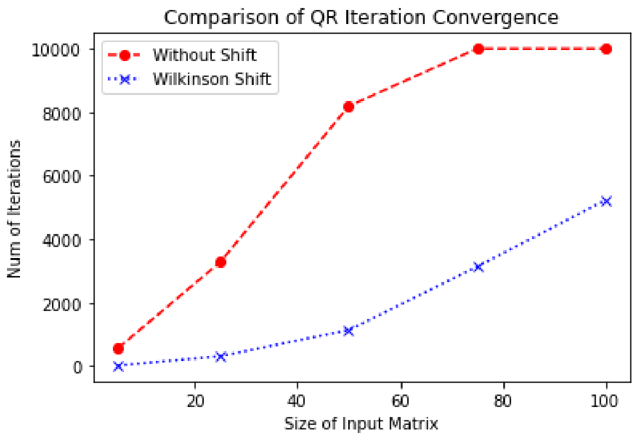
- Function QR_Iteration_WilkinsonShift() seems to be working as expected with both real and complex matrices, with real eigenvalues or complex conjugate pairs
- Same issue with special matrices -> diagonal dominant with ones in lower triangular

Visualization of Convergence Comparison with Wilkinson's Shift

In [17]:
```python
# TAKES QUITE A WHILE - RERUN ONLY IF NEEDED!

size = [5,25,50,75,100]
no_shift = []
Wilkinson = []

for n in size:
    A = np.random.rand(n,n)+1j*np.random.rand(n,n)
    print('\n\nRandom Complex >>> Input shape = ',A.shape)
    eig,_ = np.linalg.eig(A)
    print('\nQR Iteration without shift:')
    eig_, i_ = QR_Iteration_Eigenvalues(A, max_iter = 10000, tol=RTOL)
    print('\nQR Iteration with WILKINSON shift:')
    eig_w, i_w = QR_Iteration_WilkinsonShift(A,max_iter = 10000, tol=RTOL)
    no_shift.append(i_)
    Wilkinson.append(i_w)
    #print('np.linalg.eig:              ', sort_(eig))
    #print('QR_Iteration:               ', sort_(eig_))
    #print('QR_Iteration_WilkinsonShift:', sort_(eig_w))

plt.plot(size, no_shift,marker='o',color='red',linestyle='dashed',label='Without Shift')
plt.plot(size, Wilkinson,marker='x',color='blue',linestyle='dotted',label='Wilkinson Shift')
plt.title('Comparison of QR Iteration Convergence')
plt.xlabel('Size of Input Matrix')
plt.ylabel('Num of Iterations')
plt.legend()
plt.show()

```

```
Random Complex >>> Input shape =  (5, 5)

QR Iteration without shift:
Iteration terminates at i=559 with tol=1e-06 reached

QR Iteration with WILKINSON shift:
Iteration terminates at i=18 with tol=1e-06 reached


Random Complex >>> Input shape =  (25, 25)

QR Iteration without shift:
Iteration terminates at i=3283 with tol=1e-06 reached

QR Iteration with WILKINSON shift:
Iteration terminates at i=312 with tol=1e-06 reached


Random Complex >>> Input shape =  (50, 50)

QR Iteration without shift:
```

```
Iteration terminates at i=8193 with tol=1e-06 reached

QR Iteration with WILKINSON shift:
Iteration terminates at i=1132 with tol=1e-06 reached


Random Complex >>> Input shape =  (75, 75)

QR Iteration without shift:
max_iter=10000 reached, max error=0.00947315121975121 vs tol=1e-06; index of non-convergence=[[16], [17], [36], [38]]

QR Iteration with WILKINSON shift:
Iteration terminates at i=3130 with tol=1e-06 reached


Random Complex >>> Input shape =  (100, 100)

QR Iteration without shift:
max_iter=10000 reached, max error=0.013843580120954222 vs tol=1e-06; index of non-convergence=[[21], [22], [37], [38], [39], [40], [41]]

QR Iteration with WILKINSON shift:
Iteration terminates at i=5223 with tol=1e-06 reached
```



Eigenvectors

Two similar matrices, A and B, with similarity transformation $B = S^{-1}AS$, have the same set of eigenvalues and their eigenvectors are related $v_B = S^{-1}v_A$.

Reference sources:

- Implementing the QR algorithm for efficiently computing matrix eigenvalues and eigenvectors - Section 4.5 (p.51) https://addi.ehu.es/bitstream/handle/10810/26427/TFG_Erana_Robles_Gorka.pdf?sequence=1 (https://addi.ehu.es/bitstream/handle/10810/26427/TFG_Erana_Robles_Gorka.pdf?sequence=1)
- Find eigenvectors of an upper triangular matrix

(1) https://math.stackexchange.com/questions/2632460/general-form-of-left-and-right-eigenvectors-of-upper-triangular-matrices (https://math.stackexchange.com/questions/2632460/general-form-of-left-and-right-eigenvectors-of-upper-triangular-matrices)

(2) https://math.stackexchange.com/questions/3947108/how-to-get-eigenvectors-using-qr-algorithm (https://math.stackexchange.com/questions/3947108/how-to-get-eigenvectors-using-qr-algorithm)

In [18]:

```
# Find Eigenvector of an upper triangular matrix given eigenvectors

def eigenvectors_upperTriangularMatrix(U,eigenvalues):

    """     Compute the eigenvectors of an upper triangular matrix, based on given eigenvalues
            Source: https://addi.ehu.es/bitstream/handle/10810/26427/TFG_Erana_Robles_Gorka.pdf?sequence=1
    Input   ----------------------------------------------
            U: 2D np.array in the form of an upper triangular matrix
            eigenvalues: 1D np.array, real eigenvalues associated with matrix U
    Return  ---------------------------------------------
            V: 2D np.array with each column represents an eigenvector, in the same order as the associated eigenvalue
    """

    n = U.shape[0]

    V = np.eye(n, dtype=U.dtype) # in case of complex matrix, enforce dtype

    for i in range(1,n,1):
        for j in range(i-1,-1,-1):
            V[j,i]= - np.dot(U[j,:],V[:,i]) / (U[j,j]-eigenvalues[i])

        V[:,i] = V[:,i]/np.linalg.norm(V[:,i]) # vector normalization

    return V


```

In [19]:

```python
# Test for eigenvectors

def test_eigenvectors(eigenvectors, v, verbose=True):

    """     Compare two sets of eigenvectors
    Input   ----------------------------------------------
            eigenvectors: 2D np.array with real or complex entries, eigenvectors from my function
            v: 2D np.array, eigenvectors from np.linalg.eig(), basis for comparison
            verbose: bool, indicating whether to print the two sets of eigenvectors and ratios of eigenvectors/v
    Return  ---------------------------------------------
            None; test results will be printed
    """


    if np.allclose(eigenvectors,v,rtol=RTOL):
        print('np.linalg.eig and my function return the same eigenvectors? True \n')

    else:
        print('np.linalg.eig and my function return the same eigenvectors? False \n')
        d = np.divide(eigenvectors,v, where=(v!=0))
        if verbose:
            print('my eigenvectors =\n',eigenvectors,'\nnp.linalg.eig =\n',v, '\nmy eigenvectors / np.linalg.eig =\n', d,'\n')
        d_ = np.divide(d,d[0], where=(d[0]!=0))
        print('Eigenvectors all close except sign and/or scaling?', np.allclose(d_,np.ones(d.shape,d.dtype),rtol=RTOL))

        # special case of upper triangular matrix
        if np.allclose(np.tril(v,-1),np.zeros(d.shape,d.dtype)):
            print('[Upper Triagular Matrix] Eigenvectors all close except signs?',
                  np.allclose(np.abs(eigenvectors),np.abs(v),rtol=RTOL),'\n')
        print('\n\n')


def test_eigenvectors_function(A,eigenvalues,function):

    """     Perform eigenvector testing on selected function; eigenvectors are compared with results from np.linalg.eig()
    Input   ----------------------------------------------
            A: input matrix, 2D np.array with real or complex entries
            function: name of the function being tested
    Return  ---------------------------------------------
            None; test results will be printed
    """

    print("------ Testing for function: {}() ------".format(function.__name__))
    eigenvectors = function(A, eigenvalues)
    w,v = np.linalg.eig(A)

    test_eigenvectors(eigenvectors, v)

```

In [20]:

```python
### Testing eigenvectors_upperTriangularMatrix() on Random Real Upper Triangular Matrices

print('\n\n')
print("====================================================================================")
print('Testing eigenvectors_upperTriangularMatrix() on Random REAL Upper Triangular Matrices')
print("====================================================================================")

for A in RANDOM_REAL_MATRICES:
    U = np.triu(A)
    print('\nREAL >>> Input shape = {}\n'.format(U.shape))
    print('U = \n',U)
    eigenvalues = np.diag(U)
    test_eigenvectors_function(U,eigenvalues,eigenvectors_upperTriangularMatrix)


### Testing eigenvectors_upperTriangularMatrix() on Random COMPLEX Upper Triangular Matrices

print('\n\n')
print("====================================================================================")
print('Testing eigenvectors_upperTriangularMatrix() on Random COMPLEX Upper Triangular Matrices')
print("====================================================================================")

for A in RANDOM_COMPLEX_MATRICES:
    U = np.triu(A)
    print('\nCOMPLEX >>> Input shape = {}\n'.format(U.shape))
    print('U = \n',U)
    eigenvalues = np.diag(U)
    test_eigenvectors_function(U,eigenvalues,eigenvectors_upperTriangularMatrix)

```

```
====================================================================================
Testing eigenvectors_upperTriangularMatrix() on Random REAL Upper Triangular Matrices
====================================================================================

REAL >>> Input shape = (3, 3)

U = 
 [[0.261 0.15  0.741]
 [0.    0.174 0.07 ]
 [0.    0.    0.583]]
------ Testing for function: eigenvectors_upperTriangularMatrix() ------
np.linalg.eig and my function return the same eigenvectors? True


REAL >>> Input shape = (5, 5)

U = 
 [[0.118 0.985 0.164 0.45  0.808]
```

```
 [0.    0.35  0.155 0.424 0.595]
 [0.    0.    0.497 0.135 0.143]
 [0.    0.    0.    0.706 0.689]
 [0.    0.    0.    0.    0.418]]
------ Testing for function: eigenvectors_upperTriangularMatrix() ------
np.linalg.eig and my function return the same eigenvectors? True


REAL >>> Input shape = (10, 10)

U =
 [[0.842 0.604 0.581 0.121 0.591 0.322 0.824 0.236 0.569 0.429]
 [0.    0.375 0.832 0.701 0.155 0.575 0.078 0.99  0.485 0.793]
 [0.    0.    0.083 0.414 0.819 0.3   0.075 0.567 0.834 0.001]
 [0.    0.    0.    0.009 0.453 0.353 0.712 0.761 0.973 0.933]
 [0.    0.    0.    0.    0.361 0.201 0.964 0.459 0.884 0.486]
 [0.    0.    0.    0.    0.    0.199 0.453 0.445 0.676 0.858]
 [0.    0.    0.    0.    0.    0.    0.784 0.963 0.477 0.747]
 [0.    0.    0.    0.    0.    0.    0.    0.897 0.068 0.115]
 [0.    0.    0.    0.    0.    0.    0.    0.    0.306 0.268]
 [0.    0.    0.    0.    0.    0.    0.    0.    0.    0.561]]
------ Testing for function: eigenvectors_upperTriangularMatrix() ------
np.linalg.eig and my function return the same eigenvectors? True




=====================================================================================
Testing eigenvectors_upperTriangularMatrix() on Random COMPLEX Upper Triangular Matrices
=====================================================================================

COMPLEX >>> Input shape = (3, 3)

U =
 [[0.9  +0.122j 0.418+0.753j 0.391+0.67j ]
 [0.   +0.j    0.32 +0.669j 0.602+0.798j]
 [0.   +0.j    0.   +0.j    0.763+0.183j]]
------ Testing for function: eigenvectors_upperTriangularMatrix() ------
np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 1.   +0.j     0.181-0.711j  0.867-0.459j]
 [ 0.   +0.j     0.679+0.j    -0.03 +0.16j ]
 [ 0.   +0.j     0.   +0.j     0.107+0.j   ]]
np.linalg.eig =
 [[ 1.   +0.j     0.734+0.j     0.981+0.j   ]
 [ 0.   +0.j     0.168+0.658j -0.102+0.127j]
 [ 0.   +0.j     0.   +0.j     0.095+0.05j ]]
my eigenvectors / np.linalg.eig =
 [[1.   +0.j    0.247-0.969j 0.884-0.468j]
 [0.   +0.j    0.247-0.969j 0.884-0.468j]
 [0.   +0.j    0.   +0.j    0.884-0.468j]]
```

```
Eigenvectors all close except sign and/or scaling? False
[Upper Triagular Matrix] Eigenvectors all close except signs? True




COMPLEX >>> Input shape = (5, 5)

U =
 [[0.893+0.697j 0.926+0.527j 0.219+0.222j 0.412+0.28j  0.684+0.401j]
 [0.   +0.j    0.   +0.605j 0.197+0.516j 0.827+0.086j 0.448+0.326j]
 [0.   +0.j    0.   +0.j    0.263+0.364j 0.416+0.594j 0.687+0.376j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.257+0.995j 0.286+0.426j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.157+0.737j]]
------ Testing for function: eigenvectors_upperTriangularMatrix() ------
np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 1.   +0.j    -0.7  -0.308j  0.17 -0.763j -0.791-0.328j  0.812-0.169j]
 [ 0.   +0.j     0.644+0.j    -0.193+0.487j  0.36 -0.289j -0.379+0.381j]
 [ 0.   +0.j     0.   +0.j     0.338+0.j     0.141-0.101j -0.064+0.066j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.151+0.j    -0.105+0.024j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.058+0.j   ]]
np.linalg.eig =
 [[ 1.   +0.j     0.765+0.j     0.782+0.j     0.857+0.j     0.829+0.j   ]
 [ 0.   +0.j    -0.59 +0.259j -0.517-0.083j -0.221+0.405j -0.449+0.295j]
 [ 0.   +0.j    -0.   +0.j     0.074+0.33j  -0.092+0.147j -0.076+0.052j]
 [ 0.   +0.j    -0.   +0.j     0.   +0.j    -0.14 +0.058j -0.108+0.002j]
 [ 0.   +0.j    -0.   +0.j     0.   +0.j    -0.   +0.j     0.057+0.012j]]
my eigenvectors / np.linalg.eig =
 [[ 1.   +0.j    -0.915-0.403j  0.217-0.976j -0.924-0.383j  0.979-0.204j]
 [ 0.   +0.j    -0.915-0.403j  0.217-0.976j -0.924-0.383j  0.979-0.204j]
 [ 0.   +0.j     0.   +0.j     0.217-0.976j -0.924-0.383j  0.979-0.204j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j    -0.924-0.383j  0.979-0.204j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.979-0.204j]]

Eigenvectors all close except sign and/or scaling? False
[Upper Triagular Matrix] Eigenvectors all close except signs? True




COMPLEX >>> Input shape = (10, 10)

U =
 [[0.685+0.497j 0.168+0.663j 0.181+0.56j  0.225+0.179j 0.347+0.007j 0.236+0.152j 0.646+1.j    0.096+0.474j 0.678+0.406j 0.228+0.469j]
 [0.   +0.j    0.253+0.336j 0.681+0.165j 0.438+0.96j  0.717+0.246j 0.117+0.129j 0.889+0.353j 0.402+0.735j 0.205+0.427j 0.34 +0.28j ]
 [0.   +0.j    0.   +0.j    0.279+0.976j 0.698+0.974j 0.234+0.915j 0.981+0.724j 0.202+0.752j 0.924+0.962j 0.867+0.988j 0.71 +0.413j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.754+0.469j 0.166+0.42j  0.632+0.601j 0.636+0.471j 0.047+0.719j 0.322+0.934j 0.134+0.476j]
```

```
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.011+0.797j 0.927+0.193j 0.572+0.391j 0.149+0.855j 0.352+0.962j 0.794+0.865j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.414+0.68j  0.225+0.995j 0.624+0.036j 0.229+0.228j 0.9  +0.424j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.038+0.901j 0.533+0.388j 0.101+0.3j   0.032+0.014j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.166+0.035j 0.798+0.694j 0.68 +0.182j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.788+0.359j 0.03 +0.419j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.721+0.572j]]
------ Testing for function: eigenvectors_upperTriangularMatrix() ------
np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 1.   +0.j    -0.472-0.682j -0.109-0.755j -0.994+0.06j   0.145+0.587j  0.506-0.757j  0.598-0.252j  0.597+0.162j  0.048+0.975j -0.211+0.972j]
 [ 0.   +0.j     0.559+0.j     0.131-0.459j  0.019+0.081j -0.037+0.381j  0.372-0.007j  0.457+0.019j -0.706-0.122j  0.141-0.126j  0.005+0.09j ]
 [ 0.   +0.j     0.   +0.j     0.437+0.j    -0.007+0.036j -0.652-0.01j   0.043+0.167j  0.064+0.586j  0.08 +0.177j  0.073-0.042j -0.022+0.035j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.021+0.j     0.005-0.121j  0.011-0.044j -0.077+0.015j  0.023-0.128j -0.036-0.045j  0.019+0.006j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.219+0.j     0.021+0.011j -0.057-0.105j -0.161+0.027j -0.006+0.004j -0.   +0.002j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.011+0.j     0.004-0.013j  0.033-0.009j -0.002+0.001j  0.   +0.001j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.006+0.j    -0.041+0.078j -0.   +0.002j  0.   +0.j   ]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.117+0.j     0.002+0.001j  0.   -0.j   ]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.002+0.j     0.   -0.j   ]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j   ]]
np.linalg.eig =
 [[ 1.   +0.j     0.829+0.j     0.762+0.j     0.996+0.j    -0.154-0.585j  0.91 +0.j     0.65 +0.j    -0.616-0.059j  0.976+0.j     0.995+0.j   ]
 [ 0.   +0.j    -0.318+0.46j   0.436+0.195j -0.014-0.082j  0.031-0.382j  0.212+0.306j  0.414+0.196j  0.717+0.j    -0.119-0.147j  0.087-0.024j]
 [ 0.   +0.j    -0.   +0.j    -0.062+0.432j  0.009-0.035j  0.652+0.j    -0.115+0.129j -0.169+0.565j -0.109-0.161j -0.038-0.075j  0.039+0.014j]
 [ 0.   +0.j    -0.   +0.j    -0.   +0.j    -0.021-0.001j -0.003+0.121j  0.042-0.015j -0.077-0.016j -0.001+0.13j  -0.047+0.034j  0.002-0.02j ]
 [ 0.   +0.j    -0.   +0.j    -0.   +0.j     0.   -0.j    -0.219+0.003j  0.002+0.024j -0.011-0.119j  0.154-0.054j  0.004+0.007j  0.002-0.j   ]
 [ 0.   +0.j    -0.   +0.j    -0.   +0.j     0.   -0.j    -0.   +0.j     0.006+0.009j  0.009-0.01j  -0.031+0.014j  0.001+0.003j  0.001-0.j   ]
 [ 0.   +0.j    -0.   +0.j    -0.   +0.j     0.   -0.j    -0.   +0.j     0.   +0.j     0.005+0.002j  0.027-0.084j  0.002+0.j     0.   -0.j   ]
 [ 0.   +0.j    -0.   +0.j    -0.   +0.j     0.   -0.j    -0.   +0.j     0.   +0.j     0.   +0.j    -0.115+0.02j   0.001-0.002j -0.   -0.j   ]
 [ 0.   +0.j    -0.   +0.j    -0.   +0.j     0.   -0.j    -0.   +0.j     0.   +0.j     0.   +0.j    -0.   +0.j     0.   -0.002j -0.   -0.j   ]
 [ 0.   +0.j    -0.   +0.j    -0.   +0.j     0.   -0.j    -0.   +0.j     0.   +0.j     0.   +0.j    -0.   +0.j     0.   +0.j    -0.   -0.j   ]]
my eigenvectors / np.linalg.eig =
 [[ 1.   +0.j    -0.569-0.822j -0.143-0.99j  -0.998+0.06j  -1.   -0.015j  0.556-0.831j  0.921-0.389j -0.985-0.17j   0.049+0.999j -0.212+0.977j]
 [ 0.   +0.j    -0.569-0.822j -0.143-0.99j  -0.998+0.06j  -1.   -0.015j  0.556-0.831j  0.921-0.389j -0.985-0.17j   0.049+0.999j -0.212+0.977j]
 [ 0.   +0.j     0.   +0.j    -0.143-0.99j  -0.998+0.06j  -1.   -0.015j  0.556-0.831j  0.921-0.389j -0.985-0.17j   0.049+0.999j -0.212+0.977j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j    -0.998+0.06j  -1.   -0.015j  0.556-0.831j  0.921-0.389j -0.985-0.17j   0.049+0.999j -0.212+0.977j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j    -1.   -0.015j  0.556-0.831j  0.921-0.389j -0.985-0.17j   0.049+0.999j -0.212+0.977j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.556-0.831j  0.921-0.389j -0.985-0.17j   0.049+0.999j -0.212+0.977j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.921-0.389j -0.985-0.17j   0.049+0.999j -0.212+0.977j]
 [ 0.   +0.j     0.   -0.j     0.   +0.j       nan+0.j     0.   +0.j     0.   +0.j     0.   +0.j    -0.985-0.17j   0.049+0.999j -0.212+0.977j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.049+0.999j -0.212+0.977j]
 [ 0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j     0.   +0.j    -0.212+0.977j]]

Eigenvectors all close except sign and/or scaling? False
[Upper Triagular Matrix] Eigenvectors all close except signs? True
```

