CPSC 320 2020S2: Tutorial 10

1 Puzzling Parades

On the last day of summer the city of Vancouver will close a section of downtown to vehicle traffic to allow for several community to groups to host various outdoor activities. Each group submits a sequence of streets that they wish to reserve for the occasion. However from past experiences of multiple groups *passing the buck* on clean-up duties, the city has learned that no two groups can be both granted their requested streets if the requests share an intersection. Given a set of requests, the city is interested maximizing the number of requests they can accept, subject to the above constraint.

An instance of Puzzling Parade consists of a directed graph $G = (V, E)$ (where edges are city streets and vertices are intersections), a set of simple paths $P_1, P_2, \cdots P_d$ of G (requested by d community groups), and a positive integer k. The problem asks: Are there at least k paths that share no node (intersection)?

1. Show that Puzzling Parade is in NP.

2 Reductions: Vertex Cover and Dominating Sets

Your task is to find a reduction from Vertex Cover to Dominating Sets, two famous graph problems:

- *Vertex Cover:* An instance is a graph $G = (V, E)$ and an integer K. The problem asks: Is there a vertex cover with at most K vertices in G? Here, a vertex cover is a subset W of V such that $|W| \leq K$, such that every edge in E has at least one endpoint in W.
- *Dominating Set:* An instance is a graph $G = (V, E)$ and an integer K. The problem asks: Is there a dominating set with at most K vertices in G? Here, a dominating set is a subset W of V such that $|W| \leq K$, such that every element of $V - W$ is joined by an edge to an element of W.

1. Give a reduction from Vertex Cover to Dominating Set. Explain why your reduction is correct, and runs in polynomial time.
2. Suppose someone tells you that they have an $O(n^6)$ algorithm to solve the Dominating Set problem on a graph with n vertices. What can you say about the time complexity $T(n, m)$ of the Vertex Cover problem on graphs with n nodes and m edges? Choose all answers that apply.

○ $T(n, m) \in O((n + m)^2)$	● $T(n, m) \in O((n + m)^6)$
○ $T(n, m) \in \Omega((n + m)^2)$	○ $T(n, m) \in \Omega((n + m)^6)$

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1. The algorithm takes as input an instance I.
In this case, a potential solution is a subset S of $\{1, \dots, d\}$ where $|S| \geq k$, and S is good if for every pair of distinct indices $i, j \in S$, the paths P_i and P_j don't share a node.
We traverse each ~~node~~ path P_i, for $i \in S$, marking nodes that are visited. If we visit a node that is already marked, there is a common intersection and we reject it. Otherwise, once all paths are traversed, if no marked node is visited twice we accept, since the solution S is good.
At each vertex is only marked once, the runtime of this algorithm is $O(n)$. (n is # of nodes of the graph).

2. Given an instance (G, k) of Vertex Cover, construct an instance (G', k) of Dominating Set by adding one node v_e for each edge $e \in E$, and connecting this node to the endpoints of e.
If $e = \{x, y\}$ then we add the edges $\{v_e, x\}$ and $\{v_e, y\}$. k remains the same.
Time to generate new nodes and edges is $O(m)$.

Suppose that G has a vertex cover W of size at most k. Since W is a vertex cover of G. ~~[illegible]~~
all nodes in $V-W$ are joined by an edge to the nodes of W and all nodes v_e are also joined by an edge to the nodes of W. So the set W is a dominating set of G'.
Conversely, let W be a dominating set of G'. Let W' be obtained by W by replacing any v_e node in W by one of e's endpoints. Then the node of W' are all nodes of G. and have an edge to all the v_e nodes. This means that every edge of G has at least one endpoint in W, and so W must be a vertex cover of G.

$T(n, m) \in O((n+m)^6)$ Because graph G' has $(n+m)$ nodes.