Observations:

- Seems to be working except for the signs of eigenvectors

In [21]:

```python
# General method for obtaining eigenvectors

def inverse_power_method_with_shift(A, s, max_iter=MAX_ITER, rtol=RTOL):

    """     Apply the inverse power method with shift to compute the eigenvectors associated with an eigenvalue which is close to the shift
    Input   ----------------------------------------------
            A: input matrix, 2D np.array wiht real or complex entries
            s: shift to be performed, which is closed to an eigenvalue
            max_iter: int, maximum number of iterations to be performed
            rtol: float, relative tolerance between eigenvalues by successive iterations
    Return  ---------------------------------------------
            x_new: 1D np.array, the eigenvector associated with the eigenvalue which is closed to the given shift
    """

    n = A.shape[0]
    A_ = A.astype(s.dtype) - s*np.eye(n,dtype=s.dtype)
    x_old = np.ones(n, dtype=s.dtype) #initialization as [1,1,1,...]

    for i in range(max_iter):
        y = np.linalg.solve(A_,x_old) # assuming this can be used? Potentially Singular Matrix Error...
        x_new = y/np.linalg.norm(y)

        if np.allclose(x_new,x_old,rtol=rtol):
            print('... num of iterations with inverse power method =',i)
            break
        x_old = x_new

    return x_new



def eigenvectors_inversePowerMethod(U,eigenvalues):

    """     Compute the eigenvectors of a block triangular matrix, based on given eigenvalues
    Input   ----------------------------------------------
            U: 2D np.array in the form of a block triangular matrix with complex eigenvalues
            eigenvalues: 1D np.array, complex eigenvalues associated with matrix U
    Return  ---------------------------------------------
            V: 2D np.array with each column represents an eigenvector, in the same order as the associated eigenvalues
    """

    n = U.shape[0]
    V = np.eye(n, dtype='complex') # only used in case of complex matrix, need to enforce dtype

    for i in range(n):
        V[:,i] = inverse_power_method_with_shift(U, eigenvalues[i])

    # print('\n ----\n',V.dtype, 'eigenvectors_inversePowerMethod=\n',V,)

    return V

```

In [22]:
```python
# Testing

print('\n\n')
print("====================================================================================")
print('Testing eigenvectors_inversePowerMethod() on Random REAL Matrices')
print("====================================================================================")

for A in RANDOM_REAL_MATRICES[-1:]:
    print('\nREAL >>> Input shape = {}\n'.format(A.shape))
    w,v = np.linalg.eig(A)
    eigenvalues = w
    test_eigenvectors_function(A,eigenvalues,eigenvectors_inversePowerMethod)


print('\n\n')
print("====================================================================================")
print('Testing eigenvectors_inversePowerMethod() on Random COMPLEX Matrices')
print("====================================================================================")

for A in RANDOM_COMPLEX_MATRICES[-1:]:
    print('\nCOMPLEX >>> Input shape = {}\n'.format(A.shape))
    w,v = np.linalg.eig(A)
    eigenvalues = w
    test_eigenvectors_function(A,eigenvalues,eigenvectors_inversePowerMethod)


print('\n\n')
print("====================================================================================")
print('Testing eigenvectors_inversePowerMethod() on Random COMPLEX Upper *BLOCK* Triangular Matrices')
print("====================================================================================")

for A in RANDOM_COMPLEX_MATRICES[-1:]:
    print('\nCOMPLEX >>> Input shape = {}\n'.format(A.shape))
    U = np.triu(A,-1)
    print('U = \n',U)
    w,v = np.linalg.eig(U)
    eigenvalues = w
    test_eigenvectors_function(U,eigenvalues,eigenvectors_inversePowerMethod)

```

```
====================================================================================
Testing eigenvectors_inversePowerMethod() on Random REAL Matrices
====================================================================================

REAL >>> Input shape = (10, 10)

------ Testing for function: eigenvectors_inversePowerMethod() ------
... num of iterations with inverse power method = 1
... num of iterations with inverse power method = 1
```

```
... num of iterations with inverse power method = 1
np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 0.331+0.j    -0.054-0.15j  -0.054+0.15j   0.071+0.129j  0.071-0.129j -0.311+0.158j -0.311-0.158j -0.294+0.j     0.33 +0.j     0.052+0.j   ]
 [ 0.348+0.j     0.41 +0.153j  0.41 -0.153j  0.264+0.081j  0.264-0.081j -0.463-0.138j -0.463+0.138j -0.629+0.j     0.176+0.j     0.328+0.j   ]
 [ 0.245+0.j    -0.303+0.335j -0.303-0.335j -0.341-0.131j -0.341+0.131j -0.108-0.036j -0.108+0.036j  0.131+0.j    -0.443+0.j     0.015+0.j   ]
 [ 0.326+0.j     0.13 +0.103j  0.13 -0.103j -0.222+0.088j -0.222-0.088j  0.153-0.043j  0.153+0.043j  0.273+0.j    -0.087+0.j    -0.237+0.j   ]
 [ 0.332+0.j     0.065-0.392j  0.065+0.392j -0.393-0.072j -0.393+0.072j  0.198-0.114j  0.198+0.114j  0.299+0.j    -0.383+0.j     0.196+0.j   ]
 [ 0.336+0.j     0.053-0.293j  0.053+0.293j  0.074-0.157j  0.074+0.157j -0.354-0.077j -0.354+0.077j -0.406+0.j     0.706+0.j    -0.795+0.j   ]
 [ 0.317+0.j     0.313+0.053j  0.313-0.053j  0.003-0.115j  0.003+0.115j  0.522+0.156j  0.522-0.156j  0.385+0.j    -0.053+0.j    -0.267+0.j   ]
 [ 0.35 +0.j    -0.217-0.002j -0.217+0.002j  0.281+0.038j  0.281-0.038j  0.283-0.008j  0.283+0.008j -0.116+0.j    -0.096+0.j     0.245+0.j   ]
 [ 0.273+0.j    -0.277-0.017j -0.277+0.017j  0.487+0.037j  0.487-0.037j -0.194+0.04j  -0.194-0.04j   0.066+0.j    -0.015+0.j    -0.013+0.j   ]
 [ 0.286+0.j     0.031+0.282j  0.031-0.282j -0.435+0.002j -0.435-0.002j -0.018-0.083j -0.018+0.083j  0.079+0.j     0.011+0.j     0.178+0.j   ]]
np.linalg.eig =
 [[-0.331+0.j    -0.075+0.14j  -0.075-0.14j   0.081+0.123j  0.081-0.123j  0.252-0.241j  0.252+0.241j -0.294+0.j     0.33 +0.j     0.052+0.j   ]
 [-0.348+0.j    -0.161-0.407j -0.161+0.407j  0.269+0.06j   0.269-0.06j   0.483-0.j     0.483+0.j    -0.629+0.j     0.176+0.j     0.328+0.j   ]
 [-0.245+0.j     0.452+0.j     0.452-0.j    -0.35 -0.104j -0.35 +0.104j  0.114+0.004j  0.114-0.004j  0.131+0.j    -0.443+0.j     0.015+0.j   ]
 [-0.326+0.j    -0.011-0.166j -0.011+0.166j -0.215+0.105j -0.215-0.105j -0.134+0.085j -0.134-0.085j  0.273+0.j    -0.087+0.j    -0.237+0.j   ]
 [-0.332+0.j    -0.334+0.215j -0.334-0.215j -0.398-0.042j -0.398+0.042j -0.157+0.166j -0.157-0.166j  0.299+0.j    -0.383+0.j     0.196+0.j   ]
 [-0.336+0.j    -0.253+0.157j -0.253-0.157j  0.062-0.163j  0.062+0.163j  0.361-0.027j  0.361+0.027j -0.406+0.j     0.706+0.j    -0.795+0.j   ]
 [-0.317+0.j    -0.171-0.268j -0.171+0.268j -0.006-0.115j -0.006+0.115j -0.545+0.j    -0.545-0.j     0.385+0.j    -0.053+0.j    -0.267+0.j   ]
 [-0.35 +0.j     0.143+0.162j  0.143-0.162j  0.283+0.017j  0.283-0.017j -0.269+0.089j -0.269-0.089j -0.116+0.j    -0.096+0.j     0.245+0.j   ]
 [-0.273+0.j     0.173+0.217j  0.173-0.217j  0.488+0.j     0.488-0.j     0.174-0.093j  0.174+0.093j  0.066+0.j    -0.015+0.j    -0.013+0.j   ]
 [-0.286+0.j     0.188-0.212j  0.188+0.212j -0.433+0.035j -0.433-0.035j  0.041+0.075j  0.041-0.075j  0.079+0.j     0.011+0.j     0.178+0.j   ]]
my eigenvectors / np.linalg.eig =
 [[-1.   -0.j    -0.67 +0.742j -0.67 -0.742j  0.997+0.076j  0.997-0.076j -0.958-0.286j -0.958+0.286j  1.   -0.j     1.   +0.j     1.   +0.j   ]
 [-1.   -0.j    -0.67 +0.742j -0.67 -0.742j  0.997+0.076j  0.997-0.076j -0.958-0.286j -0.958+0.286j  1.   -0.j     1.   +0.j     1.   +0.j   ]
 [-1.   -0.j    -0.67 +0.742j -0.67 -0.742j  0.997+0.076j  0.997-0.076j -0.958-0.286j -0.958+0.286j  1.   +0.j     1.   -0.j     1.   +0.j   ]
 [-1.   -0.j    -0.67 +0.742j -0.67 -0.742j  0.997+0.076j  0.997-0.076j -0.958-0.286j -0.958+0.286j  1.   +0.j     1.   -0.j     1.   -0.j   ]
 [-1.   -0.j    -0.67 +0.742j -0.67 -0.742j  0.997+0.076j  0.997-0.076j -0.958-0.286j -0.958+0.286j  1.   +0.j     1.   -0.j     1.   +0.j   ]
 [-1.   -0.j    -0.67 +0.742j -0.67 -0.742j  0.997+0.076j  0.997-0.076j -0.958-0.286j -0.958+0.286j  1.   -0.j     1.   +0.j     1.   -0.j   ]
 [-1.   -0.j    -0.67 +0.742j -0.67 -0.742j  0.997+0.076j  0.997-0.076j -0.958-0.286j -0.958+0.286j  1.   +0.j     1.   -0.j     1.   -0.j   ]
 [-1.   -0.j    -0.67 +0.742j -0.67 -0.742j  0.997+0.076j  0.997-0.076j -0.958-0.286j -0.958+0.286j  1.   -0.j     1.   -0.j     1.   +0.j   ]
 [-1.   -0.j    -0.67 +0.742j -0.67 -0.742j  0.997+0.076j  0.997-0.076j -0.958-0.286j -0.958+0.286j  1.   +0.j     1.   -0.j     1.   -0.j   ]
 [-1.   -0.j    -0.67 +0.742j -0.67 -0.742j  0.997+0.076j  0.997-0.076j -0.958-0.286j -0.958+0.286j  1.   +0.j     1.   +0.j     1.   +0.j   ]]

Eigenvectors all close except sign and/or scaling? True





=================================================================================
Testing eigenvectors_inversePowerMethod() on Random COMPLEX Matrices
=================================================================================

COMPLEX >>> Input shape = (10, 10)

------ Testing for function: eigenvectors_inversePowerMethod() ------
np.linalg.eig and my function return the same eigenvectors? False
```

```
my eigenvectors =
 [[ 0.203-0.12j   0.114+0.01j   0.108-0.411j -0.072-0.042j -0.332+0.048j -0.057-0.057j -0.174-0.344j  0.126-0.125j -0.16 +0.196j -0.206-0.021j]
 [ 0.223-0.17j  -0.211+0.16j  -0.111+0.113j -0.066+0.203j -0.507+0.005j  0.469-0.019j  0.393-0.085j -0.18 -0.117j  0.448+0.133j -0.403+0.513j]
 [ 0.358-0.204j  0.261+0.097j -0.227+0.054j -0.35 -0.063j  0.483-0.119j -0.261-0.197j  0.027-0.229j -0.122+0.156j -0.21 -0.012j  0.171-0.53j ]
 [ 0.248-0.135j  0.035+0.221j  0.113-0.256j -0.026-0.27j  -0.102+0.092j  0.201-0.087j  0.21 -0.111j  0.09 +0.555j  0.132-0.358j  0.127+0.075j]
 [ 0.349-0.167j -0.315+0.006j -0.179-0.347j -0.423-0.186j  0.034+0.067j -0.407-0.24j  -0.5  +0.076j -0.146-0.479j  0.344+0.181j -0.129+0.155j]
 [ 0.25 -0.202j -0.192-0.267j  0.153-0.049j  0.262+0.389j  0.33 +0.144j -0.332+0.245j -0.115-0.053j  0.067+0.108j -0.217-0.084j  0.19 -0.072j]
 [ 0.201-0.103j -0.157+0.009j  0.085+0.098j  0.253+0.127j -0.068-0.259j -0.086-0.15j  -0.117+0.089j  0.206-0.117j -0.083+0.13j   0.066-0.127j]
 [ 0.263-0.176j  0.172-0.174j  0.282+0.402j -0.121+0.017j  0.06 +0.003j  0.132+0.342j -0.177+0.133j  0.224-0.147j -0.108+0.118j -0.064+0.068j]
 [ 0.237-0.225j  0.578-0.255j -0.174+0.081j  0.407-0.128j -0.141-0.167j  0.216+0.101j  0.298+0.369j -0.393+0.018j  0.235-0.074j  0.058+0.126j]
 [ 0.25 -0.206j -0.297+0.098j -0.101+0.422j -0.073-0.175j  0.143+0.292j  0.001-0.063j  0.075+0.09j   0.123+0.048j -0.38 -0.259j  0.14 -0.223j]]
np.linalg.eig =
 [[ 0.236-0.004j  0.101+0.055j -0.275-0.325j -0.075+0.036j  0.333-0.045j  0.078+0.021j  0.121+0.366j -0.103-0.145j -0.097+0.234j  0.111+0.175j]
 [ 0.278-0.038j -0.258+0.062j  0.029+0.155j  0.132+0.168j  0.507+0.j    -0.395+0.255j -0.401+0.025j -0.144+0.159j  0.468+0.j     0.652+0.j   ]
 [ 0.412+0.j     0.2  +0.194j -0.086+0.217j -0.248+0.256j -0.484+0.114j  0.325+0.037j -0.061+0.222j  0.134+0.145j -0.205+0.048j -0.523+0.193j]
 [ 0.282+0.006j -0.057+0.216j -0.145-0.239j -0.239-0.129j  0.103-0.091j -0.129+0.177j -0.224+0.078j  0.562+0.j     0.025-0.381j -0.02 -0.146j]
 [ 0.386+0.028j -0.291-0.121j -0.387-0.053j -0.391+0.246j -0.034-0.067j  0.473+0.j     0.506+0.j    -0.497+0.068j  0.381+0.076j  0.201+0.005j]
 [ 0.318-0.052j -0.068-0.322j  0.048-0.154j  0.469+0.j    -0.329-0.148j  0.161-0.379j  0.106+0.07j   0.117-0.049j -0.232-0.019j -0.174-0.105j]
 [ 0.225+0.01j  -0.148-0.055j  0.129-0.014j  0.247-0.139j  0.066+0.259j  0.15 +0.085j  0.129-0.07j  -0.082-0.222j -0.043+0.149j -0.14 +0.027j]
 [ 0.316-0.022j  0.227-0.09j   0.491+0.j    -0.054+0.11j  -0.06 -0.004j -0.287-0.228j  0.194-0.105j -0.109-0.245j -0.07 +0.144j  0.093+0.008j]
 [ 0.317-0.078j  0.632+0.j    -0.033+0.189j  0.121-0.41j   0.139+0.168j -0.237+0.023j -0.24 -0.409j -0.045+0.39j   0.204-0.138j  0.063-0.123j]
 [ 0.319-0.055j -0.311-0.03j   0.288+0.325j -0.186-0.037j -0.14 -0.294j  0.031+0.054j -0.06 -0.1j    0.067-0.114j -0.438-0.14j  -0.262+0.027j]]
my eigenvectors / np.linalg.eig =
 [[ 0.869-0.495j  0.915-0.404j  0.574+0.819j  0.559+0.829j -1.   +0.01j  -0.861-0.508j -0.989+0.15j   0.16 +0.987j  0.959+0.284j -0.618+0.786j]
 [ 0.869-0.495j  0.915-0.404j  0.574+0.819j  0.559+0.829j -1.   +0.01j  -0.861-0.508j -0.989+0.15j   0.16 +0.987j  0.959+0.284j -0.618+0.786j]
 [ 0.869-0.495j  0.915-0.404j  0.574+0.819j  0.559+0.829j -1.   +0.01j  -0.861-0.508j -0.989+0.15j   0.16 +0.987j  0.959+0.284j -0.618+0.786j]
 [ 0.869-0.495j  0.915-0.404j  0.574+0.819j  0.559+0.829j -1.   +0.01j  -0.861-0.508j -0.989+0.15j   0.16 +0.987j  0.959+0.284j -0.618+0.786j]
 [ 0.869-0.495j  0.915-0.404j  0.574+0.819j  0.559+0.829j -1.   +0.01j  -0.861-0.508j -0.989+0.15j   0.16 +0.987j  0.959+0.284j -0.618+0.786j]
 [ 0.869-0.495j  0.915-0.404j  0.574+0.819j  0.559+0.829j -1.   +0.01j  -0.861-0.508j -0.989+0.15j   0.16 +0.987j  0.959+0.284j -0.618+0.786j]
 [ 0.869-0.495j  0.915-0.404j  0.574+0.819j  0.559+0.829j -1.   +0.01j  -0.861-0.508j -0.989+0.15j   0.16 +0.987j  0.959+0.284j -0.618+0.786j]
 [ 0.869-0.495j  0.915-0.404j  0.574+0.819j  0.559+0.829j -1.   +0.01j  -0.861-0.508j -0.989+0.15j   0.16 +0.987j  0.959+0.284j -0.618+0.786j]
 [ 0.869-0.495j  0.915-0.404j  0.574+0.819j  0.559+0.829j -1.   +0.01j  -0.861-0.508j -0.989+0.15j   0.16 +0.987j  0.959+0.284j -0.618+0.786j]
 [ 0.869-0.495j  0.915-0.404j  0.574+0.819j  0.559+0.829j -1.   +0.01j  -0.861-0.508j -0.989+0.15j   0.16 +0.987j  0.959+0.284j -0.618+0.786j]]

Eigenvectors all close except sign and/or scaling? True




==================================================================================
Testing eigenvectors_inversePowerMethod() on Random COMPLEX Upper *BLOCK* Triangular Matrices
==================================================================================

COMPLEX >>> Input shape = (10, 10)

U =
 [[0.685+0.497j 0.168+0.663j 0.181+0.56j  0.225+0.179j 0.347+0.007j 0.236+0.152j 0.646+1.j    0.096+0.474j 0.678+0.406j 0.228+0.469j]
 [0.508+0.872j 0.253+0.336j 0.681+0.165j 0.438+0.96j  0.717+0.246j 0.117+0.129j 0.889+0.353j 0.402+0.735j 0.205+0.427j 0.34 +0.28j ]
 [0.   +0.j    0.4  +0.483j 0.279+0.976j 0.698+0.974j 0.234+0.915j 0.981+0.724j 0.202+0.752j 0.924+0.962j 0.867+0.988j 0.71 +0.413j]
```

```
 [0.   +0.j    0.   +0.j    0.347+0.289j 0.754+0.469j 0.166+0.42j  0.632+0.601j 0.636+0.471j 0.047+0.719j 0.322+0.934j 0.134+0.476j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.215+0.961j 0.011+0.797j 0.927+0.193j 0.572+0.391j 0.149+0.855j 0.352+0.962j 0.794+0.865j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.629+0.281j 0.414+0.68j  0.225+0.995j 0.624+0.036j 0.229+0.228j 0.9  +0.424j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.527+0.245j 0.038+0.901j 0.533+0.388j 0.101+0.3j   0.032+0.014j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.205+0.451j 0.166+0.035j 0.798+0.694j 0.68 +0.182j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.797+0.195j 0.788+0.359j 0.03 +0.419j]
 [0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.   +0.j    0.062+0.735j 0.721+0.572j]]
------ Testing for function: eigenvectors_inversePowerMethod() ------

np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[-0.153-0.476j -0.227-0.105j  0.065+0.077j  0.127+0.002j  0.02 +0.49j   0.422-0.08j   0.146-0.592j -0.453-0.44j  -0.638+0.003j  0.387-0.51j ]
 [-0.294-0.516j  0.564+0.496j -0.677-0.356j -0.288+0.284j -0.205+0.367j -0.615+0.195j  0.398+0.148j  0.192+0.055j -0.612+0.136j  0.213-0.552j]
 [-0.138-0.538j -0.235-0.355j  0.45 +0.303j  0.412+0.199j -0.309-0.364j -0.002-0.3j   -0.25 +0.234j -0.076+0.481j -0.41 +0.088j  0.1  -0.42j ]
 [-0.143-0.191j -0.049+0.202j -0.07 -0.178j -0.018-0.037j -0.148-0.173j  0.252+0.366j -0.059+0.285j  0.4  +0.04j  -0.02 +0.112j -0.147-0.069j]
 [-0.038-0.151j  0.161-0.018j  0.042+0.15j  -0.521-0.172j  0.213-0.38j  -0.267-0.101j -0.055-0.347j -0.107-0.036j  0.033+0.044j -0.105-0.002j]
 [-0.048-0.065j -0.067+0.015j -0.064-0.136j  0.039+0.421j -0.013-0.006j  0.023-0.142j  0.104-0.197j -0.066-0.268j  0.069+0.03j  -0.092+0.043j]
 [-0.021-0.019j  0.123-0.071j  0.023+0.085j  0.017-0.3j    0.141+0.276j  0.061+0.018j  0.054+0.095j  0.064-0.056j  0.045+0.01j  -0.043+0.035j]
 [-0.005-0.004j -0.254-0.076j  0.121-0.012j -0.096-0.136j  0.038+0.079j  0.008+0.027j  0.001+0.11j  -0.001+0.089j  0.009-0.012j  0.02 +0.019j]
 [-0.003+0.j     0.144+0.033j -0.067-0.j     0.018+0.089j  0.01 -0.064j -0.034-0.038j  0.052-0.14j   0.114+0.003j  0.   -0.017j  0.03 +0.012j]
 [-0.001-0.j    -0.034-0.069j  0.018+0.032j  0.056-0.016j -0.055-0.045j  0.022+0.036j -0.122+0.112j -0.214-0.025j  0.007-0.014j  0.02 +0.027j]]
np.linalg.eig =
 [[ 0.49 +0.102j -0.24 +0.071j -0.093-0.038j -0.122+0.038j  0.49 +0.j    -0.426-0.051j  0.609+0.j     0.631+0.j     0.638+0.j     0.64 +0.j   ]
 [ 0.594+0.j     0.751+0.j     0.764+0.j     0.185-0.36j   0.359+0.22j   0.645+0.j    -0.049+0.422j -0.176+0.094j  0.612-0.133j  0.568-0.165j]
 [ 0.536+0.147j -0.411-0.111j -0.539-0.059j -0.454-0.06j  -0.376+0.294j -0.089+0.287j -0.287-0.187j -0.28 -0.398j  0.41 -0.086j  0.395-0.174j]
 [ 0.237-0.03j   0.096+0.184j  0.145+0.125j  0.029+0.029j -0.179+0.141j -0.129-0.425j -0.291+0.011j -0.314+0.25j   0.021-0.111j -0.034-0.159j]
 [ 0.15 +0.042j  0.109-0.12j  -0.107-0.113j  0.549+0.j    -0.371-0.229j  0.224+0.177j  0.323-0.137j  0.102-0.049j -0.033-0.044j -0.061-0.085j]
 [ 0.08 -0.01j  -0.04 +0.056j  0.12 +0.091j -0.169-0.387j -0.007+0.013j -0.065+0.128j  0.216+0.054j  0.234+0.146j -0.069-0.03j  -0.09 -0.047j]
 [ 0.027-0.008j  0.046-0.134j -0.06 -0.064j  0.078+0.29j   0.282-0.13j  -0.053-0.036j -0.08 +0.075j -0.007+0.085j -0.045-0.01j  -0.054-0.013j]
 [ 0.006-0.003j -0.241+0.111j -0.101+0.066j  0.134+0.099j  0.08 -0.035j  0.   -0.029j -0.106+0.028j -0.061-0.065j -0.01 +0.012j -0.003+0.028j]
 [ 0.001-0.003j  0.13 -0.07j   0.059-0.031j -0.045-0.079j -0.064-0.013j  0.021+0.046j  0.148+0.017j -0.084+0.077j -0.   +0.017j  0.009+0.031j]
 [ 0.001-0.001j -0.071-0.029j -0.031-0.02j  -0.048+0.033j -0.047+0.053j -0.01 -0.041j -0.138-0.092j  0.171-0.132j -0.007+0.014j -0.01 +0.032j]]
my eigenvectors / np.linalg.eig =
 [[-0.495-0.869j  0.751+0.661j -0.885-0.465j -0.949-0.314j  0.041+0.999j -0.953+0.303j  0.239-0.971j -0.717-0.697j -1.   +0.005j  0.605-0.796j]
 [-0.495-0.869j  0.751+0.661j -0.885-0.465j -0.949-0.314j  0.041+0.999j -0.953+0.303j  0.239-0.971j -0.717-0.697j -1.   +0.005j  0.605-0.796j]
 [-0.495-0.869j  0.751+0.661j -0.885-0.465j -0.949-0.314j  0.041+0.999j -0.953+0.303j  0.239-0.971j -0.717-0.697j -1.   +0.005j  0.605-0.796j]
 [-0.495-0.869j  0.751+0.661j -0.885-0.465j -0.949-0.314j  0.041+0.999j -0.953+0.303j  0.239-0.971j -0.717-0.697j -1.   +0.005j  0.605-0.796j]
 [-0.495-0.869j  0.751+0.661j -0.885-0.465j -0.949-0.314j  0.041+0.999j -0.953+0.303j  0.239-0.971j -0.717-0.697j -1.   +0.005j  0.605-0.796j]
 [-0.495-0.869j  0.751+0.661j -0.885-0.465j -0.949-0.314j  0.041+0.999j -0.953+0.303j  0.239-0.971j -0.717-0.697j -1.   +0.005j  0.605-0.796j]
 [-0.495-0.869j  0.751+0.661j -0.885-0.465j -0.949-0.314j  0.041+0.999j -0.953+0.303j  0.239-0.971j -0.717-0.697j -1.   +0.005j  0.605-0.796j]
 [-0.495-0.869j  0.751+0.661j -0.885-0.465j -0.949-0.314j  0.041+0.999j -0.953+0.303j  0.239-0.971j -0.717-0.697j -1.   +0.005j  0.605-0.796j]
 [-0.495-0.869j  0.751+0.661j -0.885-0.465j -0.949-0.314j  0.041+0.999j -0.953+0.303j  0.239-0.971j -0.717-0.697j -1.   +0.005j  0.605-0.796j]
 [-0.495-0.869j  0.751+0.661j -0.885-0.465j -0.949-0.314j  0.041+0.999j -0.953+0.303j  0.239-0.971j -0.717-0.697j -1.   +0.005j  0.605-0.796j]]

Eigenvectors all close except sign and/or scaling? True
```

Observations:

- Seems to be working except for the signs and/or scaling of eigenvectors

In [28]:

```python
# Modified function to take Hessenberg matrix as an input and added optionality to output Q matrix for eigenvector calculation

def eigen_solver(H, S, max_iter, tol, eigvec, shift, tridiagonal):

    """     Compute the eigenvalues and eigenvectors of a Hessenberg matrix
    Input   ---------------------------------------------
            H: 2D np.array in the form of a Hessenberg matrix with real or complex entries
            S: similarity transformation associated with the Hessenberg reduction, H = S @ A @ S.T where A is the input matrix
            max_iter: int, maximum number of iterations to be performed
            tol: float, relative tolerance between eigenvalues by successive iterations
            eigenvec: bool, indicating whether eigenvectors are to be calculated
            shift: options - None or 'Wilkinson'
            tridiagonal: bool, indicating whether the input H is a triadiagonal matrix (i.e. original input A is symmetric)
    Return  ---------------------------------------------
            eigvals_new: 1D np.array, eigenvalues
            eigenvectors: 2D np.array with each column represents an eigenvector, in the same order as the associated eigenvalues;
                          if eigvec is False, then this default to a zero matrix
    """

    n = H.shape[0]

    H_old = H.copy() # H will NOT be updated inplace - this is necessary because we are applying different methods on the same H matrix
    # H_old = H # for optimizing storage if no need to display comparison of methods

    eigvals_old = diagonal_block(H)

    if eigvec:
        T = np.eye(n,dtype=eigvals_old.dtype)

    if not shift:
        print('>>> QR Iteration WITHOUT shift')
    else:
        print('>>> QR Iteration with {} shift'.format(shift))

    for i in range(max_iter):

        if shift == 'Wilkinson':
            datatype, lam = Wilkinson_shift(H_old)
            #print(">>> QR Iteration with Wilkinson's Shift: lambda = {}, {}".format(lam, datatype))

        if shift:
            # H_old = H_old - lam * np.eye(n,dtype=datatype) #naive implementation O(2n^2)
            for j in range(n):
                H_old[j,j] -= lam #elementwise only O(n)

        H_new, G_iter = QR_Givens(H_old,eigvec,True, tridiagonal) # updating H_old in-place # default QR_Givens(H,eigvec=False, inplace=True, tridiag

        if shift:
            # H_new = H_new +  lam * np.eye(n,dtype=datatype) #naive implementation O(2n^2)
            for j in range(n):
                H_new[j,j] += lam #elementwise only O(n)

```

```
        eigvals_new = diagonal_block(H_new)
        H_old = H_new # necessary to reset H_old = H_new


        # if eigenvectors are required
        if eigvec:
            T = G_iter @ T   # Q = I.Q1.Q2.Q3...Qk; Q1 = G1*; T*= Q = G1*...Gk* ==> T = Gk...G1.I

        # Test for convergence: relative tol = 1- w / w' for each eigenvalue
        if np.allclose(eigvals_new, eigvals_old, rtol=tol):
            print('Iteration terminates at i={} with tol={} reached'.format(i,tol))
            break

        if i == max_iter-1:
            err = np.max(abs((eigvals_new - eigvals_old)) / abs(eigvals_old))
            idx = np.argwhere(np.isclose(eigvals_new, eigvals_old, rtol=tol)==False).tolist()
            print('max_iter={} reached, max error={} vs tol={}; index of non-convergence={}'.format(max_iter, err, tol, idx))

        eigvals_old = eigvals_new

    # Sort eigenvalues in descending order
    eigvals_new = sort_(eigvals_new)

    # Compute eigenvectors if required
    eigenvectors = np.zeros((n,n),dtype=eigvals_new.dtype)

    if eigvec:
        #print(H_new.dtype, 'H = \n',H_new)
        if eigvals_new.dtype == 'complex':
            #print('complex eigenvalues -',eigvals_new)
            eig_U = eigenvectors_inversePowerMethod(H_new,eigvals_new)
        else:
            #print('real eigenvalues - ',eigvals_new)
            eig_U = eigenvectors_upperTriangularMatrix(H_new,eigvals_new)

        for i in range(n):
            eigenvectors[:,i] = S.conjugate().T @ T.conjugate().T @ eig_U[:,i]

    return eigvals_new, eigenvectors



def sort_eigen(w, v):

    """     Sort eigenvalues and eigenvectors
    Input   ----------------------------------------------
            w: 1D np.array containing eigenvalues
            v: 2D np.array containing eigenvectors in each column
    Return  ---------------------------------------------
            w_sorted: 1D np.array, eigenvalues sorted in descending order
            v_sorted: 2D np.array, eigenvectors sorted in the same order as its correspondong eigenvalue in w_sorted
    """
```

```

    w_sorted = np.flip(np.sort(w)) #same as sort_(w) # sort eigenvectors in descending order
    w_sorted_ix = np.flip(np.argsort(w))
    # w_sorted_ix = [np.argwhere(w==x).item() for x in w_sorted] # this does not deal with repeating eigenvalues ...

    v_sorted = np.empty_like(v)

    for i,j in enumerate(w_sorted_ix):
        v_sorted[:,i] = v[:,j]

    return w_sorted, v_sorted


```

In [29]:

```
print('\n\n')
print("===================================================================================")
print('Testing eigen_solver() on Random REAL Matrices')
print("===================================================================================")

for A in RANDOM_REAL_MATRICES[1:2]:
    print('\nREAL >>> Input shape = {}\n'.format(A.shape))

    w,v = np.linalg.eig(A)
    w_sorted, v_sorted = sort_eigen(w, v)

    H, S = Hessenberg(A,eigvec=True)

    eigenvalues, eigenvectors = eigen_solver(H, S, max_iter=MAX_ITER, tol=RTOL, eigvec=True, shift=None, tridiagonal=False)
    test_eigenvalues(eigenvalues,w_sorted)
    test_eigenvectors(eigenvectors,v_sorted)


    eigenvalues, eigenvectors = eigen_solver(H, S, max_iter=MAX_ITER, tol=RTOL, eigvec=True, shift='Wilkinson',tridiagonal=False)
    test_eigenvalues(eigenvalues,w_sorted)
    test_eigenvectors(eigenvectors,v_sorted)

```

```


=====================================================================================
Testing eigen_solver() on Random REAL Matrices
=====================================================================================

REAL >>> Input shape = (5, 5)

>>> QR Iteration WITHOUT shift
Iteration terminates at i=28 with tol=1e-06 reached
... num of iterations with inverse power method = 1

my eigenvalues: [ 2.116+0.j     0.207+0.229j  0.207-0.229j  0.073+0.j    -0.515+0.j   ]
np.linalg.eig:  [ 2.116+0.j     0.207+0.229j  0.207-0.229j  0.073+0.j    -0.515+0.j   ]

np.linalg.eig and my eigenvalues close? True

np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 0.496+0.j    -0.055+0.215j -0.055-0.215j -0.116+0.j    -0.549+0.j   ]
 [ 0.409+0.j    -0.117+0.129j -0.117-0.129j -0.222+0.j     0.221+0.j   ]
 [ 0.384+0.j     0.492-0.16j   0.492+0.16j   0.369+0.j     0.398+0.j   ]
 [ 0.576+0.j    -0.537-0.511j -0.537+0.511j -0.681+0.j    -0.576+0.j   ]
 [ 0.328+0.j     0.275+0.167j  0.275-0.167j  0.581+0.j     0.4  +0.j   ]]
np.linalg.eig =
```

```
 [[ 0.496+0.j     0.109+0.194j  0.109-0.194j -0.116+0.j     0.549+0.j   ]
 [ 0.409+0.j     0.004+0.174j  0.004-0.174j -0.222+0.j    -0.221+0.j   ]
 [ 0.384+0.j     0.246-0.455j  0.246+0.455j  0.369+0.j    -0.398+0.j   ]
 [ 0.576+0.j    -0.741+0.j    -0.741-0.j    -0.681+0.j     0.576+0.j   ]
 [ 0.328+0.j     0.314-0.068j  0.314+0.068j  0.581+0.j    -0.4  +0.j   ]]
my eigenvectors / np.linalg.eig =
 [[ 1.   +0.j     0.725+0.689j  0.725-0.689j  1.   -0.j    -1.   +0.j   ]
 [ 1.   +0.j     0.725+0.689j  0.725-0.689j  1.   -0.j    -1.   -0.j   ]
 [ 1.   +0.j     0.725+0.689j  0.725-0.689j  1.   +0.j    -1.   -0.j   ]
 [ 1.   +0.j     0.725+0.689j  0.725-0.689j  1.   -0.j    -1.   +0.j   ]
 [ 1.   +0.j     0.725+0.689j  0.725-0.689j  1.   +0.j    -1.   -0.j   ]]

Eigenvectors all close except sign and/or scaling? True



>>> QR Iteration with Wilkinson shift

C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:32: ComplexWarning: Casting complex values to real discards the imaginary part

Iteration terminates at i=14 with tol=1e-06 reached
... num of iterations with inverse power method = 1
... num of iterations with inverse power method = 1

my eigenvalues: [ 2.116+0.j     0.207+0.229j  0.207-0.229j  0.073+0.j    -0.515+0.j   ]
np.linalg.eig:  [ 2.116+0.j     0.207+0.229j  0.207-0.229j  0.073+0.j    -0.515+0.j   ]

np.linalg.eig and my eigenvalues close? True

np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 0.496+0.j    -0.213+0.062j -0.213-0.062j -0.116+0.j     0.549+0.j   ]
 [ 0.409+0.j    -0.171-0.035j -0.171+0.035j -0.222+0.j    -0.221+0.j   ]
 [ 0.384+0.j     0.387+0.343j  0.387-0.343j  0.369+0.j    -0.398+0.j   ]
 [ 0.576+0.j     0.169-0.722j  0.169+0.722j -0.681+0.j     0.576+0.j   ]
 [ 0.328+0.j    -0.006+0.321j -0.006-0.321j  0.581+0.j    -0.4  +0.j   ]]
np.linalg.eig =
 [[ 0.496+0.j     0.109+0.194j  0.109-0.194j -0.116+0.j     0.549+0.j   ]
 [ 0.409+0.j     0.004+0.174j  0.004-0.174j -0.222+0.j    -0.221+0.j   ]
 [ 0.384+0.j     0.246-0.455j  0.246+0.455j  0.369+0.j    -0.398+0.j   ]
 [ 0.576+0.j    -0.741+0.j    -0.741-0.j    -0.681+0.j     0.576+0.j   ]
 [ 0.328+0.j     0.314-0.068j  0.314+0.068j  0.581+0.j    -0.4  +0.j   ]]
my eigenvectors / np.linalg.eig =
 [[ 1.   +0.j    -0.228+0.974j -0.228-0.974j  1.   -0.j     1.   +0.j   ]
 [ 1.   +0.j    -0.228+0.974j -0.228-0.974j  1.   -0.j     1.   -0.j   ]
 [ 1.   +0.j    -0.228+0.974j -0.228-0.974j  1.   +0.j     1.   -0.j   ]
 [ 1.   +0.j    -0.228+0.974j -0.228-0.974j  1.   -0.j     1.   +0.j   ]
 [ 1.   +0.j    -0.228+0.974j -0.228-0.974j  1.   +0.j     1.   -0.j   ]]

Eigenvectors all close except sign and/or scaling? True
```

```
C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:44: ComplexWarning: Casting complex values to real discards the imaginary part
C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:51: ComplexWarning: Casting complex values to real discards the imaginary part
```

In [30]:

```
print('\n\n')
print("=====================================================================================")
print('Testing eigen_solver() on Random COMPLEX Matrices')
print("=====================================================================================")

for A in RANDOM_COMPLEX_MATRICES[1:2]:
    print('\nCOMPLEX >>> Input shape = {}\n'.format(A.shape))

    w,v = np.linalg.eig(A)
    w_sorted, v_sorted = sort_eigen(w, v)

    H, S = Hessenberg(A,eigvec=True)

    eigenvalues, eigenvectors = eigen_solver(H, S, max_iter=MAX_ITER, tol=RTOL, eigvec=True, shift=None,tridiagonal=False)
    test_eigenvalues(eigenvalues,w_sorted)
    test_eigenvectors(eigenvectors,v_sorted)


    eigenvalues, eigenvectors = eigen_solver(H, S, max_iter=MAX_ITER, tol=RTOL, eigvec=True, shift='Wilkinson',tridiagonal=False)
    test_eigenvalues(eigenvalues,w_sorted)
    test_eigenvectors(eigenvectors,v_sorted)
```

```
=====================================================================================
Testing eigen_solver() on Random COMPLEX Matrices
=====================================================================================

COMPLEX >>> Input shape = (5, 5)

>>> QR Iteration WITHOUT shift
Iteration terminates at i=935 with tol=1e-06 reached

my eigenvalues: [ 2.251+2.287j  0.132+0.508j -0.182+0.806j -0.189-0.474j -0.442+0.271j]
np.linalg.eig:  [ 2.251+2.287j  0.132+0.508j -0.182+0.806j -0.189-0.474j -0.442+0.271j]

np.linalg.eig and my eigenvalues close? True

np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 0.51 +0.131j -0.036+0.524j -0.424+0.002j  0.254+0.395j  0.178-0.149j]
 [ 0.325+0.253j -0.415-0.312j  0.325-0.567j  0.15 -0.652j -0.52 +0.098j]
 [ 0.396+0.217j  0.018-0.456j  0.222-0.04j  -0.028+0.334j -0.379-0.102j]
 [ 0.343+0.253j -0.013-0.21j   0.236+0.496j -0.147-0.3j   -0.02 -0.171j]
 [ 0.385+0.138j  0.397+0.209j -0.177+0.096j -0.325+0.042j  0.587+0.371j]]
np.linalg.eig =
 [[ 0.527+0.j     0.525+0.j    -0.213-0.367j -0.327+0.337j  0.071-0.221j]
```

```
 [ 0.377+0.164j -0.283+0.436j  0.654+0.j     0.669+0.j    -0.387+0.361j]
 [ 0.437+0.111j -0.457+0.014j  0.145+0.173j -0.331+0.048j -0.375+0.117j]
 [ 0.396+0.16j  -0.209+0.027j -0.313+0.451j  0.26 -0.211j -0.108-0.134j]
 [ 0.407+0.038j  0.181-0.41j  -0.171-0.106j -0.114-0.308j  0.694+0.j   ]]
my eigenvectors / np.linalg.eig =
 [[ 0.968+0.249j -0.069+0.998j  0.497-0.868j  0.225-0.974j  0.845+0.535j]
 [ 0.968+0.249j -0.069+0.998j  0.497-0.868j  0.225-0.974j  0.845+0.535j]
 [ 0.968+0.249j -0.069+0.998j  0.497-0.868j  0.225-0.974j  0.845+0.535j]
 [ 0.968+0.249j -0.069+0.998j  0.497-0.868j  0.225-0.974j  0.845+0.535j]
 [ 0.968+0.249j -0.069+0.998j  0.497-0.868j  0.225-0.974j  0.845+0.535j]]

Eigenvectors all close except sign and/or scaling? True



>>> QR Iteration with Wilkinson shift
Iteration terminates at i=12 with tol=1e-06 reached

my eigenvalues: [ 2.251+2.287j  0.132+0.508j -0.182+0.806j -0.189-0.474j -0.442+0.271j]
np.linalg.eig:  [ 2.251+2.287j  0.132+0.508j -0.182+0.806j -0.189-0.474j -0.442+0.271j]

np.linalg.eig and my eigenvalues close? True

np.linalg.eig and my function return the same eigenvectors? False

my eigenvectors =
 [[ 0.356+0.389j  0.507+0.137j  0.362+0.221j -0.029+0.469j -0.233+0.001j]
 [ 0.134+0.389j -0.386+0.347j -0.575+0.312j  0.508-0.436j  0.463+0.257j]
 [ 0.213+0.397j -0.444-0.106j -0.209-0.083j -0.22 +0.252j  0.227+0.32j ]
 [ 0.149+0.4j   -0.209-0.028j  0.06 -0.546j  0.06 -0.329j -0.094+0.144j]
 [ 0.247+0.326j  0.282-0.349j  0.201+0.012j -0.287-0.159j -0.214-0.66j ]]
np.linalg.eig =
 [[ 0.527+0.j     0.525+0.j    -0.213-0.367j -0.327+0.337j  0.071-0.221j]
 [ 0.377+0.164j -0.283+0.436j  0.654+0.j     0.669+0.j    -0.387+0.361j]
 [ 0.437+0.111j -0.457+0.014j  0.145+0.173j -0.331+0.048j -0.375+0.117j]
 [ 0.396+0.16j  -0.209+0.027j -0.313+0.451j  0.26 -0.211j -0.108-0.134j]
 [ 0.407+0.038j  0.181-0.41j  -0.171-0.106j -0.114-0.308j  0.694+0.j   ]]
my eigenvectors / np.linalg.eig =
 [[ 0.676+0.737j  0.966+0.26j  -0.879+0.477j  0.759-0.652j -0.309-0.951j]
 [ 0.676+0.737j  0.966+0.26j  -0.879+0.477j  0.759-0.652j -0.309-0.951j]
 [ 0.676+0.737j  0.966+0.26j  -0.879+0.477j  0.759-0.652j -0.309-0.951j]
 [ 0.676+0.737j  0.966+0.26j  -0.879+0.477j  0.759-0.652j -0.309-0.951j]
 [ 0.676+0.737j  0.966+0.26j  -0.879+0.477j  0.759-0.652j -0.309-0.951j]]

Eigenvectors all close except sign and/or scaling? True
```

Observations

- Seems to be working for real and complex matrices

In [ ]:
```
1
```

References for Francis Double-Shift QR Algorithm

- https://www.cs.cornell.edu/~bindel/class/cs6210-f12/notes/lec28.pdf (https://www.cs.cornell.edu/~bindel/class/cs6210-f12/notes/lec28.pdf)
- https://people.inf.ethz.ch/arbenz/ewp/Lnotes/chapter4.pdf (https://people.inf.ethz.ch/arbenz/ewp/Lnotes/chapter4.pdf)
- https://people.eecs.berkeley.edu/~wkahan/Math128/Parlett.pdf (https://people.eecs.berkeley.edu/~wkahan/Math128/Parlett.pdf)
- https://www.math.usm.edu/lambers/mat610/class0331.pdf (https://www.math.usm.edu/lambers/mat610/class0331.pdf)
- http://math.ntnu.edu.tw/~min/matrix_computation/QR_alg_ch2_1.pdf (http://math.ntnu.edu.tw/~min/matrix_computation/QR_alg_ch2_1.pdf)
- https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.215.3553&rep=rep1&type=pdf (https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.215.3553&rep=rep1&type=pdf)
- https://www.youtube.com/watch?v=RSm_Mqi0aSA (https://www.youtube.com/watch?v=RSm_Mqi0aSA)

Background

Implicit double shifts

Apply 1st shift s_1 on H_0:

- $H_0 - s_1 I = Q_1 R_1$ ......(1)
- $H_1 = R_1 Q_1 + s_1 I$ ......(2)

Apply 2nd shift s_2 on H_1:

- $H_1 - s_2 I = Q_2 R_2$ ......(3)
- $H_2 = R_2 Q_2 + s_2 I$ ......(4)

Now combining the two shifts (without explicitly forming H_1), from (2) and (3):

- $R_1 Q_1 + s_1 I = Q_2 R_2 + s_2 I$

Apply Q1 from left and R1 from right on both sides and expand terms:

- $Q_1(R_1 Q_1)R_1 + Q_1(s_1 I)R_1 = Q_1(Q_2 R_2)R_1 + Q_1(s_2 I)R_1$

Substitude $Q_1 R_1$ from (1) and re-arrange:

- $(Q_1 Q_2)(R_2 R_1) = (H_0 - s_1 I)^2 + (H_0 - s_1 I)(s_1 - s_2) = (H_0 - s_1 I)(H_0 - s_2 I)$

This is effectively the QR decomposition of matrix $M = (H_0 - s_1 I)(H_0 - s_2 I)$, i.e. performing two shifts on H_0 to obtain H_2:

- $H_2 = (Q_1 Q_2)^* H_0 (Q_1 Q_2)$

When s_1 and s_2 are a conjugate pair of a complex eigenvalue λ, matrix M is real, thus avoiding QR decomposition in complex arithmatics

- $M = H_0^2 - 2Re(\lambda)H + |\lambda|^2 I$

A few observations:

- Naive implementation involves forming M explicitly by matrix multiplication, which is computational expensive ~$O(n^3)$
- Note also the matrix M is *not* in Hessenberg form; it has two lower sub-diagonals due to the multiplication of two Hessenberg matrices.
- Implicit Q-Theream ...
- Need to restore M to Hessenberg by Givens rotation

Implicit Q thereom

- "The gist of the implicit Q theorem is that if Q.T AQ = H and Z.T AZ = G are both unreduced Hessenberg matrices and Q and Z have the same first column, the G and H are “essentially equal” in the sense that G = DHD with D = diag(±1, . . . , ±1)." -- Golub and van Loan [5, p.347]

Bulge chasing not implemented

- Instead of forming M in its entirety, we only form its first column, which being a second degree polynomial of a Hessenberg matrix, has only three nonzero entries.
- Compute a Householder reflection P0 that makes Me1 a multiple of e1.
- Then, we compute P0.T H P0, which is no longer Hessenberg, because P0 operates on the first three rows and columns of H.
- Finally, we apply a series of Householder reflections P1, P2, . . . , Pn−2 that restore Hessenberg form
- Because the reflections P1, P2, . . . , Pn−2 do not affect the first row (when applied on the left) or column (when applied on the right),
- it follows that if we define Z˜ = P0 P1 P2 · · · Pn−2, then Z and Z˜ have the same first column
- Since both matrices implement similarity transformations that preserve the Hessenberg form of H, it follows from the Implicit Q Theorem that
- Z and Z˜ are essentially equal, and that they essentially produce the same updated matrix H2
- This variation of a Hessenberg QR step is called a Francis QR Step.
- (Source: [https://www.math.usm.edu/lambers/mat610/class0331.pdf (https://www.math.usm.edu/lambers/mat610/class0331.pdf)](https://www.math.usm.edu/lambers/mat610/class0331.pdf))

In [ ]:

```
##############
### IGNORE ###
##############

# Naive implementation of Francis Double-shift QR Algorithm

def construct_M_full(H,lam_re,lam_im):

    """     Naive implementation of the full matrix M = (H-s1.I)(H-s2.I) where s1 and s2 are complex conjugate pair

    Input   ----------------------------------------------
            H: 2D np.array in the form of a Hessenberg matrix with real entries
            lam_re: real part of a complex eigenvalue
            lam_im: imaginary part of a complex eigenvalue
    Return  ---------------------------------------------
            M: 2D np.array with only real entries
    """

    I = np.eye(H.shape[0])

    M = H@H - 2*lam_re*H + (lam_re**2+lam_im**2)*I  #Note: H@H complexity = O(n^3) expensive!
    # RuntimeWarning: overflow encountered in matmul
    # RuntimeWarning: invalid value encountered in matmul
    #print(M)

    return M



# for eigenvalues only

def QR_Iteration_DoubleShift(A, max_iter, tol):

    """     Perform QR iteration with double shift to obtain eigenvalues
    Input   ----------------------------------------------
            A: 2D np.array with real or complex entries
            max_iter: int, maximum number of iterations to be performed
            tol: float, relative tolerance between eigenvalues by successive iterations
    Return  ---------------------------------------------
            eigvals_new: 1D np.array with real or complex entries
    """

    n = A.shape[0]

    # step 1: tranform A to Hessenberg
    H, _ = Hessenberg(A)

    # step 2: QR iteration
    H_old = H.copy()
    eigvals_old = diagonal_block(H)

    for i in range(max_iter):
```

```

        # calculate Wilkinson shift
        datatype, lam = Wilkinson_shift(H_old)
        # print(">>> Wilkinson_shift = {},{}".format(lam,datatype))

        if datatype == 'complex':
            M = construct_M_full(H_old,lam.real,lam.imag)
            H_old, _ = Hessenberg(M) # Hessenberg(A,eigvec=False, inplace=True)
            H_new, _ = QR_Givens(H_old) # updating H_old in-place ## default QR_Givens(H,eigvec=False, inplace=True, tridiagonal=False)

        else:
            H_old = H_old - lam * np.eye(n)
            H_new, _ = QR_Givens(H_old) +  lam * np.eye(n) # updating H_old in-place

        eigvals_new = diagonal_block(H_new)

        H_old = H_new

        # Test for convergence: relative tol = 1- w / w' for each eigenvalue
        if np.allclose(eigvals_new, eigvals_old, rtol=tol):
            print('Iteration terminates at i={} with tol={} reached'.format(i,tol))
            break

        if i == max_iter-1:
            err = np.max(abs((eigvals_new - eigvals_old)) / abs(eigvals_old))
            idx = np.argwhere(np.isclose(eigvals_new, eigvals_old, rtol=tol)==False).tolist()
            print('max_iter={} reached, max error={} vs tol={}; index of non-convergence={}'.format(max_iter, err, tol, idx))

        eigvals_old = eigvals_new

    return eigvals_new


```

In [ ]:

```
print('\n\n')
print("================================================================================")
print('Testing QR_Iteration_DoubleShift on Random REAL Matrices')
print("================================================================================")

for A in RANDOM_REAL_MATRICES[1:-1]:
    print('\nREAL >>> Input shape = {}\n'.format(A.shape))

    test_eigenvalues_function(A,QR_Iteration_Eigenvalues)
    test_eigenvalues_function(A,QR_Iteration_WilkinsonShift)
    test_eigenvalues_function(A,QR_Iteration_DoubleShift)
```

Observations:

- Double-shift does not work **#TODO**

- Overflow issues with repeated matrix multiplication H@H in order to get M in the iteration
- Need implement element-wise Bulge-chasing (based on implicit Q)

In [ ]:

```
##############
### IGNORE ###
##############

def eigen_solver_real(H, S, max_iter, tol, eigvec):

    """     Compute the eigenvalues and eigenvectors of a Hessenberg matrix with real entries
            with Wilkinson' shift (if real) or double shift (if complex)
    Input   ---------------------------------------------
            H: 2D np.array in the form of a Hessenberg matrix with real entries
            S: similarity transformation associated with the Hessenberg reduction, H = S @ A @ S.T where A is the input real matrix
            max_iter: int, maximum number of iterations to be performed
            tol: float, relative tolerance between eigenvalues by successive iterations
            eigenvec: bool, indicating whether eigenvectors are to be calculated
    Return  ---------------------------------------------
            eigvals_new: 1D np.array, eigenvalues
            eigenvectors: 2D np.array with each column represents an eigenvector, in the same order as the associated eigenvalues;
                          if eigvec is False, then this default to a zero matrix
    """

    datatype, lam = Wilkinson_shift(H_old)

    if datatype == 'complex':
        shift = 'Double-'
    else:
        shift = "Wilkinson's "
    print(">>> QR Iteration with Real Shift: lambda = {}, {} -> Applying {}Shift".format(lam, datatype, shift))

    pass # TODO


def eigen_solver_realsymmetric(H, S, max_iter, tol, eigvec):
    pass # TODO


```

In [ ]:

```

```

Visualization

In [31]:

```python
# Visualization

def plot_eigenvalues_real(w_sorted,eigenvalues):

    '''Plotting line chart for two sets of REAL eigenvalues in comparison
    Input   ---------------------------------------------
            w_sorted: eigenvalues from np.linalg.eig(), sorted in descending order by magnitude
            eigenvalues: eigenvalues from my function, sorted in descending order by magnitude
    Return  ---------------------------------------------
            None; chart on display
    '''
    plt.plot(w_sorted, color='red', marker='o', linestyle='dashed', label = 'np.linalg.eig()')
    plt.plot(eigenvalues, color='green', marker='x', linestyle='dotted', label='my eigenvalues')
    plt.title('Real Eigenvalues')
    plt.legend()
    plt.show()


def plot_eigenvalues_complex(w_sorted,eigenvalues):

    '''Plotting line chart for two sets of COMPLEX eigenvalues in comparison
    Input   ---------------------------------------------
            w_sorted: eigenvalues from np.linalg.eig(), sorted in descending order by magnitude
            eigenvalues: eigenvalues from my function, sorted in descending order by magnitude
    Return  ---------------------------------------------
            None; two charts on display, one for real parts and the other for imaginary parts
    '''
    f, ax = plt.subplots(1,2,figsize=(10,5))
    ax[0].plot(w_sorted.real, color='red', marker='o', linestyle='dashed', label = 'np.linalg.eig()')
    ax[0].plot(eigenvalues.real, color='green', marker='x', linestyle='dotted', label='my eigenvalues')
    ax[0].set_title('Complex Eigenvalues - Real')
    ax[0].legend()


    x = np.arange(len(eigenvalues))  # the label locations
    width = 0.02  # the width of the bars
    ax[1].bar(x - width/2, w_sorted.imag, width, color='red',label = 'np.linalg.eig()')
    ax[1].bar(x + width/2, eigenvalues.imag, width,color='green', label='my eigenvalues')
    ax[1].set_title('Complex Eigenvalues - Imaginary')
    ax[1].legend()

    plt.show()
```

Attampt at Object-Oriented Programming

In [32]:

```
# Object-Oriented Programming: implement algorithm 1.4.14 in a way, that H is either Hessenberg or tridiagonal, either real or complex, but encapsulat

class Matrix:

    # General matrix class for both complex and real matrices

    def __init__(self, A, calc_eigvec=False):
        self.calc_eigvec = calc_eigvec
        self.A = A
        self.n = A.shape[0]
        self.dtype = A.dtype

        self.is_symmetric = False
        if np.allclose(A,A.T) or np.allclose(A, A.conjugate().T):
            self.is_symmetric = True

        self.is_hessenberg = False
        zeros = np.zeros([self.n,self.n],dtype=self.dtype)
        if np.allclose(np.tril(self.A, -2),zeros):
            self.is_hessenberg = True
            self.H = self.A
            self.S = None
        else:
            self.H,self.S = Hessenberg(self.A, self.calc_eigvec)

        self.is_tridiagonal = False
        tridiag = self.A - np.tril(self.A,-2) - np.triu(self.A,2)
        if self.is_hessenberg and np.allclose(np.triu(self.A,2),zeros) and not np.allclose(tridiag,zeros):
            self.is_tridiagonal = True

    def __eq__(self, other):
        return np.allclose(self.A,other.A)

    def __repr__(self):
        return self.A

    def __str__(self):
        return 'Matrix dtype={}, shape={}x{}, is_symmetric={}, is_hessenberg={}, is_tridiagonal={} \n{}'.format(self.dtype,self.n,self.n,self.is_symme


    # Wilkson's shift in complex arithmatics
    def eigen_solver(self, shift=None):
        # shift = None or 'Wilkinson'
        tridiagonal = self.is_symmetric
        # tridiagonal = False
        print('activiating tridigonal:', tridiagonal)

        if self.calc_eigvec:
            print('\n ------------ Processing both eigenvalues and eigenvectors ------------ ')
            self.eigenvalues, self.eigenvectors = eigen_solver(self.H, self.S, MAX_ITER, RTOL, self.calc_eigvec,shift,tridiagonal)
            #eigen_solver(H, S, max_iter, tol, eigvec, shift, tridiagonal)
        else:
```

```
            print('\n ------------ Processing eigenvalues only ------------ ')
            self.eigenvalues, _ = eigen_solver(self.H, self.S, MAX_ITER, RTOL, self.calc_eigvec,shift,tridiagonal)

    def eigen_test(self, verbose):
        w,v = np.linalg.eig(self.A)
        w_sorted, v_sorted = sort_eigen(w, v)

        print('\n ------------ Checking Eigenvalues ------------')
        test_eigenvalues(self.eigenvalues,w_sorted,verbose)
        if self.calc_eigvec:
            print('\n ------------ Checking Eigenvectors ------------')
            test_eigenvectors(self.eigenvectors,v_sorted,verbose)
        if verbose:
            print(self.eigenvalues.dtype)

            if self.eigenvalues.dtype == 'complex128':
                plot_eigenvalues_complex(w_sorted,self.eigenvalues)
            else:
                plot_eigenvalues_real(w_sorted,self.eigenvalues)
```

In [ ]:

```
##############
### IGNORE ###
##############


class RealMatrix(Matrix):

    def __init__(self, A, calc_eigvec=False):
        super().__init__(A, calc_eigvec)


    # double-shifts for complex eigenvalues
    def eigen_solver(self):

        if self.calc_eigvec:
            print('\n ------------ Processing both eigenvalues and eigenvectors ------------ ')
            self.eigenvalues, self.eigenvectors = eigen_solver_real(self.H, self.S, max_iter=MAX_ITER, tol=RTOL, eigvec=self.calc_eigvec)

        else:
            print('\n ------------ Processing eigenvalues only ------------ ')
            self.eigenvalues, _ = eigen_solver_real(self.H, self.S, max_iter=MAX_ITER, tol=RTOL, eigvec=self.calc_eigvec)


class SymmetricRealMatrix(RealMatrix):

    def __init__(self, A, calc_eigvec=False):
        super().__init__(A, calc_eigvec)


    # transformation to tridiagonal form with optimized storage
    def eigen_solver(self):

        if self.calc_eigvec:
            print('\n ------------ Processing both eigenvalues and eigenvectors ------------ ')
            self.eigenvalues, self.eigenvectors = eigen_solver_realsymmetric(self.H, self.S, max_iter=MAX_ITER, tol=RTOL, eigvec=self.calc_eigvec)

        else:
            print('\n ------------ Processing eigenvalues only ------------ ')
            self.eigenvalues, _ = eigen_solver_realsymmetric(self.H, self.S, max_iter=MAX_ITER, tol=RTOL, eigvec=self.calc_eigvec)

```

In [ ]:

```

```

Matric Market

from scipy.io import mmread

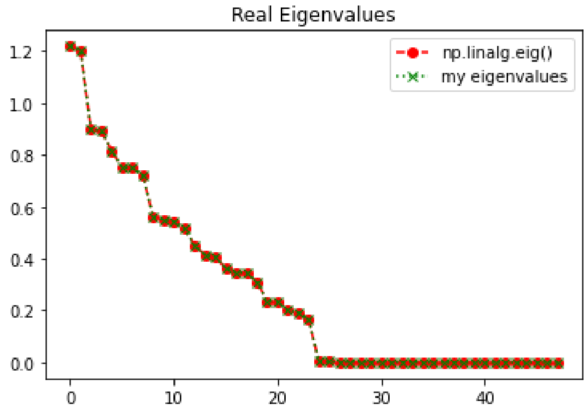
- BCSSTK01 (real symmetric positive definite, 48 by 48, 224 entries), Small test problem; source: https://math.nist.gov/MatrixMarket/data/Harwell-Boeing/bcsstruc1/bcsstruc1.html (https://math.nist.gov/MatrixMarket/data/Harwell-Boeing/bcsstruc1/bcsstruc1.html)
- CK104 (real unsymmetric, 104 by 104, 992 entries) The matrices have several multiple eigenvalues and clustered eigenvalues. source: https://math.nist.gov/MatrixMarket/data/NEP/chuck/ck104.html (https://math.nist.gov/MatrixMarket/data/NEP/chuck/ck104.html)
- QC324 (complex symmetric indefinite, 324 x 324, 26730 entries) H2PLUS: Model of H2+ in an Electromagnetic Field source: https://math.nist.gov/MatrixMarket/data/NEP/h2plus/h2plus.html (https://math.nist.gov/MatrixMarket/data/NEP/h2plus/h2plus.html)
- MHD1280 (complex unsymmetric, 1280 by 1280, 47906 entries) MHD: Alfven Spectra in Magnetohydrodynamicssource: https://math.nist.gov/MatrixMarket/data/NEP/mhd/mhd.html (https://math.nist.gov/MatrixMarket/data/NEP/mhd/mhd.html)

- RBS480A (real unsymmetric, 480 by 480, 17088 entries) The computational task is to compute the real eigenvalues and the corresponding eigenvectors (most of the eigenvalues are complex). source: https://math.nist.gov/MatrixMarket/data/NEP/robotics/robotics.html (https://math.nist.gov/MatrixMarket/data/NEP/robotics/robotics.html)
- QH1484 (real unsymmetric, 1484 by 1484, 6110 entries) This set of matrices if from the application of the Hydro-Quebec power systems' small-signal model. In the application, one wants to compute all eigenvalues a + b i in a box of the complex plane. https://math.nist.gov/MatrixMarket/data/NEP/quebec/quebec.html (https://math.nist.gov/MatrixMarket/data/NEP/quebec/quebec.html)
- ORANI678 (real unsymmetric, 2529 by 2529, 90158 entries) Australian Economic Models Economic model of Australia, 1968-69 data source: https://math.nist.gov/MatrixMarket/data/Harwell-Boeing/econaus/orani678.html (https://math.nist.gov/MatrixMarket/data/Harwell-Boeing/econaus/orani678.html)

In [33]:
```python
def normalize(A):
    """ normalize a matrix to between [-1,1]
    """
    scaler = max(abs(A.max()),abs(A.min()))
    return A/scaler, scaler


def runtest_matrixmarket(source, matrix_class,shift):
    """
    Testing on matrix from matrix market
    Input   ----------------------------------------------
            source: str, filename in the form of 'xxxn.mtx.gz'
            matrix_class: options = [Matrix,RealMatrix, SymmetricRealMatrix]; currently only Matrix option is available
            shift: None or 'Wilkinson'
    Return  ---------------------------------------------
            None; print test results
    """

    # read data from matrix market format
    A = mmread(source)
    print("From Matrix Market: \nmatrix source={}, matrix type={}, shape={}, data type={}, max={}, min={}"
          .format(source, type(A), A.shape,A.dtype, A.max(),A.min()))

    # convert to np.array
    A = np.asarray(A.todense())

    # normalization
    A, s = normalize(A)

    # wrap into a matrix class object
    a = matrix_class(A, calc_eigvec=False)
    print('is_symmetric={}', a.is_symmetric)

    # run test on eigen_solver()
    a.eigen_solver(shift)
    a.eigen_test(verbose=True)
```

In [34]:
```
# real symmetric 48x48

source = 'matricmarket/bcsstk01.mtx.gz'
matrix_class = Matrix

runtest_matrixmarket(source, matrix_class,shift=None)

```

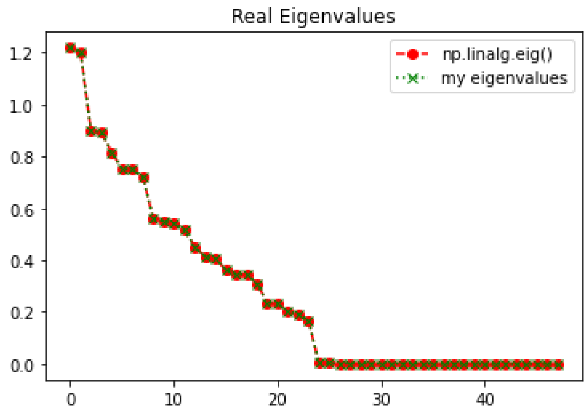
```
From Matrix Market:
matrix source=matricmarket/bcsstk01.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=(48, 48), data type=float64, max=2472387301.98, min=
-109779731.332
is_symmetric={} True
activiating tridigonal: True

 ------------ Processing eigenvalues only ------------
>>> QR Iteration WITHOUT shift

C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:32: ComplexWarning: Casting complex values to real discards the imaginary part

Iteration terminates at i=181 with tol=1e-06 reached

 ------------ Checking Eigenvalues ------------

my eigenvalues: [1.22  1.201 0.898 0.893 0.816 0.752 0.75  0.722 0.561 0.551 0.544 0.516 0.452 0.415 0.407 0.362 0.346 0.346 0.31  0.236 0.234 0.2   0.19
3 0.167 0.003 0.003 0.002 0.002 0.002 0.002 0.002 0.002 0.002 0.002 0.001 0.001 0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.   ]
np.linalg.eig:  [1.22  1.201 0.898 0.893 0.816 0.752 0.75  0.722 0.561 0.551 0.544 0.516 0.452 0.415 0.407 0.362 0.346 0.346 0.31  0.236 0.234 0.2   0.19
3 0.167 0.003 0.003 0.002 0.002 0.002 0.002 0.002 0.002 0.002 0.002 0.001 0.001 0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.   ]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [ 0. -0. -0.  0. -0. -0.  0.  0. -0.  0.  0. -0.  0. -0.  0. -0. -0.  0.  0. -0. -0.  0. -0. -0.  0. -0.  0. -0. -0. -0.  0. -0.  0.  0. -0. -0. -0. -0.
0.  0.  0. -0.  0. -0.  0.  0. -0.  0.]

max abs diff =  6.407067543401368e-06



float64
```


Real Eigenvalues
np.linalg.eig()
my eigenvalues
1.2
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40

In [38]:

```
# real symmetric 48x48

source = 'matricmarket/bcsstk01.mtx.gz'
matrix_class = Matrix

runtest_matrixmarket(source, matrix_class,shift='Wilkinson')

```

```
From Matrix Market:
matrix source=matricmarket/bcsstk01.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=(48, 48), data type=float64, max=2472387301.98, min=
-109779731.332
is_symmetric={} True
activiating tridigonal: True

 ------------ Processing eigenvalues only ------------
>>> QR Iteration with Wilkinson shift

C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:32: ComplexWarning: Casting complex values to real discards the imaginary part

Iteration terminates at i=174 with tol=1e-06 reached

 ------------ Checking Eigenvalues ------------

my eigenvalues: [1.22  1.201 0.898 0.893 0.816 0.752 0.75  0.722 0.561 0.551 0.544 0.516 0.452 0.415 0.407 0.362 0.346 0.346 0.31  0.236 0.234 0.2   0.19
3 0.167 0.003 0.003 0.002 0.002 0.002 0.002 0.002 0.002 0.002 0.002 0.001 0.001 0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.   ]
np.linalg.eig:  [1.22  1.201 0.898 0.893 0.816 0.752 0.75  0.722 0.561 0.551 0.544 0.516 0.452 0.415 0.407 0.362 0.346 0.346 0.31  0.236 0.234 0.2   0.19
3 0.167 0.003 0.003 0.002 0.002 0.002 0.002 0.002 0.002 0.002 0.002 0.001 0.001 0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.    0.   ]

np.linalg.eig and my eigenvalues close? True

float64
```



Observation

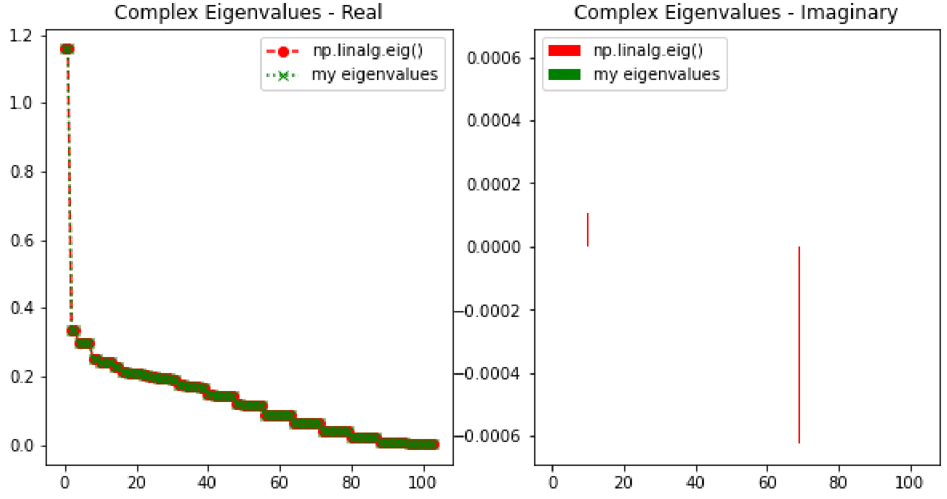
- General QR iteration without shift seems to be working for small real symmetric matrix
- QR iteration with Wilkinson shift seems to be working for small real symmetric matrix
- Similar convergence rate...

In [35]:

```python
MAX_ITER = 10000 # increased max iteration

# real unsymmetric matrix 104x104 - WIHTOUT SHIFT

source = 'matricmarket/ck104.mtx.gz'
matrix_class = Matrix

runtest_matrixmarket(source, matrix_class,shift=None)

```

```
From Matrix Market:
matrix source=matricmarket/ck104.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=(104, 104), data type=float64, max=4.7481454523109, min
=-2.6168329561089
is_symmetric={} False
activiating tridigonal: False

 ------------ Processing eigenvalues only ------------
>>> QR Iteration WITHOUT shift

C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:32: ComplexWarning: Casting complex values to real discards the imaginary part

Iteration terminates at i=2982 with tol=1e-06 reached

 ------------ Checking Eigenvalues ------------

my eigenvalues: [1.159+0.j    1.159-0.j    0.336+0.j    0.336+0.j    0.299+0.j    0.299+0.j    0.297+0.j    0.297+0.j    0.252+0.j    0.252+0.j    0.2
4 +0.j    0.24 -0.j    0.24 +0.j    0.24 -0.j    0.228+0.j    0.228-0.j    0.216+0.j    0.216+0.j    0.211+0.j    0.211-0.j    0.211+0.j    0.211-0.j
0.204+0.j
 0.204+0.j    0.198+0.j    0.198+0.j    0.194+0.j    0.194-0.j    0.194+0.j    0.194-0.j    0.189+0.j    0.189+0.j    0.175+0.j    0.175-0.j    0.172+
0.j    0.172+0.j    0.172+0.j    0.172+0.j    0.167+0.j    0.167+0.j    0.148+0.j    0.148+0.j    0.145+0.j    0.145+0.j    0.145+0.j    0.145+0.j
 0.142+0.j    0.142-0.j    0.119+0.j    0.119+0.j    0.117+0.j    0.117+0.j    0.117+0.j    0.117+0.j    0.116+0.j    0.116+0.j    0.089+0.j    0.089+
0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089-0.j    0.089+0.j    0.089-0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.062+0.001j
 0.062-0.001j 0.062+0.001j 0.062-0.001j 0.04 +0.j    0.04 +0.j    0.039+0.j    0.039+0.j    0.039+0.j    0.039+0.j    0.039-0.j    0.039+0.j    0.021+
0.j    0.021+0.j    0.021+0.j    0.021-0.j    0.02 +0.j    0.02 +0.j    0.02 +0.j    0.02 +0.j    0.008+0.j    0.008+0.j    0.008+0.j    0.008+0.j
 0.008+0.j    0.008+0.j    0.007+0.j    0.007+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+
0.j   ]
np.linalg.eig:  [1.159+0.j    1.159+0.j    0.336+0.j    0.336+0.j    0.299+0.j    0.299+0.j    0.297+0.j    0.297+0.j    0.252+0.j    0.252+0.j    0.2
4 +0.j    0.24 -0.j    0.24 +0.j    0.24 -0.j    0.228+0.j    0.228+0.j    0.216+0.j    0.216+0.j    0.211+0.j    0.211-0.j    0.211+0.j    0.211-0.j
0.204+0.j
 0.204+0.j    0.198+0.j    0.198+0.j    0.194+0.j    0.194-0.j    0.194+0.j    0.194-0.j    0.189+0.j    0.189+0.j    0.175+0.j    0.175-0.j    0.172+
0.j    0.172+0.j    0.172+0.j    0.172+0.j    0.167+0.j    0.167+0.j    0.148+0.j    0.148-0.j    0.145+0.j    0.145+0.j    0.145+0.j    0.145+0.j
 0.142+0.j    0.142+0.j    0.119+0.j    0.119+0.j    0.117+0.j    0.117+0.j    0.117+0.j    0.117+0.j    0.116+0.j    0.116+0.j    0.089+0.j    0.089+
0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089-0.j    0.089+0.j    0.089-0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.062+0.001j
 0.062-0.001j 0.062+0.001j 0.062-0.001j 0.04 +0.j    0.04 +0.j    0.039+0.j    0.039+0.j    0.039+0.j    0.039-0.j    0.039+0.j    0.039-0.j    0.021+
0.j    0.021+0.j    0.021+0.j    0.021+0.j    0.02 +0.j    0.02 -0.j    0.02 +0.j    0.02 +0.j    0.008+0.j    0.008+0.j    0.008+0.j    0.008+0.j
 0.008+0.j    0.008+0.j    0.007+0.j    0.007+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+
0.j   ]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [-0.+0.j     0.-0.j    -0.+0.j    -0.+0.j     0.+0.j    -0.+0.j    -0.+0.j    -0.+0.j     0.+0.j    -0.+0.j    -0.-0.j    -0.+0.j    -0.-0.j    -0.+
```

```
0.j    -0.+0.j    -0.-0.j    -0.+0.j    -0.+0.j    -0.+0.j    -0.-0.j    -0.+0.j    -0.-0.j    -0.+0.j    -0.+0.j    -0.+0.j    -0.+0.j    -0.+0.j
 -0.-0.j    -0.+0.j    -0.-0.j    -0.+0.j    -0.+0.j     0.+0.j     0.-0.j     0.+0.j    -0.+0.j     0.+0.j    -0.+0.j    -0.+0.j     0.+0.j    -0.-0.
j    -0.+0.j     0.+0.j    -0.+0.j     0.+0.j    -0.+0.j    -0.+0.j    -0.-0.j     0.+0.j    -0.+0.j     0.+0.j     0.+0.j    -0.+0.j    -0.+0.j
  0.+0.j     0.+0.j     0.+0.j    -0.+0.j     0.+0.j     0.+0.j     0.-0.j     0.+0.j    -0.+0.j    -0.-0.j    -0.+0.j    -0.+0.j     0.+0.j    -0.+0.
j     0.+0.j     0.-0.j    -0.-0.j    -0.+0.j    -0.+0.j    -0.+0.j     0.+0.j    -0.+0.j     0.-0.j     0.+0.001j  0.-0.001j -0.+0.j     0.+0.j
 -0.+0.j    -0.+0.j    -0.-0.j     0.-0.j    -0.+0.j     0.+0.j     0.+0.j     0.+0.j    -0.+0.j    -0.+0.j    -0.+0.j     0.+0.j    -0.+0.j    -0.+0.
j     0.+0.j    -0.+0.j     0.+0.j    -0.+0.j    -0.+0.j     0.+0.j     0.+0.j    -0.+0.j     0.+0.j   ]

max abs diff =  0.0007594189357278833



complex128
```

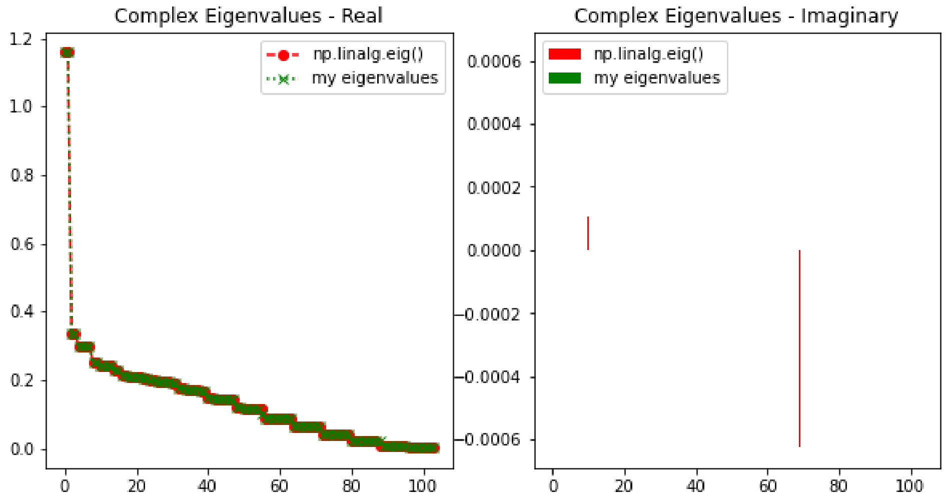


In [36]:

```
# real unsymmetric matrix 104x104 - WILKINSON SHIFT

source = 'matricmarket/ck104.mtx.gz'
matrix_class = Matrix

runtest_matrixmarket(source, matrix_class,shift='Wilkinson')

```

```
From Matrix Market:
matrix source=matricmarket/ck104.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=(104, 104), data type=float64, max=4.7481454523109, min
=-2.6168329561089
is_symmetric={} False
activiating tridigonal: False

 ------------ Processing eigenvalues only ------------
>>> QR Iteration with Wilkinson shift

C:\ProgramData\Anaconda3\lib\site-packages\ipykernel_launcher.py:32: ComplexWarning: Casting complex values to real discards the imaginary part

max_iter=10000 reached, max error=0.0029218219274107005 vs tol=1e-06; index of non-convergence=[[67], [69]]

 ------------ Checking Eigenvalues ------------

my eigenvalues: [1.159+0.j    1.159-0.j    0.336+0.j    0.336+0.j    0.299+0.j    0.299+0.j    0.297+0.j    0.297+0.j    0.252+0.j    0.252+0.j    0.2
4 +0.j    0.24 -0.j    0.24 +0.j    0.24 -0.j    0.228+0.j    0.228-0.j    0.216+0.j    0.216+0.j    0.211+0.j    0.211-0.j    0.211+0.j    0.211-0.j
0.204+0.j
 0.204+0.j    0.198+0.j    0.198+0.j    0.194+0.j    0.194-0.j    0.194+0.j    0.194-0.j    0.189+0.j    0.189+0.j    0.175+0.j    0.175+0.j    0.172+
0.j    0.172+0.j    0.172+0.j    0.172+0.j    0.167+0.j    0.167+0.j    0.148+0.j    0.148+0.j    0.145+0.j    0.145+0.j    0.145+0.j    0.145+0.j
 0.142+0.j    0.142+0.j    0.119+0.j    0.119+0.j    0.117+0.j    0.117+0.j    0.117+0.j    0.116+0.j    0.116+0.j    0.101+0.j    0.089+0.j    0.089+
0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089-0.j    0.089+0.j    0.089-0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.062+0.001j
 0.062-0.001j 0.062+0.001j 0.062-0.001j 0.04 +0.j    0.04 +0.j    0.039+0.j    0.039+0.j    0.039+0.j    0.039-0.j    0.039+0.j    0.039-0.j    0.023+
0.j    0.021+0.j    0.021+0.j    0.021+0.j    0.021+0.j    0.02 +0.j    0.02 +0.j    0.02 +0.j    0.02 +0.j    0.008+0.j    0.008+0.j    0.008+0.j
 0.008+0.j    0.008+0.j    0.007+0.j    0.007+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+
0.j   ]
np.linalg.eig:  [1.159+0.j    1.159+0.j    0.336+0.j    0.336+0.j    0.299+0.j    0.299+0.j    0.297+0.j    0.297+0.j    0.252+0.j    0.252+0.j    0.2
4 +0.j    0.24 -0.j    0.24 +0.j    0.24 -0.j    0.228+0.j    0.228+0.j    0.216+0.j    0.216+0.j    0.211+0.j    0.211-0.j    0.211+0.j    0.211-0.j
0.204+0.j
 0.204+0.j    0.198+0.j    0.198+0.j    0.194+0.j    0.194-0.j    0.194+0.j    0.194-0.j    0.189+0.j    0.189+0.j    0.175+0.j    0.175-0.j    0.172+
0.j    0.172+0.j    0.172+0.j    0.172+0.j    0.167+0.j    0.167+0.j    0.148+0.j    0.148-0.j    0.145+0.j    0.145+0.j    0.145+0.j    0.145+0.j
 0.142+0.j    0.142+0.j    0.119+0.j    0.119+0.j    0.117+0.j    0.117+0.j    0.117+0.j    0.117+0.j    0.116+0.j    0.116+0.j    0.089+0.j    0.089+
0.j    0.089+0.j    0.089+0.j    0.089+0.j    0.089-0.j    0.089+0.j    0.089-0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.063+0.j    0.062+0.001j
 0.062-0.001j 0.062+0.001j 0.062-0.001j 0.04 +0.j    0.04 +0.j    0.039+0.j    0.039+0.j    0.039+0.j    0.039-0.j    0.039+0.j    0.039-0.j    0.021+
0.j    0.021+0.j    0.021+0.j    0.021+0.j    0.02 +0.j    0.02 -0.j    0.02 +0.j    0.02 +0.j    0.008+0.j    0.008+0.j    0.008+0.j    0.008+0.j
 0.008+0.j    0.008+0.j    0.007+0.j    0.007+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+0.j    0.001+
0.j   ]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [-0.   +0.j  0.   -0.j -0.   +0.j -0.   +0.j -0.   +0.j -0.   +0.j  0.   +0.j -0.   +0.j -0.   +0.j  0.   +0.j  0.   +0.j  0.   -0.j  0.   +0.j  0.
-0.j -0.   +0.j -0.   -0.j -0.   +0.j -0.   +0.j  0.   -0.j  0.   +0.j  0.   +0.j  0.   -0.j  0.   +0.j -0.   +0.j -0.   +0.j -0.   +0.j -0.   -0.j
 -0.   +0.j -0.   -0.j -0.   +0.j  0.   +0.j -0.   +0.j -0.   -0.j -0.   +0.j -0.   +0.j -0.   +0.j  0.   +0.j  0.   +0.j -0.   +0.j -0.   +0.j  0.
```

```
-0.j  0.   +0.j  0.   +0.j  0.   +0.j -0.   +0.j -0.   +0.j  0.   +0.j  0.   +0.j -0.   +0.j  0.   +0.j -0.   +0.j -0.   +0.j -0.   +0.j -0.001+0.j
  0.   +0.j -0.014+0.j -0.   +0.j  0.   +0.j -0.   +0.j -0.   +0.j -0.   -0.j -0.   +0.j -0.   -0.j -0.   +0.j  0.   +0.j -0.   +0.j -0.   +0.j -0.
+0.j -0.   +0.j -0.   -0.j -0.   +0.j -0.   -0.j -0.   +0.j  0.   +0.j -0.   +0.j -0.   +0.j  0.   +0.j  0.   -0.j  0.   +0.j  0.   -0.j  0.002+0.j
  0.   +0.j  0.   +0.j  0.   +0.j  0.   -0.j -0.   +0.j  0.   +0.j  0.   +0.j  0.012+0.j -0.   +0.j  0.   +0.j  0.   +0.j  0.   +0.j -0.   +0.j -0.
+0.j  0.   +0.j  0.   +0.j  0.   +0.j -0.   +0.j -0.   +0.j  0.   +0.j  0.   +0.j  0.   +0.j  0.   +0.j]

max abs diff =  0.014099740514612172



complex128
```



Observation

- General QR iteration without shift seems to be working for real unsymmetric matrix
- However, applying Wilkinson's shift seems to converge slower than without shift... **WHY??**

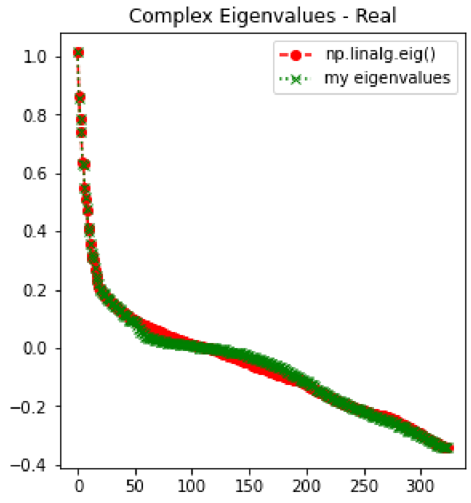
In [37]:

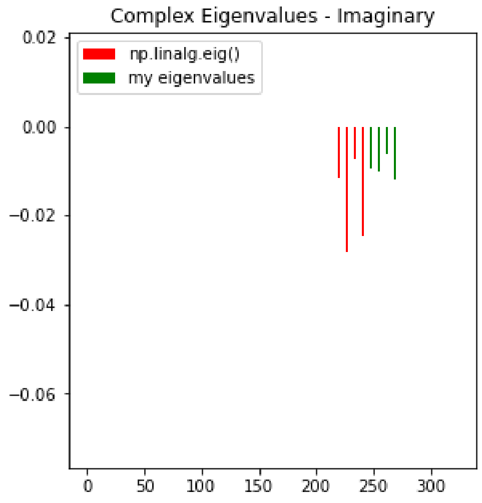
```python
MAX_ITER = 10000

# complex symmetric matrix - WIHTOUT SHIFT

source = 'matricmarket/qc324.mtx.gz'
matrix_class = Matrix

runtest_matrixmarket(source, matrix_class,shift=None)

```

```
From Matrix Market:
matrix source=matricmarket/qc324.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=(324, 324), data type=complex128, max=(1.4957127687965-
0.087585823232133j), min=(-0.40837438550439-0.017204228197979j)
is_symmetric={} True
activiating tridigonal: True

 ------------ Processing eigenvalues only ------------
>>> QR Iteration WITHOUT shift
max_iter=10000 reached, max error=0.034575504317016725 vs tol=1e-06; index of non-convergence=[[52], [54], [142], [144], [178], [180], [268], [270], [27
1]]

 ------------ Checking Eigenvalues ------------

my eigenvalues: [ 1.014-0.063j  0.859-0.064j  0.786-0.072j  0.741-0.07j   0.632-0.069j  0.628-0.06j   0.548-0.058j  0.519-0.048j  0.48 -0.056j  0.41 -0.0
42j  0.401-0.048j  0.354-0.054j  0.324-0.039j  0.312-0.042j  0.305-0.041j  0.275-0.039j  0.248-0.034j  0.235-0.039j  0.218-0.038j  0.203-0.033j  0.2  -0.
038j
  0.193-0.039j  0.189-0.036j  0.184-0.026j  0.182-0.028j  0.17 -0.027j  0.168-0.03j   0.167-0.031j  0.164-0.028j  0.155-0.027j  0.152-0.023j  0.148-0.032
j  0.144-0.03j   0.141-0.031j  0.141-0.03j   0.137-0.032j  0.133-0.028j  0.131-0.033j  0.124-0.037j  0.115-0.033j  0.112-0.037j  0.112-0.03j
  0.111-0.043j  0.108-0.027j  0.101-0.025j  0.095-0.017j  0.094-0.023j  0.092-0.021j  0.091-0.017j  0.091-0.025j  0.088-0.031j  0.085-0.032j  0.079-0.042
j  0.066-0.042j  0.064-0.048j  0.056-0.043j  0.056-0.049j  0.051-0.04j   0.046-0.045j  0.04 -0.04j   0.039-0.036j  0.036-0.036j  0.033-0.042j
  0.032-0.03j   0.031-0.036j  0.031-0.037j  0.031-0.038j  0.028-0.03j   0.027-0.031j  0.027-0.034j  0.026-0.023j  0.025-0.036j  0.024-0.025j  0.022-0.024
j  0.021-0.032j  0.021-0.022j  0.021-0.018j  0.02 -0.024j  0.018-0.019j  0.017-0.016j  0.016-0.019j  0.016-0.032j  0.015-0.022j  0.015-0.019j
  0.015-0.015j  0.015-0.02j   0.014-0.017j  0.014-0.015j  0.014-0.018j  0.013-0.016j  0.013-0.021j  0.012-0.01j   0.012-0.017j  0.012-0.019j  0.011-0.019
j  0.011-0.02j   0.01 -0.008j  0.009-0.015j  0.009-0.014j  0.008-0.007j  0.008-0.014j  0.008-0.027j  0.007-0.013j  0.006-0.01j   0.005-0.006j
  0.003-0.003j  0.002-0.009j  0.002-0.008j  0.002-0.011j  0.002-0.008j  0.001-0.014j  0.   -0.016j -0.   -0.004j -0.   -0.01j  -0.   -0.006j -0.001-0.013
j -0.001-0.002j -0.002-0.002j -0.002-0.004j -0.002-0.005j -0.003-0.003j -0.004-0.009j -0.005-0.006j -0.005-0.007j -0.006-0.006j -0.006-0.001j
 -0.007-0.007j -0.007-0.004j -0.008-0.006j -0.009-0.007j -0.009-0.008j -0.012-0.008j -0.013-0.012j -0.015-0.007j -0.015-0.007j -0.015-0.016j -0.016-0.008
j -0.017-0.01j  -0.017-0.01j  -0.018-0.01j  -0.018-0.008j -0.018-0.014j -0.019-0.01j  -0.02 -0.01j  -0.02 -0.01j  -0.021-0.013j -0.021-0.016j
 -0.021-0.014j -0.027-0.014j -0.029-0.012j -0.03 -0.015j -0.031-0.012j -0.032-0.011j -0.035-0.013j -0.036-0.012j -0.036-0.015j -0.037-0.01j  -0.038-0.01j
-0.038-0.013j -0.041-0.016j -0.044-0.009j -0.044-0.008j -0.046-0.014j -0.046-0.017j -0.047-0.011j -0.049-0.01j  -0.051-0.013j -0.052-0.016j
 -0.053-0.017j -0.055-0.011j -0.056-0.022j -0.058-0.017j -0.059-0.012j -0.06 -0.022j -0.063-0.007j -0.064-0.013j -0.064-0.016j -0.067-0.017j -0.069-0.012
j -0.071-0.018j -0.072-0.024j -0.074-0.011j -0.077-0.018j -0.08  -0.017j -0.082-0.019j -0.083-0.013j -0.089-0.021j -0.089-0.017j -0.094-0.02j
 -0.094-0.021j -0.095-0.018j -0.097-0.018j -0.098-0.013j -0.099-0.019j -0.107-0.026j -0.109-0.011j -0.111-0.024j -0.112-0.017j -0.114-0.009j -0.114-0.009
j -0.115-0.015j -0.117-0.016j -0.119-0.022j -0.124-0.019j -0.126-0.006j -0.128-0.013j -0.128-0.021j -0.131-0.02j  -0.141-0.015j -0.141-0.013j
 -0.144-0.013j -0.148-0.012j -0.148-0.016j -0.149-0.012j -0.149-0.009j -0.153-0.02j  -0.156-0.007j -0.157-0.01j  -0.16  -0.019j -0.164-0.014j -0.167-0.013
j -0.168-0.015j -0.169-0.005j -0.171-0.001j -0.172-0.013j -0.178-0.012j -0.18  -0.013j -0.181-0.008j -0.182-0.012j -0.183-0.023j -0.186-0.012j
 -0.186-0.02j  -0.187-0.011j -0.189-0.017j -0.192-0.009j -0.193-0.009j -0.196-0.016j -0.196-0.015j -0.201-0.016j -0.204-0.013j -0.205-0.013j -0.207-0.004
j -0.208-0.009j -0.209-0.012j -0.21  -0.014j -0.211-0.013j -0.211-0.012j -0.212-0.007j -0.217-0.009j -0.218-0.016j -0.22  -0.018j -0.22  -0.015j
 -0.222-0.014j -0.222-0.014j -0.227-0.013j -0.227-0.01j  -0.229-0.005j -0.23  -0.008j -0.234-0.012j -0.235-0.014j -0.235-0.01j  -0.237-0.009j -0.239-0.006
j -0.239-0.004j -0.24  -0.012j -0.243-0.013j -0.244-0.009j -0.245-0.014j -0.246-0.006j -0.246-0.012j -0.249-0.008j -0.25  -0.012j -0.251-0.015j
 -0.253-0.016j -0.254-0.008j -0.256-0.013j -0.256-0.012j -0.259-0.013j -0.26  -0.011j -0.261-0.014j -0.267-0.011j -0.267-0.015j -0.268-0.01j  -0.269-0.009
```

```
j -0.272-0.014j -0.272-0.006j -0.274-0.008j -0.275-0.007j -0.278-0.011j -0.284-0.003j -0.285-0.013j -0.287-0.014j -0.288-0.008j -0.288-0.011j
 -0.289-0.009j -0.292+0.006j -0.295-0.004j -0.297-0.004j -0.299-0.004j -0.302-0.005j -0.302+0.002j -0.302-0.002j -0.306+0.007j -0.308-0.004j -0.309-0.006
j -0.315-0.003j -0.316-0.013j -0.317-0.001j -0.318+0.006j -0.319-0.005j -0.323-0.005j -0.323-0.009j -0.325+0.004j -0.332-0.004j -0.333+0.004j
 -0.335+0.005j -0.336-0.002j -0.337+0.007j -0.338+0.004j -0.34 +0.009j -0.341-0.002j -0.341+0.008j -0.342+0.017j -0.343+0.003j]
np.linalg.eig:  [ 1.014-0.064j  0.859-0.063j  0.785-0.064j  0.739-0.064j  0.637-0.064j  0.631-0.063j  0.549-0.063j  0.511-0.064j  0.474-0.062j  0.41 -0.0
6j   0.408-0.064j  0.355-0.058j  0.32 -0.063j  0.307-0.055j  0.301-0.045j  0.267-0.053j  0.245-0.062j  0.232-0.05j   0.208-0.038j  0.202-0.048j  0.201-0.
032j
  0.193-0.03j   0.184-0.029j  0.181-0.06j   0.177-0.028j  0.176-0.046j  0.17 -0.027j  0.164-0.026j  0.159-0.035j  0.158-0.025j  0.154-0.044j  0.152-0.024
j  0.146-0.023j  0.14 -0.022j  0.136-0.042j  0.135-0.022j  0.13 -0.021j  0.127-0.031j  0.126-0.058j  0.125-0.021j  0.12 -0.04j   0.12 -0.02j
  0.115-0.019j  0.111-0.019j  0.108-0.038j  0.106-0.019j  0.103-0.02j   0.1  -0.018j  0.098-0.037j  0.096-0.017j  0.092-0.017j  0.092-0.035j  0.088-0.016
j  0.087-0.034j  0.085-0.015j  0.082-0.033j  0.081-0.015j  0.079-0.055j  0.078-0.014j  0.077-0.033j  0.074-0.014j  0.073-0.045j  0.072-0.032j
  0.071-0.013j  0.068-0.013j  0.067-0.031j  0.064-0.012j  0.062-0.03j   0.061-0.012j  0.058-0.011j  0.057-0.03j   0.055-0.011j  0.053-0.029j  0.052-0.01j
0.049-0.01j   0.048-0.028j  0.046-0.009j  0.044-0.009j  0.043-0.027j  0.041-0.008j  0.039-0.027j  0.038-0.008j  0.038-0.053j  0.036-0.007j
  0.034-0.026j  0.033-0.007j  0.031-0.006j  0.03 -0.025j  0.028-0.006j  0.026-0.006j  0.025-0.024j  0.024-0.005j  0.021-0.005j  0.021-0.024j  0.019-0.004
j  0.017-0.004j  0.016-0.023j  0.015-0.004j  0.014-0.003j  0.012-0.022j  0.012-0.003j  0.01 -0.003j  0.008-0.002j  0.008-0.022j  0.007-0.002j
  0.006-0.002j  0.004-0.001j  0.004-0.021j  0.003-0.05j   0.003-0.001j  0.002-0.001j  0.001-0.j     0.   -0.j    -0.   -0.02j  -0.   -0.j    -0.001-0.j
-0.002-0.j    -0.004-0.j    -0.004-0.02j  -0.006-0.j    -0.008-0.j    -0.008-0.019j -0.011-0.j    -0.012-0.018j -0.014-0.j    -0.016-0.018j
 -0.017-0.j    -0.02 -0.017j -0.021-0.j    -0.021-0.038j -0.024-0.017j -0.025-0.j    -0.027-0.048j -0.027-0.016j -0.027-0.032j -0.03 -0.j    -0.031-0.015
j -0.034-0.015j -0.035-0.j    -0.036-0.03j  -0.038-0.014j -0.04 -0.j    -0.041-0.014j -0.044-0.029j -0.045-0.013j -0.046-0.j    -0.048-0.013j
 -0.051-0.012j -0.051-0.028j -0.052-0.j    -0.052-0.046j -0.054-0.011j -0.058-0.011j -0.058-0.j    -0.058-0.027j -0.061-0.01j  -0.064-0.01j  -0.065-0.026
j -0.065-0.j    -0.066-0.009j -0.069-0.009j -0.07 -0.035j -0.071-0.025j -0.072-0.008j -0.074-0.044j -0.075-0.008j -0.077-0.024j -0.078-0.008j
 -0.08 -0.007j -0.083-0.007j -0.083-0.023j -0.085-0.006j -0.087-0.006j -0.088-0.022j -0.09 -0.005j -0.092-0.005j -0.093-0.042j -0.094-0.022j -0.094-0.004
j -0.096-0.004j -0.098-0.004j -0.099-0.021j -0.1  -0.003j -0.102-0.031j -0.102-0.003j -0.104-0.003j -0.104-0.021j -0.105-0.002j -0.107-0.002j
 -0.108-0.04j  -0.109-0.002j -0.109-0.02j  -0.11 -0.001j -0.111-0.001j -0.112-0.001j -0.113-0.j    -0.114-0.019j -0.118-0.019j -0.121-0.038j -0.122-0.019
j -0.125-0.02j  -0.129-0.018j -0.13 -0.037j -0.133-0.017j -0.137-0.017j -0.137-0.035j -0.14 -0.016j -0.142-0.034j -0.144-0.015j -0.147-0.033j
 -0.147-0.015j -0.151-0.014j -0.152-0.033j -0.154-0.014j -0.157-0.032j -0.158-0.013j -0.161-0.013j -0.162-0.031j -0.164-0.012j -0.166-0.03j  -0.167-0.012
j -0.171-0.011j -0.171-0.03j  -0.174-0.011j -0.176-0.029j -0.177-0.01j  -0.179-0.01j  -0.181-0.028j -0.182-0.009j -0.185-0.009j -0.185-0.027j
 -0.188-0.008j -0.19 -0.027j -0.191-0.008j -0.193-0.007j -0.195-0.026j -0.196-0.007j -0.198-0.006j -0.199-0.025j -0.2  -0.006j -0.203-0.006j -0.203-0.024
j -0.205-0.005j -0.207-0.005j -0.208-0.024j -0.209-0.004j -0.211-0.004j -0.212-0.023j -0.213-0.004j -0.215-0.003j -0.216-0.022j -0.217-0.003j
 -0.219-0.003j -0.22 -0.002j -0.221-0.022j -0.222-0.002j -0.223-0.002j -0.224-0.001j -0.225-0.021j -0.226-0.001j -0.227-0.001j -0.228-0.j    -0.228-0.j
-0.229-0.02j  -0.229-0.j    -0.23 -0.j    -0.231-0.j    -0.233-0.j    -0.233-0.02j  -0.234-0.j    -0.237-0.j    -0.237-0.019j -0.239-0.j
 -0.241-0.018j -0.242-0.j    -0.245-0.018j -0.246-0.j    -0.248-0.017j -0.25 -0.j    -0.252-0.017j -0.254-0.j    -0.256-0.016j -0.258-0.j    -0.259-0.015
j -0.263-0.015j -0.263-0.j    -0.266-0.014j -0.269-0.j    -0.27 -0.014j -0.273-0.013j -0.274-0.j    -0.277-0.013j -0.28 -0.012j -0.28 -0.j
 -0.283-0.011j -0.286-0.011j -0.287-0.j    -0.289-0.01j  -0.292-0.01j  -0.294-0.j    -0.295-0.009j -0.298-0.009j -0.301-0.008j -0.304-0.008j -0.306-0.008
j -0.309-0.007j -0.311-0.007j -0.314-0.006j -0.316-0.006j -0.318-0.005j -0.321-0.005j -0.323-0.004j -0.325-0.004j -0.327-0.004j -0.329-0.003j
 -0.331-0.003j -0.332-0.003j -0.334-0.002j -0.336-0.002j -0.337-0.002j -0.339-0.001j -0.34 -0.001j -0.341-0.001j -0.342-0.j   ]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [-0.   +0.001j -0.001-0.002j  0.001-0.008j  0.002-0.006j -0.005-0.005j -0.003+0.003j -0.001+0.005j  0.008+0.016j  0.005+0.006j  0.   +0.018j -0.007+0.01
6j -0.001+0.004j  0.003+0.024j  0.005+0.013j  0.004+0.003j  0.008+0.014j  0.003+0.028j  0.003+0.011j  0.011+0.j     0.002+0.015j -0.001-0.006j
  0.001-0.008j  0.004-0.007j  0.003+0.033j  0.005-0.001j -0.006+0.019j -0.003-0.004j  0.003-0.005j  0.006+0.007j -0.003-0.002j -0.003+0.02j  -0.004-0.008
j -0.002-0.007j  0.001-0.009j  0.005+0.012j  0.002-0.01j   0.003-0.007j  0.004-0.002j -0.002+0.021j -0.01 -0.013j -0.008+0.003j -0.008-0.01j
 -0.004-0.024j -0.003-0.007j -0.007+0.013j -0.011+0.002j -0.009-0.003j -0.008-0.003j -0.007+0.02j  -0.005-0.008j -0.004-0.014j -0.007+0.003j -0.01 -0.026
j -0.021-0.008j -0.021-0.032j -0.026-0.01j  -0.026-0.034j -0.028+0.016j -0.031-0.031j -0.037-0.008j -0.036-0.022j -0.036+0.008j -0.039-0.01j
 -0.039-0.016j -0.036-0.023j -0.036-0.006j -0.034-0.026j -0.034+0.001j -0.034-0.02j  -0.031-0.023j -0.031+0.007j -0.03 -0.025j -0.029+0.004j -0.03 -0.014
j -0.028-0.023j -0.027+0.006j -0.026-0.009j -0.024-0.016j -0.026+0.009j -0.023-0.007j -0.022+0.007j -0.022-0.024j -0.023+0.031j -0.02 -0.012j
 -0.019+0.011j -0.018-0.013j -0.016-0.011j -0.016+0.01j  -0.015-0.012j -0.013-0.011j -0.013+0.004j -0.011-0.005j -0.01 -0.012j -0.009+0.005j -0.008-0.015
j -0.006-0.016j -0.006+0.015j -0.006-0.012j -0.005-0.011j -0.004+0.015j -0.004-0.012j -0.002-0.024j -0.002-0.011j -0.002+0.011j -0.002-0.004j
 -0.002-0.002j -0.002-0.007j -0.002+0.013j -0.001+0.039j -0.001-0.007j -0.001-0.013j -0.001-0.016j -0.   -0.004j -0.   +0.01j   0.   -0.006j  0.   -0.013
j  0.001-0.002j  0.002-0.002j  0.002+0.016j  0.003-0.005j  0.005-0.003j  0.004+0.01j   0.006-0.006j  0.007+0.011j  0.008-0.006j  0.01 +0.017j
```

```
  0.01 -0.007j  0.012+0.013j  0.013-0.006j  0.013+0.031j  0.014+0.009j  0.013-0.008j  0.014+0.036j  0.013+0.009j  0.012+0.025j  0.014-0.016j  0.014+0.007
j  0.018+0.005j  0.017-0.01j   0.018+0.021j  0.02 +0.006j  0.022-0.014j  0.022+0.004j  0.025+0.019j  0.025+0.003j  0.024-0.013j  0.027-0.003j
  0.03 -0.002j  0.025+0.014j  0.023-0.012j  0.022+0.031j  0.023-0.001j  0.025-0.j      0.023-0.013j  0.023+0.015j  0.025-0.004j  0.027-0.001j  0.027+0.016
j  0.027-0.013j  0.025-0.006j  0.026+0.j      0.026+0.027j  0.025+0.01j   0.026-0.008j  0.027+0.032j  0.026-0.002j  0.026+0.011j  0.026-0.009j
  0.027-0.01j   0.028-0.004j  0.027+0.001j  0.027-0.011j  0.028-0.006j  0.028+0.001j  0.027-0.002j  0.028-0.008j  0.029+0.025j  0.026+0.005j  0.025-0.008
j  0.026-0.014j  0.026-0.021j  0.024+0.01j   0.024-0.014j  0.022+0.014j  0.02 -0.016j  0.021-0.01j   0.015+0.j      0.016-0.015j  0.013-0.018j
  0.014+0.019j  0.014-0.016j  0.012+0.002j  0.012-0.011j  0.012-0.018j  0.005-0.026j  0.005-0.01j   0.003-0.005j  0.006+0.003j  0.006+0.029j  0.008+0.01j
0.01 +0.005j  0.012+0.003j  0.012+0.015j  0.008-0.002j  0.011+0.01j   0.009+0.021j  0.012-0.005j  0.01 +0.014j  0.003+0.001j  0.005+0.02j
  0.004+0.002j  0.003+0.002j  0.004+0.017j  0.005+0.002j  0.007+0.023j  0.005-0.007j  0.005+0.006j  0.005+0.021j  0.005-0.007j  0.002+0.016j  0.   -0.001
j  0.002-0.004j  0.003+0.024j  0.002+0.009j  0.004+0.015j -0.001-0.002j -0.   -0.003j -0.001+0.02j   0.001-0.003j  0.002-0.015j -0.   +0.015j
  0.002-0.012j  0.003+0.015j  0.002-0.01j   0.001-0.002j  0.002+0.017j -0.   -0.009j  0.002-0.008j -0.002+0.009j -0.004-0.007j -0.002-0.007j -0.004+0.02j
-0.003-0.004j -0.002-0.007j -0.003+0.01j  -0.002-0.008j  0.   -0.008j  0.   +0.016j -0.004-0.006j -0.003-0.013j -0.003+0.004j -0.003-0.012j
 -0.003-0.012j -0.002-0.012j -0.006+0.009j -0.005-0.008j -0.006-0.003j -0.006-0.007j -0.01 +0.009j -0.009-0.013j -0.008-0.01j  -0.009-0.009j -0.011-0.006
j -0.011+0.017j -0.011-0.012j -0.013-0.013j -0.013-0.009j -0.013-0.014j -0.013+0.014j -0.012-0.012j -0.012-0.008j -0.013+0.007j -0.012-0.015j
 -0.012+0.003j -0.012-0.008j -0.011+0.005j -0.01 -0.012j -0.011+0.004j -0.011-0.011j -0.009+0.003j -0.013-0.011j -0.011+0.001j -0.01 -0.01j  -0.009+0.007
j -0.009+0.j    -0.008-0.006j -0.007+0.006j -0.006-0.007j -0.009+0.003j -0.011+0.01j  -0.011-0.013j -0.01 -0.002j -0.008+0.004j -0.007-0.011j
 -0.006+0.002j -0.006+0.017j -0.008-0.004j -0.008+0.007j -0.006+0.006j -0.008-0.005j -0.006+0.011j -0.004+0.007j -0.006+0.016j -0.005+0.004j -0.002+0.001
j -0.006+0.004j -0.004-0.006j -0.003+0.005j -0.002+0.012j -0.   +0.j    -0.002-0.j    -0.   -0.004j  0.   +0.008j -0.005-0.j    -0.004+0.007j
 -0.004+0.008j -0.004+0.j    -0.002+0.009j -0.003+0.006j -0.003+0.011j -0.002-0.001j -0.001+0.009j -0.001+0.017j -0.001+0.004j]

max abs diff =  0.04387580291749045


complex128
```





Observation

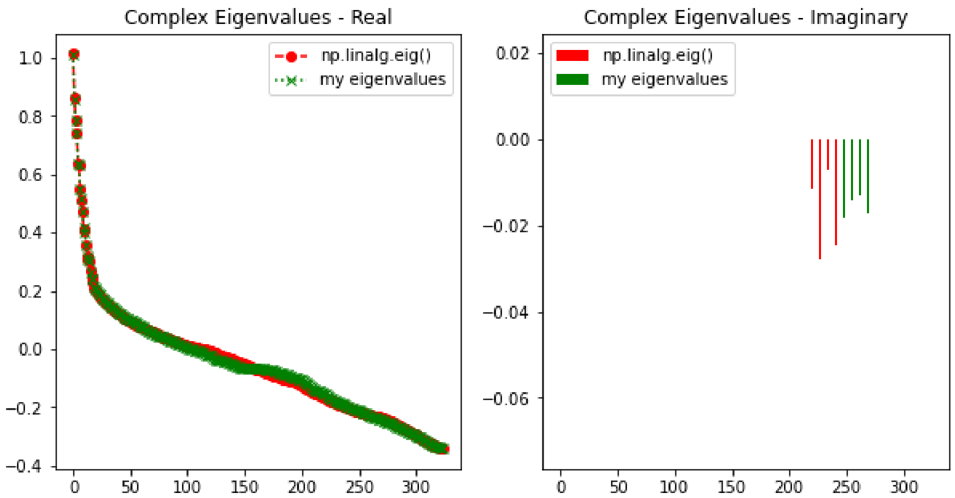
- General QR iteration without shift seems to be working for complex symmetric matrix

In [39]:
```python
MAX_ITER = 10000

# complex symmetric matrix - WILKINSON SHIFT

source = 'matricmarket/qc324.mtx.gz'
matrix_class = Matrix
runtest_matrixmarket(source, matrix_class,shift='Wilkinson')

```

```
From Matrix Market:
matrix source=matricmarket/qc324.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=(324, 324), data type=complex128, max=(1.4957127687965-
0.087585823232133j), min=(-0.40837438550439-0.017204228197979j)
is_symmetric={} True
activiating tridigonal: True

 ------------ Processing eigenvalues only ------------
>>> QR Iteration with Wilkinson shift
Iteration terminates at i=9047 with tol=1e-06 reached

 ------------ Checking Eigenvalues ------------

my eigenvalues: [ 1.013-0.063j  0.859-0.064j  0.788-0.067j  0.741-0.071j  0.635-0.061j  0.634-0.072j  0.547-0.058j  0.517-0.061j  0.471-0.053j  0.414-0.0
47j  0.402-0.046j  0.357-0.035j  0.317-0.036j  0.309-0.038j  0.304-0.044j  0.265-0.04j   0.243-0.042j  0.241-0.037j  0.213-0.034j  0.206-0.031j  0.202-0.
033j
  0.196-0.037j  0.191-0.036j  0.188-0.03j   0.184-0.039j  0.178-0.04j   0.171-0.031j  0.168-0.028j  0.164-0.031j  0.158-0.036j  0.156-0.024j  0.152-0.026
j  0.147-0.03j   0.142-0.022j  0.141-0.031j  0.14 -0.027j  0.137-0.018j  0.134-0.021j  0.13 -0.025j  0.125-0.035j  0.122-0.038j  0.12 -0.026j
  0.118-0.028j  0.114-0.027j  0.112-0.029j  0.11 -0.028j  0.107-0.031j  0.104-0.031j  0.102-0.032j  0.098-0.037j  0.098-0.02j   0.097-0.019j  0.096-0.016
j  0.094-0.02j   0.091-0.02j   0.088-0.024j  0.088-0.013j  0.085-0.017j  0.082-0.021j  0.077-0.021j  0.076-0.025j  0.076-0.023j  0.07 -0.028j
  0.067-0.035j  0.065-0.036j  0.062-0.044j  0.061-0.039j  0.059-0.035j  0.055-0.046j  0.054-0.037j  0.052-0.045j  0.049-0.047j  0.049-0.035j  0.046-0.03j
0.046-0.03j   0.045-0.05j   0.043-0.023j  0.043-0.033j  0.042-0.035j  0.042-0.014j  0.037-0.025j  0.037-0.007j  0.034-0.024j  0.03 -0.026j
  0.029-0.024j  0.029-0.019j  0.028-0.017j  0.026-0.022j  0.025-0.024j  0.021-0.026j  0.021-0.026j  0.017-0.023j  0.016-0.023j  0.015-0.028j  0.015-0.017
j  0.011-0.022j  0.01 -0.021j  0.009-0.025j  0.008-0.027j  0.006-0.023j  0.003-0.008j  0.002-0.001j  0.   -0.029j -0.   -0.017j -0.   -0.006j
 -0.002-0.012j -0.003-0.041j -0.004-0.016j -0.005-0.045j -0.005-0.011j -0.007-0.017j -0.009-0.007j -0.011-0.021j -0.013-0.009j -0.013-0.011j -0.015-0.015
j -0.021-0.011j -0.022-0.017j -0.022-0.017j -0.023-0.015j -0.023-0.011j -0.025-0.012j -0.031-0.01j  -0.031-0.012j -0.036-0.008j -0.037-0.008j
 -0.038-0.012j -0.038-0.014j -0.039-0.011j -0.04 -0.007j -0.041-0.009j -0.043-0.008j -0.043-0.01j  -0.043-0.006j -0.044-0.005j -0.047-0.008j -0.047-0.013
j -0.047-0.007j -0.052-0.016j -0.053-0.014j -0.054-0.01j  -0.055-0.007j -0.056-0.011j -0.058-0.007j -0.061-0.013j -0.063-0.005j -0.065-0.007j
 -0.065-0.004j -0.065-0.005j -0.065-0.005j -0.067-0.007j -0.067-0.005j -0.067-0.004j -0.068-0.005j -0.068-0.004j -0.068-0.009j -0.068-0.005j -0.068-0.004
j -0.068-0.005j -0.068-0.004j -0.068-0.004j -0.068-0.003j -0.068-0.005j -0.068-0.007j -0.068-0.006j -0.068-0.004j -0.069-0.009j -0.069-0.003j
 -0.069-0.004j -0.07 -0.006j -0.07 -0.j    -0.07 -0.011j -0.071-0.007j -0.072-0.012j -0.072-0.007j -0.073-0.008j -0.076-0.011j -0.078-0.014j -0.079-0.012
j -0.083-0.013j -0.084-0.018j -0.084-0.011j -0.084-0.009j -0.085-0.008j -0.088-0.011j -0.088-0.017j -0.088-0.009j -0.089-0.013j -0.09 -0.011j
 -0.091-0.018j -0.092-0.016j -0.092-0.013j -0.097-0.011j -0.098-0.018j -0.098-0.015j -0.098-0.017j -0.1   -0.017j -0.105-0.018j -0.106-0.011j -0.106-0.02j
-0.107-0.014j -0.109-0.011j -0.113-0.013j -0.115-0.019j -0.116-0.021j -0.119-0.016j -0.121-0.025j -0.126-0.021j -0.133-0.011j -0.133-0.008j
 -0.137-0.01j  -0.137-0.017j -0.142-0.005j -0.143-0.013j -0.143-0.004j -0.145-0.007j -0.146-0.017j -0.149-0.006j -0.15 -0.004j -0.155-0.006j -0.155-0.008
j -0.156-0.011j -0.162-0.011j -0.165-0.016j -0.166-0.019j -0.173-0.018j -0.173-0.008j -0.173-0.011j -0.173-0.006j -0.174-0.016j -0.181-0.007j
 -0.181-0.019j -0.183-0.018j -0.186-0.014j -0.188-0.011j -0.189-0.016j -0.19 -0.017j -0.19 -0.01j  -0.193-0.015j -0.196-0.025j -0.2   -0.005j -0.201-0.016
j -0.202-0.023j -0.202-0.021j -0.203-0.014j -0.204-0.022j -0.207-0.01j  -0.208-0.019j -0.209-0.018j -0.21 -0.022j -0.212-0.023j -0.215-0.02j
 -0.217-0.011j -0.22 -0.017j -0.221-0.01j  -0.222-0.014j -0.223-0.019j -0.227-0.015j -0.227-0.017j -0.228-0.018j -0.232-0.016j -0.232-0.015j -0.234-0.013
j -0.236-0.017j -0.236-0.02j  -0.237-0.016j -0.239-0.011j -0.241-0.019j -0.243-0.018j -0.244-0.017j -0.244-0.017j -0.245-0.009j -0.246-0.011j
 -0.249-0.01j  -0.25  -0.017j -0.253-0.007j -0.254-0.013j -0.254-0.018j -0.256-0.014j -0.258-0.007j -0.261-0.011j -0.263-0.011j -0.263-0.015j -0.267-0.015
j -0.269-0.012j -0.271-0.008j -0.272-0.019j -0.274-0.012j -0.275-0.014j -0.277-0.011j -0.279-0.009j -0.28  -0.008j -0.284-0.01j  -0.285-0.003j
 -0.288-0.005j -0.289+0.003j -0.291-0.005j -0.292+0.001j -0.293-0.005j -0.295+0.003j -0.298-0.008j -0.301-0.001j -0.301-0.004j -0.302-0.004j -0.305-0.006
j -0.308+0.002j -0.312-0.007j -0.315-0.009j -0.316-0.002j -0.317-0.003j -0.321-0.004j -0.323+0.004j -0.323+0.003j -0.329+0.002j -0.331+0.002j
```

```
 -0.332+0.004j -0.336+0.005j -0.337+0.004j -0.337+0.003j -0.338+0.007j -0.338+0.011j -0.34 +0.009j -0.341+0.007j -0.343+0.02j ]
np.linalg.eig:  [ 1.014-0.064j  0.859-0.063j  0.785-0.064j  0.739-0.064j  0.637-0.064j  0.631-0.063j  0.549-0.063j  0.511-0.064j  0.474-0.062j  0.41 -0.0
6j   0.408-0.064j  0.355-0.058j  0.32 -0.063j  0.307-0.055j  0.301-0.045j  0.267-0.053j  0.245-0.062j  0.232-0.05j   0.208-0.038j  0.202-0.048j  0.201-0.
032j
  0.193-0.03j   0.184-0.029j  0.181-0.06j   0.177-0.028j  0.176-0.046j  0.17 -0.027j  0.164-0.026j  0.159-0.035j  0.158-0.025j  0.154-0.044j  0.152-0.024
j  0.146-0.023j  0.14 -0.022j  0.136-0.042j  0.135-0.022j  0.13 -0.021j  0.127-0.031j  0.126-0.058j  0.125-0.021j  0.12 -0.04j   0.12 -0.02j
  0.115-0.019j  0.111-0.019j  0.108-0.038j  0.106-0.019j  0.103-0.02j   0.1  -0.018j  0.098-0.037j  0.096-0.017j  0.092-0.017j  0.092-0.035j  0.088-0.016
j  0.087-0.034j  0.085-0.015j  0.082-0.033j  0.081-0.015j  0.079-0.055j  0.078-0.014j  0.077-0.033j  0.074-0.014j  0.073-0.045j  0.072-0.032j
  0.071-0.013j  0.068-0.013j  0.067-0.031j  0.064-0.012j  0.062-0.03j   0.061-0.012j  0.058-0.011j  0.057-0.03j   0.055-0.011j  0.053-0.029j  0.052-0.01j
0.049-0.01j   0.048-0.028j  0.046-0.009j  0.044-0.009j  0.043-0.027j  0.041-0.008j  0.039-0.027j  0.038-0.008j  0.038-0.053j  0.036-0.007j
  0.034-0.026j  0.033-0.007j  0.031-0.006j  0.03 -0.025j  0.028-0.006j  0.026-0.006j  0.025-0.024j  0.024-0.005j  0.021-0.005j  0.021-0.024j  0.019-0.004
j  0.017-0.004j  0.016-0.023j  0.015-0.004j  0.014-0.003j  0.012-0.022j  0.012-0.003j  0.01 -0.003j  0.008-0.002j  0.008-0.022j  0.007-0.002j
  0.006-0.002j  0.004-0.001j  0.004-0.021j  0.003-0.05j   0.003-0.001j  0.002-0.001j  0.001-0.j      0.   -0.j     -0.   -0.02j  -0.   -0.j     -0.001-0.j
-0.002-0.j    -0.004-0.j    -0.004-0.02j  -0.006-0.j    -0.008-0.j    -0.008-0.019j -0.011-0.j    -0.012-0.018j -0.014-0.j    -0.016-0.018j
 -0.017-0.j    -0.02 -0.017j -0.021-0.j    -0.021-0.038j -0.024-0.017j -0.025-0.j    -0.027-0.048j -0.027-0.016j -0.027-0.032j -0.03 -0.j    -0.031-0.015
j -0.034-0.015j -0.035-0.j    -0.036-0.03j  -0.038-0.014j -0.04 -0.j    -0.041-0.014j -0.044-0.029j -0.045-0.013j -0.046-0.j    -0.048-0.013j
 -0.051-0.012j -0.051-0.028j -0.052-0.j    -0.052-0.046j -0.054-0.011j -0.058-0.011j -0.058-0.j    -0.058-0.027j -0.061-0.01j  -0.064-0.01j  -0.065-0.026
j -0.065-0.j    -0.066-0.009j -0.069-0.009j -0.07 -0.035j -0.071-0.025j -0.072-0.008j -0.074-0.044j -0.075-0.008j -0.077-0.024j -0.078-0.008j
 -0.08 -0.007j -0.083-0.007j -0.083-0.023j -0.085-0.006j -0.087-0.006j -0.088-0.022j -0.09 -0.005j -0.092-0.005j -0.093-0.042j -0.094-0.022j -0.094-0.004
j -0.096-0.004j -0.098-0.004j -0.099-0.021j -0.1  -0.003j -0.102-0.031j -0.102-0.003j -0.104-0.003j -0.104-0.021j -0.105-0.002j -0.107-0.002j
 -0.108-0.04j  -0.109-0.002j -0.109-0.02j  -0.11 -0.001j -0.111-0.001j -0.112-0.001j -0.113-0.j    -0.114-0.019j -0.118-0.019j -0.121-0.038j -0.122-0.019
j -0.125-0.02j  -0.129-0.018j -0.13 -0.037j -0.133-0.017j -0.137-0.017j -0.137-0.035j -0.14 -0.016j -0.142-0.034j -0.144-0.015j -0.147-0.033j
 -0.147-0.015j -0.151-0.014j -0.152-0.033j -0.154-0.014j -0.157-0.032j -0.158-0.013j -0.161-0.013j -0.162-0.031j -0.164-0.012j -0.166-0.03j  -0.167-0.012
j -0.171-0.011j -0.171-0.03j  -0.174-0.011j -0.176-0.029j -0.177-0.01j  -0.179-0.01j  -0.181-0.028j -0.182-0.009j -0.185-0.009j -0.185-0.027j
 -0.188-0.008j -0.19 -0.027j -0.191-0.008j -0.193-0.007j -0.195-0.026j -0.196-0.007j -0.198-0.006j -0.199-0.025j -0.2  -0.006j -0.203-0.006j -0.203-0.024
j -0.205-0.005j -0.207-0.005j -0.208-0.024j -0.209-0.004j -0.211-0.004j -0.212-0.023j -0.213-0.004j -0.215-0.003j -0.216-0.022j -0.217-0.003j
 -0.219-0.003j -0.22 -0.002j -0.221-0.022j -0.222-0.002j -0.223-0.002j -0.224-0.001j -0.225-0.021j -0.226-0.001j -0.227-0.001j -0.228-0.j    -0.228-0.j
-0.229-0.02j  -0.229-0.j    -0.23 -0.j    -0.231-0.j    -0.233-0.j    -0.233-0.02j  -0.234-0.j    -0.237-0.j    -0.237-0.019j -0.239-0.j
 -0.241-0.018j -0.242-0.j    -0.245-0.018j -0.246-0.j    -0.248-0.017j -0.25 -0.j    -0.252-0.017j -0.254-0.j    -0.256-0.016j -0.258-0.j    -0.259-0.015
j -0.263-0.015j -0.263-0.j    -0.266-0.014j -0.269-0.j    -0.27 -0.014j -0.273-0.013j -0.274-0.j    -0.277-0.013j -0.28 -0.012j -0.28 -0.j
 -0.283-0.011j -0.286-0.011j -0.287-0.j    -0.289-0.01j  -0.292-0.01j  -0.294-0.j    -0.295-0.009j -0.298-0.009j -0.301-0.008j -0.304-0.008j -0.306-0.008
j -0.309-0.007j -0.311-0.007j -0.314-0.006j -0.316-0.006j -0.318-0.005j -0.321-0.005j -0.323-0.004j -0.325-0.004j -0.327-0.004j -0.329-0.003j
 -0.331-0.003j -0.332-0.003j -0.334-0.002j -0.336-0.002j -0.337-0.002j -0.339-0.001j -0.34 -0.001j -0.341-0.001j -0.342-0.j   ]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [-0.001+0.001j -0.001-0.001j  0.002-0.003j  0.002-0.006j -0.001+0.003j  0.003-0.009j -0.002+0.006j  0.007+0.004j -0.003+0.009j  0.005+0.013j -0.006+0.01
8j  0.003+0.022j -0.003+0.027j  0.001+0.018j  0.003+0.001j -0.002+0.012j -0.002+0.02j   0.009+0.013j  0.006+0.004j  0.005+0.017j  0.001-0.002j
  0.003-0.007j  0.007-0.008j  0.007+0.029j  0.007-0.011j  0.002+0.006j  0.001-0.004j  0.004-0.002j  0.005+0.004j  0.001-0.011j  0.002+0.019j  0.001-0.002
j  0.002-0.007j  0.001+0.001j  0.006+0.011j  0.005-0.005j  0.007+0.003j  0.007+0.01j   0.004+0.032j -0.   -0.014j  0.002+0.002j  0.   -0.006j
  0.003-0.009j  0.003-0.008j  0.004+0.009j  0.004-0.008j  0.003-0.011j  0.004-0.012j  0.004+0.004j  0.002-0.02j   0.006-0.004j  0.005+0.016j  0.007+0.j
0.007+0.014j  0.006-0.004j  0.006+0.009j  0.007+0.002j  0.006+0.038j  0.004-0.007j  0.   +0.012j  0.002-0.011j  0.003+0.021j -0.002+0.004j
 -0.004-0.022j -0.002-0.024j -0.005-0.013j -0.004-0.027j -0.003-0.004j -0.006-0.034j -0.005-0.026j -0.005-0.016j -0.006-0.037j -0.004-0.006j -0.006-0.02j
-0.004-0.021j -0.003-0.022j -0.004-0.014j -0.001-0.024j -0.001-0.007j  0.001-0.006j -0.001+0.002j -0.001+0.001j -0.004+0.029j -0.006-0.019j
 -0.005+0.002j -0.004-0.013j -0.002-0.011j -0.004+0.004j -0.003-0.018j -0.005-0.021j -0.004-0.001j -0.007-0.018j -0.006-0.018j -0.006-0.004j -0.004-0.013
j -0.006-0.019j -0.006+0.002j -0.006-0.021j -0.006-0.024j -0.006-0.j    -0.009-0.005j -0.008+0.001j -0.008-0.027j -0.008+0.004j -0.007-0.004j
 -0.007-0.011j -0.008-0.04j  -0.008+0.005j -0.008+0.005j -0.008-0.01j  -0.009-0.016j -0.01 -0.007j -0.011-0.02j  -0.013+0.011j -0.013-0.011j -0.014-0.015
j -0.018-0.011j -0.018-0.017j -0.018+0.003j -0.017-0.015j -0.015-0.011j -0.016+0.008j -0.02 -0.01j  -0.019+0.007j -0.022-0.008j -0.021+0.01j
 -0.02 -0.012j -0.018+0.004j -0.018-0.011j -0.019+0.031j -0.018+0.007j -0.018-0.008j -0.016+0.038j -0.016+0.01j  -0.017+0.027j -0.017-0.008j -0.016+0.003
j -0.013+0.008j -0.017-0.016j -0.017+0.016j -0.016+0.005j -0.015-0.007j -0.015+0.003j -0.014+0.022j -0.016+0.j    -0.017-0.005j -0.017+0.005j
 -0.014+0.008j -0.014+0.022j -0.014-0.005j -0.014+0.039j -0.013+0.006j -0.01 +0.007j -0.009-0.005j -0.009+0.023j -0.007+0.001j -0.004+0.005j -0.003+0.022
```

```
j -0.003-0.005j -0.001+0.005j  0.001+0.005j  0.002+0.032j  0.003+0.02j   0.004+0.001j  0.006+0.038j  0.006+0.004j  0.008+0.015j  0.009+0.004j
  0.011+0.003j  0.013+0.001j  0.013+0.023j  0.015-0.005j  0.016-0.002j  0.016+0.011j  0.018-0.002j  0.019-0.003j  0.017+0.03j   0.016+0.008j  0.016-0.007
j  0.013-0.009j  0.014-0.014j  0.015+0.01j   0.016-0.005j  0.017+0.023j  0.014-0.008j  0.016-0.014j  0.016+0.012j  0.016-0.011j  0.018-0.01j
  0.017+0.022j  0.017-0.014j  0.017+0.007j  0.013-0.01j   0.013-0.017j  0.015-0.014j  0.015-0.016j  0.013+0.003j  0.013+0.001j  0.015+0.028j  0.016-0.001
j  0.018+0.006j  0.02 +0.007j  0.018+0.024j  0.017-0.001j  0.02 -0.004j  0.018+0.019j  0.019-0.009j  0.016+0.013j  0.011+0.004j  0.013+0.025j
  0.011+0.005j  0.014-0.002j  0.009+0.027j  0.012+0.001j  0.014+0.028j  0.013+0.006j  0.015-0.004j  0.013+0.025j  0.014+0.008j  0.012+0.024j  0.012+0.003
j  0.015-0.j     0.009+0.018j  0.008-0.006j  0.01 +0.01j   0.004-0.007j  0.007+0.002j  0.008+0.017j  0.009+0.003j  0.011-0.008j  0.005+0.02j
  0.007-0.01j   0.007+0.009j  0.005-0.006j  0.005-0.004j  0.006+0.01j   0.006-0.01j   0.008-0.004j  0.006+0.01j   0.004-0.019j  0.002+0.001j  0.003+0.008
j  0.003-0.018j  0.005-0.016j  0.005+0.009j  0.005-0.018j  0.004-0.006j  0.004+0.004j  0.004-0.015j  0.005-0.019j  0.005-0.j     0.002-0.017j
  0.001-0.008j -0.    -0.015j -0.001+0.012j -0.    -0.012j  0.001-0.017j -0.002-0.013j -0.002+0.004j -0.003-0.017j -0.005-0.015j -0.005-0.015j -0.006-0.013
j -0.007+0.003j -0.007-0.02j  -0.007-0.016j -0.008-0.011j -0.008-0.019j -0.01 +0.002j -0.009-0.017j -0.007-0.017j -0.009+0.01j  -0.007-0.011j
 -0.008+0.008j -0.008-0.017j -0.008+0.011j -0.008-0.013j -0.006-0.001j -0.006-0.014j -0.006+0.009j -0.007-0.011j -0.007+0.005j -0.004-0.015j -0.008+0.j
-0.006+0.002j -0.007-0.008j -0.005-0.005j -0.005-0.012j -0.005-0.j    -0.004+0.002j -0.005-0.009j -0.004+0.005j -0.005+0.002j -0.004-0.003j
 -0.005+0.007j -0.003+0.014j -0.004-0.005j -0.002+0.011j -0.001+0.005j -0.001+0.003j -0.003+0.002j -0.003+0.008j -0.    +0.004j  0.002+0.004j  0.002+0.002
j  0.001+0.009j -0.001+0.j    -0.001-0.002j  0.001+0.004j  0.002+0.002j -0.    +0.001j  0.    +0.008j  0.002+0.007j -0.002+0.005j -0.002+0.005j
 -0.002+0.007j -0.004+0.007j -0.003+0.006j -0.001+0.005j -0.001+0.009j  0.    +0.012j -0.    +0.01j   0.    +0.008j -0.001+0.02j ]

max abs diff =  0.04126956850298427



complex128
```



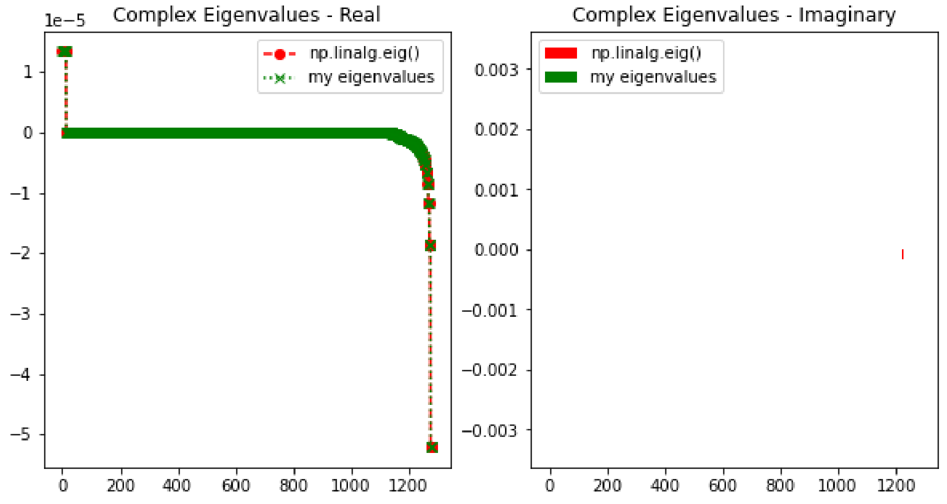
Observation

- QR iteration with Wilkinson's shift seems to be working for complex symmetric matrix
- Slightly better convergence than the general method without shift
- Max iteration increased to 1e4

In [40]:

```python
MAX_ITER = 100000

# Complex unsymmetric
source = 'matricmarket/mhd1280a.mtx.gz'
matrix_class = Matrix
runtest_matrixmarket(source, matrix_class,shift=None)

```

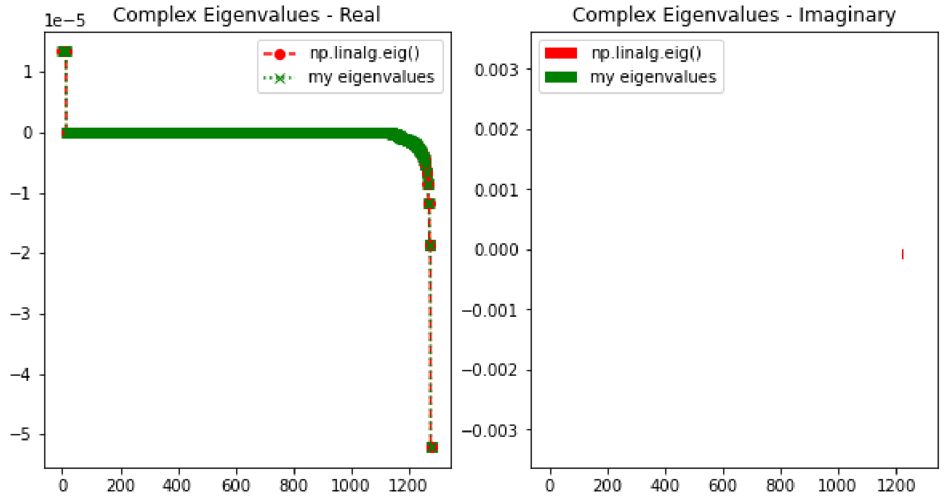
```
From Matrix Market:
matrix source=matricmarket/mhd1280a.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=(1280, 1280), data type=complex128, max=(74619.3808+
3.84011238e-11j), min=(-15906.7974-2.73633283e-07j)
is_symmetric={} False
activiating tridigonal: False

 ------------ Processing eigenvalues only ------------
>>> QR Iteration WITHOUT shift
Iteration terminates at i=33607 with tol=1e-06 reached

 ------------ Checking Eigenvalues ------------

my eigenvalues: [ 0.-0.j     0.-0.j     0.-0.j    ... -0.-0.003j -0.-0.003j -0.+0.003j]
np.linalg.eig:  [ 0.+0.j     0.+0.j     0.+0.j    ... -0.-0.003j -0.-0.003j -0.+0.003j]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [0.-0.j 0.-0.j 0.-0.j ... 0.+0.j 0.+0.j 0.-0.j]

max abs diff =  0.00017325703855534271



complex128
```


Complex Eigenvalues - Real
1e−5
np.linalg.eig()
my eigenvalues
Complex Eigenvalues - Imaginary
np.linalg.eig()
my eigenvalues

In [41]:

```
MAX_ITER = 100000

# Complex unsymmetric
source = 'matricmarket/mhd1280a.mtx.gz'
matrix_class = Matrix
runtest_matrixmarket(source, matrix_class,shift='Wilkinson')

```

```
From Matrix Market:
matrix source=matricmarket/mhd1280a.mtx.gz, matrix type=<class 'scipy.sparse.coo.coo_matrix'>, shape=(1280, 1280), data type=complex128, max=(74619.3808+3.84011238e-11j), min=(-15906.7974-2.73633283e-07j)
is_symmetric={} False
activiating tridigonal: False

 ------------ Processing eigenvalues only ------------
>>> QR Iteration with Wilkinson shift
Iteration terminates at i=13234 with tol=1e-06 reached

 ------------ Checking Eigenvalues ------------

my eigenvalues: [ 0.-0.j     0.-0.j     0.+0.j    ... -0.-0.003j -0.-0.003j -0.+0.003j]
np.linalg.eig:  [ 0.+0.j     0.+0.j     0.+0.j    ... -0.-0.003j -0.-0.003j -0.+0.003j]

np.linalg.eig and my eigenvalues close? False

my eigenvalues - np.linalg.eig =
 [ 0.-0.j  0.-0.j  0.+0.j ...  0.+0.j -0.+0.j -0.-0.j]

max abs diff =  0.00015237568085155927



complex128
```



Observation

- Increased max iteration number to 1e5
- General QR iteration without shift seems to be working for complex unsymmetric matrix
- This demonstrates that QR iteration with Wilkinson shift indeed converges faster

In [ ]:

```
1
```

Checklist / To-Dos:

Option 1

General complex matrices with transformation to Hessenberg form

- Hessenberg form via Housholder Tranformation ***[Done]***
- QR Iteration with Givens Rotation ***[Done]***
- Test for convergence ***[Done]***
- Modify Householder to accommodate for complex matrices ***[Done]***
- Modify Givens to accommodate for complex matrices ***[Done]***
- Diagonal blocks for real matrix with complex conjugate eigenvalues ***[Done]***
- Use Wilkinson shifts to accelerate convergence ***[Done]***
- Investigate convergence - comparison visualization ***[Done]***

- Compute not only eigenvalues but also eigenvectors ***[Done]***
- Apply your program to several suitable matrices from the Matrix Market. Use suitable library functions for reading the matrices. ***[Done]***
- Object-oriented programming: is it possible to implement algorithm 1.4.14 in a way, that H is either Hessenberg or tridiagonal, either real or complex, but encapsulate the differences inside the QR-decompositions applied in every step _ element-wise application (special case of tridiagonal) ***[Done]***

Option 2

General real matrices with transformation to Hessenberg form and double shifts for complex eigenvalues

- Use double shifts to avoid complex arithmetic _ Matrix multiplication ***[Done]***
- "Bulge-chasing" for element-wise computation **#NOT IMPLEMENTED**

Option 3

Symmetric real matrices with transformation to tridiagonal form with optimized storage

- Use deflation when the subdiagonal element in the last row is sufficiently small (implementation may be hard if not option 3)
- Use deflation**, if any other subdiagonal element is small (seems very hard)
- Add SVD capability to option 3

Additional Considerations

- Well-chosen tests for correctness
- Investigation into convergence
- Write very well structured code
- Well-prepared jupyter notebooks with code, accompanying text, and results

---

In [ ]:

```
1
```